Part III — Topics in Set Theory

Based on lectures by B Löwe

Notes taken by myself

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0 Introduction

In 1900 at the ICM in Paris, Hilbert tried to predict and steer the mathematics of the coming century, presenting a list of 23 problems he thought would be the most important.
Problem 1: Is there an uncountable set $A \subseteq \mathbb{R}$ not in bijection with $\mathbb{R}$? Cantor thought not, and formulated
Continuum Hypothesis:

$$\forall A \subseteq \mathbb{R}(A \preceq \mathbb{N} \vee A \sim \mathbb{R})$$

The answer is: The axioms of set theory do not answer this question. More precisely, if ZFC is consistent, then so are $\mathsf{ZFC} + \mathsf{CH}$ and $\mathsf{ZFC} + \neg\mathsf{CH}$.

Comments on incompleteness:

1. Gödel's First Incompleteness Theorem: If a theory T is 'reasonable' (recursive, consistent, and strong enough), then T is not complete, i.e. there is a sentence φ such that $T \nvdash \varphi$ and $T \nvdash \neg\varphi$.

2. Gödel's Second Incompleteness Theorem: The statement $\mathrm{Cons}(T)$ is one such φ. This means that all consistency proofs in set theory must be <u>relative</u> consistency proofs, i.e. of the form "$\mathrm{Cons}(T) \to \mathrm{Cons}(S)$".

3. In order to prove statements like $\mathrm{Cons}(T) \to \mathrm{Cons}(S)$, we use Gödel's Completeness Theorem: T syntactically proves φ ($T \vdash \varphi$) iff T semantically implies φ. Apply this to $\varphi = \perp$ to get "T is consistent $\iff$ T has a model".
So we prove $\mathrm{Cons}(T) \to \mathrm{Cons}(S)$ by taking a model of T and constructing a model of S from it.
In our case, start with a model M of ZFC and construct models M_1 of $\mathsf{ZFC}+\mathsf{CH}$, M_2 of $\mathsf{ZFC}+\neg\mathsf{CH}$. For this, we need a lot of control over both M and the 'construction mechanism'. In particular, M shouldn't be strange.
[Since ZFC should not prove $\mathrm{Cons}(\mathsf{ZFC})$, by Gödel 2 $\mathsf{ZFC} + \neg\,\mathrm{Cons}(\mathsf{ZFC})$ is consistent. So there is a model M of $\mathsf{ZFC} + \neg\,\mathrm{Cons}(\mathsf{ZFC})$ by Gödel's completeness theorem, i.e. a model M which does not contain any models of ZFC. This M has to be weird, e.g. in the sense that the naturals in M can't be the real naturals, since then the proof of $\neg\,\mathrm{Cons}(\mathsf{ZFC})$ would be

an actual proof. We don't want models like this!]

First goal of the course: Use basic model theory of set theory to 'construct' (with appropriate additional assumptions) models M of ZFC that are 'nice'.

1 Von Neumann hierarchy

Assume that we're working in some "big" model M of ZFC.

Definition. $V_0 := \emptyset$
$V_{\alpha+1} = \mathcal{P}(V_\alpha)$
$V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha$
$x \in V \iff x \in V_\alpha$ for some α.

We can use this to prove $\mathrm{Cons}(\mathsf{ZF}^-) \to \mathrm{Cons}(\mathsf{ZF})$.

Definition. Z^- contains the axioms of Pairing, Power Set, Union, Separation, Extensionality, and Infinity.
$\mathsf{ZF}^- = \mathsf{Z}^- +$ Replacement
$\mathsf{Z} = \mathsf{Z}^- +$ Foundation
$\mathsf{ZF} = \mathsf{ZF}^- +$ Foundation.

Theorem.
1. If $M \models \mathsf{Z}^-$, then $V \models \mathsf{Z}$
2. If $M \models \mathsf{ZF}^-$, then $V \models \mathsf{ZF}$
3. If $M \models \mathsf{ZF}$, then $M = V$.

Definition. $\rho(x) := \min\{\alpha \mid x \in V_{\alpha+1}\}$ is the (Miramanoff) 'rank' of x. Assuming ZF, ρ is defined for all sets x.

Some key properties of V_α:

1. V_α is a transitive set, i.e. $x \in y \in V_\alpha \to x \in V_\alpha$. Equivalently, $V_\alpha \subseteq \mathcal{P}(V_\alpha)$.
2. For $\alpha \leq \beta$, $V_\alpha \subseteq V_\beta$.
3. $V_\alpha = \{x \mid \rho(x) < \alpha\}$
4. $x \in y \to \rho(x) < \rho(y)$
5. $\rho(x) = \sup\{\rho(y) + 1 \mid y \in x\}$
6. $\mathrm{Ord} \cap V_\alpha = \alpha$
7. $\rho(\alpha) = \alpha$

We can imagine this hierarchy as a big V, with successive levels. (Some might claim this is why it called V, but this is 'fake news'.) This picture is completely wrong with regards to size, since each level is much bigger than the previous level.

We might try to find λ such that V_λ is 'closed' under the axioms of set theory, providing us with a nice model of set theory. Looking at this more carefully, we see that for $x, y \in V_\alpha$:

Pairing: $\{x, y\} \subseteq V_\alpha$ so $\{x, y\} \in \mathcal{P}(V_\alpha) = V_{\alpha+1}$.
Union: $\bigcup x \in V_\alpha$. (In fact, this follows from $x \in V_{\alpha+1}$.)
Power set: $\mathcal{P}(x) \subseteq V_{\alpha+1}$, so $\mathcal{P}(x) \in V_{\alpha+2}$. (In fact, $\mathcal{P}(x) \in V_{\alpha+1}$)
Separation: If φ is a formula and $p_1, \ldots, p_n \in V_\alpha$ are parameters, then $\{y \in x \mid \varphi(y, p_1, \ldots, p_n)\} \in V_{\alpha+1}$. (I suppose φ should be relativised to our model V_λ.)
Putting all this together, we get: if $\lambda > \omega$ is a limit ordinal, then $V_\lambda \models \mathsf{Z}$.
In particular, $V_{\omega+\omega} \models \mathsf{Z}$. But we know abstractly that it shouldn't satisfy Replacement. Our goal is to understand why.

Proposition. (ZFC)
$\mathsf{CH} \iff 2^{\aleph_0} = \aleph_1$

Proof. First: what does $2^{\aleph_0} = \aleph_1$ mean? Usually, $X^Y = \{f : Y \to X\}$, and if X, Y are cardinals, we write X^Y for the unique cardinal κ such that $\kappa \sim X^Y$.
In ZFC, we usually define $\aleph$'s by $\aleph_0 := \omega$, $\aleph_{\alpha+1} :=$ least ordinal γ which does not inject into $\aleph_\alpha$, $\aleph_\lambda := \bigcup_{\alpha<\lambda} \aleph_\alpha$ for λ a limit. In particular, $2^{\aleph_0} = \aleph_\alpha$ for some α. By Cantor, $\alpha \geq 1$. Suppose $2^{\aleph_0} > \aleph_1$, so there is a bijection $f : \mathbb{R} \to \aleph_\alpha$ for some $\alpha > 1$. Since $\aleph_1 \subseteq \aleph_\alpha$, consider $A := f^{-1}\aleph_1$. Clearly $f|_A : A \to \aleph_1$ is a bijection, so $A \sim \aleph_1$. So $\neg\mathsf{CH}$.
The direction $2^{\aleph_0} = \aleph_1 \to \mathsf{CH}$ should be clear. $\square$

We saw last time: If $\lambda > \omega$ is a limit ordinal, then $V_\lambda \models \mathsf{Z}$. What about replacement?
Axiom schema of replacement: Let φ be a forumula with $n + 2$ free variables. Suppose for every $x, p_1, \ldots, p_n$, $\varphi(x, y, p_1, \ldots, p_n) \wedge \varphi(x, y', p_1, \ldots, p_n) \to y = y'$. Then $\forall p_1 \ldots p_n \forall X \exists r \forall y (y \in r \iff \exists x \in X : \varphi(x, y, p_1, \ldots, p_n))$.
V_ω is a very nice object, but it's no good for set theory (doesn't have infinite sets). $V_{\omega+\omega}$ is the first candidate for a model of ZF. But consider $\varphi(x, y) :=$ '$x \in \mathbb{N} \wedge y = \omega + x$. ('Parameter-free definition of the map $f : n \mapsto \omega + n$.')
$f''\mathbb{N} = \{\omega + n \mid n \in \mathbb{N}\}$ has rank $\omega + \omega$, so it is not in $V_{\omega+\omega}$.
Let's analyse this argument: if λ is a limit ordinal, $\alpha < \lambda$, and there is a definable function $f : \alpha \to \lambda$ with $f''\alpha$ unbounded in λ, then $V_\lambda \not\models$ Replacement.
Recall: If λ is any ordinal, we say that a set $C \subseteq \lambda$ is <u>cofinite</u> in λ, if $\forall \alpha \in \lambda$, $\exists \beta \in C$ with $\beta \geq \alpha$.

Definition. $cf(\lambda) := \min\{\alpha \mid \exists f : \alpha \to \lambda,\ f''\alpha \text{ cofinite}\}$ is the cofinality of λ.

Note:

1. $cf(\lambda) \leq \lambda$
2. $cf(\lambda) < \lambda$ if λ is not a cardinal (Since $cf(\lambda)$ is a cardinal.)
3. $cf(\lambda) = cf(\aleph_\lambda)$ if λ is a limit.

A cardinal κ is called <u>regular</u> if $cf(\kappa) = \kappa$, and <u>singular</u> if $cf(\kappa) < \kappa$.
A cardinal κ is an <u>$\aleph$-fixed point</u> if $\kappa = \aleph_\kappa$. Since $\aleph_\kappa \geq \kappa$, if κ (or equivalently,

$\aleph_\kappa$) is not an aleph fixed point and κ is a limit ordinal, then $cf(\aleph_\kappa) = cf(\kappa) \leq \kappa < \aleph_\kappa$.
Do $\aleph$-fixed points exist? Consider $\alpha_0 := \aleph_0$, $\alpha_{n+1} := \aleph_{\alpha_n}$, $\alpha := \bigcup_{n\in\mathbb{N}} \alpha_n$. Then $\alpha = \bigcup_{n\in\omega} \alpha_n = \bigcup_{n\in\omega} \alpha_{n+1} = \bigcup_{n\in\omega} \aleph_{\alpha_n} = \aleph_\alpha$. But α is clearly singular, since $n \mapsto \alpha_n$ is cofinal in α. $cf(\alpha) = \omega < \alpha$. What about V_{α_1}?

Theorem. (ZFC)
Every successor cardinal $\aleph_{\alpha+1}$ is regular.

Proof. Suppose $f : \aleph_\alpha \to \aleph_{\alpha+1}$ is cofinal, i.e. $\aleph_{\alpha+1} = \bigcup_{\gamma\in\aleph_\alpha} f(\gamma)$. Use AC ~~(or just well-ordering?)~~ to pick surjections $s_\gamma : \aleph_\alpha \to f(\gamma)$, and let $s : \aleph_\alpha^2 \to \aleph_{\alpha+1}$, $(\gamma, \delta) \mapsto s_\gamma(\delta)$. s is surjective by cofinality, but $\aleph_\alpha^2 \sim \aleph_\alpha$ (by 'Hessenberg'), so $\aleph_\alpha \succeq \aleph_{\alpha+1}$. □

However,

Theorem. For any α, $V_{\aleph_{\alpha+1}} \not\models$ Replacement.

Proof. We need the following lemma:

Lemma. There is a definable surjection from $\mathcal{P}(\aleph_\alpha) \times \mathcal{P}(\aleph_\alpha \times \aleph_\alpha)$ onto $\aleph_{\alpha+1}$. (This is really just an explicit form of Hartog's theorem: $\forall x \exists \alpha (\alpha \not\preceq x)$. The smallest such α is called the Hartogs-Aleph of x: $\aleph(x)$. E.g. $\aleph_{\alpha+1} = \aleph(\aleph_\alpha)$)
Proof of lemma: For $A \subseteq \aleph_\alpha$, $R \subseteq \aleph_\alpha \times \aleph_\alpha$, $(A, R) \mapsto \begin{cases} \alpha & (A,R) \cong (\alpha, \in) \\ 0 & \text{otherwise} \end{cases}$

The theorem should now be clear. □

This means we need to look for regular limit cardinals. (Which must be aleph fixed points.) But what if $2^{\aleph_0}$ is already 'really big', in the sense that $2^{\aleph_0} > \kappa$ where κ is a regular limit?

Definition. A cardinal κ is called a strong limit if $\forall \lambda < \kappa (2^\lambda < \kappa)$, and inaccessible if it's a regular strong limit. (Note that $\aleph_0$ is inaccessible according to this definition.)

Definition.

$$\beth_0 := \omega$$
$$\beth_{\alpha+1} := 2^{\beth_\alpha}$$
$$\beth_\lambda := \bigcup_{\alpha<\lambda} \beth_\alpha$$

Remark. κ is a strong limit iff $\kappa = \beth_\lambda$ for some limit ordinal λ.
By induction and Cantor, $\aleph_\alpha \leq \beth_\alpha$ for all α.
CH $\iff \aleph_1 = \beth_1$.
GCH $\iff \forall \alpha (\aleph_\alpha = \beth_\alpha) \iff \forall \kappa (2^\kappa = \aleph(\kappa))$
If GCH, then every limit cardinal is a strong limit, so then every regular limit cardinal is inaccessible.

Lemma. If κ is inaccessible and $\alpha < \kappa$, then $|V_\alpha| < \kappa$.

Proof. Let us prove $\forall \alpha < \kappa (|V_\alpha| < \kappa)$ by induction.
1. $\alpha = 0 : V_\alpha = \emptyset$ and $0 < \kappa$
2. Suppose $|V_\alpha| < \kappa$ and consider $V_{\alpha+1}$. $|V_{\alpha+1}| = 2^{|V_\alpha|} < \kappa$ since κ is a strong limit cardinal.
3. Let $\lambda < \kappa$ be a limit ordinal and suppose $\forall \alpha < \lambda$, $|V_\alpha| < \kappa$. Then $|V_\lambda| = |\bigcup_{\alpha<\lambda} V_\alpha| \leq \sum_{\alpha<\lambda} |V_\alpha| < \kappa$ since κ is regular.

□

Theorem. If κ is inaccessible, then $V_\kappa \models$ Replacement.

Proof. Take a functional sentence φ, and $X, p_1, \ldots, p_n \in V_\kappa$. We need to show $\{y \in V_\kappa \mid \exists x \in X \ \varphi(x, y, p_1, \ldots, p_n)\} =: R \in V_\kappa$.
$X \in V_\kappa = \bigcup_{\alpha<\kappa} V_\alpha$, so there is an $\alpha < \kappa$ such that $X \in \alpha$. Since V_α is transitive, $X \subseteq V_\alpha$. So $|X| \leq |V_\alpha| < \kappa$ by our lemma. By functionality, we have $|R| \leq |X| < \kappa$. If $y \in R$, then $y \in V_\kappa = \bigcup_{\alpha<\kappa} V_\alpha$, so we find for every (x, y) such that $\varphi(x, y, \ldots, p_n)$ some α_x such that $y \in V_{\alpha_x}$.
Consider $A := \{\alpha_x \mid x \in X\}$. $A \subseteq \kappa$, $|A| \leq |X| < \kappa$, so by regularity there is some $\gamma < \kappa$ such that $A \subseteq \gamma$. So $R \subseteq V_\gamma$, and $R \in V_{\gamma+1} \subseteq V_\kappa$. □

Definition. $\mathsf{IC} :=$ "$\exists$ inaccessible $\kappa > \omega$".

Theorem. If ZFC is consistent, then $\mathsf{ZFC} \nvdash \mathsf{IC}$.

Proof. If $\mathsf{ZFC} \vdash \mathsf{IC}$, then $\mathsf{ZFC} \vdash \mathsf{ZFC} + \mathsf{IC} \vdash \mathrm{Cons}(\mathsf{ZFC})$, so ZFC is inconsistent by Gödel's second incompleteness theorem. □

By 'general logic' (Löwenheim-Skolem),

$$\mathsf{ZFC} + \mathsf{IC} \vdash \text{“there is a countable model of } \mathsf{ZFC}\text{.”}$$

Let's explore what those countable models look like.

Definition. A substructure $M \subseteq N$ is an elementary substructure, written $M \preceq N$, if for every first-order formula $\phi = \varphi(a_1, \ldots, a_n)$ with parameters a_i in M, $N \models \phi \iff M \models \phi$. In particular, M and N satisfy the same first-order sentences.

Theorem. Tarski–Vaught test (TVT)
Let $M \subseteq N$ be a substructure. Then $M \preceq N$ iff for every formula φ with $n+1$ free variables and every $p_1, \ldots, p_n \in M$ such that $N \models \exists x \ \varphi(x, p_1, \ldots, p_n)$, there is $m \in M$ such that $N \models \varphi(m, p_1, \ldots, p_n)$.

If $N := (V_\kappa, \in) \models \mathsf{ZFC}$, construct a countable elementary substructure as follows:
$M_0 := \emptyset$. If M_i is constructed, $\Phi := \{(\varphi, p_1, \ldots, p_n) \mid N \models \exists x \ \varphi(x, p_1, \ldots, p_n)\}$.
(I guess here φ is 'relativised' (?) to N, and is represented by its Gödel number. In order to show that M_i exists by separation, we need to show that we can express $N \models \varphi$ using a formula in ZFC. But this should be straightforward: Just define it recursively.)
For $(\varphi, p_1, \ldots, p_n) \in \Phi$, take 'witness' $x_{\varphi,p_1,\ldots,p_n}$ such that $N \models \varphi(x_{\varphi,p_1,\ldots,p_n}, p_1, \ldots, p_n)$.

Let $M_{i+1} := \{x_t \mid t \in \Phi\} \cup M_i$, and take $M = \bigcup_{i\in\mathbb{N}} M_i$.
Claim 1: $|M| \leq \aleph_0$. (Note $|\Phi| \leq \aleph_0 |M_i^n| = \aleph_0$ if M_i is countable.)
Claim 2: By TVT, $M \preceq V_\kappa$.
So $(M, \in) \models \mathsf{ZFC}$.
M will have lots of really big ordinals, but also quite a lot of holes. (I.e. very much non-transitive.)

2 Lecture 4: Notes taken by Gheehyun Nahm

If κ is inaccessible, then $V_\kappa \models$ ZFC.

Lowenheim-Skolem $\to$ Skolem hull inside V_κ

M countable elementary submodel

ZERO(x) : $\forall z(\neg z \in x)$ (Note that $x \neq \emptyset$ is not a formula of the language of set theory)

ONE(x) : $\forall z(z \in x \leftrightarrow \text{ZERO}(x))$ (Again, $x = \{\emptyset\}$ is not a formula in the language of set theory)

We can do this since logic allows definition by formula (there is an algorithmic way to transform...)

Since our (unique) witness in V_κ should be also in M, we can show that $0 \in M$, $1 \in M$, and more generally that $n \in \mathbb{N} \to n \in M$ (what we said is that we can prove, for all $n \in \mathbb{N}$, that $n \in M$, but we can also prove that as well). Also other ordinals (or sets) uniquely identified by a formula in V_κ is in M. M has huge gaps, since $\aleph_1 \cap M$ is countable, so $\alpha = \sup(\aleph_1 \cap M) < \aleph_1$ but $\aleph_1 \in M$, so $\forall \beta(\alpha < \beta < \aleph_1 \to \beta \notin M)$.

Why are gaps bad?

Simple example: Let $M = V_\kappa \backslash \{1\}$. Then $(M, \in)$ is a structure for the language of set theory. Most likely, most axioms of ZFC are broken.

$0 \in M$, $1 \notin M$, $\{0, 1\} = 2 \in M$. What is true in M?

$M \models 0 \in 2$, $M \models \text{ZERO}(0)$, $M \models \text{ONE}(2)$. In particular, the following does not hold:

$$\forall x \in M(M \models \text{ONE}(x) \Leftrightarrow V_\kappa \models \text{ONE}(x))$$

This is called "absoluteness" and the example shows that ONE is not absolute between M and V_κ.

Thus gaps are bad since the "meaning" of things are not preserved.

This means: we should be looking for models without gaps, i.e., transitive models.

Thus we use Moskowski collapse.

If $(M, \mathcal{R})$ is a structure such that $\mathcal{R}$ is extensional and well-founded, then the following recursive construction yields a transitive set T such that $(M, \mathcal{R})$ is isomorphic to $(T, \in)$ via a function F where $F(x) := \{F(y) : y\mathcal{R}x\}$.

What does this applied to our stupid model give? $F(0) = 0$, $F(2) = 1$, $\cdots$.

By Mostowski, find T transitive such that $(T, \in) \simeq (M, \in)$ where M was countable and $M \models$ ZFC. So, T is a countable transitive model of ZFC.

How does this look like? By the same argument, there should be a countable limit of ordinals in T. This model is quite different from the model we had before, since in the original model, $\aleph_1$ in the real sense is the same with what our model believes $\aleph_1$ is. However, in our new transitive model, this cannot be the case.

We will see later that countable transitive models of ZFC are very nice (Absoluteness).

Detour: Other transitive models

Recall our definition: $M_0 = \emptyset$, $M_{i+1} = M_i \cup \{x_{\varphi,p_1,\cdots,p_n} : (\varphi, p_1, \cdots, p_n) \in \Phi\}$ where $\Phi = \{(\varphi, p_1 \cdots, p_n) : p_1, \cdots, p_n \in M_i \text{ and } V_\kappa \models \exists x \varphi(x, p_1, \cdots, p_n)\}$

Instead of this, we may define $M'_{i+1} = V_{\alpha_{i+1}}$ where $\alpha_{i+1} = \sup\{\rho(x_{\varphi,p_1,\cdots,p_n}) : (\varphi, p_1, \cdots, p_n) \in \Phi\}$. Note that in this definition, since κ is inaccessible, everything is small compared to κ, thus everything works well. Let $\alpha = \bigcup \alpha_i$. By TVT, $V_\alpha \preccurlyeq V_\kappa$. However, $\alpha < \kappa$ since $\mathrm{cf}(\alpha) = \aleph_0 < \kappa = \mathrm{cf}(\kappa)$. Thus the least γ such that $V_\gamma \models$ ZFC must have cofinality $\aleph_0$ (it does not have to be inaccessible!).

Definition. We call κ a worldly cardinal if $V_\kappa \models$ ZFC.

We just showed that if there are worldly cardinals, then there are those which are not inaccessible.

Idea. Try to give a simpler argument that ZFC $\nvdash$ IC.

Attempt. Assume ZFC $\vdash$ IC. Let ι be the least inaccessible cardinal. Then $V_\iota \models$ ZFC. By assumption, $V_\iota \models$ IC, i.e. $V_\iota \models \exists \kappa(\kappa$ is inaccessible). We want to deduce that $\kappa < \iota$, but V_ι may lie. This only gives a contradiction if the formula "x is inaccessible" is absolute for V_ι. We do know that it is not absolute for arbitrary transitive models (since we gave a counterexmaple). However, this is true for V_ι: see exercise 9.

Definition. Let $M \subseteq N$ be two $L_\in$-structures, and let φ be a formula in $n+1$ free variables. We say φ is absolute between M and N if $\forall p_1, \cdots, p_n, x \in M$,

$$M \models \varphi(x, p_1, \cdots, p_n) \leftrightarrow N \models \varphi(x, p_1, \cdots, p_n).$$

If $N = V$, then we say that φ is absolute for M.

We have seen that the formula ONE is not absolute between $M = V_\kappa \backslash \{1\}$ and V_κ.

Proposition.

1. If φ, ψ are absolute between M and N, then so are $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \to \psi$, $\neg\varphi$.
2. $x \in y$, $x = y$ are absolute for M.
3. If φ is quantifier-free, then φ is absolute for M.

However this is not that exciting since almost all statements in set theory has a quantifier: even stating that something is the emptyset needs a quantifier.

2.1 Notes taken by me again

Last lecture, we learnt that non-transitive models are bad. We had a model M containing 0 and 2 but not 1, and saw that ONE was not absolute for M ($M \models ONE(2)$).

Guess: transitive models are better.

We proved: If there is a set model of ZFC, then there is a transitive set model of ZFC. (I guess "set model" means in particular $\in$ is $\in$?)

Also: If κ is inaccessible, then there are lots of α such that $V_\alpha \models$ ZFC.

We observed that

1. the formulas $x \in y$, $x = y$ are absolute for abitrary models.
2. if φ, ψ are absolute, then so are $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \to \psi$ and $\neg\varphi$.

So any quantifier-free formula is absolute.
But even "$x = \emptyset$" has a quantifier.

Definition. If φ is a formula, we define $\exists x \in y : \varphi$ as $\exists x(x \in y \wedge \varphi)$. This is bounded quantification.

Definition. Δ_0 is the smallest class of formulas which

1. contains atomic formulas $x \in y$, $x = y$
2. is closed under $\wedge$, $\vee$, $\neg$
3. and is closed under bounded quantification: if φ is in Δ_0, then so is $\exists x \in y : \varphi$.

Note that $\forall x \in y(x = y)$ is not Δ_0, but $\neg\exists x \in y(\neg x = y)$ is logically equivalent, i.e. equivalent in predicate calculus. Generally, formulas can be equivalent or not in different contexts, e.g. $\exists y(y = \emptyset \wedge y \in x)$ and $\forall y(y = \emptyset \to y \in x)$ are both equivalent to $\emptyset \in x$ modulo set theory, but not in predicate logic.
$\forall x \in y(x = y)$ is equivalent to $x = \emptyset$ modulo $\mathsf{Replacement}$, but not probably not modulo ZFC^-.

Definition. Let T be any theory; we say φ is Δ_0^T if there is a Δ_0 formula ψ such that $T \vdash \varphi \leftrightarrow \psi$.

Proposition. Let $M \subseteq N$ be models of T, and φ, ψ formulas such that $T \vdash \varphi \leftrightarrow \psi$. If φ is absolute between M and N, then so is ψ.

Proof. $M \models \psi \iff M \models \varphi \iff N \models \varphi \iff N \models \psi$.

□

Theorem. Suppose $M \subseteq N$ are transitive models of some theory T and φ is a Δ_0^T formula. Then φ is absolute between M and N.

Proof. We just need to show this for Δ_0 by the previous proposition. To do this, we use induction, using the definition of Δ_0. We have already dealt with atomic formulas and logical connectives.
We just need to show that if ψ is absolute, then so is $\varphi := \exists x \in y : \psi$. Let ψ have $n + 2$ free variables, and pick parameters $p_1, \ldots, p_n \in M$, $y \in M$.

$$\begin{aligned}
&M \models \exists x \in y : \psi(x, y, p_1, \ldots, p_n)\\
\iff &M \models \exists x(x \in y \wedge \psi(x, y, p_1, \ldots, p_n))\\
\iff &\text{there is } m \in M \text{ s.t. } M \models m \in y \wedge \psi(m, y, p_1, \ldots, p_n)\\
\iff &\text{there is } m \in M \text{ s.t. } M \models m \in y \text{ and } M \models \psi(m, y, p_1, \ldots, p_n)\\
\iff &\text{there is } m \text{ s.t. } M \models m \in y \text{ and } M \models \psi(m, y, p_1, ..., p_n)\\
\iff &\text{there is } m \text{ s.t. } N \models m \in y \text{ and } N \models \psi(m, y, p_1, ..., p_n)\\
\iff &\text{there is } m \in N \text{ s.t. } N \models m \in y \text{ and } N \models \psi(m, y, p_1, ..., p_n)\\
\iff &N \models \exists x \in y : \psi(x, y, p_1, ..., p_n)
\end{aligned}$$

$\square$

Some Δ_0^{PL} formulas (PL being just predicate logic): $x \in y$, $x = y$, $x \subseteq y$.
If we have objects or functions defined by a formula, we say that the object/function is Δ_0^T if the formula is. For example, $\emptyset$ is Δ_0^{PL}, being defined by $\neg\exists z \in x(z = z)$. Note that e can define things in different ways, so a Δ_0 concept can still be defined in non-Δ_0 ways.
Other Δ_0 concepts: (some of which might require basic set theory)

1. $\{x\}$ $\qquad (x \in y \wedge \neg\exists z \in y(\neg z = x))$
2. $\{x, y\}$ $\qquad (x \in z \wedge y \in z \wedge \neg\exists w \in z(\neg w = x \wedge \neg w = y))$
3. $(x, y) := \{\{x\}, \{x, y\}\}$ $\qquad (\exists a, b \in X \dots)$
4. $x \cup y$, $\bigcup x$
5. $x \cap y$, $x \setminus y$
6. $x \subseteq y$
7. "x is transitive"
8. $x \times y$
9. "x is a function"
10. "x is an injection"
11. $x = \mathrm{dom}(y)$
12. $x = \mathrm{ran}(y)$.

Now we have reconstructed some of the foundations of mathematics, but we're missing some nice concepts, such as "x is the set of natural numbers", or "x is an ordinal".
What is an ordinal? Usually, we say that x is an ordinal if x is a transitive set such that $(x, \in)$ is a well-ordering. "Well-ordering" is a problem, since we can't refer to power sets. Luckily, if we assume regularity, this is equivalent to "x is a transitive set such that $(x, \in)$ is a total/linear order", which is Δ_0.
$(\forall y \in x\ \forall z \in y(z \in x) \wedge \forall y \in x\ \forall z \in x(y = z \vee y \in z \vee z \in y))$
So "x is an ordinal" is Δ_0^{ZFC} (but not necessarily $\Delta_0^{\mathsf{ZFC}^-}$), and so is "$x$ is a limit ordinal" $(\exists y \in x(x = x) \wedge \forall y \in x(y \cup \{y\} \in x))$.
As a consequence, transitive models of ZFC cannot disagree about what $\mathbb{N}$ is, and so they agree on what 'finite' and 'finite sequence of naturals' mean. So they agree on what formulas and proofs are. This provides further evidence that transitive models are nice, unlike some pathological examples that one might encounter in logic.

We have seen that Δ_0^T formulas are absolute between transitive models of T, and absolute for transitive models, and we have seen some examples of Δ_0^{ZFC} concepts.
However, to construct models of weird sentences, we need non-absolute sentences. (If say CH were absolute, then we wouldn't be able to find a transitive

model of CH unless CH were true in our original model.)
Consider the formula $\varphi(x) :=$ "x is a cardinal", i.e. "x is an ordinal and $\forall \alpha \in x \ \forall f : \alpha \to x(f \text{ is not surjective})$. Why might this not be Δ_0? We have unbounded quantification $\forall f$.
Let M be a countable transitive model of ZFC. Let α be the ordinal such that $M \models \alpha = \aleph_1$, where $\alpha = \aleph_1$ is an abbreviation of "$\alpha \not\preceq \mathbb{N} \wedge \forall \beta < \alpha(\beta \preceq \mathbb{N})$".
Since M is countable, there is some $\gamma < \aleph_1$ such that $\mathrm{Ord} \cap M = \gamma$. So $\alpha < \gamma$. Thus $\alpha \neq \aleph_1$. So the formula "$x = \aleph_1$" is not absolute for M. Also, $M \models \varphi(\alpha)$, even though $\varphi(\alpha)$ is false, so φ is not absolute for M.
However, there is a clear assymmetry here, so our argument won't work if we turn it around. Indeed:
If κ is a cardinal, then for every transitive model $M \ni \kappa$, $M \models \varphi(\kappa)$.

Proof. Suppose not: $M \models \exists \alpha < \kappa \ \exists f : \alpha \to \kappa(f \text{ is a surjection})$. Interpreting this, we see that there is $\alpha \in M \cap \kappa = \kappa$ and $f \in M$ such that $f : \alpha \to \kappa$, such that $M \models f$ is a surjection, which contradicts κ being a cardinal. □

Definition. φ is <u>downwards absolute</u> if for all $p_1, \ldots, p_n, x \in M$ such that $\varphi(x, p_1, \ldots, p_m)$, $M \models \varphi(x, p_1, \ldots, p_n)$, and <u>upwards absolute</u> if $\varphi(x, p_1, \ldots, p_m) \leftarrow M \models \varphi(x, p_1, \ldots, p_n)$, i.e. if $\neg\varphi$ is downwards absolute.

Syntactically, a formula is called Σ_1 if it is of the form $\exists x \varphi$ with φ Δ_0, and Π_1 if it is of the form $\forall x \varphi$ for φ Δ_0.
It is called Σ_1^T or Π_1^T if it is equivalent to a Σ_1 or Π_1 formula modulo T.
Our previous argument shows that Σ_1^T formulas are upwards absolute for transitive models of T, and that Π_1^T formulas are downwards absolute for transitive models of T.

Example.

1. "x is a cardinal" is Π_1^{ZFC}
2. "x is a regular cardinal" is Π_1^{ZFC} [replace "surjection" in φ by "cofinal"]

Suppose κ is a regular limit, and M is transitive, $\kappa \in M$. We want to show that $M \models \kappa$ is a regular limit. By the above, $M \models \kappa$ is a regular cardinal. Suppose $M \models \kappa$ is a successor. Let $\lambda < \kappa$ be such that $M \models \lambda^+ = \kappa$. Thus take any cardinal μ, $\lambda < \mu < \kappa$. By downwards absoluteness $M \models \mu$ is a cardinal, contradiction. So $M \models \kappa$ is a limit cardinal, and hence $M \models \kappa$ is a regular limit. So "κ is a regular limit" is downwards absolute.
(In fact, "κ is a limit cardinal" is downwards absolute.)

Note in a transitive model $M \ni \aleph_1$, the real $\aleph_1$ could very well be $\aleph_\alpha$ for some limit $\alpha > 1$, but not for $\alpha = \omega$, since $\aleph_1$ is regular and being a regular cardinal is downwards absolute. (Π_1^{ZFC})

ES1: Being an inaccessible cardinal is absolute between V_α and V.
Second proof of $\mathsf{ZFC} \nvdash \mathsf{IC}$: Suppose $\mathsf{ZFC} \vdash \mathsf{IC}$. Let ι be the least inaccessible cardinal in a given model $V \models \mathsf{ZFC}$. By previous theorem, $V_\iota \models \mathsf{ZFC}$, so $V_\iota \models \mathsf{ZFC} + \mathsf{IC}$. Take κ such that $V_\iota \models \kappa$ is inaccessible. By ES1Q9, κ is inaccessible, contradicting minimality of ι.

A "major" worry: Remember how we proved that for limit ordinals λ, $V_\lambda \models \mathsf{Pair}$. We did this by observing that if $x, y \in V_\lambda$, there is $\alpha < \lambda$ such that

$x, y \in V_\alpha$ and so $\{x,y\} \in V_{\alpha+1} \in V_\lambda$. But the axiom of pairing does not say "the universe is closed under pairing": In order to make this semantic argument into an actual proof of $V_\lambda \models \mathsf{Pair}$, we need to show that the formula describing $p = \{x,y\}$ is absolute for V_λ: $p = \{x,y\} \iff \forall z(z \in p \iff z = x \wedge z = y)$, which can be made Δ_0^{ZFC}.
What about power set? $\psi(x,p) \iff p = \mathcal{P}(x) \iff \forall y(y \subseteq x \leftrightarrow y \in p) \iff \forall y(\forall z(z \in y \to z \in x) \leftrightarrow y \in p)$
Let M be a countable transitive model of ZFC. Let p be the set such that $M \models p = \mathcal{P}(\mathbb{N})$. Since M is countable in V, there is a bijection between p and $\mathbb{N}$. So p can't be $\mathbb{N}$ by Cantor. So "$x = \mathcal{P}(\mathbb{N})$" is not absolute for M.
Generally, $M \models \psi(x,p)$ implies $p \subseteq \mathcal{P}(x)$. In our case, we made sure that V_λ contained the real $\mathcal{P}(x)$, so we're fine.

Next time: (end of absoluteness) Lemma $\to$ if φ is defined by transfinite recursion via absolute formulas, then φ is absolute.

Hello, good morning. Let's finish our disccussion of absoluteness. What we've seen so far is that absoluteness is a fundamental, and is closely related to transitive models. In particular, Δ_0^T-formulas are absolute for transitive models of T. Σ_1^T-formulas are upwards absolute, and Π_1^T-formulas are downwards absolute. (If the smaller model has a witness of a Σ_1^T-formula, then so does the bigger model.)
Note that absolute is equivalent to upwards $\wedge$ downwards absolute.

Definition. A concept is Δ_1^T if it is both Σ_1^T and Π_1^T.

Proposition. The formula expressing "(X,R) is a well-order" is absolute for transitive models of ZFC.

Proof. The canonical formula is "(X,R) is a linear order and $\forall z(\emptyset \neq z \subseteq X \implies z$ has a R-minimal element". The only unbounded quantifier is $\forall z$, so this formula is Π_1^{ZFC}. So "(X,R) is a well order" is downwards absolute.
However, $\mathsf{ZF} \vdash (X,R)$ is a well-order $\iff \exists(\alpha,f) : \alpha$ is an ordinal $\wedge\ f : (\alpha, \in) \cong (X,R)$. Here the only unbounded quantifier (modulo ZFC) is $\exists(\alpha,f)$, so "(X,R) is a well-order" is Σ_1^{ZFC} and hence upwards absolute.
Thus "(X,R) is a well-order" is absolute for transitive models. $\square$

Proposition. Suppose M is a transitive model of set theory, and φ is functional in M [i.e. for every $x \in M$ there is exactly one $y \in M$ such that $M \models \varphi(x,y)$], and absolute for M.
Suppose that F is a function from the ordinals into M recursively defined by $F(\alpha) := y \iff M \models \varphi(F \restriction \alpha, y)$. Then F is defined by a formula Φ which is absolute for M.

Proof. Recall from the proof of the recursion theorem that F is defined by

$$\begin{aligned} F(\alpha) = y &\iff \exists f(\operatorname{dom} f \ni \alpha \wedge f \text{ satisfies the rec. eq. } \wedge f(\alpha) = y) \\ &\iff \forall f(\operatorname{dom} f \ni \alpha \wedge f \text{ satisfies the rec. eq.} \to f(\alpha) = y) \end{aligned}$$

The first definition is Σ_1 and the second definition is Π_1, so the concept is absolute. $\square$

Remark: Relativisation. If ψ is a formula with $n+1$ free variables, $p_1, \ldots, p_n$ parameters, then we can recursively define the relativisation of a formula to $\psi, p = (p_1, \ldots, p_n)$ by:

$$(x \in y)^{\psi,p} := x \in y$$
$$(x = y)^{\psi,p} := x = y$$
$$(\varphi \wedge \varphi')^{\psi,p} := \varphi^{\psi,p} \wedge \varphi'^{\psi,p}$$
$$(\neg\varphi)^{\psi,p} := \neg\varphi^{\psi,p}$$
$$(\exists x \varphi)^{\psi,p} := \exists x(\psi(x,p) \wedge \varphi)$$

Proposition. If M is a transitive model of set theory, then $M \models \varphi \leftrightarrow \varphi^M$.

The proof is just interpreting both statements semantically.

Next goal: Show that CH is consistent.
Recall that CH is essentially a 'minimality statement': it says that $2^{\aleph_0}$ takes the least possible value.

Idea: Build up a "definable power set".
Not good idea: Replace $\mathcal{P}(x)$ by the set of definable subsets of x, since the second thing should be countable.
Definability is dangerous: $D := \{\alpha < \aleph_1 \mid \alpha \text{ is definable}\}$ is countable, so $\aleph_1 \setminus D \neq \emptyset$, so define $\delta := \min(\aleph_1 \setminus D)$. δ is the smallest non-definable ordinal, but isn't the above a definition of δ?

Lemma. There can be no absolute formula Φ such that $\Phi(x) \iff \exists$ formula φ such that φ defines x.

Proof. If there was, then δ would be defined by $\neg\Phi(\delta) \wedge \forall\alpha < \delta(\Phi(\alpha))$. Thus $\Phi(\delta)$, a contradiction. □

Note that this is sort of 'proof by cheating': We haven't talked about where things are definable, i.e. what 'ϕ defines x' means.
Observation: "definable" is underdetermined. What we mean is "definable in a structure". If we formulate our lemma properly, we will actually need absoluteness in the assumption.

Gödel had the idea of internalising definability by the following definition: $x \in y$, $x = y$, $\varphi \wedge \psi$, $\neg\varphi$, $\exists x\varphi$ are the ways to construct formulas in the language of set theory. Corresponding to this, we can consider, for $i, j < n$:

$$\mathrm{Diag}_\in(A, n, i, j) := \{s \in A^n \mid s(i) \in s(j)\}$$
$$\mathrm{Diag}_=(A, n, i, j) := \{s \in A^n \mid s(i) = s(j)\}$$
$$R, S \mapsto R \cap S$$
$$R \mapsto A^n \setminus R$$
$$\mathrm{Proj}(A, R, n) := \{s \in A^n \mid \exists t \in R(s = t \restriction n)\}$$

And define

$$D_0(A,n) := \{\mathrm{Diag}_\in(A,n,i,j) \mid i,j<n\} \cup \{\mathrm{Diag}_=(A,n,i,j) \mid i,j<n\}$$

$$D_{k+1}(A,n) := D_k(A,n) \cup \{R\cap S \mid R,S \in D_k(A,n)\}$$
$$\cup\ \{A^n \setminus R \mid R \in D_k(A,n)\} \cup \{\mathrm{Proj}(A,R,n) \mid R \in D_k(A,n+1)\}$$

Then define $\mathrm{Def}(A,n) := \bigcup_{k\in\mathbb{N}} D_k(A,n)$.
Note that by construction, $\mathrm{Def}(A,n)$ should contain precisely all n-ary relations definable within A in the language of set theory, so now we've internalised definability, using only set theoretic operations and no logic mumbo-jumbo.

Hello, good morning. Let us remind ourselves where we are. Goal: construct model of $\mathsf{ZFC}+\mathsf{CH}$. V_α won't help us, since $\mathcal{P}(\mathbb{N}) \in V_\alpha$. We need to 'thin out' $\mathcal{P}(\mathbb{N})$, by only considering certain subsets. We need 'definable' subsets, but definability seems like a tricky concept. Problem: There is no formula defining "definability". Last time, we fixed A,n, and defined $\mathrm{Def}(A,n)$.
Even more concretely: We can explicitly enumerate $\mathrm{Def}(A,n)$ by:

$$\mathrm{En}(2^i3^j,A,n) := \mathrm{Diag}_\in(A,n,i,j)$$

$$\mathrm{En}(2^i3^j5,A,n) := \mathrm{Diag}_=(A,n,i,j)$$
$$\mathrm{En}(2^i3^j5^2,A,n) := A^n \setminus \mathrm{En}(j,A,n)$$
$$\mathrm{En}(2^i3^j5^3,A,n) := \mathrm{En}(i,A,n) \setminus \mathrm{En}(j,A,n)$$
$$\mathrm{En}(2^i3^j5^4,A,n) := \mathrm{Proj}(A,n,\mathrm{En}(i,A,n+1))$$

$\mathrm{En}(m,A,n) = \emptyset$ for all other m.

Proposition. The range of $En(,A,n)$ is $\mathrm{Def}(A,n)$.

Proof. Induction! □

Proposition. If φ is a 'concrete' formula, then $\{s\in A^n \mid \varphi^A(s_0,\ldots,s_{n-1})\} \in \mathrm{Def}(A,n)$.

Proposition. $|\mathrm{Def}(A,n)| \le \aleph_0$

This means that we can't just let $\mathcal{P}(n) = \mathrm{Def}(\mathbb{N},1)$.

Gödel's idea: Assume the existence of ordinals and build the power set by defining it (minimally) relative to ordinal parameters.

Intuitively, x is ordinal definable if there is φ and $\alpha_1,\ldots,\alpha_n \in \mathrm{Ord}$, such that $\forall z(z=x \leftrightarrow \varphi(z,\alpha_1,\ldots,\alpha_n))$.
There is a problem with this, since this involves truth and truth is undefinable.

Definition. Formally, let x be any set, and $\alpha := \rho(x)$. Then x is ordinal definable if

$$\exists\beta>\alpha\,\exists n\,\exists\alpha : n\to\beta\,\exists R\in\mathrm{Def}(V_\beta,n{+}1)\forall z\in V_\beta(z=x\leftrightarrow(\alpha_0,\ldots,\alpha_{n-1},z)\in R)$$

We define OD to be the class of ordinal definable sets. It's easy to check that
(1): $\text{Ord} \subseteq \text{OD}$
(2): $x, y \in \text{OD} \to \{x, y\} \in \text{OD}$
So there might be hope that OD is a model of set theory.

We observe that if $x \in \text{OD}$, then there is a sequence of ordinals $\beta, n, \alpha_0, \ldots, \alpha_{n-1}, m$, such that $R = \text{En}(m, V_\beta, n+1)$, which witnesses that $x \in \text{OD}$. This is surprising, since we didn't make any assumptions on our original model of ZF, but now we have a class surjection from $\text{Ord}^{<\omega}$ onto OD, given by mapping $(\alpha_0, \ldots, \alpha_{n-1}, \beta, \gamma_1, \gamma_2)$ to:
If $\gamma_1 = n$ and $\gamma_2 \in \mathbb{N}$ and $\forall z \in V_\beta (x = z \leftrightarrow (\alpha_0, \ldots, \alpha_{n-1}, z \in \text{En}(\gamma_2, V_\beta, \gamma_1 + 1))$, then $F(\alpha_0, \ldots, \alpha_{n-1}, \beta, \gamma_1, \gamma_2) := x$, and $F(\alpha_0, \ldots, \alpha_{n-1}, \beta, \gamma_1, \gamma_2) := \emptyset$ otherwise.

Definition. If $s, t \in \text{Ord}^{<\omega}$, define $s \lhd t$ by $\max(s) < \max(t) \vee (\max(s) = (len(s) < len(t) \vee (len(s) = len(t) \wedge s <_{lex} t)))$.

Proposition. $(\text{Ord}^{<\omega}, \lhd) \cong (\text{Ord}, \in)$

[By Mostowski]

Together, this means that there is a class surjection from Ord onto OD.

Thus if $\text{OD} \models \mathsf{ZF}$, then by the above, $\text{OD} \models \mathsf{ZFC}$. So if $V \models \mathsf{ZF}$, then $\text{OD}^V \models \mathsf{ZFC}$, and this would be a relative consistency proof of AC.

Question: $\text{OD} \models \mathsf{Extensionality}$?
This would follow from OD being transitive. But this seems very unlikely: since V_α is ordinal-definable, OD is transitive $\iff$ OD = V. So either get a model with gaps, or we just get the model we started with.
Thus we will have to refine our construction.

Hello, good morning. So, what are we currently doing? We're looking at OD, the class of ordinal definable sets. The intuition behind this is an informal notion of ordinal definability aiming at breaking the problem of countability.
From our definition of OD, it follows that if a is 'informally ordinal defined' by $x = a \leftrightarrow \varphi^{V_\beta}(\alpha_1, \ldots, \alpha_n, x)$ for some formula φ and $\beta > \alpha_1, \ldots, \alpha_n, \rho(a)$, then by our previous lemma about Def, $a \in \text{OD}$.
Observation: If a is informally ordinal definable by an absolute formula (absolute for sufficiently large V_β)), then $a \in \text{OD}$.
Conseqences: The formulas defining $\{x, y\}$, $\bigcup x$, $\mathcal{P}(x)$ are all of this type, so OD is closed under Pairing, Union, and Power set. Similarly, the formula $x = V_\gamma$ is of this type.

Corollary. For all γ, V_γ is ordinal definable.

Theorem. The following are equivalent:

1. $V = \text{OD}$
2. OD is transitive

3. OD $\models$ Extensionality

4. (Also, $V = \mathrm{HOD}$)

Proof. $1 \implies 2 \implies 3$ are easy.
For $2 \implies 1$: By corollary, $V_\gamma \in \mathrm{OD}$. Assuming transitivity, $V_\gamma \subseteq \mathrm{OD}$ for all γ, and hence $V = \mathrm{OD}$.
For $3 \implies 1$, we claim that $\mathrm{OD} \cap V_\gamma \in \mathrm{OD}$. Assuming OD $\models$ Extensionality, it follows that $V_\gamma = V_\gamma \cap \mathrm{OD}$, so that again $V = \mathrm{OD}$. The fact that $\mathrm{OD} \cap V_\gamma \in \mathrm{OD}$ is on example sheet 2. $\square$

The problem with OD is that is is so big that when we take the transitive closure, we get all of V. So we need to thin it out.
Idea: If $x \in \mathrm{OD}$, only include it if $x \subseteq \mathrm{OD}$. (Of course this is not quite right, since we want the elements of elements of x to be in OD, etc.)

Definition. A set x is heredetarily OD, or HOD, if the transitive closure $\mathrm{tcl}(x) \subseteq \mathrm{OD}$.

Clearly

1. $\mathrm{Ord} \subseteq \mathrm{HOD} \subseteq \mathrm{OD}$

2. HOD is transitive

3. If $x \in \mathrm{OD}$, $x \subseteq \mathrm{HOD}$, then $x \in \mathrm{HOD}$.

Is HOD a model of set theory?
Extensionality follows from transitivity. Pair: For $x, y \in \mathrm{HOD} \subseteq \mathrm{OD}$, $\{x, y\} \in \mathrm{OD}$ and $\{x, y\} \subseteq \mathrm{HOD}$, so $\{x, y\} \in \mathrm{HOD}$. (Also pairs are absolute...)
PowerSet: If $x \in \mathrm{HOD}$, then $\mathcal{P}(x) \in \mathrm{OD}$. So $\mathcal{P}(x) \cap \mathrm{HOD} \in \mathrm{OD}$, and $\mathcal{P}(x) \cap \mathrm{HOD} \subseteq \mathrm{HOD}$, so $\mathcal{P}(x) \cap \mathrm{HOD} \in \mathrm{HOD}$. And $\mathrm{HOD} \models \mathcal{P}(x) = \mathcal{P}(x) \cap \mathrm{HOD}$, so HOD $\models$ PowerSet.
Separation, Replacement are both on example sheet 2.
Infinity and Union are clear.
AC: Last time, we found a definable surjection from Ord onto OD, and this gives a surjection $\pi : \mathrm{Ord} \twoheadrightarrow \mathrm{HOD}$. This shows that

$$V \models \text{there is a global well-ordering of HOD}$$

But to show HOD $\models$ AC, we need to show $\mathrm{HOD} \models \forall x \exists R \exists \alpha : (x, R) \cong (\alpha, \in)$. To construct R given $x \in \mathrm{HOD}$, note that $x \subseteq \mathrm{HOD}$, so each $y \in x$ has a 'birthdate' α_y, the minimal ordinal such that $\pi(\alpha_y) = y$. Order x by $yRz \leftrightarrow \alpha_y < \alpha_z$. Since π is defined by a formula without parameters, this relation R is HOD. By construction, $V \models (x, R)$ is a well-order. By absoluteness of wellfoundedness, $\mathrm{HOD} \models (x, R)$ is a well-order.

Corollary. Cons(ZF) $\implies$ Cons(ZFC)

Hello. Let's recall where we are.
Our goal is to determine whether CH is provable.
The technique we've been using is that of inner models: Given $V \models$ ZFC, we consider $(M, \in)$, with $M \subseteq V$.
First attempt: try to 'cut off' V: find α such that $V_\alpha \models$ ZFC and check the

value of CH there. The problem with this is that α will need to be so large that it will inherit the value of $2_0^{\aleph}$. So if $V \models 2^{\aleph_0} > \aleph_1$, then $V_\alpha \models 2^{\aleph_0} > \aleph_1$, and if $V \models 2^{\aleph_0} = \aleph_1$, then $V_\alpha \models 2^{\aleph_0} = \aleph_1$.
Second attempt: building countable elementary substructures. This might seem promising since countability means that $\mathcal{P}(\mathbb{N})$ inside M will be very different from $\mathcal{P}(\mathbb{N})$ inside V. But there is a big problem: Since this is an elementary submodel, we can't change the truth value of CH in this way.
Third attempt: Use the idea of "the definable power set" in the most minimal way to ensure that: (a) $\mathcal{P}(\mathbb{N})$ is uncountable, and (b) it stays as small as possible.
Attempt 3a: Ordinal definability: x is ordinal definable if $\exists(\beta, \alpha_1, \ldots, \alpha_n, m)$ such that m encodes a formula φ and $V_\beta \models \varphi(\alpha_1, \ldots, \alpha_n, z) \leftrightarrow z = x$. The problem with this definition is that it refers to V_β, and V_β is not absolute for transitive models of set theory: If M is a countable transitive model of ZFC, the formula $x = V_\alpha$ is $\forall z(z \in x \leftrightarrow \rho(z) < \alpha)$. We've seen that $\rho(z)$ is absolute (since it is defined recursively), so this might look like a Δ_0 formula. The problem is the $\leftarrow$: $\forall z(z \in x \to \rho(z) < \alpha)$ is Δ_0, but $\forall z(\rho(z) < \alpha \leftarrow z \in x)$ is in general not. So if $M \subseteq N$, then $V_\alpha^M \subseteq V_\alpha^N$, with equality iff $V_\alpha^N = M$. So $V_{\omega+1}^M \subsetneq V_{\omega+1}$, since $V_{\omega+1}$ is uncountable. So, in V, since $V_{\omega+1}$ is ordinal definable, there is a formula φ and some $\beta, \alpha_1, \ldots, \alpha_n$ such that $x = V_{\omega+1} \leftrightarrow V_\beta \models \varphi(\alpha_1, \ldots, \alpha_n, x)$. More precisely, $\beta = \omega + \omega$, $n = 1$, $\alpha_1 = \omega + 1$, and so $x = V_{\omega+1} \leftrightarrow V_{\omega+\omega} \models \varphi(\alpha, x)$ where φ is the formula above. Let m be the code for φ. Then $(\omega + \omega, \omega + 1, m)$ describes $V_{\omega+1}$ in V, the the same tuple describes $V_{\omega+1}^M \neq V_{\omega+1}$ in M. So the set defined by $(\omega + \omega, \omega + 1, m)$ is not determined absolutely.
Attempt 3b: "Just define all of your parameters as you work along the ordinals":

Definition. The definable powerset of X is

$$\mathcal{D}(X) := \{Z \subseteq X \mid \exists n\, \exists s \in X^n\, \exists R \in \mathrm{Def}(X, n{+}1) : Z = \{x \in X \mid (s_1, \ldots, s_n, x) \in R\}\}$$

The constructible hierarchy is defined by

$$L_0 := \emptyset$$

$$L_{\alpha+1} := \mathcal{D}(L_\alpha)$$

$$L_\lambda := \bigcup_{\alpha<\lambda} L_\alpha$$

Let's look at some properties of this:

1. $\mathcal{D}(X) \subseteq \mathcal{P}(X)$
2. If X is transitive, then $X \subseteq \mathcal{D}(X)$. (If $x \in X$, then x is defined, in X, by $\{z \in X \mid z \in x\}$, which corresponds to a binary relation $R \in \mathrm{Def}(X, 2)$...)
3. $\emptyset, X \in \mathcal{D}(X)$
4. If X is infinite, then

$$\begin{aligned}|\mathcal{D}(X)| &\leq \sum_{n\in\mathbb{N}} |X^n||\mathrm{Def}(X, n+1)| \\ &\leq |\aleph_0|^2|X| = |X|\end{aligned}$$

5. If X is transitive and infinite, then $|\mathcal{D}(X)| = |X|$.

6. For each α, L_α is transitive. (By induction on α. This is clear for $\alpha = 0$ or α limit. So assume L_β is transitive and consider $L_{\beta+1} = \mathcal{D}(L_\beta)$. By 2, $L_\beta \subseteq L_{\beta+1}$. Now if $x \in y \in L_{\beta+1} = \mathcal{D}(L_\beta) \subseteq \mathcal{P}(L_\beta)$, then $x \subseteq L_\beta$, so $y \in L_\beta \subseteq L_{\beta+1}$, so $y \in L_{\beta+1}$.)

7. The same argument shows that $\forall \alpha < \beta : L_\alpha \subseteq L_\beta$.

   Definition.
$$L = \bigcup_{\alpha \in \mathrm{Ord}} L_\alpha$$
   is the constructible universe, or Gödel's constructible universe. (Note that $x \in L$ is actually a formula, since L_α is defined by a forumla.)

   If $x \in L$, define $\rho_L(x) := \min\{\alpha \mid x \in L_{\alpha+1}\}$.

8. $L_\alpha = \{x \in L \mid \rho_L(x) < \alpha\}$

9. $\mathrm{Ord} \cap L_\alpha = \alpha$. Proof by induction: $\alpha = 0$ or limit is trivial. So assume $\mathrm{Ord} \cap L_\alpha = \alpha$. Since $L_{\alpha+1} \subseteq \mathcal{P}(L_\alpha)$, we get $\mathrm{Ord} \cap L_\alpha \subseteq \alpha + 1$. So we just need to show $\alpha \in L_{\alpha+1}$. By induction hypothesis, $\alpha \subseteq L_\alpha$. We just need to show $\alpha \in \mathcal{D}(L_\alpha)$. But α is defined as $\{x \in L_\alpha \mid x \text{ is an ordinal}\}$.

   Corollary. L_{ω_1} is uncountable.

10. If α is infinite, then $|L_\alpha| = |\alpha|$. Proof by induction: $L_\omega = V_\omega$, so $|L_\omega| = |V_\omega| = |\omega|$. By 9, $\alpha \subseteq L_\alpha$, so $|\alpha| \leq |L_\alpha|$. We just need to show $|L_\alpha| \leq \alpha$. If α is a limit, then $|L_\alpha| = |\bigcup_{\beta<\alpha} L_\beta| \leq |\alpha \times \alpha| = |\alpha|$. If $\alpha = \beta + 1$, then $|L_\alpha| = |\mathcal{D}(L_\beta)| = |L_\beta| = |\beta| = |\alpha|$ by the above.

Theorem. Assuming ZF, then for every axiom φ of ZF, φ^L holds. In other words, $V \models \mathsf{ZF} \implies L \models \mathsf{ZF}$.

Proof. Extensionality and Foundation follow from transitivity.
Infinity follows from $\omega \in L_{\omega+1} \subseteq L$.
Pair: Suppose $x, y \in L_\gamma$ for some γ ($\gamma = 1 + \max(\rho_L(x), \rho_L(y))$). $\{x, y\} \subseteq L_\gamma$ and is defined in L_γ by $\{z \in L_\gamma \mid z = x \vee z = y\}$, so $\{x, y\} \in L_{\gamma+1}$.
Separation: Suppose $x \in L_\gamma, p \in L_\gamma^n$, and φ is a formula. We want to show that $X_\varphi^L := \{z \in x \mid \varphi^L(z, p)\} \in L$. It might seem like all we can show is $X_\varphi^{L_\gamma} := \{z \in x \mid \varphi^{L_\gamma}(z, p)\} \in L_{\gamma+1}$. Since φ is arbitrary, we don't know that $X_\varphi^L, X_\varphi^{L_\gamma}$ are the same. For this, we need the reflection theorem:

Theorem. Reflection theorem for L:
Suppose Φ is a finite set of formulas, and α is an ordinal. Then there is $\beta > \alpha$ such that $\forall \varphi \in \Phi$, φ is absolute between L_β and L.

(See example sheet/class 2.)
So find the 'reflection ordinal' $\beta > \gamma$ such that φ is absolute between L and L_β: $X_\varphi^L = X_\varphi^{L_\beta} \in \mathcal{D}(L_\beta) = L_{\beta+1} \subseteq L$.

PowerSet and Union are on example sheet 2.
Replacement: Suppose Φ is a functional formula in L. Let $X \in L$, $p \in L^n$. We

want to find Y such that $y \in Y \leftrightarrow \exists x \in X : \varphi^L(x, y, p)$. For $x \in X$, let $\alpha_x := \rho_L(y)$ where $\Phi^L(x, y, p)$. Let $\lambda := \sup\{\alpha_x \mid x \in X\} \in \mathrm{Ord}$ (using $\mathsf{Replacement}$ in V). Now separate Y from L_λ by the formula $\exists x \in X : \Phi^L(x, y, p)$, using separation in L. $\square$

Consider the Axiom of Constructibility: $V = L$, i.e. $\forall x \exists \alpha (x \in L_\alpha)$.
Cosider also $V = OD$: $\forall x : \delta x$ where $\delta(x) :\iff \exists \beta, n, \alpha_1, \ldots, \alpha_n, m(m$ codes $\varphi \wedge x \in V_\beta \wedge \forall z \in V_\beta : z = x \leftrightarrow V_\beta \models \varphi(\alpha_1, \ldots, \alpha_n, z)$, and $V = HOD$: $\forall x \forall y \in \mathrm{tcl}(x) : \delta(y)$. Since $\delta(x)$ refers to V_β which is very much not absolute, it should not be that surprising that $HOD \models V \neq HOD$ is consistent. However, $L \models V = L$.

Okay. So $V = L$ is really an abbreviation of $\forall x \exists \alpha (x \in L_\alpha)$, where L_α is defined recursively by $L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha; L_{\alpha+1} = \mathcal{D}(L_\alpha)$, and $D(x)$ is is defined using a lot of bounded quantifiers and Def, which is defined recursively using $\mathrm{Diag}_\in$, Proj, etc.

Observation: 'being in the power set' is absolute, i.e. $x \in \mathcal{P}(y)$, i.e. $x \subseteq y$ is an absolute formula, for transitive models of enough of set theory, but $x = \mathcal{P}(y)$ is not.
In this sense, $x = \mathrm{Def}(y, n)$ is absolute, so if M is transitive, $y \in M$, then $\mathrm{Def}(y, n) \in M$.
Apply the closure of absoluteness under recursive definitions and bounded quantification to conclude that $x = \mathcal{D}(y)$ is absolute, and hence also

Proposition. The formula $x = L_y$ is absolute for transitive models of some finite fragment of ZF.
In particular, if $\alpha \in M$, then $L_\alpha \in M$. (Since $L_\alpha^M = L_\alpha$.)

Theorem. Minimality of L:
Suppose that $M \subseteq N$ are transitive models of ZF such that $M \cap \mathrm{Ord} = N \cap \mathrm{Ord}$, i.e. M contains all the ordinals of N. Then $L^N \subseteq M$.

Proof. Clear from the above and transitivity of M. $\square$

Corollary. $\mathrm{HOD}^L = L$

Corollary. $L \models V = L = \mathrm{OD} = \mathrm{HOD}$

Corollary. $L \models \mathsf{AC}$

Note that the global well-ordering of L, unlike that of HOD, is absolute.

Corollary. $\mathrm{Cons}(\mathsf{ZF}) \to \mathrm{Cons}(\mathsf{ZFC} + V = \mathrm{OD})$

Let's analyse the proof of the minimality theorem:
By compactness, find a finite subset $T \subseteq \mathsf{ZF}$ which proves that L_α exists for all α, and extend it finitely to $T' \subseteq \mathsf{ZF}$ such that T' proves that the definitions of $\rho, \mathrm{Def}, \mathcal{D}, L_\alpha$ are absolute for transitive models f T'
Let $\varphi := \forall \alpha \exists \beta (\alpha \in \beta)$: For transitive M, $M \models \varphi$ iff $M \cap \mathrm{Ord}$ is a limit ordinal or Ord. Finally, take $S := T' \cup \{\varphi, V = L\}$. Note that S is finite.

Theorem. If M is transitive, and $M \models S$, then either $M = L_\lambda$ for some limit ordinal λ, or $M = L$.

Proof. By choice of T, for every $M \cap \mathrm{Ord}$, $L_\alpha \in M$. Since M is transitive, $L_\alpha \subseteq M$, so $\bigcup_{\alpha \in M \cap \mathrm{Ord}} L_\alpha \subseteq M$. Since $V = L \in S'$, $M \models \forall x \exists \alpha (x \in L_\alpha)$, so $M = \bigcup_{\alpha \in M \cap \mathrm{Ord}} L_\alpha$. Since $\varphi \in S$, we know that $M \cap \mathrm{Ord}$ is either Ord, in which case $M = L$, or $M \cap \mathrm{Ord} = \lambda$ for some limit, in which case $M = L_\lambda$. $\square$

(I guess we don't really need φ, if we allow $M = L_{\alpha+1}$?)
What we've shown is a lot stronger than L_λ being definable: This is a way to characterise (among transitive models) L_λ inside L_λ. Compare this to how we can define finite structures externally, but we can't characterise them internally.

Theorem. $V = L \to (\mathsf{G})\mathsf{CH}$, i.e. $L \models \mathsf{GCH}$.

Observation: If $V = L$, $A \subseteq \mathbb{N}$, then $\rho_L(A)$ is some ordinal.

Theorem. Condensation Lemma:
If $V = L$, $A \subseteq \mathbb{N}$, then $\rho_L(A) < \aleph_1$.

Corollary. $\mathcal{P}(\mathbb{N}) \subseteq L_{\omega_1}$

Corollary. $2^{\aleph_0} \leq |L_{\omega_1}| = \aleph_1$

Proof. Outline:
Step 1: Find countable $M \preceq L$ such that $A \in M$; actually, just find countable $M \subseteq L$, with $A \in M$, satisfying S.
Step 2: Observe that $M \models S$.
Step 3: Take Mostowski collapse $M' \cong M$ with M' transitive, and note $M' \models S$.
Step 4: Apply above theorem (and use M' being a set) to get $M' = L_\lambda$ for some limit λ.
Step 5: Use countability of M' to get $\lambda < \omega_1$.
Step 6: Argue that $A \in L_\lambda \subseteq L_{\omega_1}$, and so $\rho_L(A) \leq \lambda < \omega_1$.
Actual proof on Monday, for GCH. $\square$

Hello. So, today we are going to finish the chapter on L.

Theorem. Condensation lemma:
Suppose κ is an infinite cardinal, $V = L$, $A \subseteq \kappa$. Then $\rho_L(A) < \kappa^+$.

Corollary. $V = L \to \mathsf{GCH}$.

Aside: Let's analyse the Mostowski collapse.
'If (M, E) is extensional and wellfounded, then there is a (unique) transitive T such that $(T, \in) \cong (M, E)$.' (We don't bother with proper classes.) Proof: We are forced to define $f(m) := \{f(x) \mid xEm\}$, and can then check that f is an isomorphism $(M, E) \to (f''M, \in)$. Note now that if E is $\in$, then transitive sets in M are fixed by f.

Proof. of CL
Consider $\kappa \cup \{A\}$: This is a transitive set of cardinality κ. Let β be such that $\kappa \cup \{A\} \in L_\beta$. Find $\alpha > \beta$ such that S is abolute between L_α and L, by Levy reflection for L. Consider the Skolem hull H of $\kappa \cup \{A\}$ inside L_α, so that $\kappa \cup \{A\} \subseteq H$, and $(H, \in) \preceq (L_\alpha, \in) \models S$, i.e. $(H, \in) \models S$. By Mostowski, collapse H to transitive T, so that $(H, \in) \cong (T, \in)$. By our previous analysis, $\kappa \cup \{A\} \subseteq T$, so $A \in T$. Being the Skolem hull, $|H| \leq |\kappa|$, and so $|T| \leq |\kappa|$. $T = L_\lambda$ for some λ since $T \models S$, so $\lambda < \kappa^+$. So $A \in T = L_\lambda \subseteq L_\kappa^+$, i.e. $\rho_L(A) < \kappa^+$. $\square$

Here we reach the end of the road: If $V = L$, then all inner models are 'L-like', so we shouldn't be able to use this method to prove consistency of $\neg\mathsf{CH}$. What about 'outer' models?

3 Outer models

For example, consider this 'relativised L-construction': For any set A, let $L_0(A) := \mathrm{tcl}(A)$, $L_{\alpha+1}(A) := \mathcal{D}(L_\alpha(A))$, $L_\lambda(A) = \bigcup_{\alpha<\lambda} L_\alpha(A)$, $L = \bigcup_\alpha L_\alpha$. E.g. if $V \neq L$ and $\aleph_1^L < \aleph_a^V$, then by ES2 there is some $A \subseteq \mathbb{N}$, $A \in V$, that encodes a surjection from $\mathbb{N}$ onto $\aleph_1^L$. Consider then $L(A)$. We would want to show $L(A) \models \mathsf{ZFC}$, and that the condensation lemma holds for $L(A)$. But then $L(A)$ won't help us satisfy $\neg\mathsf{CH}$!

We need a technique that takes a transitive set model $(M, \in)$ and a set $A \notin M$, and constructs a model N of ZFC such that $M \cup \{A\} \subseteq N$, and such that N "keeps the nature of A".

For example, suppose M is a countable transitive model of ZFC. Then for each ordinal κ such that $M \models \kappa$ is a cardinal, there is $A \in V$, $A \subseteq \mathbb{N}$, such that A witnesses κ being countable. Consider $\alpha := \aleph_1^M, \beta := \aleph_2^M$, so that $\alpha, \beta < \aleph_1$, and find in V some injection $i : \beta \to \mathbb{R}$, and 'add' this to M, to get something like $M(i)$. Then $M(i) \models \exists$ injection $\beta \to \mathbb{R}$, and hence $M(i) \models |\mathbb{R}| \geq \beta = \aleph_2^M$. But if one of the reals we added is a wellordering of order type α, then $M(i) \models \alpha$ is not a cardinal, and so $M(i) \models \beta \leq \aleph_1$. So we need to be able to control our construction well enough that we can trick our stupid model M without M noticing.

Given a countable transitive model M of ZFC, and a set x that we like, we want a way to construct a model $M[x]$, such that

1. $M[x] \models \mathsf{ZFC}$
2. $M[x]$ still makes sure that x has the properties we like

So in this case, we want to add lots of reals (at least $\aleph_2^M$), but need to avoid reals that encode a wellordering of order type $\aleph_1^M$ or $\aleph_2^M$. $L(x)$ is good at modelling ZFC and adding lots of stuff, but we can't control things like $\aleph_1^{L[x]} = \aleph_1$. We need a construction that gives us more control.

3.1 Forcing

We will work with partial orders $(\mathbb{P}, \leq, 1)$, where $(\mathbb{P}, \leq)$ is partial order and 1 is the largest element of $\mathbb{P}$.

Definition. These are called forcing partial orders, or forcings. Theire elements are called conditions. We say that if $p \leq q$, then p is at least as strong as q.

(Note that the ordering is reversed in the Jerusalem convention.)
So 1 is the weakest condition.

Definition. $C \subseteq \mathbb{P}$ is a chain if $(C, \leq)$ is a total order, and $p, q \in \mathbb{P}$ are compatible if there is $r \in \mathbb{P}$ with $r \leq p, r \leq q$. Otherwise p, q are incompatible, and we write $p \perp q$.
$A \subseteq \mathbb{P}$ is an antichain if $\forall p \neq q \in A (p \perp q)$.

We can think of the size of an antichain as measuring how for away a partial order is from being a chain.

Definition. $D \subseteq \mathbb{P}$ is dense if $\forall p \in \mathbb{P}\ \exists d \in D (d \leq p)$.
D is dense below $q \in \mathbb{P}$ if $\forall p \leq q\ \exists d \in D (d \leq p)$
$F \subseteq \mathbb{P}$ is a filter if

1. $q \geq p \in F \implies q \in F$ i.e. F is upwards closed
2. $\forall p, q \in F\ \exists r \in F (r \leq p \wedge r \leq p)$, i.e. p, q are compatible inside F

Consider a family of dense sets $\mathcal{D}$. We say that G is $\mathbb{P}$-generic for $\mathcal{D}$ if

1. G is a filter
2. $D \cap G \neq \emptyset$ for all $D \in \mathcal{D}$.

Why is this an interseting notion?
Let x, y be any sets with x infinite and y non-empty. Let

$$\mathrm{Fn}(x, y) := \{p \mid p \text{ is a function} \wedge |p| < \omega \wedge \mathrm{dom}(p) \subseteq x \wedge \mathrm{ran}(p) \subseteq y\}$$

Define a partial order on $Fn(x, y)$ by $p \leq q$ iff $p \supseteq q$, and $1 := \emptyset$.
Note that $p \perp q$ iff $\exists \xi \in \mathrm{dom}(p) \cap \mathrm{dom}(q) (p(\xi) \neq q(\xi))$. I.e. p, q are compatible iff they agree on the intersection of their domains. In particular, if the intersection of their domains is empty, then they are compatible.
For $\xi \in x$, let $D_\xi = \{p \in \mathrm{Fn}(x, y) \mid \xi \in \mathrm{dom}(p)\}$. Note that D_ξ is dense (since y is non-empty). Similarly, if $\eta \in y$, let $R_\eta = \{p \in \mathrm{Fn}(x, y) \mid \eta \in \mathrm{ran}(p)\}$. This is dense since x is infinite.

Proposition. If $\mathcal{D} := \{D_\xi \mid \xi \in x\} \cup \{R_\eta \mid \eta \in y\}$, and G is $\mathrm{Fn}(x, y)$-generic over $\mathcal{D}$, then G defines a surjection from x onto y.

Proof. Let $f = \bigcup G \subseteq x \times y$. f is a function since elements of G are compatible. Since G intersects D_ξ for all $\xi \in x$, $\mathrm{dom} f = x$. Since G intersects R_η for all $\eta \in y$, $\mathrm{ran} f = y$. So F is indeed a surjection from x onto y. □

Theorem. If $\mathcal{D}$ is a countable set of sets dense in $\mathbb{P}$, then there is a $\mathbb{P}$-generic for $\mathcal{D}$.

Proof. Write $\mathcal{D} = \{D_n \mid n \in \mathbb{N}\}$. Let $p_0 := 1$. Consider D_0. Since D_0 is dense, there is $p \in D_0$ such that $p \leq 1$. Let $p_1 := 1$. In general, suppose that for $i \leq n$, we have $p_{i+1} \in D_i$ with $p_j \leq p_i$ for all $i \leq j \leq n$. Then consider D_n: by density, there is $p_{n+1} \in D_n$, $p_{n+1} \leq p_n$. So, assuming AC, we get a chain $\{p_i \mid i \in \mathbb{N}\}$. Letting $G = \{q \in \mathbb{P} \mid \exists i \in \mathbb{N} (q \geq p_i)\}$, we see that G is a filter since the p_i are compatible, and by construction, G is an up-set, and $p_{n+1} \in G \cap D_n$ for all n, so G is $\mathbb{P}$-generic for $\mathcal{D}$. □

(It seems like this construction is quite widely applicable.)

Now, if M is a countable transitive model of set theory, and $(\mathbb{P}, \leq, 1) \in M$, then $\mathcal{D}_m := \{D \subseteq \mathbb{P} \mid D \text{ dense in } \mathbb{P}\} \cap M$ is countable.

Corollary. In this situation, there is a $\mathbb{P}$-generic for $\mathcal{D}_M$.

Definition. G is $\underline{\mathbb{P}\text{-generic over } M}$ if G is $\mathbb{P}$-generic for $\mathcal{D}_M$.

Theorem. Suppose $(\mathbb{P}, \leq, 1) \in M$ is any partial order with the property

$$\forall p \in \mathbb{P}\ \exists q_1, q_2 \leq p (q_1 \perp q_2)$$

Suppose that G is $\mathbb{P}$-generic over M. Then $G \notin M$.

Proof. Suppose $G \in M$. Then $\mathbb{P} \setminus G \in M$.
Claim: $\mathbb{P} \setminus G$ is dense. Proof of claim: Let $p \in \mathbb{P}$ be arbitrary. By assumption, pick $q_1, q_2 \leq p$ with $q_1 \perp q_2$. We can't have $q_1, q_2 \in G$, because a filter can't contain incompatible elements. So wlog $q_1 \in \mathbb{P} \setminus G$. So $\mathbb{P} \setminus G$ is dense.
Now if G is $\mathbb{P}$-generic, then $G \cap (\mathbb{P} \setminus G) \neq \emptyset$, which is absurd. □

This tells us that we want to look at posets that sort of branch out indefinitely.

3.2 $\mathbb{P}$-names

Motivation: the von Neumann hierarchy. In this construction, $\mathcal{P}(V_\alpha)$ construct new sets as characterstic functions on the old sets. We could phrase this construction as

$$V'_0 := \emptyset$$
$$V'_{\alpha+1} := \{f \mid \mathrm{dom}(f) = V_\alpha, \mathrm{ran}(f) \subseteq 2\}$$
$$V'_\lambda = \bigcup_{\alpha<\lambda} V'_\alpha$$

But now extensionality sort of breaks bown when we compare objects form different levels.

Definition. We define $\mathbb{P}$-names by recursion:

$$\begin{aligned} V_0^{\mathbb{P}} &:= \emptyset \\ V_{\alpha+1}^{\mathbb{P}} &:= \{f \mid \mathrm{dom}(f) \subseteq V_\alpha^{\mathbb{P}}, \mathrm{ran}(f) \subseteq \mathbb{P}\} \\ V_\lambda^{\mathbb{P}} &:= \bigcup_{\alpha<\lambda} V_\alpha^{\mathbb{P}} \\ V^{\mathbb{P}} &:= \bigcup_{\alpha \in \mathrm{Ord}} V_\alpha^{\mathbb{P}} \end{aligned}$$

is the class of $\mathbb{P}$-names.

This construction, if $\mathbb{P}$ is a Boolean algebra, is also known as "Boolean-valued models". E.g., if $\mathbb{B} = \{0, 1, L, R\}$ is the 4-element Boolean algebra, we get $V_0^{\mathbb{B}} = \emptyset$, $V_1^{\mathbb{B}} = \{\emptyset\}$, $V_2^{\mathbb{B}} = \{\emptyset \mapsto 1, \emptyset \mapsto 0, \emptyset \mapsto L, \emptyset \mapsto R\}$. We interpret $\emptyset \mapsto 1$ as $\{\emptyset\}$, $\emptyset \mapsto 0$ as $\emptyset$, and $\emptyset \mapsto L$ as somewhere in between. After choosing a filter on $\mathbb{B}$, we will have decided whether $\emptyset \mapsto L$ is $\emptyset$ or $\{\emptyset\}$. But even without knowing the filter, we know that at one of $\emptyset \mapsto L$, $\emptyset \mapsto R$ will be empty.

Good morning. 'Generalising this to the partial order case':

Definition.

$$\begin{aligned} \mathrm{Name}_0^{\mathbb{P}} &:= \emptyset \\ \mathrm{Name}_{\alpha+1}^{\mathbb{P}} &:= \mathcal{P}(\mathrm{Name}_\alpha \times \mathbb{P}) \\ \mathrm{Name}_\lambda^{\mathbb{P}} &:= \bigcup_{\alpha<\lambda} \mathrm{Name}_\alpha^{\mathbb{P}} \\ \mathrm{Name}^{\mathbb{P}} &:= \bigcup_\alpha \mathrm{Name}_\alpha^{\mathbb{P}} \end{aligned}$$

Where $\mathrm{Name}^{\mathbb{P}}$ is the class of all names.

Lemma. If X is any set such that all elements of X are pairs (σ, p) with σ a name, and $p \in \mathbb{P}$, then X is a name.

Proof. Use replacement on X to find α with $X \in \mathrm{Name}_\alpha^{\mathbb{P}}$. $\square$

Interpreting names

If $\mathbb{P}$ is a partial order, and D is any filter on $\mathbb{P}$, then we define the valuation function for names by recursion:
if $\sigma \in \mathrm{Name}^{\mathbb{P}}$, then

$$\mathrm{val}(\sigma, D) := \{\mathrm{val}(\tau, D) \mid \exists p \in D((\tau, p) \in \sigma)\}$$

For example, $\mathrm{val}(\emptyset, D) = \{\mathrm{val}(\tau, D) \mid \exists p \in D((\tau, p) \in \emptyset)\} = \emptyset$.
Consider $\{(\emptyset, 1)\} =: \sigma$. $\mathrm{val}(\sigma, D) = \{\mathrm{val}(\tau, D) \mid \exists p \in D((\tau, p) = (\emptyset, 1)\} = \{\mathrm{val}(\emptyset, D)\} = \{\emptyset\}$, since 1 is in every filter. Similarly,

$$\mathrm{val}(\{(\emptyset, p)\}, D) = \begin{cases} \{\emptyset\} & p \in D \\ \emptyset & p \notin D \end{cases}$$

,

$$\mathrm{val}(\{(\emptyset, p), (\emptyset, q)\}, D) = \begin{cases} \{\emptyset\} & p \in D \vee q \in D \\ \emptyset & p \notin D \wedge q \notin D \end{cases}$$

If $\mathbb{P}$ does not have greatest lower bounds, then this is an important name.

Definition. For $x \in V$, $\check{x} := \{(\check{y}, 1) \mid y \in x\}$.

Note that this definition is valid by foundation.
$\mathrm{val}(\check{x}, D) = \{\mathrm{val}(\check{y}, D) \mid y \in x\}$. Claim: $\mathrm{val}(\check{x}, D) = x$. Proof: By induction. Suppose x is $\in$-minimal violating the claim, so that $x \neq \mathrm{val}(\check{x}, 1)$, but $\mathrm{val}(\check{y}, D) = y \ \forall y \in x$. But this is clearly absurd.
So every $x \in V$ has a 'canonical name'.

Now suppose that M is a transitive set model of ZFC, and $(\mathbb{P}, \leq, 1) \in M$. Then this construction can be carried out inside M, so we get a collection (in fact a set) of $\mathbb{P}$-names in M, denoted by $M^{\mathbb{P}}$.
Observation: Being a $\mathbb{P}$-name in M is absolute for transitive models containing M, so we don't need to write $(M^{\mathbb{P}})^M$. Similarly, if any transitive model contains M as a subclass, and D as an element, then for $\sigma \in M^{\mathbb{P}}$, $\mathrm{val}(\sigma, D)$ is absolute for that model. This justifies the following definition

Definition.
$$M[D] := \{\mathrm{val}(\sigma, D) \mid \sigma \in M^{\mathbb{P}}\}$$

Some properties: If M is a transitive set model, then $M[D]$ is a set. By the recursive definition of val, $M[D]$ is transitive. If $x \in M$, then $\check{x} \in M^{\mathbb{P}}$, and $\mathrm{val}(\check{x}, D) = x$ (given $1 \in D$). So $M \subseteq M[D]$.
In M, let $\Gamma := \{(\check{p}, p) \mid p \in \mathbb{P}\}$. Now $\mathrm{val}(\Gamma, D) = \{\mathrm{val}(\check{p}, D) \mid p \in D\} = \{p \mid p \in D\} = D$. So $D \in M[D]$.
So $M[D]$ is a transitive set with $M \cup \{D\} \subseteq M[D]$.

Corollary. If N is any transitive model of ZFC with $M \subseteq N$, $D \in N$, then $M[D] \subseteq N$.

But is $M[D]$ a model of ZFC?
By transitivity, $M[D] \models$ Extensionality and $M[D] \models$ Foundation. We have seen that $\emptyset \in M[D]$.
Pair: If $x, y \in M[D]$, find $\sigma, \tau \in M^{\mathbb{P}}$ such that $x = \mathrm{val}(\sigma, D), y = \mathrm{val}(\tau, D)$. Define $\mathrm{up}(\sigma, \tau) = \{(\sigma, 1), (\tau, 1)\} \in M^{\mathbb{P}}$ by Pair in M. Then

$$\mathrm{val}(\mathrm{up}(\sigma, \tau)) = \{\mathrm{val}(\sigma, D), \mathrm{val}(\tau, D)\} = \{x, y\}$$

So $M[D] \models$ Pair. On ES3, we prove $M[D] \models$ Union.
PowerSet is trickier.

(Henceforth let's write G instead of D, since G will almost always be generic in applications.)
'If we've done the rest', $M[G] \models$ Infinity follows from $\omega \in M[G]$. So we just need to show $M[G] \models$ PowerSet, Separation, Replacement, AC.
$M[G] \models$ AC should follow immediately from $M \models$ AC: A well-ordering on (a subset of) $M^{\mathbb{P}}$ induces a well-ordering on the corresponding subset of $M[G]$.
Now consider PowerSet. Given $x \in M[G]$, take σ with $x = \mathrm{val}(\sigma, G)$. We want a name ρ such that ρ becomes a name for the power set of x. 'As usual (assuming we'll have Separation later)', it's enough to find ρ such that $\forall y \in M[G](y \subseteq x \to \mathrm{val}(\rho, G)$. Let
$$\rho := \{(\tau, 1) \mid \mathrm{dom}(\tau) \subseteq \mathrm{dom}(\sigma)\}$$
where dom is the usual domain of a binary relation. Now, suppose $y \in M[G], y \subseteq x$. Let $y = \mathrm{val}(\mu, G)$. Note that $y \subseteq x$ means $z \in y \to z \in x$, i.e. $z = \mathrm{val}(\zeta, G) \in y$, then $\exists \zeta' \in \mathrm{dom}(\sigma)$ such that $z = \mathrm{val}(\zeta', G)$. So it's possible that $\mathrm{dom}(\mu) \cap \mathrm{dom}(\sigma) = \emptyset$. In particular, we could have $(\mu, 1) \notin \rho$. But 'morally', $(\mu, 1)$ should be in ρ, i.e. there should be something in ρ with the same valuation as μ. But why?

Definition. The <u>forcing language</u> $\mathcal{L}_\in(M^{\mathbb{P}})$ is $\mathcal{L}_\in$ augmented with one constant symbol for each name in $M^{\mathbb{P}}$.

E.g. if μ, σ are names, then "μ represents/is a subset of σ" is a sentence in $\mathcal{L}_\in(M^{\mathbb{P}})$, namely $\forall z(z \in \mu \to z \in \sigma)$. (Let's try not to confuse this with its interpretation in M.)
If $\varphi = \varphi(\tau_1, \ldots, \tau_n)$ is a sentence of $\mathcal{L}(M^{\mathbb{P}})$, and G is $\mathbb{P}$-generic, then φ can be interpreted in $M[G]$ by $\varphi(\mathrm{val}(\tau_1, G), \ldots, \mathrm{val}(\tau_1, G))$. We abbreviate this as $M[G] \models \varphi$.

Definition. For $p \in \mathbb{P}$, $\varphi \in \mathcal{L}_\in(M^\mathbb{P})$, $p \Vdash \varphi$ means $\forall G(G$ $\mathbb{P}$-generic over $M \wedge p \in G \to M[G] \models \varphi)$.

How can M know anything about the relation $p \Vdash \varphi$?
Consider $\mathbb{P} := \mathrm{Fn}(x, y)$. We proved that if G is $\mathbb{P}$-generic over M, then G defines a surjection form x to y. More concretely, we can give a name for this surjection:

$$\tau := \{(\mathrm{op}(\check{\xi}, \check{\eta}), p) \mid p \in \mathbb{P} \wedge \xi \in x \wedge \eta \in y \wedge \xi \in \mathrm{dom}(p) \wedge p(\xi) = \eta\}$$

where $\mathrm{op}(\sigma, \tau) := \mathrm{up}(\mathrm{up}(\sigma, \sigma), \mathrm{up}(\sigma, \tau))$.
Then $\mathrm{val}(\tau, G) = \bigcup G = f_G : x \twoheadrightarrow y$. So '$f_G : x \twoheadrightarrow y$ can be expressed in $\mathcal{L}_\in(M^\mathbb{P})$ by $\Phi := \tau$ is a function and $\mathrm{dom}(\tau) = \check{x}$ and $\mathrm{ran}(\tau) = \check{y}$.
We proved $1 \Vdash \Phi$.
If p is a non-empty function with, say, $p(\xi) = \eta$, then $p \Vdash \Phi \wedge f_G(\xi) = \eta$, where '$f_G(\xi) = \eta$' is indeed expressible in $\mathcal{L}_\in(M^\mathbb{P})$.

Theorem. Forcing theorem
Suppose M is a (countalbe) transitive model of ZFC and $(\mathbb{P}, \leq, 1) \in M$. Then

1. The relation $p \Vdash \varphi$ is definable in M.
2. If G is $\mathbb{P}$-generic over M, and φ is a $\mathcal{L}_\in(M^\mathbb{P})$-sentence, then

$$M[G] \models \varphi \iff \exists p \in G(p \Vdash \varphi)$$

I.e. truth in the generic model is not 'random': every true sentence is forced to be true. This may come as a surprise.

Let's use the forcing theorem to prove $M[G] \models \mathsf{PowerSet}$:
Let $x \in M[G]$, say $x = \mathrm{val}(\sigma, G)$. We defined $\rho = \{(\tau, 1) \mid \mathrm{dom}(\tau) \subseteq \mathrm{dom}(\sigma)\}$. Given $y \subseteq x, y \in M[G]$, say $y = \mathrm{val}(\mu, G)$, how can we relate μ and τ? We want to prove that there is some $\mu^\star \in \mathrm{dom}(\rho)$ such that $\mathrm{val}(\mu^\star, G) = \mathrm{val}(\mu, G)$. Let

$$\mu^\star = \{(\pi, p) \mid p \in \mathbb{P} \wedge \pi \in \mathrm{dom}(\sigma) \wedge p \Vdash \pi \in \mu\}$$

This is in $M^\mathbb{P}$ by FT1. By construction, $\mu^\star \in \mathrm{dom}(\rho)$, so $\mathrm{val}(\mu^\star, G) \in \mathrm{val}(\rho, G)$. We just need to show $\mathrm{val}(\mu^\star, G) = \mathrm{val}(\mu, G)$. For $\subseteq$, given $z \in \mathrm{val}(\mu^\star, G)$, there is $\pi \in \mathrm{dom}(\sigma)$, $p \Vdash \pi \in \mu$, $p \in G$, and $z = \mathrm{val}(\pi, G)$. By FT2, we get $M[G] \models \pi \in \mu$, i.e. $\mathrm{val}(\pi, G) \in \mathrm{val}(\mu, G)$. I.e. $z \in \mathrm{val}(\mu, G)$, i.e. $\mathrm{val}(\mu^\star, G) \subseteq \mathrm{val}(\mu, G)$.
Now suppose $z \in \mathrm{val}(\mu, G) = y \subseteq x = \mathrm{val}(\sigma, G)$. Take π, and $p \in G$, such that $(\pi, p) \in \sigma$ and $z = \mathrm{val}(\pi, G)$. We have $\mathrm{val}(\pi, G) \in \mathrm{val}(\mu, G)$, i.e. $M[G] \models \pi \in \mu$. FT2 gives us $p' \in G$, with $p' \Vdash \pi \in \mu$. Thus $(\pi, p') \in \mu^\star$, so $\mathrm{val}(\mu^\star, G) \ni \mathrm{val}(\pi, G) = z$.

Proof. of Forcing Theorem

1. Goal: Define a relation $\Vdash^\star$ – the syntactic forcing relation, in contrast to $\Vdash$, the semantic forcing relation – which is definable in M. This is done in two recursions: First we define $p \Vdash^\star \tau = \sigma$, $p \Vdash^\star \tau \in \sigma$. Then, assuming

$p \Vdash^\star \varphi, p \Vdash^\star \psi$ are defined, define $p \Vdash^\star \varphi \wedge \psi, \neg\varphi, \exists x\varphi$.

(a) Let's define $=$ recursively, since $\mathrm{Name}^{\mathbb{P}}$ is well-founded,

$$\begin{aligned} p \Vdash^\star \tau_1 = \tau_2 \iff \forall(\pi_1, p_1) \in \tau_1 : \\ \{q \leq p \mid q \leq p_1 \implies \exists(\pi_2, p_2) \in \tau_2(q \leq p_2 \wedge q \Vdash^\star \pi_1 = \pi_2)\} \\ \text{is dense below } p \wedge \\ \forall(\pi_1, p_1) \in \tau_2 : \\ \{q \leq p \mid q \leq p_1 \implies \exists(\pi_2, p_2) \in \tau_1(q \leq p_2 \wedge q \Vdash^\star \pi_1 = \pi_2)\} \\ \text{is dense below } p \end{aligned}$$

(Really we defined $p \Vdash^\star \tau_1 = \tau_2 \iff (p \Vdash^\star \tau_1 \subseteq \tau_2) \wedge (p \Vdash^\star \tau_2 \subseteq \tau_1))$
(b) And then $\in$,

$$\begin{aligned} p \Vdash^\star \tau_1 \in \tau_2 \iff \\ \{q \leq p \mid \exists(\pi, s) \in \tau_2(q \leq s \wedge q \Vdash^\star \pi = \pi_1)\} \\ \text{is dense below } p \end{aligned}$$

As for logical connectives,
Since $p \Vdash \varphi \wedge \psi \iff p \Vdash \varphi \wedge p \Vdash \psi$, we have to define
(c) $p \Vdash^\star \varphi \wedge \psi \iff (p \Vdash^\star \varphi) \wedge (p \Vdash^\star \psi)$.
$p \Vdash \neg\varphi \implies p \nVdash \varphi$, but the converse does not hold. Instead, we will define
(d) $p \Vdash^\star \neg\varphi \iff \forall q \leq p(q \nVdash^\star \varphi)$.
(e):

$$p \Vdash^\star \exists x\varphi \iff \left\{q \leq p \mid \exists\sigma \in \mathrm{Name}^{\mathbb{P}} \left(q \Vdash^\star \varphi\left[\frac{\sigma}{x}\right]\right)\right\} \text{ is dense below } p$$

Remark. Our definition of $p \Vdash^\star \varphi$ is almost absolute between models containing $(\mathbb{P}, \leq, 1)$, with the exception of our use of $\mathrm{Name}^{\mathbb{P}}$. If we replace $\mathrm{Name}^{\mathbb{P}}$ by $M^{\mathbb{P}}$, we get a relation $p \Vdash^\star_M \varphi$, which is absolutely definable in the parameter M, and the property $p \Vdash^\star_M \varphi \iff (p \Vdash^\star \varphi)^M$.
By induction, $q \leq p$, $p \Vdash \psi$ implies $q \Vdash \psi$.

□

We can now restate the forcing theorem

Theorem. Forcing Theorem
Let M be a transitive model of ZFC, $(\mathbb{P}, \leq, 1) \in M$, G $\mathbb{P}$-generic over M, φ a sentence of $\mathcal{L}_\epsilon(M^{\mathbb{P}})$. Then the following are equivalent:

1. $M[G] \models \varphi$
2. $\exists p \in G(p \Vdash^\star \varphi)^M$

Proof. We prove this by two inductions with five equivalences to show: 1. $\implies$ 2. and 2. $\implies$ 1. for each of $(a) - (e)$.

1. Let's start with $\wedge$. Assume 1. $\iff$ 2. for both φ and ψ. If $M[G] \models \varphi \wedge \psi$, then $M[G] \models \varphi$ and $M[G] \models \psi$, so take $p, q \in G$ with $p \Vdash_M^\star \varphi$, $q \Vdash_M^\star \psi$. Since G is a filter, take $r \leq p, q, r \in G$. Then $r \Vdash_M^\star \varphi \wedge r \Vdash_M^\star \psi$, so $r \Vdash_M^\star \varphi \wedge \psi$, as desired. Conversely, suppose there is $p \in G$ such that $p \Vdash_M^\star \varphi \wedge \psi$. Then $p \Vdash_M^\star \varphi \wedge p \Vdash_M^\star \psi$, so $M \models \varphi$ and $M[G] \models \psi$, i.e. $M[G] \models \psi$.

2. Moving on to $\neg$: Suppose that 1. $\iff$ 2. for φ, and first $M[G] \models \neg\varphi$. Let
$$D := \{p \in \mathbb{P} \mid (p \Vdash_M^\star \varphi) \vee (p \Vdash_M^\star \neg\varphi)^M\}$$
Claim: D is dense. Proof: take $q \in \mathbb{P}$. If $q \Vdash_M^\star \neg\varphi$, then $q \in D$, and we're happy. If $q \nVdash_M^\star \neg\varphi$, then by definition there is $r \leq q$ such that $r \Vdash_M^\star \varphi$. So $r \in D$. So D is dense.
By genericity of G, there is some $p \in D \cap G$. Thus either $p \Vdash_M^\star \varphi$, or $p \Vdash_M^\star \neg\varphi$. But if $p \Vdash_M^\star \varphi$, then $M[G] \models \varphi$ by induction hypothesis, which is false. So $p \Vdash_M^\star \neg\varphi$, as desired.
Conversely, suppose $p \in G$, $p \Vdash_M^\star \neg\varphi$. Suppose for the sake of contradiction that $M[G] \models \varphi$. By IH, take $q \in G$ with $q \Vdash_M^\star \varphi$. Since G is a filter, take $r \leq p, q, r \in G$. So $r \Vdash_M^\star \varphi, r \Vdash_M^\star \neg\varphi$. But this contradicts the definition of $\Vdash_M^\star \neg\varphi$.

3. $\exists$: IH: suppose 1. $\iff$ 2. for $\varphi\left[\frac{\sigma}{x}\right]$ for every name σ. Suppose $M[G] \models \exists x\varphi$. Take $X = \text{val}(\sigma, G) \in M[G]$ such that $M[G] \models \varphi\left[\frac{\sigma}{x}\right]$. By IH, take $p \in G$ such that $p \Vdash_M^\star \varphi\left[\frac{\sigma}{x}\right]$. Now the set
$$\{r \leq p \mid \exists\tau : r \Vdash_M^\star \varphi\left[\frac{\tau}{x}\right]\}$$
is not only dense below p, it is all of $\{r \leq p\}$. So $p \Vdash_M^\star \exists x\varphi$.
Conversely, suppose $p \in G$ and $p \Vdash_M^\star$. This means that $\{r \leq p \mid \exists\sigma : r \Vdash_M^\star \varphi\left[\frac{\sigma}{x}\right]\} =: D$ is dense below p. Since G is generic and $p \in G$, there is $r \in G \cap D$ (union with things incomparable to p). Pick σ such that $r \Vdash_M^\star \varphi\left[\frac{\sigma}{x}\right]$. By IH, $M[G] \models \varphi\left[\frac{\sigma}{x}\right]$. So $M[G] \models \exists x\varphi$.

4. $\in$: Our induction hypothesis is that 1. $\iff$ 2. for all formulas of the form $\sigma = \tau$.
Now $M[G] \models \tau_1 = \tau_2 \iff \text{val}(\tau_1, G) = \text{val}(\tau_2, G)$. Suppose $p \in G, p \Vdash^\star \tau_1 \in \tau_2$. (Note that we may now omit M.) So $D := \{q \leq p \mid \exists(\pi, s) \in \tau_2(q \leq s \wedge q \Vdash^\star \pi = \tau_1\}$ is dense below p. Take $q \in D \cap G$. $q \in D$ is witessed by $(\pi, s) \in \tau_2$ such that $q \leq s$ and $q \Vdash^\star \pi = \tau_1$, so $M[G] \models \pi = \tau_1$ by IH. Now $s \in G$, so $\text{val}(\tau_1, G) = \text{val}(\pi, G) \in \text{val}(\tau_2, G)$, i.e. $M[G] \models \tau_1 \in \tau_2$, as desired.
Conversely, suppose $M[G] \models \tau_1 \in \tau_2$, i.e. $\text{val}(\tau_1, G) \in \text{val}(\tau_2, G)$. Take $(\pi, s) \in \tau_2$ such that $s \in G$ and $\text{val}(\pi, G) = \text{val}(\tau_1, G)$. By IH, take $q \in G$ such that $q \Vdash^\star \pi = \tau_1$. Then take $p \leq q, s, p \in G$. Now $p \Vdash^\star \pi = \tau_1$, and $p \leq s$, so $p \Vdash^\star \tau_1 \in \tau_2$ by definition. As before, the set $\{r \leq p \mid \exists(\pi, s) \in \tau_2(r \leq s \wedge r \Vdash^\star \pi = \tau_1)\}$ is not only dense below p: it is all of $\{r \leq p\}$. So $p \Vdash^\star \tau_1 \in \tau_2$.

5. $\tau_1 = \tau_2$: We proceed by induction on the rank of τ_1, τ_2. IH: 1. $\iff$ 2. for '$\pi_1 = \pi_2$' of π_1, π_2 of lower rank than τ_1, τ_2.

Let's start with $2 \implies 1$. So suppose we have $p \Vdash^\star \tau_1 = \tau_2$. We need to show $\mathrm{val}(\tau_1, G) = \mathrm{val}(\tau_2, G)$. By the symmetry of the definition of $\tau_1 = \tau_2$, it suffices to show $\mathrm{val}(\tau_1, G) \subseteq \mathrm{val}(\tau_2, G)$. Take $x \in \mathrm{val}(\tau_1, G)$. So there is $(\pi_1, s_1) \in \tau_1$ such that $s_1 \in G$ and $x = \mathrm{val}(\pi_1, G)$. take $r \in G, r \leq p, s_1$. Since $r \leq p$, $p \Vdash^\star \tau_1 = \tau_2$, the set $D_{\pi_1, s_1} = \{q \leq p \mid q \leq s_1 \implies \exists(\pi_2, p_2) \in \tau_2 (q \leq p_2 \wedge q \Vdash^\star \pi_1 = \pi_2\}$ is dense below r. So take $q \in G \cap D_{\pi_1, s_1}, q \leq s_1$. Take $(\pi_2, p_2) \in \tau_2$ such that $q \leq p_2$ and $q \Vdash^\star \pi_1 = \pi_2$. By IH, $M[G] \models \pi_1 = \pi_2$, i.e. $\mathrm{val}(\pi_1, G) = \mathrm{val}(\pi_2, G)$. Now $p_2 \in G$, so $x = \mathrm{val}(\pi_1, G) = \mathrm{val}(\pi_2, G) \in \mathrm{val}(\tau_2, G)$, as desired.
Conversely, suppose $\mathrm{val}(\tau_1, G) = \mathrm{val}(\tau_2, G)$. Let

$$\begin{aligned} D := \{r \mid \text{either } (\alpha')\ \exists(\pi_1, s_1) \in \tau_1 \\ (r \leq s_1 \wedge \forall(\pi_2, s_2) \in \tau_2 \forall q ((q \leq s_2 \wedge q \Vdash^\star \pi_1 = \pi_2 \to q \perp r)) \text{, or} \\ (\beta')\ \exists(\pi_2, s_2) \in \tau_2 \\ (r \leq s_2 \wedge \forall(\pi_1, s_1) \in \tau_1 \forall q ((q \leq s_1 \wedge q \Vdash^\star \pi_1 = \pi_2 \to q \perp r)) \text{, or} \\ (\gamma)\ r \Vdash^\star \tau_1 = \tau_2\} \end{aligned}$$

Claim 1: D is dense. Suppose $p \in \mathbb{P}$ is arbitrary. Either $p \Vdash^\star \tau_1 = \tau_2$, then $p \in D$ by (γ), or $p \nVdash^\star \tau_1 = \tau_2$. In this case, one of the conditions for $p \Vdash^\star \pi_1 = \pi_2$ is violated. Wlog suppose it is the first condition, and that for $(\pi_1, p_1) \in \tau_1$, the set D_{π_1, s_1} is not dense below p. So take $r \leq p$ such that

$$\forall q \leq r\, (q \leq s_1 \wedge \forall(\pi_2, s_2) \in \tau_2 (\neg(q \Vdash^\star \pi_1 = \pi_2 \wedge q \leq s_2)))$$

In particular, $r \leq s_1$. Fix $(\pi_2, s_2) \in \tau_2$ and q such that $q \leq s_2 \wedge q \Vdash^\star \pi_1 = \pi_2$. If q and r are compatible, then find $q' \leq q, r$, so that $q' \leq r, q' \leq s_2, q' \Vdash^\star \pi_1 = \pi_2$, a contradiction. So $q \perp r$. Thus D is dense.
Now take $r \in G \cap D$.
Claim 2: No $r \in G$ can satisfy (α') or (β'). By symmetry, we just need to consider (α'). Suppose r does satisfy (α'). So take $(\pi_1, s_1) \in \tau_1$, such that $r \leq s_1$ and $\forall(\pi_2, s_2) \in \tau_2 \forall q \leq s_2 (q \Vdash^\star \pi_1 = \pi_2 \to q \perp r)$. $r \in G$ gives $s_1 \in G$, so $\mathrm{val}(\pi_1, G) \in \mathrm{val}(\tau_1, G) = \mathrm{val}(\tau_2, G)$. So take $(\pi_2, s_2) \in \tau_2$ such that $s_2 \in G$ and $\mathrm{val}(\pi_1, G) = \mathrm{val}(\pi_2, G)$. By IH, some $p \in G$ forces $p \Vdash^\star \pi_1 = \pi_2$. Take $q \leq p, r, s_2; q \in G$. This gives a contradiction, proving claim 2.
So we must have $(\gamma) : r \Vdash^\star \tau_1 = \tau_2$, as desired.

□

Theorem. Suppose M is a transitive model of ZFC, $(\mathbb{P}, \leq, 1) \in M$, and M has the property that for every $p \in \mathbb{P}$, there is a $\mathbb{P}$-generic filter G over M with $p \in G$.
Then for every $p \in \mathbb{P}$ and φ a sentence of $\mathcal{L}_\in(M^\mathbb{P})$,

$$p \Vdash_M \varphi \iff (p \Vdash^\star \varphi)^M.$$

Proof. If $p \Vdash^\star_M \varphi$, and $G \ni p$ is $\mathbb{P}$-generic over M, then by FT $M[G] \models \varphi$, so $p \Vdash_M \varphi$.
Conversely, if $p \Vdash_M \varphi$, consider

$$D := \{r \leq p \mid r \Vdash^\star_M \varphi\}.$$

Claim: this is dense below p. Proof: Let $q \le p$. By assumption, take G $\mathbb{P}$-generic over M with $q \in G$. Then $p \in G$ as well, so $M[G] \models \varphi$. So by FT, there is $r \in G$ with $r \Vdash^\star_M \varphi$. Finally, take $s \le q, r, s \in G$. Then $s \in D$, as desired. Finally, from the definition of $\Vdash^\star$, it now follows that $p \Vdash^\star_M \varphi$. $\square$

Remark: The assumption '$\forall p \in \mathbb{P} \exists G \ni p$' in particular holds for countable transitive models M of ZFC. (When constructing the generic, just start with p instead of 1.)

We haven't actually proved this, but there is are equivalences

1. $p \Vdash^\star \varphi$
2. $\forall q \le p (q \Vdash^\star \varphi)$
3. $\{r \mid r \Vdash^\star \varphi\}$ is dense below p.

Theorem. Generic Model Theorem

$$M[G] \models \mathsf{ZFC}$$

Proof. We have showed $M[G] \models \mathsf{Pair}, \mathsf{Union}, \mathsf{Foundation}, \mathsf{Extensionality}, \mathsf{Infinity}$, and $\mathsf{PowerSet}$ (modulo FT and $\mathsf{Separation}$)). So we need to do $\mathsf{Separation}$, $\mathsf{Replacement}$, and AC.
$\mathsf{Separation}$: Suppose $x, p_1, \dots, p_n \in M[G]$, $\varphi^{M[G]}$ is a formula in $n+1$ free variables. We want

$$\{z \in x \mid \varphi^{M[G]}(z, p_1, \dots, p_n)\} =: S \in M[G].$$

So we need to find a name ρ with $\mathrm{val}(\rho, G) = S$. Take names $\sigma, \tau_1, \dots, \tau_n$ for $x, p_1, \dots, p_n$. Let

$$\rho := \{(\pi, p) \mid \pi \in \mathrm{dom}(\sigma) \wedge p \Vdash (\varphi(\pi, \tau_1, \dots, \tau_n) \wedge \pi \in \sigma)\}$$

Claim: let $S = \mathrm{val}(\rho, G)$.
Let $z \in \mathrm{val}(\rho, G)$. Take $(\pi, p) \in \rho$ with $p \in G$ and $z = \mathrm{val}(\pi, G)$. Then $p \Vdash \varphi(\pi, \tau_1, \dots, \tau_n) \wedge \pi \in \sigma$, so by FT $M[G] \models \varphi(\pi, \tau_1, \dots, \tau_n) \wedge \pi \in \sigma$, i.e. $M[G] \models \varphi(z, \dots, p_n) \wedge z \in x$.
Conversely, let $z \in S$. Take $(\pi, p) \in \sigma$ such that $p \in G$ and $z = \mathrm{val}(\pi, G)$ and $M[G] \models \varphi(z, \dots, p_n) \dots$.
$\mathsf{Replacement}$: We will actually show "$\mathsf{Collection}$": If Φ is a formula such that $M[G] \models \forall x \exists y \Phi(x, y)$. Then given $X \in M[G]$ we will find $Y \in M[G]$ such that $\forall x \in X \exists y \in Y : M[G] \models \Phi(x, y)$.
To do this, take a name σ for X. If $\pi \in \mathrm{dom}(\sigma)$, either:

1. $\forall \mu \in M^{\mathbb{P}}\ \forall p : p \not\Vdash \Phi(\pi, \mu)$
2. $\exists \mu\ \exists p : p \Vdash \Phi(\pi, \mu)$

Let $\alpha_\pi = 0$ in the first case, and otherwise let α_π be the least α_π such that there is $\mu \in V^{;}_{\alpha_\pi}$ and p such that $p \Vdash \Phi(\pi, \mu)$. So the map $\alpha : \pi \mapsto \alpha_\pi$ is definable in M. By replacement, this is a set, so let λ be $\sup\{\alpha_\pi \mid \pi \in \mathrm{dom}(\sigma)\} \in \mathrm{Ord}^M$. Consider $T = V_\lambda \cap M^{\mathbb{P}}$, and let $\rho = \{(\pi, 1) \mid \pi \in T\}$. If $Y := \mathrm{val}(\rho, G)$. Then Y has the desired property: If $x \in X$, then find $\pi \in \mathrm{dom}(\sigma)$ such that

$x = \mathrm{val}(\pi, G)$. By the assumption about Φ, find $y \in M[G]$ such that $M[G] \models \Phi(x, y)$. Thus FT implies that we are not in case 1. So take μ, p with $\mu \in M^M_{\alpha_\pi}$ and $p \Vdash \Phi(\pi, \mu)$. Then $\mathrm{val}(\mu, G) \in \mathrm{val}(\rho, G)$.
AC: We show that

$$M[G] \models \forall X \ \exists \alpha \ \exists f : \alpha \twoheadrightarrow X.$$

If $X \in M[G]$, take a name σ for X. Since $M \models \mathsf{AC}$, we can find $\alpha, g \in M$ with $g : \alpha \twoheadrightarrow \mathrm{dom}(\sigma)$. Then define f with $\mathrm{dom}(f) = \alpha$ by $f(\beta) := \mathrm{val}(g(\beta), G)$. $\square$