Analysis of PDEs

Introduction

For $U \subset \mathbb{R}^n$ is open, *partial differential equation* of order k, a system of PDEs.

Data and Well-Posedness

well-posedness?

Notations) Let $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$ be a multi-index(where $\mathbb{N} = \{0, 1, 2, 3, \cdots\}$). Then define : $|\alpha|$, $D^\alpha f(x)$, x^α, $\alpha!$, $\beta \leq \alpha$.

Classifying PDEs

linear, semi-linear, quasi-linear, fully non-linear

Cauchy-Kovalevskaya Theorem

Theorem) (Picard-Lindelöf) Suppose there exist $r, K > 0$ s.t. $B_r(u_0) = \{w \in \mathbb{R}^n : |w - u_0| < r\}$ and $|f(x) - f(y)| \leq K|x - y|$ for all $x, y \in B_r(x_0)$. Then there exists $\epsilon > 0$(depending in r and K) and a unique C^1-function $u : (-\epsilon, \epsilon) \to U$ solving

$$\dot{u}(t) = f(u(t)), \quad u(0) = u_0 \in U \tag{1}$$

with $u : I \subset \mathbb{R} \to U$.

Motivation for Cauchy-Kovalevskaya?
formal power series solution

Theorem) (Cauchy-Kovalevskaya, for the case of ODEs) The formal power series solution(to be constructed) converges to a solution of (1) in a neighbourhood of $t = 0$ if f is real analytic(to be defined) at u_0.
(to be followed from a general result.)

Definition) real analytic function $f : U \to \mathbb{R}$, $U \subset \mathbb{R}^n$.

• Last lecture : $U \subset \mathbb{R}^n$ open, $f : U \to \mathbb{R}$ is real analytic at $x_0 \in U$ if $\exists f_\alpha \in \mathbb{R}$, $r > 0$ s.t.

$$f(x) = \sum_\alpha f_\alpha (x - x_0)^\alpha \quad \forall |x - x_0| < r$$

Properties of real analytic functions

- f is real analytic at x_0 if and only if $\exists s > 0$ and $C, \rho > 0$ such that:

$$\sup_{|x - x_0| < s} \left| D^\alpha f(x) \right| \leq C \frac{|\alpha|!}{\rho^{|\alpha|}}$$

- If f is RA(real analytic) at x_0, it is RA for all x close enough to x_0.

- If $f : U \to \mathbb{R}$ is real analytic everywhere on a connected set U, then f is determined by its values on any open subset of U. (Or by its Taylor expansion at a single point.)

(proofs in ES1)

Example : If $r > 0$ set

$$f(x) = \frac{r}{r - (x_1 + \cdots + x_n)} \quad \text{for } |x| < r/\sqrt{n}$$

(Verify it is RA and find its Taylor expansion)

Definition) $g \gg f$ (majorises), majorant

Lemma)

(i) If $g \gg f$ and g converges for $|x| < r$ then f also converges (absolutely) for $|x| < r$.

(ii) If f converges for $|x| < r$, then for any $s \in (0, r/\sqrt{n})$, f has a majorant that converges for $|x| < s/\sqrt{n}$.(n is the dimension of the space)

Remark : If $f = (f^1, \cdots, f^m)$ and $g = (g^1, \cdots, g^m)$ are formal power series, then we say

$$g \gg f \quad \text{if} \quad g^i \gg f^i \quad i = 1, \cdots, m$$

Cauchy-Kovalevskaya for First Order Systems

As coordinates on $\mathbb{R}^n$ we take $(x', t) = x$ where $x' = (x_1, \cdots, x_{n-1}) \in \mathbb{R}^{n-1}$, $t = x^n \in \mathbb{R}$. Set $B_r^n = \{t^2 + |x'|^2 < r^2\}$, $B_r^{n-1} = \{|x'| < r, t = 0\}$

We consider a system of equations for unknown $\underline{u}(x) \in \mathbb{R}^m$,

$$\begin{array}{ll} \underline{u}_t = \sum_{j=1}^{n-1} \underline{\underline{B}}_j(\underline{u}, x') \cdot \underline{u}_{x_j} + \underline{c}(\underline{u}, x') & \text{on } B_r^n \\ \underline{u} = 0 & \text{on } B_r^{n-1} \end{array} \tag{2}$$

(Note we assume $\underline{\underline{B}}_j$ and $\underline{u}$ do not depend explicitly on t. why do we note lose any generality by assuming this?)

Write $\underline{\underline{B}}_j = ((b_j^{kl}))$ and $\underline{c} = (c^1, \cdots, c^m)^T$. Then in components (2) reads: write out

Theorem) (Cauchy-Kovalevskaya) Assume $\{\underline{\underline{B}}_j\}_{j=1}^{n-1}$ and $\underline{c}$ are real analytic. Then for sufficiently small $r > 0$ there exists a unique real analytic function $\underline{u} : B_r^n \to \mathbb{R}^m$ solving the problem (2).

Reduction to a First Order System

Example)

Consider the PDE problem for $u : \mathbb{R}^3 \to \mathbb{R}$

$$\begin{aligned} u_{tt} &= uu_{xy} - u_{xx} + u_t \\ u\big|_{t=0} &= u_0 \\ u_t\big|_{t=0} &= u_1 \end{aligned} \tag{3}$$

where $u_0, u_1 : \mathbb{R}^2 \to \mathbb{R}$ are given real analytic functions (near 0).

Write out how we do the reduction to a First order system and apply Cauchy-Kovalevskaya.

Note : Which fact does this procedure rely on?

How can we generalize this to solve the quasilinear problem :

$$\sum_{|\alpha|=k} a_\alpha(D^{k-1}u, \cdots, u, x) D^\alpha u + a_0(D^{k-1}u, \cdots, u, x) = 0 \quad \text{for } |x| < r$$

$$u = \frac{\partial u}{\partial x_n} = \cdots = \frac{\partial^{k-1} u}{\partial x_n^{k-1}} = 0 \qquad \text{for } |x'| < r,\, x_n = 0$$

(called a Cauchy problem)?

Cauchy Problems for Quasilinear Equations with Data on a Surface

Real analysis hypersurface

Let γ be the unit normal to Σ and suppose u solves

$$\sum_{|\alpha|=k} a_\alpha(D^{k-1}u, \cdots, u, x)D^\alpha u + a_0(D^{k-1}u, \cdots, u, x) = 0 \quad \text{in } B_\epsilon(x) \tag{4}$$
$$u = \gamma^i \partial_i u = \cdots = (\gamma^i \partial_i)^{k-1} = 0 \quad \text{on } \Sigma$$

How do we translate this into a Cauchy problem on B_r^n?

Definition) suface Σ non-characteristic at $x \in \Sigma$ for a problem † (derive the relation to satisfy in terms of (†))

Theorem) Suppose $\Sigma \subset \mathbb{R}^n$ is a real analytic hypersurface. If Σ is non-characteristic for (4) at $x \in \Sigma$, there exists a unique real analytic solution to (4) in a neighbourhood of x.

Characteristic Surfaces for 2nd Order Linear PDE

Consider the linear operator

$$Lu = \sum_{i,j=1}^n a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i \frac{\partial u}{\partial x^i} + cu$$

with $a_{ij}, b_i, c : \mathbb{R}^n \to \mathbb{R}$ and the Cauchy problem

$$Lu = f, \quad u = \sum_{i=1}^n \xi^i \frac{\partial u}{\partial x^i} = 0 \quad \text{on } \Pi_\xi = \{\xi \cdot x = 0\}$$

-Condition for Π_ξ to be characteristic, a principal symbol of L, elliptic operator,

Criticisms/Shortcomings of Cauchy-Kovalevskaya

What are thy?

Elliptic Boundary Value Problems

Dirichlet Problem (for laplace equation)

Hölder and Sobolev Spaces

Hölder spaces

$U \subset \mathbb{R}^n$ open, $C^k(U)$, $C^k(\overline{U})$, Hölder continuity with exponent γ, Hölder seminorm $[u]_{C^{0,\gamma}}(\overline{U})$, $C^{k,\gamma}(\overline{U})$ Hölder norm $\|u\|_{C^{k,\gamma}(\overline{U})}$

The Spaces $L^p(U)$, $L^p_{\mathbf{loc}}(U)$

$U \subset \mathbb{R}^n$ open suppose $1 \leq p < \infty$. L^p for $p \in [1, \infty]$. Why are these spaces complete?
$L^p_{\text{loc}}(U)$ space.

Weak Derivatives

Definition) α^{th} weak derivative of $u \in L^1_{\text{loc}}$

$\star$ Check that if $D^\alpha u = v$, then v is indeed also a weak derivative of u.

Lemma) Suppose $v, \tilde{v} \in L^1_{\text{loc}}(U)$ are both weak α-derivatives of $u \in L^1_{\text{loc}}(U)$. Then $v = \tilde{v}$ almost everywhere, *i.e.* weak derivative is unique.

Definition) Sobolev space, H^k, $W^{k,p}$-norm, $W_0^{k,p}$-space.

We will find out that these spaces will be useful in fining solutions of PDEs. In particular, the H^k spaces will be useful.

Example : Let $U = B_1(0) = \{|x| < 1\} \subset \mathbb{R}^n$. Set $u(x) = |x|^{-\lambda}$ for $x \in U\backslash\{0\}$ and $\lambda > 0$. show : $u \in W^{1,p}(U) \quad \Leftrightarrow \quad \lambda < \frac{n-p}{p}$.

Theorem) For each $k = 1, 2, \cdots$ and $1 \leq p \leq \infty$. Then the space $W^{k,p}(U)$ is a Banach space.

Approximation of Functions in Sobolev Spaces

Convolution and Smoothing

Definition) standard mollifier, ϵ-mollification.
Let $U_\epsilon = \{x \in U | \text{dist}(x, \partial U) > \epsilon\}$.

Theorem) (Properties of Mollifiers)

(i) $f^\epsilon \in C^\infty(U_\epsilon)$ and $D^\alpha f^\epsilon = \int_U D_x^\alpha \eta_\epsilon(x-y) f(y) dy$.

(ii) $f^\epsilon \to f$ almost everywhere as $\epsilon \to 0$.

(iii) If $f \in C^0(U)$, then $f^\epsilon \to f$ uniformly on compact subsets of U.

(iv) If $1 \leq p < \infty$ and $f \in L^p_{\text{loc}}(U)$ then $f^\epsilon \to f$ in $L^p_{\text{loc}}(U)$, i.e.

$$\|f^\epsilon - f\|_{L^p(V)} \to 0 \quad \forall V \subset\subset U$$

(proved in handout)

Lemma) Assume $u \in W^{k,p}(U)$ for some $1 \leq p < \infty$. Set $u^\epsilon = \eta_\epsilon * u$ in U_ϵ. Then

(i) $u^\epsilon \in C^\infty(U_\epsilon) \quad \forall \epsilon > 0$

(ii) If $V \subset\subset U$, then $u^\epsilon \to u$ in $W^{k,p}(V)$

We can do better :

Theorem) (Global approximation by smooth functions) Suppose $U \subset \mathbb{R}^n$ is open and *bounded*, and suppose $u \in W^{k,p}(U)$ for some $1 \leq p < \infty$. Then there exists functions $u_n \in C^\infty(U) \cap W^{k,p}(U)$ such that

$$u_n \to u \quad \text{in } W^{k,p}(U)$$

Note, we do not assert $u_n \in C^\infty(\overline{U})$. When can this result go bad?

Definition) $C^{k,\alpha}$ domain

Theorem) Suppose $U \subset \mathbb{R}^n$ is a $C^{0,1}$ domain (U has Lipshitz boundary). Let $u \in W^{k,p}(U)$ for some $1 \leq p < \infty$. Then there exist functions $u_m \in C^\infty(\overline{U})$ such that $u_m \to u$ in $W^{k,p}(U)$.

Theorem) (Extension of Sobolev functions) Suppose $U \subset \mathbb{R}^n$, open, bounded, is a $C^{1,0}$ domain. Choose a bounded V such that $U \subset\subset V$. Then there exists a bounded linear operator $E : W^{1,p}(U) \to W^{1,p}(\mathbb{R}^n)$ such that for each $u \in W^{1,p}(U)$:

(i) $Eu = u$ almost everywhere in U.

(ii) Eu has support in V.

(iii) $\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(U)}$ where C only depends on U, V and p.

We call Eu an **extension of u to $\mathbb{R}^n$**. This is not unique.

Lemma) Suppose $U = B_r(0) \cap \{x_n > 0\}$. Suppose $u \in C^1(\overline{\{x_n > 0\}})$. We can find an $Eu \in C^1(\mathbb{R}^n)$ such that

$$\|Eu\|_{W^{1,p}(B_r(0))} \leq C\|u\|_{W^{1,p}(U)}$$

for some constant $C > 0$.

Lemma) Suppose $U \subset \mathbb{R}^n$, bounded, open C^1-domain. Suppose $u \in C^1(\overline{U})$. Then $\exists \overline{u} \in C^1_c(\mathbb{R}^n)$ that depends linearly on u and that

$$\|\overline{u}\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(U)} \quad u = \overline{u} \text{ on } U$$

We can repeat our argument to show a result for extensions of functions in $W^{1,p}(U)$ where U is a C^k domain, using a suitable higher order reflections.

Trace theorem

Theorem) *(Trace Theorem)* Assume $U \subset \mathbb{R}^n$ is open, bounded C^1 domain. There exists a bounded linear operator

$$T : W^{1,p}(U) \to L^p(\partial U) \quad 1 \leq p < \infty$$

such that

(i) $Tu = u\big|_{\partial U}$ if $u \in W^{1,p}(U) \cap C(\overline{U})$

(ii) $\|Tu\|_{L^p(\partial U)} \leq C\|u\|_{W^{1,p}(U)}$ for all $u \in W^{1,p}(U)$ where $C = C(U,p)$ only depends on U and p.

The trace map $T : W^{1,p}(U) \to L^P(\partial U)$ is not surjective.

Note : One can show without difficulty that if $u \in W^{1,p}_0(U)$ then $Tu = 0$. The converse is also true : if $u \in W^{1,p}(U)$ and $Tu = 0$ then $u \in W^{1,p}_0(U)$.

Sobolev Inequalities, Embeddings

Theorem) *(Sobolev-Gagliardo-Nirenberg, or SGN)* Assume $n > p$. We have $W^{1,p}(\mathbb{R}^n) \subset L^{p^*}(\mathbb{R}^n)$ with $p^* = \frac{np}{n-p} > p$, and $\exists C > 0$ depending only on n, p such that $\forall u \in W^{1,p}(\mathbb{R}^n)$,

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|Du\|_{L^p(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)}$$

(makes use of next lemma)

Lemma) *(projection lemma)* Let $n \geq 2$ and $f_1, \cdots, f_n \in L^{n-1}(\mathbb{R}^{n-1})$. For any $1 \leq i \leq n$, denote $\tilde{x}_i = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n) \in \mathbb{R}^{n-1}$ (remove i^{th} component from $(x_1, \cdots, x_n) \in \mathbb{R}^n$) and

$$f : \mathbb{R}^n \to \mathbb{R} \quad f(t) = f_1(\tilde{x}_1)f_2(\tilde{x}_2)\cdots f_n(\tilde{x}_n)$$

Then $f \in L^1(\mathbb{R}^n)$ with

$$\|f\|_{L^1(\mathbb{R}^n)} \leq \prod_{i=1}^{n}\|f_i\|_{L^{n-1}(\mathbb{R}^{n-1})}$$

In fact, in the example sheet(Exercise 2.9), you will see that this family of inequality, bounding $\|u\|_q$ by $\|Du\|_p$, can only exist for only particular pair of exponents (p, q) satisfying $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$

Corollary 1) Let $U \subset \mathbb{R}^n$ be open, bounded C^1-domain, and $1 \leq p < n$. Then $W^{1,p}(U) \subset L^{p^*}(U)$ (where p^* is as before) and $\exists\ C(p, n, U)$ such that

$$\|u\|_{L^{p^*}(U)} \leq C(p, n, U)\|u\|_{W^{1,p}(U)} \quad \forall u \in W^{1,p}(U)$$

Corollary 2) *(Poincaré Inequality)* Suppose $U \subset \mathbb{R}^n$ be open and bounded. Suppose $u \in W^{1,p}_0(U)$ for some $1 \leq p < n$. Then we have the following estimate.

$$\|u\|_{L^q(U)} \leq C\|Du\|_{L^p(U)} \quad \forall q \in [1, p^*)$$

where $C = C(p,q,n,U)$. In particular,

$$\|u\|_{L^p(U)} \le C\|Du\|_{L^p(U)}$$

Now suppose $n < p < \infty$. Then naively, we might expect a function in $W^{1,p}(\mathbb{R}^n)$ to be better than L^∞. In fact, we have

Theorem) *(Morrey's Inequality)* Suppose $n < p < \infty$. Then $\exists C = C(p,n)$ such that

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \le C\|u\|_{W^{1,p}(\mathbb{R}^n)} \quad \forall u \in C_c^1(\mathbb{R}^n)$$

where $\gamma = 1 - \frac{n}{p}$. (interpretation?)

Corollary) Let $n < p < \infty$. Suppose $u \in W^{1,p}(U)$. For $U \subset \mathbb{R}^n$ open, bounded C^1-domain. (boundedness is in fact not necessary.) Then $\exists u^* \in C^{0,1-\frac{n}{p}}(U)$ such that $u = u^*$ almost everywhere, and

$$\|u^*\|_{C^{0,1-\frac{n}{p}}(U)} \le C\|u\|_{W^{1,p}(U)}$$

for some $C = C(n,p,U)$.

By iterating these results, it is possible to establish similar embedding results for $W^{k,p}(\mathbb{R}^n)$ into $W^{k',p'}(\mathbb{R}^n)$ for $k' < k, p' > p$ or $C^{k',\gamma}(\mathbb{R}^n)$ for $k' < k$.

For example, we have $u \in W^{2,2}(\mathbb{R}^3)$ then $u \in C^{0,1/2}(\mathbb{R}^3)$ (prove how this is done).

Second Order Elliptic Equations

Let $U \subset \mathbb{R}^n$ and consider the operator

$$Lu = -\sum_{i,j=1}^n \left(a^{ij}(x)u_{x_j}\right)_{x_i} + \sum_{i=1}^n b^i(x)u_{x_i} + c(x)u \quad (\textbf{Divergence form})$$

where a^{ij}, b^i, c are given functions on U. Typically we will assume they are at least L^∞, but sometimes we will require more.

Definition) Elliptic / uniformly elliptic operator L.

Assume L is uniformly elliptic

We consider the boundary value problem

$$\begin{cases} Lu = f & \text{in } U \\ u\big|_{\partial U} = 0 \end{cases} \tag{5}$$

where U is *always* open bounded C^1-domain.

Definition) A weak solution $u \in H_0^1(U)$, the form $B[u,v]$.

Theorem) *(Lax-Milgram)* Let H be a (real) Hilbert space, with inner product $(\cdot,\cdot)$ and suppose $B : H \times H \to \mathbb{R}$ is a bilinear mapping such that... (state and prove)

Energy Estimates

Suppose $a^{ij}, b^i, c \in L^\infty(U)$ for U open, bounded and $B[u,v]$ as before.

Theorem) There exist $\alpha, \beta > 0$ and $\gamma \ge 0$ such that

(i) $\big|B[u,v]\big| \le \alpha\|u\|_{H^1(U)}\|v\|_{H^1(U)}$ for all $u, v \in H^1(U)$ *and*

(ii) $\beta\|u\|^2_{H^1(U)} \le B[u,u] + \gamma\|u\|^2_{L^2(U)}$ (**Gårding Inequality**)

Remark : If $B[\cdot,\cdot]$ is a bilinear form corresponding to an operator with $b^i = 0$, $c \geq 0$ then we can deduce Gårding's inequality holds with $\gamma = 0$ for $u \in H_0^1(U)$ by modifying the above proof with choosing ϵ appropriately.

Theorem) *(First Existence Theorem for Weak Solutions)* Let $U \subset \mathbb{R}^n$ be open, bounded and L be as before. Then there exists $\gamma \geq 0$ such that for any $\mu \geq \gamma$ and any $f \in L^2(U)$ there exists a unique weak solution to the boundary value problem(BVP) :

$$\begin{cases} Lu + \mu u = f & \text{in } U \quad \cdots\cdots(\dagger) \\ u = 0 & \text{on } \partial U \end{cases}$$

Moreover, $\|u\|_{H^1(U)} \leq C\|f\|_{L^2(U)}$ for some $C = C(L, U, \mu)$

Definition) weak convergence (in a Hilbert space)

Theorem) Let H be a separable Hilbert space and suppose $(u_n)_{n=1}^\infty$ is a bounded sequence, $u_n \in H$, $\|u_n\| \leq K$ for all n. Then $(u_n)_{n=1}^\infty$ admits a weakly convergent subsequence.

Lemma) *(Poincaré revisited)* Suppose $u \in H^1(\mathbb{R}^n)$. Let

$$Q = [\xi_1, \xi_1 + L] \times [\xi_2, \xi_2 + L] \times \cdots \times [\xi_n, \xi_n + L]$$

be a cube of side length L. Then we have:

$$\|u\|^2_{L^2(Q)} \leq \frac{1}{|Q|}\Big(\int_Q u dx\Big)^2 + \frac{n}{2}|L|^2\|Du\|^2_{L^2(Q)}$$

Note, this is equivalent to saying $\|u - \overline{u}\|^2_{L^2(Q)} \leq \frac{n}{2}|L|^2\|Du\|^2_{L^2(Q)}$ where $\overline{u} = \frac{1}{|Q|}\int_Q u(x)dx$.

Theorem) *(Rellich-Kondrachov)* Suppose $U \subset \mathbb{R}^n$ is open, bounded C^1-domain. Let $(u_m)_{m=1}^\infty$ be a sequence in $H^1(U)$ with

$$\|u_m\|_{H^1(U)} \leq K$$

Then there exists $u \in H^1(U)$ and a subsequence $(u_{m_j})_{j=1}^\infty$ such that u_{m_j} tends to u weakly in $H^1(U)$ and strongly in $L^2(U)$, i.e.

$$u_{m_j} \to u \quad \text{in } L^2(U), \quad u_{m_j} \xrightarrow{weak} u \quad \text{in } H^1(U)$$

Remark : Could replace $H^1(U)$ with $H_0^1(U)$ everywhere and the result will hold. Then we could drop C^1 regularity of ∂U (follows from the proof)

Definition) A bounded linear operator $K : H \to H$ being compact,.

Theorem) *(Fredholm alternative for compact operators)* Let $K : H \to H$ be a compact operator. Then (state) (see Linear Analysis for proof)

Formal adjoint $L^\dagger$

If $b \in C^1(U)$, this is an elliptic operator itself, otherwise we have to understand this as a formal expression through the following definition.

Definition) We say $v \in H_0^1(U)$ is a weak solution of the adjoint problem if...

With this setting, along with the additional assumption that a^{ij} is *uniformly elliptic*, we have the following result.

Theorem) *(Fredhold alternative for elliptic BVP)* Consider

$$\begin{cases} Lu = f & \text{in } U, \quad f \in L^2(U) \\ u = 0 & \text{on } \partial U \end{cases} \tag{6}$$

Then *either*

(a) for each $f \in L^2(U)$, (12) admits a *unique* weak solution, *or*

(b) there exist a weak solution to

$$\begin{cases} Lu = 0 & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases} \tag{7}$$

with $u \neq 0$.

If (b) holds, the dimension of the space $N \subset H_0^1(U)$ of weak solutions to (13) is finite and equals the dimension of $N^* \subset H_0^1(U)$, the space of weak solutions to the homogeneous adjoint problem

$$\begin{cases} L^\dagger v = 0 & \text{in } U \\ v = 0 & \text{on } \partial U \end{cases}$$

Finally (12) has a solution *iff*

$$(f, v)_{L^2(U)} = 0 \quad \forall v \in N^*$$

Spectrum of Elliptic Operators

Suppose $A : H \to H$ is a bounded linear operator on a Hilbert space H. Then
Definition) $\rho(A)$, $\sigma(A)$(spectrum), point spectrum, eigenvalue, eigenvector.

Theorem) *(Spectrum of a compact operator)* Assume $\dim(H) = \infty$ and $K : H \to K$ is compact and H is separable, then

(i) $0 \in \sigma(K)$,

(ii) $\sigma(K) - \{0\} = \sigma_p(K) - \{0\}$ *and*

(iii) either $\sigma(K) - \{0\}$ is finite or $\sigma(K) - \{0\}$ is a seqeunce tending to 0.

If moreover K is symmetric, $K = K^\dagger$, then there exists a countable orthonormal basis of H consisting of eigenvectors.

Theorem) *(Spectrum of L)* Let L, B, U be as in the last theorem. Then

(i) there exists an at most countable set $\Sigma \subset \mathbb{R}$ such that the BVP

$$\begin{cases} Lu = \lambda u + f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases} \qquad \cdots\cdots\cdots (\Diamond)$$

has a unique weak solution for each $f \in L^2(U)$ *iff* $\lambda \notin \Sigma$.

(ii) If Σ is infinite then $\Sigma = \{\lambda_k\}_{k=1}^\infty$(i.e. is at most countably infinite) and up to reordering, $\lambda_k \nearrow \infty$ as $k \to \infty$.

(iii) To each $\lambda \in \Sigma$, there is a attached a finite dimensional space

$$\mathcal{E}(\lambda) = \{u \in H_0^1(U) : u \text{ is a weak soution of } Lu = \lambda u \text{ in } U, u = 0 \text{ on } U\}$$

We say $\lambda \in \Sigma$ is an **eigenvalue of** L and $u \in \mathcal{E}(\lambda)$ is the **corresponding eigenfunction**.

Theorem) *(Spectrum of symmetric elliptic operators)* Suppose L is a symmetric uniformly elliptic operator $Lu = -\sum_{i,j=1}^n (a^{ij} u_{x_i})_{x_j} + cu$ on $U \subset \mathbb{R}^n$ open, bounded, C^1 domian. Then we can represent the eigenvalues of L as

$$\lambda_1 \leq \lambda_2 \leq \cdots$$

where each eigenvalue appears multiple times according to its multiplicity $(\dim(\mathcal{E}(\lambda)))$, and there exists an orthonormal basis $\{w_k\}_{k=1}^\infty$ for $L^2(U)$ with $w_k \in H_0^1(U)$ an eigenfunction of L corresponding to λ_k, i.e.

$$\begin{cases} Lw_k = \lambda_k w_k & \text{in } U \\ w_k = 0 & \text{on } \partial U \end{cases}$$

Elliptic Regularity

Suppose $U \subset \mathbb{R}^n$ is open, and $V \subset\subset U$. For $0 < |h| < \text{dist}(V, \partial U)$, we define the difference quotients

$$\Delta_i^h u(x) = \frac{u(x + he_i) - u(x)}{h} \quad i = 1, 2, \cdots, n$$

and define

$$\Delta^h u(x) = (\Delta_1^h u(x), \Delta_2^h u(x), \cdots, \Delta_n^h u(x))$$

Note $\Delta_i^h(x)u \in H^1(V)$ if $u \in H^1(U)$.

Lemma) Suppose $u \in L^2(U)$. Then $u \in H^1(V)$ with $\|Du\|_{L^2(V)} \leq K$ *iff* $\|\Delta^h u\|_{L^2(V)} \leq K$ for some $K \geq 0$ and all $0 < |h| < \frac{1}{2}\text{dist}(V, \partial U)$.

Theorem) *(Interior Regularity)* Suppose L is a uniformly elliptic operator on U, $a^{ij} \in C^1(U)$, $b^i, c \in L^\infty(U)$ and $f \in L^2(U)$. Suppose further that $u \in H^1(U)$ satisfies

$$B[u, v] = (f, v) \quad \forall v \in H_0^1(U) \quad \cdots\cdots\cdots (\star)$$

Then $u \in H^2_{\text{loc}}(U)$ and for each $V \subset\subset U$ we have the estimate

$$\|u\|_{H^2(V)} \leq C(\|f\|_{L^2(U)} + \|u\|_{L^2(U)})$$

where $C = C(U, V, L)$ does not depend on f.

Theorem) *(Higher interior regularity)* Let m be a non-negative integer, assume $a^{ij}, b^i, c \in C^{m+1}(U)$ and $f \in H^m(U)$(or $\in L^2(U) \cap H^m_{\text{loc}}(U)$). Suppose $u \in H^1(U)$ satisfies $B[u, v] = (f, v)_{L^2(U)}$ for all $v \in H_0^1(U)$. Then in fact $u \in H^{m+2}_{\text{loc}}(U)$ and for each $V \subset\subset U$ we have

$$\|u\|_{H^{m+2}(V)} \leq C\big(\|f\|_{H^m(U)} + \|u\|_{L^2(U)}\big)$$

where $C = C(U, V, L)$ does not depend on u or f.
(proof in ES4)

Remarks :

- Note this is a local result
- This result allows us to understand the equation as holding pointwise almost everywhere. Let $v \in C_c^\infty(U)$ and $B[u, v] = (f, v)_{L^2(U)}$. Since $u \in H^2_{\text{loc}}(U)$, we can integrate by parts to find $\int_U (Lu - f)v dx = 0$. This holds for any $v \in C_c^\infty(U)$, so $Lu = f$ almost everywhere. If m is large enough, $f \in H^m(U)$ implies $u \in C^2_{\text{loc}}(U)$ and solution is classical. (**Exercise** : figure out how large m should be, using Sobolev embedding).

Theorem) *(Boundary H^2 regularity)* Assume $a^{ij} \in C^1(\overline{U})$, $b^i, c \in L^\infty(U)$ and $f \in L^2(U)$. Suppose $u \in H_0^1(U)$ is a weak solution of $Lu = f$ in U, $u = 0$ on $\partial U(\diamond)$ and finally assume ∂U is C^2. Then $u \in H^2(U)$ and we have the estimate

$$\|u\|_{H^2(U)} \leq C\big(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}\big)$$

If the BVP has a *unique* solution for each $f \in L^2(U)$, we can drop the $\|u\|_{L^2(U)}$ term from RHS.

We can still do better.

Theorem) *(Higher boundary regularity)* Let $m \in \mathbb{N}$, assume $a^{ij}, b^i, c \in C^{m+1}(\overline{U})$, $f \in H^m(U)$ and ∂U is C^{m+2}. Then if $u \in H_0^1(U)$ is the weak solution of

$$\begin{cases} Lu = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$$

We in fact have $u \in H^{m+2}(U)$ with

$$\|u\|_{H^{m+1}(U)} \leq C(\|f\|_{H^m(i)} + \|u\|_{L^2(U)})$$

and can drop $\|u\|_{L^2(U)}$ if solution for the BVP exists for all $f \in L^2(U)$.

Initial-Boundary Value Problems for Wave Equations

Suppose $U \subset \mathbb{R}^n$ is open with C^1-boundary. We define $U_T, \Sigma_t, \partial^* U_T$, Lu (where $a^{ij}, b^i, b, c \in C^1(\overline{U}_T)$). Assume a^{ij} satisfy the uniform ellipticity condition.

We consider the initial-boundary value problem(IBVP).

Definition) weak solution of the IBVP
(Note that, we do not say $\partial_t u = \psi'$ on Σ_0 in trace sense, because $\partial_t u$ is just a L^2-function while we do not have trace theorem for L^2 functions.)

Theorem) A weak solution to a IBVP, if it exists, is unique. (would be useful to evoke the motivation of the proof)

Theorem) Given $\psi \in H_0^1(U)$, $\psi' \in L^2(U)$ and $f \in L^2(U_T)$, there exists a weak solution $u \in H^1(U_T)$ and

$$\|u\|_{H^1(U_T)} \leq C(\|\psi\|_{H^1(U)} + \|\psi'\|_{L^2(U)} + \|f\|_{L^2(U_T)}) \tag{8}$$

for some $C = C(U, T, a^{ij}, a^i, b, c)$ not depending on u.

Improved Regularity for the hyperbolic IBVP

We define for a Banach space X, $L^p((0,T);X) = \{u : (0,T) \to X : \|u\|_{L^p((0,T);X)} < +\infty\}$ with norm $\|u\|_{L^p((0,T);X)}$

Theorem) *(Higher Regularity for IBVP)* If $a^{ij}, b^i, b, c \in C^{k+1}(\overline{U}_T)$, ∂U is C^{k+1} and

$$\begin{aligned} &\partial_t^i u\big|_{\Sigma_0} \in H_0^1(U), \quad i = 0, \cdots, k \\ &\partial_t^{k+1} u\big|_{\Sigma_0} \in L^2(U) \\ &\partial_t^i f \in L^2((0,T); H^{k-i}(U)), \quad i = 0, \cdots, k \end{aligned}$$

Then $u \in H^{k+1}(U_T)$ and

$$\partial_t^i u \in L^\infty((0,T); H^{k+1-i}(U)), \quad i = 0, \cdots, k+1$$

(proof in handout)

Note that since $u_{tt}|_{\Sigma_0} = (f - Lu)|_{\Sigma_0}$ etc, the conditions on Σ_0 can be reduced tot he requirements that

$$\psi \in H^{k+1}(U), \quad \psi' \in H^k(U)$$

together with some compatibility condition hold at $\partial\Sigma_0$. (how do we do this? find the compatibility condition)

Finite propagation speed and solutions on unbounded domains

Let $S_0 \subset U$ be an open set with smooth boundary and let

$$D = \{(t,x) \in U_T : x \in S_0, t \in (0, \tau(x))\}$$

where $\tau : S_0 \to \mathbb{R}$ is a smooth function vanishing at ∂S_0. We say $S' = \{(\tau(x), x) : x \in S_0\} \subset U_T$ is **space-like** if (state)

Theorem) If S_0, D, S' are as above with S' *space-like* and $u \in H^1(U_T)$ is a weak solution to the IBVP (**??**). Then $u|_D$ depends only on $\psi|_{S_0}, \psi'|_{S_0}$ and $f|_D$.

This implies in particular that signals propagate at finite speed : suppose

$$\sum_{i,j} a^{ij}\xi_i\xi_j \leq \mu|\xi|^2 \quad \forall (x,t) \in U_T,\ \xi \in \mathbb{R}^n$$

Then no signal propagates faster than $\sqrt{\mu}$. (state how we can formalize this)
Using this property, we can construct solutions on *unbounded domains* by reducing locally to a bounded problem and using this uniqueness result - state how I do this.