Analysis of Partial Differential Equations
Exercise sheet II (Chapter 2)

1. Recall the *Liouville theorem* for analytic functions on the whole complex plane. Does the theorem hold true for analytic functions on the real line?

2. We have proved in lectures that f is real analytic on an open set $\mathcal{U}$ of the real line iff for any compact set $K \subset \mathcal{U}$ there are constants $C(K), r > 0$ such that

$$\forall\, x \in K, \quad |f^{(n)}(x)| \leq C(K)\frac{n!}{r^n}.$$

Prove a similar statement with several variables: f is real analytic on an open set $\mathcal{U}$ of $\mathbb{R}^\ell$ iff for any compact set $K \subset \mathcal{U}$ there are constants $C(K), r > 0$ such that

$$\forall\, \mathbf{x} \in K, \quad |\partial_x^\alpha f(\mathbf{x})| \leq C(K)\frac{\alpha!}{r^{|\alpha|}}.$$

Hint. Prove and use the multinomial identities for $\mathbf{x} = (x_1, \ldots, x_\ell)$, $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ and $m \in \mathbb{N}$:

$$(x_1 + \cdots + x_\ell)^m = \sum_{|\alpha|=m} \frac{\mathbf{x}^\alpha m!}{\alpha!} \quad \text{and} \quad \sum_{\beta \geq \alpha} \frac{\beta!}{(\beta-\alpha)!}\mathbf{x}^{\beta-\alpha} = \partial_{\mathbf{x}}^\alpha \left(\prod_{j=1}^{\ell} \frac{1}{1-x_j} \right) = \frac{\alpha!}{(1-x_1)^{1+\alpha_1} \cdots (1-x_\ell)^{1+\alpha_\ell}},$$

where we have used the standard multinomial notations: $\mathbf{x}^\alpha = x_1^{\alpha_1} \cdots x_\ell^{\alpha_\ell}$, $\alpha! = \alpha_1! \cdots \alpha_\ell!$ and $\partial_{\mathbf{x}}^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_\ell}^{\alpha_\ell}$.

3. Using the method of characteristics, solve the following PDE:

$$\begin{cases} u\,\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} = 1 & (x, y) \in \mathbb{R}^2, \\ u\,(x, x) = 0 & x \in \mathbb{R}. \end{cases} \tag{1}$$

Explore what happens when $u\,(x, x) = 0$ in (1) is replaced by $u\,(x, x) = 1$.

Method of characteristics: In this method, we try to solve a first order PDE like (1) by converting the PDE into an appropriate system of ODEs. For any $(x, y) \in \mathbb{R}^2$, we would like to find a curve in $\mathbb{R}^2$ which passes through (x, y) and the hypersurface upon which we are given the data (the line $x = y$ happens to be the hypersurface above). In order to find the curve, we introduce a dummy parameter $s \in \mathbb{R}$ and define $x := x(s), y := y(s)$ and $z(s) := u(x(s), y(s))$. Then, we write a system of ODE for $(x(s), y(s), z(s))$ as dictated by the PDE and the data. The system is then solved to arrive at a curve in $\mathbb{R}^2$ and the value of the solution u along the curve.

4. Show that the line $\{t = 0\}$ is characteristic for the heat equation:

$$\frac{\partial u}{\partial t}(t, x) = \frac{\partial^2 u}{\partial x^2}(t, x) \text{ for } (t, x) \in \mathbb{R}^2. \tag{2}$$

Show further that there does not exist an analytic solution $u(t, x)$ of (2) with

$$u(0, x) = \frac{1}{1 + x^2}.$$

5. (Cauchy-Kovalevskaya Theorem for system of ODEs) Suppose $b > 0$ and $\mathbf{F} : \mathbf{u}_0 + (-b, b)^d \to \mathbb{R}^d$ be real analytic in a neighbourhood of $\mathbf{u}_0$. Let $\mathbf{u}(t)$ be the unique C^1 solution to the following system of ODEs:

$$\mathbf{u}'(t) = \mathbf{F}(t, \mathbf{u}(t)), \quad \mathbf{u}(0) = \mathbf{u}_0 \in \mathbb{R}^d,$$

on $(-a, a)$ for some $a > 0$ with $\mathbf{u}((-a, a)) \subset \mathbf{u}_0 + (-b, b)^d$. Using the *method of majorants*, show that $\mathbf{u}(t)$ is analytic in a neighbourhood of 0.
Note: This exercise is analogous to the scalar ODE case treated during the lectures.

6. Consider the reduced setting for Cauchy-Kovalevskaya theorem for PDEs:

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial x_\ell} = \displaystyle\sum_{j=1}^{\ell-1} \mathbf{b}_j(\mathbf{u}, \bar{x}) \frac{\partial \mathbf{u}}{\partial x_j} + \mathbf{b}_0(\mathbf{u}, \bar{x}), \quad x \in \mathcal{U} \\ \mathbf{u} = 0 \quad \text{on} \quad \Gamma, \end{cases} \tag{3}$$

with matrix-valued functions $\mathbf{b}_j : \mathbb{R}^m \times \mathbb{R}^{\ell-1} \mapsto \mathcal{M}_{m\times m}$ and vector-valued function $\mathbf{b}_0 : \mathbb{R}^m \times \mathbb{R}^{\ell-1} \mapsto \mathbb{R}^m$ which are locally analytic around $(0, 0)$, and where $\bar{x} = (x_1, \dots, x_{\ell-1})$. Using similar calculations as for the system of ODEs (Question 5) on all entries of $\mathbf{b}_j$, $j = 0, \dots, \ell - 1$ (which depend on $m + \ell - 1$ variables), find $C, r > 0$ such that

$$g(z_1, \dots, z_m, x_1, \dots, x_{\ell-1}) = \frac{Cr}{r - (x_1 + \dots x_{\ell-1}) - (z_1 + \cdots + z_m)}$$

is a majorant of all these entries.

7. Let g be the majorant function obtained in Question 6. Define $\mathbf{b}_j^* := g\mathbf{M}_1$, $j = 1, \dots, \ell - 1$, and $\mathbf{b}_0^* := g\mathbf{U}_1$, where $\mathbf{M}_1$ is the $m \times m$-matrix with 1 in all enteries and $\mathbf{U}_1$ is the m-vector with 1 in all enteries. Check that the solution $\mathbf{v} = (v_1, \dots, v_m)$ to

$$\begin{cases} \dfrac{\partial \mathbf{v}}{\partial x_\ell} = \displaystyle\sum_{j=1}^{\ell-1} \mathbf{b}_j^*(\mathbf{v}, \bar{x}) \frac{\partial \mathbf{u}}{\partial x_j} + \mathbf{b}_0^*(\mathbf{v}, \bar{x}) \\ \mathbf{v} = 0 \quad \text{on} \quad \Gamma, \end{cases}$$

can be searched in the form $v_1 = \cdots = v_m =: w$, and

$$w = w(x_1 + x_2 + \cdots + x_{\ell-1}, x_\ell) = w(\xi, x_\ell), \quad \xi := x_1 + \cdots + x_{\ell-1}.$$

8. Let $t = x_\ell$ and $\xi = x_1 + x_2 + \cdots + x_{\ell-1}$. Suppose $w(t, \xi)$ defines the solution to the majorant problem as in Question 7. Show that $w(t, \xi)$ satisfies the following PDE:

$$\partial_t w = \frac{Cr}{r - \xi - \gamma_1 w} \left(\gamma_2 \partial_\xi w + 1\right), \quad w(\xi, 0) = 0, \quad t, \xi \in \mathbb{R}, \tag{4}$$

with $\gamma_2 = (\ell - 1)m$ and $\gamma_1 = m$, and that for $\ell \geq 3$ the solution is given by

$$w(\xi, t) = \frac{1}{\ell m} \left((r - \xi) - \sqrt{(r - \xi)^2 - 2\ell m C r t} \right).$$

Hint. Use the method of Characteristics to solve (4) as in Question 3.

9. (Hadamard's example: amplified version) Consider the problem

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} = 0, \quad u(0, x) = \phi(x), \quad \frac{\partial u}{\partial t}(0, x) = \psi(x).$$

(a) For a given $\varepsilon > 0$ and an integer $k > 0$, construct initial data ϕ and ψ such that

$$\|\phi\|_\infty + \|\phi^{(1)}\|_\infty + \cdots + \|\phi^{(k)}\|_\infty + \|\psi\|_\infty + \|\psi^{(1)}\|_\infty + \cdots + \|\psi^{(k)}\|_\infty < \varepsilon$$

and

$$\|u(\cdot,\varepsilon)\|_\infty \geq \frac{1}{\varepsilon}.$$

(b) Repeat the exercise with the condition on the initial data replaced by

$$\forall\, k \geq 0, \quad \|\phi^{(k)}\|_\infty + \|\psi^{(k)}\|_\infty < \varepsilon.$$

10. For $u = u(x,y)$ on $\mathbb{R}^2$ consider the PDE

$$\partial^2_{xx}u + 2x\partial^2_{xy}u + y\partial^2_{yy}u + (\partial_x u)^2 - u\partial_y u = 0.$$

Determine the regions in $\mathbb{R}^2$ where the above PDE is elliptic, parabolic or hyperbolic and sketch them.