3. Symmetric Markov Diffusion Operators

3.1. Markov Triples

3.1.1. Initial Structure

Definition) A Markov triple (E, μ, Γ), relative to an algebra $\mathcal{A}_0$.
Usually, it is reasonable to assume that $\mathcal{A}_0$ is stable under composition with a smooth function vanishing at 0.

3.1.2. Diffusion Property

Definition) *(Diffusion property)* A Markvo triple (E, μ, Γ) has diffusion property if

$$\Gamma(\Psi(f_1, \cdots, f_k), g) = \sum_{i=1}^{k} \partial_i \Psi(f_1, \cdots, f_k)\Gamma(f_k, g)$$

3.1.3. Diffusion Operators

Given triple (E, μ, Γ), L is the diffusion operator if

$$\int_E gLf d\mu = -\int_E \Gamma(f, g) d\mu$$

Then, we have $\int_E Lf\mu = 0$, i.e. the invariance of measure μ. Moreover, for any smooth function $\Psi : \mathbb{R}^k \to \mathbb{R}$ with $\Psi(0) = 0$, has

$$L(\Psi(f_1, \cdots, f_k)) = \sum_{i=1}^{k} \partial_i \Psi(f_1, \cdots, f_k)Lf_i + \sum_{i,j=1}^{k} \partial_{ij} \Psi(f_1, \cdots, f_k)\Gamma(f_i, f_j)$$

3.1.4. Dirichlet Form and Domains

The next step is to understand to which natrual domain the diffusion operator L is extended. First we will construct $D(\mathcal{E})$ and next construct $D(L)$.
Recall, we defined

$$\mathcal{E}(f, g) = \int_E \Gamma(f, g) d\mu = -\int_E fLgd\mu, \quad f, g \in \mathcal{A}_0$$

Then we have Cauchy-Schwarz inquality, $|\mathcal{E}(f, g)| \leq \mathcal{E}(f)^{1/2}\mathcal{E}(g)^{1/2}$.