Analysis of PDEs

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Example Classes : Dr. Ivan Moyano

Texts : (1)Evans. PDEs, (2)Rauch, PDEs, (3)F.John, PDEs, (4)Gilberg + Raudinger, Elliptic PDE, (5) Ladyzhenskay, The Boundary Value Problems of Mathematical Physics.

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(5th October 2018, Friday)

Introduction

Suppose $U \subset \mathbb{R}^n$ is open. A *partial differential equation* of order k is an expression of the following form:

$$F(x, u(x), Du(x), \cdots, D^{(k)}u(x)) = 0 \tag{1}$$

Here, $F : U \times \mathbb{R} \times \mathbb{R}^n \times \cdots \times \mathbb{R}^{n^k} \to \mathbb{R}$ is a given function and $u : U \to \mathbb{R}$ is the 'unknown'. We say $u \in C^k(U)$ is a classical solution of 1 if 1 is satisfied on U when we substitute u into the expression.

We could also consider the case where $u : U \to \mathbb{R}^p$ and F takes values in $\mathbb{R}^q$, then we speak of a *system of PDE's*.

Examples)

1. The Transport Equation: Suppose $V : \mathbb{R}^{n+2} \to \mathbb{R}^n$ is given.

$$\frac{\partial u}{\partial t}(x,t) + V(x,t,u(t,x)) \cdot D_x u(x,t) = f(x,t) \quad \text{for } x \in \mathbb{R}^n$$

is a PDE for $u : \mathbb{R}^{n+1} \to \mathbb{R}$. This describes evolution of some chemical produced at rate $f(x,t)$ and being advected by a flow of velocity $V(x,t,u(t,x))$.

2. The Laplace and Poisson Equations:

$$\Delta u(x) = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i \partial x_j}(x) = 0 \quad \text{(Laplace Equation)}$$

This describes:

+ Electrostatic potential in empty space
+ Static distribution of heat in a solid body
+ Applications to steady flows in 2D
+ Connections to complex analysis

$$\Delta u(x) = f(x) \quad \text{some given } f : \mathbb{R}^n \to \mathbb{R} \quad \text{(Poisson's Equation)}$$

This describes:

+ Electric field produced by charge distribution f
+ Gravitational field in Newton's Theory(f is mass density)

3. Heat/Diffusion Equation:

$$\frac{\partial u}{\partial t} = \Delta u$$

This describes evolution of temperature in a solid homogeneous body.

4. Wave Equation:

$$-\frac{\partial^2 u}{\partial t^2} + \Delta u = 0$$

This describe:

+ Displacement of a stretched string (dimension=1)
+ Ripples on surface of water (dimension=2)
+ Density of air in a sound wave (dimension=3)

5. Maxwell's Equations: With $E, B : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$,

$$\nabla \cdot E = \rho \quad \nabla \cdot B = 0$$
$$\nabla \times E + \frac{\partial B}{\partial t} = 0 \quad \nabla \times B - \frac{\partial E}{\partial t} = J$$

ρ, J are charge density/current respectively, are given.

6. Ricci Flow:

$$\partial_t g_{ij} = -2R_{ij}$$

where g_{ij} is a Riemannian metric, R_{ij} is its Ricci curvature.

7. Minimal Surface Equation: For $u : \mathbb{R}^2 \to \mathbb{R}$,

$$\mathrm{div}\left(\frac{Du}{\sqrt{1 - |Du|^2}}\right) = 0$$

Condition for the graph $\{(x, y, u(x, y))\}$ to locally extremise area.

8. Eikonal Equation: for $U \subset \mathbb{R}^3$ and $u : U \to \mathbb{R}$

$$|Du| = 1$$

Level sets parametrise a wave-front moving according to the ray theory of light.

9. Schrödinger's Equation: For $u : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{C} \equiv \mathbb{R}^2$,

$$i\frac{\partial u}{\partial t} + \Delta u - Vu = 0$$

for $V : \mathbb{R}^3 \to \mathbb{R}$ given. u is the wavefunction of a quantum mechanical particle moving in a potential V.

10. Einstein's Equations for General Relativity:

$$R_{\mu\nu}[g] = 0$$

where g is Lorentzian metric. $R_{\mu\nu}$ is Ricci tensor. This describes gravitational field in vacuum.

-. There are Many more examples.

Data and Well-Posedness

In all examples, there is extra information required beyond the equation. We call this the *data*. An important question is what data is appropriate. We typically ask of a PDE problem that:

a) A solution exists,

b) for given data the solution is unique,

c) the solution depends on the data continuously.

If these hold, we say the problem is 'well-posed'. To make these precise, we have to (usually) specify function spaces for the data and solution to belong to.

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8th October, Monday

Let $U \subset \mathbb{R}^n$, $u : U \to \mathbb{R}$ be unknown. Then our system of interest will be

$$F(x; u, Du, \cdots, D^k u) = 0 \tag{2}$$

Notations) Let $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$ be a multi-index(where $\mathbb{N} = \{0, 1, 2, 3, \cdots\}$). Then we let:

- $D^\alpha f(x) = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$ where $|\alpha| = \alpha_1 + \cdots + \alpha_n$ is the order of α.
- For $x \in \mathbb{R}^n$, $x^\alpha = x_1^{\alpha_1} \times \cdots \times x_n^{\alpha_n}$
- $\alpha! = \alpha_1! \cdots \alpha_n!$.
- For $\beta = (\beta_1, \cdots, \beta_n)$, $\beta \leq \alpha$ is equivalent to having $\beta_k \leq \alpha_k$ for all k.

Classifying PDEs

- We say (2) is **linear** if F is a linear function of u and its derivatives. We can write (2) as

$$\sum_{|\alpha \leq k} a_\alpha(x) D^\alpha u(x) = f(x)$$

- We say (2) is **semi-linear** if it is of the form

$$\sum_{|\alpha \leq k} a_\alpha(x) D^\alpha u(x) + a_0(x; u(x), \cdots, D^{k-1}u(x)) = 0$$

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- We say (2) is **quasi-linear** if it is of the form

$$\sum_{|\alpha \leq k} a_\alpha\big(x; u(x), \cdots, D^{k-1}u(x)\big) D^\alpha u(x) + a_0(x; u(x), \cdots, D^{k-1}u(x)) = 0$$

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- We say (2) is **fully non-linear** if its not linear, semi-linear, nor quasi-linear

Examples)

- $\Delta u = f$ is linear
- $\Delta u = u^3$ is semi-linear
- $uu_{xx} + u_x u_{yy} = f$ is quasi-linear
- $u_{xx}u_{yy} - u_{xy}^2 = f$ is fully non-linear.

Cauchy-Kovalevskaya Theorem

For motivation, we recall some ODE theory. Fix $U \subset \mathbb{R}^n$, and assume $f : U \to \mathbb{R}^n$ is given. Consider the ODE

$$\dot{u}(t) = f(u(t)), u(0) = u_0 \in U \tag{3}$$

with $u : I \subset \mathbb{R} \to U$.

Theorem) (Picard-Lindelöf) Suppose there exist $r, K > 0$ s.t. $B_r(u_0) = \{w \in \mathbb{R}^n : |w - u_0| < r\}$ and $|f(x) - f(y) \leq K|x - y|$ for all $x, y \in B_r(x_0)$. Then there exists $\epsilon > 0$(depending in r and K) and a unique C^1-function $u : (-\epsilon, \epsilon) \to U$ solving (3).
proof) Use U solves (3), then

$$u(t) = u_0 + \int_0^t f(u(s))ds \tag{4}$$

and conversely, if U is C^0 and solves (4), then in fact U is C^1 by FTC, and u solves (3).(in context of PDEs, this is called *weak formulation*)
Then our solution, if exists, is a fixed point of the map $B : w \mapsto u_0 + \int_0^t f(w(s))ds$.(use Banach fixed point theorem)

Observations:

- We start by reformulating the problem in a weak form and find a unique C^0 solution. Then C^1 the regularity follows a posteriori.
- to construct the fixed point map, we solve the linear problem $\dot{w}(t) = f(w(t))$.

Lets consider an alternative approach to solving (3). Assuming f is differentiable, we have

$$\begin{aligned}
u^{(1)}(t) &= f(u(t)) \\
u^{(2)}(t) &= f'(u(t))\dot{u}(t) \\
u^{(3)}(t) &= f''(u(t))(\dot{u}(t))^2 + f'(u(t))\ddot{u}(t) \\
&\vdots \\
u^{(k)}(t) &= f_k(u(t), \dot{u}(t), \cdots, u^{(k-1)})(t))
\end{aligned}$$

So in principle, given $u(0) = u_0$, we can determine $u_k = u^{(k)}(0)$ for all $k \geq 0$. *Formally* at least, we can write

$$u(t) = \sum_{k=0}^{\infty} u_k t^k / k! \tag{5}$$

ignoring the issues of convergence. Call this a **formal power series solution**. When will this agree with the Picard-Lindelöf solution we have constructed?

Theorem) (Cauchy-Kovalevskaya, for the case of ODEs) The series in (5) converges to a solution of (3) in a neighbourhood of $t = 0$ if f is real analytic at u_0.
-This will follow from a more general result later.

Definition) Let $U \subset \mathbb{R}^n$ be open and suppose $f : U \to \mathbb{R}$. f is called **real analytic** near $x_0 \in U$ if $\exists r > 0$ and constants f_α(α are multi-indices) such that

$$f(x) = \sum_\alpha f_\alpha (x - x_0)^\alpha \quad \text{for } x \in B_r(x_0)$$

Note: if f is real analytic, then it is C^∞. Furthermore, the constants f_α are given by $f_\alpha = D^\alpha f(x_0)/\alpha!$. Thus f equals its Taylor expansion about x_0, in a neighbourhood of x_0.

$$f(x) = \sum_\alpha \frac{D^\alpha f(x_0)}{\alpha!}(x - x_0)^\alpha \quad \text{for } x \in B_r(x_0)$$

By translation, we usually assume $x_0 = 0$

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(10th October, Wednesday)

• Last lecture : $U \subset \mathbb{R}^n$ open, $f : U \to \mathbb{R}$ is real analytic at $x_0 \in U$ if $\exists f_\alpha \in \mathbb{R}$, $r > 0$ s.t.

$$f(x) = \sum_\alpha f_\alpha (x - x_0)^\alpha \quad \forall |x - x_0| < r$$

Properties of real analytic functions

- f is real analytic at x_0 if and only if $\exists s > 0$ and $C, \rho > 0$ such that:

$$\sup_{|x-x_0|<s} \left| D^\alpha f(x) \right| \le C \frac{|\alpha|!}{\rho^{|\alpha|}}$$

- If f is RA(real analytic) at x_0, it is RA for all x close enough to x_0.
- If $f : U \to \mathbb{R}$ is real analytic everywhere on a connected set U, then f is determined by its values on any open subset of U. (Or by its Taylor expansion at a single point.)

Example : If $r > 0$ set

$$f(x) = \frac{r}{r - (x_1 + \cdots + x_n)} \quad \text{for } |x| < r/\sqrt{n}$$

Then for $|x| < r/\sqrt{n}$,

$$f(x) = \frac{1}{1 - (x_1 + \cdots + x_n)/r} = \sum_{k=0}^\infty \left(\frac{x_1 + \cdots + x_n}{r} \right)^k = \sum_{k=0}^\infty \frac{1}{r^k} \sum_{|\alpha|=k} \binom{|\alpha|}{\alpha} x^\alpha$$
$$= \sum_\alpha \frac{|\alpha|!}{r^{|\alpha|}\alpha!} x^\alpha$$

by multinomial theorem. This is valid for $|x_1 + \cdots + x_n|/r < 1$, which holds for $|x| < r/\sqrt{n}$. In fact, on this domain, the series converges absolutely. Indeed :

$$\sum_\alpha \frac{|\alpha|!}{r^{|\alpha|}\alpha!} |x|^\alpha = \sum_{k=0}^\infty \left(\frac{|x_1| + \cdots + |x_n|}{r} \right)^k < \infty$$

since $|x_1| + \cdots + |x_n| \le |x|\sqrt{n} < r$.

Definition) Let $f = \sum_\alpha f_\alpha x^\alpha$, $g = \sum_\alpha g_\alpha x^\alpha$ be two formal power series. We say g **majorises** f, written $g \gg f$ if

$$\left| f_\alpha \right| \le g_\alpha$$

for all α, and say that g is a **majorant** of f.

Lemma)

(i) If $g \gg f$ and g converges for $|x| < r$ then f also converges (absolutely) for $|x| < r$.

(ii) If f converges for $|x| < r$, then for any $s \in (0, r/\sqrt{n})$, f has a majorant that converges for $|x| < s/\sqrt{n}$.(n is the dimension of the space)

proof)

(i) We note that

$$\sum_\alpha \left| f_\alpha x^\alpha \right| \le \sum_\alpha \left| f_\alpha \right| |x_1|^{\alpha_1} \cdots |x_n|^{\alpha_n}$$
$$\le \sum_\alpha g_\alpha \tilde{x}^\alpha$$

where $\tilde{x} = (|x_1|, \cdots, |x_n|)$. Now $|\tilde{x}| = |x| < r$ so $\sum_\alpha g_\alpha \tilde{x}^\alpha$ converges, hence $\sum_\alpha \left| f_\alpha x^\alpha \right|$ converges. Hence f converges on $|x| < r$ absolutely.

(ii) Pick s s.t. $0 < s\sqrt{n} < r$, and set $y = s(1, \cdots, 1)$. Then $|y| = s\sqrt{n} < r$. Hence $\sum_\alpha f_\alpha y^\alpha$ converges. A convergent series has bounded terms, $\exists C > 0$ s.t. $\left|f_\alpha y^\alpha\right| \leq C$ for all α, and therefore

$$\left|f_\alpha\right| \leq \frac{C}{y_1^{\alpha_1} \cdots y_n^{\alpha_n}} = \frac{C}{s^{|\alpha|}} \leq \frac{C|\alpha|!}{s^\alpha \alpha!}$$

But then $g(x)$ defined by

$$g(x) = \frac{Cs}{s - (x_1 + \cdots + x_n)} = C \sum_\alpha \frac{|\alpha|!}{s^\alpha \alpha!} x^\alpha$$

majorises f and converges for $|x| < s/\sqrt{n} < r/n$.

(End of proof) $\square$

Remark : If $f = (f^1, \cdots, f^m)$ and $g = (g^1, \cdots, g^m)$ are formal power series, then we say

$$g \gg f \quad \text{if} \quad g^i \gg f^i \quad i = 1, \cdots, m$$

Cauchy-Kovalevskaya for First Order Systems

We will study a problem that generalises the Cauchy problem for ODEs we have already discussed.

As coordinates on $\mathbb{R}^n$ we take $(x', t) = x$ where

$$x' = (x_1, \cdots, x_{n-1}) \in \mathbb{R}^{n-1}, \quad t = x^n \in \mathbb{R}$$

Set

$$B_r^n = \{t^2 + |x'|^2 < r^2\}, \quad B_r^{n-1} = \{|x'| < r, t = 0\}$$

We consider a system of equations for unknown $\underline{u}(x) \in \mathbb{R}^m$. More concretely, we seek a solution to

$$\begin{aligned} \underline{u}_t &= \sum_{j=1}^{n-1} \underline{\underline{B}}_j(\underline{u}, x') \cdot \underline{u}_{x_j} + \underline{c}(\underline{u}, x') \quad \text{on } B_r^n \\ \underline{u} &= 0 \quad \text{on } B_r^{n-1} \end{aligned} \tag{6}$$

where $\underline{u}_{x_j} = \partial u / \partial x_j$ etc. We assume that we are given the real analytic functions

$$\begin{aligned} \underline{\underline{B}}_j &: \mathbb{R}^m \times \mathbb{R}^{n-1} \to \mathrm{Mat}(m \times m) \\ \underline{c} &: \mathbb{R}^m \times \mathbb{R}^{n-1} \to \mathbb{R}^m \end{aligned}$$

(these functions do not have to defined on the entire space, but just have to be defined on $\mathbb{R}^n \times B_r^{n-1}$)
Note we assume $\underline{\underline{B}}_j$ and $\underline{u}$ do not depend explicitly on t. We can always introduce u^{m+1} satisfying $\partial_t u^{m+1} = 1$, $u^{m+1} = 0$ on B_r^{n-1} and extending the system.

We will write $\underline{\underline{B}}_j = ((b_j^{kl}))$ and $\underline{c} = (c^1, \cdots, c^m)^T$. Then in components (6) reads:

$$u_t^k = \sum_{j=1}^{n-1} \sum_{l=1}^{m} b_j^{kl}(\underline{u}, x') u_{x_j}^l + c^k(\underline{u}, x') \quad k = 1, \cdots, m$$

Examples : Take $m = 2$, write $\underline{u} = (f, g)^T$.

(a)

$$\begin{cases} f_t = g_x + F \\ g_t = f_x \end{cases}$$

together imply $f_{tt} - f_{xx} = F_t$

(b)

$$\begin{cases} f_t = -g_x + F \\ g_t = f_x \end{cases}$$

together imply $f_{tt} + f_{xx} = F_t$. (Note $F = 0$ gives Cauchy-Riemann equation)

Theorem) (Cauchy-Kovalevskaya) Assume $\{\underline{\underline{B}}_j\}_{j=1}^{n-1}$ and $\underline{c}$ are real analytic. Then for sufficiently small $r > 0$ there exists a unique real analytic function $\underline{u} : B_r^n \to \mathbb{R}^m$ solving the problem (6).

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(12th October, Friday)

Theorem) (Cauchy-Kovalevskaya) Assume $\{\underline{\underline{B}}_j\}_{j=1}^{n-1}$ and $\underline{c}$ are real analytic. Then for sufficiently small $r > 0$ there exists a unique real analytic function $\underline{u} : B_r^n \to \mathbb{R}^m$ solving the problem (6).

proof)

1. The strategy will be to write

$$\underline{u}(x) = \sum_\alpha \underline{u}_\alpha x^\alpha \tag{7}$$

and compute coefficients

$$\underline{u}_\alpha = \frac{D^\alpha \underline{u}(0)}{\alpha!}$$

in terms of $\underline{\underline{B}}_j$, $\underline{c}$ and show that the series (7) converges on B_r^n for r small enough.

2. As $\underline{\underline{B}}_j$ and $\underline{c}$ are real analytic, we can write

$$\begin{aligned} \underline{\underline{B}}_j(z, x') &= \sum_{\gamma,\delta} \underline{\underline{B}}_{j,\gamma,\delta} z^\gamma (x')^\delta \quad \gamma \in \mathbb{N}^m, \delta \in \mathbb{N}^{n-1} \text{ multiindices} \\ \underline{c}(z, x') &= \sum_{\gamma,\delta} \underline{c}_{\gamma,\delta} z^\gamma (x')^\delta \end{aligned}$$

where these power series converge for $|z|^2 + |x'|^2 < s^2$, wlog $s > r$. Thus:

$$\begin{aligned} \underline{\underline{B}}_{j,\gamma,\delta} &= \frac{D_z^\delta D_{x'}^\delta \underline{\underline{B}}_j(0,0)}{\gamma!\delta!} \\ \underline{c}_{\gamma,\delta} &= \frac{D_z^\delta D_{x'}^\delta \underline{c}(0,0)}{\gamma!\delta!} \end{aligned} \tag{8}$$

3. Since $\underline{u} \equiv 0$ on $\{t = x^n = 0\}$, we have

$$\underline{u}_\alpha = \frac{D^\alpha \underline{u}(0)}{\alpha!} = 0$$

for all multi-indices α with $\alpha_n = 0$.
Now, we use the evolution equation (6) to deduce

$$\underline{u}_{x_n}(0) = \underline{u}_t(0) = \sum_{j=1}^{n-1} \underline{\underline{B}}_j(\underline{u}(0), 0)\underline{u}_{x_j}(0) + \underline{c}(\underline{u}(0), 0) = \underline{c}(0, 0)$$

Fix $i \in \{1, 2, \cdots, n-1\}$, differentiate (6) with respect to x^i :

$$\begin{aligned} \underline{u}_{tx_i} = \sum_{j=1}^{n-1} \Big[\partial_{x_i} \underline{\underline{B}}_j(\underline{u}, x')\underline{u}_{x_j} + \big(\sum_{i=1}^{m} \partial_{z_i} \underline{\underline{B}}_j(\underline{u}, x') \frac{\partial u^i}{\partial x^j} \underline{u}_{x_j} \big) + \underline{\underline{B}}_j(\underline{u}, x')\underline{u}_{x_i x_j} \Big] \\ + \partial_{x_i} \underline{c}(\underline{u}, x') + \sum_{i=1}^{m} \partial_{z_l} \underline{c}(\underline{u}, x') \frac{\partial u^l}{\partial x^i} \end{aligned}$$

$$\underline{u}_{tx_i}(0) = \partial_{x_i} \underline{c}(0, 0)$$

Iterating this, we deduce $D^\alpha \underline{u}(0) = D^\delta \underline{c}(\underline{0}, 0)$ where $\alpha = (\delta, 1)$.

4. Now, suppose $\alpha = (\delta, 2)$, for $\delta \in \mathbb{N}^{n-1}$. Then

$$\begin{aligned}
D^\alpha u^k &= D^\delta(u^k_{x_n x_n}) = D^\delta(u^k_t)_t \\
&= D^\delta\Big(\sum_{j=1}^{n-1}\sum_{l=1}^{m} b^{kl}_j u^l_{x_j} + c^k\Big)_t \\
&= D^\delta\Big(\sum_{j=1}^{n-1}\sum_{i=1}^{m}\Big[b^{kl}_j u^l_{x_j t} + \sum_{p=1}^{m}(b^{kl}_j)_{z_p} u^l_{x_j} u^p_t\Big] + \sum_{p=1}^{m} c^k_{z_p} u^p_t\Big)
\end{aligned}$$

so

$$D^\alpha u^k(0) = D^\alpha\Big(\sum_{j=1}^{n-1}\sum_{i=1}^{m} b^{kl}_j u^l_{x_j t} + \sum_{p=1}^{m} c^k_{z_p} u^p_t\Big)\Big|_{x=0,\underline{u}=0}$$

Now crucially, the expression on the right can be expanded to produce a polynomial with non-negative coefficients involving derivative of $\underline{\underline{B}}_j$ and $\underline{c}$, and derivatives $D^\beta \underline{u}$ where $\beta_n \leq 1$. More generally, for each multi-index α and each $k \in \{1, \cdots, n\}$, we can compute

$$D^\alpha u^k(0) = p^k_\alpha\Big(D^\alpha_z D^\delta_{x'} \underline{\underline{B}}_j, D^\alpha_z D^\delta_{x'} \underline{c}, D^\beta \underline{u}\Big)\big|_{x=0,\underline{u}=0}$$

where $\beta_n \leq \alpha_n - 1$ and p^k_α is some polynomial in its arguments with non-negative coefficients. Equivalently, for each α, k

$$u^k_\alpha = q^k_\alpha(\underline{\underline{B}}_{j,\alpha,\delta}, \underline{c}_{\gamma,\delta}, u_\beta)$$

where q^k_α is a polynomial with non-negative coefficients, with $\beta_n \leq \alpha_n - 1$.

5. We have shown that if a solution exists, we can compute all derivatives at 0 in terms of known quantities. We will construct a series which majorises the formal sum $\sum_\alpha u_\alpha x^\alpha$.

   First suppose

$$\underline{\underline{B}}^*_j \gg \underline{\underline{B}}_j \quad \underline{c}^* \gg \underline{c}$$

where

$$\begin{aligned}
\underline{\underline{B}}^*_j &= \sum_{\gamma,\delta} \underline{\underline{B}}^*_{j,\gamma,\delta} z^\gamma (x')^\delta \\
\underline{c}^* &= \sum_{\gamma,\delta} \underline{c}^*_{\gamma,\delta} z^\gamma (x')^\delta
\end{aligned}$$

Assume these converge for $|z|^2 + |x'|^2 < s^2$ (decrease s if necessary). For all j, γ, δ, k, l,

$$0 \leq |B^{kl}_{j,\gamma,\delta}| \leq (B^*)^{kl}_{j,\gamma,\delta}, \quad 0 \leq |c^k_{\gamma,\delta}| \leq (c^*)^{kl}_{\gamma,\delta}$$

We consider the modified problem:

$$\begin{aligned}
&\underline{u}^*_t = \sum_{j=1}^{n-1} \underline{\underline{B}}^*_j(\underline{u}^*, x')\underline{u}^*_{x_j} + \underline{c}^*(\underline{u}^*, x') \quad \text{for } |x| < r \\
&\underline{u}^* = \underline{0} \quad \text{on } B^{n-1}_r
\end{aligned}$$

As above, seek a real analytic solution

$$\underline{u}^* = \sum_\alpha \underline{u}^*_\alpha x^\alpha \quad \text{where } \underline{u}^*_\alpha = \frac{D^\alpha \underline{u}(0)}{\alpha!}$$

6. We claim $0 \leq |u^k_\alpha| \leq (u^*)^k_\alpha$ for all $\alpha \in \mathbb{N}^n$.

   We do this by proof by induction on α_n.

   For $\alpha_n = 0$, $u^*_\alpha = u_\alpha = 0$

For the induction step: (for $\beta_\alpha \leq \alpha_n - 1$)

$$\begin{aligned}|u^k_\alpha| &= |q^k_\alpha(\underline{\underline{B}}_{j,\gamma,\delta}, \underline{c}_{\gamma,\delta}, \underline{u}_\beta)| \\ &\leq q^k_\alpha(|B^{kl}_{j,\gamma,\delta}|, |C^k_{\gamma,\delta}|, |u^k_\beta|) \\ &\leq q^k_\alpha((B^*)^{kl}_{j,\gamma,\delta}, (c^*)^k_{\gamma,\delta}, (u^*)^k_\beta) \\ &= (u*)^k_\alpha\end{aligned}$$

Using positivity of coefficients of q and induction assumption. Thus $\underline{u}^* \gg \underline{u}$. Remains to show we can find $\underline{\underline{B}}^*_j$, $\underline{c}^*$ s.t. a solution $\underline{u}^*$ exists and converges near 0.

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(15th October, Monday)

Last lecture :

- a formal power series solution $\underline{u} = \sum_\alpha \underline{u}_\alpha x^\alpha$ exists.
- If $\underline{\underline{B}}^*_j \gg \underline{\underline{B}}_j$, $\underline{c}^* \gg \underline{c}$ and $\underline{u}^*$ satisfies

$$\begin{aligned}&\underline{u}^*_t = \sum_{j=1}^{n-1} \underline{\underline{B}}^*_j(\underline{u}^*, x')\underline{u}^*_{x_j} + \underline{c}^*(\underline{u}^*, x') \quad \text{for } |x| < r \\ &\underline{u}^* = \underline{0} \quad \text{on } B^{n-1}_r\end{aligned}$$

then the power series for $\underline{u}^* = \sum_\alpha \underline{u}^*_\alpha x^\alpha$.

proof, continued) To complete the proof, it suffices to show that for any $\underline{\underline{B}}_j$, $\underline{c}$, we can find $\underline{\underline{B}}^*_j$, $\underline{c}^*_j$ such that the corresponding $\underline{u}^*_j$ is a convergent series.

We make a particular choice for $\underline{\underline{B}}^*_j$, $\underline{c}^*$. For this we recall from an earlier lemma that

$$\begin{aligned}\underline{\underline{B}}^*_j &= \frac{Cr}{r - (x_1 + \cdots + x_{n-1}) - (z_1 + \cdots + z_m)} \begin{pmatrix} 1 & 1 & \cdots 1 \\ \vdots & & \ddots \vdots \\ 1 & 1 & \cdots 1 \end{pmatrix} \\ \underline{c}^* &= \frac{Cr}{r - (x_1 + \cdots + x_{n-1}) - (z_1 + \cdots + z_m)} (1, \cdots, 1)^T\end{aligned}$$

will majorise $\underline{\underline{B}}_j$, $\underline{c}$, provided C is large enough, r is small enough and $\underline{\underline{B}}^*_j$, $\underline{c}^*$ are given by convergent series for $|x'|^2 + |z|^2 < r^2$. With these choices of majorants, the modified equation takes the form :

$$\begin{aligned}(u^*)^k_t &= \frac{Cr}{r - (x_1 + \cdots + x_{n-1}) - ((u^*)^1 + \cdots + (u^*)^m)} (\sum_{j,l} \left(u^*\right)^l_{x_j} + 1\big) \quad \text{for } |x'|^2 + t^2 < r^2 \\ u^* &= 0 \qquad\qquad \text{for } t = 0, |x'| < r\end{aligned}$$

This problem has an explicit solution.

$$\underline{u}^* = v^*(1, \cdots, 1)^T$$

where

$$v^* = \frac{1}{mn}\Big(r - (x_1 + \cdots + x_{n-1}) - \sqrt{(r - (x_1 + \cdots + x_{n-1}))^2 - 2nmCrt}\Big)$$

(*Check this is indeed the solution!!*) v^* is real analytic for $|x'|^2 + t^2 < r^2$, provided r is small enough. Hence $\underline{u}^*$ is given by a convergent series since $\underline{u}^* \gg \underline{u}$. Our formal power series for $\underline{u}$ converges.

Initial condition hold for $\underline{u}$ since

$$\underline{u}_\alpha = \underline{0} \quad \text{if} \quad \alpha_n = 0$$

Moreover, the functions $\underline{u}_t$ and $\sum_{j=1}^{n-1} \underline{\underline{B}}_j(\underline{u}, x')\underline{u}_{x_j} + \underline{c}(\underline{u}, x')$ are both real analytic near 0 and by construction, have the same Taylor expansion. Hence they must agree on a neighbourhood of 0, so the equation holds in some ball about 0.

(End of proof) □

Reduction to a First Order System

Example)

Consider the PDE problem for $u : \mathbb{R}^3 \to \mathbb{R}$

$$\begin{aligned} u_{tt} &= u u_{xy} - u_{xx} + u_t \\ u\big|_{t=0} &= u_0 \\ u_t\big|_{t=0} &= u_1 \end{aligned} \tag{9}$$

where $u_0, u_1 : \mathbb{R}^2 \to \mathbb{R}$ are given real analytic functions (near 0).
First note that $f = u_0 + t u_1$ is analytic in a neighbourhood of $0 \in \mathbb{R}^3$ and $f\big|_{t=0} = u_0$, $f_t\big|_{t=0} = u_1$.
Set $w = u - f$, then

$$\begin{aligned} w_{tt} &= w w_{xy} - w_{xx} + w_t + f w_{xy} + f_{xy} w + F \\ w\big|_{t=0} &= w_t\big|_{t=0} = 0 \end{aligned}$$

where $F = f f_{xy} - f_{xx} + f_t - f_{tt}$.
Let $(x, y, t) = (x^1, x^2, x^3)$ and set $\underline{u} = (w, w_x, w_y, w_t) = (u^1, u^2, u^3, u^4)$. Then

$$\begin{aligned} u^1_{x^3} &= w_t = u^4 \\ u^2_{x^3} &= w_{xt} = u^4_{x_1} \\ u^3_{x^3} &= w_{yt} = u^4_{x_2} \\ u^4_{x^3} &= w_{tt} = u^1 u^2_{x^2} - u^2_{x^1} + u^4 + f u^2_{x^2} + f_{xy} u^1 + F \end{aligned}$$

Now, defining:

$$\underline{\underline{B}}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \underline{\underline{B}}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ u_1 + f & 0 & 0 & 0 \end{pmatrix}$$

$$\underline{c} = (u^4, 0, 0, u^4 + f_{xy} u^1 + F)^T$$

The system of equations is in the form

$$\underline{u}_{x_2} = \sum_{j=1} 2 \underline{\underline{B}}_j \underline{u}_{x_j} + \underline{c}$$

where $\underline{\underline{B}}_j$, $\underline{c}$ are real analytic near 0. By Cauchy-Kovalevskaya, a real analytic solution to (9) exists near 0.

Note : this procedure relied on

(a) being able to solve for u_{tt},

(b) u_{tt} depending on at most two derivatives of u (in a quasilinear fashion)

More generally, suppose we wish to solve the quasilinear problem :

$$\begin{aligned} &\sum_{|\alpha|=k} a_\alpha(D^{k-1}u, \cdots, u, x) D^\alpha u + a_0(D^{k-1}u, \cdots, u, x) = 0 \quad \text{for } |x| < r \\ &u = \frac{\partial u}{\partial x_n} = \cdots = \frac{\partial^{k-1} u}{\partial x_n^{k-1}} = 0 \qquad \text{for } |x'| < r,\ x_n = 0 \end{aligned}$$

called a **Cauchy problem**.
We introduce

$$\underline{u} = \left(u, \frac{\partial u}{\partial x_n}, \cdots, D^\alpha u, \cdots\right)_{|\alpha| \le k-1} = (u^1, \cdots, u^m)$$

$\underline{u}$ contains all derivative of u up to order $k-1$. Wlog, (by changing the order if necessary) put $u^m = \partial^{k-1}u/\partial x_n^{k-1}$. For $j < m$, we can compute $\partial u^j/\partial x^n$ in terms of $\partial u^l/\partial x^p$ for some $l \in \{1, \cdots, m\}$ and $p < n$.

To computed $\partial u^m/\partial x_n$ we need to use the equation. Suppose that

$$a_{(0,\cdots,0,k)}(0,\cdots,0) \neq 0$$

Then we can write the equation as :

$$\frac{\partial^k u}{\partial x_n^k} = \frac{-1}{a_{(0,\cdots,k)}(D^{k-1}u,\cdots,u,x)}\Big[\sum_{|\alpha|=k,\alpha_n<k} a_\alpha D^\alpha u + a_0\Big]$$

Assuming a_α are real analytic, the denominator will be non-zero near the origin. The RHS can be written in terms of $\frac{\partial u^l}{\partial x^p}$ for $p < n$ and $\underline{u}$. We see we can write the equation as a first ordered system for $\underline{u}$, *provided*(this condition is important! would come back to this later)

$$a_{(0,\cdots,k)}(0,\cdots,0) \neq 0 \quad \text{(non-characteristic condition)}$$

In this case we can apply Cauchy-Kovalevskaya.

==

(17th October, Wednesday)

(Problem sheet 1 handed out. Example classes sign-up. First example class (probably) at Thur/Fri next week)

Cauchy Problems for Quasilinear Equations with Data on a Surface

We say $\Sigma \subset \mathbb{R}^n$ is a real analytic **hypersurface** near $x \in \Sigma$ if there exists $\epsilon > 0$ and a real analytic map $\Phi : B_\epsilon(x) \to U \subset \mathbb{R}^n$ where $U = \Phi(B_\epsilon(x))$ such that

- Φ is bijective, and the inverse $\Phi^{-1} : U \to B_\epsilon(x)$ is real analytic.
- $\Phi(\Sigma \cap B_\epsilon(x)) = \{x_n = 0\} \cap U$.

We think of Φ as 'straightening out the boundary'.

There are many examples, e.g. $\{|x| = 1\}$.

Let γ be the unit normal to Σ and suppose u solves

$$\sum_{|\alpha|=k} a_\alpha(D^{k-1}u,\cdots,u,x)D^\alpha u + a_0(D^{n-1}u,\cdots,u,x) = 0 \quad \text{in } B_\epsilon(x) \tag{10}$$
$$u = \gamma^i\partial_i u = \cdots = (\gamma^i\partial_i)^{k-1} = 0 \quad \text{on } \Sigma$$

Define $v(y) = u(\Phi^{-1}(y)) \quad \Leftrightarrow \quad u(x) = v(\Phi(x))$. Note

$$\frac{\partial u}{\partial x^i} = \frac{\partial v}{\partial y^j}\frac{\partial \Phi^j}{\partial x^i}$$
$$\frac{\partial^2 u}{\partial x^i \partial x^k} = \frac{\partial v^2}{\partial y^j \partial y^i}\frac{\partial \Phi^j}{\partial x^i}\frac{\partial \Phi^l}{\partial x^k} + \frac{\partial v}{\partial y^j}\frac{\partial^2 \Phi^j}{\partial^i \partial x^k} \quad \text{etc.}$$

So we can compute $D^\alpha u$ as a linear combination of $D^\beta v$ for $|\beta| \leq |\alpha|$, with coefficients depending on Φ. So if u solves (10), then v will solve

$$\sum_{|\alpha|=k} b_\alpha(D^{k-1}v,\cdots,v,x)D^{\alpha v} + b_0(D^{k-1}v,\cdots,v,x) = 0$$

Moreover,

$$v\big|_{x_n=0} = u\big|_\Sigma = 0$$
$$\partial_i v\big|_{x_n=0} = (D\Phi^-1)_{ij}\partial_j u\big|_\Sigma = 0$$

and proceeding similarly for $\partial^{k-1}v/(\partial x^n)^{k-1}$, we have

$$v = \frac{\partial v}{\partial x^n} = \cdots = \partial^{k-1}v/(\partial x^n)^{k-1} = 0 \quad \text{on } \{x_n = 0\}$$

We can solve this, provided

$$b_{(0,\cdots,0,k)}(0,0,\cdots,0,y) \neq 0 \quad \text{on } \{x_n = 0\}$$

Note if $|\alpha| = k$,

$$D^\alpha u = \frac{\partial^k v}{\partial y_n^k}(D\Phi^n)^\alpha + (\text{terms not involving } \frac{\partial^k v}{\partial y_n^k})$$

So the coefficient of $\partial^k v/\partial y_n^k$ in

$$\sum_{|\alpha|=k} a_\alpha(D^{k-1}u,\cdots,u,x)D^\alpha u + a_0(D^{n-1}u,\cdots,u,x) = 0$$

is

$$b_{(0,\cdots,k)} = a_\alpha(D\Phi^n)^\alpha$$

But $\Sigma = \{\Phi^n = 0\}$ so $D\Phi^n \propto \gamma$. Therefore,

$$b_{(0,\cdots,k)} \neq 0 \quad \Leftrightarrow \quad \sum_{|\alpha|=k} a_\alpha(D\Phi^n)^\alpha \neq 0 \quad \Leftrightarrow \quad \sum_{|\alpha|=k} a_\alpha\gamma^\alpha \neq 0$$

Definition) Σ is a non-characteristic at $x \in \Sigma$ for the problem (10) provided

$$\sum_{|\alpha|=k} a_\alpha(0,\cdots,0,x)\gamma^\alpha(x) \neq 0$$

Finally, we have a more general version of Cauchy-Kovalevskaya.

Theorem) (Cauchy-Kovalevskaya Redux) Suppose $\Sigma \subset \mathbb{R}^n$ is a real analytic hypersurface. IfΣ is non-characteristic for (10) at $x \in \Sigma$, there exists a unique real analytic solution to (10) in a neighbourhood of x.

> **proof)** We have already seen that we can solve the problem for v uniquely, then $u(x) = v(\Phi(x))$ is the unique solution for (10)
>
> *(End of proof)* □

Characteristic Surfaces for 2nd Order Linear PDE

Consider the linear operator

$$Lu = \sum_{i,j=1}^{n} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i\frac{\partial u}{\partial x^i} + cu$$

with $a_{ij}, b_i, c : \mathbb{R}^n \to \mathbb{R}$.

Consider the Cauchy problem

$$\begin{aligned} &Lu = f \\ &u = \sum_{i=1}^{n}\xi^i\frac{\partial u}{\partial x^i} = 0 \quad \text{on } \Pi_\xi = \{\xi\cdot x = 0\} \end{aligned}$$

Π_ξ is characteristic at $x \in \mathbb{R}^n$ if :

$$\sigma_p(\xi,x) = \sum_{i,j=1}^{n} a_{ij}\xi^i\xi^j = 0$$

σ_p is the **principal symbol** of L.

- If $\sigma_p(\xi, x) > 0$ for all x, $\xi \neq 0$, then no plane is characteristic, and such operations are called **elliptic**.

Let us restrict to the case where a_{ij}, b_i, c are constants. Suppose $b_i = c = 0$ and Π_ξ is characteristic. Then

$$u(x) = e^{i\lambda\xi\cdot x}$$

solve $Lu = 0$ for any λ. By taking λ large, we can construct solutions to $Lu = 0$ whose derivative (in the ξ direction) is a large as we like. In particular, Lu is very regular, but u need not be. In the elliptic setting, this cannot happen.