Percolation and Random Walks on Graphs-revision note

1. Percolation

1.1. Definition of the model

Definition) Bond/site Percolation on a graph $G = (V, E)$ and parameter $p \in [0, 1]$.
-What is the probability space and the σ-algebra? What is the probability measure?
-denote the state by random variable $\eta_p \in \{0, 1\}^E$.

Definition)$x \leftrightarrow y$, $\mathcal{C}(x)$

Definition 1.1) Coupling of two probability measures μ and ν.

1.2. Coupling of percolation processes

Definition) Percolation modelled via uniform random variables.

Lemma 1.2.) The probability $\theta(p) = \mathbb{P}_p(|\mathcal{C}(0)| = \infty)$ is an increasing function of p.

1.3. Phase transition

Definition)$p_c(d)$

Theorem 1.4.) For all $d \geq 2$ we have $p_c(d) \in (0, 1)$.
-uses Σ_n, the number of open self-avoiding walks of length n and σ_n, the number of self-avoiding walks of length n.
-come back to proof after **Definition 1.11.** Note that, the number of dual circuits of length n that surrounds 0 is at most $n4^n$ using the following argument - a closed circuit surrounding 0 should pass at least one point among $\{(1, 0), \cdots, (n, 0)\}$ so choose this as a start point, then there are at most 4^n ways to proceed from this point, so the number is bounded by $n4^n$.

1.3.1. Self-avoiding walks

Lemma 1.5.) Let σ_n be the number of self-avoiding paths of length n> Then for all m, n we have

$$\sigma_{n+m} \leq \sigma_n \sigma_m$$

Corollary 1.6.) There is a constant λ so that

$$\lim_{n\to\infty} \frac{\log \sigma_n}{n} = \lambda$$

-**Remark:** the corollary tells us that $\sigma_n = e^{n\lambda(1+o(1))}$. Define $\kappa = e^\lambda$.

Improved versions of the corollary includes: **Theorem 1.9.)**(Hammersley and Welsh) For all d the number of self-avoiding walks σ_n satisfies

$$\sigma_n \leq \exp(c_d\sqrt{n})(\kappa_d)^n$$

where c_d is a positive constant.

Theorem 1.10.)(Hutchcroft) For all d we have

$$\sigma_n \leq \exp(o(\sqrt{n}))\kappa^n$$

-We do not prove 1.9. and 1.10.

1.3.2. Existence and uniqueness of the infinite cluster

Definition 1.11.) The dual of a planar graph G.
-Remark : The $\mathbb{Z}^2$ lattice is isomorphic to its dual, i.e. has duality property.
-We may prove **Theorem 1.4.** using duality of $\mathbb{Z}^2$ lattice.

Lemma 1.13.) Let A_∞ be the event that there exists an infinite cluster. Then we have the following dichotomy:

(a) If $\theta(p) = 0$, then $\mathbb{P}_p(A_\infty) = 0$.

(b) If $\theta(p) > 0$, then $\mathbb{P}_p(A_\infty) = 1$.

Theorem 1.14.) Let N be the number of infinite clusters. For all $p > p_c$ we have that

$$\mathbb{P}_p(N = 1) = 1$$

-Refers to the fact that N is translational invariant, hence therefore is a.s. a constant.
(From Exercise 4. - proof : Let X be a random variable that is translational invariant. Let $\Omega_x = \{\omega \in \Omega : X(\omega) = x\}$ and that $p_y = \mathbb{P}(\Omega_y) > 0$ for some y. As X is translational invariant, one has $\Omega_y = T_a(\Omega_y)$ for each $a \in \mathbb{Z}^2$. !!!! I don't have any idea.
It is sufficient to show that translation map acts as an ergodic map on the lattice)
-proving that the number of clusters is not ∞ is difficult. Once assuming this, use the above fact to complete proof.(the remaining part is still hard)
-Why do we have $\#\{\text{vertices of degree} \geq 3\} \leq \#\text{leaves}$? This is because we may make injection from the set $\{\text{vertices of degree} \geq 3\}$ to the set of leaves by modifying the paths appropriately.

1.4. Correlation inequalities

Let $G = (V, E)$ be a graph, $\Omega = \{0, 1\}^E$ be endowed with the σ-algebra $\mathcal{F}$ generated by the cylinder sets and with the usual probability measure with parameter $p \in [0, 1]$.

Definition 1.16.) For configurations $\omega, \omega' \in \Omega$, $\omega' \geq \omega$.
-this defines a partial order on Ω.
-increasing/decreasing random variable X/event A.

Example) The event $\{|\mathcal{C}()| = \infty\}$ is increasing.

Theorem 1.18.) If N is an increasing random variable and $p_1 \leq p_2 <$ then

$$\mathbb{E}_{p_1}[N] \leq \mathbb{E}_{p_2}[N]$$

Similarly if A is an increasing event, then

$$\mathbb{P}_{p_1}(A) \leq \mathbb{P}_{p_2}(A)$$

-proved by coupling

Theorem 1.19)(FKG inequality) Let X and Y be two increasing variable on $(\Omega, \mathcal{F})$ such that $\mathbb{E}_p[X^2] < \infty$ and $\mathbb{E}_p[Y^2] < \infty$. Then

$$\mathbb{E}_p[XY] \geq \mathbb{E}_p[X]\mathbb{E}_p[Y]$$

-Another way of writing this is

$$\mathbb{P}_p(A|B) \geq \mathbb{P}_p(A)$$

-The theorem tells us that whenever two random variables are increasing, then they are positively correlated.

Example:

- conditioning on $x \leftrightarrow y$ increases the probability of having $u \leftrightarrow v$ for any x, y, u, v.
- Let G be a graph and for every vertex x we define

  $$p_c(x) = \sum\{p \in [0,1] : \mathbb{P}_p(|\mathcal{C}(x)| = \infty) = 0\}$$

  Then we get $p_c(x) = p_c(y)$ for all x, y.(draw contradiction by assuming that for some p, we have $\mathbb{P}_p(|\mathcal{C}(x)| = \infty) = 0$ but $\mathbb{P}_p(|\mathcal{C}(y)| = \infty) > 0$)

Definition 1.24) $[\omega]_S$ for $\omega \in \Omega = \{0,1\}^E$ and $S \subset E$.
-The disjoint occurrence $A \circ B$ for events A, B.

Theorem 1.25)(BK inequality) Let F be a *finite set* and $\Omega = \{0,1\}^F$. For all increasing events A, B, we have

$$\mathbb{P}_p(A \circ B) \leq \mathbb{P}_p(A)\mathbb{P}_p(B)$$

Theorem 1.26)(Reimer's inequality) Let F b e a finite set and $\Omega = \{0,1\}^F$. For all A and B we have(without assuming that they are increasing)

$$\mathbb{P}_p(A \circ B) \leq \mathbb{P}_p(A)\mathbb{P}_p(B)$$

-A generalized version of BK inequality, not proving

Theorem 1.27) Suppose that $\chi(p) = \mathbb{E}_p[|\mathcal{C}(0)|] < \infty$. Then there exists a positive constant c so that for all $n \geq 1$ we have

$$\mathbb{P}_p(0 \leftrightarrow \partial\mathcal{B}_n) \leq e^{-cn}$$

where $\mathcal{B}_n = \{-n, \cdots, n]^d$ is the box with diameter $2n + 1$.
-uses BK inequality for proof.

1.5. Russo's formula

Definition 1.28) A pivotal edge e for A an event and ω a percolation configuration.
-The event $\{e$ is pivotal for $A\}$ is equal to $\{\omega : e$ is pivotal for $(A, \omega)\}$.

Example Let A be the event that 0 is in an infinite cluster. Then an edge e is pivotal for A if the removal of e leads to a finite component containing the origin.

Theorem 1.30) (Russo's formula) Let A be an increasing event that depends only on the states of a finite number of edges. Then

$$\frac{d}{dp}\mathbb{P}_p(A) = \mathbb{E}_p[N(A)]$$

where $N(A)$ is the number of pivotal edges for A.

Remark : If A is an increasing event depending on an infinite number of edges, then

$$\liminf_{\delta \to 0} \frac{\mathbb{P}_{p+\delta}(A) - \mathbb{P}_p(A)}{\delta} \geq \mathbb{E}_p[N(A)]$$

-Why do we need equation (1.8)?

Corollary 1.32) Let A be an increasing event depending on the states of m edges and $p \leq q$ be in [0,1]. Then

$$\mathbb{P}_q(A) \leq (\frac{q}{p})^m \mathbb{P}_p(A)$$

1.6. Subcritical phase

In this section we focus on $p < p_c$. In this case we know that there is no infinite cluster a.s. However, oone can ask what is the size of the cluster of 0. How do probabilities like $\mathbb{P}_p(|\mathcal{C}(0)| \geq n)$ decay in n?
Write $\mathcal{B}_n = [-n, n]^d \cap \mathbb{Z}^d$.

Theorem 1.33) Let $d \geq 2$. Then the following are true.

(a) If $p < p_c$, then there exists a constant c so that for all $n \geq 1$, we have

$$\mathbb{P}_p(0 \leftrightarrow \partial \mathcal{B}_n) \leq e^{-cn}$$

(b) If $p > p_c$< then

$$\theta(p) = \mathbb{P}_b(0 \leftrightarrow \infty) \geq \frac{p - p_c}{p(1 - p_c)}$$

proof) Define

$$\varphi_p(S) = p \sum_{(x,y) \in \partial S} \mathbb{P}_p(0 \overset{S}{\longleftrightarrow} x)$$

and

$$\tilde{p}_c = \sup\{p \in [0,1] : \exists \text{a finite set } S \text{ s.t. } 0 \in S \text{ with } \varphi_p(S) < 1\}$$

We prove the theorem with p_c replaced by $\tilde{p}_c$, and from the results of the theorem, it follows that $p_c = \tilde{p}_c$.

(a) Let $\mathcal{C} = \{x \in S : 0 \overset{S}{\longleftrightarrow} x\}$. Then

$$\begin{aligned}
\mathbb{P}_p(0 \leftrightarrow \partial \mathcal{B}_{kL}) &= \mathbb{P}_p(\cup_{(x,y) \in \partial S} \cup_{A \subset S} 0 \overset{S}{\longleftrightarrow} x, (x,y) \text{ is open}, \mathcal{C} = A, y \overset{A^c}{\longleftrightarrow} \partial \mathcal{B}_{kL}) \\
&\leq \sum_{(x,y) \in \partial S} \sum_{A \subset S} \mathbb{P}_p(0 \overset{S}{\longleftrightarrow} x, (x,y) \text{ is open}, \mathcal{C} = A, y \overset{A^c}{\longleftrightarrow} \partial \mathcal{B}_{kL}) \\
&= p \sum_{(x,y) \in \partial S} \sum_{A \subset S} \mathbb{P}_p(0 \overset{S}{\longleftrightarrow} x, \mathcal{C} = A) \mathbb{P}_p(y \overset{A^c}{\longleftrightarrow} \partial \mathcal{B}_{kL}) \\
&\leq p \sum_{(x,y) \in \partial S} \sum_{A \subset S} \mathbb{P}_p(0 \overset{S}{\longleftrightarrow} x, \mathcal{C} = A) \mathbb{P}_p(0 \leftrightarrow \partial \mathcal{B}_{(k-1)L}) \\
&= p \sum_{(x,y) \in \partial S} \mathbb{P}_p(0 \overset{S}{\longleftrightarrow} x) \mathbb{P}_p(0 \leftrightarrow \partial \mathcal{B}_{(k-1)L}) \\
&= \varphi_p(S) \mathbb{P}_p(0 \leftrightarrow \partial \mathcal{B}_{(k-1)L})
\end{aligned}$$

Iterating this inequality, we have

$$\mathbb{P}_p(0 \leftrightarrow \partial \mathcal{B}_{kL}) \leq (\varphi_p(S))^{k-1}$$

and hence has exponential decay.

(b) By Russo's formula, we have

$$\begin{aligned}
\frac{d}{dp} \mathbb{P}_p(0 \leftrightarrow \partial \mathcal{B}_n) &= \sum_{e \in \mathcal{B}_n} \mathbb{P}_p(e \text{ is pivotal for } \{0 \leftrightarrow \partial \mathcal{B}_n\}) \\
&= \frac{1}{1-p} \sum_{e \in \mathcal{B})n} \mathbb{P}_p(e \text{ is pivotal}, 0 \not\leftrightarrow \partial \mathcal{B}_n)
\end{aligned}$$

Define

$$\mathcal{S} = \{x \in \mathcal{B}_n : x \not\leftrightarrow \partial \mathcal{B}_n\}$$

Then

$$\begin{aligned}
\frac{d}{dp}\mathbb{P}_b(0 \leftrightarrow \partial\mathcal{B}_n) &= \frac{1}{1-p}\sum_{e\in\mathcal{B}_n}\sum_{A\subset\mathcal{B}_n, 0\in A}\mathbb{P}_p(e \text{ is pivotal}, \mathcal{S}=A) \\
&= \frac{1}{1-p}\sum_{A\subset\mathcal{B}_n, 0\in A}\sum_{(x,y)\in\partial A}\mathbb{P}_p(0 \overset{A}{\leftrightarrow} x, \mathcal{S}=A) \\
&= \frac{1}{1-p}\sum_{A\subset\mathcal{B}_n, 0\in A}\sum_{(x,y)\in\partial A}\mathbb{P}_p(0 \overset{A}{\leftrightarrow} x)\mathbb{P}_p(\mathcal{S}=A) \\
&= \frac{1}{p(1-p)}\sum_{A\subset\mathcal{B}_n, 0\in A}\varphi_p(A)\mathbb{P}_p(\mathcal{S}=A) \\
&\geq \frac{1}{p(1-p)}\inf_{S\subset\mathcal{B}_n, 0\in S}\varphi_p(S)\mathbb{P}_p(0 \nleftrightarrow \partial\mathcal{B}_n)
\end{aligned}$$

and therefore

$$\frac{d}{dp}\mathbb{P}_p(0 \leftrightarrow \partial\mathcal{B}_n) \geq \frac{1}{p(1-p)}\inf_{S\subset\mathcal{B}_n, 0\in S}\varphi_p(S)(1-\mathbb{P}_p(0 \leftrightarrow \partial\mathcal{B}_n))$$

For $p > \tilde{p}_c$, integrating from $\tilde{p}_c$ to p gives

$$1-\mathbb{P}_p(0 \leftrightarrow \partial\mathcal{B}_n) \leq -\frac{1}{p(1-p)}(1-\mathbb{P}_{\tilde{p}_c}(0 \leftrightarrow \partial\mathcal{B}_n))\exp(-\frac{p-\tilde{p}_c}{p(1-p)}) \leq \exp(-\frac{p-\tilde{p}_c}{p(1-p)})$$

and we have the desired inequality as $n \to \infty$.

•**Remark :** We have assumed that $p < p_c$ but not $\theta(p) = 0$. For $d = 2$, at the critical probability $1/2$, we do not have this exponential decay.

•**Remark :** The probability $\mathbb{P}_p(0 \leftrightarrow \partial\mathcal{B}_n)$ is at least p^n, and hence we cannot hope for a faster convergence than exponential decay.

•**Remark :** The theorem tells us that

$$\mathbb{P}_p(|\mathcal{C}(0)| \geq n) \leq \mathbb{P}_p(0 \leftrightarrow \partial\mathcal{B}_{n^{1/d}}) \leq \exp(-cn^{1/d})$$

However, this bound can be replace by $\exp(-cn)$.

1.7. Supercritical phase in $\mathbb{Z}^2$

Theorem 1.37.) For bond percolation on $\mathbb{Z}^2$ the probability $p_c = 1/2$ and $\theta(1/2) = 0$.

1.8. Russo Seymour Welsh theorem

Let $\mathcal{B}(kl, l) = [-l, (2k-1)l] \times [-l, l]$ and $\mathcal{B}(l) = \mathcal{B}(l, l)$. Denote $\mathrm{LR}(kl, l)$ the event that there exists a left to right crossing of $\mathcal{B}(kl, l)$ and write $\mathrm{LR}(l)$ for a crossing of $\mathcal{B}(l)$. Also, let $A(l) = \mathcal{B}(3l)\backslash\mathcal{B}(l)$. Write $O(l)$ for the event that there is an open circuit in $A(l)$ containing 0 in its interior.

Theorem 1.38)(RSW) Suppose that $\mathbb{P}_p(\mathrm{LR}(l)) = \alpha$. Then

$$\mathbb{P}_p(O(l)) \geq (\alpha(1-\sqrt{1-\alpha})^4)^{12}$$

Lemma 1.40.) Suppose that $\mathbb{P}_p(\mathrm{LR}(l)) = \alpha$. Then

$$\mathbb{P}_p(\mathrm{LR}(\frac{3}{2}l, l)) \geq (1-\sqrt{1-\alpha})^3$$

Lemma 1.41.) We have

$$\begin{aligned}
\mathbb{P}_p(\mathrm{LR}(2l, l)) &\geq \mathbb{P}_p(\mathrm{LR}(l))\Big(\mathbb{P}_p(\mathrm{LR}(3l/2, l))\Big)^2 \\
\mathbb{P}_p(\mathrm{LR}(3l, l)) &\geq \mathbb{P}_p(\mathrm{LR}(l))\big(\mathbb{P}_p(\mathrm{LR}(2l, l))\big)^2
\end{aligned}$$

$$\mathbb{P}_p(O(l)) \geq (\mathbb{P}_p(\mathrm{LR}(3l, l)))^4$$

1.9 Power law inequalities at the critical point

Theorem 1.42.) There exist finite positive constants $\alpha_1, A_1, \alpha_2, A_2, \alpha_3, A_3, \alpha_4, A_4$ so that for all $n \geq 1$ we have

$$\begin{aligned}\frac{1}{2\sqrt{2}} \leq& \mathbb{P}_{1/2}(0 \leftrightarrow \partial \mathcal{B}_n) \leq A_1 n^{-\alpha_1} \\ \frac{1}{2\sqrt{2}} \leq& \mathbb{P}_{1/2}(|\mathcal{C}(0)| \geq n) \leq A_2 n^{-\alpha_2} \\ & \mathbb{E}[|\mathcal{C}(0)|^{\alpha_3}] < \infty\end{aligned}$$

Moreover, for all $p > 1/2$ we have

$$\theta(p) \leq A_4(p - \frac{1}{2})^{\alpha_4}$$

Lemma 1.44.) Let $O(l)$ be as in the previous section. Then there exists a positive constant ζ such that for all $l \geq 1$ we have

$$\mathbb{P}_{1/2}(O(l)) \geq \zeta$$

1.10. Grimmett Marstrand theorem

1.11. Conformal invariance of crossing probabilities $p = p_c$