Analysis of Partial Differential Equations
Exercise sheet III (Chapter 3)

1. (Sobolev embedding) Let $\Omega \subset \mathbb{R}^d$ be a C^1 bounded domain. Prove that for all $s \in \mathbb{N}$ with $s > d/2$, there exists a constant $C > 0$ such that

$$\|u\|_{C^{s-d/2}(\Omega)} \leq C\|u\|_{H^s(\Omega)}. \tag{1}$$

Hint: *Construct an 'Extension Operator' $T : H^1(\Omega) \to H^1(\mathbb{R}^d)$ such that for all $u \in H^1(\Omega)$, we have: (i) $Tu|_\Omega = u$, (ii) $\|Tu\|_{L^2(\mathbb{R}^d)} \leq C\|u\|_{L^2(\Omega)}$, (iii) $\|Tu\|_{H^1(\mathbb{R}^d)} \leq C\|u\|_{H^1(\Omega)}$, where C depends only on Ω. In order to construct the above extension operator, use the local charts to straighten (rectify) the boundary and use the 'partition of unity'. Once we have the extension operator, apply the result proved in lectures for functions defined in $\mathbb{R}^d$ and reproduce all the arguments.*

2. In the lectures, we have seen that $W^{1,p}(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d)$ for $p > d$. Give an example of a function $f \in H^1(R^2)$ but $f \notin L^\infty(\mathbb{R}^2)$.

3. (Poincaré-Wirtinger) Let $\Omega \in \mathbb{R}^d$ be an open, bounded domain. Show that there exists a constant $C(\Omega)$, depending on the domain, such that

$$\forall\, u \in H^1(\Omega), \qquad \int_\Omega |u - \bar{u}|^2 \,\mathrm{d}x \leq C(\Omega) \int_\Omega |\nabla u|^2 \,\mathrm{d}x, \tag{2}$$

where $\bar{u} = \dfrac{1}{|\Omega|} \displaystyle\int_\Omega u(x) \,\mathrm{d}x$.

4. (Hardy) Let $\Omega \subset \mathbb{R}^d$ be a C^1 bounded domain. Define the distance function $d : \Omega \to \mathbb{R}_+$ by $d(x) := \mathrm{dist}(x, \partial\Omega)$. Show that there exists a constant $C > 0$ such that

$$\left\|\frac{u}{d}\right\|_{L^2(\Omega)} \leq C\|\nabla u\|_{L^2(\Omega)} \text{ for all } u \in H_0^1(\Omega). \tag{3}$$

5. (Rellich) Let $\Omega \in \mathbb{R}^d$ be an open and bounded domain. Prove that any sequence uniformly bounded in $H^1(\Omega)$ is relatively compact in $L^2(\Omega)$ i.e., if $\{u_n\} \subset H^1(\Omega)$ is a sequence such that $\|u_n\|_{H^1(\Omega)} \leq C$ for some constant C independant of n, then there exists a subsequence $\{u_{\varphi(n)}\}$ (with $\varphi : \mathbb{N} \to \mathbb{N}$ strictly increasing) and a limit function $u \in L^2(\Omega)$ such that

$$\left\|u_{\varphi(n)} - u\right\|_{L^2(\Omega)} \to 0 \qquad \text{as } \; n \to \infty.$$

6. (Riesz-Fréchet-Kolmogorov) Let $\Omega \in \mathbb{R}^d$ be open.
• First consider $\omega \subset\subset \Omega$ i.e., ω open with $\bar{\omega} \subset \Omega$. Consider a bounded $\mathcal{G} \subset L^2(\Omega)$. Suppose

$$\begin{aligned} &\forall\, \varepsilon > 0,\ \exists\, \delta > 0,\ \delta < \mathrm{dist}(\omega, \partial\Omega) \text{ such that} \\ &\forall\, h \in \mathbb{R}^d,\ |h| < \delta \ \text{ and } \forall\, u \in \mathcal{G}, \qquad \int_\omega |u(x+h) - u(x)|^2 \,\mathrm{d}x \leq \varepsilon. \end{aligned} \tag{4}$$

Prove that $\mathcal{G}|_\omega$ is relatively compact in $L^2(\omega)$. (Notation: $\mathcal{G}|_\omega$ denotes the elements of $\mathcal{G}$ restricted to ω).

• Second, assume that $\Omega = \mathbb{R}^d$ and consider a bounded $\mathcal{G} \subset L^2(\mathbb{R}^d)$. Suppose

$$\forall\, \varepsilon > 0,\ \exists\, \delta > 0 \text{ such that } \forall\, h \in \mathbb{R}^d,\ |h| < \delta \ \text{ and } \forall\, u \in \mathcal{G},\quad \int_{\mathbb{R}^d} |u(x+h) - u(x)|^2 \,\mathrm{d}x \le \varepsilon. \tag{5}$$

In addition to (5), suppose also that

$$\forall\, \varepsilon > 0,\ \exists\, \omega \subset\subset \mathbb{R}^d \text{ such that } \|u\|_{L^2(\mathbb{R}^d \setminus \omega)} < \varepsilon \ \forall\, u \in \mathcal{G}. \tag{6}$$

Then, deduce that $\mathcal{G}$ is relatively compact in $L^2(\mathbb{R}^d)$.

7. (Homogenization) Consider the following one dimensional Dirichlet problem:

$$\begin{cases} -\dfrac{\partial}{\partial x}\Big(a\left(\dfrac{x}{\varepsilon}\right)\dfrac{\partial u_\varepsilon}{\partial x}\Big) = f \text{ in } (0,1), \\ u_\varepsilon(0) = u_\varepsilon(1) = 0, \end{cases} \tag{7}$$

where a is a smooth, positive function which is 1-periodic and $f \in L^2(0,1)$.
(a) For any fixed $\varepsilon > 0$, prove existence and uniqueness of $u_\varepsilon \in H_0^1(0,1)$ for the Dirichlet problem (7).
(b) Consider a sequence of solutions $\{u_\varepsilon\}$ associated with (7) for $\varepsilon \to 0$. Show that $u_\varepsilon \rightharpoonup u_0$ in $H_0^1(0,1)$ as $\varepsilon \to 0$. Show also that the limit function u_0 satisfies:

$$\begin{cases} -\dfrac{\partial}{\partial x}\Big(\bar{a}\dfrac{\partial u_0}{\partial x}\Big) = f \text{ in } (0,1), \\ u_0(0) = u_0(1) = 0, \end{cases} \tag{8}$$

for some constant $\bar{a}$.
(c) How is $\bar{a}$ represented in terms of the periodic function a?
Hint: *Use Rellich Theorem and Riemann-Lebesgue Lemma.*

8. (Fredholm Alternative) Let $u \in H^1(\Omega)$ be a weak solution of the following Neumann problem:

$$\begin{cases} b(x) \cdot \nabla u - \nabla \cdot (A(x)\nabla u) = f & \text{in } \Omega, \\ -A(x)\nabla u \cdot n = g & \text{on } \partial\Omega. \end{cases} \tag{9}$$

where $f \in L^2(\Omega)$, $g \in H^1(\Omega)$. The coefficient $A(x)$ be a symmetric matrix such that there exist $\alpha_0 > 0$, $\alpha_1 > 0$ with

$$\alpha_0|\xi|^2 \le A_{ij}(x)\xi_i\xi_j \le \alpha_1|\xi|^2 \qquad \text{for a.e. } x \in \Omega,\ \forall\, \xi \in \mathbb{R}^d. \tag{10}$$

and $b(x) \in L^\infty(\Omega)$ satisfies $\nabla \cdot b = 0$ in Ω and $b \cdot n = 0$ on $\partial\Omega$. Prove that (9) has a unique weak solution modulo an additive constant if and only if the source terms satisfy the following compatibility condition:

$$\int_\Omega f(x)\,\mathrm{d}x = \int_{\partial\Omega} g(x)\,\mathrm{d}\sigma(x), \tag{11}$$

where $\mathrm{d}\sigma(x)$ is the surface measure on $\partial\Omega$.
Hint: *Employ Poincaré-Wirtinger inequality and use Lax-Milgram Lemma in the quotient space $H^1(\Omega)/\mathbb{R}$.*

9. (Mean Value Theorem) Let $\Omega \subset \mathbb{R}^d$ be open. A function $u \in C^2(\Omega)$ is said to be *harmonic* if $\Delta u = 0$ in Ω. Suppose u is harmonic in Ω. Let $x_0 \in \Omega$ and $r > 0$ so that the closed ball $\bar{B}(x_0, r) \subset \Omega$. Show that:

$$u(x_0) = \frac{1}{r^{d-1}\omega_d} \int_{S(x_0,r)} u(y)\mathrm{d}\sigma(y), \tag{12}$$

where ω_d is the surface area of the unit sphere in $\mathbb{R}^d$ and $S(x_0, r)$ is the sphere of radius r centered at x_0.

10. (Liouville Theorem) Prove that every bounded harmonic function on the whole space $\mathbb{R}^d$ should be a constant. Deduce that any harmonic function $v(x)$ on the whole space $\mathbb{R}^d$ which satisfies $|v(x)| \to 0$ as $x \to \infty$ should be zero.

11. (Dirichlet Principle) Let $\Omega \in \mathbb{R}^d$ be an open, bounded domain. For a source term $f \in L^2(\Omega)$, show that solving for $u \in H_0^1(\Omega)$ satisfying

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{13}$$

is the same as solving for $u \in H_0^1(\Omega)$, the following minimization problem:

$$F(u) = \inf_{v \in H_0^1(\Omega)} F(v), \tag{14}$$

where

$$F(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 \, \mathrm{d}x - \int_\Omega f v \, \mathrm{d}x.$$

12. (Helmholtz decomposition) Let $\Omega \subset \mathbb{R}^d$ be an open and bounded domain. Suppose $b(x) \in (L^2(\Omega))^d$ is a vector field in Ω. Show that there exists $u \in H_0^1(\Omega)$ and $v \in (L^2(\Omega))^d$ such that

$$b(x) = \nabla u(x) + v(x)$$

with

$$\nabla \cdot v = 0 \quad \text{in } \Omega \quad \text{and} \quad \int_\Omega \nabla u \cdot v \, \mathrm{d}x = 0.$$

13. (Cacciopoli) Suppose $\Omega \subset \mathbb{R}^d$ be open. Let $x_0 \in \Omega$ and $0 < \rho < \bar{\rho}$ such that the ball $B(x_0, \bar{\rho}) \subset \Omega$. Suppose $u \in H^1(\Omega)$ satisfies

$$-\Delta u + \mathbf{b} \cdot \nabla u + a\, u = 0 \text{ in } \Omega, \tag{15}$$

where $a \in \mathbb{R}$ and $\mathbf{b} \in \mathbb{R}^d$. Show that there exists a constant C such that

$$\int_{B(x_0,\rho)} |\nabla u|^2 \, \mathrm{d}x \leq \frac{C}{(\bar{\rho} - \rho)^2} \int_{B(x_0,\bar{\rho})} |u|^2 \, \mathrm{d}x. \tag{16}$$

Take $a = 0$ and $\mathbf{b} = 0$ in (15). Deduce from (16) that

$$\forall\, k \in \mathbb{N}, \quad \|u\|^2_{H^k(B(x_0,\rho))} \leq C(\rho, \bar{\rho}, k) \|u\|^2_{L^2(B(x_0,\bar{\rho}))} \tag{17}$$

and

$$\forall\, k \in \mathbb{N}, \quad \|u\|^2_{C^k(B(x_0,\rho))} \leq C(\rho, \bar{\rho}, k) \|u\|^2_{L^2(B(x_0,\bar{\rho}))}. \tag{18}$$

What can we infer from (18)?
Hint: *Use 'Cut-off functions' as was done in lectures for the interior regularity results.*

14. (Maximum Principle - Divergence Form) Let $A(x)$ be a symmetric matrix (i.e., $A_{ij} = A_{ji}$) such that there exist $\alpha_0 > 0$, $\alpha_1 > 0$ with

$$\alpha_0 |\xi|^2 \leq A_{ij}(x)\xi_i\xi_j \leq \alpha_1 |\xi|^2 \qquad \text{for a.e. } x \in \Omega,\ \forall\, \xi \in \mathbb{R}^d. \tag{19}$$

Also, let $c(x) \in L^\infty(\Omega)$ and $c(x) \geq \lambda > 0$. Suppose $u \in H^1(\Omega)$ verifies in the weak sense:

$$-\nabla \cdot (A(x)\nabla u) + c\, u \geq 0 \text{ on } \Omega, \tag{20}$$

which means in an explicit way that we have:

$$\int_\Omega A(x)\nabla u \cdot \nabla \phi \, \mathrm{d}x + \int_\Omega c\, u \phi \, \mathrm{d}x \geq 0, \quad \forall\, \phi \in C_0^\infty(\Omega) \text{ with } \phi \geq 0 \text{ in } \Omega. \tag{21}$$

Show that

$$\inf_{x \in \Omega} u(x) = \inf_{x \in \partial\Omega} u(x)$$

Hint: *Use the density of* $C_0^\infty(\Omega)$ *in* $H_0^1(\Omega)$. *Take* $-(u - \inf_{x\in\partial\Omega} u(x))^-$ *as test function. We have used the following notation:* $h^- := \min(0, -h)$.

15. (Parabolic Equations) Let $\Omega \subset \mathbb{R}^d$ be open. We consider the initial-boundary value problem (IBVP):

$$\begin{cases} \dfrac{\partial u}{\partial t} + Pu = f & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times [0,T], \\ u = g & \text{on } \Omega \times \{t = 0\}, \end{cases} \tag{22}$$

where P is a second order partial differential operator in divergence form:

$$Pu := -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\Big(a_{ij}(t,x)\frac{\partial u}{\partial x_j}\Big) + \sum_{i=1}^{d} b_i(t,x)\frac{\partial u}{\partial x_i} + c(t,x)u.$$

Define a time dependent bilinear form for $u, v \in H_0^1(\Omega)$ and for a.e. $t \in [0,T]$:

$$B[u,v;t] := \sum_{i,j=1}^{d} \int_\Omega a_{ij}(t,x)\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i}\mathrm{d}x + \sum_{i=1}^{d}\int_\Omega b_i(t,x)\frac{\partial u}{\partial x_i}v\mathrm{d}x + \int_\Omega c(t,x)uv\mathrm{d}x. \tag{23}$$

We give the following definition for a weak solution of the IBVP (22): A function $u \in L^2((0,T); H_0^1(\Omega))$ with $u' \in L^2((0,T); H^{-1}(\Omega))$ is a *weak solution* to (22) if

$$\begin{aligned} &\text{(i) } \langle u', v\rangle + B[u,v;t] = (f,v) \text{ for each } v \in H_0^1(\Omega) \text{ and for a.e. } t \in [0,T] \\ &\text{and (ii) } u(0) = g, \end{aligned} \tag{24}$$

where $\langle\cdot,\cdot\rangle$ is the dual product between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$ whereas $(\cdot,\cdot)$ is the standard L^2 inner product. The idea is to construct approximate solutions to (22) by considering an orthonormal basis $\{\varphi_k\}_{k=1}^\infty$ of $L^2(\Omega)$. Define approximations for $n \in \mathbb{N}$:

$$u_n(t) := \sum_{k=1}^{n} d_n^k(t)\varphi_k, \tag{25}$$

where the coefficient functions $d_n^k : [0,T] \to \mathbb{R}$ for $k = 1, \cdots, n$ are chosen such that:

$$d_n^k(0) = (g, \varphi_k) \text{ for } k = 1, \cdots, n \qquad \text{and} \qquad (u_n', \varphi_k) + B[u_n, \varphi_k; t] = (f, \varphi_k). \tag{26}$$

(**15.a**) Using an existence result from the theory of ODEs, show that for each $n \in \mathbb{N}$ there exists a unique function $u_n(t)$ of the form (25) satisfying (26).
Hint: *Use the Cauchy-Lipschitz Theorem.*

The next task is to consider the finite dimensional approximations u_n for $n \in \mathbb{N}$ and pass to the limit as $n \to \infty$. In order to apply some compactness results, we need to derive uniform (in n) estimates on $\{u_n\}$. The next question addresses this aspect.

(**15.b**) There exists a constant C depending only on $\Omega, T, a_{ij}, b_i, c$ such that

$$\|u_n\|_{L^\infty([0,T];L^2(\Omega))} + \|u_n\|_{L^2((0,T);H_0^1(\Omega))} + \|u_n'\|_{L^2((0,T);H^{-1}(\Omega))} \le C\Big(\|f\|_{L^2((0,T);L^2(\Omega))} + \|g\|_{L^2(\Omega)}\Big). \tag{27}$$

(**15.c**) Using the apriori estimates (27) and compactness results, arrive at the associated limit function with the approximate solutions (25). Show that the limit function is indeed a weak solution of (22) in the sense of (24) by passing to the limit in (26).
Hint: *You may assume the following: If* $v_n \rightharpoonup v$ *in* $L^2((0,T); H_0^1(\Omega))$ *and* $v_n' \rightharpoonup w$ *in* $L^2((0,T); H^{-1}(\Omega))$, *then* $w = v'$.

(**15.d**) Show that a weak solution of (22) is unique.
Hint: *Assume that* $f = g = 0$ *in (22) and show that the solution* $u = 0$.