Analysis of PDEs

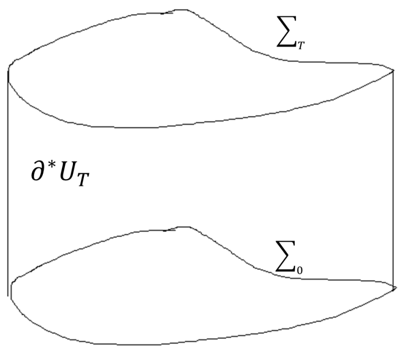
(23rd November, Friday)

Initial-Boundary Value Problems for Wave Equations

Suppose $U \subset \mathbb{R}^n$ is open with C^1-boundary. We define

$$U_T = U \times (0,T), \quad \Sigma_t = U \times \{t\}, \quad \partial^* U_T = \partial U \times [0,T]$$

So $\partial U_T = \Sigma_0 \sqcup \Sigma_T \sqcup \partial^* U_T$. We define



$$Lu = -\sum_{i,j=1}^{n} (a^{ij} u_{x_i})_{x_j} + \sum_{i=1}^{n} b^i u_{x_i} + b u_t + cu$$

where $a^{ij}, b^i, b, c \in C^1(\overline{U}_T)$. Further assume a^{ij} satisfy the uniform ellipticity condition

$$\sum_{i,j=1}^{n} a^{ij}(x,t)\xi_i \xi_j \geq \theta |\xi|^2$$

for some $\theta > 0$, all $(x,t) \in U_T$, $\xi \in \mathbb{R}^n$.

The **initial-boundary value problem(IBVP)** we consider is:

$$\begin{cases} u_{tt} + Lu = f & \text{in } U_T \\ u = \psi, \ u_t = \psi' & \text{on } \Sigma_0 \\ u = 0 & \text{on } \partial^* U_T \end{cases} \tag{1}$$

e.g. The model in our mind is solving wave equation on a string given boundary conditions. If $L = -\Delta$, $f = 0$, this is the wave equation on a bounded domain with specified initial conditions.

As with the elliptic boundary value problem, we first find a weak formulation of the problem. Suppose $u \in C^2(\overline{U}_T)$ is a solution of (1) and multiply the equation by $v \in C^2(\overline{U}_T)$ satisfying $v = 0$ on $\partial^* U_T \cup \Sigma_T$. Then integrate over U_T.

$$\int_0^T dt \int_U dx(u_{tt}v + Luv) = \int_0^T dt \int_U dx f v$$

Integrating by parts, we find

$$\int_{U_T} \Big(-u_t v_t + \sum_{i,j} a^{ij} u_{x_i} v_{x_j} + \sum_i b^i u_{x_i} v + b u_t v + cuv \Big) dxdt - \int_{\Sigma_0} \psi' v dx = \int_{U_T} f v dxdt \qquad (2)$$

Conversely, if (2) holds for all $v \in C^2(\overline{U}_T)$ which vanish on $\Sigma_T \cup \partial^* U_T$ and $u \in C^2(\overline{U}_T)$ satisfies $u = \psi$ on Σ_0, $u = 0$ on $\partial^* U_T$, undoing the integration by parts gives

$$\int_{U_T} (u_{tt}v + Lv - fv)dxdt + \int_{\Sigma_0} (u_t - \psi')v dx = 0$$

Taking $v \in C_c^\infty(U_T)$, the Σ_0 term vanishes and we deduce $u_{tt} + Lu = f$ in U_T. This implies

$$\int_{\Sigma_0} (u_t - \psi')v dx = 0 \quad \forall v \in C_c^\infty(\Sigma_0) \quad \Rightarrow \quad u_t = \psi'$$

The expression (2) makes sense if $u \in H^1(U_T)$, $v \in H^1(U_T)$. This motivates the definition :

Definition) Suppose $f \in L^2(U_T)$, $\psi \in H_0^1(\Sigma_0)$, $\psi \in L^2(\Sigma_0)$ and $a^{ij}, b^i, b, c \in C^1(\overline{U}_T)$ with a^{ij} satisfying uniform ellipticity condition in U_T. We say $u \in H^1(U_T)$ is a weak solution of the IBVP (1) if

$$\begin{cases} u = \psi & \text{on } \Sigma_0 & \text{in the trace sense} \\ u = 0 & \text{on } \partial^* U_T & \text{in the trace sense} \end{cases}$$

and (2) holds for all $v \in H^1(U_T)$ with $v = 0$ on $\Sigma_T \cup \partial^* U_T$ in the trace class.

Note that, we could not say $\partial_t u = \psi'$ on Σ_0 in trace sense, because $\partial_t u$ is just a L^2-function while we do not have trace theorem for L^2 functions.

We cannot use Lax-Milgrim theorem as it is. But we can do something different to show unique existence of the solution in a different way.

Theorem) A weak solution to (1), if it exists, is unique.

Motivation : Suppose we consider the standard wave equation

$$u_{tt} - \Delta u = 0 \quad \text{in } U_T$$

with the initial and boundary conditions as in (1). Assume $u \in C^2(U_T)$. To show the solution is unique, sufficient to consider $\psi = \psi' = 0$. Multiply by u_t and integrate over $x \in U$.

$$\int_U u_{tt}u_t - \Delta u \cdot u_t dx = \int_U u_{tt}u_t + Du \cdot Du_t dx = \frac{d}{dt} \int_U \frac{1}{2}u_t^2 + \frac{1}{2}|Du|^2 dx$$

So if $u = u_t = 0$ initially, then

$$\int_{\Sigma_t} \frac{1}{2}u_t^2 + |Du|^2 dx = 0 \quad \forall t \in (0, T)$$

and therefore $u = 0$ in U_T.

We work in the same spirit for the general case where $u \in H^1(U_T)$, but we have to be more careful when doing this.

proof of theorem) Note that by linearity, sufficient to prove that if $\psi = 0$, $\psi' = 0$, $f = 0$ then $u = 0$. We want to use u_t as a test function but it is not regular enough (does not vanish on Σ_T). Take

$$v(x,y) = \int_t^T e^{-\lambda s} u(x,s)ds$$

for $\lambda \in \mathbb{R}$ we choose later. We find $v \in H^1(U_T)$, $v = 0$ on $\partial^* U_T \cup \Sigma_T$ and $v_t = -e^{-\lambda t} u \in H^1(U_T)$. Putting this into (2) with $\psi = \psi' = f = 0$, we have

$$\int_{U_T} \Big[u_t u e^{-\lambda t} - \sum_{i,j} a^{ij} v_{tx_i} v_{x_j} e^{\lambda t} + \sum_i b^i u_{x_i} v - b v^2 e^{\lambda t} + (c-1)uv - v v_t e^{\lambda t} \Big] dxdt = 0$$

Rewriting,

$$\begin{aligned}
\textbf{(A)} = \int_{U_T} \Big[&\frac{d}{dt} \Big(\frac{1}{2} u^2 e^{-\lambda t} - \frac{1}{2} \sum_{i,j} a^{ij} v_{x_i} v_{x_j} e^{\lambda t} - \frac{1}{2} v^2 e^{\lambda t} \Big) \\
&+ \frac{\lambda}{2} \Big(u^2 e^{-\lambda t} + \frac{1}{2} \sum_{i,j} a^{ij} v_{x_i} v_{x_j} e^{\lambda t} + v^2 e^{\lambda t} \Big) \Big] dxdt \\
= \int_{U_T} \Big[&\frac{1}{2} \sum_{i,j} \dot{a}^{ij} v_{x_i} v_{x_j} e^{\lambda t} - \sum_i b^i u_{x_i} v + b v^2 e^{\lambda t} - (c-1)uv \Big] dxdt = \textbf{(B)}
\end{aligned}$$

and

$$\begin{aligned}
\textbf{(A)} = &\int_{\Sigma_T} \frac{1}{2} u^2 e^{-\lambda T} dx + \int_{\Sigma_0} \Big(\frac{1}{2} \sum_{i,j} v_{x_i} v_{x_j} + \frac{1}{2} v^2 \Big) \\
&+ \frac{\lambda}{2} \int_{U_T} \Big(u^2 e^{-\lambda t} + \frac{1}{2} \sum_{i,j} a^{ij} v_{x_i} v_{x_j} + v^2 e^{\lambda t} \Big) dxdt
\end{aligned}$$

and (using AM-GM inequality and that a, b, c are of C^1)

$$\textbf{(B)} \leq C \int_{U_T} u^2 e^{-\lambda t} + (\sum_{i,j} a^{ij} v_{x_i} v_{x_j} + v^2) e^{\lambda t} dxdt$$

for some constant C independent of λ. Putting these together and taking λ large enough, we have

$$(\lambda - 2C) \int_{U_T} u^2 e^{-\lambda t} + (\sum_{i,j} a^{ij} v_{x_i} v_{x_j} + v^2) e^{\lambda t} dxdt \leq 0$$

With $\lambda - 2C \geq 0$, we have $u \equiv 0$

(End of proof) $\square$