The Point-Vortex Method for Periodic Weak Solutions of the 2-D Euler Equations

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Abstract

Almost-sure convergence of a subsequence of the vorticity to a weak solution is proven for the point-vortex method for 2-D, inviscid, incompressible fluid flow. Here "almost-sure" is with respect to sequences of random components included in the initial position and strength of each vortex. The initial vorticity is assumed to be periodic and, depending on the initialization scheme, to lie in $L \log L$ or L^p with $p > 2$. The randomization of the initial data is not needed when the initial vorticity is nonnegative; such initial data also need not be periodic, and is only required to be a bounded measure lying in H^{-1}.

All these results are also valid for the "vortex-blob" method with the smoothing parameter vanishing at an arbitrary rate.

The sense in which solutions of point-vortex dynamics are weak solutions of the Euler equations is also discussed.

1. Introduction

Since the vorticity of an incompressible, inviscid, two-dimensional fluid is simply convected by the velocity field it creates, the fluid flow may be viewed as a continuum superposition of moving point vortices. The dynamics of a finite number of point vortices can therefore be seen both as an idealization of fluid dynamics and as a method for approximating it. This viewpoint dates back at least to [39] and [45], who used this point-vortex method to investigate the roll-up of a vortex sheet. But larger-scale, more accurate calculations of vortex sheets made possible by the advent of computers showed chaotic behavior of the vortices instead of the theoretically and experimentally predicted roll-up, so the point-vortex method fell into disrepute ([4], [15], [37], and references therein). Various methods for numerically stabilizing the method were proposed ([12], [37]), of which the vortex-blob method ([13], [15]) has been the most widely studied (e.g., [1], [29]). In this method, the singular kernel of the Biot-Savart law determining the velocity in terms of the vorticity is smoothed; this is equivalent to replacing the point vortices by small, rigid regions ("blobs") of nonzero vorticity ([13]). Although the first applications of vortex methods were to vortex-sheet calculations, they have since been used to approximate a wide variety of flows (see, e.g., [43]). A justification of the point-vortex method will be given here for various classes of nonsmooth initial data, including that used in [39] and [45].

The first convergence proofs for a vortex method covered the vortex-blob method for smooth flows (the review article [30] contains numerous references). Convergence is obtained by estimating the difference between the exact and ap-

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proximate particle trajectories and velocities. The same approach was recently used to rehabilitate the point-vortex method, at least in the context of smooth flows, by showing that it, too, converges for such flows [27]. Such a direct approach is limited to cases in which the uniqueness of the exact solution has been demonstrated; so far, this requires that the initial vorticity be in L^∞ ([46]) or be an analytic, nearly horizontal, and nearly uniform vortex sheet ([44], [8]). This approach has indeed been extended to these cases: [11] demonstrates the convergence of the vortex-blob method for bounded vorticity, and [7] and [31] prove convergence for sufficiently small times of both the vortex-blob and point-vortex methods for such vortex sheets. However, because of the smallness restrictions, these last results may not apply to calculations of vortex-sheet roll-up.

For more general initial data with unbounded vorticity, a different method is needed. A general approach for proving the existence of solutions to nonlinear equations and/or the convergence of a particular method of approximation is the following: Let $N(u) = 0$ denote the equation(s) to be solved, where $N(u)$ may be a weak formulation and includes the initial conditions. A sequence of approximations u_ε satisfy $N(u_\varepsilon) = E_\varepsilon$. We first show "stability," that is, that the u_ε are uniformly bounded in some norm strong enough to imply compactness in some space so that a subsequence of the u_ε converges in some (possibly weak) sense to some u. Restricting attention to such a subsequence and using this bound and convergence (and possibly additional estimates), we then show "nonlinear continuity" (i.e., that $N(u_\varepsilon)$ converges to $N(u)$) and "consistency" (i.e., that E_ε tends to 0). Taking the limit along the subsequence of the equation $N(u_\varepsilon) = E_\varepsilon$ then yields the desired equation $N(u) = 0$. Note that the terms *stability* and *consistency* are used here in a slightly different sense than in the direct approach. Thus, the truncation error in the consistency step here is the amount by which the approximate solutions fail to satisfy the exact equation, instead of the amount by which exact solutions fail to satisfy the approximate equation.

The first application of this approach to the 2-D incompressible Euler equations with unbounded initial vorticity used the approximations obtained by smoothing the initial data or adding viscosity ([20]). Stability is obtained by noting that these approximations do not increase L^p-type norms of the vorticity and have time derivatives uniformly bounded in some negative Sobolev space; consistency then follows easily. Furthermore, elliptic estimates show that weak-$*$ convergence of the vorticity in both the space $\mathcal{M}$ of measures and L^p with $p > 1$ implies strong convergence of the velocity in L^2_{loc}, which yields nonlinear continuity. Technical refinements allow replacing L^p by any norm stronger than $L(\log L)^{1/2}$, so that the initial data need only be in L^1 and also finite in such a norm ([9]).

The main difficulty in extending these results to vortex methods is the loss of stability: L^p norms of the vorticity are no longer preserved. Of course, even a uniform L^p-bound growing in time would suffice to show stability, and subsequently also convergence. [3] demonstrates such a bound for the vortex-blob method when the initial vorticity lies in L^p with $p > 1$, under a very strong restriction on the smoothing parameter ε that is proportional to the diameter of the vortex blobs. Letting N denote the number of vortex blobs, it is required that

$\varepsilon \geqq \frac{c}{\sqrt{\log N}}$ instead of an algebraic rate of decay as in other vortex-blob results. Although practical numerical issues will not be addressed in this paper, it should be noted that since smoothing introduces a consistency error, (strong) restrictions on ε are undesirable. Although it is not apparent how to improve the worst-case analysis of [3], one can hope that such bad cases are rare in some sense and then seek a probabilistic bound.

Such a probabilistic vorticity bound was recently obtained by [6] for a modified vortex-blob method for periodic flows in which ε is allowed to decay at an algebraic rate in $1/N$. The modifications are that the approximate velocity field is frozen over periods of duration Δt and that a random "rezoning"—that is, a new selection of the vortex-blob locations by sampling of the vorticity function—is made in each time period. It is shown that each time step is equivalent to sampling the solution of a linearized vorticity equation. That equation preserves the L^p norms of the vorticity, while Cramér's theorem on large deviations implies that sampling is very unlikely to increase that norm significantly. The Borel-Cantelli lemma then implies that, for almost all sequences of samplings, that norm does not increase significantly up to any given time T for sufficiently large N. When this almost-sure stability holds and the initial data lie in L^p with $p > 4/3$, then for an appropriate choice of the relative sizes of Δt and ε, the consistency errors tend to 0, yielding convergence of a subsequence to a solution of the Euler equations [6]. Furthermore, for a different choice of the relative size of those parameters, a subsequence tends to a solution of the Navier-Stokes equations [6]; as noted in [6], the above-mentioned techniques from probability theory were used earlier ([26]) in the proof of convergence for smooth solutions of the "random vortex" method for the Navier-Stokes equations, in which a random walk is used to simulate diffusion.

Meanwhile, [17] proved a nonlinear continuity theorem for the 2-D Euler equations under weaker hypotheses: It suffices that the total mass of the vorticity considered as a measure be uniformly bounded, that the time derivative be uniformly bounded in some negative Sobolev space, and that no vorticity "concentration" occur, that is, that the maximum total mass of the vorticity approximants in balls of radius r tends uniformly to 0 with r. The sense in which continuity holds depends on the available velocity bounds: If the approximate velocities are uniformly bounded in L^2_{loc}, then a velocity subsequence converges weakly to a solution of the usual weak velocity formulation of the Euler equations; even when this condition does not hold, a vorticity subsequence converges weak-$*$ to a solution of Delort's new weak vorticity formulation. The two formulations are equivalent whenever both are defined, so that the limit velocity satisfies the weak velocity formulation whenever it lies in L^2_{loc}.

Furthermore, by using the approximations obtained by smoothing the initial data or the Navier-Stokes equations, this theorem can be used to prove the existence of solutions when the initial vorticity is a nonnegative Radon measure of finite mass and compact support that belongs to the negative Sobolev space H^{-1} ([17]): Conservation of the L^1-norm of the vorticity approximants yields the desired uniform vorticity bound, while an L^2 velocity estimate yields the velocity bound, and for nonnegative vorticity the latter bound can be shown to imply that no vorticity concentration occurs. [33] then used this continuity theorem to show convergence of the vortex-blob method under the same conditions on the initial data. Conservation of the vortex Hamiltonian, which is formally a discrete version of the L^2-norm of the velocity, replaces the L^2 velocity estimate and so allows the continuity theorem to be applied; the consistency error is then estimated in the usual weak velocity formulation. The smoothing parameter is allowed to vanish at the algebraic rate $\frac{c}{\sqrt{N}}$.

Convergence results will be proven here for several versions of the point-vortex method under varying hypotheses on the initial vorticity ω_0. All these results also cover the corresponding versions of the vortex-blob method with the smoothing parameter vanishing at an arbitrary rate. The only modification made is randomizing the scheme for choosing the vortex strengths and initial positions.

There are two well-known schemes for choosing the initial vortex strengths and locations for vortex methods. The first scheme places vortices at the centers x_j of the cells of a uniform grid, with strengths equal to the area of the cell times $\omega_0(x_j)$ or times the value at x_j of a mollification of ω_0; a special case of the latter possibility is to take the strength to be the total vorticity in the cell. In the second scheme, the vortex positions x_j are chosen randomly according to the uniform distribution on the period cell Ω of the initial vorticity (or on a set Ω containing the support of the initial vorticity when it has compact support), and the strength of the vortex at x_j is $\frac{\mu(\Omega)}{N}\omega_0(x_j)$, where μ denotes two-dimensional Lebesgue measure; a mollification of ω_0 is sometimes used here, too. Random versions of both of these schemes will be treated here: Small random offsets will be added to the strengths of the vortices, and in the first scheme small random offsets will be added to their initial locations as well. Furthermore, for the first scheme a mollification of ω_0 must be used; its necessity is clear from the fact that the initial vorticity under consideration is only defined almost everywhere. It will be shown that when the initial vorticity is periodic and in L^1, then for almost every sequence of offsets, some subsequence of the approximate vorticities converges weak-$*$ to a solution ω of the weak vorticity formulation; if in addition the initial vorticity lies in L^p with $p > 2$, then the velocity corresponding to ω is in L^2 and hence is a solution of the weak velocity formulation.

We will also consider a scheme in which all vortices of a given sign have the same size, for which the proofs are somewhat simpler. For this scheme it will be shown that if the initial vorticity is periodic and lies in $L \log L$, then for almost every sequence of offsets some subsequence of the approximate vorticities converges weak-$*$ to an ω such that the corresponding velocity satisfies the weak velocity formulation.

When the initial vorticity ω_0 is a nonnegative measure of finite mass and compact support lying in H^{-1}, convergence of a vorticity subsequence to a solution is obtained for all nonnegative sequences of initial vorticities having uniformly compact support, tending weak-$*$ to ω_0, and such that the point-vortex Hamiltonian is uniformly bounded. An appropriate version of the first initialization scheme satisfies these conditions. Also, there is no difficulty in treating the periodic case as well. That the vorticity in the calculations of [39] and [45] was of a single sign shows that the method used was in fact convergent in this sense. This result can be reconciled with the chaotic behavior observed in later calculations by noting that, even restricted to a convergent subsequence, the result here does not imply convergence of the contour joining the vortices, which was the criterion by which those calculations were judged.

Although the convergence results for the vortex-blob and modified vortex-blob methods cited above include the L^2_{loc} convergence of the approximate velocities, such convergence cannot hold for the point-vortex method since its approximate velocities do not lie in that space. We will, however, obtain such convergence almost-surely for smoothed velocities, with the smoothing parameter vanishing at the algebraic rate $\varepsilon = c\frac{(\log N)^{3/2}}{\sqrt{N}}$ for those cases in which the limit velocity is shown to be almost-surely in L^2_{loc}. As in [17] and [33], the case of nonnegative initial vorticity that is a bounded measure lying in H^{-1} is exceptional in that only weak L^2_{loc} convergence will be obtained; however, in this case it will be shown here that it suffices for ε to vanish at an arbitrarily fast algebraic rate $\frac{c}{N^k}$. In particular, we obtain almost-sure L^2_{loc} velocity convergence for the vortex-blob method with the smoothing parameter vanishing at an algebraic rate, since that method is obtained from the point-vortex method simply by smoothing the velocity. In general, however, one can distinguish between the velocity smoothing needed to obtain L^2_{loc} convergence from the smoothing actually used in the numerical method. The results here show that no smoothing is needed for stability of the numerical method; if L^2_{loc} velocity convergence is desired it can be obtained via postprocessing.

These results will be proven using the general approach described above. Since the total vorticity is conserved by vortex methods and the time derivative is uniformly bounded in some negative Sobolev space, the weak-$*$ convergence of a vorticity subsequence in the space of functions bounded in time with values in the space of measures is assured. A variant of Delort's theorem ([42], lemma 3.7) will be used to obtain nonlinear continuity, and Delort's weak vorticity formulation will be used to show consistency. As explained in Section 2, solutions of point-vortex dynamics are exact solutions of that formulation; this is not the case for the weak velocity formulation on account of the omission of infinite self-interaction terms ([28]). Thus, consistency holds trivially for the point-vortex method just as it does for the approximation by smooth solutions of the Euler equations. Furthermore, the use of this formulation simplifies considerably the proof of consistency of the vortex-blob method.

In order to apply Delort's theorem, we also need to prove that concentration does not occur, and the bulk of this paper is devoted to demonstrating this. For the approximation by smooth solutions, this absence of concentration holds when the initial vorticity lies merely in L^1 [18] because smooth Euler flows preserve two-dimensional Lebesgue measure. The analogous property for point-vortex dynamics is the conservation of the $2N$-dimensional phase-space volume of the flow

(Liouville's theorem). Some interesting consequences of this conservation were noted already in [5]. The fact that conservation of phase-space volume is used in the convergence proof suggests that a symplectic method (see [10], [25], and [41]) be used in numerical calculations and convergence proofs for time discretizations of the point-vortex method applied to weak solutions.

Phase-space volume conservation does not actually forbid vorticity concentration but does imply that it is a rare event: If some fraction μ of N vortices of given strengths congregate at a fixed time t in regions whose diameters are $o(1)$ as $N \to \infty$, then the vortices occupy a $2\mu N$-dimensional phase-space volume $o(1)^{\mu N}$. (Since concentration occurs whenever the diameter of the region of congregation tends to 0 with N, no rate can be imposed on the $o(1)$ size of that region.) Since the flow is periodic, the remaining vortices lie in a $2(1-\mu)N$-dimensional region of volume at most $c^{2(1-\mu)N}$, so that the $2N$-dimensional phase-space volume in which concentration of μN vortices occurs is $o(1)^{\mu N}$. By considering congregation of smaller and smaller fractions of the vortices in suitably chosen smaller and smaller regions, one can show that the phase-space volume in which any concentration occurs is $o(1)^N$. By the preservation of the $2N$-dimensional phase-space volume, the set of initial positions occupied by vortices that concentrate at a fixed later time also occupies a volume $o(1)^N$; an extension of the argument shows that the same is true of the set of initial conditions that lead to concentration at a set of times having positive measure.

Hence, if the total phase-space volume of a set of initial conditions approximating a given initial vorticity is at least δ^N for some positive δ, then congregation of μN vortices within a ball of fixed, sufficiently small diameter $r(\delta, \mu)$ will become exceedingly rare as $N \to \infty$. In fact, the Borel-Cantelli lemma then implies that for almost all sequences of initial data such congregation does not occur infinitely often, which of course implies that concentration does not occur.

For what initial data can this argument be used? The study of sets having measure at most δ^N with $\delta < 1$ is known as "large-deviations theory" in probability. That theory has previously been used in the study of vortex methods in [6] and [26] in order to obtain sharp upper bounds on certain probabilities; its use here is essential since we will need to show that lower bounds for certain probabilities exceed upper bounds for others. Furthermore, the results obtained here for the velocity depend critically on the form of the rate function governing large deviations of the empirical law of independent, identically distributed random variables.

For the case in which all vortices of the same sign have the same strength, large-deviations theory says, roughly speaking, that the set of point-vortex approximations to ω_0 has phase volume $e^{-N\iint |\omega_0| \log |\omega_0|}$. A direct calculation in Section 4, undertaken to obtain a sharp bound on the subexponential factor, yields the same conclusion. (Similar calculations occur in the study of statistical equilibrium states of the Euler flow [14].) Thus, the initial data must lie in $L \log L$. In particular, vortex sheets are not allowed; this can also be seen directly by noting that the phase volume of initial conditions approximating a vortex sheet must be $o(1)^N$ just like the phase volume of a concentration, since in both cases $O(N)$ vortices lie in a region of area $o(1)$.

Furthermore, large-deviations theory also says roughly that the set of all approximants to the set of all ω such that $\iint |\omega| \log |\omega| > L$ has total phase volume e^{-LN}. This allows us to replace the old "badness criterion" that μN vortices congregate within a set of diameter $r(\delta, \mu)$ by the condition that an approximation to $\iint |\omega| \log |\omega|$ derived from the vortices be larger than some large but fixed L (depending on $\iint |\omega_0| \log |\omega_0|$). The Borel-Cantelli lemma then shows that for almost all sequences of initial approximants the approximation to $\iint |\omega| \log |\omega|$ exceeds L only for finitely many N. This implies that no concentration occurs and furthermore shows that the limit vorticity ω satisfies $\iint |\omega| \log |\omega| \leqq L$ at almost all times. This bound in turn implies that the limit velocity u lies in L^2_{loc} and so satisfies the weak velocity formulation, since $\|u\|_{L^2_{\text{loc}}}$ can be bounded in terms of the total mass of ω as a measure together with $\iint |\omega| (\log |\omega|)^\alpha$ for $\alpha > \frac{1}{2}$ ([9] and Section 5).

Since the second large-deviations result shows that the total phase volume of approximants to all ω_0 not lying in $L \log L$ is $o(1)^N$, the above method fails even to show that there exists some such ω_0 for which at least one sequence of approximants does not concentrate. Nevertheless, since the point-vortex flow preserves the value of the Hamiltonian $H \equiv \frac{1}{4\pi} \sum_{j \neq k} \omega_j \omega_k \log(\frac{1}{|x_j - x_k|})$, and "most" point-vortex initial data approximating ω_0 have values of H close to $\mathcal{H}(\omega_0) \equiv \frac{1}{4\pi} \iint\iint \omega_0(x) \omega_0(y) \log(\frac{1}{|x-y|})$, one might hope to preclude concentration for some such ω_0 by restricting to the region of phase space having H close to $\mathcal{H}(\omega_0)$. Clearly this will work only if nearly all configurations involving congregation have values of H far from $\mathcal{H}(\omega_0)$. The one case in which this strategy works is when ω_0 is nonnegative (or nonpositive), since whenever vortices all of the same sign concentrate then H tends to $+\infty$. Hence no concentration can ever occur in this case, so the phase-space volume argument is not even needed; any set of initial vortices approximating ω_0 for which H approximates $\mathcal{H}(\omega_0)$ may be used, and the proof is much simpler. This point-vortex convergence result for nonnegative initial vorticity emphasizes the fact that the same mechanism underlies Delort's [17] existence result for weak solutions having nonnegative initial vorticity and the global existence result for point-vortex dynamics with nonnegative vorticities (e.g., [36], p. 139): The conservation of $\mathcal{H}$ or its formal equivalent $\frac{1}{2} \iint v^2$ and of its discretization H prevents vorticity concentration. In fact, the absence of vorticity concentration for nonnegative vortex sheets was noted already in [4] based on this analogy.

The weak vorticity formulation, vortex methods, and Delort's theorem will be reviewed in Section 2, where it will also be shown that (1) the point-vortex method is always consistent and (2) the vortex-blob method is consistent as long as the nonconcentration condition holds and the smoothing parameter tends to 0 (at an arbitrary rate). The convergence theorem for nonnegative vorticity will then be proven in Section 3. The initial-volume estimates and concentration-volume estimates will be presented in Sections 4 and 5, respectively. Using these, the convergence theorems for periodic initial vorticity will be given in Section 6.

In the final section, a remark will be made about the possibility of extending some of the results to the case of compactly supported initial vorticity. Also, the lemmas needed to prove the theorem will be phrased so as to include the compactly supported case; for that reason, the velocity will be said to lie in L^2_{loc} even though that space is just $L^2(\Omega)$ in the periodic case.

2. The Vorticity Formulation and Point Vortices

The incompressible Euler equations $u_t + (u \cdot \nabla)u + \nabla p = 0$, $\nabla \cdot u = 0$, may be written in weak velocity form as ([20])

$$\int \iint \{\phi_t \cdot u + u \cdot (u \cdot \nabla)\phi\}\, dx\, dt = 0, \tag{2.1}$$

$$\int \iint u \cdot \nabla \eta \, dx\, dt = 0, \tag{2.2}$$

for all test functions ϕ and η such that $\nabla \cdot \phi = 0$. For smooth solutions in two space dimensions decaying at infinity, the equations may also be written in terms of the vorticity $\omega \equiv \nabla^{\perp} \cdot u \equiv (-\partial_2, \partial_1) \cdot u$ as

$$\omega_t + (u \cdot \nabla)\omega = 0, \tag{2.3}$$

where ω determines the velocity u via the Biot-Savart law

$$\begin{aligned} u = K * \omega = \iint K(x-y)\omega(y) &= \iint \frac{1}{2\pi} \nabla^{\perp} \left[\log(|x-y|)\right] \\ &= \iint \frac{1}{2\pi} \frac{(x-y)^{\perp}}{|x-y|^2} \omega(y). \end{aligned} \tag{2.4}$$

Writing (2.3) in weak form, eliminating u in favor of ω via (2.4), and symmetrizing the kernel of the integral quadratic in ω yields [42]

$$\begin{aligned} &W(\omega; \psi) \\ &\equiv \int \iint \psi_t(t,x)\omega_{(t)}(x)\, dt \\ &+ \int \iint \iint \frac{1}{4\pi} \frac{(x-y)^{\perp} \cdot (\nabla\psi(t,x) - \nabla\psi(t,y))}{|x-y|^2} \omega_{(t)}(x)\omega_{(t)}(y)\, dt \\ &= 0, \end{aligned} \tag{2.5}$$

where the vorticity is written as $\omega_{(t)}(x)$ to indicate that for fixed t it is considered to be a measure in the spatial variables.

As first proven in [17] without recourse to the explicit formula, for every C_0^∞ function ψ, the kernel

$$H_\psi(t,x,y) \equiv \tfrac{1}{2} K(x-y) \cdot (\nabla\psi(t,x) - \nabla\psi(t,y)) \tag{2.6}$$

appearing in the weak vorticity formulation (2.5) is bounded, is continuous for $x \neq y$, and tends to 0 at infinity; these properties also follow easily from formula (2.6) ([42]). Using these properties, it is shown in (1.2.20) of [17] that $\iint \iint H_\psi(t,x,y)\mu(x)\nu(y)$ is well-defined as long as at least one of the measures μ or ν is a continuous measure, that is, assigns no mass to single points. However, if both include delta functions, then that integral is not defined because $H_\psi(t,x,y)$ is not defined when $x = y$. Nevertheless, this difficulty can be overcome simply by assigning a value to H_ψ on that diagonal. For the purposes of this paper, let us define H_ψ to equal 0 on the diagonal or, equivalently, restrict the domain of integration to the complement of the diagonal. The contribution to (2.5) from the discrete part of ω is now well-defined since H_ψ is defined pointwise throughout the domain of integration and bounded. Since (1.2.20) of [17] shows that when at least one measure is continuous then $\mu(x)\nu(y)$ assigns no mass to the diagonal, our modification makes no difference for vorticities containing no delta functions. In particular, smooth solutions of the Euler equations still satisfy (2.5).

What about solutions of point-vortex dynamics? The strengths ω_j of the vortices do not change, while their positions x_j satisfy

$$\frac{dx_j}{dt} = \sum_{k\neq j} \omega_k K(x_j - x_k). \tag{2.7}$$

Thus, the measure $\omega_{(t)}(x) \equiv \sum_j \omega_j \delta(x - x_j(t))$ satisfies

$$\begin{aligned}
\int \iint \psi_t(t,x)\omega_{(t)}(x)\,dt &= \int \sum_j \omega_j \psi_t(t,x_j(t))\,dt \\
&= \int \frac{d}{dt}\sum_j \omega_j \psi(t,x_j(t))\,dt \\
&\quad - \int \sum_j \omega_j \nabla\psi(t,x_j)\cdot \frac{dx_j}{dt}\,dt \\
&= 0 - \int \sum_j \omega_j \nabla\psi(t,x_j)\cdot \sum_{k\neq j} \omega_k \frac{(x_k - x_j)^\perp}{2\pi|x_k - x_j|^2} \\
&= -\int \sum_{\substack{j,k \\ k\neq j}} \omega_j\omega_k H_\psi(t,x_j,x_k)\,dt \\
&= -\int \iint \iint H_\psi(t,x,y)\omega_{(t)}(x)\omega_{(t)}(y)\,dt
\end{aligned}$$

where the final equality depends on the above extension of the definition of H_ψ to equal 0 on the diagonal. Thus, under that extended definition of H_ψ the solutions of point-vortex dynamics are exact solutions of (2.5).

Although extending the definition of H may seem artificial, it is done in a uniform way for all solutions. Note also that [35], theorem 2.1, showed that as long as the point vortices don't collide, the solutions of point-vortex dynamics are the weak-$*$ limits of classical weak solutions in which the vorticity is nonzero in small regions ("patches"). Furthermore, even though the self-interaction terms are not defined until the definition of H_ψ is extended, those terms are not infinite as in the weak velocity formulation [28] but are bounded by a constant (depending on ψ) times ω_j^2; in particular, for N point vortices of maximum strength $o(1)$ whose absolute strengths sum to $O(1)$, the sum of all such terms is $o(1)$ as N tends to infinity. Thus, extending the definition of H_ψ is a convenience, not a necessity.

Furthermore, considering point vortices to be solutions of the weak vorticity formulation allows us to extend their dynamics beyond collisions simply by merging vortices that collide into a single vortex whose strength is the algebraic sum of the colliding vortices. Clearly this defines a solution of (2.5) for times less than and for times greater than the collision time, and the resulting vorticity ω is continuous in time in the weak-$*$ topology of measures, so that there is no contribution to the integrals in (2.5) from the "jump" at the collision time. Of course, this extended notion of point-vortex dynamics is horribly nonunique since the time-reversibility of the Euler equations implies that a single vortex can split equally well into several vortices at any time. Fortunately, uniqueness is easily restored by imposing the "entropy" condition that the total mass of the vortices be nonincreasing in time, since vortex collisions, and hence also vortex splitting, must involve vortices of opposite signs. Thus, any splitting of one vortex into several would increase the sums of the absolute values of the vortex strengths, that is, the total mass of the measure ω, in violation of the above entropy condition.

Although point-vortex collisions can therefore be permitted, we will use the the fact that they almost never occur [36] to allow us to avoid them in the point-vortex method, since such collisions would complicate the use of the preservation of phase-space volume.

In the periodic case, all the above remains true as long as the kernel K in (2.4) is replaced by the kernel giving the velocity in terms of the vorticity in the periodic case, and the definition of H_ψ is modified accordingly.

The kernel for the periodic case can be written as

$$\sum_{m,n\in\mathbb{Z}} K\left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} m \\ n \end{pmatrix}\right) = \sum_{m,n\in\mathbb{Z}} \frac{(y-n,\ m-x)}{(x-m)^2+(y-n)^2}$$

provided that that sum converges. Let us show that in the sense of the limit as $R \to \infty$ of the sum of those terms with $n^2+m^2 \leqq R$ it does indeed converge. For that purpose it suffices to show that the sum can be rewritten as a sum of terms that are $O(\frac{1}{(m^2+n^2)^{3/2}})$. Since the two components of the sum transform into each other under the symmetry $x \leftrightarrow -y$, $m \leftrightarrow -n$, it suffices to show this for one of them.

First take the average of the terms (m,n) and $(-m,-n)$. A calculation shows that the first component of the sum becomes

$$\sum_{m,n\in\mathbb{Z}} \left[\frac{(m^2-n^2)y-2mnx}{(m^2+n^2)^2} + O\left(\frac{1}{(m^2+n^2)^{3/2}}\right)\right].$$

Taking the average of the terms (m, n) and $(n, -m)$ of the transformed sum reduces it to the desired form.

By making the transformation $(m, n) \to (-m, -n)$ together with $(x, y) \to (-x, -y)$, we see that the periodic kernel, which will still be denoted K, remains odd, so that the symmetrization step leading to (2.5) still yields that equation, and the kernel H_ψ can still be written in the form (2.6) using the K for the periodic case.

Define z_{per} to be $z - \binom{m}{n}$ where (m, n) is chosen so as to minimize the norm of the result, and let $|z|_{\text{per}} = \min_{m,n\in\mathbb{Z}} |z - (m, n)|$ be that minimum. Then the kernel $K(x - y)$ for the periodic case is the whole-space K evaluated at $(x - y)_{\text{per}}$ plus a smooth function. Since the test function ψ is now periodic, $|\nabla\psi(x) - \nabla\psi(y)| \leqq \|\psi\|_{C^2} |x - y|_{\text{per}}$. Hence H_ψ is still bounded for all t, z, y and is still continuous for $|x - y|_{\text{per}} \geqq \delta > 0$.

The vortex-blob method is defined by replacing the kernel K in (2.7) by a mollified kernel $J_{\varepsilon(N)}K \equiv \phi_{\varepsilon(N)} * K$, where ϕ is a smooth function whose integral equals 1, $\phi_\varepsilon(x) = \frac{1}{\varepsilon^2}\phi(\frac{x}{\varepsilon})$, and $\varepsilon(N)$ tends to 0 as $N \to \infty$. We will still let ω_N denote the the weighted sum of point masses $\sum \omega_j \delta(x - x_j(t))$, so that the velocity is $\phi_\varepsilon * K * \omega_N$. Although the actual vorticity corresponding to this velocity is $\phi_{\varepsilon(N)} * \omega_N$, we will continue to refer to ω_N as the vorticity. This is just a convenient convention, since if we show that ω_N tends weak-$*$ to a solution ω, then so does $\phi_{\varepsilon(N)} * \omega_N$.

The mollified kernel $\phi_\varepsilon * K$ is just the "velocity" corresponding to the "vorticity" ϕ_ε. When ϕ is a radial function, then that velocity is given explicitly by $[\phi_\varepsilon * K](x) = \frac{x^\perp}{|x|^2} \iint_{|y|<|x|} \phi_\varepsilon(y)\,dy = \frac{x^\perp}{|x|^2} \iint_{|y|<\frac{|x|}{\varepsilon}} \phi(y)\,dy$ ([20], 1.14). Hence

$$\sup_{|x|\geqq\delta} |K(x) - [\phi_\varepsilon * K](x)| = \sup_{|x|\geqq\delta} \left| \frac{x^\perp}{|x|^2} \iint_{|y|>\frac{|x|}{\varepsilon}} \phi(y)\,dy \right| \leqq \frac{1}{\delta} \left| \iint_{|y|>\frac{\delta}{\varepsilon}} \phi(y)\,dy \right| .$$

This implies that there exists a $\delta(\varepsilon)$ tending to 0 with ε such that

$$\lim_{\varepsilon\to 0} \sup_{|x|\geqq\delta(\varepsilon)} |K(x) - [\phi_\varepsilon * K](x)| = 0, \tag{2.8}$$

which will be convenient in the proof of consistency of the method. Since the kernel in the periodic case is the convergent sum of translates of the kernel for the whole-space case, the same is true of the mollified kernel, and this implies that (2.8) still holds in the periodic case provided that we replace x by x_{per}.

We now turn to Delort's theorem. The variant (lemma 3.7 in [42]) that we need for the point-vortex method says that a sequence of approximate vorticities ω_N has a subsequence converging weak-$*$ in $L^\infty([0, T], \mathcal{M})$ to a solution of the weak vorticity formulation (2.5) provided that the following four conditions hold:

(1) The ω_N are uniformly bounded in $L^\infty([0, T], \mathcal{M})$ (stability, part 1).
(2) The ω_N are equicontinuous in some negative Sobolev space (stability, part 2).
(3) $W(\omega_N; \psi) \to 0$ for every test function ψ (consistency).

(4) For almost all $t \in [0,T]$,

$$\lim_{N\to\infty} \limsup_{r\to 0} \sup_{n>N} \sup_{x} \iint_{|y-x|<r} |\omega_N(t,y)| = 0 \tag{2.9}$$

(nonconcentration).

In particular, the theorem says that stability plus nonconcentration implies nonlinear continuity; that is, $W(\text{weak-}*\lim \omega_N;\psi) = \lim W(\omega_N;\psi)$.

Since a uniform L^p vorticity bound is no longer required, it is easy to prove the stability and consistency of vortex methods in this sense, although consistency for the vortex-blob method will only be obtained under the assumption that nonconcentration holds.

LEMMA 2.1. *Assume that the sum of the absolute strengths of the vortices is bounded uniformly in N at time* 0. *Then both the point-vortex and vortex-blob methods satisfy the above stability conditions* (1) *and* (2). *In addition, the point-vortex method satisfies the consistency condition* (3). *Furthermore, if the nonconcentration condition* (4) *holds, then the vortex-blob method using a radial mollifier with smoothing parameter $\varepsilon(N)$ tending to* 0 *also satisfies the consistency condition* (3).

Proof: The total mass of the vorticity produced by the point-vortex method equals its value at time 0 and hence satisfies (1). Since point-vortex dynamics yields exact solutions of the weak vorticity formulation, (3) certainly holds. Furthermore, the equation $W(\omega_N;\psi) = 0$ together with the uniform bound on the total vorticity implies that the time derivative of the vorticity is uniformly bounded in $W^{-2,1}$ (e.g., lemma 3.2 in [42]), which yields (2).

Since the strengths of the vortices remain fixed in the vortex-blob method too, we again obtain (1). Next, the vortex-blob vorticity ω_N satisfies

$$\int\iint \psi_t(t,x)\omega_N(t,x) + \int\iint\iint H_\psi^{\varepsilon(N)}(t,x,y)\omega_N(t,x)\omega_N(t,y)\,dt = 0,$$

where H_ψ^ε is defined similarly to H_ψ but the mollified kernel $[J_\varepsilon K](x-y)$ is used in place of K. This still shows that the time derivative of the vorticity is uniformly bounded in $W^{-2,1}$, again yielding (2).

Finally, although the T in Delort's theorem may be $+\infty$, for every test function ψ there is a τ such that $\psi = 0$ for $t > \tau$. Let $\delta(\varepsilon)$ be the function given in (2.8).

Then

$$
\begin{aligned}
|W(\omega_N)| &= \left| \int_0^T \iint \iint \left(H_\psi(t,x,y) - H_\psi^{\varepsilon(N)}(t,x,y)\right) \right. \\
&\qquad \left. \times\, \omega_N(t,x)\omega_N(t,y)\,dt \right| \\
&\leqq \left| \int_0^T \iint \iint_{|y-x|<\delta(\varepsilon(N))} \left(H_\psi(t,x,y) - H_\psi^{\varepsilon(N)}(t,x,y)\right) \right. \\
&\qquad \left. \times\, \omega_N(t,x)\omega_N(t,y)\,dt \right| \\
&\quad + \left| \int_0^T \iint \iint_{|y-x|\geqq\delta(\varepsilon(N))} \left(H_\psi(t,x,y) - H_\psi^{\varepsilon(N)}(t,x,y)\right) \right. \\
&\qquad \left. \times\, \omega_N(t,x)\omega_N(t,y)\,dt \right| \\
&\leqq cT \sup_{|z|\geqq\delta(\varepsilon(N))} |K(z) - [\phi_\varepsilon * K](z)|\, \|\omega_N\|^2_{L^\infty([0,T]\mathcal{M})} \\
&\quad + c\int_0^T \sup_x \iint_{|x-y|<\delta(\varepsilon(N))} |\omega_N(t,y)|\,dt\,.
\end{aligned}
\tag{2.10}
$$

By (2.8) plus the uniform boundedness of the total mass of ω_N, the first term on the rightmost side of (2.10) tends to 0 as $N \to \infty$. For each fixed t, assumption (2.9) implies that for arbitrary $\delta(N)$ tending to 0,

$$
\lim_{N\to\infty} \sup_x \iint_{|x-y|<\delta(N)} |\omega_N(t,y)| = 0\,,
$$

and by the bounded convergence theorem this implies that the second term on the rightmost side of (2.10) also tends to 0 as $N \to \infty$.

Since the initial vorticities we will consider are all bounded measures, any reasonable initialization scheme will yield vortices whose absolute strengths are uniformly summable. Thus, in order to show that both the point vortex and the vortex blob have a subsequence that converges weak-$*$ to a solution of the weak vorticity formulation, it suffices to show that the nonconcentration condition (2.9) holds for almost all t. Although this step, too, is easy for the case of nonnegative vorticity, it is much more difficult to verify this condition when the initial data is merely assumed to be in $L\log L$ or L^p, and most of the rest of this paper is devoted to that task. First, however, we will do the nonnegative case.

3. Nonnegative Vorticity

When the vorticity is nonnegative, then the velocity decays like $\frac{1}{|x|}$ at infinity, and so cannot lie in L^2. It may, however, lie in L^2_{loc}, and we will start by relating this condition to two others; parts of this lemma appear in [17], [33], and [34]. Define $\mathscr{H}(\omega) \equiv \iint\iint \log \frac{1}{|x-y|}\omega(x)\omega(y)$, $\log^{\pm} x \equiv (\log x)^{\pm} \equiv \max\{\pm \log x, 0\}$, and $\mathscr{H}^{+}(\omega) \equiv \iint\iint \log^{+} \frac{1}{|x-y|}\omega(x)\omega(y)$.

LEMMA 3.1. *Let ω be a nonnegative measure of finite mass and compact support, and let $u = K * \omega$ be the velocity corresponding to the vorticity ω. Then the following are equivalent:*

(1) *ω is in H^{-1}.*
(2) *u is in L^2_{loc}.*
(3) *$\mathscr{H}^{+}(\omega) < \infty$.*

Furthermore, even without the assumption that ω has compact support, (3) *implies* (2) *and*

$$\|u\|_{L^2_{\text{loc}}} \leqq c\left[\mathscr{H}^{+}(\omega) + \left(\iint \omega(x)\right)^2\right]. \tag{3.1}$$

Proof: We will show that condition (2) is equivalent to each of the others. Let ϕ be a nonnegative test function that is identically 1 on the support of ω; we introduce ϕ into the integral, defining $\mathscr{H}^{+}(\omega)$ in order to ensure that there is no contribution from infinity when integrating by parts. Thus,

$$\begin{aligned}
\mathscr{H}^{+}(\omega) &= \iint\iint \phi(x)\log^{+}\frac{1}{|x-y|}\omega(x)\omega(y)\\
&= \iint\iint \phi(x)\log\frac{1}{|x-y|}\omega(x)\omega(y)\\
&\quad + \iint\iint \phi(x)\log^{-}\frac{1}{|x-y|}\omega(x)\omega(y)\\
&= \iint\iint \phi(x)\log\frac{1}{|x-y|}\nabla^{\perp}\cdot u(x)\omega(y)\\
&\quad + \iint\iint \phi(x)\log^{-}\frac{1}{|x-y|}\omega(x)\omega(y)\\
&= c\iint\iint \omega(y)\phi(x)u(x)\cdot K(x-y)\\
&\quad + \iint\iint \omega(y)\log\frac{1}{|x-y|}u(x)\cdot\nabla^{\perp}\phi(x)\\
&\quad + \iint\iint \phi(x)\log^{-}\frac{1}{|x-y|}\omega(x)\omega(y)
\end{aligned} \tag{3.2}$$

$$
\begin{aligned}
&= c \iint |u(x)|^2 \phi(x) \\
&+ \iint \iint \phi(x) \log^- \frac{1}{|x-y|} \omega(x)\omega(y) \\
&+ \iint \iint \omega(y) \log \frac{1}{|x-y|} u(x) \cdot \nabla^\perp \phi(x).
\end{aligned}
$$

Since $\log^- \frac{1}{|x-y|}$ is bounded for x and y in the support of ω, the term

$$\iint \iint \phi(x) \log^- \frac{1}{|x-y|} \omega(x)\omega(y)$$

is bounded in absolute value by a constant depending on ϕ times $\left(\iint \omega\right)^2$. Also, for $p < 2$, K is in L^p_{loc}, and hence $\|u\|_{L^p_{\text{loc}}} \leqq c \iint \omega$, while $\log \frac{1}{|x-y|}$ considered as a function of x is in the complementary L^q on the support of ϕ, uniformly in y for y in the support of ω. Hence the last term on the right side of the last equation in (3.2) has a bound of the same form. Since ϕ may be identically 1 on an arbitrarily large region, (3.2) therefore shows that conditions (2) and (3) are equivalent. When ω does not have compact support, pick a ball $B_R(0)$ in which we want to estimate the L^2-norm of u and divide ω into the part supported in $B_{2R}(0)$ plus the part supported on the complement of that set. The velocity corresponding to the latter is bounded on $B_R(0)$ by a constant times its total mass, while the former has compact support, which yields (3.1).

To show that condition (2) is also equivalent to (1), note that we can reduce to the case when ω is no longer nonnegative but satisfies $\iint \omega = 0$ since there exists a smooth, compactly supported $\tilde{\omega}$ such that $\iint \tilde{\omega} = \iint \omega$, and clearly $\tilde{\omega} \in L^2 \subset H^{-1}$, so ω is in H^{-1} if and only if $\omega - \tilde{\omega}$ is.

Assuming now that $\iint \omega = 0$, we have for x outside a ball B containing the support of ω that

$$u(x) = \frac{1}{2\pi} \iint \frac{(x-y)^\perp}{|x-y|^2} \omega(y) = \frac{1}{2\pi} \iint \left[\frac{(x-y)^\perp}{|x-y|^2} - \frac{x^\perp}{|x|^2} \right] \omega(y) = O\left(\frac{1}{|x|^2}\right).$$

Since u is bounded for $x \in \mathbb{R}^2 \setminus B$, this means that u lies in $L^2(\mathbb{R}^2 \setminus B)$. Hence, u is in $L^2(\mathbb{R}^2)$ if and only if u lies in L^2_{loc}.

Next, for every ψ in H^1 tending to 0 at infinity,

$$\iint \psi\omega = \iint u \cdot \nabla^\perp \psi. \tag{3.3}$$

Since the set of such ψ is dense in H^1, this shows that if u lies in L^2, then ω is in H^{-1}. Suppose, on the other hand, that u does not lie in L^2; in Fourier space this says that $\frac{\hat{\omega}(k)}{|k|}$ is not in L^2. By the compact support and vanishing mean of ω, $\hat{\omega}(k) = O(|k|)$, so the problem is not due to a neighborhood of 0, and hence

$g(k) \equiv \frac{\hat{\omega}(k)}{|k|}\chi_{\{|k|>\delta\}}$ is not in L^2 either. Since L^2 is self-dual, this implies that there exists a bounded sequence $g_j(k)$ of elements in L^2 such that $\iint g(k)g_j(k) \to \infty$, and we can assume that g_j vanishes for $|k| < \delta$ since g does. This implies that $\psi_j(k) = \frac{g_j(k)}{|k|}$ is a bounded sequence in H^1 for which $\iint \psi_j\omega \to \infty$, which shows that ω does not lie in H^{-1}.

LEMMA 3.2. *Let ω_0 be a nonnegative measure of finite mass and compact support lying in H^{-1}. Divide a square containing the support of ω_0 into N squares $\Omega_{j,N}$ of equal diameter $d(N) = O(\frac{1}{\sqrt{N}})$, with centers $x_{j,N}$. Let the $\Omega_{j,N}$ include part of their boundary in such a way that they are disjoint and their union contains the support of ω_0. Let $\omega_{j,N}$ be the total vorticity of ω_0 in $\Omega_{j,N}$. Then $\omega_N \equiv \sum_{j=1}^N \omega_{j,N}\delta(x-x_{j,N})$ converges weak-$*$ in the sense of measures to ω_0 as $N \to \infty$, and $H_N \equiv \sum_{j\neq k} \omega_{j,N}\omega_{k,N} \log \frac{1}{|x_{j,N}-x_{k,N}|}$ is uniformly bounded.*

Proof: On the space of nonnegative measures having a given total mass, the Kantorovich-Rubinshtein-Vasershtein (KRV) metric is defined by $d(\mu,\nu) = \inf_{dP} \iint\iint |x-y|\,dP(x,y)$ ([21], [36]), where the infimum is taken over all measures $dP(x,y)$ such that

$$\iint\iint f(x)\,dP(x,y) = \iint f(x)\mu(x)$$

and

$$\iint\iint f(y)\,dP(x,y) = \iint f(y)\nu(y)\,.$$

Such a dP will be called a *joint measure* for μ and ν. For such measures, the topology determined by the metric is equivalent to the weak-$*$ topology. Let $dP(x,y) = \sum_j \omega_0(x)\upharpoonright_{\Omega_{j,N}} \cdot \delta(y-x_{j,N})$. Then dP is a joint measure for ω_0 and ω_N, so $d(\omega_0,\omega_N) \leqq \iint\iint |x-y|\,dP(x,y) \leqq d(N)\iint \omega_0$, which shows that ω_N converges weak-$*$ to ω_0. Furthermore, for $j \neq k$, $|x_{j,N} - x_{k,N}| \geqq c \sup_{x\in\Omega_{j,N},y\in\Omega_{k,N}} |x-y|$ with c a fixed constant less than 1. Hence, for $d(N) \leqq 1$,

$$\begin{aligned}
H_N &= \sum_{j\neq k} \omega_{j,N}\omega_{k,N} \log \frac{1}{|x_{j,N}-x_{k,N}|} \\
&= \sum_{j\neq k} \iint_{\Omega_j}\iint_{\Omega_k} \omega_0(x)\omega_0(y)\left[\log\frac{|x-y|}{|x_{j,N}-x_{k,N}|} + \log\frac{1}{|x-y|}\right] \\
&\leqq \sum_{j\neq k} \iint_{\Omega_j}\iint_{\Omega_k} \omega_0(x)\omega_0(y)\left[\log\frac{1}{c} + \log\frac{1}{|x-y|}\right] \\
&\leqq \iint\iint \omega_0(x)\omega_0(y)\left[\log\frac{1}{c} + \log\frac{1}{|x-y|}\right] \\
&= \log\frac{1}{c}\left(\iint \omega_0\right)^2 + \mathscr{H}(\omega_0),
\end{aligned}$$

and this is finite by Lemma 3.1.

Lemma 3.2 shows that there exists initial data satisfying the hypotheses of the following theorem:

THEOREM 3.3. *Let ω_0 be a nonnegative Radon measure of finite total mass having compact support and lying in H^{-1}. Suppose that the point-vortex initial vorticity $\omega_{N,0} = \sum_{j=1}^{N} \omega_{j,N}\delta(x - x_{j,N})$ is nonnegative, is compactly supported in a set independent of N, tends weak-$*$ to ω_0, and satisfies*

$$H_N \equiv \sum_{j \neq k} \omega_{j,N}\omega_{k,N} \log \frac{1}{|x_{j,N} - x_{k,N}|} \leqq c < \infty . \tag{3.4}$$

Then the solution of the point-vortex method with initial data $\omega_{N,0}$ exists for all time and has a subsequence that converges weak-$*$ in the sense of measures to a solution of the weak vorticity formulation having initial data ω_0. Furthermore, $u \equiv K * \omega$ lies in L^2_{loc} and hence is a solution of the weak velocity formulation with initial data $K * \omega_0$.

Proof: Let us first show that no collisions occur and no vortices escape to infinity in finite time, so that the point-vortex approximants ω_N are defined for all time. Up to a constant factor, H_N is the Hamiltonian of the point-vortex dynamics ([36], section 4.2) and thus is a conserved quantity. The center of vorticity $M_N = \sum_{j=1}^{N} \omega_{j,N}x_j$ and the moment of inertia $I_N = \sum_{j=1}^{N} \omega_{j,N}x_j^2$ are also conserved ([36], section 4.2), and the total vorticity $W_N = \sum_{j=1}^{n} \omega_{j,N}$ clearly is also. The combination $2(W_N I_N - M_N^2)$ can be rearranged to yield the conserved quantity $P_N \equiv \sum_{j\neq k} \omega_{j,N}\omega_{k,N}|x_j - x_k|^2$. Adding this to H_N yields the conserved quantity

$$J_N \equiv \sum_{j \neq k} \omega_j\omega_k \left[|x_j - x_k|^2 + \log \frac{1}{|x_j - x_k|} \right].$$

Now the function $z^2 + \log \frac{1}{z}$ is nonnegative, and so are the $\omega_{j,N}$ by assumption, so J_N is nonnegative. Furthermore, P_N is bounded at time 0 uniformly in N on account of the uniform compact support of the initial data, and hence so is J_N. Thus $0 \leqq J_N \leqq c$. The fact that each term in the sum defining J_N is nonnegative and would tend to $+\infty$ if the corresponding pair of vortices collided implies that vortex collisions never occur, while the conservation of I_N implies that no vortices escape to infinity.

Next, by Lemma 2.1, in order to show that a subsequence converges weak-$*$ to a solution of the weak vorticity formulation, it suffices to show that the nonconcentration condition (2.9) holds for all t. To prove this, note that

$$
\begin{aligned}
\frac{J_N}{\log\frac{2}{r}} &\geqq \sup_x \sum_{|x_j-x|<r,|x_k-x|<r,j\neq k} \omega_{j,N}\omega_{k,N} \\
&= \sup_x \left[\left(\sum_{|x_j-x|<r} \omega_{j,N}\right)^2 - \sum_{|x_j-x|<r} \omega_{j,N}^2\right] \\
&\geqq \sup_x \left(\sum_{|x_j-x|<r} \omega_{j,N}\right)\left[\left(\sum_{|x_j-x|<r} \omega_{j,N}\right) - \max_j \omega_{j,N}\right] \\
&\geqq \left[\left(\sup_x \sum_{|x_j-x|<r} \omega_{j,N}\right) - \max_j \omega_{j,N}\right]^2 .
\end{aligned} \tag{3.5}
$$

Solving this inequality for $\sup_x \sum_{|x_j-x|<r} \omega_{j,N}$ yields

$$\sup_x \sum_{|x_j-x|<r} \omega_{j,N} \leqq \sqrt{\frac{J_N}{\log\frac{2}{r}}} + \max_j \omega_{j,N} ,$$

and this implies that (2.9) holds for all t since the fact that $\omega_{N,0}$ converges weak-$*$ to ω_0 implies that the maximum point-vortex strength tends to 0 as N goes to infinity.

Finally, let ω be the weak-$*$ limit of the ω_N. Then $\omega_N(t,x)\omega_N(t,y)$ converges weak-$*$ to $\omega(t,x)\omega(t,y)$ (e.g., [42], lemma 3.2). Let $\eta_\delta(r)$ be a nonnegative continuous function that vanishes for $r \leqq \delta$ and is identically 1 for $r \geqq 2\delta$. Then, since the nonconcentration estimate (2.9) implies that ω has no pure-point part,

$$
\begin{aligned}
\mathcal{J}(\omega) &\equiv \iint\iint \omega(t,x)\omega(t,y)\left[|x-y|^2 + \log\frac{1}{|x-y|}\right] \\
&= \lim_{\delta\to 0} \iint\iint \omega(t,x)\omega(t,y)\left[|x-y|^2 + \log\frac{1}{|x-y|}\right]\eta_\delta(|x|)\eta_\delta(\tfrac{1}{|x|}) \\
&\leqq \underline{\lim}_N J_N \\
&\leqq c .
\end{aligned} \tag{3.6}
$$

By Lemma 3.1, this implies that $u \equiv K * \omega$ lies in L^2_{loc} since $\mathcal{J}$ dominates $\mathcal{H}^+$.

To treat the case of the vortex-blob method, define $L_\varepsilon = \phi_\varepsilon * \log\frac{1}{|\cdot|}$. Then the Hamiltonian for the vortex-blob method is a constant times

$$H_N^\varepsilon \equiv \sum_{j\neq k} \omega_{j,N}\omega_{k,N} L_\varepsilon(x_j - x_k) .$$

Note that we can still omit the self-interaction term since it is just a constant K_N; we therefore will not need to assume that K_N is bounded uniformly in N. Similarly, define J_N^ε and $\mathcal{J}^\varepsilon$ by substituting $L_\varepsilon(z)$ for $\log \frac{1}{|z|}$. We first obtain some estimates for $L_\varepsilon(x)$.

LEMMA 3.4. *Let ϕ be a continuous radial mollifier such that*

$$\int_0^\infty s|\phi(s)\log s|\,ds < \infty\,. \tag{3.7}$$

Then

$$|L_\varepsilon(x)| \leqq c\left(1 + |\log \tfrac{1}{|x|}|\right), \tag{3.8}$$

and furthermore

$$L_\varepsilon(x) \leqq c(1 + \log \tfrac{1}{\varepsilon})\,. \tag{3.9}$$

Proof: Since $\frac{1}{2\pi}\log\frac{1}{|x|}$ is the fundamental solution for $-\Delta$, L_ε satisfies

$$\Delta L_\varepsilon = -2\pi\phi_\varepsilon\,. \tag{3.10}$$

Since ϕ is radial, so is L_ε, and hence (3.10) simplifies to $(rL_\varepsilon'(r))' = -2\pi r\phi_\varepsilon(r)$. Hence

$$L_\varepsilon'(r) = -\frac{2\pi}{r}\int_0^r s\phi_\varepsilon(s)\,ds = -\frac{2\pi}{r}\int_0^{\frac{r}{\varepsilon}} s\phi(s)\,ds\,. \tag{3.11}$$

In order to integrate (3.11), we need to know the value of $L_\varepsilon(0)$. Now

$$\begin{aligned} L_\varepsilon(0) &= \iint \phi_\varepsilon(-y)\log\frac{1}{|y|}\,dy \\ &= 2\pi\int_0^\infty s\phi_\varepsilon(s)\log\frac{1}{s}\,ds \\ &= 2\pi\int_0^\infty S\phi(S)\log\frac{1}{\varepsilon S}\,dS \\ &= c\log\frac{1}{\varepsilon} + c\,. \end{aligned} \tag{3.12}$$

Integrating (3.11) from 0 to r, substituting for $L_\varepsilon(0)$ from (3.12), and integrating by parts yields

$$\frac{L_\varepsilon(r)}{2\pi} = \log\frac{1}{r}\int_0^{\frac{r}{\varepsilon}} s\phi(s)\,ds + \int_{\frac{r}{\varepsilon}}^\infty s\phi(s)\log\tfrac{1}{\varepsilon s}\,ds \tag{3.13}$$

since the integral obtained from the integration by parts partially cancels the integral from (3.12). From (3.11) we see that the maximum value of L_ε occurs at a point $c\varepsilon$ where $\int_0^c s\phi(s)\,ds = 0$; (3.13) then implies that $L_\varepsilon = c\log\frac{1}{\varepsilon} + c$, which proves (3.9). Furthermore, (3.9) shows that (3.8) holds for $r \leqq \sqrt{\varepsilon}$, while for $r > \sqrt{\varepsilon}$ we can obtain (3.8) from (3.13) since then

$$\left|\log\tfrac{1}{\varepsilon}\int_{\frac{r}{\varepsilon}}^{\infty} s\phi(s)\,ds\right| \leqq 2\int_{\frac{1}{\sqrt{\varepsilon}}}^{\infty} s|\phi(s)\log s|\,ds \leqq c\,.$$

COROLLARY 3.5. *Theorem* 3.3 *remains valid when the point-vortex method is replaced by the the vortex-blob method using a radial mollifier with smoothing parameter* $\varepsilon(N)$ *tending to* 0 *at an arbitrary rate.*

Proof: Since the vortex-blob velocity is Lipschitz, collisions are impossible, while the moment of inertia I_N is still conserved since the mollifier is radial, and hence the ω_N exist for all time. By Lemma 3.4 plus the compact support of the initial data, the uniform boundedness of H_N implies that H_N^ε is also uniformly bounded. The other conclusions of the theorem are therefore obtained for the vortex-blob method by replacing $\log\frac{1}{|z|}$ by $L_{\varepsilon(N)}(|z|)$ and J_N by $J_N^{\varepsilon(N)}$ in (3.5), replacing $\mathcal{J}$ by $\mathcal{J}^{\varepsilon(N)}$ and $\log\frac{1}{|z|}$ by $L_{\varepsilon(N)}(|z|)$ in (3.6), and taking the limit as $\varepsilon(N) \to 0$.

For simplicity, we will only prove the weak convergence of the smoothed velocities under some additional assumptions.

COROLLARY 3.6. *Let the initial data be given by the method of Lemma* 3.2, *and choose a radial nonnegative function* η *whose integral equals* 1. *For the point-vortex method, if* $\varepsilon \geqq \frac{c}{N^k}$ *for some* k *and tends to* 0 *as* $N \to \infty$, *then the smoothed velocity* $\eta_{\varepsilon(N)} * K * \omega_N$ *converges weakly in* L^2_{loc} *along the subsequence for which the vorticity convergence is obtained to a solution of the weak velocity formulation having initial data* $K * \omega_0$.

For the vortex-blob method, assume that the function ϕ *used in the mollifier satisfies* $\phi = \eta * \eta$ *for some nonnegative radial* η. *Then the same convergence result holds for the velocity of the vortex-blob method under the same conditions on* $\varepsilon(N)$.

Proof: By the convergence result obtained for the vorticity, it suffices to show that the smoothed velocity of the point-vortex method and the velocity of the vortex-blob method are uniformly bounded in L^2_{loc}. Since the velocity $\phi_\varepsilon * K * \omega_N$ of the vortex-blob method equals $\eta_\varepsilon * (\eta_\varepsilon * K * \omega_N)$, it suffices to show that $\eta_\varepsilon * K * \omega_N$ is uniformly bounded in L^2_{loc}. By Lemma 3.1 it therefore suffices to show that in each case $\mathcal{H}^+(\eta_\varepsilon * \omega_N)$ is uniformly bounded. Since $\mathcal{H}^+ \leqq \mathcal{J}$, it also suffices to show that $\mathcal{J}(\eta_\varepsilon * \omega_N)$ is uniformly bounded.

Recall that a radial mollifier of any harmonic function equals that function. Although the function $h(x) = |x|^2$ is not harmonic, it satisfies $\Delta|x|^2 = 4$. Hence

$$\Delta(\phi_\varepsilon * h) = \phi_\varepsilon * (\Delta h) = \phi_\varepsilon * 4 = 4.$$

Since $\phi_\varepsilon * h$ is radial, this implies that $\phi_\varepsilon * h = c(\varepsilon) + h$; a calculation then shows that $c(\varepsilon) = c\varepsilon^2$.

Hence, letting $\phi = \eta * \eta$ also in the point-vortex case,

$$\begin{aligned}
\mathcal{J}(\eta_\varepsilon * \omega_N) &= \iint\iint [\eta_\varepsilon * \omega_N](x)[\eta_\varepsilon * \omega_N](y)\left(|x-y|^2 + \log\frac{1}{|x-y|}\right) \\
&= \sum_{j,k} \omega_{j,N}\omega_{k,N}\left[\eta_\varepsilon * \eta_\varepsilon * \left(|\cdot|^2 + \log\frac{1}{|\cdot|}\right)\right](x_j - x_k) \\
&= \sum_{j,k} \omega_{j,N}\omega_{k,N}\left[\phi_\varepsilon * \left(|\cdot|^2 + \log\frac{1}{|\cdot|}\right)\right](x_j - x_k) \\
&= \sum_{j,k} \omega_{j,N}\omega_{k,N}\left(|x_j - x_k|^2 + c\varepsilon^2 + L_\varepsilon(x_j - x_k)\right) \\
&= c\varepsilon^2\left(\sum_j \omega_{j,N}\right)^2 + J_N^\varepsilon + L_\varepsilon(0)\sum_j \omega_{j,N}^2.
\end{aligned}$$

Now J_N^ε is conserved and uniformly bounded in the case of the vortex-blob method, while in the case of the point-vortex method Lemma 3.4 shows that it can be bounded in terms of the uniformly bounded conserved quantities J_N and the total vorticity.

It therefore suffices to show that $L_\varepsilon(0)\sum_j \omega_{j,N}^2$ is uniformly bounded. Note that now this self-interaction term cannot be omitted since it is included in $\mathcal{J}$ of the mollification of ω_N. Since we have calculated $L_\varepsilon(0) = c\log\frac{1}{\varepsilon} + c$ in the proof of Lemma 3.4, let us now estimate the rate at which $\sum_j \omega_{j,N}^2$ tends to 0 as $N \to \infty$:

$$\begin{aligned}
\sum_j \omega_{j,N}^2 \log\frac{1}{d(N)} &= \sum_j\left[\iint_{\Omega_{j,N}} \omega_0(x)\right]^2 \log\frac{1}{d(N)} \\
&\leqq \sum_j \iint_{\Omega_{j,N}}\iint_{\Omega_{j,N}} \omega_0(x)\omega_0(y)\log\frac{1}{|x-y|} \\
&\leqq \sum_j \iint_{\Omega_{j,N}}\iint_{\Omega_{j,N}} \omega_0(x)\omega_0(y) \\
&\qquad \times\left[|x-y|^2 + \log\frac{1}{|x-y|}\right] \\
&\leqq \iint\iint \omega_0(x)\omega_0(y)\left[|x-y|^2 + \log\frac{1}{|x-y|}\right] \\
&\equiv \mathcal{J}(\omega_0) \leqq c < \infty.
\end{aligned} \tag{3.14}$$

Thus, $\sum_j \omega_{j,N}^2 \leqq \frac{c}{\log \frac{1}{d(N)}}$. It therefore suffices to have $\frac{\log \frac{1}{\varepsilon}}{\log \frac{1}{d(N)}} \leqq c$, and this will be true provided that $\varepsilon \geqq \frac{c}{N^k}$ for some c and k.

4. Initial-Volume Estimates

We now begin considering the case when the initial data need not be nonnegative. As explained in the Introduction, our strategy for showing that concentration almost never occurs involves showing that the ratio of the phase-space volume in which concentration may occur to the phase-space volume of approximations to the initial data rapidly tends to 0 as the number of vortices increases to infinity. In this section we will derive the estimates for the approximations to the initial data.

For each initialization scheme considered, we will show that the point-vortex initial vorticity ω_{init} converges weak-$*$ to ω_0 and then estimate the total volume of phase space occupied by subsets of the allowable initial data having the same set of vortex strengths, counting multiplicities. This is the relevant volume for the application of Liouville's theorem mentioned in the Introduction, because changing the set of vortex strengths changes the Hamiltonian.

We first consider the case in which all vortices of the same sign have the same strength and many vortices are placed in each cell. The positive and negative vortices will be allowed to have different absolute strengths $\frac{\omega^\pm}{N}$, since, as discussed in Section 7, this would be needed to avoid collisions in the case of compactly supported initial data.

The *equal-strength vortex initialization scheme* is defined as follows: Divide a period cell Ω (or a fixed rectangle Ω containing the support of ω_0 in the compactly supported case) into M squares Ω_j of equal area $\frac{c_1}{M}$ with centers X_j. Let $J(x)$ be a function of compact support lying in L^∞ such that $\iint J(x)\,dx = 1$, and define $J_d(x) = \frac{1}{d^2} J(x/d)$. Choose a function $d(M)$ tending to 0 as $M \to \infty$ and such that $d(M) \geqq \frac{c}{\sqrt{M}}$, and let $\omega_{j,M}^\pm = \frac{c_1}{M}\left[J_{d(M)} * \omega_0^\pm\right](X_j)$, where $f^\pm \equiv \sup(0, \pm f)$. Let the numbers of positive and of negative vortices to be placed in the j^{th} cell be $n_{j,M,N}^\pm \equiv [\![\frac{N\omega_{j,M}^\pm}{\omega^\pm}]\!]$, where $[\![\cdot]\!]$ is the least-integer function, and let $N^\pm \equiv \sum_j n_{j,M,N}^\pm$ be the total number of positive and negative vortices. Let A be a set of positive measure lying in the square $[-\frac{1}{2}, \frac{1}{2}]^2$, and let the rescaled and translated set $A_{j,M} = \{x \mid \sqrt{\frac{M}{c_1}}(x - X_j) \in A\}$ be the portion of the j^{th} cell in which the vortices are to be placed. Randomly pick $N^+ + N^-$ points in A independently, each according to the uniform distribution, and for each j and both choices of the sign $\pm$, rescale $n_{j,M,N}^\pm$ of those points so that they lie in $A_{j,M}$, and place vortices of strength $\pm\frac{\omega^\pm}{N}$ at those points.

Remark 4.1. In this and the other initialization schemes, neither the fact that all the cells and all the sets $A_{j,M}$ have the same shape nor the fact that the convolution factors $J_{d(M)}$ are all the same is important.

LEMMA 4.2. *Let M and N tend to infinity with $\frac{M}{N} \to 0$. If ω_0 is in L^1 and has compact support, then for every random choice of the vortex positions, the initial vorticity ω_{init} of the equal-strength initialization scheme converges weak-$*$ in the sense of measures to ω_0.*

Proof: For any $\phi \in C^0$, define

$$\delta(\phi, r) = \sup_{\substack{|x-y| \leqq r \\ x,y \in \Omega}} |\phi(x) - \phi(y)| .$$

Now $\frac{c_1}{M} J_{d(M)}(x - X_j)$ and the indicator function $\chi_{\Omega_j}(x)$ both have integral $\frac{c_1}{M}$ and L^1-norm $O(\frac{1}{M})$, and the maximum distance between points in their supports is $O(\frac{1}{\sqrt{M}} + d(M))$, so

$$\left| \iint \left[\frac{c_1}{M} J_{d(M)}(x - X_j) - \chi_{\Omega_j}(x) \right] \phi(x) \right| \leqq O(\tfrac{1}{M})\, \delta(\phi, O(\tfrac{1}{\sqrt{M}} + d(M))) .$$

Upon summing over j we find that $\sum_{j=1}^{M} \frac{c_1}{M} J_{d(M)}(x - X_j)$ tends weak-$*$ in the sense of measures as $M \to \infty$ to the indicator function of Ω. Furthermore, $\sum_j \frac{c_1}{M} J_{d(M)}(x - X_j)$ is uniformly bounded in L^∞ because at every point the sum contains at most $O(1 + (d(M)M)^2)$ terms of size $O(\frac{1}{d(M)^2 M})$; the condition $d(M) \geqq \frac{c}{\sqrt{M}}$ is needed only to obtain this L^∞-bound. Hence $\sum_{j=1}^{M} \frac{c_1}{M} J_{d(M)}(x - X_j)$ also converges weak-$*$ in L^∞; that is, $\iint \sum_j \frac{c_1}{M} J_{d(M)}(x - X_j) f(x)$ tends to $\iint_\Omega f$ for every f in L^1.

Let x_k denote the random position of the k^{th} vortex. Then for $\phi \in C^0$,

$$\begin{aligned}
&\iint \phi(x)\, [\omega_{\text{init}} - \omega_0] \\
&\quad = \sum_k \frac{\omega^\pm}{N} \phi(x_k) - \iint \phi \omega_0 \\
&\quad = \sum_\pm \sum_j \Big[\frac{\omega^\pm n^\pm_{j,M,N}}{N} \left(\phi(X_j) + \delta(\phi, O(\tfrac{1}{\sqrt{M}})) \right) \\
&\qquad\qquad - \iint \frac{c_1}{M} J_{d(M)}(x - X_j) \omega_0^\pm(x) \phi(x) \Big] + o(1) \qquad (4.1) \\
&\quad = \sum_\pm \sum_j \Big[\frac{\omega^\pm}{N} \Big[\frac{N}{\omega^\pm} \iint \frac{c_1}{M} J_{d(M)}(x - X_j) \omega_0^\pm(x) \Big] \\
&\qquad\qquad \times \left(\phi(X_j) + \delta(\phi, O(\tfrac{1}{\sqrt{M}})) \right) \\
&\qquad\qquad - \iint \frac{c_1}{M} J_{d(M)}(x - X_j) \omega_0^\pm(x) \phi(x) \Big] + o(1) .
\end{aligned}$$

Now $\frac{c}{N}[\![\frac{N}{c}$ anything$]\!]$ = anything + $O(\frac{1}{N})$; since there are $O(M)$ such terms, the total error caused by eliminating the $[\![\cdot]\!]$ is $O(\frac{M}{N})$, which tends to 0 by assumption. The final expression in (4.1) can therefore be written as

$$\begin{aligned}(4.2)\qquad &\iint \sum_{\pm}\sum_{j=1}^{M} \frac{c_1}{M} J_{d(M)}(x - X_j)\omega_0^{\pm}(x)\\ &\times \left(\phi(X_j) - \phi(x) + \delta(\phi, O(\tfrac{1}{\sqrt{M}}))\right) + o(1).\end{aligned}$$

For x in the support of $J_{d(M)}(x - X_j)$ we have $x - X_j = O(d(M))$, so $\phi(X_j) - \phi(x)$ is $O(\delta(\phi, O(d(M))))$. Since $\sum_{j=1}^{M} \frac{c_1}{M} J_{d(M)}(x - X_j)$ is uniformly bounded in L^∞, (4.2) tends to 0.

LEMMA 4.3. *Let ω_0 have compact support and be in $L \log L$, and assume that $M = o(N)$. Let I be the set of initial vortex positions produced by the equal-strength initialization scheme, allowing arbitrary renumberings of vortices having the same strength; that is, I is the set of points $(x_1^+, \dots, x_{N^+}^+, x_1^-, \dots, x_{N^-}^-)$ in $(\mathbb{R}^2)^{N^+ + N^-}$ such that for every j the number of $x_i^{\pm}$ lying in $A_{j,M}$ is precisely $n_{j,M,N}^{\pm}$; in particular, every point lies in some $A_{j,M}$.*

Then the $2(N^+ + N^-)$-dimensional volume of I is at least $\delta^{N^+ + N^-}$ for some positive δ.

Remarks:

(1) A vortex calculation with the vortices renumbered yields the same answer as the original calculation, so the renumbering of vortices is purely a theoretical tool. In this and the next initialization scheme, such a permutation of the vortices changes the spatial regions that different vortices occupy, thereby increasing the total region of phase space that may be occupied. However, in the final initialization scheme permutations would not affect the region a vortex occupies, and so will not be considered.

(2) The vortex initial data ω_{init} would still converge weak-$*$ to ω_0 if we allowed the number of vortices in each cell to differ from $n_{j,M,N}^{\pm}$ by $o(\frac{N}{M})$, and the resulting initial phase volume, being larger, would still be at least δ^N. The fact that this larger volume does not suffice to relax the condition that ω_0 lie in $L \log L$ can be seen either from large-deviations theory or a direct calculation.

(3) Although the number of vortices in a single cell may be large, the total number of vortices is $O(N)$, which can be any superlinear function of the number M of cells since $\frac{M}{N}$ may tend to 0 at an arbitrary rate. Thus, the number of vortices needed here to obtain the same accuracy in approximating the initial data is larger than the number needed in the scheme with one vortex per cell only by a logarithmic or even more slowly growing factor.

Proof: It suffices to show that the $2N^+$-dimensional volume of I^+, the set of allowed values of $(x_1^+, \dots, x_{N^+}^+)$, is at least δ^{N^+}, since the calculation for I^- is the same. If $\omega_0 \leqq 0$ then $N^+ = 0$ and the result holds trivially, so assume that ω_0 takes on positive values somewhere.

There are $\frac{(N^+)!}{\prod_{j=1}^M (n_{j,M,N}^+)!}$ ways of assigning the correct number of vortices to each cell. If $n_{j,M,N}^+$ equals 0, then the corresponding factorial is 1 by definition and so can be omitted from the product. Using Stirling's formula $n! = n^{n+1/2}e^{-n}e^{O(1)}$, the number of possible assignments can be written as

$$(4.3) \qquad \frac{(\sum n_{j,M,N}^+)^{(\sum n_{j,M,N}^+ + 1/2)}}{\prod_{n_{j,M,N}^+ \neq 0} (n_{j,M,N}^+)^{n_{j,M,N}^+ + 1/2}} \cdot O(1)^{-\min(M,N^+)}$$

since the exponential factors arising from the e^{-n} in Stirling's formula cancel while the number of terms $e^{O(1)}$ in the denominator is bounded by the maximum number of nonempty cells.

Maximizing $\prod_{j=1}^M n_j$ subject to the constraint $\sum_{j=1}^M n_j = N_{\text{tot}}$ and then maximizing the result over M shows that for all M, $\prod_{j=1}^M n_j \leqq (e^{1/e})^{\sum n_j}$. Applying this to the factor $\frac{1}{\sqrt{\prod n_{j,M,N}}}$ in (4.3), we obtain that the number of possible ways to assign the vortices is at least

$$\left(\frac{\sum n_{j,M,N}^+}{\prod_{n_{j,M,N}^+ \neq 0} (n_{j,M,N}^+)^{n_{j,M,N}^+/N^+}} \right)^{N^+} \cdot c^{-N^+}.$$

Now every vortex placed in cell j may be put anywhere in $A_{j,M}$. Hence, to every assignment of the vortices to specific cells there corresponds a phase volume of $O(\frac{1}{M})^{N^+}$, and the volumes for different assignments are disjoint because at least one vortex is placed in a different disjoint cell. Note that for the purpose of determining the phase volume, different vortices having the same strength are still distinguishable. Thus, the total phase volume is bounded from below by

$$\left(\frac{N^+}{cM \prod_{n_{j,M,N}^+ \neq 0} (n_{j,M,N}^+)^{n_{j,M,N}^+/N^+}} \right)^{N^+}.$$

After taking the logarithm of this expression, we see that it suffices to show that $\log N^+ - \log M - \sum_{j=1}^M \frac{n_{j,M,N}^+}{N^+} \log(n_{j,M,N}^+) \geqq -c > -\infty$ for some constant c independent of M and N. Using the fact that $\sum n_{j,M,N}^+ = N^+$, this condition can be written as

$$(4.4) \qquad \sum_{j=1}^M \frac{n_{j,M,N}^+}{N^+} \log\left(\frac{M n_{j,M,N}^+}{N^+} \right) \leqq c < \infty .$$

Recall that $n_{j,M,N} = [\![\frac{cN}{M} \iint J_{d(M)}(x - X_j)\omega_0^+(x)]\!]$, which can be written as $\frac{cN}{M}\left(\iint J_{d(M)}(x - X_j)\omega_0^+(x) + o(1)\right)$ since $\frac{M}{N} = o(1)$. Hence, the first paragraph of the proof of Lemma 4.2 shows that $N^+ = cN\left(\iint \omega_0^+ + o(1)\right)$. Thus, since $c \leqq \iint \omega_0^+ + o(1) \leqq C$, the left side of (4.4) is bounded by

$$\begin{aligned} c_2 + \frac{c_3}{M} \cdot \sum_{j=1}^{M} &\left(\iint J_{d(M)}(x - X_j)\omega_0^+(x) + o(1)\right) \\ &\times \log\left(\iint J_{d(M)}(x - X_j)\omega_0^+(x) + o(1)\right). \end{aligned} \tag{4.5}$$

Next, the $o(1)$ terms may be omitted from (4.5) if we increase the c_i there by $o(1)$. To see this, just use the uniform (Hölder) continuity of $x \log x$ for $0 \leqq x \leqq c$ and the fact that $\frac{y \log y}{x \log x} \leqq 1 + O(y - x)$ when $y \geqq x \geqq c > 1$ and $y - x \leqq c$, which follows from the mean value theorem $y \log y = x \log x + (1 + \log z)(y - x)$.

Finally, since $x \log x$ is convex, and the integral of J_r equals 1, Jensen's inequality and the uniform L^∞-boundedness of $\frac{1}{M}\sum J_{d(M)}(x - X_j)$ imply that

$$\begin{aligned} &\frac{c}{M}\sum \iint J_{d(M)}(x - X_j)\omega_0^+(x) \log\left(\iint J_{d(M)}(x - X_j)\omega_0^+(x)\right) \\ &\leqq \frac{c}{M}\sum \iint J_{d(M)}(x - X_j)\omega_0^+ \log(\omega_0^+) \\ &\leqq c \iint \omega_0^+ \log(\omega_0^+). \end{aligned}$$

Remark 4.4. A slightly weaker version of the lemma, in which the assumption $M = o(N)$ is tightened to $M \leqq c\frac{N}{\log N}$, can be obtained from the more general large-deviations result [19], lemma 2.1.9: The phase volume of I^+ equals c^{-N} times the probability that when N^+ vortices are each placed randomly, according to the uniform distribution, in one of M cells, the number of x_i^+ lying in the j^{th} cell is precisely $n_{j,M,N}^+$. Taking the lower bound of [19], lemma 2.1.9, with μ being the uniform distribution yields condition (4.4) with $\frac{M \log(N^+ + 1)}{N^+}$ added to the left side, so the conclusion of the lemma is obtained as long as this added term is assumed to be uniformly bounded.

We now consider the random versions of the classical initialization schemes, beginning with the *one-vortex-per-cell* scheme, which is defined as follows: Divide a period cell Ω (or a rectangle Ω containing the support of ω_0) into N equal squares Ω_j of equal area $\frac{c_1}{N}$ with centers X_j. Let $J(x)$ be a bounded function of integral 1 having compact support, let $d(N)$ be a function tending to 0 as $N \to \infty$ and such that $d(N) \geqq \frac{C}{\sqrt{N}}$, and define $w_{j,N} = c_1 \iint J_{d(N)}(x - X_j)\omega_0(x)$. Let A be a set of positive measure lying in the square $[-\frac{1}{2}, \frac{1}{2}]^2$, define $A_{j,N} = \{x \mid \sqrt{\frac{N}{c_1}}(x - X_j) \in A\}$, and let δ_1 be a fixed number. Then for $1 \leqq j \leqq N$, randomly pick a point in A independently according to the uniform distribution, and let x_j be the rescaled point lying in $A_{j,N}$. Then pick an offset s_j in $[-\frac{1}{2}, \frac{1}{2}]$ randomly with the uniform distribution, and place a vortex of strength $\frac{\omega_{j,N}}{N} \equiv \frac{w_{j,N}+\delta_1 s_j}{N}$ at x_j.

LEMMA 4.5. *Let ω_0 be in L^1 and have compact support. Then for every random choice of the vortex positions, the initial vorticity ω_{init} of the one-vortex-per-cell initialization scheme converges weak-$*$ in the sense of measures to ω_0 as $N \to \infty$ for almost every sequence of the offsets s_j.*

Proof: The proof of Lemma 4.2 shows that $\frac{1}{N}\sum w_{j,N}\delta(x - x_j)$ converges weak-$*$ to ω_0, so it suffices to show that $S_N \equiv \frac{\delta_1}{N}\sum_{j=1}^{N} s_j\delta(x - x_j)$ converges weak-$*$ to 0. Now for any choice of the x_j and any continuous ϕ, the expectation of $\iint \phi S_N$ is 0 and its variance is $O(\frac{1}{N})$, so that expression tends to 0 almost surely. A countable union of sets of measure zero still has measure zero, so, with probability 1, $\iint \phi_j S_N$ tends to 0 for every ϕ_j in a countable set, which can be chosen to be dense in $C^0(\Omega)$. Since the total mass of S_N is bounded uniformly in N, this implies that $\iint \phi S_N \to 0$ for all continuous ϕ: For any such ϕ, $\iint \phi S_N = \iint(\phi - \phi_j)S_N + \iint \phi_j S_N$. For each ϕ_j the second term is $o(1)$ as $N \to \infty$, while a subsequence ϕ_j can be chosen that tends in C^0 to ϕ, so that the first term is $o(1)$, uniformly in N, as j tends to infinity along this subsequence.

LEMMA 4.6. *Suppose that ω_0 has compact support and lies in L^1. For each infinite sequence $\{s_j\}$, let $I_N(\{s_j\})$ be the set of points $(x_1, \dots, x_N)$ in $\mathbb{R}^{2N}$ such that for some permutation σ, $x_{\sigma(j)} \in A_{j,N}$ for all j and $|\omega_{\sigma(j)} - w_j| \leqq \frac{1}{2}\delta_1$ for all j. Then there exists a positive δ such that for almost all sequences $\{s_j\}$ in $[-\frac{1}{2}, \frac{1}{2}]^\infty$ the $2N$-dimensional volume of $I(\{s_j\})$ is at least δ^N for all sufficiently large N.*

Proof: Fix a sequence $\{s_j\}$; this fixes the strengths of the vortices. The regions $A_{j,N}$ are disjoint, and for different permutations at least some of the particles occupy different regions, so the total volume is the volume $O(\frac{1}{N})^N$ for each permutation times the number of allowable permutations. Thus, in order for the total volume to be δ^N as desired, the number of permutations must be $(\delta N)^N$. Furthermore, by the Borel-Cantelli lemma, in order to show that the volume V_N of I_N is at least δ^N for N sufficiently large, it suffices to show that $\sum_N \text{Prob}(V_N < \delta^N) < \infty$. Here and later, Prob$(A)$ denotes the probability of an event A, and $E(B)$ denotes the expectation of a random variable B.

We will divide the vortices into sets of size $g_i N$ that can each be permuted freely among themselves, plus a set of at most bN that cannot be permuted; here g_i and b depend on N, as will be made clear shortly. Thus, the total number of permutations will be at least

$$\prod (g_iN)! \geqq \prod \left(\frac{g_iN}{e}\right)^{g_iN}$$
$$= \left(\prod (g_i)^{g_i}\right)^N \left(\frac{N}{e}\right)^{\sum g_iN} = \left(\prod (g_i)^{g_i}\right)^N \left(\frac{N}{e}\right)^{N-bN}$$

Since $N^{-bN} = e^{-bN\log N}$, we will indeed have $(\delta_3 N)^N$ permutations provided that $b \leqq \frac{c}{\log N}$ and $\prod (g_i)^{g_i} \geqq c$. In particular, we see that we can afford to leave $\frac{cN}{\log N}$ vortices fixed.

Define the normalized strength of a vortex with strength $\frac{\omega_{j,N}}{N}$ to be $\omega_{j,N}$, and divide the normalized vortex-strength axis into intervals I_k of length, say, $\frac{\delta_1}{2}$. For simplicity, we will only consider permutations among vortices j whose expected normalized strengths w_j lie in the same interval I_k. To see how many intervals need to be considered, note that the number of vortices with unnormalized strengths greater than v is at most $\frac{1}{v}$ times the sum of the unnormalized strengths of the vortices, which is uniformly bounded because of the L^∞-bound on $\sum \frac{c_1}{N} J_{d(N)}(x - X_j)$ and the L^1-bound for ω_0. Hence at most $cN/\log N$ of the w_j are greater than $\log N$, so only $O(\log N)$ intervals I_k need be considered.

We will divide each of these $O(\log N)$ intervals in half up to $\log_2 (O(\log N)^2)$ times, and consider a subset of the w_i that lie in the subinterval. Since each interval is divided into at most $O(\log N)^2$ subintervals, there will be at most $O(\left(\log N\right)^3)$ subintervals, which means that we can afford to neglect those subintervals for which we consider fewer than $\frac{cN}{\left(\log N\right)^4}$ of the w_j. We will use a large-deviations theorem in each subinterval considered in order to determine the exceptional set of sequences $\{s_j\}$; ignoring those subintervals having a relatively small number of the w_j under consideration ensures that the probability of that set will be extremely small.

Let J_j denote the interval $[w_j - \frac{1}{2}\delta_1, {}_j + \frac{1}{2}\delta_1]$ of possible values of ω_j. There exists a "common interval" that must be contained in every J_j such that w_j lies in the "current interval" I_k. The set of ω_j lying in this common interval can be permuted arbitrarily, and $\omega_{\sigma(j)}$ will remain in J_j. Since the length of J_j is twice that of I_k, it turns out that this common interval is just I_k itself. Thus, for any $w_j \in I_k$, ω_j has probability $\frac{1}{2}$ of lying within the common interval, and so the expected proportion of such ω_j is $\frac{1}{2}$. In this and all later stages we will ignore those (sub)intervals in which fewer than $\frac{cN}{(\log N)^4}$ of the w_j are under consideration. Hoeffding's inequality from large-deviations theory ([38], appendix B) says that if $a \leqq y_i \leqq b$ and the y_i are independent, then for $\eta > 0$

$$\text{Prob}\left(\sum_{j=1}^{M} y_i \leqq ME(y_i) - \eta\right) \leqq e^{-2\eta^2/M(b-a)^2}. \tag{4.6}$$

Letting y_j equal 1 if ω_j lies in the common interval and 0 otherwise, we have $0 \leqq y_i \leqq 1$. Taking η to be a constant times the number of ω_j under consideration, we find that the probability that the actual proportion of the ω_j lying in the common interval will be less than the expected proportion by at least a fixed amount is exponentially small in the number of w_j under consideration. Thus, letting μ be a positive number less than $\frac{1}{2}$, there is a probability at most $e^{-cN/(\log N)^4}$ that the proportion of ω_j under consideration lying in the common interval is less than μ.

We now repeat the following process: Place those $\{s_j\}$ for which the proportion is less than μ into the exceptional set. For nonexceptional $\{s_j\}$, take all possible permutations of the ω_j under consideration that lie in the common interval. Then divide the current interval into two equal halves, and for each half consider those w_j lying in that half for which the corresponding ω_j did not lie in the common interval of the previous stage. Reducing the size of the current interval in which w_j must lie from $2L$, say, to L increases the size of the corresponding common interval by L. Hence by induction the total length of the portion of J_j outside the common interval also equals L. Therefore the probability that an ω_j not lying in the old common interval will lie in the new one is $\frac{1}{2}$. Hence the probability is at most $e^{-cN/(\log N)^4}$ that the proportion of such ω_j is less than μ.

For nonexceptional sequences $\{s_j\}$, the total number of w_j from all intervals I_k remaining under consideration after p steps is less than $N(1-\mu)^p$. This will be less than $\frac{cN}{\log N}$ provided that $p = O(\log\log N)$. Since $\left(\frac{1}{1-\mu}\right)^p = \frac{\log N}{c}$, the total number of subintervals of all the I_k is at most

$$O(\log N)\cdot 2^p = O(\log N)\left(\frac{1}{1-\mu}\right)^{p\log_{1/(1-\mu)}(2)} = O(\log N)^{1+\log_{1/(1-\mu)}(2)}$$
$$= O(\log N)^3$$

as claimed, provided that μ is sufficiently close to $\frac{1}{2}$.

The total probability that a sequence of length N is exceptional is at most the number of times the large-deviations theory was used multiplied by the probability of being exceptional each time, that is, $O((\log N)^3)\cdot e^{-O(N/(\log N)^4)} = e^{-O(N/(\log N)^4)}$, which is clearly summable.

What is the minimum number of permutations that our scheme yields? At each step, all possible permutations are taken of some fraction $\nu \geqq \mu$ of the vortices under consideration, and the remaining vortices are divided into two subgroups with proportions α and $1-\alpha$. Actually, this is only done provided that at least $\frac{N}{(\log N)^4}$ vortices are under consideration. However, there are at most $c\,(\log N)^3$ subintervals in which this condition fails to hold, so ignoring this limitation in calculating the minimum possible number of permutations increases that number by at most a factor $c\,(\log N)^3\cdot\left(\frac{N}{(\log N)^4}\right)!$, which is $(1+o(1))^N$ and so cannot change our determination that the number of permutations is $(\delta N)^N$.

Claim: The number of permutations is minimized when ν is always taken to equal μ (to within the nearest multiple of 1 over the number of vortices under consideration), and α is always taken to equal $\frac{1}{2}$ (to within the nearest multiple of 1 over the number of remaining vortices).

Proof of Claim: The number of permutations of a set of n elements is $n!$. Clearly this number is reduced if we make n smaller, and the product $n! \cdot m!$ is reduced if we replace n by $n-k$ and m by $m+k$ provided that $n-k \geqq m+k$. Start from the smallest subintervals and work backwards. Suppose that at some stage, the minimum number of permutations from any later stages, for any given values of the ν's and α's from the current and any earlier stages, is always obtained when the ν's and α's of those later stages satisfy $\nu = \mu$ and $\alpha = \frac{1}{2}$. Consider the value of α for the current stage given the values of earlier ν's including that from the current stage and values of earlier α's. These choices determine the number n of vortices to be divided among the two subintervals of the current stage. By the induction hypothesis, the number of vortices permuted in each subinterval at this and later stages is proportional to α and $1 - \alpha$, respectively, with the same constants of proportionality in corresponding subintervals. Therefore, if $\alpha \neq \frac{1}{2}$, then we can make it equal $\frac{1}{2}$ (as nearly as possible) by transferring vortices from each subsubinterval of the subinterval having more vortices to the corresponding subsubinterval of the other subinterval. After the transfer the number of vortices is equal (or different by 1), the product of the factorials is reduced. Similarly, the values of ν and α from earlier stages determines the number m of vortices under consideration at the beginning of the current stage. By the induction hypothesis, we may assume that the total number of vortices permuted in subintervals of the current interval is $(1-\nu)m\mu \sum_{j=0}^{p-1}(1-\mu)^j$, where p is the number of stages after the current one. Regardless of the value of p, this total is less than $(1-\nu)n$. Therefore, if $\nu > \mu$, then by transferring $k < (\nu - \mu)n$ vortices from the current subinterval to its subsubintervals and then removing $(\nu - \mu)n - k$ additional vortices, we can change ν into μ (as nearly as possible). This process reduces the product of the factorials, since the number of vortices in the current interval after the transfer is greater than the number of vortices in any of its subintervals. Hence, by finite induction the minimum number of permutations is obtained when every ν equals μ and every $\alpha = \frac{1}{2}$.

For the minimizing values of ν and α, the number of vortices under consideration in the subintervals at each stage is $\frac{1-\mu}{2}$ of the number under consideration at the previous stage, and the number of such subintervals is, of course, double the number at the previous stage. Recall that the number p of stages satisfies $(1 - \mu)^p = \frac{c}{\log N}$. Hence, letting n_k be the number of w_j in I_k, the total number of permutations among vortices whose expected strengths lie in I_k is

$$\prod_{j=0}^{p-1}\left(\left[\left(\frac{1-\mu}{2}\right)^j \mu n_k\right]!\right)^{2^j}$$

$$
\begin{aligned}
&> \prod_{j=0}^{p-1}\left(\frac{\left(\frac{1-\mu}{2}\right)^j \mu n_k}{e}\right)^{(1-\mu)^j \mu n_k} \\
&= \left(\frac{\mu n_k}{e}\right)^{\left(\sum_{j=0}^{p-1}(1-\mu)^j\right)\mu n_k} \cdot \left(\frac{1-\mu}{2}\right)^{\left(\sum_{j=0}^{p-1} j(1-\mu)^j\right)\mu n_k} \\
&\geqq (c_1 n_k)^{\left(1-(1-\mu)^p\right)n_k} \cdot c_2^{c_3 n_k} \\
&\geqq (c_4 n_k)^{n_k - c n_k/\log N}
\end{aligned}
$$

by the formula for the geometric series and the fact that the other sum converges.

Recalling that $N^{-cN/\log N} \geqq c_1^N$, we see that for N sufficiently large

$$
\begin{aligned}
\prod_k (c_4 n_k)^{n_k - c n_k/\log N} &\geqq c^{-N} \prod \left(\frac{n_k}{N}\right)^{n_k - c n_k/\log N} \\
&\geqq c^{-N} \prod \left(\frac{n_k}{N}\right)^{\frac{1}{2} n_k} \\
&\geqq \left[c e^{c \sum \frac{n_k}{N} \log \frac{N}{n_k}}\right]^{-N}.
\end{aligned}
$$

By minimizing the function $c + dx - x\log\frac{1}{x}$ on the interval $[0, 1]$, we find that for such x, $x \log\frac{1}{x} \leqq c + dx$ provided that c and d are positive and $ce^{d+1} \geqq 1$. The constants $c = \frac{1}{k^2}$ and $d = 2\log|k|$ satisfy these inequalities for $k > 1$, so

$$
\begin{aligned}
\sum_k \frac{n_k}{N} \log \frac{N}{n_k} &\leqq c + \sum_{|k|>1} \left[\frac{1}{k^2} + c\log|k| \frac{n_k}{N}\right] \\
&\leqq c + \sum_k \frac{|k|}{N} n_k \\
&\leqq c + c \iint |\omega_0|,
\end{aligned}
$$

which shows that the number of permutations is indeed at least $(\delta N)^N$.

The final initialization scheme to be considered, the *sampling initialization scheme*, is defined as follows: Let Ω be a period cell (or a set containing the support of ω_0), let $\mu(\Omega)$ be its Lebesgue measure, and let δ_1 be a fixed number. Pick N points x_j randomly in Ω and N offsets s_j randomly in $[-\frac{1}{2}, \frac{1}{2}]$, each according to the uniform distribution, and let $\omega_{\text{init}} \equiv \frac{\mu(\Omega)}{N} \sum_{j=1}^N \left(\omega_0(x_j) + \delta_1 s_j\right) \delta(x - x_j)$. Although a proof of the phase-space volume estimate could be built for this scheme along the lines of the proof for the one-vortex-per-cell scheme, the fact that here the vortex strengths and positions are chosen independently makes possible a quick probabilistic, rather than combinatorial, proof.

LEMMA 4.7. *Assume that ω_0 is in L^1. Then for almost all pairs of random sequences $\{x_j\}, \{s_j\}$, the initial vorticity ω_{init} of the sampling initialization scheme converges weak-$*$ in the sense of measures to ω_0 as $N \to \infty$.*

Proof: For every continuous ϕ, the expectation of $\mu(\Omega)(\omega_0(x_j) + \delta_1 s_j)\phi(x_j)$ equals $\iint \phi\omega_0$. The strong law of large numbers (e.g., [23], section VII.8) therefore shows that, with probability 1, $\iint \phi_j\omega_{\text{init}}$ converges to $\iint \phi_j\omega_0$ for every ϕ_j in a given countable set, which may be chosen to be dense in C^0. Similarly, the expectation of $\mu(\Omega)|\omega_0(x_j) + \delta_1 s_j|$ is bounded by a constant plus $\iint |\omega_0|$, so the strong law of large numbers says that the total mass of ω_{init} almost surely converges to its expected value, and so in particular is almost surely uniformly bounded. As in the proof of Lemma 4.5, these together imply that, with probability 1, $\iint \phi\omega_{\text{init}}$ converges to $\iint \phi\omega_0$ for all ϕ in C^0.

LEMMA 4.8. *Suppose that ω_0 has compact support and lies in L^1. For each pair of infinite sequences $\{s_j\}$, $\{x_j\}$, let $I_N(\{s_j\}, \{x_j\})$ be the set of points $(y_1, \dots, y_N)$ in $\mathbb{R}^{2N}$ such that*

$$|\omega_0(x_j) + \delta_1 s_j - \omega_0(y_j)| \leqq \frac{\delta_1}{2} \tag{4.7}$$

for $1 \leqq j \leqq N$. Then there exists a positive δ such that for almost all infinite pairs of sequences $\{x_j\}$, $\{s_j\}$ in $\Omega^\infty \times [-\frac{1}{2}, \frac{1}{2}]^\infty$, the $2N$-dimensional volume of $I_N(\{s_j\}, \{x_j\})$ is at least δ^N for all sufficiently large N.

Proof: Since condition (4.7) that determines the region in which y_j must lie does not depend on the values of other y_i's, the set $I_N(\{s_j\}, \{x_j\})$ is just the product of two-dimensional regions. Letting a_j denote the area of the region for y_j, we can write the volume V_N of $I_N(\{s_j\}, \{x_j\})$ as

$$V_N = \prod_{j=1}^{N} a_j = \left[\mu(\Omega)e^{\frac{1}{N}\sum_{j=1}^{N}\log(a_j/\mu(\Omega))}\right]^N .$$

Furthermore, since the region in which y_j must lie depends only on x_j and s_j, the areas a_j of those regions are independent and identically distributed. Hence, by the strong law of large numbers, in order to show that for almost all pairs of infinite sequences $\{x_j\}$, $\{s_j\}$ we have $V_N \geqq \delta^N$ for N sufficiently large, it suffices to show that the expectation of $\log(a_j/\mu(\Omega))$ is finite. Since $\log \frac{a}{\mu(\Omega)}$ is nonpositive, this will be true as long as that expectation is not $-\infty$. Let $\lambda(v) \equiv \frac{\mu(\{x|\omega_0(x)<v\}\cap\Omega)}{\mu(\Omega)}$ be the normalized distribution function of ω_0. Then the expectation of $\log \frac{a}{\mu(\Omega)}$ is

$$\int_{v=-\infty}^{\infty}\int_{s=-\frac{1}{2}}^{\frac{1}{2}} \log(\lambda(v + \delta_1 s + \tfrac{1}{2}\delta_1) - \lambda(v + \delta_1 s - \tfrac{1}{2}\delta_1))\,ds\,d\lambda(v)\,. \tag{4.8}$$

Make the change of variables $r = \delta_1 s + v$ and $w = v$, where the new outer variable r runs over $\mathbb{R}$. Since the inner integral is now just $\int_{r-\frac{1}{2}\delta_1}^{r+\frac{1}{2}\delta_1} d\lambda(w)$, (4.8) becomes

$$
\begin{aligned}
&-\frac{1}{\delta_1}\int_{r=-\infty}^{\infty}\left[\lambda(r+\tfrac{1}{2}\delta_1)-\lambda(r-\tfrac{1}{2}\delta_1)\right] \\
&\qquad\times\log\frac{1}{\lambda(r+\frac{1}{2}\delta_1)-\lambda(r-\frac{1}{2}\delta_1)}\,dr \\
&\qquad= -\frac{1}{\delta_1}\int_{r=0}^{\delta_1}\sum_{k=-\infty}^{\infty}\left[\lambda(r+(k+\tfrac{1}{2})\delta_1)-\lambda(r+(k-\tfrac{1}{2})\delta_1)\right] \\
&\qquad\quad\times\log\frac{1}{\lambda(r+(k+\frac{1}{2})\delta_1)-\lambda(r+(k-\frac{1}{2})\delta_1)}\,dr,
\end{aligned}
\tag{4.9}
$$

where the identity $\log L \equiv -\log\frac{1}{L}$ has been used to emphasize the fact that the integrand in (4.9) is nonnegative. Using once more the estimate $x\log\frac{1}{x} \leqq \frac{1}{k^2} + 2(\log|k|)x$ for $|k| > 1$ and $0 \leqq x \leqq 1$, we find that the sum in (4.9) is bounded by

$$
\begin{aligned}
&c+\sum_{|k|>1}\left[\frac{1}{k^2}+c(\log|k|)\left(\lambda(r+(k+\tfrac{1}{2})\delta_1)-\lambda(r+(k-\tfrac{1}{2})\delta_1)\right)\right] \\
&\leqq c+\sum_{k}|k|\left(\lambda(r+(k+\tfrac{1}{2})\delta_1)-\lambda(r+(k-\tfrac{1}{2})\delta_1)\right) \\
&\leqq c+c\int|k|\,d\lambda(k)\leqq c+c\iint|\omega_0|\,.
\end{aligned}
$$

Remark 4.9. The exceptional probability for pairs of sequences of length N is e^{-cN}. To see this, we will show that the logarithmic moment-generating function

$$
\Lambda(\alpha) = \log E(e^{\alpha\log[a_j/\mu(\Omega)]}) = \log E\left(\left[\frac{a_j}{\mu(\Omega)}\right]^{\alpha}\right)
$$

is finite for some negative α. By the change of variables leading from (4.8) to (4.9), the expectation here equals

$$
\frac{1}{\delta_1}\int_{r=0}^{\delta_1}\sum_{k=-\infty}^{\infty}\left(\lambda(r+(k+\tfrac{1}{2})\delta_1)-\lambda(r+(k-\tfrac{1}{2})\delta_1)\right)^{1+\alpha}dr\,. \tag{4.10}
$$

Take, for example, $\alpha = -1/3$ and use the estimate $x^{2/3} \leqq \frac{1}{k^2} + c|k|x$ for $k \neq 0$ to bound the sum in (4.10) by

$$
\begin{aligned}
&c+\sum_{|k|>0}\left[\frac{1}{k^2}+c|k|\left(\lambda(r+(k+\tfrac{1}{2})\delta_1)-\lambda(r+(k-\tfrac{1}{2})\delta_1)\right)\right] \\
&\leqq c+c\iint|\omega_0|\,.
\end{aligned}
$$

This finiteness of $\Lambda(\alpha_0)$ for some negative α_0 is known to have pleasant consequences (e.g., [19], lemma 2.2.5(b)). Here we will just need the elementary consequence that

$$\Lambda^*(y) \equiv \sup_{\alpha} \left[\alpha y - \Lambda(\alpha)\right] \geqq \alpha_0 y - \Lambda(\alpha_0) \geqq \text{some } c_0 > 0$$

for y less than or equal to some y_0. Hence by Cramér's theorem ([19], theorem 2.2.3), the probability that $\sum_j \log \frac{a_j}{\mu(\Omega)} \leqq N y_0$ is at most $e^{-(1+o(1))c_0 N}$.

5. Concentration-Volume Estimates

We now turn to estimating the phase-space volume in which concentration may occur.

If $\omega(t,x)$ is a solution of the vorticity equation, then so is $w_k(t,x) \equiv \omega(t,kx)$ for any k; by the appropriate choice of k we can normalize the area of the torus Ω in which the vortices move to equal 1. The phase volume of a set A in Ω^k is then just the probability that the $\{x_j\}_{j=1}^k$ lie in A when the x_j are placed randomly and independently in Ω according to the uniform distribution.

Let us first consider the case when all the vortices of a given sign have the same strength; the probability that either the set of positive vortices lies in A^+ or the set of negative vortices lies in A^- is less than the sum of the individual probabilities of those occurrences. We are interested in sets A^+ of all sequences $\{x_j\}_{j=1}^{N^+}$ such that the corresponding vorticity $\omega_N^+ \equiv \frac{\omega^+}{N} \sum_{j=1}^{N^+} \delta(x-x_j)$ lies in some subset $\mathscr{A}$ of the set $\mathscr{M}(\Omega)$ of measures on Ω; the probability of A is then the probability that ω_N^+ lies in $\mathscr{A}$. The x_j here are not the original locations of the vortices but random locations chosen according to the uniform distribution on Ω. It is convenient to work with the normalized vorticity $w_N \equiv \frac{N}{\omega^+ N^+} \omega_N^+ = \frac{1}{N^+} \sum_{j=1}^{N^+} \delta(x-x_j)$ that lies in the set $\mathscr{M}_1(\Omega)$ of nonnegative measures having total mass 1. Since we showed in the proof of Lemma 4.3 that $\frac{N^+}{N}$ is bounded away from 0 and ∞, the two may be used interchangeably; in particular, any probability that is e^{-cN^+} is $e^{-c_1 N}$, and vice versa. Sanov's theorem (e.g., [19], 6.2.10) then says that the probability that $w_N \in \mathscr{A} \subseteq \mathscr{M}_1(\Omega)$ is at most

$$e^{-(1-o(1))N^+ \inf_{\nu \in \overline{\mathscr{A}}} H(\nu \mid dx)}, \tag{5.1}$$

where

$$H(\nu \mid dx) = \begin{cases} \iint_\Omega f \log f \, dx & \text{if } \nu = f\,dx \text{ is absolutely continuous with} \\ & \qquad\text{respect to Lebesgue measure} \\ +\infty & \text{otherwise.} \end{cases}$$

We wish to find "exceptional" sets for which the infimum in (5.1) is arbitrarily large, so that the probabilities of those sets are at most δ^{N^+} with δ arbitrarily small.

Furthermore, we would like concentration to be impossible when w_N lies in the complement of any fixed exceptional set. Now the set $\overline{\mathscr{A}}$ over which H is to be minimized is the closure of $\mathscr{A}$ in $\mathscr{M}_1(\Omega)$ in the weak-$*$ topology of measures. Therefore, the infimum obtained when $\mathscr{A}$ is $\mathscr{A}(L) \equiv \{\nu \mid H(\nu \mid dx) \geqq L\}$ is not L but 0, since even the set of finite linear combinations of Dirac masses is weak-$*$ dense in the set of measures and so includes in its closure the measure $\nu = 1\,dx$ for which $H(\nu \mid dx) = 0$. This problem occurs because even when the x_j are well-distributed so that w_N looks approximately like $1\,dx$ when averaged over a fixed finite-length scale, if we look at arbitrarily small-length scales, then w_N always looks exceptional since it is a sum of Dirac masses. A truly exceptional w_N must be exceptional even on some fixed finite-length scale. This suggests taking the exceptional set to be $\mathscr{A}(L, \varepsilon(N))$, where $\mathscr{A}(L,\varepsilon) \equiv \{\nu \mid J_\varepsilon \nu \in \mathscr{A}(L)\} = \{\nu \mid H(J_\varepsilon \nu \mid dx) \geqq L\}$, J_ε is a mollifier, and $\varepsilon(N)$ is to be chosen later. We will show that $\mathscr{A}(L,\varepsilon)$ is closed and that $H(\nu \mid dx) \geqq H(J_\varepsilon \nu \mid dx)$, which together imply that $\inf_{\nu \in \overline{\mathscr{A}(L,\varepsilon)}} H(\nu \mid dx) \geqq L$, so that the probability of $\mathscr{A}(L,\varepsilon) \leqq e^{-(1-o(1))NL}$ Furthermore, we will also show that the nonconcentration condition (2.9) holds when w_N lies in the complement of $\mathscr{A}(L, \varepsilon(N))$, provided that $\varepsilon(N)$ tends to 0 as N goes to infinity.

The mollifier J_ε is defined for periodic functions by $[J_\varepsilon w](x) = \iint_{\mathbb{R}^2} \phi(z) w(x - \varepsilon z)\,dz$, where ϕ is a smooth, nonnegative function of compact support whose integral equals 1. The change of variables $y = x - \varepsilon z$ transforms this into $[J_\varepsilon w](x) = \iint_{\mathbb{R}^2} \frac{1}{\varepsilon^2} \phi(\frac{x-y}{\varepsilon}) w(y)\,dy$. Since w is periodic, it is useful to convert this last integral into an integral over one period cell Ω: Let p_i be the period of w with respect to x_i and define $v_1 = \binom{p_1}{0}$, $v_2 = \binom{0}{p_2}$. Then

$$[J_\varepsilon w](x) = \sum_{k_1,k_2 \in \mathbb{Z}} \iint_\Omega \frac{1}{\varepsilon^2} \phi\left(\frac{x - y - k_1 v_1 - k_2 v_2}{\varepsilon}\right) w(y)\,dy. \tag{5.2}$$

Now for x and y in Ω, $|x_i - y_i| \leqq p_i$. Since ϕ has compact support, this implies that for ε sufficiently small the only nonzero terms in (5.2) are those with $|k_i| \leqq 1$, that is,

$$\begin{aligned}[J_\varepsilon w](x) &= \iint_\Omega \phi_\varepsilon(x - y) w(y)\,dy \\ &\equiv \iint_\Omega \frac{1}{\varepsilon^2} \sum_{k_i \in \{-1,0,1\}} \phi\left(\frac{x - y - k_1 v_1 - k_2 v_2}{\varepsilon}\right) w(y)\,dy.\end{aligned}$$

Similarly, when ν is a measure, then $J_\varepsilon \nu = \iint \phi(z) \nu(x + \varepsilon z)\,dz$; that is, $\iint g J_\varepsilon \nu = \iint \phi(z) \left(\iint g(x) \nu(x + \varepsilon z)\right)\,dz$. Just as for functions, the variable x can be transferred from ν to ϕ by changes of variables, so $J_\varepsilon \nu$ is a smooth function times dx.

LEMMA 5.1. *There exists a function $\varepsilon(N)$ tending to 0 as $N \to \infty$ such that for N sufficiently large, the probability that $w_N \in \mathscr{A}(L, \varepsilon(N))$ is at most $e^{-c(L-k)N}$*

Furthermore, when $w_N \notin \mathscr{A}(L, \varepsilon)$ then

$$\sup_x \iint_{|y-x|<r} w_N(y) \leqq \frac{c(L+k)}{\log \frac{c}{r+c\varepsilon}} . \tag{5.3}$$

Proof: Let f_ν denote the Radon-Nykodym derivative of ν when it is absolutely continuous with respect to Lebesgue measure; then $\nu = f_\nu \, dx$. The set $f_{J_\varepsilon \mathscr{M}_1(\Omega)}$ is a bounded subset of C^1 and hence a precompact set in C^0. Therefore, if ν_j converges weak-$*$ in $\mathscr{M}_1(\Omega)$ to some ν, then $f_{J_\varepsilon \nu_j}$ converges to $f_{J_\varepsilon \nu}$ in C^0 and hence $H(J_\varepsilon \nu_j \mid dx)$ converges to $H(J_\varepsilon \nu \mid dx)$, which shows that $\mathscr{A}(L, \varepsilon)$ is a closed set. Therefore, by Sanov's theorem (e.g., [19], 6.2.10), the probability that $\omega \in \mathscr{A}(L, \varepsilon)$ is at most

$$e^{-N(1-o_\varepsilon(1)) \inf_{\nu \in \mathscr{A}(L,\varepsilon)} H(\nu \mid dx)} , \tag{5.4}$$

where $o_\varepsilon(1)$ denotes a function of N and ε that, for each fixed ε, converges to 0 when $N \to \infty$. Now if ν is not absolutely continuous with respect to Lebesgue measure, then $H(J_\varepsilon \nu \mid dx) < \infty = H(\nu \mid dx)$, while if $\nu = f_\nu \, dx$, then the convexity of $h(x) \equiv x \log x$ implies that

$$\begin{aligned} H(J_\varepsilon \nu \mid dx) &= \iint h\left(\iint \phi(z) f_\nu(x + \varepsilon z) \, dz\right) dx \\ &\leqq \iint \iint \phi(z) h(f_\nu(x + \varepsilon z)) \, dz \, dx = \iint h(f_\nu(x)) \, dx = H(\nu \mid dx), \end{aligned}$$

so the infimum in the exponent of (5.4) is at least L. Since for each fixed ε the expression $o_\varepsilon(1)$ converges to 0 as $N \to \infty$, for every ε there is an $N(\varepsilon)$ such that that expression is at most $\frac{1}{2}$ for $N \geqq N(\varepsilon)$. Letting $\varepsilon(N)$ equal $\min\{\delta > 0 \mid N \geqq N(\delta)\}$ yields the first conclusion of the lemma.

Next, since the mollifier J_ε spreads mass over a radius $c\varepsilon$,

$$\iint_{|x-y| \leqq r} \nu(y) \leqq \iint_{|x-y| \leqq r + c\varepsilon} J_\varepsilon \nu .$$

Also, for all nonnegative f having integral 1, $\iint f\,(\log f)^{-}\,dx \leqq c$. Hence, for $\nu \in \mathcal{M}_1(\Omega)\setminus\mathcal{A}(L,\varepsilon)$,

$$
\begin{aligned}
L + c &\geqq \iint f_{J_\varepsilon\nu}\log^{+} f_{J_\varepsilon\nu}\,dy \\
&\geqq \sup_x \iint\limits_{|x-y|<r} f_{J_\varepsilon\nu}\log^{+} f_{J_\varepsilon\nu}\,dy \\
&\geqq \sup_x \iint\limits_{|x-y|<r} f_{J_\varepsilon\nu}\log f_{J_\varepsilon\nu}\,dy = \pi r^2 \sup_x \fint\!\!\fint\limits_{|x-y|<r} f_{J_\varepsilon\nu}\log f_{J_\varepsilon\nu}\,dy \\
&\geqq \pi r^2 \sup_x \fint\!\!\fint\limits_{|x-y|<r} f_{J_\varepsilon\nu}\,dy\,\log\left(\ \fint\!\!\fint\limits_{|x-y|<r} f_{J_\varepsilon\nu}\,dy\right) \\
&= \sup_x\left[\log\frac{1}{\pi r^2}\iint\limits_{|x-y|<r} f_{J_\varepsilon\nu}\,dy + \iint\limits_{|x-y|<r} f_{J_\varepsilon\nu}\,dy\,\log\left(\ \iint\limits_{|x-y|<r} f_{J_\varepsilon\nu}\,dy\right)\right] \\
&\geqq \log\frac{1}{\pi r^2}\sup_x \iint\limits_{|x-y|<r} f_{J_\varepsilon\nu}\,dy - c \geqq \log\frac{1}{\pi r^2}\sup_x\left[\ \iint\limits_{|x-y|<r-c\varepsilon} \nu\right] - c,
\end{aligned}
$$

which implies the second conclusion of the lemma.

Since, as we will see later, vorticities in $L\log L$ give rise to velocities in L^2_{loc}, the parameter $\varepsilon(N)$ determines how much the velocity must be smoothed in order to be uniformly bounded in L^2_{loc}. How slowly must $\varepsilon(N)$ tend to 0? Since $J_\varepsilon w$ is a weighted average of w over a set of measure $O(\varepsilon^2)$, if $\varepsilon \ll \frac{1}{\sqrt{N}}$, then $J_\varepsilon w$ will be 0 except on a set of small measure, on which it will be very large. Hence, no matter how well the x_j are distributed, the integral $\iint_\Omega f_{J_\varepsilon w}\log f_{J_\varepsilon w}$ will be large, which means that the criterion that that integral be large gives no information and the $o_\varepsilon(1)$ term must be nearly 1. This shows that $\varepsilon(N)$ must be at least $O(\frac{1}{\sqrt{N}})$. Up to a logarithmic factor, this is also sufficient, as we will now show:

LEMMA 5.2. *The function $\varepsilon(N)$ in Lemma 5.1 can be taken to be $\frac{(\log N)^{3/2}}{\sqrt{N}}$.*

Proof: By exercise 6.2.19 of [19],

$$
\operatorname{Prob}(w_N \in \mathcal{A}(L,\varepsilon)) \leqq \inf_{\delta>0}\left\{m(\mathcal{M}_1(\Omega),\delta)e^{-N^{+}\inf_{\nu\in(\mathcal{A}(L,\varepsilon))^\delta} H(\nu\,|\,dx)}\right\}, \tag{5.5}
$$

where $m(\mathcal{M}_1(\Omega),\delta)$ is the number of balls of size δ needed to cover $\mathcal{M}_1(\Omega)$, and $A^\delta = \{\nu \mid \exists\mu \in A,\ d(\nu,\mu) \leqq \delta\}$ is the δ-neighborhood of A in a metric d compatible with the weak-$*$ topology restricted to $\mathcal{M}_1(\Omega)$. We will use the Kantorovich-Rubinshtein-Vasershtein ([21], [36]) metric rather than the Levy-Prokhorov metric used in [19]. The KRV metric is defined in the periodic case by $d(\mu,\nu) =$

$\inf_{dP} \iint_\Omega \iint_\Omega |x-y|_{\text{per}} dP(x,y)$, where $|\cdot|_{\text{per}}$ is the distance function on the torus Ω (i.e., $|x|_{\text{per}} = \inf_{k\in\mathbb{Z}^2} |x-k|$) and the infimum is taken over all measures $dP(x,y)$ such that $\iint\iint f(x)\,dP(x,y) = \iint f(x)\mu(x)$ and $\iint\iint f(y)\,dP(x,y) = \iint f(y)\nu(y)$. Such a dP will be called a *joint measure* for μ and ν. If dP_i are joint measures for μ_i and ν_i, $i=1,2$, then $\alpha dP_1 + (1-\alpha)dP_2$ is a joint measure for $\alpha\mu_1 + (1-\alpha)\mu_2$ and $\alpha\nu_1 + (1-\alpha)\nu_2$. This implies the convexity condition ([19], 6.1.5)

$$d(\alpha\mu_1 + (1-\alpha)\mu_2, \alpha\nu_1 + (1-\alpha)\nu_2) \leqq \max\{d(\mu_1,\nu_1), d(\mu_2,\nu_2)\},$$

while all the other conditions needed to obtain (5.5) follow from the fact that $\mathcal{M}_1(\Omega)$ is compact in the weak topology. A continuum version of this convexity of joint measures says that if $dP(x,y)$ is a joint measure for μ and ν, then $\iint_\Omega \phi(z)dP(x-\varepsilon z,\ y-\varepsilon z)\,dz$ is a joint measure for $J_\varepsilon\mu$ and $J_\varepsilon\nu$, since

$$\begin{aligned}
&\iint_\Omega\iint_\Omega f(x)\iint_\Omega \phi(z)dP(x-\varepsilon z,\ y-\varepsilon z)\,dz \\
&= \iint_\Omega \phi(z)\left[\iint_\Omega\iint_\Omega f(x)dP(x-\varepsilon z,\ y-\varepsilon z)\right] dz \\
&= \iint_\Omega \phi(z)\left[\iint_\Omega f(x)\,d\mu(x-\varepsilon z)\right] dz \\
&= \iint_\Omega f(x)\left[\iint_\Omega \phi(z)\,d\mu(x-\varepsilon z)\,dz\right] \\
&= \iint_\Omega f(x)J_\varepsilon\mu(x),
\end{aligned}$$

with a similar calculation holding when f depends only on y. This implies that $d(J_\varepsilon\mu, J_\varepsilon\nu) \leqq d(\mu,\nu)$. Next,

$$\begin{aligned}
&\iint_\Omega |f_{J_\varepsilon\mu}(z) - f_{J_\varepsilon\nu}(z)|\,dz \\
&= \iint_\Omega \left|\iint_\Omega \phi_\varepsilon(x-z)\,d\mu(x) - \iint_\Omega \phi_\varepsilon(y-z)\,d\nu(y)\right| dz \\
&= \iint_\Omega \left|\iint_\Omega\iint_\Omega [\phi_\varepsilon(x-z) - \phi_\varepsilon(y-z)]\,dP(x,y)\right| dz \\
&\leqq \min_{dP}\iint_\Omega\iint_\Omega \left[\iint_\Omega |\phi_\varepsilon(x-z) - \phi_\varepsilon(y-z)|dz\right] dP(x,y) \\
&\leqq \frac{c}{\varepsilon}\min_{dP}\iint_\Omega\iint_\Omega |x-y|_{\text{per}}\,dP(x,y) \\
&= \frac{c}{\varepsilon}d(\mu,\nu).
\end{aligned}$$

Also, $\|f_{J_\varepsilon\mu}\|_{L^\infty} \leqq \frac{c}{\varepsilon}$. Hence, if $\nu \in \mathscr{A}(L,\varepsilon)^\delta$, then there exists a $\mu \in \mathscr{A}(L,\varepsilon)$ such that $d(\mu,\nu) \leqq \delta$, and so

$$\begin{aligned} H(J_\varepsilon\nu \mid dx) &= H(J_\varepsilon\mu \mid dx) - \iint_\Omega \left[f_{J_\varepsilon\mu}\log f_{J_\varepsilon\mu} - f_{J_\varepsilon\nu}\log f_{J_\varepsilon\nu}\right] dx \\ &\geqq L - \iint_\Omega \left[f_{J_\varepsilon\mu}\log^+ f_{J_\varepsilon\mu} - f_{J_\varepsilon\nu}\log^+ f_{J_\varepsilon\nu}\right] dx - c \\ &\geqq L - c - \left(1 + \log\frac{c}{\varepsilon}\right)\iint_\Omega |f_{J_\varepsilon\mu} - f_{J_\varepsilon\nu}| \\ &\geqq L - c - \left(1 + \log\frac{c}{\varepsilon}\right)\frac{c\delta}{\varepsilon}. \end{aligned}$$

Recalling that $H(\nu \mid dx) \geqq H(J_\varepsilon\nu \mid dx)$, we therefore obtain that if $\varepsilon \geqq \frac{\delta}{c}\log\frac{1}{\delta}$, then $\inf_{\nu\in\mathscr{A}(L,\varepsilon)^\delta} H(\nu \mid dx) \geqq L - c_1$.

On the other hand, we can cover Ω by $O(\frac{1}{\delta^2})$ rectangles of diameter δ. As in exercise 6.2.19 in [19], we can approximate any measure $\nu \in \mathscr{M}_1(\Omega)$ to within $c\delta$ by a measure consisting of a sum of delta functions at the centers of the rectangles, with weights that are integer multiples of $1/[\![1/\delta^3]\!]$: Let the weight of the delta function in each cell be the greatest integer multiple of $1/[\![1/\delta^3]\!]$ that does not exceed the total mass of ν in the rectangle except that the weight of one delta function $\delta(x - x_l)$ is increased if necessary so that the sum of the weights equals 1; call the resulting measure μ. Write the restriction of ν to each rectangle as $\nu_i + r_i$, where the total mass of μ_i equals the weight of the delta function in that rectangle. Then $dP(x,y) \equiv \sum_i \nu_i(x)\delta(y-x_i) + (\sum_i r_i(x))\delta(y-x_l)$ is a joint measure for ν and μ, and

$$\iint_\Omega\iint_\Omega |x-y|\,dP(x,y) \leqq \sum_i \delta\|\nu_i\|_{\mathscr{M}} + c\sum r_i \leqq c\delta + c\frac{1}{\delta^2}\delta^3 \leqq c\delta.$$

This shows that, just as for the Levy-Prokhorov metric, any $\nu \in \mathscr{M}_1(\Omega)$ can be approximated to within $c\delta$ by a sum of delta functions at $O(\frac{1}{\delta^2})$ fixed locations, with weights that are integer multiples of $O(\delta^3)$ and sum to 1. By [19], exercise 6.2.19, this implies that $m(\mathscr{M}_1(\Omega),\delta) \leqq e^{\frac{c}{\delta^2}\log\frac{1}{\delta}}$. Hence, $m(\mathscr{M}_1(\Omega),\delta) \leqq e^{cN}$ provided that $\delta \geqq \sqrt{\frac{\log N}{N}}$. Putting this bound together with the bound for ε in terms of δ shows that we may take $\varepsilon = \frac{(\log N)^{3/2}}{N^{1/2}}$.

We now consider the empirical absolute vorticity $|\omega_N| \equiv \frac{1}{N}\sum_{j=1}^N |\omega_j|\delta(x-x_j)$ obtained from the point-vortex method using one of the classical initialization schemes. In this case the probability that $|\omega_N|$ lies in $\mathscr{A}(L,\varepsilon(N))$ with $\varepsilon(N) \to 0$ cannot be as small as $e^{-cN(L-k)}$ unless we place some restriction on the ω_j. To see this, note that for any $\omega_0 \in L^1$ the results of Section 4 imply that there are sets A_N of probability δ^N in Ω^N on which ω_N converges weak-$*$ to ω_0. But by 6.2.3 and 6.2.13 in [19], the function $H(\nu \mid dx)$ is a rate function on $\mathscr{M}_1(\Omega)$

with the weak-$*$ topology and so is weak-$*$ lower semicontinuous. Hence when $\omega_0 \in (L^1 \setminus L \log L)$, the minimum of $H(J_\varepsilon(N)\nu \mid dx)$ on A_N tends to infinity. Thus, on the one hand we can no longer conclude that estimate (5.3) holds except with probability $e^{-c(L-k)N}$, but on the other hand we are also not limited to initial vorticities lying in $L \log L$.

Of course, this new situation does not contradict the results of Lemma 5.1 since $H(\nu \mid dx)$ is no longer the rate function governing the large deviations of $|\omega_N|$. There does not seem to be an explicit formula like $H(\nu \mid dx)$ for the rate function for $|\omega_N|$, so we will have to resort to estimates for it. We will look for two different estimates under different conditions on ω_0, because the conclusions of Lemma 5.1 serve two different purposes. First of all, (5.3) shows that concentration does not occur, and for this purpose it suffices to replace the right side of (5.3) by any function of $r + c\varepsilon$ tending to zero at 0. Such an estimate will be obtained with probability $e^{-c(L-k)N}$ for any ω_0 in L^1. Furthermore, the condition that ω_N lie in $\mathscr{A}(L, \varepsilon(N))$ for all N implies that its weak-$*$ limit lies in $L \log L$, and this in turn implies, as we will see later, that the limit velocity lies in L^2_{loc}. In order for the limit velocity to be in L^2_{loc}, it suffices for the limit vorticity to lie in $L\,(\log L)^\alpha$ with $\alpha > \frac{1}{2}$, and this will be obtained when the initial vorticity ω_0 lies in L^p with $p > 2$.

As in the derivation of the initial-volume estimates, the case when the ω_j are obtained via sampling is simpler and so will be considered here first. For any nonnegative convex function Q, define $\mathscr{A}_Q(L, \varepsilon) = \{\nu \in \mathscr{M}_1(\Omega) \mid H_Q(J_\varepsilon \nu\, dx) \geqq L\}$, where

$$H_Q(\nu \mid dx) = \begin{cases} \iint_\Omega Q(f)\, dx & \text{if } \nu = f\,dx \text{ is absolutely continuous with respect to Lebesgue measure} \\ +\infty & \text{otherwise.} \end{cases}$$

LEMMA 5.3. *For every $\omega_0 \in L^1$ there exists an $\varepsilon(N)$ tending to 0 as $N \to \infty$ and a nonnegative superlinear convex function Q such that for almost all sequences of initial offsets $\{x_j, s_j\} \in \Omega^\infty \times [-\frac{1}{2}, \frac{1}{2}]^\infty$ that determine the ω_j via the sampling initialization scheme, the probability of the set of $\{z_j\}$ for which the vorticity $\omega_N = \sum_{j=1}^N \omega_j \delta(x - z_j)$ lies in $\mathscr{A}_Q(L, \varepsilon(N))$ is at most $e^{-cN(L-k)}$. Also, there exists a $d(r)$ tending to 0 with r such that when $\omega_N \notin \mathscr{A}_Q(L, \varepsilon)$ then*

$$\sup_x \iint_{|y-x|<r} |\omega_N(y)| \leqq c(L + k) d(r + c\varepsilon). \tag{5.6}$$

Furthermore, when ω_0 lies in L^p with $p > 2$, then $Q(x)$ may be taken to be $x(\log(e + x))^{1-\frac{1}{p}}$.

Proof: By the abstract Gärtner-Ellis theorem ([19], 4.5.1, 4.5.20) the probability that $|\omega_N| \equiv \frac{1}{N} \sum |\omega_j| \delta(x - z_j) \in \mathscr{A}$ is at most

$$e^{-(1-o(1))N \inf_{\nu \in \overline{\mathscr{A}}} \Lambda^*(\nu)}, \tag{5.7}$$

where

$$\Lambda^*(\nu) = \sup_{\alpha \in C^0} \left[\iint \alpha\nu - \Lambda(\alpha) \right],$$

provided that for all $\alpha \in C^0$ the limit

$$\Lambda(\alpha) \equiv \lim_{N\to\infty} \frac{1}{N} \log E_{\{z_j\}} \left(e^{\sum_j |\omega_j| \alpha(z_j)} \right) \tag{5.8}$$

exists (the value $+\infty$ is allowed). Since the z_j are independent, the limit in (5.8) can be written as

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} \log \left(\iint e^{|\omega_j|\alpha(x)}\, dx \right). \tag{5.9}$$

The sum in (5.9) is just the sample mean of the random variable

$$v_j \equiv \log \left(\iint e^{|\omega_j|\alpha(x)}\, dx \right).$$

Since the v_j are independent and identically distributed and $|v_j| \leqq \|\alpha\|_{C^0}|\omega_j|$, which has finite expectation, the strong law of large numbers shows that the sum in (5.9) converges almost surely to the expectation of $|v_j|$. Thus, for almost all sequences of initial offsets,

$$\Lambda(\alpha) = \int_{v=-\infty}^{\infty} \int_{s=-\frac{1}{2}}^{\frac{1}{2}} \log \left(\int e^{|v+\delta_1 s|\alpha(x)}\, dx \right) ds\, d\lambda(v), \tag{5.10}$$

where $\lambda(v)$ is the distribution function of ω_0. From now on we restrict our attention to those sequences for which (5.10) holds.

Let $\Lambda^*(\nu) \equiv \sup_{\alpha\in C^0} [\iint \alpha\nu - \Lambda(\nu)]$ be the Legendre transform of Λ. Now $\iint(\alpha + c)\nu - \Lambda(\alpha + c) = \iint \alpha\nu - \Lambda(\alpha) + c(\iint \nu - E(|\omega_j|))$ so that if $\iint \nu \neq E(|\omega_j|)$ then taking $c \to \pm\infty$ yields $\Lambda^*(\nu) = +\infty$. When $\omega(t,x)$ is a solution of the Euler equations, then so is $c\omega(ct,x)$, which allows us to normalize $E(|\omega_j|)$ to equal 1. Also, if ν is not nonnegative, then by taking $\alpha \to -\infty$ on some set on which ν has negative mass again shows that $\Lambda^*(\nu) = +\infty$. This is not surprising since upon multiplying $|\omega_N|$ by $1 + o(1)$ we can make it lie in $\mathcal{M}_1(\Omega)$; from now on we will assume that this has been done.

Now any ω_0 in L^1 satisfies

$$\iint \Phi(|\omega_0(x)|)\, dx < \infty \tag{5.11}$$

for some nonnegative convex function Φ such that

$$\lim_{t\to+\infty} \frac{\Phi(t)}{t} = +\infty, \tag{5.12}$$

$$\lim_{t\to 0^+} \frac{\Phi(t)}{t} = 0, \tag{5.13}$$

and

$$\Phi(t + \delta_1) \leqq c\Phi(t) + c. \tag{5.14}$$

To see this, let k_j be any sequence of positive numbers increasing to infinity, let a_j be any summable nonnegative sequence, let t_j be a sequence of positive numbers tending to infinity such that $\iint_{|\omega_0|>t_j} k_j|\omega_0| \leqq a_j$, and define $\Phi(t) = \sum_j k_j \left(\min(|t|, t_{j+1}) - t_j\right)^+$. Then Φ is continuous and even, vanishes for $0 \leqq t \leqq t_1$, and is affine with slope k_j on $[t_j, t_{j+1}]$. Hence Φ satisfies $\Phi(t) < k_j t$ on $[t_j, t_{j+1}]$, which implies (5.11), and satisfies $\Phi(t) \geqq k_j t - c_j$ on $[t_j, \infty)$, which implies (5.12). Next, by successively increasing the t_j in order of increasing j, if necessary, we can obtain for $j > 1$ the estimates $\Phi(t_j) \geqq \frac{k_{j+1}}{c}$ and $t_{j+1} \geqq t_j + \delta_1$. Then for $j > 1$, if $t_j \leqq t < t_{j+1}$, then $\Phi(t+\delta_1) \leqq \Phi(t) + k_{j+1}\delta_1 \leqq (1 + c\delta_1)\Phi(t)$, so that (5.14) holds. Conditions (5.12) and (5.13) are standard conditions for convex functions used to define Orlicz spaces (e.g., [32], (1.15), (1.16)), while (5.14) is a slight variant of such a condition. The existence of such a Φ has been proven and used previously in [9]. Let $\Phi^*(x) \equiv \sup_y\{xy - \Phi(y)\}$ be the Legendre transform of Φ, so that $xy \leqq \Phi(x) + \Phi^*(y)$. Since Φ is even, for $x > 0$ the value $y = y_*(x)$ at which the supremum defining $\Phi^*(x)$ occurs is nonnegative. Condition (5.13) then implies that $y_*(x)$ is strictly positive for positive x, and this in turn implies that Φ^* is strictly increasing for $x > 0$, since $\Phi^*(x + \delta) \geqq \Phi^*(x) + \delta y_*(x)$.

Let $(\Phi^*)^{-1}$ be the inverse function of the restriction of Φ^* to nonnegative numbers. Then by letting α tend to $(\Phi^*)^{-1}(\log^+ f)$ we obtain for any $\nu \in \mathcal{M}_1(\Omega)$

$$\begin{aligned}
\Lambda^*(\nu) &\geqq \iint f_\nu (\Phi^*)^{-1}(\log^+ f_\nu)\,dx \\
&\quad - \int_{v=-\infty}^{\infty} \int_{s=-\frac{1}{2}}^{\frac{1}{2}} \log\left(\int e^{|v+\delta_1 s|(\Phi^*)^{-1}(\log^+ f_\nu)}\,dx\right) ds\,d\lambda(v) \\
&\geqq \iint f_\nu (\Phi^*)^{-1}(\log^+ f_\nu)\,dx \\
&\quad - \int_{v=-\infty}^{\infty} \int_{s=-\frac{1}{2}}^{\frac{1}{2}} \log\left(\int e^{\Phi(|v+\delta_1 s|)+\log^+ f_\nu}\,dx\right) ds\,d\lambda(v) \\
&\geqq \iint f_\nu (\Phi^*)^{-1}(\log^+ f_\nu)\,dx - k
\end{aligned} \tag{5.15}$$

where the last inequality follows from the boundedness of $\iint f_\nu\,dx = 1$ and of $\iint 1\,dx$, (5.11), and (5.14).

Now, although the function $q(x) \equiv x(\Phi^*)^{-1}(\log^+ x)$ may not be convex, it is larger than some nonnegative convex superlinear function. To see this, note that the function $g(x) \equiv (\Phi^*)^{-1}(\log^+ x)$ is nonnegative and increases to $+\infty$ with x. Therefore $q'(x) = g(x) + xg'(x)$ also tends to $+\infty$ with x. Hence for all n there exists a $c(n)$ such that $q(x) \geqq nx - c(n)$. This implies that the convexification $Q(x) \equiv q(x)^{**}$ satisfies $\frac{Q(x)}{x} \geqq n - \frac{c(n)}{x}$, so $\lim_{x\to\infty} \frac{Q(x)}{x} = +\infty$. As is well-known (e.g. [19], 4.5.10(c)), $q(x)^{**} \leqq q(x)$, and the fact that $q(x)$ is nonnegative implies that q^{**} is too, so $Q(x)$ is the desired nonnegative convex superlinear function smaller than $q(x)$.

Since $Q(f) \leqq q(f)$, (5.15) implies that $\Lambda^*(f\,dx) \geqq \iint Q(f)\,dx - k$. Just as for $\mathscr{A}(L,\varepsilon)$ in the proof of Lemma 5.1, the set $\mathscr{A}_Q(L,\varepsilon)$ is closed, and the normalized $|\omega_N|$ are trivially exponentially tight since they are supported on the compact set $\mathscr{M}_1(\Omega)$, so by the abstract Gärtner-Ellis theorem ([19], 4.5.20) the probability that the normalized $|\omega_N|$ lies in $\mathscr{A}_Q(L,\varepsilon)$ is at most $e^{-(1-o_\varepsilon(1))N(L-k)}$.

Furthermore, for $\nu \in \mathscr{M}_1(\Omega)\setminus A_Q(L,\varepsilon)$,

$$
\begin{aligned}
\sup_x \iint_{|x-y|<r} \nu(y) &\leqq \sup_x \iint_{|x-y|<r+c\varepsilon} J_\varepsilon \nu(y) \\
&= \sup_x \left[\iint_{\substack{|x-y|<r+c\varepsilon,\\ \{f_{J_\varepsilon\nu}\leqq k\}}} f_{J_\varepsilon\nu}(y)\,dy + \iint_{\substack{|x-y|<r+c\varepsilon,\\ \{f_{J_\varepsilon\nu}>k\}}} f_{J_\varepsilon\nu}(y)\,dy \right] \\
&\leqq \inf_k ck(r+c\varepsilon)^2 + \frac{1}{Q(k)/k} \iint Q(f_{J_\varepsilon\nu})\,dy \\
&\leqq \inf_k ck(r+c\varepsilon)^2 + \frac{L}{Q(k)/k} \equiv d(r+c\varepsilon).
\end{aligned}
$$

Since $\lim_{k\to\infty} \frac{Q(k)}{k} = +\infty$, $d(r)$ tends to 0 with r.

Finally, when ω_0 lies in L^p, we can take $\Phi(x)$ to be $\frac{|x|^p}{p}$, so that $\Phi^*(x) = \frac{|x|^q}{q}$, where $\frac{1}{p} + \frac{1}{q} = 1$. Hence $(\Phi^*)^{-1}(y) = (qy)^{\frac{1}{q}} = cy^{1-\frac{1}{p}}$, which means that $Q(x)$ can be taken to be $x(\log(e+x))^{1-\frac{1}{p}}$ since it suffices for $Q(x)$ to be less than $x(\Phi^*)^{-1}(\log^+ x) + c$. This yields the final claim of the lemma for $1 < p < \infty$.

Remark 5.4. Although the special case $|w|\alpha \leqq c_p|w|^{\frac{1}{1-p}} + \alpha^{\frac{1}{p}}$ of the estimate $xy \leqq \Phi(x) + \Phi^*(y)$ used in the above proof may seem crude, the result obtained is essentially sharp in the sense that

$$
(5.16)\quad \Lambda^*(\nu) \geqq c \iint f_\nu \left[\log^+ f_\nu\right]^p dx - k \quad \text{for all } \nu \text{ only when } \omega_0 \text{ is in } L^{\frac{1}{1-p}}.
$$

Since, as will be shown later, $\omega \in L\left(\log^+ L\right)^p$ with $p < \frac{1}{2}$ does not imply that u lies in L^2_{loc}, the result of the lemma cannot be improved to $\omega_0 \in L^r$ with $r < 2$.

To see that (5.16) is true, we will use the duality lemma for the Legendre transform ([19], 4.5.8) to write

$$\Lambda(\alpha) = \sup_{\nu \in \mathcal{M}(\Omega)} \left[\iint \alpha\nu - \Lambda^*(\nu) \right].$$

Note that Λ is continuous, and a direct calculation or using 4.5.3 in [19] shows that it is convex, so the hypotheses of that lemma hold. Recall that $\Lambda^*(\nu) = +\infty$ unless ν is nonnegative and has total mass equal to the expectation of $|\omega_j|$. Furthermore, if the estimate for Λ^* from (5.16) holds, then $\Lambda^*(\nu) = \infty$ unless ν is absolutely continuous with respect to Lebesgue measure. Hence, when that estimate holds then

$$\Lambda(\alpha) \leqq k + \sup_{\substack{f \geqq 0 \\ \iint f\,dx = 1}} \left(\iint \alpha f\,dx - c \iint f\,[\log^+ f]^p\,dx \right). \tag{5.17}$$

Next, for any nonnegative increasing function h and nonnegative functions f and g, $fh(g) \leqq fh(f) + gh(g)$, since the first term on the right exceeds the left when $g \leqq f$ while the second term on the right exceeds the left when $f \leqq g$. Hence, taking α in (5.17) to have the form $c\,[\log^+ g(x)]^p$, we obtain

$$\begin{aligned} \Lambda(c\,[\log^+ g]^p) &\leqq k + c \sup_{\substack{f \geqq 0 \\ \iint f\,dx = 1}} \left(\iint f\,[\log^+ g]^p - f\,[\log^+ f]^p \right) \\ &\leqq k + c \iint g\,[\log^+ g]^p\,dx \end{aligned} \tag{5.18}$$

When (5.18) holds for continuous functions then by approximation it holds for all functions for which the right side is finite. Therefore, in order to show that (5.16) is true, it suffices to find a g such that $\iint g\,[\log^+ g]^p\,dx < \infty$ but $\Lambda(c\,[\log^+ g]^p) = +\infty$ for all $c > 0$ unless ω_0 lies in L^p.

Take g to equal $\frac{e^j}{j^3}$ on a set A_j of measure e^{-j} for each positive integer j. Then g is in $L\,(\log L)^p$ and is even in $L \log L$. Since $\log g = j - 3\log j \geqq \frac{1}{4} j$ on A_j, we have $\iint e^{cv[\log g]^j} \geqq \sum_j e^{c_p(cv)j^p - j}$. Estimating the sum by its largest term, which occurs for $j \approx (c_p p(cv))^{\frac{1}{1-p}}$, shows that the integral is at least $e^{C_p(cv)^{\frac{1}{1-p}}}$. Since v is, to within a constant, the absolute value of w, taking the logarithm of this integral and then performing the integrals over w and s in $\Lambda(\alpha)$ shows that

$$\Lambda(c\,[\log^+ g]^p) \geqq C \int |w|^{\frac{1}{1-p}}\,d\lambda(w) - k = C\|\omega_0\|_{L^{\frac{1}{1-p}}} - k.$$

By combining this argument with the estimate used in the proof of the lemma, we can obtain the following lemma:

LEMMA 5.5. *Let a sequence of weights w_j be obtained by sampling a nonnegative function ω according to a measure $\mu \in \mathcal{M}_1$ having a nonzero continuous part, where $\omega \in L^1(\mu)$, and then let $S_N \equiv \frac{1}{N}\sum_{j=1}^{N} w_j \delta(x - x_j)$ be the weighted empirical law of a sequence of independent x_j identically distributed according to μ. Then for almost all sequences w_j the S_N satisfy the large-deviations principle with the rate function*

$$\Lambda^*(\nu) \equiv \sup_{\alpha \in C^0} \left[\int \alpha\nu - \int \log\left(\int e^{w\alpha(x)} \mu(x) \right) d\lambda_\omega(w) \right],$$

where $\lambda_\omega(w) = \mu(\{\omega < w\})$, and this rate function satisfies the estimate $\Lambda^*(\nu) \geqq c \int f_\nu \left[\log^+ f_\nu\right]^p \mu - k$ for $0 < p \leqq 1$ if and only if ω is in $L^{\frac{1}{1-p}}(\mu)$.

The case when $\omega \in L^\infty$ but is not identically constant can be obtained by taking α's tending to $\frac{\log f_\nu}{\|\omega_N\|_{L^\infty}}$. The fact that the measure has a continuous part is assumed in order to assure that sets like the A_k exist, and so the result still holds for certain pure-point measures. However, a calculation shows that the obvious modification of the above proof does not work if the masses of the points decay like $e^{-e^{e^n}}$.

By combining the proofs of Lemmas 5.3 and 5.2, we find that the $\varepsilon(N)$ in Lemma 5.3 is allowed to decay at the same rate as in Lemma 5.1. To see this, note that the estimate for the $|\omega_n|$ of Lemma 5.3 parallel to (5.5) is, by 4.5.5 in [19],

$$\text{Prob}(|\omega_N| \in \mathscr{A}_Q(L, \varepsilon)) \leqq \inf_{\delta > 0} \left\{ m(\mathcal{M}_1(\Omega), \delta) e^{-N \inf_{\nu \in (\mathscr{A}_Q(L,\varepsilon))^\delta} \Lambda_N^*(\nu)} \right\}, \tag{5.19}$$

where

$$\begin{aligned} \Lambda_N^*(\nu) &\equiv \sup_{\alpha \in C^0} \left\{ \iint \alpha\nu - \frac{1}{N} \log\left(E_{\{z_j\}} \left(e^{\sum_{j=1}^{N} |\omega_j| \alpha(z_j)} \right) \right) \right\} \\ &= \sup_{\alpha \in C^0} \left\{ \iint \alpha\nu - \frac{1}{N} \sum_{j=1}^{N} \log\left(\iint e^{|\omega_j|\alpha(x)}\, dx \right) \right\}. \end{aligned}$$

The argument from Lemma 5.3 still shows that $\Lambda_N^*(\nu) \geqq \iint Q(f_\nu)\, dx - k$ as long as

$$\frac{1}{N} \sum_{j=1}^{N} \Phi(|\omega_j|) \leqq c. \tag{5.20}$$

Since the expected value of the left side of (5.20) is finite, for almost all infinite sequences of offsets that left side converges to its finite expected value and so indeed satisfies (5.20). Hence the infimum in (5.19) will be at least $L - k_1$ as long as the infimum over the same set of $H_Q(\nu \mid dx)$ is at least $L - k_2$, and this follows from the proof of Lemma 5.2 upon substituting H_Q for H there.

Furthermore, this new argument does not require that the limit (5.8) exist; it only requires that the bound (5.20) hold. It is not obvious that that limit exists for the one-vortex-per-cell scheme, but the bound (5.20) is easily obtained:

$$\begin{aligned}
\frac{1}{N}\sum_{j=1}^{N}\Phi(|\omega_j|) &= \frac{1}{N}\sum_{j=1}^{N}\Phi\left(\left|\iint J_{d(N)}(x-x_j)[\omega_0(x)+\delta_1 s_j]\right|\right) \\
&\leqq \frac{1}{N}\sum_{j=1}^{N}\Phi\left(\iint J_{d(N)}(x-x_j)|\omega_0(x)+\delta_1 s_j|\right) \\
&\leqq \frac{1}{N}\sum_{j=1}^{N}\iint J_{d(N)}(x-x_j)\Phi(|\omega_0(x)+\delta_1 s_j|) \\
&\leqq c\iint \Phi(|\omega_0(x)+\delta_1 s_j|) \\
&\leqq c\iint \Phi(|\omega_0(x)|)+c\,.
\end{aligned}$$

Here the second inequality follows from the convexity of Φ, the third from the L^∞-boundedness of $\frac{1}{N}\sum_j J_{d(N)}(x-x_j)$ proven in the previous section, and the last from (5.14). Note, too, that the constant c_1 in the definition of ω_j equals 1 here because of the normalization of the area of Ω; the need to use this fact could be eliminated by picking the t_j to obtain the strengthened version $\Phi(c_1(t+\delta_1)) \leqq c\Phi(t)+c$ of (5.14).

Hence we obtain the following lemma:

LEMMA 5.6. *For both the one-vortex-per-cell and sampling initialization schemes, the conclusions of Lemma* 5.3 *hold with* $\varepsilon(N) = \frac{(\log N)^{3/2}}{\sqrt{N}}$.

Let us now show that, as promised, the velocity will lie in L^2_{loc} provided that the vorticity ω satisfies $\iint |\omega|(\log(e+|\omega|))^\alpha < \infty$ for some $\alpha > \frac{1}{2}$:

LEMMA 5.7. *If* $\iint |\omega|(\log(e+|\omega|))^\alpha < \infty$ *for some* $\alpha > \frac{1}{2}$, *then* $u \equiv K * \omega$ *is in* L^2_{loc}. *Furthermore, the map from* ω *to* u *is compact in the sense that if* ω_n *converges weak-$*$ in the sense of measures to* ω *and the integrals* $\iint |\omega_n|(\log(e+|\omega_n|))^\alpha$ *are uniformly bounded, then the* $u_n \equiv K * \omega_n$ *converge strongly in* L^2_{loc} *to* $u \equiv K * \omega$.

Proof: Let η be a compactly supported continuous function that is identically 1 in a neighborhood of the origin. Then

$$
\begin{aligned}
u(x) &= \iint K(x-y)\omega(y)\,dy \\
&= \iint (1-\eta(|x-y|))K(x-y)\omega(y)\,dy \\
&\quad + \iint \eta(|x-y|)K(x-y)\omega(y)\,dy \\
&\equiv u_1(x) + u_2(x)\,.
\end{aligned}
\tag{5.21}
$$

Since $(1-\eta(|x-y|))K(x-y)$ is bounded and ω is in L^1, $u_1(x)$ is in L^∞, and hence certainly in L^2_{loc}.

In order to obtain the first conclusion of the lemma, it therefore suffices to show that $u_2(x)$ also lies in L^2_{loc}. Let Ω be a compact set. Then

$$
\begin{aligned}
&\iint_\Omega |u_2(x)|^2\,dx \\
&= \iint_\Omega \left|\iint \eta(|x-y|)K(x-y)\omega(y)\,dy\right|^2 dx \\
&\leqq \iint_\Omega \left[\iint_{|x-y|\leqq c} \frac{1}{|x-y|}|\omega(y)|\,dy\right]^2 dx \\
&= \iint_\Omega \left[\iint_{|x-y|\leqq c} \left\{\frac{1}{|x-y|\left(\log\frac{1}{|x-y|}\right)^\alpha}\sqrt{|\omega(y)|(\log(e+|\omega|))^\alpha}\right\}\right. \\
&\quad \left.\times\left\{(\log\frac{1}{|x-y|})^\alpha\sqrt{\frac{|\omega(y)|}{(\log(e+|\omega|))^\alpha}}\right\}dy\right]^2 dx \\
&\leqq \iint_\Omega \left(\iint_{|x-y|\leqq c} \frac{1}{|x-y|^2\left(\log\frac{1}{|x-y|}\right)^{2\alpha}}|\omega(y)|(\log(e+|\omega|))^\alpha\,dy\right. \\
&\quad \left.\times\iint_{|x-y|\leqq c}\left(\log\frac{1}{|x-y|}\right)^{2\alpha}\frac{|\omega(y)|}{(\log(e+|\omega|))^\alpha}\,dy\right)dx \\
&\leqq \iint_\Omega\iint_{|x-y|\leqq c} \frac{1}{|x-y|^2\left(\log\frac{1}{|x-y|}\right)^{2\alpha}}|\omega(y)|(\log(e+|\omega|))^\alpha\,dy\,dx \\
&\quad \times\sup_x\iint_{|x-y|\leqq c}\left(\log\frac{1}{|x-y|}\right)^{2\alpha}\frac{|\omega(y)|}{(\log(e+|\omega|))^\alpha}\,dy \\
&\equiv I_1\cdot\sup_x I_2\,,
\end{aligned}
\tag{5.22}
$$

Switching the order of integration in the integral I_1 and then changing the variable of the new inner integral from x to $z \equiv x - y$ allows us to estimate I_1 by $\iint_{|y|\leqq k} |\omega(y)|(\log(e + |\omega(y)|))^\alpha \, dy$ times $\iint_{|z|\leqq c} \frac{1}{|z|^2\left(\log \frac{1}{|z|}\right)^{2\alpha}} \, dz$, and both of these are finite since $z \in \mathbb{R}^2$ and $2\alpha > 1$.

Now let $\Phi(t) = t(\log(e+t))^{2\alpha}$; a calculation shows that the maximum of $st - \Phi(t)$ occurs at a point t where $s \geqq (\log(e + t))^{2\alpha}$, that is, where $t = t_*(s) \leqq e^{s^{(1/2\alpha)}}$, so that $\Phi^*(s) = \max_t\{st - \Phi(t)\} \leqq st_*(s) \leqq se^{s^{(1/2\alpha)}}$. After applying the resulting inequality

$$st \leqq se^{s^{(1/2\alpha)}} + t(\log(e + t))^{2\alpha}$$

to $s = (\log \frac{1}{|x-y|})^{2\alpha}$ and $t = \frac{|\omega(y)|}{(\log(e+|\omega|))^\alpha}$, we find I_2 is bounded by the sum of the integrals of $\frac{(\log \frac{1}{|x-y|})^{2\alpha}}{|x-y|}$ and $c + |\omega(y)|(\log|\omega(y)|)^\alpha$. Both of these are locally integrable, so $u_2(x)$ is indeed in L^2_{loc}.

Now assume that the ω_n converge weak-$*$ in the sense of measures to ω and that $\iint |\omega_n|(\log(e + \omega_n|))^\alpha \leqq c$. Then the ω_n are uniformly integrable and hence weakly convergent in L^1 to ω. Since $t(\log(e + t))^\alpha$ is convex and nonnegative, the functional $\iint |\omega_n|(\log(e+\omega_n|))^\alpha$ is lower semicontinuous under weak convergence in L^1 (e.g., [16], theorem 3.4). Thus, $\iint |\omega|(\log(e+|\omega|))^\alpha \leqq c$, and hence $u = K * \omega$ lies in L^2_{loc}.

Hence, upon replacing $\eta(|x-y|)$ in (5.21) by $\eta(\frac{|x-y|}{\varepsilon})$ and then repeating (5.22) for $\omega_n - \omega$, we find that

$$\|u_n - u\|_{L^2_{\mathrm{loc}}} \leqq c \left| \iint K(|x-y|)\left(1 - \eta\left(\frac{|x-y|}{\varepsilon}\right)\right)(\omega_n(y) - \omega(y)) \right| + c \iint_{|z|\leqq c\varepsilon} \frac{1}{|z|^2\left(\log \frac{1}{|z|}\right)^{2\alpha}} \, dz. \tag{5.23}$$

The integrand of the second integral is in L^1, so that integral tends to 0 with ε independently of n, while for fixed ε the first integral tends to 0 as $n \to \infty$ since $K \cdot (1 - \eta)$ is continuous.

Remark 5.8. Lemma 5.7 is sharp in the sense that when the hypothesis of that lemma holds with $\alpha < \frac{1}{2}$, then the velocity need not be in L^2_{loc}. To see this, pick a β such that $\alpha < \beta < \frac{1}{2}$, and let $\omega(x) = \frac{1}{|x|^2(\log \frac{1}{|x|})^{1+\beta}}$ for $r < e$ and equal 0 otherwise. Since $\log(e + \omega) \leqq c \log \frac{1}{|x|}$ when $\omega > 0$, we find that $\omega(\log \omega)^\alpha \leqq \frac{c}{|x|^2(\log \frac{1}{|x|})^{1+\beta-\alpha}}$, which is locally integrable. Since ω depends only on $|x|$, the velocity is given by the formula $u(x) = \frac{x^\perp}{|x|^2} \int_0^{|x|} s\omega(s)\, ds$ (1.14 in [20]), and a calculation shows that u is not locally square integrable. In the case $\alpha = \frac{1}{2}$, an argument based on (5.11) shows that the velocity does belong to L^2_{loc} [9], although the embedding is not compact.

6. The Convergence Theorem

Now that we have estimated the initial and concentration phase-space volumes, we are ready to prove the convergence theorem by showing that concentration almost never occurs. Recall that for the equal-strength and one-vortex-per-cell initialization schemes, the random locations are chosen in a set A and then rescaled to lie in Ω to give the vortex locations, while in the sampling initialization scheme the random vortex locations are chosen directly in Ω. Normalized Lebesgue measure defines a probability measure on A (respectively, Ω), for which the product measure defines a probability measure on A^∞ (respectively, Ω^∞).

THEOREM 6.1. *If ω_0 is periodic and lies in $L\log L$, then for almost all sequences $\{y_j\}$ in A^∞ the solutions ω_N of the point-vortex method with initial data given by the equal-strength initialization scheme exist for all time and have a subsequence that converges weak-$*$ in $L^\infty([0,\infty),\mathcal{M})$ to a solution of the weak vorticity formulation having initial data ω_0. Furthermore, the corresponding velocity is in L^2_{loc} and hence is a solution of the weak velocity formulation.*

If ω_0 is periodic and in L^1, then for almost all sequences in $A^\infty \times [-\frac{1}{2},\frac{1}{2}]^\infty$ (respectively, $\Omega^\infty \times [-\frac{1}{2},\frac{1}{2}]^\infty$) the solutions ω_N of the point-vortex method with the initial data given by the one-vortex-per-cell scheme (respectively, the sampling scheme) exist for all time and have a subsequence that converges weak-$*$ in $L^\infty([0,\infty),\mathcal{M})$ to a solution of the weak vorticity formulation having initial data ω_0. Furthermore, if ω_0 lies in L^p with $p > 2$, then the corresponding velocity is in L^2 and hence a solution of the weak velocity formulation.

Proof: We will first show that the ω_N almost always exist for all time. Theorem 2.2 of [22] says that for each fixed N and fixed set $\{\omega_j\}$ of vortex strengths, the set $B(\{\omega_j\})$ of those initial position-sets $\{x_j\} \in \mathbb{R}^{2N}$ for which collisions ever occur in the resulting point-vortex dynamics has zero $2N$-dimensional Lebesgue measure. Let us show that the set of finite random sequences (of length N) for which collisions occur also has measure zero. For the equal-strength initialization scheme this is immediate, since the map from random sequences to vortex positions is affine and invertible.

For the other initialization schemes the set of random sequences maps not to just the set of vortex positions but to the set of vortex positions and strengths. Now the set of positions for which a collision does occur is the countable intersection over rational ε and rational t of the finite union over pairs $\{k,l\}$ of vortices of those initial position-sets $B_{\varepsilon,t,\{k,l\}}(\{\omega_j\})$ for which the given pair of vortices are at a distance less than ε at time t under the vortex-blob dynamics with smoothing parameter ε ([22]). By the continuity of solutions of smooth ODEs on parameters in the ODE and the initial data, each set

$$\mathcal{B}_{\varepsilon,t,\{k,l\}} \equiv \{\{x_j\},\{\omega_j\} \mid \{x_j\} \in B_{\varepsilon,t,\{k,l\}}(\{\omega_j\})\}$$

is open. Hence the set $\mathcal{B} \equiv \{\{x_j\},\{\omega_j\} \mid \{x_j\} \in B(\{\omega_j\})\}$, being a countable intersection of a union of the $\mathcal{B}_{\varepsilon,t,\{k,l\}}$, is measurable and is, in fact, a Borel set. Therefore the indicator function of that set is measurable and is even Borel measurable. (To see that some argument like the above is needed in order to show the measurability of $\mathcal{B}$, note that under the assumption that the continuum hypothesis holds there exist countable sets E_x for which the set $E \equiv \{(x, y) \mid y \in E_x\}$ is not Lebesgue measurable ([24], chapter 2, exercise 47).) Since the dynamics on the complement of $\mathcal{B}$ are the same as the vortex-blob dynamics for some value of the smoothing parameter depending on the initial data [22], the sets considered later are the intersection over rational ε of the corresponding sets for the vortex-blob dynamics and so are also measurable.

For the one-vortex-per-cell initialization scheme the map from the random points in A to the vortex positions is affine and invertible, and so is the map from the random offsets to the vortex strengths. Therefore, for each fixed offset sequence in $[-\frac{1}{2}, \frac{1}{2}]^N$, the set of random finite sequences in A^N for which collisions ever occur has measure zero. By the above argument, the set of all random sequences in $[-\frac{1}{2}, \frac{1}{2}]^N \times A^N$ for which collisions ever occur is measurable, so by Fubini's theorem that set has measure zero.

For the sampling initialization scheme, the image of a random sequence $\{x_j, s_j\} \in \Omega^N \times [-\frac{1}{2}, \frac{1}{2}]^N$ is $\{x_j, \omega_j = \omega_0(x_j) + \delta_1 s_j\}$, and even though this map is not smooth, it formally has a constant, nonzero Jacobian determinant, so that the Fubini argument still works: As shown above, the indicator function $\chi_{B(\{\omega_j\})}(\{x_j\})$ is Borel measurable, while the function from the random sequence to the vortex strengths is measurable, so the composition of the former with the latter in that order, namely $\chi_{B(\{\omega_0(x_j)+\delta_1 s_j\})}(\{x_j\})$, is measurable (e.g., [40], chapter 3, problem 17). Hence, by Tonelli's theorem and the translation invariance of Lebesgue measure, the measure of $\mathcal{B}$ equals

$$\begin{aligned}
&\int_{\Omega^N} \int_{[-\frac{1}{2},\frac{1}{2}]^N} \chi_{B(\{\omega_0(x_j)+\delta_1 s_j\})}(\{x_j\}) \prod_j ds_j \prod_j dx_j \\
&= \frac{1}{\delta_1^N} \int_{\Omega^N} \int_{[-\frac{1}{2},\frac{1}{2}]^N + \{\omega_0(x_j)\}} \chi_{B(\{\omega_j\})}(\{x_j\}) \prod_j d\omega_j \prod_j dx_j \\
&= \frac{1}{\delta_1^N} \int_{[-\infty,\infty]^N} \left[\int_{\Omega^N \cap \{|\omega_0(x_j) - \omega_j| \leqq \frac{1}{2}\delta_1\}} \chi_{B(\{\omega_j\})}(\{x_j\}) \prod_j dx_j \right] \prod_j d\omega_j \\
&= \frac{1}{\delta_1^N} \int_{[-\infty,\infty]^N} 0 \prod_j d\omega_j \\
&= 0.
\end{aligned}$$

For later use, let us note that this also shows that the map from finite random sequences to vortex positions—and strengths where applicable—is measure-preserving up to a constant factor. This means that the probability of a set of sequences equals the measure of the image of that set divided by the measure of the image of all sequences.

Thus, for each initialization scheme, the set of length-N random sequences for which collisions ever occur has Lebesgue measure zero for every N. Since the product of a set of measure zero with a set of measure one has measure zero for the product measure, for every N the measure of the set of infinite random sequences for which collisions occur in the N-vortex system equals 0. Hence their union also has measure zero, that is, for almost all infinite random sequences no collisions ever occur for any value of N.

Second, by Lemma 2.1, in order to show that a subsequence of the ω_N converges weak-$*$ to a solution of the weak vorticity formulation, it suffices to show that the nonconcentration condition (2.9) holds for almost all t. We will show that this is true for almost all random sequences.

Let $X(\{\omega_j\}_{j=1}^N)$ be the set of all initial positions $\{x_j\}_{j=1}^N$ such that some length-N random sequence produces the initial positions and strengths $\{x_j\}_{j=1}^N$, $\{\omega_j\}_{j=1}^N$. By the results of Section 4, almost all random sequences produce vortex strengths $\{\omega_j\}$ such that the $2N$-dimensional Lebesgue measure of $X(\{\omega_j\}_{j=1}^N)$ is at least δ^N for all sufficiently large N. By the results of Section 5, for L sufficiently large the $2N$-dimensional Lebesgue measure of the set of vortex positions for which the normalized vorticity lies in $\mathscr{A}_Q(L, \varepsilon(N))$ is at most $\left(\frac{\delta}{2}\right)^N$. Actually, this is only true for almost all infinite random sequences, but that suffices for the argument. Here $Q(t) = t \log t$ for the equal-strength scheme and is defined in the proof of Lemma 5.3 for the other schemes. Let us call a set of N initial positions bad if the resulting normalized vorticity lies in $\mathscr{A}_Q(L, \varepsilon(N))$ for a set of times in $[0, N]$ having measure at least $\frac{1}{N}$, and good otherwise. By Theorem 2.2 of [22], the point-vortex flow restricted to the set of initial conditions for which no collisions occur is measure-preserving, so the measure of the bad initial positions is at most $N^2\left(\frac{\delta}{2}\right)^N$. Thus, for almost all random sequences, the probability, conditioned on the value of the vortex strengths, that the initial data is bad for the N-vortex system is at most $N^2\left(\frac{1}{2}\right)^N$. Since this value does not depend on the value of the conditioning variable, the unconditional probability to be bad for the N-vortex system is also at most $N^2\left(\frac{1}{2}\right)^N$. Hence, by the Borel-Cantelli lemma, almost all random sequences yield initial data that is good for all sufficiently large N.

Let $f_N(t)$ be the indicator function of the set of times at which the normalized vorticity lies in $\mathscr{A}_Q(L, \varepsilon(N))$. Then the above says that for almost all random sequences, f_N converges in measure to the function 0. This implies that a subsequence f_{N_j} converges to 0 for almost all times; that is, for almost all times t there exists an $N(t)$ such that the vorticity does not lie in $\mathscr{A}_Q(L, \varepsilon(N))$ for those N in the subsequence such that $N > N(t)$. By the results of Section 5, this implies that for almost all t condition (2.9) indeed holds.

Third, the results of Section 4 show that for almost all random sequences the point-vortex initial data converges weak-$*$ in the sense of measures to ω_0, so by applying lemma 3.4 in [42] to the convergent subsequence obtained above, we find that for almost all random sequences the solution has ω_0 for its initial data.

Finally, since the limit vorticity is also the weak-$*$ limit of the mollified point-vortex vorticities with parameter $\varepsilon(N)$ tending to 0, the results of Section 5 show that whenever ω_0 is in $L \log L$ for the equal-strength scheme or in L^p with $p > 2$ for the other schemes, then the velocity corresponding to the limit vorticity lies in L^2_{loc} and hence is a solution of the weak velocity formulation.

COROLLARY 6.2. *Theorem* 6.1 *remains valid when the point-vortex method is replaced by the vortex-blob method using a radial mollifier with the smoothing parameter* $\varepsilon(N)$ *tending to* 0 *at an arbitrary rate.*

Also, let $u_N = K * \omega_N$, *where* $\omega_N = \sum \omega_j \delta(x - x_j(t))$ *is the vorticity in either the point-vortex or vortex-blob method. If* $\varepsilon \geqq \frac{(\log N)^{3/2}}{\sqrt{N}}$ *and tends to* 0 *as* $N \to \infty$, *then the smoothed velocity* $J_{\varepsilon(N)} u_N$ *converges strongly in* L^2_{loc} *along the subsequence for which the vorticity convergence is obtained to a solution of the weak velocity formulation having initial data* $K * \omega_0$.

In particular, such convergence holds for the velocity of the vortex-blob method provided that $\varepsilon(N) \geqq \frac{(\log N)^{3/2}}{\sqrt{N}}$ *and tends to* 0 *as* $N \to \infty$.

Proof: Since the vortex-blob velocity is Lipschitz, collisions are impossible, and hence the ω_N exist for all time. Furthermore, the vortex-blob method is also Hamiltonian and so preserves phase-space volume, while the phase-space estimates in Sections 4 and 5 are time independent and so do not depend on the particular system used to create the dynamics. These estimates therefore remain valid for the vortex-blob method, and hence the weak-$*$ convergence to a solution follows as before. The remaining parts of the corollary follow from Theorem 6.1, the first part of the corollary, and Lemmas 5.2, 5.6, and 5.7.

7. Remark on Compactly Supported Initial Vorticity

The only point where the periodicity of the flow is really needed is to ensure that the vortices not involved in concentration have a phase-space volume of at most c^N. To see this, note first that for a given set $\{\omega_j\}_{j=1}^N$ of vortex strengths, no vortices will escape to infinity in finite time, and the set of initial conditions for which collisions ever occur will still have measure zero provided that for all subsets S of $\{1, 2, \dots, N\}$,

$$\sum_{j \in S} \omega_j \neq 0 \tag{7.1}$$

([36], section 4.2, corollaries 2.1 and 2.2). Since for fixed N the set of vortex-strength sets $\{\omega_j\}$ for which (7.1) fails is the union of a finite number of hypersurfaces, (7.1) holds for the one-vortex-per-cell and sampling initialization schemes for almost all length-N random sequences. It therefore holds for all N for almost all infinite random sequences. For the equal-strength initialization scheme, (7.1) will hold for all N provided that we let the ratio $\frac{\omega^+}{\omega^-}$ of the strengths of the positive and negative vortices be irrational.

Next, if we restrict our attention to those vortices lying in a region with area R, then the results of Section 5 still hold, even phrased in terms of the phase-space volume rather than the probability, provided that the functions Q, ε, and d there are allowed to depend on R also. Summing over all possibilities of which vortices lie in the region gives at worst an acceptable factor of 2^N. For each given configuration of the vortices in a square centered at the origin having sufficiently large area R, consider the phase-space volume occupied by the vortices outside the region. If the latter volume is at most c^N and we let R tend to infinity with N sufficiently slowly, then the Borel-Cantelli argument from the proof of Theorem 6.1 still shows that the probability of concentration in any finite region of the plane is zero. Since H_ψ in this case decays to 0 at infinity, this would suffice to conclude that the weak-$*$ limit is a solution.

For example, for the vortex-blob method with smoothing parameter depending algebraically on $\frac{1}{N}$, the velocity is bounded in L^∞ by some power of N, so for $0 \leqq t \leqq T$ all the vortices lie in a region of area $O(c + cTN^\alpha)$. Hence the above condition on the phase-space volume of the faraway vortices would hold if the number of vortices outside some circle with sufficiently large radius $r(T)$ was at most $c\frac{N}{\log N}$.

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