Analysis of PDEs

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Example Classes : Dr. Ivan Moyano

Texts : (1)Evans. PDEs, (2)Rauch, PDEs, (3)F.John, PDEs, (4)Gilberg + Raudinger, Elliptic PDE, (5) Ladyzhenskay, The Boundary Value Problems of Mathematical Physics.

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(5th October 2018, Friday)

Introduction

Suppose $U \subset \mathbb{R}^n$ is open. A *partial differential equation* of order k is an expression of the following form:

$$F(x, u(x), Du(x), \cdots, D^{(k)}u(x)) = 0 \tag{1}$$

Here, $F : U \times \mathbb{R} \times \mathbb{R}^n \times \cdots \times \mathbb{R}^{n^k} \to \mathbb{R}$ is a given function and $u : U \to \mathbb{R}$ is the 'unknown'. We say $u \in C^k(U)$ is a classical solution of 1 if 1 is satisfied on U when we substitute u into the expression.

We could also consider the case where $u : U \to \mathbb{R}^p$ and F takes values in $\mathbb{R}^q$, then we speak of a *system of PDE's*.

Examples)

1. The Transport Equation: Suppose $V : \mathbb{R}^{n+2} \to \mathbb{R}^n$ is given.

$$\frac{\partial u}{\partial t}(x,t) + V(x,t,u(t,x)) \cdot D_x u(x,t) = f(x,t) \quad \text{for } x \in \mathbb{R}^n$$

   is a PDE for $u : \mathbb{R}^{n+1} \to \mathbb{R}$. This describes evolution of some chemical produced at rate $f(x,t)$ and being advected by a flow of velocity $V(x,t,u(t,x))$.

2. The Laplace and Poisson Equations:

$$\Delta u(x) = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i \partial x_j}(x) = 0 \quad \text{(Laplace Equation)}$$

   This describes:

   + Electrostatic potential in empty space
   + Static distribution of heat in a solid body
   + Applications to steady flows in 2D
   + Connections to complex analysis

$$\Delta u(x) = f(x) \quad \text{some given } f : \mathbb{R}^n \to \mathbb{R} \quad \text{(Poisson's Equation)}$$

   This describes:

+ Electric field produced by charge distribution f
+ Gravitational field in Newton's Theory(f is mass density)

3. Heat/Diffusion Equation:

$$\frac{\partial u}{\partial t} = \Delta u$$

This describes evolution of temperature in a solid homogeneous body.

4. Wave Equation:

$$-\frac{\partial^2 u}{\partial t^2} + \Delta u = 0$$

This describe:

+ Displacement of a stretched string (dimension=1)
+ Ripples on surface of water (dimension=2)
+ Density of air in a sound wave (dimension=3)

5. Maxwell's Equations: With $E, B : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$,

$$\begin{aligned} \nabla \cdot E = \rho \quad & \nabla \cdot B = 0 \\ \nabla \times E + \frac{\partial B}{\partial t} = 0 \quad & \nabla \times B - \frac{\partial E}{\partial t} = J \end{aligned}$$

ρ, J are charge density/current respectively, are given.

6. Ricci Flow:

$$\partial_t g_{ij} = -2R_{ij}$$

where g_{ij} is a Riemannian metric, R_{ij} is its Ricci curvature.

7. Minimal Surface Equation: For $u : \mathbb{R}^2 \to \mathbb{R}$,

$$\operatorname{div}\left(\frac{Du}{\sqrt{1 - |Du|^2}}\right) = 0$$

Condition for the graph $\{(x, y, u(x, y))\}$ to locally extremise area.

8. Eikonal Equation: for $U \subset \mathbb{R}^3$ and $u : U \to \mathbb{R}$

$$|Du| = 1$$

Level sets parametrise a wave-front moving according to the ray theory of light.

9. Schrödinger's Equation: For $u : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{C} \equiv \mathbb{R}^2$,

$$i\frac{\partial u}{\partial t} + \Delta u - Vu = 0$$

for $V : \mathbb{R}^3 \to \mathbb{R}$ given. u is the wavefunction of a quantum mechanical particle moving in a potential V.

10. Einstein's Equations for General Relativity:

$$R_{\mu\nu}[g] = 0$$

where g is Lorentzian metric. $R_{\mu\nu}$ is Ricci tensor. This describes gravitational field in vacuum.

-. There are Many more examples.

Data and Well-Posedness

In all examples, there is extra information required beyond the equation. We call this the *data*. An important question is what data is appropriate. We typically ask of a PDE problem that:

a) A solution exists,

b) for given data the solution is unique,

c) the solution depends on the data continuously.

If these hold, we say the problem is 'well-posed'. To make these precise, we have to (usually) specify function spaces for the data and solution to belong to.

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8th October, Monday

Let $U \subset \mathbb{R}^n$, $u : U \to \mathbb{R}$ be unknown. Then our system of interest will be

$$F(x; u, Du, \cdots, D^k u) = 0 \tag{2}$$

Notations) Let $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$ be a multi-index(where $\mathbb{N} = \{0, 1, 2, 3, \cdots\}$). Then we let:

- $D^\alpha f(x) = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$ where $|\alpha| = \alpha_1 + \cdots + \alpha_n$ is the order of α.
- For $x \in \mathbb{R}^n$, $x^\alpha = x_1^{\alpha_1} \times \cdots \times x_n^{\alpha_n}$
- $\alpha! = \alpha_1! \cdots \alpha_n!$.
- For $\beta = (\beta_1, \cdots, \beta_n)$, $\beta \leq \alpha$ is equivalent to having $\beta_k \leq \alpha_k$ for all k.

Classifying PDEs

- We say (2) is **linear** if F is a linear function of u and its derivatives. We can write (2) as

$$\sum_{|\alpha \leq k} a_\alpha(x) D^\alpha u(x) = f(x)$$

- We say (2) is **semi-linear** if it is of the form

$$\sum_{|\alpha \leq k} a_\alpha(x) D^\alpha u(x) + a_0(x; u(x), \cdots, D^{k-1} u(x)) = 0$$

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- We say (2) is **quasi-linear** if it is of the form

$$\sum_{|\alpha \leq k} a_\alpha\big(x; u(x), \cdots, D^{k-1} u(x)\big) D^\alpha u(x) + a_0(x; u(x), \cdots, D^{k-1} u(x)) = 0$$

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- We say (2) is **fully non-linear** if its not linear, semi-linear, nor quasi-linear

Examples)

- $\Delta u = f$ is linear
- $\Delta u = u^3$ is semi-linear
- $u u_{xx} + u_x u_{yy} = f$ is quasi-linear
- $u_{xx} u_{yy} - u_{xy}^2 = f$ is fully non-linear.

Cauchy-Kovalevskaya Theorem

For motivation, we recall some ODE theory. Fix $U \subset \mathbb{R}^n$, and assume $f : U \to \mathbb{R}^n$ is given. Consider the ODE

$$\dot{u}(t) = f(u(t)), u(0) = u_0 \in U \tag{3}$$

with $u : I \subset \mathbb{R} \to U$.

Theorem) (Picard-Lindelöf) Suppose there exist $r, K > 0$ s.t. $B_r(u_0) = \{w \in \mathbb{R}^n : |w - u_0| < r\}$ and $|f(x) - f(y) \leq K|x - y|$ for all $x, y \in B_r(x_0)$. Then there exists $\epsilon > 0$(depending in r and K) and a unique C^1-function $u : (-\epsilon, \epsilon) \to U$ solving (3).
proof) Use U solves (3), then

$$u(t) = u_0 + \int_0^t f(u(s))ds \tag{4}$$

and conversely, if U is C^0 and solves (4), then in fact U is C^1 by FTC, and u solves (3).(in context of PDEs, this is called *weak formulation*)
Then our solution, if exists, is a fixed point of the map $B : w \mapsto u_0 + \int_0^t f(w(s))ds$.(use Banach fixed point theorem)

Observations:

- We start by reformulating the problem in a weak form and find a unique C^0 solution. Then C^1 the regularity follows a posteriori.
- to construct the fixed point map, we solve the linear problem $\dot{w}(t) = f(w(t))$.

Lets consider an alternative approach to solving (3). Assuming f is differentiable, we have

$$\begin{aligned}
u^{(1)}(t) &= f(u(t)) \\
u^{(2)}(t) &= f'(u(t))\dot{u}(t) \\
u^{(3)}(t) &= f''(u(t))(\dot{u}(t))^2 + f'(u(t))\ddot{u}(t) \\
&\vdots \\
u^{(k)}(t) &= f_k(u(t), \dot{u}(t), \cdots, u^{(k-1)})(t))
\end{aligned}$$

So in principle, given $u(0) = u_0$, we can determine $u_k = u^{(k)}(0)$ for all $k \geq 0$. *Formally* at least, we can write

$$u(t) = \sum_{k=0}^{\infty} u_k t^k / k! \tag{5}$$

ignoring the issues of convergence. Call this a **formal power series solution**. When will this agree with the Picard-Lindelöf solution we have constructed?

Theorem) (Cauchy-Kovalevskaya, for the case of ODEs) The series in (5) converges to a solution of (3) in a neighbourhood of $t = 0$ if f is real analytic at u_0.
-This will follow from a more general result later.

Definition) Let $U \subset \mathbb{R}^n$ be open and suppose $f : U \to \mathbb{R}$. f is called **real analytic** near $x_0 \in U$ if $\exists r > 0$ and constants f_α(α are multi-indices) such that

$$f(x) = \sum_\alpha f_\alpha (x - x_0)^\alpha \quad \text{for } x \in B_r(x_0)$$

Note: if f is real analytic, then it is C^∞. Furthermore, the constants f_α are given by $f_\alpha = D^\alpha f(x_0)/\alpha!$. Thus f equals its Taylor expansion about x_0, in a neighbourhood of x_0.

$$f(x) = \sum_\alpha \frac{D^\alpha f(x_0)}{\alpha!}(x - x_0)^\alpha \quad \text{for } x \in B_r(x_0)$$

By translation, we usually assume $x_0 = 0$

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(10th October, Wednesday)

• Last lecture : $U \subset \mathbb{R}^n$ open, $f : U \to \mathbb{R}$ is real analytic at $x_0 \in U$ if $\exists f_\alpha \in \mathbb{R}$, $r > 0$ s.t.

$$f(x) = \sum_\alpha f_\alpha (x - x_0)^\alpha \quad \forall |x - x_0| < r$$

Properties of real analytic functions

- f is real analytic at x_0 if and only if $\exists s > 0$ and $C, \rho > 0$ such that:

$$\sup_{|x-x_0|<s} \left| D^\alpha f(x) \right| \leq C \frac{|\alpha|!}{\rho^{|\alpha|}}$$

- If f is RA(real analytic) at x_0, it is RA for all x close enough to x_0.
- If $f : U \to \mathbb{R}$ is real analytic everywhere on a connected set U, then f is determined by its values on any open subset of U. (Or by its Taylor expansion at a single point.)

Example : If $r > 0$ set

$$f(x) = \frac{r}{r - (x_1 + \cdots + x_n)} \quad \text{for } |x| < r/\sqrt{n}$$

Then for $|x| < r/\sqrt{n}$,

$$f(x) = \frac{1}{1 - (x_1 + \cdots + x_n)/r} = \sum_{k=0}^\infty \left(\frac{x_1 + \cdots + x_n}{r} \right)^k = \sum_{k=0}^\infty \frac{1}{r^k} \sum_{|\alpha|=k} \binom{|\alpha|}{\alpha} x^\alpha$$
$$= \sum_\alpha \frac{|\alpha|!}{r^{|\alpha|}\alpha!} x^\alpha$$

by multinomial theorem. This is valid for $|x_1 + \cdots + x_n|/r < 1$, which holds for $|x| < r/\sqrt{n}$. In fact, on this domain, the series converges absolutely. Indeed :

$$\sum_\alpha \frac{|\alpha|!}{r^{|\alpha|}\alpha!} |x|^\alpha = \sum_{k=0}^\infty \left(\frac{|x_1| + \cdots + |x_n|}{r} \right)^k < \infty$$

since $|x_1| + \cdots + |x_n| \leq |x|\sqrt{n} < r$.

Definition) Let $f = \sum_\alpha f_\alpha x^\alpha$, $g = \sum_\alpha g_\alpha x^\alpha$ be two formal power series. We say g **majorises** f, written $g \gg f$ if

$$\left| f_\alpha \right| \leq g_\alpha$$

for all α, and say that g is a **majorant** of f.

Lemma)

(i) If $g \gg f$ and g converges for $|x| < r$ then f also converges (absolutely) for $|x| < r$.

(ii) If f converges for $|x| < r$, then for any $s \in (0, r/\sqrt{n})$, f has a majorant that converges for $|x| < s/\sqrt{n}$.(n is the dimension of the space)

proof)

(i) We note that

$$\sum_\alpha \left| f_\alpha x^\alpha \right| \leq \sum_\alpha \left| f_\alpha \right| |x_1|^{\alpha_1} \cdots |x_n|^{\alpha_n}$$
$$\leq \sum_\alpha g_\alpha \tilde{x}^\alpha$$

where $\tilde{x} = (|x_1|, \cdots, |x_n|)$. Now $|\tilde{x}| = |x| < r$ so $\sum_\alpha g_\alpha \tilde{x}^\alpha$ converges, hence $\sum_\alpha \left| f_\alpha x^\alpha \right|$ converges. Hence f converges on $|x| < r$ absolutely.

(ii) Pick s s.t. $0 < s\sqrt{n} < r$, and set $y = s(1, \cdots, 1)$. Then $|y| = s\sqrt{n} < r$. Hence $\sum_\alpha f_\alpha y^\alpha$ converges. A convergent series has bounded terms, $\exists C > 0$ s.t. $\left|f_\alpha y^\alpha\right| \leq C$ for all α, and therefore

$$\left|f_\alpha\right| \leq \frac{C}{y_1^{\alpha_1} \cdots y_n^{\alpha_n}} = \frac{C}{s^{|\alpha|}} \leq \frac{C|\alpha|!}{s^\alpha \alpha!}$$

But then $g(x)$ defined by

$$g(x) = \frac{Cs}{s - (x_1 + \cdots + x_n)} = C \sum_\alpha \frac{|\alpha|!}{s^\alpha \alpha!} x^\alpha$$

majorises f and converges for $|x| < s/\sqrt{n} < r/n$.

(End of proof) $\square$

Remark : If $f = (f^1, \cdots, f^m)$ and $g = (g^1, \cdots, g^m)$ are formal power series, then we say

$$g \gg f \quad \text{if} \quad g^i \gg f^i \quad i = 1, \cdots, m$$

Cauchy-Kovalevskaya for First Order Systems

We will study a problem that generalises the Cauchy problem for ODEs we have already discussed.

As coordinates on $\mathbb{R}^n$ we take $(x', t) = x$ where

$$x' = (x_1, \cdots, x_{n-1}) \in \mathbb{R}^{n-1}, \quad t = x^n \in \mathbb{R}$$

Set

$$B_r^n = \{t^2 + |x'|^2 < r^2\}, \quad B_r^{n-1} = \{|x'| < r, t = 0\}$$

We consider a system of equations for unknown $\underline{u}(x) \in \mathbb{R}^m$. More concretely, we seek a solution to

$$\begin{aligned} \underline{u}_t &= \sum_{j=1}^{n-1} \underline{\underline{B}}_j(\underline{u}, x') \cdot \underline{u}_{x_j} + \underline{c}(\underline{u}, x') \quad \text{on } B_r^n \\ \underline{u} &= 0 \quad \text{on } B_r^{n-1} \end{aligned} \tag{6}$$

where $\underline{u}_{x_j} = \partial u / \partial x_j$ etc. We assume that we are given the real analytic functions

$$\begin{aligned} \underline{\underline{B}}_j :& \mathbb{R}^m \times \mathbb{R}^{n-1} \to \mathrm{Mat}(m \times m) \\ \underline{c} :& \mathbb{R}^m \times \mathbb{R}^{n-1} \to \mathbb{R}^m \end{aligned}$$

(these functions do not have to defined on the entire space, but just have to be defined on $\mathbb{R}^n \times B_r^{n-1}$)
Note we assume $\underline{\underline{B}}_j$ and $\underline{u}$ do not depend explicitly on t. We can always introduce u^{m+1} satisfying $\partial_t u^{m+1} = 1$, $u^{m+1} = 0$ on B_r^{n-1} and extending the system.

We will write $\underline{\underline{B}}_j = ((b_j^{kl}))$ and $\underline{c} = (c^1, \cdots, c^m)^T$. Then in components (6) reads:

$$u_t^k = \sum_{j=1}^{n-1} \sum_{l=1}^{m} b_j^{kl}(\underline{u}, x') u_{x_j}^l + c^k(\underline{u}, x') \quad k = 1, \cdots, m$$

Examples : Take $m = 2$, write $\underline{u} = (f, g)^T$.

(a)

$$\begin{cases} f_t = g_x + F \\ g_t = f_x \end{cases}$$

together imply $f_{tt} - f_{xx} = F_t$

(b)

$$\begin{cases} f_t = -g_x + F \\ g_t = f_x \end{cases}$$

together imply $f_{tt} + f_{xx} = F_t$. (Note $F = 0$ gives Cauchy-Riemann equation)

Theorem) (Cauchy-Kovalevskaya) Assume $\{\underline{\underline{B}}_j\}_{j=1}^{n-1}$ and $\underline{c}$ are real analytic. Then for sufficiently small $r > 0$ there exists a unique real analytic function $\underline{u} : B_r^n \to \mathbb{R}^m$ solving the problem (6).

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(12th October, Friday)

Theorem) (Cauchy-Kovalevskaya) Assume $\{\underline{\underline{B}}_j\}_{j=1}^{n-1}$ and $\underline{c}$ are real analytic. Then for sufficiently small $r > 0$ there exists a unique real analytic function $\underline{u} : B_r^n \to \mathbb{R}^m$ solving the problem (6).

proof)

1. The strategy will be to write
$$\underline{u}(x) = \sum_\alpha \underline{u}_\alpha x^\alpha \tag{7}$$
and compute coefficients
$$\underline{u}_\alpha = \frac{D^\alpha \underline{u}(0)}{\alpha!}$$
in terms of $\underline{\underline{B}}_j$, $\underline{c}$ and show that the series (7) converges on B_r^n for r small enough.
2. As $\underline{\underline{B}}_j$ and $\underline{c}$ are real analytic, we can write
$$\begin{aligned} \underline{\underline{B}}_j(z, x') &= \sum_{\gamma,\delta} \underline{\underline{B}}_{j,\gamma,\delta} z^\gamma (x')^\delta \qquad \gamma \in \mathbb{N}^m, \delta \in \mathbb{N}^{n-1} \text{ multiindices} \\ \underline{c}(z, x') &= \sum_{\gamma,\delta} \underline{c}_{\gamma,\delta} z^\gamma (x')^\delta \end{aligned}$$
where these power series converge for $|z|^2 + |x'|^2 < s^2$, wlog $s > r$. Thus:
$$\begin{aligned} \underline{\underline{B}}_{j,\gamma,\delta} &= \frac{D_z^\delta D_{x'}^\delta \underline{\underline{B}}_j(0,0)}{\gamma!\delta!} \\ \underline{c}_{\gamma,\delta} &= \frac{D_z^\delta D_{x'}^\delta \underline{c}(0,0)}{\gamma!\delta!} \end{aligned} \tag{8}$$
3. Since $\underline{u} \equiv 0$ on $\{t = x^n = 0\}$, we have
$$\underline{u}_\alpha = \frac{D^\alpha \underline{u}(0)}{\alpha!} = 0$$
for all multi-indices α with $\alpha_n = 0$.
Now, we use the evolution equation (6) to deduce
$$\underline{u}_{x_n}(0) = \underline{u}_t(0) = \sum_{j=1}^{n-1} \underline{\underline{B}}_j(\underline{u}(0), 0)\underline{u}_{x_j}(0) + \underline{c}(\underline{u}(0), 0) = \underline{c}(0, 0)$$
Fix $i \in \{1, 2, \cdots, n-1\}$, differentiate (6) with respect to x^i :
$$\begin{aligned} \underline{u}_{tx_i} = \sum_{j=1}^{n-1} \Big[&\partial_{x_i} \underline{\underline{B}}_j(\underline{u}, x')\underline{u}_{x_j} + \big(\sum_{i=1}^m \partial_{z_i} \underline{\underline{B}}_j(\underline{u}, x') \frac{\partial u^i}{\partial x^j} \underline{u}_{x_j} \big) + \underline{\underline{B}}_j(\underline{u}, x')\underline{u}_{x_i x_j} \Big] \\ &+ \partial_{x_i} \underline{c}(\underline{u}, x') + \sum_{i=1}^m \partial_{z_l} \underline{c}(\underline{u}, x') \frac{\partial u^l}{\partial x^i} \\ \underline{u}_{tx_i}(0) = \partial_{x_i}&\underline{c}(0, 0) \end{aligned}$$
Iterating this, we deduce $D^\alpha \underline{u}(0) = D^\delta \underline{c}(\underline{0}, 0)$ where $\alpha = (\delta, 1)$.

4. Now, suppose $\alpha = (\delta, 2)$, for $\delta \in \mathbb{N}^{n-1}$. Then

$$\begin{aligned} D^\alpha u^k &= D^\delta(u^k_{x_n x_n}) = D^\delta(u^k_t)_t \\ &= D^\delta\Big(\sum_{j=1}^{n-1}\sum_{l=1}^{m} b^{kl}_j u^l_{x_j} + c^k\Big)_t \\ &= D^\delta\Big(\sum_{j=1}^{n-1}\sum_{i=1}^{m}\Big[b^{kl}_j u^l_{x_j t} + \sum_{p=1}^{m}(b^{kl}_j)_{z_p} u^l_{x_j} u^p_t\Big] + \sum_{p=1}^{m} c^k_{z_p} u^p_t\Big) \end{aligned}$$

so

$$D^\alpha u^k(0) = D^\alpha\Big(\sum_{j=1}^{n-1}\sum_{i=1}^{m} b^{kl}_j u^l_{x_j t} + \sum_{p=1}^{m} c^k_{z_p} u^p_t\Big)\Big|_{x=0,\underline{u}=0}$$

Now crucially, the expression on the right can be expanded to produce a polynomial with non-negative coefficients involving derivative of $\underline{\underline{B}}_j$ and $\underline{c}$, and derivatives $D^\beta \underline{u}$ where $\beta_n \leq 1$. More generally, for each multi-index α and each $k \in \{1, \cdots, n\}$, we can compute

$$D^\alpha u^k(0) = p^k_\alpha\Big(D^\alpha_z D^\delta_{x'} \underline{\underline{B}}_j, D^\alpha_z D^\delta_{x'} \underline{c}, D^\beta \underline{u}\Big)\big|_{x=0,\underline{u}=0}$$

where $\beta_n \leq \alpha_n - 1$ and p^k_α is some polynomial in its arguments with non-negative coefficients. Equivalently, for each α, k

$$u^k_\alpha = q^k_\alpha(\underline{\underline{B}}_{j,\alpha,\delta}, \underline{c}_{\gamma,\delta}, u_\beta)$$

where q^k_α is a polynomial with non-negative coefficients, with $\beta_n \leq \alpha_n - 1$.

5. We have shown that if a solution exists, we can compute all derivatives at 0 in terms of known quantities. We will construct a series which majorises the formal sum $\sum_\alpha u_\alpha x^\alpha$.

   First suppose

$$\underline{\underline{B}}^*_j \gg \underline{\underline{B}}_j \quad \underline{c}^* \gg \underline{c}$$

where

$$\begin{aligned} \underline{\underline{B}}^*_j &= \sum_{\gamma,\delta} \underline{\underline{B}}^*_{j,\gamma,\delta} z^\gamma (x')^\delta \\ \underline{c}^* &= \sum_{\gamma,\delta} \underline{c}^*_{\gamma,\delta} z^\gamma (x')^\delta \end{aligned}$$

Assume these converge for $|z|^2 + |x'|^2 < s^2$ (decrease s if necessary). For all j, γ, δ, k, l,

$$0 \leq |B^{kl}_{j,\gamma,\delta}| \leq (B^*)^{kl}_{j,\gamma,\delta}, \quad 0 \leq |c^k_{\gamma,\delta}| \leq (c^*)^{kl}_{\gamma,\delta}$$

We consider the modified problem:

$$\begin{aligned} &\underline{u}^*_t = \sum_{j=1}^{n-1} \underline{\underline{B}}^*_j(\underline{u}^*, x')\underline{u}^*_{x_j} + \underline{c}^*(\underline{u}^*, x') \quad \text{for } |x| < r \\ &\underline{u}^* = \underline{0} \quad \text{on } B^{n-1}_r \end{aligned}$$

As above, seek a real analytic solution

$$\underline{u}^* = \sum_\alpha \underline{u}^*_\alpha x^\alpha \quad \text{where } \underline{u}^*_\alpha = \frac{D^\alpha \underline{u}(0)}{\alpha!}$$

6. We claim $0 \leq |u^k_\alpha| \leq (u^*)^k_\alpha$ for all $\alpha \in \mathbb{N}^n$.

   We do this by proof by induction on α_n.

   For $\alpha_n = 0$, $u^*_\alpha = u_\alpha = 0$

For the induction step: (for $\beta_\alpha \leq \alpha_n - 1$)

$$\begin{aligned}|u^k_\alpha| &= |q^k_\alpha(\underline{\underline{B}}_{j,\gamma,\delta}, \underline{c}_{\gamma,\delta}, \underline{u}_\beta)| \\ &\leq q^k_\alpha(|B^{kl}_{j,\gamma,\delta}|, |C^k_{\gamma,\delta}|, |u^k_\beta|) \\ &\leq q^k_\alpha((B^*)^{kl}_{j,\gamma,\delta}, (c^*)^k_{\gamma,\delta}, (u^*)^k_\beta) \\ &= (u*)^k_\alpha\end{aligned}$$

Using positivity of coefficients of q and induction assumption. Thus $\underline{u}^* \gg \underline{u}$. Remains to show we can find $\underline{\underline{B}}^*_j$, $\underline{c}^*$ s.t. a solution $\underline{u}^*$ exists and converges near 0.

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(15th October, Monday)

Last lecture :

- a formal power series solution $\underline{u} = \sum_\alpha \underline{u}_\alpha x^\alpha$ exists.
- If $\underline{\underline{B}}^*_j \gg \underline{\underline{B}}_j$, $\underline{c}^* \gg \underline{c}$ and $\underline{u}^*$ satisfies

$$\begin{aligned}&\underline{u}^*_t = \sum_{j=1}^{n-1} \underline{\underline{B}}^*_j(\underline{u}^*, x')\underline{u}^*_{x_j} + \underline{c}^*(\underline{u}^*, x') \quad \text{for } |x| < r \\ &\underline{u}^* = \underline{0} \quad \text{on } B^{n-1}_r\end{aligned}$$

then the power series for $\underline{u}^* = \sum_\alpha \underline{u}^*_\alpha x^\alpha$.

proof, continued) To complete the proof, it suffices to show that for any $\underline{\underline{B}}_j$, $\underline{c}$, we can find $\underline{\underline{B}}^*_j$, $\underline{c}^*_j$ such that the corresponding $\underline{u}^*_j$ is a convergent series.

We make a particular choice for $\underline{\underline{B}}^*_j$, $\underline{c}^*$. For this we recall from an earlier lemma that

$$\begin{aligned}\underline{\underline{B}}^*_j &= \frac{Cr}{r - (x_1 + \cdots + x_{n-1}) - (z_1 + \cdots + z_m)} \begin{pmatrix} 1 & 1 & \cdots 1 \\ \vdots & & \ddots \vdots \\ 1 & 1 & \cdots 1 \end{pmatrix} \\ \underline{c}^* &= \frac{Cr}{r - (x_1 + \cdots + x_{n-1}) - (z_1 + \cdots + z_m)} (1, \cdots, 1)^T\end{aligned}$$

will majorise $\underline{\underline{B}}_j$, $\underline{c}$, provided C is large enough, r is small enough and $\underline{\underline{B}}^*_j$, $\underline{c}^*$ are given by convergent series for $|x'|^2 + |z|^2 < r^2$. With these choices of majorants, the modified equation takes the form :

$$\begin{aligned}(u^*)^k_t &= \frac{Cr}{r - (x_1 + \cdots + x_{n-1}) - ((u^*)^1 + \cdots + (u^*)^m)} (\sum_{j,l} (u^*)^l_{x_j} + 1) \quad \text{for } |x'|^2 + t^2 < r^2 \\ u^* &= 0 \qquad\qquad \text{for } t = 0, |x'| < r\end{aligned}$$

This problem has an explicit solution.

$$\underline{u}^* = v^*(1, \cdots, 1)^T$$

where

$$v^* = \frac{1}{mn}\Big(r - (x_1 + \cdots + x_{n-1}) - \sqrt{(r - (x_1 + \cdots + x_{n-1}))^2 - 2nmCrt}\Big)$$

(*Check this is indeed the solution!!*) v^* is real analytic for $|x'|^2 + t^2 < r^2$, provided r is small enough. Hence $\underline{u}^*$ is given by a convergent series since $\underline{u}^* \gg \underline{u}$. Our formal power series for $\underline{u}$ converges.

Initial condition hold for $\underline{u}$ since

$$\underline{u}_\alpha = \underline{0} \quad \text{if} \quad \alpha_n = 0$$

Moreover, the functions $\underline{u}_t$ and $\sum_{j=1}^{n-1} \underline{\underline{B}}_j(\underline{u}, x')\underline{u}_{x_j} + \underline{c}(\underline{u}, x')$ are both real analytic near 0 and by construction, have the same Taylor expansion. Hence they must agree on a neighbourhood of 0, so the equation holds in some ball about 0.

(End of proof) □

Reduction to a First Order System

Example)

Consider the PDE problem for $u : \mathbb{R}^3 \to \mathbb{R}$

$$\begin{aligned} u_{tt} &= uu_{xy} - u_{xx} + u_t \\ u\big|_{t=0} &= u_0 \\ u_t\big|_{t=0} &= u_1 \end{aligned} \tag{9}$$

where $u_0, u_1 : \mathbb{R}^2 \to \mathbb{R}$ are given real analytic functions (near 0).
First note that $f = u_0 + tu_1$ is analytic in a neighbourhood of $0 \in \mathbb{R}^3$ and $f\big|_{t=0} = u_0$, $f_t\big|_{t=0} = u_1$.
Set $w = u - f$, then

$$\begin{aligned} w_{tt} &= ww_{xy} - w_{xx} + w_t + fw_{xy} + f_{xy}w + F \\ w\big|_{t=0} &= w_t\big|_{t=0} = 0 \end{aligned}$$

where $F = ff_{xy} - f_{xx} + f_t - f_{tt}$.
Let $(x, y, t) = (x^1, x^2, x^3)$ and set $\underline{u} = (w, w_x, w_y, w_t) = (u^1, u^2, u^3, u^4)$. Then

$$\begin{aligned} u^1_{x^3} &= w_t = u^4 \\ u^2_{x^3} &= w_{xt} = u^4_{x_1} \\ u^3_{x^3} &= w_{yt} = u^4_{x_2} \\ u^4_{x^3} &= w_{tt} = u^1 u^2_{x^2} - u^2_{x^1} + u^4 + fu^2_{x^2} + f_{xy}u^1 + F \end{aligned}$$

Now, defining:

$$\underline{\underline{B}}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \underline{\underline{B}}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ u_1 + f & 0 & 0 & 0 \end{pmatrix}$$

$$\underline{c} = (u^4, 0, 0, u^4 + f_{xy}u^1 + F)^T$$

The system of equations is in the form

$$\underline{u}_{x_2} = \sum_{j=1} 2\underline{\underline{B}}_j \underline{u}_{x_j} + \underline{c}$$

where $\underline{\underline{B}}_j$, $\underline{c}$ are real analytic near 0. By Cauchy-Kovalevskaya, a real analytic solution to (9) exists near 0.

Note : this procedure relied on

(a) being able to solve for u_{tt},

(b) u_{tt} depending on at most two derivatives of u (in a quasilinear fashion)

More generally, suppose we wish to solve the quasilinear problem :

$$\begin{aligned} &\sum_{|\alpha|=k} a_\alpha(D^{k-1}u, \cdots, u, x)D^\alpha u + a_0(D^{k-1}u, \cdots, u, x) = 0 \quad \text{for } |x| < r \\ &u = \frac{\partial u}{\partial x_n} = \cdots = \frac{\partial^{k-1} u}{\partial x_n^{k-1}} = 0 \qquad \text{for } |x'| < r,\ x_n = 0 \end{aligned}$$

called a **Cauchy problem**.
We introduce

$$\underline{u} = (u, \frac{\partial u}{\partial x_n}, \cdots, D^\alpha u, \cdots)_{|\alpha| \le k-1} = (u^1, \cdots, u^m)$$

$\underline{u}$ contains all derivative of u up to order $k-1$. Wlog, (by changing the order if necessary) put $u^m = \partial^{k-1}u/\partial x_n^{k-1}$. For $j < m$, we can compute $\partial u^j/\partial x^n$ in terms of $\partial u^l/\partial x^p$ for some $l \in \{1, \cdots, m\}$ and $p < n$.

To computed $\partial u^m/\partial x_n$ we need to use the equation. Suppose that

$$a_{(0,\cdots,0,k)}(0,\cdots,0) \neq 0$$

Then we can write the equation as :

$$\frac{\partial^k u}{\partial x_n^k} = \frac{-1}{a_{(0,\cdots,k)}(D^{k-1}u,\cdots,u,x)}\Big[\sum_{|\alpha|=k,\alpha_n<k} a_\alpha D^\alpha u + a_0\Big]$$

Assuming a_α are real analytic, the denominator will be non-zero near the origin. The RHS can be written in terms of $\frac{\partial u^l}{\partial x^p}$ for $p < n$ and $\underline{u}$. We see we can write the equation as a first ordered system for $\underline{u}$, *provided*(this condition is important! would come back to this later)

$$a_{(0,\cdots,k)}(0,\cdots,0) \neq 0 \quad \text{(non-characteristic condition)}$$

In this case we can apply Cauchy-Kovalevskaya.

==
(17th October, Wednesday)

(Problem sheet 1 handed out. Example classes sign-up. First example class (probably) at Thur/Fri next week)

Cauchy Problems for Quasilinear Equations with Data on a Surface

We say $\Sigma \subset \mathbb{R}^n$ is a real analytic **hypersurface** near $x \in \Sigma$ if there exists $\epsilon > 0$ and a real analytic map $\Phi : B_\epsilon(x) \to U \subset \mathbb{R}^n$ where $U = \Phi(B_\epsilon(x))$ such that

- Φ is bijective, and the inverse $\Phi^{-1} : U \to B_\epsilon(x)$ is real analytic.
- $\Phi(\Sigma \cap B_\epsilon(x)) = \{x_n = 0\} \cap U$.

We think of Φ as 'straightening out the boundary'.

There are many examples, e.g. $\{|x| = 1\}$.

Let γ be the unit normal to Σ and suppose u solves

$$\sum_{|\alpha|=k} a_\alpha(D^{k-1}u,\cdots,u,x)D^\alpha u + a_0(D^{k-1}u,\cdots,u,x) = 0 \quad \text{in } B_\epsilon(x) \tag{10}$$
$$u = \gamma^i\partial_i u = \cdots = (\gamma^i\partial_i)^{k-1} = 0 \quad \text{on } \Sigma$$

(note that the boundary condition is equivalent to having $D^\alpha u = 0$ for all $|\alpha| < k$ on Σ.)

Define $v(y) = u(\Phi(y)) \quad \Leftrightarrow \quad u(x) = v(\Phi^{-1}(x))$. Note

$$\frac{\partial v}{\partial x^i} = \frac{\partial u}{\partial y^j}\frac{\partial \Phi^j}{\partial x^i}$$
$$\frac{\partial^2 v}{\partial x^i \partial x^k} = \frac{\partial u^2}{\partial y^j \partial y^i}\frac{\partial \Phi^j}{\partial x^i}\frac{\partial \Phi^l}{\partial x^k} + \frac{\partial u}{\partial y^j}\frac{\partial^2 \Phi^j}{\partial^i \partial x^k} \quad \text{etc.}$$

So we can compute $D^\alpha u$ as a linear combination of $D^\beta v$ for $|\beta| \leq |\alpha|$, with coefficients depending on Φ. So if u solves (10), then v will solve

$$\sum_{|\alpha|=k} b_\alpha(D^{k-1}v,\cdots,v,x)D^\alpha v + b_0(D^{k-1}v,\cdots,v,x) = 0$$

Moreover,

$$v\big|_{x_n=0} = u\big|_\Sigma = 0$$
$$\partial_i v\big|_{x_n=0} = (D\Phi)_{ij}\partial_j u\big|_\Sigma = 0$$

and proceeding similarly for $\partial^{k-1}v/(\partial x^n)^{k-1}$, we have each $D^\beta v$ for $|\beta| < k$ as a linear combination of $D^\alpha u(|\alpha| < k)$ and hence $D^\beta v = 0$ for each $|\alpha| < k$. Hence, we have (check that this is an equivalent condition)

$$v = \frac{\partial v}{\partial x^n} = \cdots = \partial^{k-1}v/(\partial x^n)^{k-1} = 0 \quad \text{on } \{x_n = 0\}$$

We can solve this, provided

$$b_{(0,\cdots,0,k)}(0,0,\cdots,0,y) \neq 0 \quad \text{on } \{x_n = 0\}$$

Note if $|\alpha| = k$,

$$D^\alpha u = \frac{\partial^k v}{\partial y_n^k}(D\Phi^n)^\alpha + (\text{terms not involving } \frac{\partial^k v}{\partial y_n^k})$$

So the coefficient of $\partial^k v/\partial y_n^k$ in

$$\sum_{|\alpha|=k} a_\alpha(D^{k-1}u,\cdots,u,x)D^\alpha u + a_0(D^{n-1}u,\cdots,u,x) = 0$$

is

$$b_{(0,\cdots,k)} = a_\alpha(D\Phi^n)^\alpha$$

But $\Sigma = \{\Phi^n = 0\}$ so $D\Phi^n \propto \gamma$. Therefore,

$$b_{(0,\cdots,k)} \neq 0 \quad \Leftrightarrow \quad \sum_{|\alpha|=k} a_\alpha(D\Phi^n)^\alpha \neq 0 \quad \Leftrightarrow \quad \sum_{|\alpha|=k} a_\alpha\gamma^\alpha \neq 0$$

Definition) Σ is a **non-characteristic** at $x \in \Sigma$ for the problem (10) provided

$$\sum_{|\alpha|=k} a_\alpha(0,\cdots,0,x)\gamma^\alpha(x) \neq 0$$

Finally, we have a more general version of Cauchy-Kovalevskaya.

Theorem) (Cauchy-Kovalevskaya Redux) Suppose $\Sigma \subset \mathbb{R}^n$ is a real analytic hypersurface. If Σ is non-characteristic for (10) at $x \in \Sigma$, there exists a unique real analytic solution to (10) in a neighbourhood of x.

proof) We have already seen that we can solve the problem for v uniquely, then $u(x) = v(\Phi(x))$ is the unique solution for (10)

(End of proof) $\square$

Characteristic Surfaces for 2nd Order Linear PDE

Consider the linear operator

$$Lu = \sum_{i,j=1}^n a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i\frac{\partial u}{\partial x^i} + cu$$

with $a_{ij}, b_i, c : \mathbb{R}^n \to \mathbb{R}$.

Consider the Cauchy problem

$$\begin{aligned} &Lu = f \\ &u = \sum_{i=1}^n \xi^i\frac{\partial u}{\partial x^i} = 0 \quad \text{on } \Pi_\xi = \{\xi \cdot x = 0\} \end{aligned}$$

Π_ξ is characteristic at $x \in \mathbb{R}^n$ if :

$$\sigma_p(\xi,x) = \sum_{i,j=1}^n a_{ij}\xi^i\xi^j = 0$$

σ_p is the **principal symbol** of L.

- If $\sigma_p(\xi, x) > 0$ for all x, $\xi \neq 0$, then no plane is characteristic, and such operations are called **elliptic**.

Let us restrict to the case where a_{ij}, b_i, c are constants. Suppose $b_i = c = 0$ and Π_ξ is characteristic. Then

$$u(x) = e^{i\lambda\xi\cdot x}$$

solve $Lu = 0$ for any λ. By taking λ large, we can construct solutions to $Lu = 0$ whose derivative (in the ξ direction) is a large as we like. In particular, Lu is very regular, but u need not be. In the elliptic setting, this cannot happen.

==
(19th October, Friday)

Criticisms/Shortcomings of Cauchy-Kovalevskaya

1. Real analyticity is (sometimes) too strong a condition. For example, if solutions of Maxwell's equations were required to be real analytic. We'd know electro-magnetic field everywhere if we could measure it in some small set. This is absurd.

2. We don't necessarily get continuous dependence on data in the form we would like.

   Example : Consider Laplace's equation on $\mathbb{R}^2$. $u_{xx} + u_{yy} = 0$, with Cauchy data $u(x,0) = \cos(kx)$, $u_y(x,0) = 0$. This has a real analytic solution

   $$u(x,y) = \cos(kx)\cosh(ky)$$

   We can check that $\sup_{x\in\mathbb{R}} |u(x,0) \leq 1|$ but $\sup_{x\in\mathbb{R}} |u(x,\epsilon)| \to \infty$ as $k \to \infty$ for all $\epsilon > 0$.

   In fact, we can require as many derivatives of u on $\{y = 0\}$ to be bounded and we can still find solutions which are arbitrarily large at $y = \epsilon$. This Cauchy problem is *not* well posed in C^k, as there is no continuous dependence on data.

These suggest the Cauchy problem for Laplace's equation is not the natural one to consider.

Elliptic Boundary Value Problems

A more natural problem arising in physics is the **Dirichlet Problem** :

$$\begin{aligned} \Delta u &= 0 \quad \text{in } U \subset \mathbb{R}^n, \quad U \text{ open, bounded} \\ u &= g \quad \text{on } \partial U \end{aligned}$$

e.g. u is electrostatic potential in a cavity whose walls are held at voltage g.

We shall develop methods to solve such problems. First we develop some technology.

Hölder and Sobolev Spaces

We need to discuss various function spaces in which to seek solutions our PDEs.

Hölder spaces

Suppose $U \subset \mathbb{R}^n$ is open. We write $u \in C^k(U)$ if $u : U \to \mathbb{R}$ is k-times differentiable at each $x \in U$ and D^α is continuous on U for all $|\alpha| \leq k$. This is not a Banach space, so we would like to restrict to a smaller complete space with a norm.

We say $u \in C^k(\overline{U})$ if $u \in C^k(U)$ and $D^\alpha u$ is uniformly continuous and bounded on U for each $|\alpha| \leq k$. We introduce a norm :

$$\|u\|_{C^k(\overline{U})} = \sum_{|\alpha|\leq k} \sup_{x\in U} |D^\alpha u(x)|$$

With this norm, $C^k(\overline{U})$ is a Banach space. (*Be aware, that $C^k(\overline{U})$ seems to be constructed from the closure $\overline{U}$, but this is not true. It is constructed from U and just depends on U. This matters when U does not have a nice boundary e.g. if U is a complement of the Cantor set$\cap[0,1]$, then $C^k(\overline{U}) \neq C^k \cap [0,1]$).

For $0 < \gamma \leq 1$, we say that u is **Hölder continuous with exponent** γ if there exists a constant $C \geq 0$ s.t.

$$|u(x) - u(y)| \leq C|x-y|^\gamma \quad \forall x, y \in U$$

We define γ^{th} **Hölder seminorm** by

$$[u]_{C^{0,\gamma}(\overline{U})} = \sup_{x,y \in U} \frac{|u(x) - u(y)|}{|x-y|^\gamma}$$

We say $u \in C^{k,\gamma}(\overline{U})$ if $u \in C^k(\overline{U})$ and $D^\alpha u$ is Hölder continuous, with exponent γ for all $|\alpha| = k$. We define a norm :

$$\|u\|_{C^{k,\gamma}(\overline{U})} = \|u\|_{C^k(\overline{U})} + \sum_{|\alpha|=k} [D^\alpha u]_{C^{0,\gamma}(\overline{U})}$$

This is again a Banach space.

The Spaces $L^p(U)$, $L^p_{\text{loc}}(U)$

For $U \subset \mathbb{R}^n$ open, suppose $1 \leq p < \infty$. We define the space $L^p(U)$ by

$$L^p(U) = \{u : U \to \mathbb{R} \text{ measurable } : \|u\|_{L^p(U)} < \infty\} / \sim$$

where

$$\|u\|_{L^p(U)} = \begin{cases} \left(\int_U |u(x)|^p dx \right)^{1/p} & \text{for } 1 \leq p < \infty \\ \\ \text{assup}_x u(x) = \inf\{C \geq 0 : |u(x)| \leq C \text{ for almost every } x\} \end{cases}$$

and the $\sim$ is an equivalence relation defined by $u_1 \sim u_2$ *if and only if* $u_1 = u_2$ almost everywhere.
$L^p(U)$ is a Banach space with norm $\|\cdot\|_{L^p(U)}$. Completeness follows from dominated convergence theorem.

We define a local versions of $L^p(u)$: we say $u \in L^p_{\text{loc}}(U)$ if $u \in L^p(V)$ for every $V \subset\subset U$ should be read **'V is compactly contained in U'**, meaning there exists a compact K s.t. $V \subset K \subset U$. Note $L^p_{\text{loc}}(U)$ is *not* a Banach space.(it is a Fréchet space)

Weak Derivatives

We would like a notion of differentiability for L^p functions. Since L^p functions like to be integrated, it makes sense to seek a definition involving integration.

Definition) Suppose $u, v \in L^1_{\text{loc}}(U)$ and α is a multi-index. We say v is a $\alpha^{\textbf{th}}$ **weak derivative** of u if

$$(-1)^{|\alpha|} \int_U u D^\alpha \phi dx = \int_U v \phi dx \quad \forall \phi \in C_c^\infty(U)$$

In other words, u, v obey the correct integration of parts formula, when integrated against a test function $\phi \in C_c^\infty(U)$.

$\star$ Check that if $D^\alpha u = v$, then v is indeed also a weak derivative of u.

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(22nd October, Monday)

(Example Classes : Group A - Thurs 2:00 - 3:30 pm, Group B - Thurs 4:00 - 5:30 pm, MR5)
(I am Group B)
(Submission in the problem class)
(Also a handout distributed - I'll try to add them in my notes!)

Last lecture : Suppose $u, v \in L^1_{\text{loc}}(U)$ and α is a multi-index. We say v is a $\alpha^{\textbf{th}}$ **weak derivative** of u if

$$(-1)^{|\alpha|} \int_U u D^\alpha \phi dx = \int_U v \phi dx \quad \forall \phi \in C_c^\infty(U)$$

In other words, u, v obey the correct integration of parts formula, when integrated against a test function $\phi \in C_c^\infty(U) = \{u \in C^k(U) \;\; \forall k = 1, 2, \cdots | \mathrm{supp}(U) \subset\subset U\}$.

If $u \in C^k(U)<$ then $D^\alpha u$ is a weak α-derivative of u for all $|\alpha| \le k$ (use integration by part for proof)

Lemma) Suppose $v, \tilde{v} \in L^1_{\mathrm{loc}}(U)$ are both weak α-derivatives of $u \in L^1_{\mathrm{loc}}(U)$. Then $v = \tilde{v}$ almost everywhere.

proof)

$$(-1)^{|\alpha|} \int_U u D^\alpha \phi dx = \int_U v \phi dx = \int_U \tilde{v} \phi dx$$

by the definition of v and $\tilde{v}$ being α-derivatives. Then

$$\int_U (v - \tilde{v}) \phi dx = 0 \quad \forall \phi \in C_c^\infty(Y) \quad \Rightarrow \quad v = \tilde{v} \quad \text{a.e.}$$

(End of proof) □

Since weak derivative is unique, we denote the α^{th} weak derivative of u by $D^\alpha u$.

Definition)

- We say $u \in L^1_{\mathrm{loc}}(U)$ belongs to the **Sobolev space** $W^{k,p}(U)$ (k is the number of derivatives we want and p is the exponent of L^p space we are working on) if $u \in L^p(U)$ and the weak derivative $D^\alpha u$ exist for all $|\alpha| \le k$ and $D^\alpha \in L^p(U)$.
- If $p = 2$ we write $H^k(U) = W^{k,2}(U)$.
- We define the $W^{k,p}$ norm by

$$\|u\|_{W^{k,p}(U)} = \begin{cases} \left(\sum_{|\alpha| \le k} \int_U |D^\alpha u|^p dx\right)^{1/p} & 1 \le p < \infty \\ \sum_{|\alpha| \le k} \|D^\alpha u\|_{L^\infty(U)} & p = \infty \end{cases}$$

  (there are various equivalent ways of defining the norm)

- We denote by $W_0^{k,p}(U)$ the completion of $C_c^\infty(U)$ in the $W^{k,p}(U)$-norm.

We will find out that these spaces will be useful in fining solutions of PDEs. In particular, the H^k spaces will be useful.

Example : Let $U = B_1(0) = \{|x| < 1\} \subset \mathbb{R}^n$. Set $u(x) = |x|^{-\lambda}$ for $x \in U \backslash \{0\}$ and $\lambda > 0$. This diverges at $x = 0$, so this is not a $C^k(U)$ function.

Note for $x \neq 0$, $D_i u = -\lambda x_i / |x|^{\lambda+2}$. By considering test functions $\phi \in C_c^\infty(U \backslash \{0\})$, if u is weakly differentiable, then the weak derivative must agree with this for $x \neq 0$. We can check $U \in L^1(U)$ if

$$\infty > \int_U |u| dx = \int_{B_1(0)} |x|^{-\lambda} dx = \omega_{n-1} \int_0^1 r^{-\lambda} r^{n-1} dr$$

where ω_{n-1} is the area of S^{n-1}. The integral is finite if $n - 1 - \lambda > -1$, or equivalently λn.

A similar computation gives $-\lambda x_i / |x|^{\lambda+2} \in L^1(U)$, and equivalently $\lambda + 1 < n$.

Now we take $\lambda < n - 1$. Now, suppose $\phi \in C_c^\infty(U)$.

$$-\int_{U \backslash B_\epsilon(0)} u \phi_{x_i} dx = \int_{U \backslash B_\epsilon(0)} u_{x_i} \phi dx - \int_{\partial B_\epsilon(0)} u \phi \nu^i dS$$

where $\underline{\nu} = (\nu^1, \cdots, \nu^n)$ is the inward normal to $\partial B_\epsilon(0)$. Then integration by parts is justified since u is smooth on $U \backslash B_\epsilon(0)$. We estimate

$$\begin{aligned} &\Big| \int_{\partial B_\epsilon(0)} u \phi \nu^i dS \Big| \le \int_{\partial B_\epsilon(0)} |u \phi| dS \\ \le &\|\phi\|_{L^\infty(U)} \int_{\partial B_\epsilon(0)} |u| dS = \|\phi\|_{L^\infty(U)} \int_{\partial B_\epsilon(0)} \epsilon^{-d} dS \\ = &\omega_{n-1} \|\phi\|_{L^\infty(U)} \epsilon^{n+1-\lambda} \xrightarrow{\epsilon \to 0} 0 \quad \text{for } \lambda < n - 1 \end{aligned}$$

Thus if $\lambda < n-1$, $|u|^\lambda$ has a i^{th} weak derivative, equal to $-\lambda x_i/|x|^{\lambda+1}$. Moreover, $|Du| = -\lambda/|x|^{\lambda+1} \in L^p(U)$ *if and only if* $p(\lambda+1) < n$. Also, $u \in L^p(U)$ *if and only if* $p\lambda < n$.
$\therefore u \in W^{1,p}(U) \quad \Leftrightarrow \quad \lambda < \frac{n-p}{p}$.

Notice that if $p > n$, we don't have an example of with $\lambda > 0$ (look back once we've done some Sobolev embeddings).

Theorem) For each $k = 1, 2, \cdots$ and $1 \leq p \leq \infty$. Then the space $W^{k,p}(U)$ is a Banach space.

We need some nice properties of weak derivatives, e.g. linearity, but we differ them to the example sheets.

proof) We just prove the $p < \infty$ case here.

1. Homogeneity and positivity of $\|\cdot\|_{W^{k,p}(U)}$ are obvious. To prove triangular inequality, recall Minkowski's inequality :

$$\Big(\sum_{i=1}^n |a_i + b_i|^p\Big)^{1/p} \leq \Big(\sum_{i=1}^n |a_i|^p\Big)^{1/p} + \Big(\sum_{i=1}^n |b_i|^p\Big)^{1/p}$$

We compute

$$\begin{aligned}
\|u+v\|_{W^{k,p}(U)} &= \Big(\sum_{|\alpha|\leq k} \|D^\alpha u + D^\alpha v\|^p_{L^p(U)}\Big)^{1/p} \\
&\leq \Big(\sum_{|\alpha|\leq k} (\|D^\alpha u\|_{L^p(U)} + \|D^\alpha v\|_{L^p(U)})^p\Big)^{1/p} \\
&\leq \Big(\sum_{|\alpha|\leq k} \|D^\alpha u\|^p_{L^p(U)}\Big)^{1/p} + \Big(\sum_{|\alpha|\leq ks} \|D^\alpha v\|^p_{L^p(U)}\Big)^{1/p} \\
&= \|u\|_{W^{k,p}(U)} + \|v\|_{W^{k,p}(U)}
\end{aligned}$$

2. For completeness, we note :

$$\|D^\alpha u\|_{L^p(U)} \leq \|u\|_{W^{k,p}(U)} \quad |\alpha| \leq k$$

If $(u_l)_l$ is a Cauchy sequence in $W^{k,p}(U)$, then

$$\|D^\alpha(u_l - u_m)\|_{L^p(U)} \leq \|u_l - u_m\|_{W^{k,p}(U)}$$

and hence $(D^\alpha u_l)_l$ is a Cauchy sequence in $L^p(U)$ for all $|\alpha| \leq k$. By completeness of $L^p(U)$, we may find $u^\alpha \in L^p(U)$ such that $D^\alpha u_l \to u^\alpha$ in $L^p(U)$ for each $|\alpha| \leq k$. In particular, let $u = u^{(0,\cdots,0)}$

Claim : $u^\alpha = D^\alpha u$ for each $|\alpha| \leq k$.
proof) To see this, let $\phi \in C_c^\infty(U)$. Then

$$(-1)^{|\alpha|} \int_U u_l D^\alpha \phi dx = \int_U D^\alpha u_l \phi dx$$

Sending $l \to \infty$, we have $D^\alpha u_l \to u^\alpha$, $u_l \to u$ in $L^p(U)$. Therefore,

$$(-1) \int_U u D^\alpha \phi dx = \int_U u^\alpha \phi dx$$

and $D^\alpha u = u^\alpha \in L^p(U)$ so $u \in W^{k,p}(U)$ and $u_l \to u$ in $W^{k,p}(U)$

(End of proof) $\square$

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(24th October, Wednesday)

A point from last time

- If $\phi \in C_c^\infty(U)$ and $u_l \to u$ in $L^p(U)$ then
$$\int_U u_l \phi dx \to \int_U u\phi dx$$
This can be shown using Hölder inequality, e.g. $|\int_U (u_l - u)\phi dx| \leq \left\|u_l - u\right\|_{L^p} \left\|\phi\right\|_{L^q} \to 0$ as $l \to \infty$.

Approximation of Functions in Sobolev Spaces

It is often useful to know that we can approximate a certain object, e.g. a function, by some nicer object. In particular, for functions in Sobolev spaces, it is useful to be able to approximate by classically differentiable functions. We will show this is possible, with some assumptions on U.

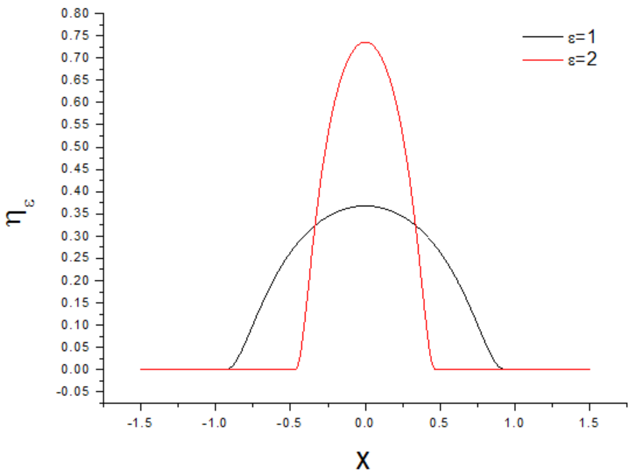
Convolution and Smoothing

A useful trick to smooth a function is **convolution with a smooth mollifier**.

Definition)

(i) Let
$$\eta(x) = \begin{cases} ce^{\frac{1}{|x|^2-1}} & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$
where c is chosen such that $\int_{\mathbb{R}^n} \eta(x)dx = 1$.

(ii) For each $\epsilon > 0$, set
$$\eta_\epsilon(x) = \frac{1}{\epsilon^n}\eta(\epsilon x)$$
the standard mollifier.



We have that $\eta_\epsilon(x) \in C^\infty(\mathbb{R}^n)$ and $\text{supp}(\eta_\epsilon) \subset B_\epsilon(0)$.

We let $U_\epsilon = \{x \in U | \text{dist}(x, \partial U) > \epsilon\}$.

Definition) If $f : U \to \mathbb{R}^n$ is locally integrable. i.e. $f \in L^1_{\text{loc}}(U)$. We define its ϵ-mollification $f^\epsilon : U_\epsilon \to \mathbb{R}$ by $f^\epsilon = \eta_\epsilon * f$, i.e.
$$f^\epsilon(x) = \int_U \eta_\epsilon(x-y)f(y)dy$$

Think of this as the average of f in a neighbourhood of x, weighted by η_ϵ shifted to have its peak at x.

Theorem) (Properties of Mollifiers)

(i) $f^\epsilon \in C^\infty(U_\epsilon)$ and $D^\alpha f^\epsilon = \int_U D_x^\alpha \eta_\epsilon(x-y) f(y)dy$.

(ii) $f^\epsilon \to f$ almost everywhere as $\epsilon \to 0$.

(iii) If $f \in C^0(U)$, then $f^\epsilon \to f$ uniformly on compact subsets of U.

(iv) If $1 \le p < \infty$ and $f \in L^p_{\text{loc}}(U)$ then $f^\epsilon \to f$ in $L^p_{\text{loc}}(U)$, i.e.

$$\left\|f^\epsilon - f\right\|_{L^p(V)} \to 0 \quad \forall V \subset\subset U$$

proof) See handout

Lemma) Assume $u \in W^{k,p}(U)$ for some $1 \le p < \infty$. Set $u^\epsilon = \eta_\epsilon * u$ in U_ϵ. Then

(i) $u^\epsilon \in C^\infty(U_\epsilon) \quad \forall \epsilon > 0$

(ii) If $V \subset\subset U$, then $u^\epsilon \to u$ in $W^{k,p}(V)$

proof) For (i), see handout.

For (ii), we claim $D^\alpha u^\epsilon = \eta_\epsilon * D^\alpha u$ for $|\alpha| \le k$ in U_ϵ, i.e. the derivative of u^ϵ is the ϵ-mollifier of the derivative of u, i.e. derivatives commute with convolutions. To see this,

$$\begin{aligned}
D^\alpha u^\epsilon(x) &= D^\alpha \int_U \eta_\epsilon(x-y)u(u)dy \\
&= \int_U D_x^\alpha \eta_x(x-y)u(y)dy \\
&= (-1)^{|\alpha|} \int_U D_y^\alpha \eta_x(x-y)u(y)dy \\
&= \int_U \eta_\epsilon(x-y) D^\alpha u(y)dy \qquad \text{(by definition of weak derivatives)} \\
&= \eta_\epsilon * D^\alpha u(x)
\end{aligned}$$

Now, fix $V \subset\subset U$. By previous theorem, $\eta_\epsilon * D^\alpha u \to D^\alpha u$ in $L^p_{\text{loc}}(U)$. So $D^\alpha u^\epsilon \to D^\alpha u$. in $L^p_{\text{loc}}(U)$ for all $|\alpha| \le k$. Thus :

$$\left\|u^\epsilon - u\right\|^p_{W^{k,p}(V)} = \sum_{|\alpha| \le k} \left\|D^\alpha u^\epsilon - D^\alpha u\right\|^p_{L^p(U)} \to 0$$

so indeed $u^\epsilon \to u$ in $W^{k,p}(U)$.

(End of proof) □

This tells us we can approximate u in the interior of U by smooth functions. We can do better :

Theorem) (Global approximation by smooth functions) Suppose $U \subset \mathbb{R}^n$ is open and *bounded*, and suppose $u \in W^{k,p}(U)$ for some $1 \le p < \infty$. Then there exists functions $u_n \in C^\infty(U) \cap W^{k,p}(U)$ such that

$$u_n \to u \quad \text{in } W^{k,p}(U)$$

Note, we do not assert $u_n \in C^\infty(\overline{U})$.

proof)

1. We have $U = \bigcup_{i=1}^\infty U_i$, where $U_i = \{x \in U : \text{dist}(x, \partial U) > 1/i\}$ for $i = 1, 2, \cdots$. Write $V_i = U_{i+3} \backslash \overline{U}_{i+1}$ and choose V_0 that is compact in U, so that we have $U = \bigcup_{i=1}^\infty V_i$. Let $\{\xi_i\}_{i=0}^\infty$ be a *partition of unity subordinate* to $\{V_i\}$ so that

$$\begin{cases} \xi_i \in C_c^\infty(V_i) \\ 0 \le \xi_i \le 1 \\ \sum_{i=0}^\infty \xi_i = 1 \quad \text{on } U \end{cases}$$

Suppose $u \in W^{k,p}(U)$. Then $\xi_i u \in W^{k,p}(U)$ and $\text{supp}(\xi_i u) \subset V_i$.

2. Fix $\delta > 0$. For each i, choose ϵ_i sufficiently small that $u^i = \eta_{\epsilon_i} * (\xi_i u)$ satisfies

$$\left\|u^i - \xi_i u\right\|_{W^{k,p}(U)} \leq \frac{\delta}{2^{i+1}} \quad i = 0, 1, 2, \cdots$$
$$\text{supp}(U_i) \subset W_i = U_{i+1} \backslash \overline{U}_i \quad i = 1, 2, \cdots$$

3. Write $v = \sum_{i=0}^{\infty} u^i$, then $v \in C^\infty(U)$ as for each $V \subset\subset U$, the sum is finite. Since $u = \sum_{i=0}^{\infty} \xi_i u$, for each $V \subset\subset U$< we have

$$\begin{aligned}\left\|v - u\right\|_{W^{k,p}(U)} &\leq \sum_{i=0}^{\infty} \left\|u^i - \xi_i u\right\|_{W^{k,p}(U)} \\ &\leq \delta \sum_{i=0}^{\infty} 2^{-i-1} = \delta\end{aligned}$$

Take supremum over $V \subset\subset U$, and conclude $v \in W^{k,p}(U)$, and $\left\|v - u\right\|_{W^{k,p}(U)} \leq \delta$.

(End of proof) □

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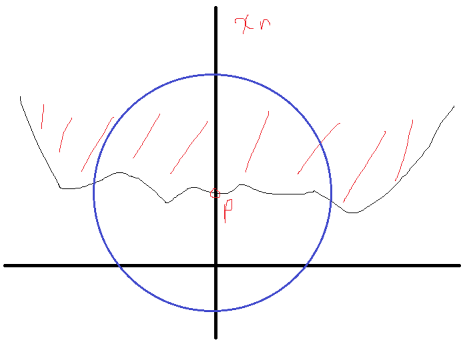
(26th October, Friday)

Last Lecture : $U \subset \mathbb{R}^n$ open and bounded, $u \in W^{k,p}(U)$, $1 \leq p < \infty$, there is a sequence $u_l \in C^\infty(U) \cap W^{k,p}(U)$ such that $u_l \to u$ in $W^{k,p}(U)$.

However, if the boundary of U is behaves badly, this approximation still is not good enough. We will extend this to an approximation result with $u_k \in C^\infty(\overline{U})$, but for this we require an assumption on ∂U.

Definition) Suppose $U \subset \mathbb{R}^n$ is open and bounded, we say U is a $C^{k,\alpha}$ **domain**, if for every $p \in \partial U = \overline{U} \backslash U$, there exists some $r > 0$ and a $C^{k,\alpha}$ function $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ such that (after possibly relabelling axes)

$$U \cap B_r(p) = \{x \in B_r(p) : x_n > \gamma(x_1, \cdots, x_{n-1})\}$$



Theorem) Suppose $U \subset \mathbb{R}^n$ is a $C^{0,1}$ domain (U has Lipshitz boundary). Let $u \in W^{k,p}(U)$ for some $1 \leq p < \infty$. Then there exist functions $u_m \in C^\infty(\overline{U})$ such that $u_m \to u$ in $W^{k,p}(U)$.

proof)

1. Fix $x^0 \in \partial U$. Since U is Lipshitz, $\exists r > 0$, $\gamma \in C^{0,1}(\mathbb{R}^{n-1})$ (after relabelling axes) s.t.

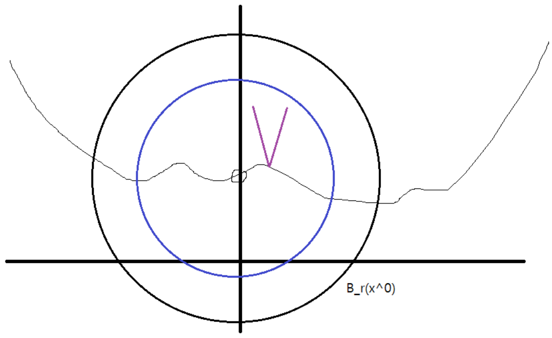
$$U \cap B_r(x^0) = \{x \in B_r(x^0) : x_n > \gamma(x_1, \cdots, x_{n-1})\}$$

Set $V = U \cap B_{r/2}(x^0)$.

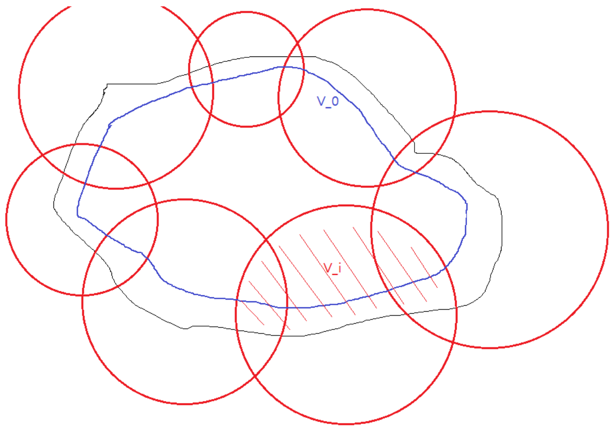
2. Define the shifted point $x^\epsilon = x + \lambda\epsilon e_n$ ($x \in V, \epsilon > 0$, $\lambda > 0$, e_n the unit vector in n-direction). For λ large enough, $B_\epsilon(x^\epsilon) \subset U \cap B_r(x^0)$ for all $x \in V$ and ϵ small enough. This is equivalent to saying that above each point, we can find a cone which remains above the graph. Also, we can choose λ uniformly on V(e.g. $\lambda \geq \left\|\gamma\right\|_{C^{0,1}}$). Define

$$u_\epsilon(x) = u(x^\epsilon) \quad x \in V$$

and set $v^{\epsilon,\tilde{\epsilon}} = \eta_{\tilde{\epsilon}} * u_\epsilon$ for $0 < \tilde{\epsilon} < \epsilon$. Clearly, we have $v^{\epsilon,\tilde{\epsilon}} \in C^\infty(\overline{V})$.



3. Fix $\delta > 0$, we estimate
$$\|v^{\epsilon,\tilde{\epsilon}} - u\|_{W^{k,p}(V)} \leq \|v^{\epsilon,\tilde{\epsilon}} - u_\epsilon\|_{W^{k,p}(V)} + \|u_\epsilon - u\|_{W^{k,p}(V)}$$
since translation is continuous in the L^p norms, we can pick $\epsilon > 0$ such that
$$\|u_\epsilon - u\|_{W^{k,p}(V)} < \delta/2$$
Having fixed $\epsilon > 0$, we can pick $\tilde{\epsilon} > 0$ such that
$$\|v^{\epsilon,\tilde{\epsilon}} - u_\epsilon\|_{W^{k,p}(V)} < \delta/2$$
by our previous theorem.
4. Now, since ∂U is compact, we can find finitely main $x_i^0 \in \partial U$ radii $r_i > 0$, sets $V_i = U \cap B_{r_i/2}(x_i^0)$ and functions $v_i \in C^\infty(\overline{V}_i)$ for $i = 1, \cdots, N$ satisfying
$$\|v_i - u\|_{W^{k,p}(V_i)} \leq \delta$$
and $\partial U \subset \bigcup_{i=1}^N V_i$. Take an open $V_0 \subset\subset U$ such that $U \subset \bigcup_{i=0}^N V_i$ and by a previous result we can find $v_0 \in C_c^\infty(U)$ such that
$$\|v_0 - u\|_{W^{k,p}(V_0)} \leq \delta$$



5. Let $\{\xi_i\}_{i=0}^N$ be a smooth partition of unity subordinated to the open sets $\{V_0, \{B_{r_i/2}(x_i^0)\}_{i=1}^N\}$ such that $\xi_i \in C_c^\infty(B_{r_i/2}(x_i^0))$, $\xi_0 \in C_c^\infty(V_0)$, $0 \leq \xi_i \leq 1$ with $\sum_{i=0}^N \xi_i = 1$ on U.

   Define $v = \sum_{i=0}^N \xi_i v_i$. Clearly $v \in C_c^\infty(\overline{U})$. Further, for $|\alpha| \leq k$,
$$\begin{aligned}\|D^\alpha v - D^\alpha u\|_{L^p(U)} &\leq \sum_{i=0}^N \|D^\alpha(\xi_i v_i) - D^\alpha(\xi_i u)\|_{L^p(V_i)} \\ &\leq C_k \sum_{i=0}^N \|v_i - u_i\|_{W^{k,p}(V_i)} \\ &\leq C_k(N+1)\delta\end{aligned}$$
As δ was arbitrary, we have the desired result.

(End of proof) $\square$

Theorem) (Extension of Sobolev functions) Suppose $U \subset \mathbb{R}^n$, open, bounded, is a $C^{1,0}$ domain. Choose a bounded V such that $U \subset\subset V$. Then there exists a bounded linear operator $E : W^{1,p}(U) \to W^{1,p}(\mathbb{R}^n)$ such that for each $u \in W^{1,p}(U)$:

(i) $Eu = u$ almost everywhere in U.

(ii) Eu has support in V.

(iii) $\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(U)}$ where C only depends on U, V and p.

We call Eu an **extension of u to $\mathbb{R}^n$**. This is not unique.

Lemma) Suppose $U = B_r(0) \cap \{x_n > 0\}$. Suppose $u \in C^1(\overline{\{x_n > 0\}})$. We can find an $Eu \in C^1(\mathbb{R}^n)$ such that

$$\|Eu\|_{W^{1,p}(B_r(0))} \leq C\|u\|_{W^{1,p}(U)}$$

for some constant $C > 0$.

proof) We define

$$\overline{u}(x) = \begin{cases} u(x) & \text{if } x_n \geq 0 \\ -3u(x_1, \cdots, x_{n-1}, -x_n) + 4u(x_1, \cdots, x_{n-1}, \frac{-x_n}{2}) & \text{if } x_n < 0 \end{cases}$$

Claim : $\overline{u} \in C^1(\mathbb{R}^n)$.
Clearly, $\overline{u} \in C^0(\mathbb{R}^n)$ as

$$\lim_{x_n \to 0} -3u(x_1, \cdots, x_{n-1}, -x_n) + 4u(x_1, \cdots, x_{n-1}, -\frac{x_n}{2}) = u(x_1, \cdots, x_{n-1}, 0)$$

Do similarly for $\partial_{x_j}\overline{u}$, $1 \leq j \leq n-1$.
For $x_n < 0$,

$$\begin{aligned} \partial_{x_n}\overline{u}(x_1, \cdots, x_n) =& 3\partial_{x_n}u(x_1, \cdots, x_{n-1}, -x_n) - 2\partial_{x_n}u(x_1, \cdots, x_{n-1}, -x_n/2) \\ &\to u_{x_n}(x_1, \cdots, x_{n-1}, 0) \quad \text{as } x_n \to 0^- \end{aligned}$$

So U, Du are continuous across $\{x_n = 0\}$.

Setting $Eu = \overline{u}$, clearly $\overline{u}$ depends on u linearly, and a computation shows $\|Eu\|_{W^{1,k}(U)} \leq C\|u\|_{W^{1,k}(U)}$

(End of proof) $\square$

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(29th October, 2018)

Recall, we were on our way to proving the following theorem.

Theorem) (Extension of Sobolev functions) Suppose $U \subset \mathbb{R}^n$, open, bounded, is a $C^{1,0}$ domain. Choose a bounded V such that $U \subset\subset V$. Then there exists a bounded linear operator $E : W^{1,p}(U) \to W^{1,p}(\mathbb{R}^n)$ such that for each $u \in W^{1,p}(U)$:

(i) $Eu = u$ almost everywhere in U.

(ii) Eu has support in V.

(iii) $\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(U)}$ where C only depends on U, V and p.

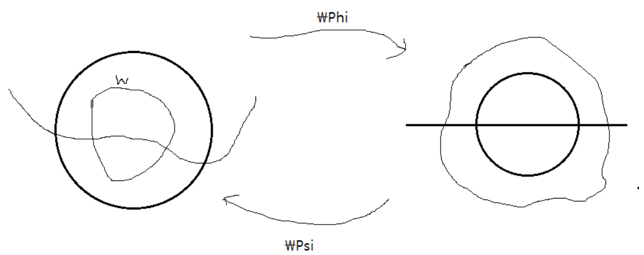
Yet we need another lemma. Be aware that in the following lines of proof, the constant C varies and might not indicate a single number.

Lemma) Suppose $U \subset \mathbb{R}^n$, bounded, open C^1-domain. Suppose $u \in C^1(\overline{U})$. Then $\exists \overline{u} \in C^1_c(\mathbb{R}^n)$ that depends linearly on u and that

$$\|\overline{u}\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(U)} \quad u = \overline{u} \text{ on } U$$

proof) We prove this lemma by reducing the setting to the setting of the previous lemma.

1. Pick $x_0 \in \partial U$. After possibly relabelling axes, $\exists r > 0$ such that
$$U \cap B_r(x_0) = \{x \in B_r(x_0) : x_n > \gamma(x_1, \cdots, x_{n-1})\}$$
for some $\gamma \in C^1(\mathbb{R}^{n-1})$. (by definition of being a C^1-domain). Define $\Phi^i(x) = y^i$ where
$$\begin{aligned} y^i &= x^i \quad \forall i = 1, \cdots, n-1 \\ y^n &= x^n - \gamma(x^1, \cdots, x^{n-1}) \end{aligned}$$
This has inverse $\Psi(y) = x$ where
$$\begin{aligned} x^i &= y^i \quad i = 1, \cdots, n-1 \\ x^n &= y^n + \gamma(y^1, \cdots, y^{n-1}) \end{aligned}$$
Clearly, $\Phi \circ \Psi = \Psi \circ \Phi = id_{B_r(x_0)}$,
$$\Phi(U \cap B_r(x_0)) \subset \{y_0 > 0\}$$
and Φ, Ψ are both C^1. If $y_0 = \Phi(X_0)$, $\exists s > 0$ such that $B_s(y_0) \subset \Phi(B_r(x_0))$. Define $W = \Phi^{-1}(B_s(y_0))$.



2. For $y \in B_s(y_0) \cap \{y_n > 0\} = B_+$, we define
$$u'(y) = u(\Psi(y))$$
By the previous lemma, $\exists \overline{u}' \in C^1(B_s(y_0))$ such that $\overline{u}' = u'$ on $B_s(y_0) \cap \{y_0 > 0\} = B_+$ and
$$\|\overline{u}'\|_{W^{1,p}(B_s(y_0))} \le C\|u'\|_{W^{1,p}(B_+)}$$
Converting back to x-coordinates, we define
$$\overline{u}(x) = \overline{u}'(\Phi(x)) \quad \forall x \in W.$$
Therefore $\overline{u}$ extends u from $U \cap W$ to W and (checking this is an exercise.)
$$\|\overline{u}\|_{W^{1,p}(W)} \le C\|u\|_{W^{1,p}(U)}$$

3. Since ∂U is compact, we can take a finite number of points $x_0^i \in \partial U$, open sets $W_i \ni x_0^i$ and extensions $\overline{u}_i$ of u to W_i such that $\bigcup_{i=1}^n W_i \supset \partial U$. Also, pick $W_0 \subset\subset U$ such that $\bigcup_{i=0}^N W_i \supset U$ and set $\overline{u}_0 = u$ on W_0.

   Pick a partition of unity subordinate to $\{w_i\}_{i=0}^N$, say $\{\xi_i\}_{i=0}^N$ such that (i) $\xi_i \in C_c^\infty(W_i)$, (ii) $0 \le \xi_i \le 1$ and (iii) $\sum_{i=0}^N \xi_i(x) = 1$ for all $x \in U$. Define
$$\overline{u} = \sum_{i=0}^N \xi_i \overline{u}_i$$
Then if $x \in U$,
$$\overline{u}(x) = \sum_{i=0}^N \xi_i(x) u(x) = u(x)$$
so $\overline{u} \in C_c^1(\mathbb{R}^n)$ is an extension of u and
$$\|\overline{u}\|_{W^{1,p}(\mathbb{R}^n)} \le C\|u\|_{W^{1,p}(U)}$$
We can also check each step that $\overline{u}$ depends linearly on u. We write $\overline{u} = Eu$.

(End of proof) □

We are finally ready to the prove the theorem based on the two previous lemmas.

proof of the extension theorem) Suppose $u \in W^{1,p}(U)$. By approximation theorem, $\exists u_m \in C^\infty(\overline{U})$ with $u_m \to u$ in $W^{1,p}(U)$ and a.e. In particular, the map E of the previous lemmas is defined on each u_m. By linearity of E,

$$\|E(u_n - u_m)\|_{W^{1,p}(\mathbb{R}^n)} = \|Eu_n - Eu_m\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u_n - u_m\|_{W^{1,p}(U)}$$

Since $(u_n)_n$ is convergent in $W^{1,p}(U)$, it is Cauchy in $W^{1,p}(U)$, we deduce that $(Eu_m)_m$ is Cauchy in $W^{1,p}(\mathbb{R}^n)$. Hence

$$Eu_m \to \tilde{u}$$

for some $F[u] \in W^{1,p}(\mathbb{R}^n)$. In fact, $\tilde{u}$ is independent of approximating sequence so we set $Fu = \tilde{u}$ and linear in u. If needs be, we can always multiply our answer by $\phi \in C_c^\infty(U)$ with $\phi = 1$ on U to fix support on Fu. We can then check F satisfies conditions (i),(ii) and (iii).

(End of proof) □

We can repeat our argument to show a result for extensions of functions in $W^{1,p}(U)$ where U is a C^k domain, using a suitable higher order reflections.

Trace theorem

Because Sobolev functions are defined only up to almost everywhere sense, we do not yet know what it means for a Sobolev function to be defined on a boundary of a domain(as a submanifold of codimension 1 has measure 0), which means that we can not state boundary value problems of PDEs properly. To amend this problem, we need the following trace theorem.

Theorem) *(Trace Theorem)* Assume $U \subset \mathbb{R}^n$ is open, bounded C^1 domain. There exists a bounded linear operator

$$T : W^{1,p}(U) \to L^p(\partial U) \quad 1 \leq p < \infty$$

such that

(i) $Tu = u\big|_{\partial U}$ if $u \in W^{1,p}(U) \cap C(\overline{U})$

(ii) $\|Tu\|_{L^p(\partial U)} \leq C\|u\|_{W^{1,p}(U)}$ for all $u \in W^{1,p}(U)$ where $C = C(U, p)$ only depends on U and p.

The operator T allows us to talk about '*the restriction of u to ∂U*' even though u is only defined almost everywhere, and ∂U is a set of measure zero.

proof)

(i) First assume $u \in C^1(\overline{U})$, and as previously, suppose $x^0 \in \partial U$ and ∂U is flat near x^0 lying in the plane $\{x_n = 0\}$. Choose $B = B_r(x^0)$ for some $r > 0$ such that

$$B_+ = B \cap \{x_n \geq 0\} \subset \overline{U}$$
$$B_- = B \cap \{x_n \leq 0\} \subset \mathbb{R}^n \backslash U$$

Set $\hat{B} = B_{r/2}(x^0)$, denote by Γ the portion of ∂U lying within $\hat{B}$. Pick $\xi \in C_c^\infty(B)$ such that

$0 \le \xi \le 1$ on B and $\xi \equiv 1$ on $\hat{B}$.

$$\begin{aligned}
\int_\Gamma |u|^p dx' &= \int_\Gamma \xi |u|^p dx' \qquad (dx' \text{ the area element of } \Gamma) \\
&\le \int_{B\cap\{x_n=0\}} \xi|u|^p dx' \\
&\le \int_0^r \frac{d}{dy_n}\Big[-\int_{B\cap\{x_n=y_n\}} \xi|u|^p dx'\Big] dy_n \\
&= -\int_{B_+} \partial_{x_n}\big(\xi|u|^p\big) dx \\
&= -\int_{B_+} \big(|u|^p \partial_{x_n}\xi + p\partial_{x_n} u|u|^{p-1}\xi\big) dx
\end{aligned}$$

Using Young's inequality($|ab| \le |a|^p/p + |b|^q/q$), we have

$$\Big|(\partial_{x_n}u)|u|^{p-1}\Big| \le \frac{|\partial_{x_n}u|^p}{p} + \frac{|u|^{q(p-1)}}{q} \quad \text{where } q = \frac{p}{1-p}$$

and therefore

$$\int_\Gamma |u|^p dx' \le C_{p,r} \int_{B^+} |u|^p + |Du|^p dx$$

and therefore $\|u\|_{L^p(\Gamma)} \le C_{p,r}\|u\|_{W^{1,p}(B_+)}$.

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(31st October, Wednesday)

proof continued)

Last lecture : $U \subset \mathbb{R}^n$ open, bounded, ∂U is C^1. If ∂U is flat near x^0 and $u \in C^\infty(\overline{Y})$. Then there exists open $\Gamma \subset \partial U$ such that

$$\int_\Gamma |u|^p dx' \le C \int_U |u|^p + |Du|^p dx$$

If $x^0 \in \partial U$, but ∂U is not flat near x^0, we use a C^1 boundary straightening function as in the proof of the extension theorem to show $\exists \Gamma \subset \partial U$ open such that

$$\int_\Gamma |u|^p dS \le C \int_U |u|^p + |Du|^p dx$$

where dS is area element of Γ not in its original coordinate but in its parametrised coordinates. But by estimating the Jacobian of the chart, this can be replaced by the area element of Γ without much difficulty.

Use compactness of ∂U with this result to show

$$\int_{\partial U} |u|^p dS \le C \int_Y |u|^p + |Du|^p dx \quad \text{hence} \quad \|Tu\|_{L^p(\partial U)} \le C\|u\|_{W^{1,p}(U)}$$

(plus partition of unity as in the previous proofs)

For general $u \in W^{1,p}(U)$, approximate by $u \in C^\infty(\overline{U})$. If $u \in W^{1,p}(U) \cap C^0(\overline{U})$ use that our approximating sequence converges uniformly to u to show $Tu = u\big|_{\partial U}$.

(See the Example sheet 2 to fill in the gaps)

We have shown that associated to $u \in W^{1,p}(U)$ is $Tu \in L^p(\partial U)$, uniquely determined by $Tu = u\big|_{\partial U}$ for $u \in C^0(\overline{U})$. But it is not true that any L^p function arises as a trace of some Sobolev function. That is, the **trace map** T is not surjective. There exists $f \in L^p(\partial U)$ such that $Tu \ne f$ for all $u \in W^{1,p}(U)$.

Note : Recall we defined $W_0^{1,p}$ to be the completion of C_c^∞ in $W^{1,p}$-norm. So one can show without difficulty that if $u \in W_0^{1,p}(U)$ then $Tu = 0$. The converse is true : if $u \in W^{1,p}(U)$ and $Tu = 0$ then $u \in W_0^{1,p}(U)$. Finally, if $u \in W^{2,p}(U)$, $D_i u \in W^{1,p}(U)$ so one can define u and Du on ∂U using T.

Sobolev Inequalities, Embeddings

(For this section, it would be useful to refer to Clément's Analysis of Functions lecture notes)

We can think of the p in L^p_{loc} as giving some measure of how 'spicky' the function can be. If $\gamma p < 1$(or equivalently $\gamma < 1/p$)

$$\|x^{-\gamma}\|^p_{L^p((0,1])} = \int_0^1 x^{-\gamma p} dx < \infty$$

For larger p, the function must tend to infinity more slowly as $x \to 0$.

The Sobolev embeddings tell us we can exchange 'differentiability' (i.e. k in $W^{k,p}$), for 'integrability' (i.e. p in $W^{k,p}$). The first result is:

Theorem) *(Sobolev-Gagliardo-Nirenberg, or SGN)* Assume $n > p$. We have $W^{1,p}(\mathbb{R}^n) \subset L^{p^*}(\mathbb{R}^n)$ with $p^* = \frac{np}{n-p} > p$, and $\exists C > 0$ depending only on n, p such that $\forall u \in W^{1,p}(\mathbb{R}^n)$,

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \le C\|Du\|_{L^p(\mathbb{R}^n)} \le C\|u\|_{W^{1,p}(\mathbb{R}^n)}$$

Lemma) Let $n \ge 2$ and $f_1, \cdots, f_n \in L^{n-1}(\mathbb{R}^{n-1})$. For any $1 \le i \le n$, denote $\tilde{x}_i = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n) \in \mathbb{R}^{n-1}$ (remove i^{th} component from $(x_1, \cdots, x_n) \in \mathbb{R}^n$) and

$$f : \mathbb{R}^n \to \mathbb{R} \quad f(t) = f_1(\tilde{x}_1)f_2(\tilde{x}_2)\cdots f_n(\tilde{x}_n)$$

Then $f \in L^1(\mathbb{R}^n)$ with

$$\|f\|_{L^1(\mathbb{R}^n)} \le \prod_{i=1}^n \|f_i\|_{L^{n-1}(\mathbb{R}^{n-1})}$$

proof) We work by induction.

For $n = 2$, $f(x_1, x_2) = f_1(x_2)f_2(x_1)$ so

$$\begin{aligned}\|f\|_{L^1(\mathbb{R}^n)} &= \int_{\mathbb{R}^2} |f(x_1, x_2)| dx_1 dx_2 = \int_{\mathbb{R}} |f_1(x_1)| dx_2 \int |f_2(x_2)| dx_1 \\ &= \|f_1\|_{L^1}\|f_2\|_{L^1}\end{aligned}$$

Suppose result holds for some $n \ge 2$. Write $f(x_1, \cdots, x_{n+1}) = f_{n+1}(\tilde{x}_{n+1})F(x)$, where $F(x) = f_1(\tilde{x}_1)\cdots f_n(\tilde{x}_n)$ and note

$$\int_{y_1,\cdots,y_n} |f(y_1, \cdots, y_n, x_{n+1})| dy_1 \cdots dy_n \le \|f_{n+1}\|_{L^n(\mathbb{R}^n)} \|F(\cdot, x_{n+1})\|_{L^{n/(n-1)}(\mathbb{R}^n)} \quad \cdots (\dagger)$$

by Hölder inequality. We apply the result for n to $|f_1|^{n/(n-1)}(\cdot, x_{n+1}) \times \cdots \times |f_n|^{n/(n-1)}(\cdot, x_{n+1})$ to find

$$\begin{aligned}\|F(\cdot, x_{n+1})\|_{L^{n/(n-1)}(\mathbb{R}^n)} =& \left\||f_1|^{\frac{n}{n-1}}(\cdot, x_{n+1}) \times \cdots \times |f_n|^{\frac{n}{n-1}}(\cdot, x_{n+1})\right\|^{\frac{n-1}{n}}_{L^1(\mathbb{R}^n)} \\ \le& \Big(\prod_{i=1}^n \left\||f_i|^{n/(n-1)}(\cdot, x_{n+1})\right\|_{L^{n-1}(\mathbb{R}^{n-1})}\Big)^{n-1/n} \\ =& \prod_{i=1}^n \|f_i(\cdot, x_{n+1})\|_{L^n(\mathbb{R}^{n-1})}\end{aligned}$$

Finally, we integrate $(\dagger)$ over x_{n+1} to get

$$\begin{aligned}\|f\|_{L^1(\mathbb{R}^{n+1})} \le& \|f_{n+1}\|_{L^n(\mathbb{R}^n)} \int \prod_{i=1}^n \|f_i(\cdot, x_{n+1})\|_{L^n(\mathbb{R}^{n-1})} dx_{n+1} \\ \le& \|f_{n+1}\|_{L^n(\mathbb{R}^n)} \prod_{i=1}^n \Big(\int_{x_{n+1}} \|f_i\|^n_{L^n(\mathbb{R}^{n-1})}(\cdot, x_{n+1}) dx_{n+1}\Big)^{1/n} \quad \text{(by Hölder)} \\ =& \|f_{n+1}\|_{L^n(\mathbb{R}^n)} \prod_{i=1}^n \|f_i\|_{L^n(\mathbb{R}^n)}\end{aligned}$$

So result hold for $n+1$, so by induction we are done.

(End of proof) $\square$

proof of Sobolev-Gagliargo-Nirenberg in case $p=1$) First assume $u \in C_c^\infty(\mathbb{R}^n)$ since u has compact support, we have

$$u(x) = \int_{-\infty}^{x_i} \partial_{x_i} u(x_1, \cdots, x_{i-1}, y, x_{i+1}, \cdots, x_n) dy$$

so that

$$|u(x)| \leq \int_{-\infty}^{\infty} |Du(x_1, \cdots, x_{i-1}, y, x_{i+1}, \cdots, x_n)| dy$$

Define $f_i(\tilde{x}_i) = \int_{-\infty}^{\infty} |Du(x_1, \cdots, x_{i-1}, y, x_{i+1}, \cdots, x_n)| dy$. Thus

$$|u(x)|^{n/n-1} \leq \prod_{i=1}^n f_i(\tilde{x}_i)^{1/(n-1)}$$

Integrating and using the previous lemma gives

$$\begin{aligned}
\Big(\|u\|_{L^{n/(n-1)(\mathbb{R}^n)}}\Big)^{\frac{n-1}{n}} &= \|u^{\frac{n}{n-1}}\|_{L^1(\mathbb{R}^n)} \\
&\leq \prod_{i=1}^n \|f_i^{\frac{1}{n-1}}\|_{L^{n-1}(\mathbb{R}^{n-1})} = \prod_{i=1}^n \|Du\|_{L^1}^{\frac{1}{n-1}} = \|Du\|_{L^1(\mathbb{R}^n)}^{\frac{n}{n-1}}
\end{aligned}$$

Since this estimate only depends on the size of u and Du, by approximation by smooth functions, this also holds for $u \in W^{1,1}(\mathbb{R}^n)$.

(End of proof) $\square$

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(2nd November, Friday)

Last lecture :

Theorem) *(Sobolev-Gagliardo-Nirenberg)* Assume $n > p$. We have $W^{1,p}(\mathbb{R}^n) \subset L^{p^*}(\mathbb{R}^n)$ with $p^* = \frac{np}{n-p} > p$, and $\exists C > 0$ depending only on n, p such that $\forall u \in W^{1,p}(\mathbb{R}^n)$,

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|Du\|_{L^p(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)}$$

We proved $p=1$ case: $\|u\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \leq \|Du\|_{L^1(\mathbb{R}^n)}$, so the constant $C=1$. The trick in this proof was to combine the integrals in various directions in a good way, using a Lemma.
In the example sheet Exercise 2.12, we show that if the set U has C^1 boundary, it satisfies an isoperimetric inequality, given an estimate of above form.

proof of the case $p > 1$) Now suppose $p > 1$. We apply the $p=1$ result to $v = |u|^\gamma$ where $\gamma > 1$ will be chosen later. We note the following fact from **Example sheet 2, Exercise 2.6**

Fact : Suppose U is bounded and $u \in W^{1,p}(U)$ for some $p \in [1, \infty)$. Also, let $F : \mathbb{R} \to \mathbb{R}$ is C^1 and F' is bounded. Then $v(x) = F(u(x))$ is in $W^{1,p}(U)$ and the weak derivative is given by $\partial_{x_i} v = F'(u)\partial_{x_i} u$ for $i = 1, \cdots, n$.

Hence, v is weakly differentiable and

$$Dv = \gamma \mathrm{sign}(u)|u|^{\gamma-1} Du$$

so

$$\begin{aligned}
\|v\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} &= \Big(\int_{\mathbb{R}^n} |u|^{\frac{\gamma n}{n-1}} dx\Big)^{\frac{n-1}{n}} \leq \gamma \int_{\mathbb{R}^n} |u|^{\gamma-1}|Du| dx \\
&\leq \gamma \Big(\int_{\mathbb{R}^n} |u|^{(\gamma-1)\frac{p}{p-1}} dx\Big)^{\frac{p-1}{p}} \Big(\int_{\mathbb{R}^n} |Du|^p dx\Big)^{1/p} \quad \text{(by Hölder)}
\end{aligned}$$

We choose γ such that

$$\frac{\gamma n}{n-1} = \frac{(\gamma-1)p}{p-1} \quad \Rightarrow \quad \gamma = \frac{p(n-1)}{n-p} > 1$$

Then $\frac{\gamma n}{n-1} = \frac{(n-1)p}{p-1} = \frac{np}{n-p} = p^*$ is as given in the statement of the theorem. Thus,

$$\Big(\int_{\mathbb{R}^n} |u|^{p^*} dx\Big)^{1/p^*} = \frac{p(n-1)}{n-p}\Big(\int_{\mathbb{R}^n} |Du|^p dx\Big)^{1/p}$$

(End of proof) $\square$

In fact, in the example sheet(Exercise 2.9), you will see that this family of inequality, bounding $\|u\|_q$ by $\|Du\|_p$, can only exist for only particular pair of exponents (p, q) satisfying

$$\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$$

Corollary 1) Let $U \subset \mathbb{R}^n$ be open, bounded C^1-domain, and $1 \le p < n$. Then $W^{1,p}(U) \subset L^{p^*}(U)$ (where p^* is as before) and $\exists\ C(p, n, U)$ such that

$$\|u\|_{L^{p^*}(U)} \le C(p, n, U)\|Du\|_{W^{1,p}(U)} \quad \forall u \in W^{1,p}(U)$$

proof) By extension theorem, $\exists$ a bounded operator $E : W^{1,p}(U) \to W^{1,p}(\mathbb{R}^n)$ such that $Eu = u$ on U and Eu has compact support. By Sobolev-Gagliardo-Nirenberg theorem, $Eu \in W^{1,p}(\mathbb{R}^n)$ implies $Eu \in L^{p^*}(\mathbb{R}^n)$ and therefore $u \in L^{p^*}(U)$ and

$$\|u\|_{L^{p^*}(U)} = \|Eu\|_{L^{p^*}(U)} \le \|Eu\|_{L^{p^*}(\mathbb{R}^n)} \le C\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \le \tilde{C}\|u\|_{W^{1,p}(U)}$$

(End of proof) $\square$

Corollary 2) *(Poincaré Inequality)* Suppose $U \subset \mathbb{R}^n$ be open and bounded. Suppose $u \in W_0^{1,p}(U)$ for some $1 \le p < n$. Then we have the following estimate.

$$\|u\|_{L^q(U)} \le C\|Du\|_{L^p(U)} \quad \forall q \in [1, p^*)$$

where $C = C(p, q, n, U)$. In particular,

$$\|u\|_{L^p(U)} \le C\|Du\|_{L^p(U)}$$

proof) Since $u \in W_0^{1,p}(U)$, $\exists\ u_m \in C_c^\infty(U)$ converging to U in $W^{1,p}(U)$. Extend u_m by zero on U^c to get $\overline{u}_m \in C_c^\infty(\mathbb{R}^n)$. Apply Sobolev-Gagliardo-Nirenberg inequality to find

$$\|u_m\|_{L^{p^*}(U)} \le C\|Du_m\|_{L^p(U)}$$

And further

$$\|u_m - u_{m'}\|_{L^{p^*}(U)} \le C\|Du_m - Du_{m'}\|_{L^p(U)}$$

Therefore, (u_m) is Cauchy in $L^{p^*}(U)$, and hence u_m converges in L^{p^*}. We can send $m \to \infty$ in $\|u_m\|_{L^{p^*}(U)} \le C\|Du_m\|_{L^p(U)}$ further to find

$$\|u\|_{L^{p^*}(U)} \le C\|Du\|_{L^p(U)}$$

Since $\mathrm{vol}(U) < \infty$, by Hölder, we can see that

$$\|u\|_{L^q(U)} \le \|Du\|_{L^p(U)} \quad \forall q \in [1, p^*]$$

(End of proof) $\square$

Now suppose $n < p < \infty$. Then naively, we might expect a function in $W^{1,p}(\mathbb{R}^n)$ to be better than L^∞. In fact, we have

Theorem) *(Morrey's Inequality)* Suppose $n < p < \infty$. Then $\exists C = C(p,n)$ such that

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)} \quad \forall u \in C_c^1(\mathbb{R}^n)$$

where $\gamma = 1 - \frac{n}{p}$. (This can be interpreted as : as p increases, we lose less and less information about differentiability in the Sobolev norm)

proof) We first establish the Hölder part of the estimate. Let Q be an open cube, sides parallel to axes of side $r > 0$ (centred at any point). Set

$$\overline{u} = \frac{1}{|Q|}\int_Q u(x)dx$$

to be the average of u on Q. Then

$$|\overline{u} - u(0)| = \Big|\frac{1}{|Q|}\int_Q u(x) - u(0)dx\Big| \leq \frac{1}{|Q|}\int_Q |u(x) - u(0)|dx$$

Note

$$u(x) - u(0) = \int_0^1 \frac{d}{dt}u(tx)dx = \sum_{i=1}^n \int_0^1 x^i \frac{\partial u}{\partial x^i}(tx)dt$$

so

$$|u(x) - u(0)| \leq r\int_0^1 \sum_{i=0}^n \Big|\frac{\partial u}{\partial x^i}(tx)\Big|dt \quad (\text{since } x \in Q \Rightarrow |x| < r)$$

Thus

$$\begin{aligned}
|\overline{u} - u(0)| &\leq \frac{r}{|Q|}\int_Q\int_0^1 \sum_{i=1}^n \Big|\frac{\partial u}{\partial x^i}(tx)\Big|dtdx \\
&= \frac{r}{|Q|}\int_0^1 \Big(\int_Q \sum_{i=1}^n \Big|\frac{\partial u}{\partial x^i}(tx)\Big|dx\Big)dt \quad (\text{Fubini}) \\
&= \frac{r}{|Q|}\int_0^1 t^{-n}\Big(\int_{tQ} \sum_{i=1}^n \Big|\frac{\partial u}{\partial x^i}(y)dy\Big|dy\Big)dt \quad (y = tx) \\
&\leq \frac{r}{|Q|}\int_0^1 t^{-n}\Big(\sum_{i=1}^n \|\frac{\partial u}{\partial x^i}\|_{L^p(Q)}|tQ|^{1/p'}\Big)dt \quad (\text{Hölder}, \frac{1}{p} + \frac{1}{p'} = 1)
\end{aligned}$$

so using the fact $|Q| = r^n$ and $|tQ| = t^n r^n$,

$$\begin{aligned}
|\overline{u} - u(0)| &\lesssim r^{1-n+\frac{n}{p'}}\|Du\|_{L^p(\mathbb{R}^n)}\int_0^1 t^{-n+\frac{n}{p'}}dt \\
&\lesssim \frac{r^{1-\frac{n}{p}}}{1-n/p}\|Du\|_{L^p(\mathbb{R}^n)}
\end{aligned}$$

Now suppose $x, y \in \mathbb{R}^n$ with $|x - y| = r/2$. Pick a box containing x, y with side r. By above,

$$|u(x) - u(y)| \leq |u(x) - \overline{u}| + |\overline{u} - u(y)| \lesssim \frac{r^{1-\frac{n}{p}}}{1-n/p}\|Du\|_{L^p(\mathbb{R}^n)}$$

and therefore

$$\frac{|u(x) - u(y)|}{|x-y|^{1-\frac{n}{p}}} \lesssim \frac{1}{1-n/p}\|Du\|_{L^p(\mathbb{R}^n)}$$

Finally, to see that u is bounded for any $x \in \mathbb{R}^n$, pick a box of side 1 containing x and estimate

$$|u(x)| \leq |\overline{u} - u(x)| + |\overline{u}| \lesssim \|Du\|_{L^p(\mathbb{R}^n)} + \|u\|_{L^p(\mathbb{R}^n)} \lesssim \|u\|_{W^{1,p}(\mathbb{R}^n)}$$

(End of proof) $\square$

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(5th November, Monday)

Last lecture :

> **Theorem)** *(Morrey's Inequality)* Suppose $n < p < \infty$. Then $\exists C = C(p,n)$ such that
>
> $$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \le C\|u\|_{W^{1,p}(\mathbb{R}^n)} \quad \forall u \in C^1_c(\mathbb{R}^n)$$
>
> where $\gamma = 1 - \frac{n}{p}$.

Morrey's inequality does not seem really useful at a first site. But from this, its follows a somewhat more useful and interesting result.

Corollary) Let $n < p < \infty$. Suppose $u \in W^{1,p}(U)$. For $U \subset \mathbb{R}^n$ open, bounded C^1-domain. (boundedness is in fact not necessary.) Then $\exists u^* \in C^{0,1-\frac{n}{p}}(U)$ such that $u = u^*$ almost everywhere, and

$$\|u^*\|_{C^{0,1-\frac{n}{p}}(U)} \le C\|u\|_{W^{1,p}(U)}$$

for some $C = C(n,p,U)$.

> **proof)** By extension theorem, $\exists \overline{u} \in W^{1,p}(\mathbb{R}^n)$ with $\overline{u} = u$ a.e. on U. We can find a sequence $u_m \in C^\infty_c(\mathbb{R}^n)$ such that $u_m \to \overline{u}$ in $W^{1,p}(\mathbb{R}^n)$ and $u_m(x) \to \overline{u}(x)$ for almost every x. By Morrey's inequality, has
>
> $$\|u_m - u_{m'}\|_{C^{0,1-\frac{n}{p}}(\mathbb{R}^n)} \le C\|u_m - u_{m'}\|_{W^{1,p}(\mathbb{R}^n)}$$
>
> and so $(u_m)_{m\ge1}$ is Cauchy in $C^{0,1-\frac{n}{p}}(\mathbb{R}^n)$. Also, since $C^{0,1-\frac{n}{p}}(\mathbb{R}^n)$ is complete, $u_m \to \overline{u}^*$ in $C^{0,1-\frac{n}{p}}(\mathbb{R}^n)$. Hence $\overline{u}^* = \overline{u}$ almost everywhere, so $u^* = \overline{u}^*\big|_U$. This satisfies our conditions.
>
> *(End of proof)* $\square$

The following diagram abstracts what we had been doing :

$$\begin{tikzcd} & u \in W^{1,p}(\mathbb{R}^n) \arrow[dl, "n>p"'] \arrow[dr, "n<p"] & \\ u \in L^{p^*}(\mathbb{R}^n) & & u \in C^{0,1-\frac{n}{p}}(\mathbb{R}^n) \end{tikzcd}$$

where the left arrow is implied by SGN inequality and the right arrow is implied by the corollary right above.

By iterating these results, it is possible to establish similar embedding results for $W^{k,p}(\mathbb{R}^n)$ into $W^{k',p'}(\mathbb{R}^n)$ for $k' < k, p' > p$ or $C^{k',\gamma}(\mathbb{R}^n)$ for $k' < k$.

For example, we have $u \in W^{2,2}(\mathbb{R}^3) \Leftrightarrow Du \in W^{1,2}(\mathbb{R}^3)$ and $u \in W^{1,2}(\mathbb{R}^3)$. This implies $Du \in L^6(\mathbb{R}^3)$ and $u \in L^6(\mathbb{R}^3)$, and therefore $u \in W^{1,6}(\mathbb{R}^3)$ and finally $u \in C^{0,1/2}(\mathbb{R}^3)$ by the corollary.

Second Order Elliptic Equations

Let $U \subset \mathbb{R}^n$ and consider the operator

$$Lu = -\sum_{i,j=1}^n \left(a^{ij}(x)u_{x_j}\right)_{x_i} + \sum_{i=1}^n b^i(x)u_{x_i} + c(x)u \quad (\textbf{Divergence form})$$

where a^{ij}, b^i, c are given functions on U. Typically we will assume they are at least L^∞, but sometimes we will require more.

Assuming $a^{ij} \in C^1(U)$ we can write L in **non-divergence form** as

$$Lu = -\sum_{i,j=1}^{n} a^{ij} u_{x_i x_j} + \sum_{i=1}^{n} \tilde{b}^i u_{x_i} + cu$$

where $\tilde{b}^i = b^i - a^{ji}_{x_i}$.

We further assume L is elliptic. This implies that L or $-L$ satisfies

$$\sum_{i,j} a^{ij} \xi_i \xi_j \geq 0 \quad \forall \xi \in \mathbb{R}^n$$

and has equality *iff* $\xi = 0$. This implies that every surface is non-characteristic.

It turns out this is not a strong enough condition in general. We shall also require **uniform ellipticity** :

Definition) L is **uniformly elliptic** if $\exists \theta \in \mathbb{R}_{>0}$ such that

$$0 \leq \theta |\xi|^2 \leq \sum_{i,j} a^{ij}(x) \xi_i \xi_j \quad \forall x \in U, \ \xi \in \mathbb{R}^n$$

Uniform ellipticity is a statement about invertibility of matrix $(a_{ij})_{ij}$. In fact, in finite dimensional case, ellipticity implies uniform ellipticity.

We consider the boundary value problem

$$\begin{cases} Lu = f & \text{in } U \\ u\big|_{\partial U} = 0 \end{cases} \tag{11}$$

where U is *always* open bounded C^1-domain.

Example : Let $L = -\Delta$, then the above problem models the electrostatic potential u, sourced by a charge distribution f, inside a cavity U whose walls are grounded.

It will turn out that form (11) is not so well suited to finding solutions. We shall reformulate the problem into a more amenable form.

Suppose $u \in C^2(\overline{U})$ is a solution of

$$Lu = -\sum_{i,j=1}^{n} \left(a^{ij}(x) u_{x_j}\right)_{x_i} + \sum_{i=1}^{n} b^i(x) u_{x_i} + c(x)u = f$$

with $u = 0$ on ∂U. Multiplying these equation by $v \in C^2(\overline{U})$ satisfying $v = 0$ on ∂U, we find :

$$\int_U vLu dx = \int_U (-v) \sum_{i,j=1}^{n} \left(a^{ij}(x) u_{x_j}\right)_{x_i} + v \sum_{i=1}^{n} b^i(x) u_{x_i} + c(x) uv dx = \int f v dx$$

Now, by the divergence theorem,

$$\begin{aligned} \int_U -\sum_{i,j} v \left(a^{ij} u_{x_j}\right)_{x_i} dx &= -\int_{\partial U} v \sum_{i,j} a^{ij} u_{x_j} dS_i + \int_U \sum_{i,j} a^{ij} u_{x_i} v_{x_j} \\ &= \int_U \sum_{i,j} a^{i,j} u_{x_j} v_{x_i} \end{aligned}$$

If we define $B[u,v] = \int_U \sum_{i,j} a^{ij}(x) u_{x_j} v_{x_i} + \sum_i b^i u_{x_i} v + cu \cdot v dx$, we deduce

$$\begin{aligned} B[u,v] &= \int_U \sum_{i,j} a^{ij}(x) u_{x_j} v_{x_i} + \sum_i b^i u_{x_i} v + cu \cdot v dx \\ &= \int_U f v dx = (f, v)_{L^2(U)} \end{aligned}$$

Conversely, suppose $u \in C^2(\overline{U})$ satisfies $u\big|_{\partial U} = 0$ and

$$B[u,v] = \big(f,v\big)_{L^2(U)} \quad \forall v \in C^2(\overline{U}),\ v\big|_{\partial U} = 0$$

We can undo the integration by parts to deduce

$$\int_U (Lu - f) v dx = 0 \quad \forall v \in C^2(\overline{U}),\ v\big|_{\partial U} = 0$$

implies $Lu = f$.

We conclude :
$u \in C^2(\overline{U})$ with $u\big|_{\partial U} = 0$ solves (11) *if and only if*

$$B[u,v] = \big(f,v\big)_{L^2(U)} \quad \text{with} \quad v\big|_{\partial U} = 0$$

Note that $B[u,v]$ in fact makes sense for $u, v \in H^1(U)$. To encode the boundary conditions, we can require u, v vanish on ∂U in the *trace sense*(as in the trace theorem), or equivalently $u, v \in H_0^1(U)$. This motivates us to define **weak solution** of a PDE.

Definition) We say that $u \in H_0^1(U)$ is a **weak solution** of the problem

$$\begin{cases} Lu = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$$

For $y \in L^2(U)$ given, if

$$B[u,v] = \big(f,v\big)_{L^2(U)} \quad \forall v \in H_0^1(U)$$

A synopsis for existential proof of a PDE : will first show that weak solutions always exist, and provided sufficient smoothness of f, the weak solution is in fact smooth.