Advanced Probability

0. Review

0.1. Measure Spaces

Let E be a set. σ-algebra on E, measurable sets, measure(countable additivity), measure space, Borel σ-algebra.

0.2. Integration of Measurable Functions

measurable function, $m\mathcal{E}^+$, simple function
Theorem 0.2.1.) (definition/characterization of integral for $m\mathcal{E}^+$-functions) Let $(E, \mathcal{E}, \mu)$ be a measure space. Then there exists a *unique* map $\tilde{\mu} : m\mathcal{E}^+ \to [0, \infty]$ such that (a),(b),(c)(to be written out) (proof only done for uniqueness of such map)

equal a.e., integrable function.

Lemma 0.2.2.) (Fatou's lemma) Let $(f_n : n \in \mathbb{N})$ be a sequence in $m\mathcal{E}^+$. Then

$$\mu(\liminf_{n\to\infty} f_n) \le \liminf_{n\to\infty} \mu(f_n)$$

Theorem 0.2.3) (Dominated Convergence) Let $(f_n : n \in \mathbb{N})$ be a sequence of measurable functions on $(E, \mathcal{E})$. Suppose $f_n(x) \to f(x)$ as $n \to \infty$ for some function f and for every $x \in E$. Suppose furthermore that $|f_n| \le g$ for all n, for some measurable function g. Then f_n is integrable for all n and so is f. Moreover, we have $\mu(f_n) \to \mu(f)$ as $n \to \infty$.

probability space

0.3. Production measure and Fubini's theorem

product σ-algebra

Theorem 0.3.1.) There exists a unique measure $\mu = \mu_1 \otimes \mu_2$ on $(E_1 \times E_2, \mathcal{E}_1 \times \mathcal{E}_2)$ such that

$$\mu(A_1 \times A_2) = \mu_1(A_1)\mu_2(A_2) \quad \forall A_1 \in \mathcal{E}_1, A_2 \in \mathcal{E}_2$$

Theorem 0.3.2.) (Fubini's theorem) Let f be a non-negative measurable function on $(E_1 \times E_2, \mathcal{E}_1 \times \mathcal{E}_2)$.

- $x_1 \in E_1$, set $f_{x_1}(x_2) = f(x_1, x_2)$, then f_{x_1} is $\mathcal{E}$-measurable for all $x_1 \in E_1$.
- Set $f_1(x_1) = \mu_2(f_{x_1})$. Then f_1 is $\mathcal{E}_1$-measurable, and $\mu_1(f_1) = \mu(f)$.

This shows

$$\int_{E_2} \Big(\int_{E_1} f(x_1, x_2)\mu_1(dx_1)\Big)\mu_2(dx_2) = \int_{E_1} \Big(\int_{E_2} f(x_1, x_2)\mu_2(dx_2)\Big)\mu_1(dx_1)$$

(How?)

Note, f being integrable is also sufficient to for Fubini's theorem.

Chapter 1. Conditional Expectation

Discrete case, Gaussiaan case of conditional expectations. Conditional density functions.

1.4. Existence and uniqueness of conditional expectation

$(\Omega, \mathcal{F}, \mathbb{P})$ is always the probability space behind.

Theorem 1.4.1.) Let X be integrable random variable and let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$. There exist a random variable Y such that

(a) Y is $\mathcal{F}$-measurable,

(b) Y is integrable and for all $A \in \mathcal{F}$, $\mathbb{E}(Y1_A) = \mathbb{E}(X1_A)$.

Moreover, if Y' is another random variable satisfying (a) and (b) then $Y' = Y$ a.s.

We will write $Y = \mathbb{E}(X|\mathcal{F})$ a.s.(because $\mathbb{E}(X|\mathcal{F})$ is not a fully specified random variable) and we say Y is a **(version of) the conditional expectation of** X **given** $\mathcal{F}$. In case $X = 1_A$ write $Y = \mathbb{P}(A|\mathcal{F})$ a.s.

An analogous statement holds with 'integrable' replaced by 'non-negative' throughout the statement.

1.5. Properties of conditional expectation

Fix an integrable random variable X and sub-σ-algebra $\mathcal{F}$. **Theorem 1.4.1** has useful consequences : (give proofs for non-obvious points)

(i) $\mathbb{E}(\mathbb{E}(X|\mathcal{F})) = \mathbb{E}(X)$

(ii) If X is $\mathcal{F}$-measurable then $\mathbb{E}(X|\mathcal{F}) = X$ a.s.

(iii) If X is independent of $\mathcal{F}$ then $\mathbb{E}(X|\mathcal{F}) = \mathbb{E}(X)$ a.s.

(iv) If $X \geq 0$ a.s. then $\mathbb{E}(X|\mathcal{F}) \geq 0$ a.s. (follows from the proof of **Theorem 1.4.1**)

(v) For all $\alpha, \beta \in \mathbb{R}$, all integrable random variables X, Y, has

$$\mathbb{E}(\alpha X + \beta Y|\mathcal{F}) = \alpha\mathbb{E}(X|\mathcal{F}) + \beta\mathbb{E}(Y|\mathcal{F}) \quad \text{a.s.}$$

Now consider a sequence of random variables $(X_n)_n$.

(vi) *(conditional monotone convergence)* If $0 \leq X_n \nearrow X$ pointwise then $\mathbb{E}(X_n|\mathcal{F}) \to \mathbb{E}(X|\mathcal{F})$ a.s.

(vii) *(Conditional Fatou's lemma)* For any non-negative random variables X_n,

$$\mathbb{E}(\liminf_{n\to\infty} X_n|\mathcal{F}) \leq \liminf_{n\to\infty} \mathbb{E}(X_n|\mathcal{F})$$

(viii) *(Conditional dominated convergence)* If $X_n(\omega) \to X(\omega)$ for all ω as $n \to \infty$ and there exist Y integrable so $|X_n| \leq Y$ for all n, then

$$\mathbb{E}(X_n|\mathcal{F}) \to \mathbb{E}(X|\mathcal{F}) \quad \text{a.s.}$$

(ix) *(Conditional Jensen)* Let $c : \mathbb{R} \to (-\infty, \infty]$ be convex. Then $c(\mathbb{E}(X|\mathcal{F})) \leq \mathbb{E}(c(X)|\mathcal{F})$ a.s.

(x) $\|\mathbb{E}(X|\mathcal{F})\|_p \leq \|X\|_p$ for all $p \in [0, \infty)$ (where $\|X\|_p = \mathbb{E}(|X|^p)$)

(xi) *(Tower property)* This is also important for martingales.

Suppose $\mathcal{H} \subset \mathcal{F} \subset \mathcal{F}$ be sub-σ-algebras. Then

$$\mathbb{E}(X|\mathcal{H}) = \mathbb{E}(\mathbb{E}(X|\mathcal{F})|\mathcal{H}) \quad \text{a.s.}$$

(xii) *(Taking out what is known)* This is related to the 'filtration' that is to be introduced in the theory of martingales.

Let Y be a bounded $\mathcal{F}$-measurable random variable. Then

$$\mathbb{E}(YX|\mathcal{F}) = Y\mathbb{E}(X|\mathcal{F}) \quad \text{a.s.}$$

(xiii) Let $\mathcal{H}$ be a σ-algebra and suppose $\mathcal{H}$ is independent of $\sigma(X, \mathcal{F})$. Then

$$\mathbb{E}(X|\sigma(\mathcal{F}, \mathcal{H})) = \mathbb{E}(X|\mathcal{H}) \quad \text{a.s.}$$

Lemma) Suppose X is integrable and $\mathbb{E}(X1_A) = 0$ for all $A \in \mathcal{A}$ where $\mathcal{A}$ is a π-system generating $\mathcal{F}$. Then $X = 0$ a.s.

We end the chapter with a lemma that is going to be useful.

Lemma 1.5.1.) Let $X \in L^1(\mathbb{P})$. Set $\mathscr{Y} = \{\mathbb{E}(X|\mathcal{F}) : \mathcal{F} \subset \mathcal{F} \text{ a sub-}\sigma\text{-algebra } \}$. Then $\mathscr{Y}$ is UI(*uniformly integrable*). That is,

$$\sup_{Y \in \mathscr{Y}} \mathbb{E}(|Y|1_{|Y| \geq \lambda}) \to 0 \quad \text{as } \lambda \to \infty$$

Chapter 2. Martingales in Discrete Time

2.1. Definitions.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.
Filtration. Random process, adapted to a filtration, natural filtration, integrable random process, martingale, super/sub-martingale.

Optional Stopping

stopping time, the algebra $\mathcal{F}_T$ for a stopping time T, stopped process.

Proposition 2.2.1.) Let X be an adapted process. Let S, T be stopping times for X. Then

(a) $S \wedge T$ is a stopping time for X.

(b) $\mathcal{F}_T$ is a σ-algebra.

(c) If $S \leq T$ then $\mathcal{F}_S \subset \mathcal{F}_T$.

(d) $X_T 1_{T<\infty}$ is an $\mathcal{F}_T$-measurable random variable.

(e) X^T is adapted.

(f) If X is integrable, then X^T is also integrable.

Theorem 2.2.2) *(Optional stopping theorem)* Let X be a super-martingale and let S, T be *bounded* stopping times with $S \leq T$ a.s. Then

$$\mathbb{E}[X_T] \leq \mathbb{E}[X_S]$$

$\star$ Note that X is a sub-martingale *if and only if* $(-X)$ is a super-martingale, and that X is a martingale *if and only if* X and $(-X)$ are super-martingales. Hence, we obtain sub-martingale and martingale versions of the theorem :

If (X_n) is a sub-martingale, $\mathbb{E}[X_T] \geq \mathbb{E}[X_S]$
If (X_n) is a martingale, $\mathbb{E}[X_T] = \mathbb{E}[X_S]$

Theorem 2.2.3.) *(OST, Part II)* Let X be an adapted integrable process. Then the followings are equivalent.

(a) X is a super-martingale.

(b) for all bounded stopping times T and stopping time S,

$$\mathbb{E}(X_T|\mathcal{F}_S) \leq X_{S \wedge T} \quad \text{a.s.},$$

(c) for all stopping times T, the stopped process X^T is a super-martingale,

(d) for all bounded stopping times T and all stopping times S with $S \leq T$ a.s,

$$\mathbb{E}(X_T) \leq \mathbb{E}(X_S)$$

$\star$ The theorem gives an inverse statement of the optional stopping theorem.

2.3. Doob's upcrossing inequality

upcrossing, upcrossing number
Theorem 2.3.1.) (Doob's upcrossing inequality) State and prove

This theorem does not seem to have any significance at the moment, but it will turn out to be important later on.

2.4. Doob's maximal inequalities.

Define $X_n^* = \sup_{k \geq n} |X_k|$

In the next two theorems, we see that the martingale(or sub-martingale) property allows us to obtain estimates on this X_n^* in terms of expectations for X_n.

Theorem 2.4.1) (Doob's maximal inequality) state and prove

Theorem 2.4.2) (Doob's L^p-inequality) State and prove

Doob's maximal and L^p inequalities have different versions which apply under the same hypothesis to

$$X^* = \sup_{n \geq 0} |X_n|$$

(state and prove)

2.5. Doob's martingale convergence theorems

We are going to study three different martingale convergence theorems. They are all important.

- We say that a random process X is L^p**-bounded** if $\sup_{n \geq 0} \left\| X_n \right\|_p < \infty$.
- We say that X is **uniformly integrable** if

$$\sup_{n \geq 0} \mathbb{E}(|X_n| 1_{|X_n| > \lambda}) \to 0 \quad \text{as } \lambda \to \infty$$

- If X is L^p bounded for some $p > 1$, then this implies that X is uniformly integrable. This again implies that X is L^1 bounded. (prove)

X is L^p-bounded, $p > 1$ $\Downarrow$ X is Uniformly integrable $\Downarrow$ X is L^1-bounded

Theorem 2.5.1) *(Almost sure martingale convergence theorem)* State and prove

Remark : Every non-negative integrable super-martingale is L^1-bounded, hence it converges a.s.

Theorem 2.5.2) *(L^1 martingale convergence theorem)* State and prove
We can think of this theorem as establishing the bijection

$$\text{unif. integrable martingale/a.s.} \quad \leftrightarrow \quad L^1(\mathcal{F}_\infty)$$

Theorem 2.5.3) *(L^p-martingale convergence theorem)* State and prove
(This is very similar to the statement of L^1-martingale convergence theorem. Indeed, the proof is also very similar.)

Theorem 2.5.5.) Let X be a uniformly integrable martingale and let T be any stopping time. Then $\mathbb{E}(X_T) = \mathbb{E}(X_0)$. Moreover, for all stopping time S and T, we have

$$\mathbb{E}(X_T|\mathcal{F}_S) = X_{S\wedge T} \quad \text{a.s.}$$

-State the significance of this theorem

(Is there a way to extend Theorem 2.2.3. in a similar way?)

Backward martingale

- A **backward filtration** $(\hat{\mathcal{F}}_n)_{n\geq 0}$ is a sequence of σ-algebras such that $\mathcal{F} \supset \hat{\mathcal{F}}_n \supset \hat{\mathcal{F}}_{n+1}$.
- This also defines $\hat{\mathcal{F}}_\infty = \bigcap_{n\geq 0} \hat{\mathcal{F}}_n$

Theorem 2.5.4.) *(Backward martingale convergence theorem)* For all $Y \in L^1(\mathcal{F})$, we have

$$\mathbb{E}(Y|\hat{\mathcal{F}}_n) \to \mathbb{E}(Y|\hat{\mathcal{F}}_\infty) \quad \text{a.s. and in } L^1 \quad \text{as } n\to\infty$$

-state why we do not need uniform integrability.

3. Applications of martingale theory

Sums of independent random variables

Let $S_n = X_1 + \cdots + X_n$, where $(X_n)_{n\geq 0}$ is a sequence of independent random variables.

Theorem 3.1.1) *(Strong Law of Large Numbers)* Let $(X_n)_{n\geq 0}$ be a sequence of independent identically distributed ($i.i.d$) integrable random variables. Set $\mu = \mathbb{E}(X_1)$. Then

$$S_n/n \to \mu \quad \text{a.s. and in } L^1$$

Corollary 3.1.2) *(Weak law of large numbers)* Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. integrable r.v.. Set $\mu = \mathbb{E}(X_1)$. Then

$$\mathbb{P}(|\frac{S_n}{n} - \mu| > \epsilon) \to 0 \quad \text{as } n\to\infty \quad \forall \epsilon > 0$$

3.2. Non-negative martingale and change of measure

Probability measure $\tilde{P}$ for a random variable with $\mathbb{E}(X) = 1$,and $\tilde{P}_n$ for $(X_n)_{n\geq 0}$.

Proposition 3.2.1.) The measures $\tilde{\mathbb{P}}_n$ are consistent. That is

$$\tilde{\mathbb{P}}_{n+1}|\mathcal{F}_n = \tilde{\mathbb{P}}_n \quad \forall n \quad \textit{iff} \quad (X_n)_{n\geq 0} \quad \text{is a martingale}$$

Moreover, there is a measure $\tilde{\mathbb{P}}$ on $\mathcal{F}$, which has a density w.r.t $\mathbb{P}$ such that

$$\tilde{\mathbb{P}}|\mathcal{F}_n = \tilde{\mathbb{P}}_n \quad \forall n \quad \textit{iff} \quad (X_n)_n \quad \text{is a uniformly integrable martingale}$$

Theorem 3.2.3) *(Radon-Nikodym theorem)* Let μ and ν be σ-finite measures on a measurable space $(E, \mathcal{E})$. Then the followings are equivalent :

(a) $\nu(A) = 0$ for all $A \in \mathcal{E}$ such that $\mu(A) = 0$, i.e. ν is **absolutely continuous** with respect to μ.

(b) There exists a measurable function f on E such that $f \geq 0$ and $\nu(A) = \mu(f 1_A)$ for all $A \in \mathcal{E}$.

The function f which is unique up to modification μ-a.e. (proof given for case where $\mathcal{E}$ *is countably generated*, so assume there is a sequence $(G_n : n \in \mathbb{N})$ of subsets of E which generates $\mathcal{E}$)

Radon-Nikodym derivative

3.3. Markov Chains

E countable set, μ probability measure on E, μ_x, $\mu(f) = \sum_x \mu_x f_x$, transition matrix, Markov chain with transition matrix P

Proposition 3.3.1) Let $(X_n)_{n\geq 0}$ be a random process in E and take $\mathcal{F}_n = \sigma(X_k : k \geq n)$. Then the following are equivalent :

(a) $(X_n)_{n\geq 0}$ is a Markov chain with initial distribution μ and transition matrix P.

(b) For all n and all $x_0, x_1, \cdots, x_n \in E$,

$$\mathbb{P}(X_0 = x_0, X_1 = x_1, \cdots, X_n = x_n) = \mu_{x_0} p_{x_0 x_1} \cdots p_{x_{n-1} x_n}$$

Proposition 3.3.2) Let E^* denote the set of sequence $x = (x_n : n \geq 0)$ taking values in E and define $X_n : E^* \to E$ by $X_n(x) = x_n$. Set $\mathcal{E} = \sigma(X_k : k \geq 0)$. Let P be a transition matrix on E. Then, for each $y \in E$, there is a unique probability measure $\mathbb{P}_y$ on $(E^*, \mathcal{E}^*)$ such that $(X_n)_{n\geq 0}$ is a Markov chain with transition matrix P and starting from y.

An example of a Markov chain in $\mathbb{Z}^d$ is the simple symmetric random walk with transition matrix

$$p_{xy} = \begin{cases} 1/2d & \text{if } |x-y| = 1 \\ 0 & \text{otherwise} \end{cases}$$

The following result shows a simple instance of a general relationship between Markov processes and martingale.

Proposition 3.3.3) Let $(X_n)_{n\geq 0}$ be an adapted process in E. TFAE :

(a) $(X_n)_{n\geq 0}$ is a Markov chain with transition matrix P.

(b) For all bounded functions f on E, the following process is a *martingale*

$$M_n^f = f(X_n) - f(X_0) - \sum_{k=0}^{n-1} (P - I) f(X_k)$$

(bounded) harmonic function

Theorem 3.3.4) The function u is bounded, harmonic in D, and $u = f$ on ∂D. Moreover, if $\mathbb{P}_x(T < \infty) = 1$ for all $x \in D$, then u is the unique bounded extension of f which is harmonic in D. (state what T and u should be)

4. Random processes in continuous time

4.1. Definitions

continuous random process($C(\mathbb{R}_{\geq 0}, \mathbb{R})$), cadlag function, cadlag random process($D(\mathbb{R}_{\geq 0}, \mathbb{R})$).
σ-algebra on $C(\mathbb{R}_{\geq 0}, \mathbb{R})$, $D(\mathbb{R}_{\geq 0}, \mathbb{R})$.
Finite dimensional distribution.

Kolmogorov's Criterion

Theorem 4.2.1) *(Kolmogorov's criterion)* state and prove

The interpretation?

We have defined and used $\mathbb{D}_n = \{k2^{-n} : k \in \mathbb{Z}^+\}$, $\mathbb{D} = \cup_{n\geq 0}\mathbb{D}_n$, $D_n = [0,1] \cap \mathbb{D}$, $D = \cup_n D_n = \mathbb{D} \cap [0,1]$.

4.3. Martingales in continuous time

We assume in this section that our probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is equipped with a continuous filtration. (define this)

$\mathcal{F}_{t^+}$, $\mathcal{N}$, usual conditions

continuous/cadlag (sup/sub-)martingale.

Define, for a cadlag random process X, X^* and $X^{(n)*}$ (defined these). The cadlag property implies $X^{(n)*} \to X^*$ as $n \to \infty$. (check!)

Theorem 4.3.1) (Doob's maximal inequality) state and prove (The proof is written at the remark right before the theorem.)

Theorem 4.3.2) (Doob's L^p inequaltiy) state and prove

Theorem 4.3.3) (Doob's upcrossing inequality) state and prove

We also have different versions of martingale convergence theorem.

Theorem 4.3.4) (a.s. martingale convergence theorem) state and prove

Theorem 4.3.5) (L^1 martingale convergence theorem) state and prove (be aware that there is an extra assumption in the statement in converse direction)

Theorem 4.3.6) (L^p martingale convergence theorem) state and prove

stopping time T, the algebra $\mathcal{F}_T$, stopped process X^T.

Proposition 4.3.7) Let S and T be stopping times and let X be a cadlag adapted process. Then

(a) $S \wedge T$ is a stopping time.

(b) $\mathcal{F}_T$ is a σ-algebra.

(c) If $S \leq T$, then $\mathcal{F}_S \subset \mathcal{F}_T$.

(d) $X_T 1_{T<\infty}$ is $\mathcal{F}_T$-measurable.

(e) X^T is adapted.

Theorem 4.3.8) *(Optional stopping theorem)* state and prove.
Moreover, if X is uniformly integrable, then (b) and (d) hold for all stopping times.

Chapter 5. Weak Convergence

5.1. Definitions

Let E be a metric space. Whenever we are talking about a metric space, the σ-algebra is given by the Borel σ-algebra.

$C_b(E)$, weak convergence of $(\mu_n)_n \in \mathcal{P}(E)$.

Theorem 5.1.1) The following are equivalent.

(a) $\mu_n \to \mu$ weakly on E

(b) $\liminf_{n\to\infty} \mu_n(U) \geq \mu(U)$ for all U open

(c) $\limsup_{\mu(F)} \leq \mu(F)$ for all F closed.

(d) $\mu_n(B) \to \mu(B)$ for all $B \in \mathcal{B}$ such that $\mu(\partial B) = 0$.(Boundary is the set of limit points of B that are not contained in B.)

(proof was an exercise)

Proposition 5.1.2) Consider the case $E = \mathbb{R}$. TFAE

(a) $\mu_n \to \mu$ weakly for some probability measure μ.

(b) $F_n(x) \to F(x)$ for all $x \in \mathbb{R}$ such that $F(x^-) = F(x)$. (Here, $F(x) = \mu((\infty, x])$ is the **distribution function** of μ.) (Sometimes called convergence of distributions)

(c) There exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and random variables X_n, X on Ω such that $X_n \sim \mu_n$, $X \sim \mu$ and $X_n \to X$ almost surely.

(refers to probability and measure notes for proof)

5.2. Prohorov's Theorem

Tight set of probability measures
Theorem 5.2.1) *(Prohorov)* state and prove

This gives a version of weakly sequential compactness of probability measures. Only going to prove this for $\mathbb{R}$.

5.3. Weak Convergence and Characteristic Functions

Take $E = \mathbb{R}^d$. Characteristic function $\phi : \mathbb{R}^d \to \mathbb{C}$ of a probability measure μ.
Lemma 5.3.1) Fix $d = 1$. For all $\lambda \in (0, \infty)$,

$$\mu(\mathbb{R} \backslash (-\lambda, \lambda)) \leq C\lambda \int_0^\lambda (1 - \mathrm{Re}(\phi(u)))du$$

where $C = (1 - \sin(1))^{-1} < \infty$.

Theorem 5.3.2) Let μ_n, μ be probability measures on $\mathbb{R}^d$ with characteristic functions ϕ_n, ϕ. Then the following are equivalent

(a) $\mu_n \to \mu$ weakly on $\mathbb{R}^d$.

(b) $\phi_n(u) \to \phi(u)$ for all $u \in \mathbb{R}^d$.

(proven only for the case $d = 1$)

In fact, the proof of the theorem implies a slightly stronger statement, which is less useful.

Theorem 5.3.3) *(Lévy's continuity theorem for characteristic functions)* Let $(\mu_n : n \in \mathbb{N})$ be a sequence of probability measures on $\mathbb{R}^n$ with characteristic functions ϕ_n. Suppose $\phi_n(u) \to \phi(u)$ for all u for some function ϕ (not necessarily a characteristic function) such that ϕ is continuous at 0. Then ϕ is the characteristic function of some probability measure μ on $\mathbb{R}^d$ and $\mu_n \to \mu$ weakly on $\mathbb{R}^d$.

6. Large Deviations

6.1. Cramérs theorem

Theorem 6.1.1) Let $(X_n : n \in \mathbb{N})$ be a sequence of integrable *i.i.d.* random variables in $\mathbb{R}$. Set $m = \mathbb{E}(X_1)$, $S_n = X_1 + \cdots + X_n$. We know $S_n/n \to \delta_m$ in probability, so if $(m - \epsilon, m + \epsilon) \cap B = \phi$ then $\mathbb{P}(S_n/n \in B) \to 0$ as $n \to \infty$. Then in fact the convergence rate is given by $\sim \exp(-n\alpha(B))$ for some α. (state this in a precise sense)
(Proof makes use of Proposition 6.1.2 and Lemma 6.1.3)

Note : ψ is always a convex function, so ψ^* is also a convex function.

Examples :

(i) $X_1 \sim N(0,1)$, then

$$\frac{1}{n}\log(\mathbb{P}(S_n \geq a)) \to -\frac{a^2}{2} \quad \forall a \geq 0$$

Can check this directly, using the fact that $S_n \sim N(0,n)$ in this case.

(ii) $X_1 \sim \text{Exp}(1)$, then

$$\frac{1}{n}\log\mathbb{P}(S_n \geq na) \to -(a-1-\log(a)) \quad \forall a \geq 1$$

On the other hand, $\text{Var}(X_1) = 1 < \infty$, so $\frac{S_n - n}{\sqrt{n}} \to N(0,1)$ by CLT. So

$$\mathbb{P}(S_n \geq n + a\sqrt{n}) \to \int_a^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx.$$

Proposition 6.1.2) Suppose X is integrable and not a.s. constant. Then

$$\psi_K(\lambda) = \log\mathbb{E}(e^{\lambda X_1} | X_1 \leq K) < \infty \quad \forall K < \infty$$
$$\textit{and} \quad \psi_K(\lambda) \nearrow \psi(\lambda) \quad \text{as } K \to \infty$$

Moreover in the case $\psi(\lambda) < \infty$ for all $\lambda \geq 0$, ψ has a continuous derivative on $[0,\infty)$ and is C^2 on $(0,\infty)$ with

$$\psi'(\lambda) = \int_{\mathbb{R}} x\mu_\lambda(dx)$$
$$\psi''(\lambda) = \text{Var}(\mu_\lambda) > 0$$

and ψ' is a homeomorphism from $[0,\infty)$ to $[m, \sup(\text{supp}(\mu))$.
(Proof in examples sheet)

Lemma 6.1.3) For all $a \geq m$, with $\mathbb{P}(X_1 > 0) > 0$ we have $\psi_K^*(a) \searrow \psi^*(a)$ as $K \to \infty$. Moreover in the case $\psi(\lambda) < \infty$ for all $\lambda \geq 0$, ψ^* is continuous at a and we have $\psi^*(a) = \lambda^* a - \psi(\lambda^*)$ where λ^* is uniquely determined by $\psi'(\lambda^*) = a$.

7. Brownian Motion

7.1. Definition

Definition) Brownian motion(starting from x), heat semigroup.

Equivalent condition for being a Brownian motion?

7.2. Wiener's theorem

$W_d, \mathcal{W}_d$, a σ-algebra on W_d.

Given a continuous process $(X_t)_{t\geq 0}$ in $\mathbb{R}^d$ on Ω, we can define

$$X : \Omega \to W_d, \quad X(\omega)(t) = X_t(\omega)$$

-State why X is $\mathcal{W}_d$-measurable.

Theorem 7.2.1.) *(Wiener)* For all $d \geq 1$ and $x \in \mathbb{R}^d$, there exist a unique probability measure μ_x on $(W_d, \mathcal{W}_d)$ such that $(x_t)_{t\geq 0}$ is a Brownian motion in $\mathbb{R}^d$ staring from x. In particular, Brownian motion exists.

7.3. Symmetries of Brownian Motion

Proposition 7.3.1) Let $(X_t)_{t\geq 0}$ be a $\text{BM}_0(\mathbb{R}^d)$ and let $\sigma \in (0,\infty)$ and $U \in O(d)$. Then the following processes are also $\text{BM}_0(\mathbb{R}^d)$.

(i) **(Scaling property)** $(\sigma X_{\sigma^{-2}t})_{t\geq 0}$,

(ii) **(Rotation invariance)** $(UX_t)_{t\geq 0}$.

In fact $\text{BM}_0(\mathbb{R}^d)$ is characterized among continuous Gaussian processes by its means and covariances,

$$\mathbb{E}(X_t) = 0, \quad \text{cov}(X_s^i, X_t^j) = \mathbb{E}(X_s^i X_t^j) = \delta_{ij}(s \wedge t)$$

7.4. Brownian Motion in a Given Filtration

Suppose given a filtration $(\mathcal{F}_t)_{t\geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{P})$. A $(\mathcal{F}_t)$-BM is...

Proposition 7.4.1) Let $X = (X_t)_{t\geq 0}$ be a BM$(\mathbb{R}^d)$ and let F be a bounded measurable function on W_d. Define

$$f(x) = \int F(\omega)\mu_x(d\omega), \quad x \in \mathbb{R}^d$$

Then f is measurable on $\mathbb{R}^d$ and $\mathbb{E}(F(X)|\mathcal{F}_0) = f(X_0)$ a.s. (recall, μ_x is the law of BM$_x$) (Was an exercise.) (Has a simple proof in the solution for the example sheet #3.)

7.5. Martingales of BM

Theorem 7.5.1) Let $(X_t)_{t\geq 0}$ be an $(\mathcal{F}_t)_{t\geq 0}$-BM in $\mathbb{R}^d$ and let $f \in C_b^2(\mathbb{R}^d)$. Define

$$M_t = f(X_t) - f(X_0) - \int_0^t \frac{1}{2}\Delta f(X_s)ds$$

Then $(M_t)_{t\geq 0}$ is an $(\mathcal{F}_t)_{t\geq 0}$-martingale.

We are in fact apply the theorem in the case where the assumption $f \in C_b^2(\mathbb{R}^d)$ does not hold. We can actually relax this condition using almost the same proof, e.g. any f of polynomial growth rate would work.

Exercise : see how much we can relax $C_b^2(\mathbb{R}^d)$.
Examples : Let $(X_t^i)_{t\geq 0}$, $i = 1, \cdots, n$ be a BM. Then... give examples!

7.6. Strong Markov Property

Theorem 7.6.1) Let $(X_t)_{t\geq 0}$ be an $(\mathcal{F}_t)_{t\geq 0}$-BM and let T be a stopping time. Then... state and prove

7.7. Properties of 1-d BM

Proposition 7.7.1) Let $(X_t)_{t\geq 0}$ be a BM$_0(\mathbb{R})$. Set $T_a = \inf\{t \geq 0 : X_t = a\}$. Then

$$\begin{aligned}
&\mathbb{P}(T_a < \infty) = 1 \quad \text{for all } a \in \mathbb{R} \quad \textit{and} \\
&\mathbb{P}(T_{-a} \leq T_b) = \frac{b}{a+b} \quad \forall a, b \geq 0 \quad \textit{and} \\
&\mathbb{E}(T_a \wedge T_b) = ab
\end{aligned}$$

Moreover T_a has a density f_a on $[0, \infty)$ given by

$$f_a(t) = \frac{a}{\sqrt{2\pi t^3}} e^{-a^2/2t} \quad t \geq 0$$

Moreover the following holds almost surely.

(a) $X_t/t \to 0$ as $t \to \infty$.

(b) $\inf_{t\geq 0} X_t = -\infty$, $\sup_{t\geq 0} X_t = \infty$.

(c) for all $s \geq 0$, there exist $t, n \geq s$ such that $X_t < 0 < X_n$.

(d) for all $s > 0$ there exist $t, n \in [0, s)$ such that $X_t < 0 < X_n$.

(proof was an exercise)

Theorem 7.7.2) Let $X \sim$ BM$_0(\mathbb{R})$. Then the following properties hold almost surely :

(a) for all $\alpha < 1/2$, $(X_t)_{t\geq 0}$ is locally Hölder continuous of exponent α.

(b) for all $\alpha > 1/2$ there is no non-trivial interval on which X is Hölder continuous of exponent α.

7.8. Recurrence and Transience of Brownian Motion

The following theorem tells about recurrence and transience in different dimensions.

Theorem 7.8.1) Let $(X_t)_{t\geq 0}$ is a $\mathbf{BM}_0(\mathbb{R}^d)$.

(a) For $d=1$, $\mathbb{P}(\{t\geq 0 : X_t = 0\}$ is unbounded$) = 1$. **(Point recurrent)**

(b) For $d=2$, and all $\epsilon > 0$, $\mathbb{P}(X_t = 0$ for some $t > 0) = 0$ but $\mathbb{P}(\{t \geq 0 : |X_t| \leq \epsilon\}$ is unbounded$) = 1$. **(Neighbourhood recurrent**, but not recurrent**)**

(c) For $d \geq 3$, $\mathbb{P}(|X_t| \to \infty$ as $t \to \infty) = 1$. **(Transient)**

7.9. Brownian Motion and the Dirichlet Problem

Let D be a connected open set in $\mathbb{R}^d$. Write ∂D for the boundary. Let $c : D \to [0,\infty)$, $f : \partial D \to [0,\infty)$ be measurable functions. For $x \in \overline{D}$, let $(X_t)_{t\geq 0}$ be a Brownian motion starting at x.

expected total cost $\phi(x)$, solution ψ of the Dirichlet problem (for Poisson equation)

Theorem 7.9.2 (a)) Let ψ be a *non-negative super-solution* of the Dirichlet Problem(DP), $\psi \in C(\overline{D}) \cap C^2(D)$, then $\psi \geq \phi$.

Theorem 7.9.2 (b)) Let ψ be a *non-negative bounded solution* of the Dirichlet Problem, $\psi \in C(\overline{D}) \cap C^2(D)$

$$\begin{cases} -\frac{1}{2}\Delta\psi = c & \text{in } D \\ \psi = f & \text{on } \partial D \end{cases}$$

and suppose that $\mathbb{E}(\psi(X_t)1_{t<T}) \to 0$ as $t \to \infty(\cdots(\dagger))$. Then $\psi = \phi$.
Remark about the condition ($\dagger$) : state

Theorem 7.9.2 (c).(I)) Assume $d \geq 3$ and $D = \mathbb{R}^d$ and c has compact support and $c \in C^2(D)$. Then $\phi \in C^2(\mathbb{R}^d)$ and $-\frac{1}{2}\Delta\phi = c$.

Theorem 7.9.4) Let D be a connected open set $\subset \mathbb{R}^d$ and let ϕ be a *non-negative* measurable function on D. Suppose

$$\phi(x) = \int_{S(x,p)} \phi(y)\sigma_{x,p}(dy)$$

whenever $B(x,p) \subset D$ where $\sigma_{x,p}$ is the uniform distribution on $S(x,p) = \{y : |x-y| = p\}$. Then *either* $\phi(x) = \infty$ for all x *or* $\phi \in C^\infty(D)$ with $\Delta\phi = 0$.
(not done in lecture)

Theorem 7.9.5) *(Blumenthal's zero-one law)* state and prove

Proposition 7.9.6) *(Brownian motion enters all cones immediately)* Let A be an open subset of S^{d-1} and let $\epsilon > 0$. Set C a cone,

$$C = \{ty : y \in A, t \in (0,\epsilon)\}$$

Let $X \sim \mathrm{BM}_0(\mathbb{R}^d)$, and define $T_C = \inf\{t \geq 0 : X_t \in T_C\}$. Then $\mathbb{P}(T_C = 0) = 1$.

See example sheet.

Some definition (**External cone condition, ECC)**) : Let $D \subset \mathbb{R}^d$ be open and connected. Then D satisfies ECC if for all $x \in \partial D$, there exist A open $\subset S^{d-1}$, $\epsilon > 0$ such that

$$\{x + ty : t \in (0,\epsilon), y \in A\} \cap D = \phi$$

This is weaker than the Lipschitz boundary condition.

Theorem 7.9.2(c)) Let D be a connected open set $\subset \mathbb{R}^d$. Let $c \in C^2(\mathbb{R}^d)$, $f \in C(\partial D)$. Set

$$\phi(x) = \mathbb{E}\Big(\int_0^T c(X_t)dt + f(X_T)1_{T<\infty}\Big) \quad \forall x \in \overline{D}$$

where $X \sim \mathrm{BM}_x(\mathbb{R}^d)$ and $T = \inf\{t \geq 0 : X_t \notin D\}$. Assume D satisfies the *external cone condition(ECC)* and that ϕ is locally bounded. Then $\phi \in C^2(D) \cap C(\overline{D})$ and $-\frac{1}{2}\Delta\phi = c$ in D and $\phi = f$ on ∂D. i.e. ϕ solves the Dirichlet problem.
-showing it satisfies the boundary condition is quite cumbersome

Lemma 7.9.3) Let D_0 be a *bounded subset* of D. Define $T_0 = \inf\{t \geq 0 : X_t \notin D_0\}$, $x \in \overline{D}$. Then $\phi(x) = \mathbb{E}\Big(\int_0^{T_0} c(X_t)dt + \phi(X_{T_0})\Big)$.

8. Poisson Random Measures

Our ambition?

8.1. Construction

Distribution $P(\lambda)$

Proposition 8.1.1.) *(Addition property)* Let $(N_k : k \in \mathbb{N})$ be a sequence of independent random variables with $N_k \sim P(\lambda_k)$ for all k. Then

$$\sum_k N_k \sim P\big(\sum_k \lambda_k\big)$$

Proposition 8.1.2.) *(Splitting property)* Suppose $N \sim P(\lambda)$. Let $(Y_n : n \in \mathbb{N})$ be a sequence of i.i.d. random variables in $\mathbb{N}$. Set $N_k = \sum_{n=1}^N 1_{Y_n = k}$ for $k \geq 1$. Then $(N_k : k \in \mathbb{N})$ is a sequence of independent $P(\lambda_k)$ random variables with $\lambda_k = \lambda \mathbb{P}(Y_1 = k)$.

Let $(E, \mathcal{E}, \mu)$ be a σ*-finite* measure space.
Poisson Ramdom measure $PRM(\mu)$, and its construction.

To show uniqueness, we need in which sense PRM is unique - to do this, define E^* and $\mathcal{E}^*$, then PRM can be though of as a random variable $(\Omega, \mathcal{F}, \mathbb{P}) \to (E^*, EE^*)$. Then such random variable is unique upto a.s. sense (unique in law).

8.2. Integrals with respect to Poisson Random Measures

Theorem 8.2.1) Let M be a PRM(μ) and let g be a measurable function on $(E, \mathcal{E})$. Assume $\mu(E) < \infty$. Define

$$M(g) = \begin{cases} \int_E g(y)M(dy) & \text{if } M(E) < \infty \\ 0 & \text{o/w} \end{cases}$$

Then $M(g)$ is a well-defined random variable and

$$\mathbb{E}(e^{inM(g)}) = \exp\Big[\int_E (e^{ing(y)} - 1)\mu(dy)\Big]$$

and if $g \in L^1(\mu)$ then $M(g) \in L^1(\mathbb{P})$ with $\mathbb{E}(M(g)) = \mu(g)$ and $\mathrm{Var}(M(g)) = \mu(g^2)$.

$\tilde{M}$, the compensated Poisson Random Measure.

Proposition 8.2.2) Assume $K(E) < \infty$ and $g \in L^1(K)$. Set

$$\tilde{M}_t(g) = \begin{cases} \int_{(0,t]\times E} g(y)\tilde{M}(ds, dy) & \text{if } \mu((0,t] \times E) < \infty \ \forall t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Then $(\tilde{M}_t(g))_{t \geq 0}$ is a *cadlag martingale* with *stationary independent increments*. Moreover,

$$\begin{aligned} &\mathbb{E}(\tilde{M}_t(g)^2) = t\int_E g(y)^2 K(dy) & \cdots\cdots\cdots (\heartsuit) \\ &\mathbb{E}(e^{iu\tilde{M}_t(g)}) = \exp\Big[t\int_E (e^{iug(y)} - 1)K(dy)\Big] & \cdots\cdots\cdots (\oplus) \end{aligned}$$

(in examples sheet)

Theorem 8.2.3) Let $g \in L^2(K)$. Take $E_n \in \mathcal{E}$ with $E_n \nearrow E$ and $K(E_n) < \infty$. Set $X_t^n = \tilde{M}_t(g1_{E_n})$. Then there exists a *cadlag martingale* $(X_t)_{t \geq 0}$ such that $\mathbb{E}(\sup_{s \leq t} |X_s^n - X_s|^2) \to 0$ as $n \to \infty$ for all $t \geq 0$.

Define $\tilde{M}_t(g) = X_t$. Then $(\tilde{M}_t(g))_{t \geq 0}$ has *stationary independent increments* and ($\heartsuit$) and ($\oplus$) still hold - so we may write, in convention,

$$\tilde{M}_t(g) = \int_{(0,t] \times E} g(y) \tilde{M}(ds, dy)$$

and call $(\tilde{M}_t(g))_{t \geq 0}$ (a version of the stochastic) integral of g with respect to $\tilde{M}$. (Although the integral does not converge absolutely!!!)

9. Lévy Processes

9.1. Definitions

Levy process, Levy triple. Levy process with Levy triple (a, b, K), and its characteristic function.

9.2. Lévy-Khinchine Theorem

Theorem 9.2.1) Let $(X_t)_{t \geq 0}$ be a Lévy process (i.e. cadlag, stationary independent increments with $X_0 = 0$.) Then there exists a Lévy triple (a, b, K) such that $\mathbb{E}(e^{iuX_t}) = e^{t\psi(u)}$ for all $t \geq 0$, $u \in \mathbb{R}$ where

$$\psi(u) \equiv \psi_{a,b,K}(u) = ibu - \frac{1}{2}au^2 + \int_{\mathbb{R}} (e^{iuy} - 1 - iuy1_{|y| \leq 1}) K(dy)$$

and $\int_{\mathbb{R}} (1 \wedge |y|^2) K(dy) < \infty$.