Attempt no more than **THREE** questions. There are **FOUR** questions in total. The questions carry equal weight and each question accounts for **40%** of the total marks.

1 This question is bookwork about real and functional analysis concepts introduced and used in the lectures.

1. Give two definitions of real analyticity of a function defined on the real line, first in terms of the convergence of the Taylor series, second in terms of the growth on the derivatives, and prove their equivalence.

2. Give an example of a function which is smooth but not real analytic on $\mathbb{R}$ (justify entirely the answer).

3. State the Liouville theorem for analytic functions in the complex plane. Is a similar statement satisfied for real analytic functions on $\mathbb{R}$?

4. Give the definition of being *separable* for a Hilbert space, and show that the space $L^2_{loc}(\mathbb{R})$ (functions square integrable on any compact interval) endowed with the inner product $\langle f, g\rangle = \lim_{R\to\infty} \frac{1}{R}\int_{-R}^{R} f(x)g(x)\,\mathrm{d}x$ defines a non-separable Hilbert space.

5. State and prove the existence and uniqueness of a projection on a closed non-empty convex subset of a Hilbert space.

6. State and prove Riesz representation theorem.

7. State and prove Lax-Milgram theorem.

2 This question is bookwork about heat, Laplace and wave equations.

1. Consider the heat equation $\partial_t u = \partial_x^2 u$ in $\mathbb{R}\times\mathbb{R}$. Show that the line $\{t = 0\}$ is characteristic and that there does not exist an analytic solution u in a neighborhood of $(0, 0)$ with $u = (1 + x^2)^{-1}$ on $\{t = 0\}$.

2. Give the formula for the wave and Schrödinger and Laplace equations and their characteristic hypersurfaces.

3. Prove the elliptic regularity principle for the Laplace equation in a smooth bounded connected domain $\mathcal{U} \subset \mathbb{R}^\ell$, $\ell \geqslant 2$.

4. Formulate the Cauchy problem for the Laplace equation. Assuming that the Cauchy data are real analytic on some real analytic Cauchy hypersurface $\Gamma \subset \mathcal{U}$, can we apply the Cauchy-Kowelevskaia theorem?

5. Assuming that the Cauchy data are C^2 but not C^3 on Γ, can we apply Cauchy-Kowalevskaia's theorem? Is there any C^2 solution locally around Γ?

6. Consider the wave equation with smooth Cauchy data on the hypersurface $\{t = 0\} \times \mathbb{R}^n$. State and prove the key "a priori estimate" seen in the lectures in the whole space domain $\mathbb{R}^n$.

7. State and prove the stronger *local* version of the previous a priori estimate, and prove as a consequence that if the Cauchy data has compact support, then the solution has compact support on each time slice. How fast can the support spread out in time?

3 This question deals with solving elliptic equations. We consider in this whole question a domain $\mathcal{U} \in \mathbb{R}^\ell$, $\ell \geqslant 1$, smooth, bounded and connected.

1. Consider the *Neumann problem* of the Poisson equation

$$\begin{aligned} -\Delta u = f \quad &\text{in } \mathcal{U} \\ \nabla_x u \cdot \mathbf{n}(x) = 0 \quad &\text{in } \partial\mathcal{U} \end{aligned}$$

with f a smooth function on $\mathcal{U}$ and where $\mathbf{n}(x)$ is the outgoing normal vector. We say that u is a weak solution to this problem if $u \in H^1(\mathcal{U})$ and

$$\forall\, v \in H^1(\mathcal{U}), \quad \int_{\mathcal{U}} \nabla u \cdot \nabla v \,\mathrm{d}x = \int_{\mathcal{U}} f v \,\mathrm{d}x.$$

(a) Prove that (1) if u is a weak solution and u is smooth on $\bar{\mathcal{U}}$ then u is a classical solution, and (2) that a classical C^2 solution is a weak solution.

(b) Prove that the weak solution is unique up to the choice of a constant.

(c) Prove the *Neumann-Poincaré inequality*

$$\forall\, v \in H^1(\mathcal{U}), \quad \int_{\mathcal{U}} (v - m[v])^2 \,\mathrm{d}x \leqslant C_P \int_{\mathcal{U}} |\nabla_x v|^2 \,\mathrm{d}x, \quad m[v] := \int_{\mathcal{U}} v \,\mathrm{d}x$$

for some constant $C_P > 0$.

Hint. Argue by contradiction and use the Rellich-Kondrachov theorem in the form that a sequence bounded in $H^1(\mathcal{U})$ is compact in $L^2(\mathcal{U})$.

(d) Prove the existence of a weak solution as soon as $\int_{\mathcal{U}} f \,\mathrm{d}x = 0$ by following the Hilbert analysis strategy we have used for the Dirichlet problem.

(e) Prove that the previous condition on f is necessary for the existence of a weak solution.

2. Consider the following boundary-value problem

$$\begin{aligned} \Delta^2 u = f \quad & \text{in } \mathcal{U} \\ u = \nabla_x u \cdot \mathbf{n}(x) = 0 \quad & \text{on } \partial\mathcal{U} \end{aligned}$$

with f a smooth function on $\mathcal{U}$ and where $\mathbf{n}(x)$ is the outgoing normal vector. We say that u is a weak H^2_0 solution to this problem if $u \in H^2_0(\mathcal{U})$ and

$$\forall\, v \in H^2_0(\mathcal{U}), \quad \int_{\mathcal{U}} \Delta u \Delta v \,\mathrm{d}x = \int_{\mathcal{U}} f v \,\mathrm{d}x.$$

(a) Prove that (1) if u is a weak solution and u is smooth on $\bar{\mathcal{U}}$ then u is a classical solution, and (2) that a classical C^4 solution is a weak solution.

(b) Prove that the weak solution is unique.

(c) Prove the existence of a weak solution by following the Hilbert analysis strategy we have used for the Dirichlet problem.

Hint. Use both the Dirichlet-Poincaré inequality proved in lectures and the Neumann-Poincaré inequality proved above.

4 This question deals with the vanishing viscosity approximation of the nonlinear transport equation.

1. Consider the equation

$$\partial_t u + \partial_x F(u) = \epsilon \partial^2_{xx} u, \quad x \in \mathbb{R},\ t \in (0, +\infty) \tag{1}$$

with $\epsilon > 0$ and F a C^2 function on $\mathbb{R}$ with F' bounded.

(a) Arguing *a priori*, i.e. assuming the existence of a global smooth solution u_ϵ decaying at infinity faster than any polynomials, perform energy estimates to establish the following estimate on the L^2 norm

$$\forall\, t \geqslant 0, \quad \int_{\mathbb{R}} u_\epsilon(t,x)^2 \,\mathrm{d}x \leqslant e^{C_0 t} \left(\int_{\mathbb{R}} u_\epsilon(0,x)^2 \,\mathrm{d}x \right)$$

and provide a formula for bounding above the constant C_0 in terms of F and ϵ.

(b) Arguing *a priori* as in the previous question, perform energy estimates to establish the following estimate on the L^2 norm of the first derivative

$$\forall\, t \geqslant 0, \quad \int_{\mathbb{R}} \left(\partial_x u_\epsilon(t,x)\right)^2 \,\mathrm{d}x \leqslant e^{C_1 t} \left(\int_{\mathbb{R}} \left(\partial_x u_\epsilon(0,x)\right)^2 \,\mathrm{d}x \right)$$

and provide a formula for bounding above the constant C_1 in terms of F and ϵ.

(c) How does these constants C_0 and C_1 behave as $\epsilon \to 0$? Can you relate it to the behavior of the solution when $\epsilon = 0$.

2. Consider again the equation

$$\partial_t u + \partial_x F(u) = \epsilon \partial^2_{xx} u, \quad x \in \mathbb{R},\ t \in (0, +\infty) \tag{2}$$

with $\epsilon > 0$, but now with F a C^2 uniformly convex function on $\mathbb{R}$.

(a) Prove that if $u_\epsilon(t,x) = v(x - \sigma t)$ is a *travelling wave* solution for some C^2 function v on $\mathbb{R}$ and $\sigma \in \mathbb{R}$, then v satisfies the implicit formula

$$\forall\, s \in \mathbb{R}, \quad s = \int_c^{v(s)} \frac{\epsilon}{F(z) - \sigma z + b}\, \mathrm{d}z$$

for some constants $b, c \in \mathbb{R}$.

(b) Assuming that v converges to u_l (resp. u_r) at $z \to -\infty$ (resp. $z \to +\infty$), prove that the travelling wave speed σ satisfies

$$\sigma = \frac{F(u_l) - F(u_r)}{u_l - u_r}.$$

(c) Assuming $u_l > u_r$ and the existence of the solution $u_\epsilon(t,x) = v(x - \sigma t)$ described in parts (a)-(b) of this question, describe the limit $\lim_{\epsilon \to 0} u_\epsilon$ and explain your answer.