Advanced Probability

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(2nd November, Friday)

Chapter 5. Weak Convergence

5.1. Definitions

Let E be a metric space. Whenever we are talking about a metric space, the σ-algebra is given by the Borel σ-algebra. Write $C_b(E)$ for the set of bounded continuous functions on E.

- Let $(\mu_n : n \in \mathbb{N})$ be a sequence of probability measures and let μ be another probability measure on E. We say that $\mu_n \to \mu$ **weakly** (as $n \to \infty$) if $\mu_n(f) \to \mu(f)$ for all $f \in C_b(\mathbb{R})$.

Theorem 5.1.1) The following are equivalent.

(a) $\mu_n \to \mu$ weakly on E

(b) $\liminf_{n\to\infty} \mu_n(U) \geq \mu(U)$ for all U open

(c) $\limsup_{\mu(F)} \leq \mu(F)$ for all F closed.

(d) $\mu_n(B) \to \mu(B)$ for all $B \in \mathcal{B}$ such that $\mu(\partial B) = 0$.(Boundary is the set of limit points of B that are not contained in B.)

proof) Exercise.

For an example, consider a sequence $(x_n)_n \subset \mathbb{R}$ such that $x_n \to 0$ as $n \to \infty$. We want to have $\delta_{x_n} \to \delta_0$. Indeed, this is true in the weak sense. However, the sequence has $\delta_{x_n}(\{0\}) = 0$ for all n, hence we should have inequality in condition (c).

We have a similar version of the theorem for the real line.

Proposition 5.1.2) Consider the case $E = \mathbb{R}$. TFAE

(a) $\mu_n \to \mu$ weakly for some probability measure μ.

(b) $F_n(x) \to F(x)$ for all $x \in \mathbb{R}$ such that $F(x^-) = F(x)$. (Here, $F(x) = \mu((\infty, x])$ is the **distribution function** of μ.) (Sometimes called convergence of distributions)

(c) There exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and random variables X_n, X on Ω such that $X_n \sim \mu_n$, $X \sim \mu$ and $X_n \to X$ almost surely.

proof) See probability and measure notes.

5.2. Prohorov's Theorem

When does a sequence of probability measures has a converging subsequence?

Let E be a metric space and $(\mu_n : n \in \mathbb{N})$ be a sequence of probability measures on E.

- We say that $(\mu_n)_n$ is **tight** if for all $\epsilon > 0$, there is a compact set $K \subset E$ such that

$$\mu_n(E \backslash K) \leq \epsilon \quad \forall n \in \mathbb{N}$$

For example, the sequence $(\delta_n)_n$ is *not* tight.

Theorem 5.2.1) Let $(\mu_n : n \in \mathbb{N})$ be a sequence of probability measures on a metric space E and suppose that $(\mu_n : n \in \mathbb{N})$ is tight. Then there exists a subsequence $(n_k)_k \subset \mathbb{N}$ and probability measure μ on E such that $\mu_{n_k} \to \mu$ weakly as $k \to \infty$.

This gives a version of weakly sequential compactness of probability measures. We are only going to prove this for $\mathbb{R}$. This theorem is hard to prove in general.(e.g. there is a method using Monge-Kantorovich metric defined for Polish spaces. For this method, see "Topics in Optimal Transport", C.Villani, Ame.Soc.Math. For the general version, see the attached note)

proof for $E = \mathbb{R}$) By a diagonal argument and by passing to a subsequence, it suffices to consider the case where $F_n(x) \to g(x)$ as $n \to \infty$ for all $x \in \mathbb{Q}$ for some $g(x) \in [0, 1]$, where F_n is the distribution function of F_n. Now $g : \mathbb{Q} \to [0, 1]$ is non-decreasing so g has a non-decreasing extension $G : \mathbb{R} \to [0, 1]$, i.e.

$$G(x) = \lim_{q \searrow x, q \in \mathbb{Q}} g(q)$$

which has only countably many discontinuities.(because there should be a rational number in each discontinuity). Now we must have

$$F_n(x) \to G(x) \quad \forall x \text{ s.t. } G \text{is continuous at } x$$

Set $F(x) = G(x^+)$, then F and G have same points of continuity, so $F_n(x) \to F(x)$ for all $x \in \mathbb{R}$.

We are only left to check that $G(x) \to 1$ as $x \to \infty$ using tightness condition.

Since $(\mu_n : n \in \mathbb{N})$ is tight, given $\epsilon > 0$, there exists $R < \infty$ such that $\mu_n(\mathbb{R} \backslash (-R, R)) \leq \epsilon$ for all ϵ so $F_n(-R) \leq \epsilon$, $F_n(R) \geq 1 - \epsilon$. So

$$\begin{aligned} F(x) \to 0 &\quad \text{as } x \to -\infty \\ F(x) \to 1 &\quad \text{as } x \to \infty \end{aligned}$$

So F is distribution function. So there exists a probability measure μ such that $\mu((-\infty, x]) = F(x)$. Then $\mu_n \to \mu$ by **Prop 5.1.2.**

(End of proof) □

5.3. Weak Convergence and Characteristic Functions

Take $E = \mathbb{R}^d$.

- For a probability measures mu on $\mathbb{R}^d$, define its **characteristic function** $\phi : \mathbb{R}^d \to \mathbb{C}$ by

$$\phi(u) = \int_{\mathbb{R}^d} e^{i\langle u, x \rangle} \mu(dx)$$

Lemma 5.3.1) Fix $d = 1$. For all $\lambda \in (0, \infty)$,

$$\mu(\mathbb{R} \backslash (-\lambda, \lambda)) \leq C\lambda \int_0^\lambda (1 - \text{Re}(\phi(u)))du$$

where $C = (1 - \sin(1))^{-1} < \infty$.

proof) Consider for $t \geq 1$. Let $A(t) = t^{-1} \int_0^t (1 - \cos v) dv$. Then

$$A(t) \geq A(0) = 1 - \sin(t)$$

(to see this, observe that $A(t)$ is the average of $(1 - \cos(v))$ on interval $(0, t)$ and divide the cases $|t| \leq \pi/2$ and $|t| \geq \pi/2$)
So $Ct^{-1} \int_0^t (1 - \cos(v)) dv \geq 1$. Substitute $v = uy$, $u = v/y$,

$$Ct^{-1} \int_0^{t/y} (1 - \cos(uy)) y du \geq 1$$

Put $t/y = 1/\lambda$, $\lambda = y/t$, $t = y/\lambda \geq 1$ to see

$$C\lambda \int_0^{1/\lambda} (1 - \cos(uy)) du \geq 1$$

whenever $t = y/\lambda \geq 1$(this was the assumption we started with). Now for general $y \in \mathbb{R}$, has

$$C\lambda \int_0^{1/\lambda} (1 - \cos(uy)) du \geq 1_{|y| \geq \lambda}$$

Now integrate with respect to μ and use Fubini.

$$\begin{aligned}
\mu(\mathbb{R} \backslash (-\lambda, \lambda)) &\leq C\lambda \int_{\mathbb{R}} \int_0^{1/\lambda} (1 - \cos(uy)) du \mu(dy) \\
&= C\lambda \int_0^{1/\lambda} \int_{\mathbb{R}} (1 - \cos(uy)) du \mu(dy)
\end{aligned}$$

(End of proof) $\square$