Mock Test - Analysis of Partial Differential Equations

Attempt all the **THREE** questions. The questions carry equal weight and each question accounts for **40%** of the total marks. The marks allotted to each sub question is mentioned in the parathesis [points].

A : This question is about real and functional analysis concepts introduced and used in the lectures.

1. Recall Hölder's inequality and Minkowski's inequality for L^p spaces and use Hölder's inequality in deriving Minkowski's inequality.

   [6 points]

2. Consider the following notion of *weak derivative* for a locally integrable function $u : \Omega \to \mathbb{R}$.

   A locally integrable function v in Ω is said to be the α^{th} weak derivative of u if

   $$\int_\Omega v\,\phi\,\mathrm{d}x = (-1)^{|\alpha|}\int_\Omega u\,\partial^\alpha\phi\,\mathrm{d}x \qquad \text{for every } \phi \in C_0^{|\alpha|}(\Omega).$$

   Present THREE different ways of defining the Sobolev space $H^s(\mathbb{R}^d)$ for $s \in \mathbb{N}$ discussed during the lectures. Show that all the three definitions are equivalent and that $H^s(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$.

   [8 points]

3. State Lax-Milgram theorem. Apply this theorem in showing that the following stationary problem admits a solution $u \in H^1(\Omega)$, unique up to the addition of a constant.

   $$\begin{cases} b(x)\cdot\nabla u - \nabla\cdot(A(x)\nabla u) = f & \text{in } \Omega, \\ -\nabla u\cdot n = g & \text{on } \partial\Omega, \end{cases}$$

   if and only if

   $$\int_\Omega f(x)\,\mathrm{d}x - \int_{\partial\Omega} g(x)\,\mathrm{d}\sigma(x) = 0,$$

   where

   $b \in L^\infty(\Omega;\mathbb{R}^d)$ such that $\nabla\cdot b = 0$ in Ω and $b\cdot n = 0$ on $\partial\Omega$,

   $A \in L^\infty(\Omega;\mathbb{R}^{d\times d})$ and $\exists\ c_1, c_2 > 0$ such that $c_1|\xi|^2 \le A(x)\xi\cdot\xi \le c_2|\xi|^2$ for a.e. $x \in \Omega$, $\forall \xi \in \mathbb{R}^d$,

   $f \in L^2(\Omega)$, $g \in H^1(\Omega)$.

   [6 points]

4. Give at least two examples of sequences converging weakly but not strongly. Show that the weak limit of a product is not in general the product of the weak limits. Show that the weak convergence plus convergence of the norm implies strong convergence.

   [6 points]

5. State and prove Arzelà-Ascoli's theorem about the compactness property of a sequence of continuous functions on a compact metric space.

   [6 points]

6. The objective of this question is to prove the Rellich-Kondrachov compactness theorem:

   Assume Ω to be a bounded open subset of $\mathbb{R}^d$ with C^1 smooth boundary. Then, $H^1(\Omega) \subset\subset L^2(\Omega)$.

   (a) State the Sobolev embedding theorem and deduce the inclusion in Rellich-Kondrachov theorem stated above.

   Admit that any sequence $\{u_n\} \subset H^1(\Omega)$ can be extended on to the whole of $\mathbb{R}^d$ i.e., $\exists\, Eu_n : \mathbb{R}^d \to \mathbb{R}$ such that $Eu_n|_\Omega = u_n$ with the compact support of each Eu_n lying in a bounded open set $K \subset \mathbb{R}^d$. For the sake of simplicity, rename Eu_n as u_n.

   (b) Consider the mollifers $\eta_\varepsilon(x) = \dfrac{1}{\varepsilon^d}\eta(x/\varepsilon)$ where

$$\eta = \begin{cases} C\,\exp\left(1/(|x|^2 - 1)\right) & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases}$$

   and $C > 0$ chosen such that $\int_{\mathbb{R}^d} \eta \,\mathrm{d}x = 1$. Define a sequence of smooth functions: $u_n^\varepsilon := \eta_\varepsilon * u_n$. Prove that $u_n^\varepsilon \to u_n$ in L^2 as $\varepsilon \to 0$, uniformly in n.

   (c) Next, show that for each fixed $\varepsilon > 0$, the sequence $\{u_n^\varepsilon\}$ is uniformly bounded and equicontinuous.

   (d) Apply Arzelà-Ascoli's theorem to obtain a convergent subsequence from $\{u_n^\varepsilon\}$ that converges uniformy in K.

   (e) Present a diagonal argument to arrive at the result of Rellich-Kondrachov theorem.

   (f) In which of the above analysis, the boundedness property of Ω is very crucial?

   [8 points]

B : This question is about classification of PDE, Cauchy-Kovalevskaya theorem, Laplace and Heat equations.

1. Find the regions in the (x, y) plane where the following PDEs are elliptic, parabolic and hyperbolic:

$$\text{(I)} \qquad (1+x)\partial^2_{xx}u + 2xy\partial^2_{xy}u + y^2\partial^2_{yy}u = 0,$$

$$\text{(II)} \qquad \partial^2_{xx}u + 2x\partial^2_{xy}u + y\partial^2_{yy}u + (\partial_x u)^2 - u\partial_y u = 0.$$

   [4 points]

2. (a) Show that the following heat equation has infinitely many C^∞ solutions

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } [0,T)\times\mathbb{R}, \\ u(0,x) = 0 & \text{for } x \in \mathbb{R}. \end{cases}$$

   (b) Do the solutions constructed above preserve sign?

   (c) Derive energy estimates for the above equation and deduce uniqueness in the L^2 setting.

   [5 points]

3. Use separation of variables to show that

$$u(x_1, x_2) = \frac{1}{n^2}\sin(nx_1)\sinh(nx_2)$$

   is a solution to the following Laplace equation:

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^2, \\ u = 0, \;\; \partial_{x_2}u = \frac{1}{n}\sin(nx_1) & \text{on } \{x_2 = 0\}. \end{cases}$$

   Discuss whether the Cauchy problem for Laplace equation is well-posed.

   [5 points]

4. Suppose that u_ε solves the problem:

$$\begin{cases} -\partial_x(a(x/\varepsilon)\partial_x u_\varepsilon) = f & \text{in } (0,1), \\ u_\varepsilon(0) = u_\varepsilon(1) = 0, \end{cases}$$

   where a is a smooth, positive function, which is 1-periodic and $f \in L^2(0,1)$.

   (a) Show that $u_\varepsilon \rightharpoonup u_0$ in $H^1_0(0,1)$ where u_0 solves

$$\begin{cases} -\bar{a}\partial_{xx}u_0 = f & \text{in } (0,1), \\ u_0(0) = u_0(1) = 0. \end{cases}$$

   (b) How is $\bar{a}$ represented in terms of the periodic function a?

   [5 points]

5. Admit the following weak maximum principles for the heat equation:

   For $u \in C^2_1(\Omega_T)\cap C(\bar{\Omega}_T)$ if $\partial_t u - \Delta u \le 0$ then $\max_{\bar{\Omega}_T} u = \max_{\Gamma_T} u$

   and if $\partial_t u - \Delta u \ge 0$ then $\min_{\bar{\Omega}_T} u = \min_{\Gamma_T} u$

   where $\Omega_T = (0,T]\times\Omega$ and $\Gamma_T = \bar{\Omega}_T - \Omega_T$.

   (a) Let u be a smooth solution of

$$\begin{cases} \partial_t u - \Delta u + cu = 0 & \text{in } (0,\infty)\times\Omega, \\ u = 0 & \text{on } [0,\infty)\times\partial\Omega, \\ u = g & \text{on } \{t=0\}\times\Omega, \end{cases}$$

where $c(x) \geq \gamma > 0$. Prove, using the above maximum principle, that there exists a constant $C > 0$ such that $|u(t,x)| \leq C \exp(-\gamma t)$.

(b) Now assume that $|c(x)| \leq M$ and $g(x) \geq 0$. Show that the solution $u(t,x) \geq 0$.

[7 points]

6. This question is about the study of Black-Scholes equation:

$$\begin{cases} \partial_t V + \frac{1}{2}\sigma^2 s^2 \partial^2_{ss} V + rs\partial_s V - rV = 0 & \text{for } (t,s) \in (0,T) \times \mathbb{R}^+, \\ V(T,s) = f(s) = \max(s-K,0) & \text{for } s \geq 0, \\ V(t,0) = 0 & \text{for } t \in [0,T]. \end{cases}$$

where s = stock price, V = option value (dependent variable), σ = volatality of stock, r = risk-free interest rate and K is some constant (the terminology is not essential for this exercise!).

(a) Make the following change of variables:

$$s = K\exp(x); \qquad t = T - \frac{2\tau}{\sigma^2},$$

and letting $v(\tau,x) = V(t,s)$, write down the PDE satisfied by $v(\tau,x)$.

(b) Compute α, β such that

$$v(\tau,x) = \exp(\alpha x + \beta\tau)u(\tau,x)$$

where u solves the following heat equation:

$$\begin{cases} \partial_\tau u - \partial^2_{xx} u = 0 & \text{for } (\tau,x) \in (0, \frac{\sigma^2 T}{2}] \times \mathbb{R}, \\ u(0,x) = \exp(-\alpha x) f(K\exp(x)) & \text{for } x \in \mathbb{R}. \end{cases}$$

[4 points]

7. (a) Give the definition of a weak solution $u \in H^1_0(\Omega)$ of the Poisson equation $\Delta u = f$ for $f \in L^2(\Omega)$.

(b) State Riesz representation theorem and use it in proving the existence of a unique $u \in H^1_0(\Omega)$ that solves the above Poisson equation.

[2 points]

8. This question is about the Riesz-Fredholm theory for a compact perturbation of the Identity. The theory can be summarized as follows:

Suppose $\mathcal{H}$ a Hilbert space and let $T : \mathcal{H} \to \mathcal{H}$ be a continuous linear operator. Suppose

$$T = \mathrm{Id} + A \text{ where } A \text{ is compact.}$$

Then, T has the following properties:

(i) $\operatorname{Ker} T$ is finite dimensional.

(ii) $\operatorname{Im} T$ is closed and $\dim(\operatorname{Ker} T) = \operatorname{codim}(\operatorname{Im} T)$.

(iii) $T|_{(\operatorname{Ker} T)^\perp} : (\operatorname{Ker} T)^\perp \to \operatorname{Im} T$ is a bijection.

(a) What can you deduce from (ii) and (iii) above if $\operatorname{Ker} T = \{0\}$.

Consider the following Elliptic problem:

$$\begin{cases} c(x)u + b(x) \cdot \nabla u - \Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

where $f \in L^2(\Omega)$, $b \in C^1(\bar{\Omega}, \mathbb{R}^d)$, $c \in C(\Omega, \mathbb{R}^+)$.

(b) In the spirit of the Riesz-Fredholm perturbation argument, taking $H^1_0(\Omega)$ to be the Hilbert space, construct an operator T for the above problem such that

$$T = \mathrm{Id} + A$$

where A is compact. Present a detailed study of $\operatorname{Ker} A$ and arguments should be given to show that $A : H^1_0 \to H^1_0$ is a bijection. Thus, we arrive at a unique solution $u \in H^1_0$ for the above elliptic problem.

[8 points]

C: This question is about wave equation and nonlinear transport equation.

1. Consider
$$\begin{cases} \partial_{tt}u - \Delta u = 0 & \text{in } (0,\infty)\times\mathbb{R}^3, \\ u = g, \quad \partial_t u = h & \text{on } \{t=0\}\times\mathbb{R}^3, \end{cases}$$
where g, h are compactly supported smooth functions. Admit the following Kirchoff representation for the solution $u(t,x)$ of the above wave equation in 3 dimensions:
$$u(t,x) = \int_{\partial B(x,t)} \Big(t\,h(y) + g(y) + \nabla g(y)\cdot(y-x)\Big)\,\mathrm{d}\sigma(y),$$
where $B(x,t)$ is a ball centered at $x \in \mathbb{R}^3$ with radius $t > 0$. Show that there exists a constant $C > 0$ such that $|u(t,x)| \le C/t$.

   [4 points]

2. Using Energy estimates, prove that there exists at most one solution for the following equations:
$$\text{(I)}\begin{cases} \partial_{tt}u - \Delta u = f & \text{in } (0,T]\times\Omega, \\ u = g & \text{on } \{t=0\}\times\partial\Omega, \\ \partial_t u = h & \text{on } \{t=0\}\times\Omega. \end{cases}$$
$$\text{(II)}\begin{cases} \dfrac{\partial^2 u}{\partial t^2} + d\dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} = f & \text{in } (0,1)\times(0,T), \\ u = 0 & \text{on } (\{0\}\times(0,T))\cup(\{1\}\times(0,T)), \\ u = g, \quad \dfrac{\partial u}{\partial t} = h & \text{on } \{t=0\}\times(0,1). \end{cases}$$
where $d \in \mathbb{R}$ is a constant and f, g, h are smooth and have compact support.

   [6 points]

3. Apply the method of characteristics to solve the following semilinear equation:
$$\begin{cases} \partial_t u + a\partial_x u = u^2 \\ u(0,x) = \cos(x). \end{cases}$$
Does the solution thus constructed exhibit any finite time blow-up?

   [4 points]

4. Compute explicitly the unique entropy solution of the following Burgers equation:
$$\partial_t u + u\,\partial_x u = 0 \quad \text{in } \mathbb{R}^+\times\mathbb{R}$$
with initial data
$$u(0,x) = \begin{cases} 1 & \text{if } x < -1 \\ 0 & \text{if } -1 < x < 0 \\ 2 & \text{if } 0 < x < 1 \\ 0 & \text{if } x > 1 \end{cases}$$
(Hint: Consider the structure of the initial data, break the given problem into multiple Riemann problems. When $u_l > u_r$, we have a shock wave with speed given by Rankine-Hugoniot condition. When $u_l < u_r$, we have a Rarefaction wave. This solution is for short time and for x near the end points of the multiple Riemann problems.)

   [4 points]

5. Let $c(x)$ be a continuous locally bounded function and $f \in C^2(\mathbb{R})$. Consider the following conservation law:

$$\partial_t u + c(x)\, \partial_x f(u) = 0 \quad \text{in } (0, \infty) \times \mathbb{R}$$

with $u(0,x) = u^{in}(x)$.

(a) By considering the weak formulation of the above conservation law and by considering the curve of discontinuity to be a regular smooth curve in the $x - t$ plane, derive the Rankine-Hugoniot condition for the above equation.

(b) Setting $f(u) = u^2/2$ and $c(x) = 1 + x^2$ with initial data

$$u(0,x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0, \end{cases}$$

find the solution to the above problem.

[4 points]

6. Let $c(x)$ be a continuously differentiable function and $f \in C^2(\mathbb{R})$. Consider the following conservation law:

$$\partial_t u + \partial_x (c(x)\, f(u)) = 0 \quad \text{in } (0, \infty) \times \mathbb{R}$$

with $u(0,x) = u^{in}(x)$.

(a) Write down the characteristics associated with the above problem and the associated Kruzkov entropy condition.

(b) If u, v are two entropy solutions of the above problem with initial data u^{in}, v^{in} respectively, then show that

$$\|u(t,\cdot) - v(t,\cdot)\|_{L^1(\mathbb{R})} \leq \|u^{in} - v^{in}\|_{L^1(\mathbb{R})}$$

(Hint: Work with Kruzkov Entropy condition (seen in lectures) for both u, v and choose test functions in the entropy condition to be supported away from zero along with the use of mollifiers.)

[6 points]

7. Consider a scalar conservation law with flux, $f \in C^2(\mathbb{R})$:

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial f(u)}{\partial x} = 0 & \text{in } \mathbb{R}^+ \times \mathbb{R}, \\ u(0,x) = u^{in}(x). \end{cases}$$

Show that if supp $u^{in} \subset [-B, +B]$ for some $B \in \mathbb{R}$, then

$$\text{supp } u(t,\cdot) \subset [-B + t \min_{x \in \mathbb{R}} f'(u^{in}(x)), +B + t \min_{x \in \mathbb{R}} f'(u^{in}(x))].$$

(Hint: Derive energy estimates and proceed as in the proof of finite speed of propagation for wave equation. At some point during the proof, you will have to use Gronwall's estimate (seen in lectures).)

[8 points]

8. Let $u \in C^2(\mathbb{R} \times [0, \infty))$ solve the initial boundary value problem for the wave equation in one dimension:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial^2 u}{\partial x^2} = 0 & \text{in } (0, \infty) \times \mathbb{R}, \\ u = g, \quad \dfrac{\partial u}{\partial t} = h & \text{on } \{t = 0\} \times \mathbb{R}. \end{cases}$$

Suppose g, h have compact support. Denote

$$\begin{cases} \text{Kinetic Energy } k(t) = \dfrac{1}{2} \displaystyle\int_{\mathbb{R}} \left(\dfrac{\partial u}{\partial t}(t,x) \right)^2 \mathrm{d}x \\ \text{Potential Energy } p(t) = \dfrac{1}{2} \displaystyle\int_{\mathbb{R}} \left(\dfrac{\partial u}{\partial x}(t,x) \right)^2 \mathrm{d}x \end{cases}$$

- Prove that the total energy $E(t) = k(t) + p(t)$ is constant in time.
- Using D'Alembert's formula for the solution $u(t, x)$ of the wave equation in one dimension, prove that $k(t) = p(t)$ for $t \geq T$ with T large enough.

[4 points]