Elliptic Partial Differential Equations

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Perron's methods

Let $\Omega \subset \mathbb{R}^d$ be an open bounded connected set.

Definition) regular point $\xi \in \partial\Omega$, barrier function.
Theorem) *(Perron)* Statement? (proof is done later)
Definition) subharmonic, superharmonic functions$\in C^2(\Omega)$.

Mean value inequality) state and prove (in versions for harmonic/subharmonic/superharmonic functions.)

Corollary 1) *(Maximum principle)* If u is subharmonic, then $\sup_{\overline{B}_R} u = \sup_{\partial B_R} u$, and if u is superharmonic, then $\inf_{\overline{B}_R} u = \inf_{\partial B_R} u$.

Corollary 2) *(Strong maximum principle)* Let u be sub(resp. super)harmonic in Ω. Assume $\exists x_0 \in \Omega$ such that $\max_{\overline{\Omega}} u = u(x_0) = M$(resp. $\min_{\overline{\Omega}} u = u(x_0)$). Then $u =$ constant.

Definition) sub/super-harmonic function in $C^0(\Omega)$

Lemma) Let u_1, u_2 be subharmonic functions Then $\max(u_1, u_2)$ is subharmonic.

Now come back to the Dirichlet problem in the ball $B = B(0, R)$,

$$\begin{cases} \triangle u = 0 & \text{in } B \\ u = \varphi & \text{on } \partial B \end{cases} \qquad \cdots\cdots\cdots \ (D)$$

Theorem) For all $\varphi \in C^0(\partial B)$,

$$u(x) = \begin{cases} \int_{\partial B} \frac{R^2-|x|^2}{dw_d R} \frac{\varphi(y)}{|x-y|^d} dS_y & , x \in B \\ \varphi(x), \quad x \in \partial B \end{cases}$$

is $C^2(B) \cap C^0(\overline{B})$ and satisfies the Dirichlet problem (D).

Interior estimate for derivatives

Theorem 2.9) *(Interior estimates for harmonic functions)* Let $\triangle u = 0$ in Ω. Let $\Omega' \subset \Omega$ compact. Then $\forall \alpha \in \mathbb{N}^d$,

$$\sup_{\Omega} |\partial^\alpha u| \leq \left(\frac{|\alpha| d}{\operatorname{dist}(\Omega', \partial\Omega)} \right)^{|\alpha|} \sup_{\Omega} |u|$$

Fact : (Exercise) $\triangle u = 0$ implies $u \in C^\infty(\Omega)$ and u real analytic.

Theorem 3.1) Let $\Omega \subset \mathbb{R}^d$, then $u \in C^0(\Omega)$ harmonic in Ω *iff* $\forall y \in \Omega$ and $\forall R > 0$, $\overline{B(y,R)} \subset \Omega$,

$$u(y) = \frac{1}{dw_d R^{d-1}} \int_{\partial B(y,R)} u(x) dS_x \quad \cdots\cdots\cdots \ (\text{MID})$$

Theorem 3.2) Given $\Omega \subset \mathbb{R}^d$ domain. $(u_n)_{n=1}^{\infty} \subset C^0(\Omega)$ such that $\triangle u_n = 0$ in Ω *and*

$$\sup_{n\in\mathbb{N}} \sup_{x\in\Omega} |u_n(x)| < \infty$$

Then $\exists (u_{n_k})_k$ such that $u_{n_k} \xrightarrow{\text{unif.}} u$ in any $\Omega' \subset \Omega$ compact and $\triangle u = 0$ in Ω.

Indication : the construction of the Perron's solution is made through a process of the form $u = \sup\{v \in C^0(\Omega)$ subharmonic, $v \leq \varphi$ on $\partial\Omega\}$. u will be the candidate for our solution of the Dirichlet problem.

Proposition 3.4) u is subharmonic and v is superharmonic in Ω, $u, v \in C^0(\Omega)$. Then

$$v \geq u \text{ on } \partial\Omega \quad \Rightarrow \quad \begin{cases} v > u & \text{in } \overline{\Omega} \quad \textit{or} \\ v \equiv u & \text{in } \overline{\Omega} \end{cases}$$

Definition) harmonic lifting of $u \in C^0(\Omega)$ in a ball B.

Lemma 3.6) Let $u \in C^0(\Omega)$ subharmonic in Ω, U a harmonic lifting of u with respect to $\overline{B} \subset \Omega$. Then U is subharmonic in Ω and $U \geq u$ in Ω.

Lemma 3.7) Given $\{u_j\}_{j=1}^N$ subharmonic functions in Ω, we have that

$$u(x) = \max\{u_j(x) : 1 \leq j \leq N\}$$

is also subharmonic in Ω.

Theorem) *(Perron)* Let $\varphi \in C^0(\partial\Omega)$ and consider the Dirichlet problem $-\triangle u = 0$ in Ω and $u = \varphi$ on $\partial\Omega$. Then

(1) The classical Dirichlet problem has a unique solution $u \in C^2(\Omega)$ if $\partial\Omega$ *regular*.

(2) If Dirichlet problem is solvable for all φ, then $\partial\Omega$ is *regular*.

Poisson equation

Consider the problem

$$\begin{cases} -\triangle u = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases} \quad \cdots\cdots\cdots \; (D)$$

where Ω is a bounded set in $\mathbb{R}^d$ and $\partial\Omega$ is regular for the $\triangle$.

First we want to find the fundamental solution : $\triangle E = \delta_{x=0}$ with $E \in C^\infty(\mathbb{R}^d \backslash \{0\})$, i.e. the fundamental solution. We have

$$E(x) = \begin{cases} \frac{1}{2}|x|, \quad d = 1 \\ \frac{1}{2\pi}\log|x|, \quad d = 2 \\ \frac{1}{dw_d(d-2)}|x|^{2-d}, \quad d \geq 3 \end{cases}$$

(show this)

Green's representation

Proposition 4.3) Assume that $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$, solving Poisson's equation

$$\begin{cases} \triangle u = -f, & f \in C^0(\Omega) \\ u = \varphi, & \varphi \in C^0(\partial\Omega) \end{cases}$$

Then

$$u(y) = \int_{\partial\Omega} \Big(\varphi \frac{\partial E}{\partial n_x}(x-y) - E(x-y)\frac{\partial u}{\partial n_x}(x)\Big) dS_x + \int_\Omega E(x-y) f(x) dx$$

Definition) Green's function.

Definition) locally Hölder function $f \in C^0(\Omega)$

Theorem 5.3) Under these hypothesis ($f \in C^0(\Omega)$, $\varphi \in C^0(\partial\Omega)$ are locally Hölder), the Dirichlet problem has a unique solution $u \in C^2\Omega \cap C^0(\overline{\Omega})$.
(needs following lemmas)

Construction of the solution of (D) is made in two steps.

1st. Given $f \in C^0(\Omega) +$ locally Hölder α, we set

$$W(x) = \int_{\mathbb{R}^d} E(x-y)f(y)dy, \quad x \in \mathbb{R}^d \quad \cdots\cdots\cdots \ (\dagger)$$

We shall prove $W \in C^2(\mathbb{R}^d)$, $W|_{\partial\Omega} \in C^0(\partial\Omega)$ and $\triangle W = f$ in Ω.

2nd. We use Perrons' theorem to solve

$$\begin{cases} \triangle\tilde{u} = 0 & \text{in } \Omega \\ \tilde{u} = \varphi - W|_{\partial\Omega} & \text{on } \partial\Omega \end{cases}$$

Then we already know $\tilde{u} \in C^2(\Omega) \cap C^0(\overline{\Omega})$ since $\varphi - W|_{\partial\Omega} \in C^0(\partial\Omega)$.
Set $u = W + \tilde{y}$, then $-\triangle u = f$ and $u|_{\partial\Omega} = \varphi$ on $\partial\Omega$.

Lemma 5.1) Under the hypothesis that $f \in L^1(\Omega) \cap L^\infty(\Omega)$, W in $(\dagger)$ is $C^1(\mathbb{R}^d)$ and

$$\partial_{x_j} W(x) = \int_{\mathbb{R}^d} \partial_{x_j} E(x-y)f(y)dy, \quad \forall x \in \mathbb{R}^d$$

Lemma 5.2) Let $f \in L^1 \cap L^\infty(\Omega)$ and $|f(x) - f(y)| \le C_{\alpha,x}|x-y|^\alpha$ locally around any $x \in \Omega$. then W as above is $C^2(\mathbb{R}^d)$ and for any $\Omega_0 \subset \mathbb{R}^d$ such that the divergence theorem holds we have

$$\begin{aligned} \partial_{x_i x_j} W(x) = &\int_{\Omega_0} \partial_{x_i x_j} E(x-y)(f(y) - f(x))dy \\ &- f(x)\int_{\partial\Omega_0} \partial_{x_j} E(x-y)(n_y \cdot e_i)dS_y =: F_{ij}(x) \quad \cdots\cdots\cdots \ (**) \end{aligned}$$

where e_i are standard bases of $\mathbb{R}^d$, and $\partial_{x_i x_j} E(x-y)$ should be understood as a distribution.

proof) Use similar cutoff function, having in addition $\partial_{ij} V_\epsilon \to F_{ij}$.

Hölder solutions for $-\triangle u = f$

Definition) $[f]_{\alpha,\Omega}$, $\|f\|_{C^{0,\alpha}(\Omega)}$, $\|f\|_{C^{k,\alpha}(\Omega)}$.
Also define $\|f\|'_{C^k(\Omega)}$, $\|f\|'_{C^{k,\alpha}(\Omega)}$ (in terms of $d = \operatorname{diam}(\Omega)$)

Lemma 6.1) Let $x_0 \in \mathbb{R}^d$, $B_2 = B(x_0, 2R)$, $B_1 = B(x_0, R)$, $f \in C^{0,\alpha}(\overline{B}_2)$, $0 < \alpha < 1$, then W defined by $W(x) = \int_\Omega E(x-y)f(y)dy$ is in $C^{2,\alpha}(B_1)$. Furthermore,

$$\begin{aligned} &\left\|D^2 W\right\|'_{C^{0,\alpha}(B_1)} \le C\left\|f\right\|'_{C^{0,\alpha}(B_2)} \\ \textit{Equivalently,} \quad &\left\|D^2 W\right\|_{C^0(B_1)} + R^\alpha[D^2 W]_{\alpha,B_1} \le C\Big(\left\|f\right\|_{C^0(\overline{B_2})} + R^\alpha[f]_{\alpha,B_2}\Big) \end{aligned}$$

Corollary 6.2) Let $u \in C_0^2(\mathbb{R}^d)$, $f \in C^{0,\alpha}(\mathbb{R})$ compactly supported and such that $-\triangle u = f$ in $\mathbb{R}^d$. Then $u \in C^{2,\alpha}(\mathbb{R}^d)$, and if $B = B(x_0, R)$ is any ball containing $\operatorname{supp}(u)$ (the support is compact by its definition), we have

$$\left\|D^2 u\right\|'_{C^{0,\alpha}(B)} \le C[f]'_{0,\alpha,B}$$

for some $C = C(d, \alpha)$ and

$$\|u\|_{C^{1,\alpha}(B)} \leq C' R^2 [f]_{0,B}$$

for some $C' = C(d)$.

Proposition 6.3) $\Omega \subset \mathbb{R}^d$ domain, $f \in C^{0,\alpha}(\Omega)$ and $u \in C^2(\Omega)$ be the solution of $-\triangle u = f$ in Ω. Then $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ and satisfies, for all balls $B_1 = (x_0, R)$, $B_2 = B(x_0, 2R) \subset \Omega$,

$$\|u\|'_{C^{2,\alpha}(B_1)} \leq C(\|u\|_{C^0(B_2)} + \|f\|'_{C^{0,\alpha}(B_2)})$$

for some $C = C(d, \alpha) > 0$.

We can prove this estimate in more general setting, when $Lu = f$ with

$$Lu = \sum_{i,j=1}^{d} a^{ij}(x) \partial^2_{x_i x_j} u + \sum_{i=1}^{d} b^i(x) \partial_{x_i} u + c(x) u, \quad u \in C^2(\Omega)$$

$a^{ji} = a^{ij}$*(symmetric)* and $\Lambda|\xi|^2 \leq a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2$ for some $\lambda, \Lambda > 0$(*(uniformly elliptic)*) and all $\xi \in \mathbb{R}^d$.

Hölder norms(II)

Definition) *(More Hölder norms)* Assume Ω is compact. For $x, y \in \Omega$, let $d_x = \operatorname{dist}(x, \partial\Omega)$, $d_y = \operatorname{dist}(y, \partial\Omega)$ and $d_{x,y} := \min\{d_x, d_y\}$. Define $[u]^*_{k,0,\Omega}$, $|u|^*_{k,\Omega}$.
Also for $u \in C^{k,\alpha}$, define $[u]^*_{k,\alpha,\Omega}$ and a norm in $C^{k,\alpha}(\overline{\Omega})$, $|u|^*_{k,\alpha,\Omega}$.
For $j \in \mathbb{N}$, also define $[u]^{(j)}_{k,0,\Omega}$, $[u]^{(j)}_{k,\alpha,\Omega}$, $|u|^{(j)}_{k,\Omega}$, $|u|^{(j)}_{k,\alpha,\Omega}$

Let L_0 only have 2nd order terms, i.e. $L_0 u = \sum a^{ij} \partial^2_{x_i x_j} u$.

Proposition 7.2) Let L_0 satisfy *uniform ellipticity* and *symmetry*, and $u \in C^2(\Omega)$ satisfy $L_0 u = f$, $f \in C^{0,\alpha}(\Omega)$. Then

$$|u|^*_{2,\alpha,\Omega} \leq C(|u|_{0,\Omega} + |f|^{(2)}_{0,\alpha,\Omega})$$

for some $C \equiv C(d, \alpha, \lambda, \Lambda) > 0$.

Theorem 7.4) *(Interior Schauder estimate for $Lu = f$)* Let $\Omega \subset \mathbb{R}^d$ be open, L be uniformly elliptic, symmetric, $f \in C^{0,\alpha}(\Omega)$ and $|a^{ij}|^{(0)}_{0,\alpha,\Omega}, |b^i|^{(1)}_{0,\alpha,\Omega}, |c|^{(2)}_{0,\alpha,\Omega} \leq \tilde{\Lambda}$. Then if $u \in C^2(\Omega)$ with $Lu = f$, we have the estimate

$$|u|^*_{2,\alpha,\Omega} \leq C(|u|_{0,\Omega} + |f|^{(2)}_{\alpha,\Omega})$$

for a constant $C = C(d, \alpha, \lambda, \tilde{\Lambda})$.
-needs following interpolation inequalities for proof.

Remark : We may assume Ω is compact, as we may take nested sequence of compact sets that covers Ω, if the constants uniform in this family of compact sets - which is indeed the case.

Lemma 1) For any $\sigma, \tau \geq 0$,

$$|fg|^{(\sigma+\tau)}_{0,\alpha,\Omega} \leq |f|^{(\sigma)}_{0,\alpha,\Omega} + |g|^{(\tau)}_{0,\alpha,\Omega}$$

Lemma 2) *(Interpolation, Hörmander)* Let $u \in C^{2,\alpha}(\Omega)$, $\Omega \subset \mathbb{R}^d$ be a domain. Then for any $\epsilon > 0$, there is a constant $C(\epsilon) > 0$ such that

$$[u]^*_{j,\beta,\Omega} \leq C(\epsilon)|u|_{0,\Omega} + \epsilon[u]^*_{2,\alpha,\Omega}$$
$$|u|^*_{j,\beta,\Omega} \leq C(\epsilon)|u|_{0,\Omega} + \epsilon[u]^*_{2,\alpha,\Omega}$$

for $j = 0, 1, 2$, $0 \leq \alpha, \beta \leq 1$ and $j + \beta \leq 2 + \alpha$.

[More generally, we can think of inequalities in the following setting : Suppose we have an inequality of form $\|u\|_{B_1} \lesssim \|u\|_{B_0}^{\theta}\|u\|_{B_2}^{1-\theta}$, where $B_2 \subset B_1 \subset B_0$ are nested Banach spaces. Then we have $\|u\|_{B_2} \leq C_\epsilon\|u\|_{B_0} + \epsilon\|u\|_{B_2} + C\|f\|_X$, so $(1-\epsilon)\|u\|_{B_2} \leq C(\epsilon)\|u\|_{B_0} + C\|f\|_X$ for small ϵ.]

Recall our hypothesis, for $L = \sum a^{ij}\partial_i\partial_j + \sum b^i\partial_i + c$,

$$\text{(H1)} \quad |a^{ij}|^{(0)}_{0,\alpha,\Omega}, |b^i|^{(1)}_{0,\alpha,\Omega}, |c|^{(2)}_{0,\alpha,\Omega} \leq \Lambda, \quad \Lambda > 0$$

$$\text{(H2)} \quad a^{ij}(x) = a^{ji}(x), \quad \sum_{i,j=1}^{d} a^{ij}(x)\xi_i\xi_j > \lambda|\xi|^2$$

and

$$(H) \quad a^{ij}, b^i, c \text{ are Hölder continouus, } a^{ij} = a^{ji}, a \text{ is uniformly elliptic, with parameter } \lambda$$

Interior Hölder estimate

Corollary) Under (H1) and (H2), the solution of $Lu = f$ satisfies that $\forall \Omega' \subset\subset \Omega$,

$$\delta|\nabla u|_{0,\Omega'} + \delta^2|D^2u|_{0,\Omega'} + \delta^{2+\alpha}[\partial^2 u]_{\alpha,\Omega'} \leq C(|u|_{0,\Omega} + |f|_{0,\alpha,\Omega})$$

for $C = C(d,\alpha,\lambda,\Lambda,\Omega)$ and $\delta = \text{dist}(\Omega', \partial\Omega)$.
(what is this a corollary of?)

Boundary and Global estimates

Definition) Domains of class $C^{2,\alpha}$), boundary portion $T \subset \partial\Omega$.

The key point in the proof of Hölder interior was to use *Interpolation Estimates*. This would be the same in boundary estimates and global estimates.

Lemma 8.1) *(Interpolation estimates on the boundary)* Let $\Omega \subset \mathbb{R}^d_+$ open in $\mathbb{R}^d_+$ with a boundary portion T on $\{x_d = 0\}$. Assume $u \in C^{2,\alpha}(\Omega \cup T)$. Then $\forall \epsilon > 0$,

$$[u]^*_{j,\beta,\Omega\cup T} \leq C_\epsilon|u|_{0,\Omega} + \epsilon[u]^*_{2,\alpha,\Omega,\cup T},$$
$$|u|^*_{j,\beta,\Omega\cup T} \leq C_\epsilon|u|_{0,\Omega} + \epsilon[u]^*_{2,\alpha,\Omega\cup T}, \quad \forall\alpha \in [0,1],\ j+\beta < 2+\alpha$$

(not proved)

Lemma 8.2) Let $\Omega \subset \mathbb{R}^d_+$, T boundary portion, and $u \in C^2(\Omega \cup T)$ bounded solution of $Lu = f$ and $u = 0$ on T under hypothesis (H1) and (H2) on $\Omega \cup T$ and $f \in C^{0,\alpha}(\Omega \cup T)$. Then

$$|u|^*_{2,\alpha,\Omega\cup T} \leq C(|u|_{0,\Omega} + |f|^{(2)}_{0,\alpha,\Omega\cup T})$$

for $C = C(d,\alpha,\lambda,\Lambda)$.
-The proof is almost the same as **Theorem 7.4**.

Definition) Let T be a boundary portion of Ω, $x, y \in \Omega$, $\overline{d}_x := \text{dist}(x, \partial\Omega\backslash T)$, $\overline{d}_{x,y} = \min(\overline{d}_x, \overline{d}_y)$. Define $[u]^*_{k,\alpha,\Omega\cup T}$, $|u|^*_{k,\alpha,\Omega\cup T}$, where $|u|^*_{k,\Omega\cup T} := |u|^*_{k,0,\alpha,\Omega\cup T}$.

Curved boundaries of Class $C^{k,\alpha}$

Consider $\psi : \Omega \to \Omega'$, $C|x-y| \leq |\psi(x) - \psi(y)| \leq \tilde{C}(x-y)$. Make change of variable $u(x) = \tilde{u}(x') = \tilde{u}(\psi(x))$ then we would have

$$C|u(x)|_{j,\beta,\Omega} \lesssim |u'(x')|_{j,\beta,\Omega'} \lesssim \tilde{C}|u(x)|_{j,\beta,\Omega}C|u(x)|_{j,\beta,\Omega\cup T} \lesssim |\tilde{u}(x')|_{j,\beta,\Omega'\cup T'} \lesssim \tilde{C}|u(x)|^*_{j,\beta,\Omega\cup T}$$
$$C|u(x)|^{(\sigma)}_{0,\beta,\Omega\cup T} \lesssim |\tilde{u}(x)|^{(\sigma)}_{0,\beta,\Omega'\cup T'} \lesssim \tilde{C}|u(x)|^{(\sigma)}_{0,\beta,\Omega\cup T}$$

by chain rule.

Lemma 8.3) Let Ω be a bounded domain of class $C^{2,\alpha}$ in $\mathbb{R}^d$, $u \in C^{2,\alpha}(\overline{\Omega})$ satisfies $Lu = f$ in Ω, $u = 0$ on $\partial\Omega$ where $f \in C^{0,\alpha}(\overline{\Omega})$ and L satisfies (H2) and

$$|a^{ij}|_{0,\alpha,\Omega},\ |b^i|_{0,\alpha,\Omega},\ |c|_{0,\alpha,\Omega} \leq \Lambda.$$

Then we have, for some $\delta > 0$ not depending x_0 such that

$$|u|_{2,\alpha,\Omega\cap B(x_0,\delta)} \leq C(|u|_{0,\Omega} + |f|_{0,\alpha,\Omega}), \quad \forall x_0 \in \partial\Omega$$

for $C = C(d, \alpha, \lambda, \Lambda, \Omega)$ but not depending on x_0.

We had interior estimate in $C^{1,\alpha}$ and boundary estimates in $C^{2,\alpha}$. If $L = \sum a^{ij}(x)\partial^2_{x_i x_j} + \sum b^i(x)\partial_{x_i} + c(x)$, is uniformly elliptic, $a^{ij} = a^{ji}$ and $|a^{ij}|_{0,\alpha,\Omega}, |b^i|_{0,\alpha,\Omega}, |c|_{0,\alpha,\Omega} < \Lambda$, then we have :

Theorem) *(Global estimates)* Let Ω has $C^{2,\alpha}$ boundary and *bounded*, $f \in C^{0,\alpha}(\overline{\Omega})$ and $u \in C^{2,\alpha}(\overline{\Omega})$ satisfies $Lu = f$ in Ω, $u = \varphi$ on $\partial\Omega$ with $\varphi \in C^{2,\alpha}(\Omega)$. Then there is $C > 0$ such that

$$|u|_{2,\alpha,\Omega} \leq C(|u|_{0,\Omega} + |\varphi|_{2,\alpha,\Omega} + |f|_{0,\alpha,\Omega})$$

where $C = C(d, \alpha, \lambda, \Lambda, \Omega) > 0$.
-proof hint : To produce Hölder interior estimate for $\Omega' \subset\subset \Omega$, choose $\sigma = \delta$ of **Lemma 8.3** and let $\Omega_\sigma = \{x \in \Omega : \text{dist}(x, \partial\Omega) > \sigma\}$. For $x, y \in \Omega$, there are three possibilities :

(1) $x, y, \in \Omega_\sigma$, then interior Hölder inequality applies

(2) $x, y \in B(x, \delta)$, then boundary Hölder inequality applies.

(3) For a boundary point $x_j x \in \Omega_\sigma$, $y \in B_{x_j,\rho}$ or $x \in B(x_j, \rho), y \in B(x)j, \rho)$ then

$$\frac{|\partial_x^2 u(x) - \partial^2 u(y)|}{|x-y|^\alpha} \leq \frac{1}{\sigma^\alpha}(|\partial^2 u(x)| + |\partial^2 u(y)|) \leq C(|u|_0 + |f|_{0,\alpha})$$

Existence of Classical solutions

Theorem) Let L be elliptic satisfying (H) and $c(x) \leq 0$. Let Ω satisfy the exterior sphere condition(*i.e.* $\forall x_0 \in \partial\Omega$, $\exists B \subset \mathbb{R}^d \backslash \Omega$, a ball, such that $B \cap \overline{\Omega} = \{x_0\}$). Assume $f \in C^{0,\alpha}(\overline{\Omega})$ and $\varphi \in C^0(\partial\Omega)$. Then the Dirichlet problem $Lu = f$ in Ω and $u = \varphi$ on $\partial\Omega$ has a unique classical solution $u \in C^0(\overline{\Omega}) \cap C^{2,\alpha}(\Omega)$.
[a difference with the previous result is that we do not have $u \in C^{2,\alpha}(\overline{\Omega})$ anymore - so we do not have linear bound of u in terms of f and φ. –¿ what does this mean???]

proof) Proof to be done in the Example Sheet. But the idea is similar to that of Poisson equation.

- First see solvability in balls - idea of harmonic lifting appplies again.
- Use maximum principles for $Lu \geq 0$(or ≤ 0) (to be done in next lecture)
- Use compactness of solutions of $Lu = f$, that is a consequence of interior estimate.

Weak/Strong Maximum Principles for $Lu = f$

As usual, $Lu = \sum a^{ij}\partial^2_{x_i x_j} u + \sum b^i \partial_{x_i} u + c(x)u = f$ in Ω, $u = \varphi$ on $\partial\Omega$.

To establish maximum principle, we need to make strong restriction on c.

Theorem) Let L be (not necessarily uniform) elliptic (that is, $a^{ij}(x)\xi_i\xi_j \geq \lambda(x)|\xi|^2$), $c = 0$ in Ω and $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ with $Lu \geq 0$ in Ω and $\beta(x) := \frac{\sup_{i=1,\cdots,d} |b^i(x)|}{\lambda(x)} \leq \beta$ for all $x \in \Omega$ (recall, λ is the ellipticity constant.) Then

$$\sup_\Omega u = \sup_{\partial\Omega} u$$

Theorem) *(Strong maximum principle, E. Hopf)* We now let L be uniformly elliptic, say $\sum_{ij} a^{ij}\xi_i\xi_j \geq \lambda|\xi|^2$ with $\lambda > 0$ uniform. Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfy $Lu \geq 0$, and assume $\max_{z\in\overline{\Omega}} u(z) = u(z_0)$. Then

(1) If $c = 0$ and $z_0 \in \Omega$, then u is constant.

(2) If $c \leq 0$, c/λ bounded, and $u(z_0) \leq 0$ for some $z_0 \in \Omega$, then u is constant.

-need a lemma

Lemma) *(Hopf)* Let L be uniformly elliiptic and $Lu \geq 0$ in Ω. Take $x_0 \in \partial\Omega$ such that

(i) u is continuous at x_0,

(ii) $u(x_0) > u(x)$ for all $x \in \Omega$,

(iii) $\partial\Omega$ satisfies the *interior sphere condition.*

Then,

(1) if $c = 0$, then $\frac{\partial u}{\partial n_x}(x_0) > 0$,

(2) if $c \leq 0$, c/λ is bounded and $u(x_0) \geq 0$, then $\frac{\partial u}{\partial n_x}(x_0) \leq 0$.

Alexandroff maximum principle

Suppose $L = \sum_{ij} \partial^2_{x_i x_j} + \sum_i b^i \partial_{x_i} + c(x)$ satisfies ellipticity condtion (*i.e.* $A = (a^{ij})^d_{i,j=1}$ positive definite in Ω) in Ω. Define $D(x) = \det(A(x))$, $D^* = D^{1/d}$, then

$$0 \leq \lambda(x) \leq D^*(x) \leq \Lambda(x)$$

where $\lambda(x)$ is the minimum eigenvalue of $A(x)$ and $\Lambda(x)$ is the maximum eighenvalue of $A(x)$. Let $u \in C^2(\Omega)$, and $\Gamma^+ = \{y \in \Omega : u(x) \leq u(y) + \nabla u(y)(x - y), \forall x \in \Omega\}$, the **upper contact set of** u
[Remark : Has $D^2 u \leq 0$ *on* Γ^+*. In particular,* u *is concave in* Ω *iff* $\Gamma^+ = \Omega$*.].*

Theorem) *(Alexandroff)* If $\Omega \subset \mathbb{R}^n$, $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$, $Lu \geq f$ in Ω with $\frac{|b|}{D^*}, \frac{f}{D^*} \in L^d(\Omega)$, $c \leq 0$ in Ω, then

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ + C \| \frac{f^-}{D^*} \|_{L^d(\Gamma^+)}$$

for some constant $C = C(d, \text{diam}(\Omega), \| \frac{b}{D^*} \|_{L^d(\Omega)})$.
-is very useful. proof uses the next lemma

Lemma) Let $g \in L^1_{loc}(\mathbb{R}^d)$, $g \geq 0$. Then $\forall u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$, we have

$$\int_{B(0,\tilde{u})} g(x)dx \leq \int_{\Gamma^+_u} g(\nabla u)|\det(D^2 u)|dx$$

for $\tilde{u} = \frac{1}{\text{diam}(\Omega)}(\sup_{\overline{\Omega}} u - \sup_{\partial\Omega} u) \geq 0$.

Remark : $\forall x \in \Gamma^+$,

$$\det(D^2 u(x)) \leq \frac{1}{D} \Big(\frac{-a^{ij}(x)\partial^2_{x_i x_j} u}{d} \Big)^d$$

We first assume the lemma and prove the theorem.

Semilinear equation

Study one particular class of semilinear equations of form

$$\begin{cases} \triangle u = f(x, u) & x \in \Omega \\ u = 0 & x \in \partial\Omega \end{cases}$$

for some function $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}, (x, \xi) \mapsto f(x, \xi)$. Here, $f(x, u)$ makes non-linearity. Note that this has no contribution from ∇u (we call those equations for which f depends on ∇u, the quasilinear equations).

Theorem) Let Ω be *bounded* and has $C^{2,\alpha}$ boundary, $f \in C^1(\overline{\Omega} \times \mathbb{R})$. Assume that there are $\underline{u}, \overline{u} \in C^{2,\alpha}(\overline{\Omega})$ satisfying

$$\begin{cases} \underline{u} \leq \overline{u} \quad \text{in } \Omega \\ \triangle \underline{u} \geq f(x, \underline{u}) \quad \text{in } \Omega, \qquad \underline{u} \leq 0 \quad \text{on } \partial\Omega \\ \triangle \overline{u} \leq f(x, \overline{u}) \quad \text{in } \Omega, \qquad \overline{u} \geq 0 \quad \text{on } \partial\Omega \end{cases}$$

Then there exists $u \in C^{2,\alpha}(\overline{\Omega})$ such that $\triangle u = f(x, u(x))$ in Ω, $u = 0$ on $\partial\Omega$ and $\underline{u} \leq u \leq \overline{u}$ in Ω. *[This is called the method of sub-&-supersolutions]*
(The proof uses a version of Arzela-Ascoli theorem - state it/ proof to be done in the example sheet)

Corollary) Let $\Omega \subset C^{2,\alpha}$ be bounded, and let $f \in C^1(\overline{\Omega} \times \mathbb{R})$ and f is bounded. Then there exists a solution $u \in C^{2,\alpha}(\overline{\Omega})$ of $\triangle u = f(x, u)$ in Ω and $u = 0$ on $\partial\Omega$.

Theorem) *(Gidas, Ni & Nirenberg)* Let $B = B(0,1) \subset \mathbb{R}^d$. Assume that $u \in C^0(\overline{B}) \cap C^2(B)$ is a positive solution ($u \geq 0$ on B) of

$$\begin{cases} \triangle u + f(u) = 0 & \text{in } B \\ u = 0 & \text{on } \partial B \end{cases}$$

Assume that f is *locally Lipschitz* in $\mathbb{R}$. Then u is radially symmetric and

$$\frac{\partial u}{\partial r}(x) < 0 \quad \text{whenever } x \neq 0.$$

(not proving)

Theorem) *(Varadham, Maximum principle in narrow domains)* Consider $Lu = \sum_{ij} a^{ij} \partial^2_{x_i x_j} u + \sum_i b^i \partial_{x_i} u + c(x)u$, a^{ij} positive definite pointwise in Ω, $|b^i| + |c| \leq \Lambda$, $\det(a^{ij}(x)) \geq \lambda$, $\delta := \operatorname{diam}(\Omega) > 0$. Assume $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ satisfies $Lu \geq 0$ in Ω and $u \leq 0$ in $\partial\Omega$. Then $\exists C_\delta = C_\delta(d, \Lambda, \lambda) > 0$ such that,

$$|\Omega| \leq C_\delta \quad \text{implies} \quad u \leq 0 \text{ in } \Omega.$$

where $|\Omega|$ is the Lebesgue measure of Ω. *[Remark : we do not need condition on sign of c]*

Theorem) *(Serrin, Comparison principle)* Suppose that $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ with $Lu \geq 0$ in Ω and $u \leq 0$ in Ω (not $\partial\Omega$, all of Ω), with L having continuous coefficients (no bounds necessary). Then

- either $u < 0$ in Ω,
- or $u \equiv 0$ in Ω.

Lemma) *(Fanghua Lin & Qing Han)* (See Section 2.6 of Han, Lin ([3])) Let Ω a bounded convex domain in the x_1-direction and symmetric with respect to $\{x_1 = 0\}$. If $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ satisfies

$$\begin{cases} \triangle u + f(u) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

with f *locally Lipschitz.* Assume that $u > 0$ in Ω, then u is symmetric with respect to x_1 direction and

$$\frac{\partial u}{\partial x_1}(x) < 0 \quad \forall x \in \Omega,\ x_1 > 0$$

Let L be an operator of form

$$L = -\sum_{i=1}^d \partial_{x_i}(a^{ij}(x)\partial_{x_i} u) + c(x) \quad (\text{so that } b^i \equiv 0)$$

and consider equation $Lu = f$ in Ω. We impose conditions

$$\begin{cases} a^{ij} \in L^\infty \cap C^0(\Omega), \\ a^{ij} = a^{ji} \\ a^{ij}(\xi)\xi_i\xi_j \geq \lambda|\xi|^2, \ \forall \xi \in \mathbb{R}^d \\ f \in L^{\frac{2d}{d+2}}(\Omega) \quad (\text{exponent chosen for Sobolev embedding}) \end{cases}$$

u is a weak solution of $Lu = f$ if

$$\int_\Omega \Big(\sum_{i,j=1}^{n} a^{ij}(x)\partial_{x_j} u \partial_{x_i}\varphi + cu\varphi\Big)dx = \int_\Omega \varphi f dx, \quad \forall \varphi \in H_0^1(\Omega)$$

We want to characterize Hölder continuity in terms of the growth of local integrals.

Let $\Omega \subset \mathbb{R}^d$ be bounded and connected. Given $u \in L^1_{loc}(\Omega)$, given $x_0 \in \Omega$, $r > 0$ such that $B(x_0, r) \subset \Omega$, we define

$$u_{x_0,r} = \frac{1}{B(x_0,r)} \int_{B(x_0,r)} u(x)dx$$

Theorem) Assume that $u \in L^2(\Omega)$ and there are $M > 0$, $\alpha \in (0,1)$.

$$\int_{B(x_0,r)} |u(x) - u_{x_0,r}|^2 dx \le M^2 r^{d+2\alpha}, \quad \forall B(x_0,r) \subset \Omega$$

Then u has continuous correction in $C^{0,\alpha}(\Omega)$ and $\forall \overline{\Omega'} \subset \Omega$, we have

$$|u|_{0,\alpha,\Omega'} \le C(M + \|u\|_{L^2(\Omega)})$$

for some $C = C(d, \alpha, \Omega, \Omega') > 0$.

Corollary) Suppose $u \in H^1_{loc}(\Omega)$ satisfies that for some $\alpha \in (0,1)$,

$$\int_{B(x_0,r)} |\nabla u|^2 dx \le M^2 r^{d-2+2\alpha}, \quad \forall B(x_0,r) \subset \Omega$$

Then $u \in C^{0,\alpha}(\Omega)$ and $\forall \Omega'$ with $\overline{\Omega'} \subset \Omega$,

$$|u|_{0,\alpha,\Omega'} \le C(M + \|u\|_{L^2(\Omega)})$$

for some $C = C(d, \alpha, \Omega', \Omega) > 0$.

We expect that if $a^{ij} \in C^0(\overline{\Omega})$, $c = c(x) \in L^d(\Omega)$, $f \in L^{\frac{2d}{d+2}}(\Omega)$ then the weak solution satisfies $u \in H^1(\Omega) \cap C^{0,\alpha}(\Omega)$. To use perturbation argument, we may write $u = v + w$ where w is the weak solution of $L_0 w = 0$ where $L_0 w := -\sum_{i,j} \partial_{x_j}(a^{ij}(x_0)\partial_{x_i} w)$ and v solves

$$\sum_{i,j=1}^{d} \int_B a^{ij}(x_0)\partial_{x_i} v \partial_{x_j}\varphi dx = \int_B (f\varphi - cu\varphi)dx + \sum_{i,j=1}^{d} \int (a^{ij}(x_0) - a^{ij}(x))\partial_{x_i} u \partial_{x_j}\varphi dx, \quad \forall \varphi \in H_0^1(B)$$

The first step would be to study the constant-coefficient case to have control on w.

Theorem) *(Caccioppoli's inequality for harmonic functions)* If $w \in C^1$ solved $L_0 w = 0$ weakly, *i.e.* it satisfies $\int_B a^{ij}(x_0)\partial_{x_i} w \partial_{x_j}\varphi dx = 0$ for all $\varphi \in H_0^1(B)$, then

$$\int_B |\nabla w|^2 \eta^2 dx \le C \int_B |\nabla \eta|^2 |w|^2 dx, \quad \forall \eta \in C_0^1(B)$$

for $C = C(\lambda, \Lambda) > 0$ where $\lambda|\xi|^2 \le \sum_{ij} a^{ij}(x_0)\xi_i\xi_j \le \Lambda|\xi|^2$.

Corollary) *(Precis version of Cacciofolli's inequality)* With same choice of w as above, for all $0 < r < R \le 1$,

$$\int_{B(0,r)} |\nabla w|^2 dx \le \frac{C}{(R-r)^2} \int_{B(0,R)} |w|^2 dx$$

[This can be thought of as a reverse of Poincaré inequality]

Proposition) Assume that w is a weak solution of $\sum_{i,j=1}^d \int_B a^{ij}\partial_{x_i} w \partial_{x_j}\varphi dx$ for all $\varphi \in H_0^1(B)$. Then for all $0 < \rho \le r$,

$$\int_{B(0,\rho)} |w|^2 dx \le C\Big(\frac{\rho}{r}\Big)^d \int_{B(0,r)} |w|^2 dx,$$

$$\int_{B(0,\rho)} |w - w_{0,\rho}|^2 dx \le C\Big(\frac{\rho}{r}\Big)^{d+2} \int_{B(0,r)} |w - w_{0,r}|^2 dx$$

where $C = C(\lambda, \Lambda)$.

Corollary) Under the previous hypothesis, we have that $\forall u \in H^1(B(x_0, r))$ and $\forall 0 < \rho \leq r$, we have

$$\int_{B(x_0,\rho)} |\nabla u|^2 dx \leq C\Big(\big(\frac{\rho}{r}\big)^d \int_{B(x_0,r)} |\nabla u|^2 dx + \int_{B(x_0,r)} |\nabla(u-w)|^2 dx\Big)$$

Theorem) Let $u \in H^1(B)$ be a weak solution of $Lu = f$.

$$\int_B \sum_{i,j=1}^d a^{ij}(x)\partial_{x_i} u \partial_{x_j} \varphi dx + \int_B c(x) u\varphi dx = \int f\varphi dx, \quad \forall \varphi \in H^1_0(B)$$

with $a^{ij} = a^{ji}$, $a^{ij} \in C^0(\overline{B})$, $c \in L^d(B)$, $f \in L^q$, $q \in (\frac{2}{d}, d)$ and $d \geq 2$. Then

$$\int_{B(x,r)} |\nabla u|^2 dx \leq C r^{d-2+2\alpha}\big(\|f\|^2_{L^q(B_1)} + \|u\|^2_{H^1}\big)$$

with $\alpha = 2 - \frac{d}{q} \in (0,1)$ and $C \equiv C(\lambda, \Lambda, \|c\|_{L^d(B)}, \tau) > 0$ where $\tau : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{0\}$ sufficiently chosen so that

$$|a^{ij}(x) - a^{ij}(y)| \leq \tau(|x-y|), \quad \forall x, y \in B$$

De Giorgi's Theorem, Part I

Let $B = B(0,1)$. Let $L = \sum a^{ij}(x)\partial_{ij} + c(x)$ (so that $b = 0$) with λ-uniformly elliptic, $a^{ij} \in L^\infty(B)$(not even continuous) and $c \in L^q(B)$ for $q > d/2$.

Definition) *(weak subsolution)* Let $u \in H^1(B)$ is a **weak subsolution** of $Lu = f$, for f given, if...

Theorem) *(De Giorgi, part I)* Under the previous hypothesis, assume in addition that $f \in L^q(B)$, $q > d/2$ and $\exists \Lambda > 0$ suhch that

$$\sup_{i,j} |a^{ij}|_{L^\infty(B)} + \|c\|_{L^q} \leq \Lambda$$

Then, if $u \in H^1(B)$ is a *weak subsolution* of $Lu = f$, then

$$u^+ \in L^\infty_{loc}(B) \quad \text{and}$$
$$\sup_{B(0,1/2)} u^+ \leq C(\|u^+\|^2_{L^2(B)} + \|f\|^2_{L^q(B)})$$

[The same bound was proved by Nash, with a method to which applies also to parabolic equations. But De Giorgi's method gives better insight.]
(the proof is *very very* long)

De Giorgi's Theorem, Part II

Set $B = B(0,1)$. We now write Lu in the *divergence form*

$$Lu = \sum_{i,j=1}^d \partial_{x_i}(a^{ij}(x)\partial_{x_j} u) + c(x)$$

Here, we assume $c = 0$. Also let $a^{ij} \in L^\infty(B)$, $a^{ij} = a^{ji}$ and $\lambda|\xi|^2 \leq \sum a^{ij}\xi_i\xi_j \leq \Lambda|\xi|^2$.

Theorem) *(De Giorgi, part II)* If u is a weak solution of $Lu = 0$ in $B(0,1)$, then $u \in C^{0,\alpha}(b)$ and

$$\sup_{x \in B(0,1/2)} |u(x)| + \sup_{x,y \in B(0,1/2)} \frac{|u(x) - u(y)|}{|x-y|^\alpha} \leq C(d, \Lambda/\lambda)\|u\|_{L^2(B)}$$

for some $\alpha = \alpha(d, \lambda/\Lambda) \in (0,1)$.

We will need three key ingredients to prove the theorem.

- Poincaré-Sobolev ienquality
- Density theorem
- Oscillation theorem

Lemma) Let $\Phi \in C^{0,1}_{loc}(\mathbb{R})$ by *convex* and $\Phi' \geq 0$. If u is a subsolution of $Lu = 0$, then we have that $v = \Phi(u)$ is also a subsolution of $Lu = 0$ whenever $v \in H^1_{loc}(B)$.

Remark : if u is a supersolution and Φ is concave, then $\Phi(u)$ is a subsolution.

Remark : if u is a subsolution, then $v = (u-k)^+$ is also a subsoltuion, with choice of $\Phi(s) = (s-k)^+$.

Proposition) *(Poincaré-Sobolev inequality)* For any $\epsilon > 0$, there is $C = C(\epsilon, d) > 0$ such that $\forall u \in H^1(B)$ satisfying $\text{meas}\{x \in B; u(x) = 0\} \geq \epsilon \cdot \text{meas}(B)$, we have

$$\int_B |u|^2 dx \leq C(\epsilon, d) \int_B |\nabla u|^2 dx$$

Proposition) *(Density theorem)* Suppose u is a positive supersolution of $Lu = 0$ in $B(0,2)$ satisfying $\text{meas}\{x \in B(0,1); u(x) \geq 1\} \geq \epsilon \cdot \text{meas}(B)$. Then there is $C = C(\epsilon, d, \Lambda/\lambda) > 0$ such that

$$\inf_{B(0,1/2)} u \geq C$$

Similarly, if u is a negative subsolution, then $\sup_{B(0,1/2)} u \leq C$.

Definition) The **oscillation** of u

Proposition) Assume that u is a bounded solution of $Lu = 0$ in $B(0,2)$, then there is $\gamma = \gamma(d, \Lambda/\lambda) \in (0,1)$ such that

$$\text{osc}_{B(0,1/2)}(u) \leq \gamma \text{osc}_{B(0,1)}(u)$$