

# Stochastic Calculus and Applications

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## 1 Introduction

### 1.1 Motivation

### 1.2 The Wiener Integral

$(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space

**Definition)** Gaussian space  $S \subset L^2(\Omega, \mathcal{F}, \mathbb{P})$

**Example :** Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space on which a sequence of independent random variables  $X_i \sim N(0, 1)$  is defined. Then the  $X_i$  are an orthonormal system in  $L^2(\Omega, \mathcal{F}, \mathbb{P})$  :

$$\mathbb{E}(X_i X_j) = 0 \quad \text{for } i \neq j \quad \text{and} \quad \mathbb{E}(X_i^2) = 1$$

and  $S = \overline{\text{span}\{X_i\}}$  is a Gaussian space. (*Exercise* : the limit in  $L^2$  of Gaussian random variables is Gaussian.)

**Proposition)** Let  $H$  be a separable Hilbert space and  $(\Omega, \mathcal{F}, \mathbb{P})$  as in the example. Then there is an isomtery  $I : H \rightarrow S$ . In particular, for every  $f \in H$ , there is a random variable  $I(f) \in S$  such that

$$I(f) \sim N(0, (f, f)_H) \quad \text{and} \quad \mathbb{E}(I(f)I(g)) = (f, g)_H$$

Moreover,  $I(\alpha f + \beta g) = \alpha I(f) + \beta I(g)$  a.s.

**Definition)** A Gaussian white noise on  $\mathbb{R}_+$

**Proposition)**

- (1) For  $A \subset \mathbb{R}_+$ , Borel,  $|A| < \infty$ ,  $WN(A) \sim N(0, |A|)$ .
- (2) For  $A, B \subset \mathbb{R}_+$  Borel,  $A \cap B = \emptyset$  then  $WN(A)$  and  $WN(B)$  are independent.
- (3) If  $A = \cup_{i=1}^{\infty} A_i$  for disjoint sets  $A_i$  with  $|A_i| < \infty$ ,  $|A| < \infty$ , then

$$WN(A) = \sum_{i=1}^{\infty} WN(A_i) \quad \text{in } L^2 \text{ and a.s.} \quad \dots\dots\dots (\star)$$

For  $t \geq 0$ , define the Brownian motion as  $B_t = WN([0, t])$ , just like the integration of white noise from 0 to  $t$ .  
- Justify that this indeed (up to a modification) a BM

### 1.3 The Lebesgue-Stieltjes Integral

**Definition)** signed measure (on  $[0, T] \subset \mathbb{R}_{\geq 0}$ ), Hahn-Jordan decomposition, total variation

**Proposition)** (*Hahn-Jordan*) For any positive measures  $\mu_1, \mu_2$  on  $[0, T]$  (we do not require them to have disjoint support), there is a signed measure  $\mu$  s.t.  $\mu = \mu_1 - \mu_2$ .

**Definition)** càdlàg function, total variation, of bounded variation on  $[0, T]$

**Proposition)**

- (i) Let  $\mu$  be a signed measure on  $[0, T]$ . Then  $a(t) = \mu([0, t])$  is càdlàg and  $|\mu|((0, t]) = v_a(0, t)$  (i.e.  $|\mu|([0, T]) = |a(0)| + v_a(0, t)$ )  
In particular,  $a \in BV([0, T])$ .

(ii) Let  $a : [0, T] \rightarrow \mathbb{R}$  be càdlàg of bounded variation. Then there is a signed measure  $\mu$  such that  $a(t) = \mu([0, t])$ .

**Definition)** Let  $a : [0, T] \rightarrow \mathbb{R}$  be càdlàg of bounded variation, Lebesgue-Stieltjes integral respect to  $a$ .

**Fact :** Let  $a : [0, T] \rightarrow \mathbb{R}$  be càdlàg and BV (bounded variation),  $h \in L^2([0, T], |da|)$ . Then

$$\left| \int_0^t h(s) da(s) \right| \leq \int_0^t |h(s)| |da(s)|$$

and the function  $h \cdot a : [0, T] \rightarrow \mathbb{R}$  is càdlàg and BV with signed measures  $h(s)da(s)$ ,  $|h(s)da(s)| = |h(s)||da(s)|$ .

**Proposition)** Let  $a : [0, T] \rightarrow \mathbb{R}$  be càdlàg and BV. Let  $h : [0, T] \rightarrow \mathbb{R}$  be *left-continuous* and bounded. Then

$$\begin{aligned} \int_0^t h(s) da(s) &= \lim_{m \rightarrow \infty} \sum_{i=1}^{n_m} h(t_{i-1}^{(m)}) (a(t_i^{(m)}) - a(t_{i-1}^{(m)})), \quad t \leq T \\ \int_0^t h(s) |da(s)| &= \lim_{m \rightarrow \infty} \sum_{i=1}^{n_m} h(t_{i-1}^{(m)}) |a(t_i^{(m)}) - a(t_{i-1}^{(m)})| \end{aligned}$$

for a sequence of subdivisions  $0 = t_0^{(m)} < \dots < t_{n_m}^{(m)} = t$  with  $\max_i |t_i^{(m)} - t_{i-1}^{(m)}| \rightarrow 0$  as  $m \rightarrow \infty$ .

**Definition)** finite variation (FV) function

## 2 Semimartingales

From now on  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  is a filtered probability space.

**Definition)** A càdlàg adapted process  $X$

*Notation :* write  $X \in \mathcal{F}$  to denote that a random variable  $X$  is measurable with respect to the sigma algebra  $\mathcal{F}$ .

### 2.1 Finite variation process

**Definition)** A finite variation process, total variation process

**Fact :** The total variation process  $V$  of càdlàg adapted process  $A$  is also càdlàg adapted and it is also increasing.

**Definition)**  $H \cdot A$  for  $A$  a finite variation process and  $H$  with an integrability condition (to be stated)

**Definition)** previsible(predictable)  $\sigma$ -algebra, predictable process.

**Definition)** simple process

**Fact :** Simple processes and their pointwise limits are predictable.

**Fact :** Adapted left-continuous processes are predictable

**Fact :** Let  $H$  be predictable. Then  $H_t \in \mathcal{F}_{t-}$  where  $\mathcal{F}_{t-} = \sigma(\mathcal{F}_s : s < t)$ . (See Example Sheet #1)

**Fact :** Let  $X$  be adapted càdlàg. Then  $X_{t-} = \lim_{s \rightarrow t-} X_s$  is left-continuous, predictable.

**Examples** Brownian motion is predictable. / A Poisson process  $(N_t)$  is *not* predictable.

**Proposition)** Let  $A$  be a finite variation process, and let  $H$  be a predictable process such that  $\int_0^t |H_s| |dA_s| < \infty$  for all  $t$  and  $\omega$ . Then  $H \cdot A$  is also a finite variation process.

### 2.2 Local Martingale

From now on, we assume that  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  satisfies the *usual conditions* (state)

**Theorem)** (*Optional Stopping Theorem, OST*) Let  $X$  be a càdlàg adapted integrable process. Then the following are equivalent : state / proof in Advanced probability

**Definition)** local martingale

**Example :**

- (i) Every martingale is a local martingale (Take  $T_n = n$  and use *OST*).
- (ii) Let  $(B_t)$  be a standard Brownian motion on  $\mathbb{R}^3$ . Then  $(X_t)_{t \geq 1} = (1/|B_t|)_{t \geq 1}$  is a local martingale, but not a martingale. (prove)

**Proposition)** Let  $X$  be a *local martingale* and  $X_t \geq 0$  for all  $t \geq 0$ . Then  $X$  is a supermartingale.

**Proposition)** Let  $X$  be a local martingale and suppose that there is  $Z \in L^1$  such that  $|X_t| \leq Z$  for all  $t \geq 0$ . Then  $X$  is a martingale. In particular, bounded local martingales are martingales.

**Fact :** Let  $X$  be a *continuous adapted process* with  $X_0 = 0$ . Then

$$S_n = \inf\{t \geq 0 : |X_t| = n\}$$

are stopping times and  $S_n \nearrow \infty$  as  $n \rightarrow \infty$ .

**Proposition)** Let  $X$  be a continuous local martingale with  $X_0 = 0$ . Then the sequence  $(S_n)$  defined above reduces  $X$ .

**Theorem)** Let  $X$  be a *continuous local martingale* with  $X_0 = 0$ . If  $X$  is also a *finite variation process*, then  $X_t = 0$  for all  $t \geq 0$  a.s.

## 2.3 $L^2$ bounded martingales

**Definition)**  $M^2, M_c^2, \|X\|_{M^2}$  - why is this a norm on  $M^2$ ?

In fact,  $(X, Y)_{M^2} = \mathbb{E}[X_\infty Y_\infty]$  is an inner product on  $M^2$  that induces the inner product - prove this.

**Proposition)**  $M^2$  is a *Hilbert space* and  $M_c^2$  is a closed subspace.

## 2.4 Quadratic Variation

**Definition)** convergent uniformly on compact intervals in probability

**Theorem)** Let  $M$  be a *continuous local martingale*. Then there exists a unique (up to indistinguishability) *continuous adapted increasing process*  $\langle M \rangle = (\langle M \rangle_t)_t$  such that (is uniquely characterized by)  $\langle M \rangle_0 = 0$  and  $M^2 - \langle M \rangle$  is a continuous local martingale.

Moreover, with  $0 = t_0^m < t_1^m < \dots$  given by  $t_i^m = 2^{-m}i$ ,

$$\langle M \rangle_t^{(m)} \xrightarrow{\text{ucp}} \langle M \rangle_t \quad \text{where} \quad \langle M \rangle_t^{(m)} = \sum_{i=1}^{\lfloor 2^m t \rfloor} (M_{t_i} - M_{t_{i-1}})^2$$

[In fact, the convergence is true for all locally finite subdivision of  $[0, \infty)$  with  $\max_i |t_i^m - t_{i-1}^m| \rightarrow 0$  as  $m \rightarrow \infty$ .]

**Definition)** quadratic variation of  $M, \langle M \rangle$ .

**Example :**  $\langle B \rangle_t = t$  for  $B$  a standard Brownian motion. - prove

**Lemma)** (*bounded case*) The theorem is true under the additional assumption  $|M_t| \leq C$  for all  $(\omega, t)$ ,  $M_t = M_{t \wedge T}$  for  $C, T$  deterministic constants.

**Lemma)** Suppose  $M$  is a continuous local martingale for which  $\langle M \rangle$  exists. Let  $T$  be a stopping time. Then  $\langle M^T \rangle$  exists and is given by  $\langle M^T \rangle_t = \langle M \rangle_{t \wedge T}$  (up to indistinguishability).

**Fact :** Let  $M$  be a continuous local martingale with  $M_0 = 0$ . Then  $M \equiv 0$  iff  $\langle M \rangle = 0$ .

**Proposition)** Let  $M \in M_c^2$  with  $M_0 = 0$ . Then  $M^2 - \langle M \rangle$  is a *uniformly integrable* martingale and

$$\|M\|_{M^2} = (\mathbb{E}[\langle M \rangle_\infty])^{1/2}$$

In particular, the norm only depends on the quadratic variation.

## 2.5 Covariation

**Definition)** For  $M$  and  $N$  continuous local martingales, define covariation  $\langle M, N \rangle$ .

**Proposition)**

- (i)  $\langle M, N \rangle$  is the unique (up to indistinguishability) finite variation process such that  $MN - \langle M, N \rangle$  is a continuous local martingale.
- (ii) We have  $\langle M, N \rangle_t^{(m)} \xrightarrow{ucp} \langle M, N \rangle_t$  where

$$\langle M, N \rangle_t^{(m)} = \sum_{i=1}^{\lfloor 2^m t \rfloor} (M_{i2^{-m}} - M_{(i-1)2^{-m}})(N_{i2^{-m}} - N_{(i-1)2^{-m}})$$

- (iii) The mapping  $M, N \mapsto \langle M, N \rangle$  is bilinear and symmetric.
- (iv) For every stopping time  $T$ ,  $\langle M^T, N^T \rangle_t = \langle M^T, N \rangle_t = \langle M, N \rangle_{T \wedge t}$ .
- (v) If  $M, N \in M_c^2$  with  $M_0 = N_0 = 0$ , then  $M_T N_t - \langle M, N \rangle$  is a uniformly integrable martingale and

$$(M, N)_{M^2} = \mathbb{E} \langle M, N \rangle_\infty$$

**Proposition)** (*Kunita-Watanabe inequality*) Let  $M$  and  $N$  be continuous local martingales and let  $H$  and  $K$  be measurable processes. Then a.s.

$$\int_0^\infty |H_s| |K_s| |d\langle M, N \rangle_s| \leq \left( \int_0^\infty |H_s|^2 d\langle M \rangle_s \right)^{1/2} \left( \int_0^\infty |K_s|^2 d\langle N \rangle_s \right)^{1/2} \dots\dots\dots \text{(KW)}$$

## 2.6 Semimartingales

**Definition)** (continuous) semimartingale, its quadratic variation.

*Exercise :* We again have limit expression

$$\langle X, Y \rangle_t^{(m)} = \sum_{i=1}^{\lfloor 2^m t \rfloor} (X_{i2^{-m}} - X_{(i-1)2^{-m}})(Y_{i2^{-m}} - Y_{(i-1)2^{-m}}) \xrightarrow{ucp} \langle X, Y \rangle_t$$

## 3 The Itô integral

### 3.1 Simple processes

**Definition)** simple process, Ito integral for simple processes respect to a  $M_c^2$ -martingale.

**Proposition)** Let  $M \in M_c^2$  and  $H \in \mathcal{E}$ . Then  $H \cdot M \in M_c^2$  and

$$\|H \cdot M\|_{M^2}^2 = \mathbb{E} \left( \int_0^\infty H_s^2 d\langle M \rangle_s \right) \quad (\text{Itô isometry for simple process})$$

What is the critical point about this proposition?

**Proposition)** Let  $M \in M_c^2$  and let  $H \in \mathcal{E}$ . Then

$$\langle H \cdot M, N \rangle = H \cdot \langle M, N \rangle, \quad \forall N \in M_c^2$$

i.e.  $\langle \int_0^\cdot H_s dM_s, N \rangle = \int_0^\cdot H_s d\langle M, N \rangle_s$ .

### 3.2 Itô isometry

**Definition)**  $L^2(M)$  for  $M \in M_c^2$ , norm and inner product in  $L^2(M)$  - why is  $(H, K)_{L^2(M)}$  finite for  $H, K \in L^2(M)$ ?

**Fact :**  $L^2(M) = L^2(\Omega \times [0, \infty), \mathcal{P}, d\mathbb{P}d\langle M \rangle)$  is a Hilbert space. (Recall  $\mathcal{P}$  is the previsible  $\sigma$ -algebra)

**Proposition)** Let  $M \in M_c^2$ . Then  $\mathcal{E}$ , the space of simple processes, is dense in  $L^2(M)$ .

**Theorem/Definition)** Let  $M \in M_c^2$ . Then

(i) The map  $H \in \mathcal{E} \mapsto H \cdot M \in M_c^2$  extends uniquely to an isometry  $L^2(M) \rightarrow M_c^2$ , the *Itô isomtery*.

(ii)  $H \cdot M$  is the unique martingale in  $M_c^2$  such that

$$\langle H \cdot M, N \rangle = H \cdot \langle M, N \rangle, \quad \forall N \in M_c^2$$

$(H \cdot M)_t = \int_0^t H_s dM_s$  is then called the **Itô integral** of  $H$  with respect to  $M$ .

**Corollary)** If  $T$  is a stopping time, then

$$(1_{[0,T]} H) \cdot M = (H \cdot M)^T = H \cdot M^T$$

**Corollary)**  $\langle H \cdot M, K \cdot N \rangle = (HK) \cdot \langle M, N \rangle$ , *i.e.*

$$\langle \int_0^\cdot H_s dM_s, \int_0^\cdot K_s dN_s \rangle_t = \int_0^t H_s K_s d\langle M, N \rangle_s$$

**Corollary)** One has, if  $t > u$ ,

$$\begin{aligned} \mathbb{E} \left( \int_0^t H_s dM_s \right) &= 0 \\ \mathbb{E} \left( \int_0^t H_s dM_s | \mathcal{F}_u \right) &= \int_0^u H_s dM_s \\ \mathbb{E} \left( \int_0^t H_s dM_s \int_0^t K_s dN_s \right) &= \mathbb{E} \left( \int_0^t H_s K_s d\langle M, N \rangle_s \right) \end{aligned}$$

**Corollary)** (*Associativity of Itô integral*) Let  $H \in L^2(M)$ . Then  $KH \in L^2(M)$  iff  $K \in L^2(H \cdot M)$  and then

$$(KH) \cdot M = K \cdot (H \cdot M)$$

### 3.3 Extension to local martingales

**Definition)** Let  $M$  be a *continuous local martingale*, define  $L_{loc}^2(M)$ .

**Theorem)** Let  $M$  be a continuous local martingale.

(i) For every  $H \in L_{loc}^2(M)$ , there is a unique (up to indistinguishability) continuous local martingale  $H \cdot M$  with  $(H \cdot M)_0 = 0$  such that

$$\langle H \cdot M, N \rangle = H \cdot \langle M, N \rangle \quad \forall N \text{ continuous local martingale}$$

(ii) If  $H \in L_{loc}^2(M)$  and  $K$  is predictable then  $K \in L_{loc}^2(H \cdot M)$  iff  $HK \in L_{loc}^2(M)$  and then

$$H \cdot (K \cdot M) = (HK) \cdot M$$

(iii) If  $T$  is a stopping time,

$$(1_{[0,T]} H) \cdot M = (H \cdot M)^T = H \cdot M^T$$

Finally, if  $M \in M_c^2$  and  $H \in L^2(M)$  then this definition is consistent with the previous one.

### 3.4 Extension to Semimartingales

**Definition)** locally bounded process

**Fact :** If  $H$  is locally bounded and predictable and if  $A$  is a finite variation process,

$$\forall t > 0, \quad \int_0^t H_s |dA_s| < \infty \quad \text{a.s.}$$

In particular, for such  $H$ , and  $M$  a continuous local martingale, it follows that  $H \in L^2_{loc}(M)$ .

**Definition)** Let  $X = X_0 + M + A$  be a continuous semimartingale, and let  $H$  be a predictable locally bounded process. Define  $H \cdot X$ .

**Proposition)** (*Stochastic Dominated Convergence Theorem, Stochastic DCT*) Let  $X$  be a continuous semimartingale, and let  $H$  be locally bounded predictable process and let  $K$  be a predictable non-negative process. Let  $t > 0$  and assume that

- (i)  $H_s^n \xrightarrow{n \rightarrow \infty} H_s$  for all  $s \in [0, t]$ .
- (ii)  $|H_s^n| \leq K_s$  for all  $s \in [0, t]$  and  $n \in \mathbb{N}$ .
- (iii)  $\int_0^t K_s^2 d\langle M \rangle + \int_0^t K_s |dA_s| < \infty$  (where  $X = X_0 + M + A$ ). [This condition is always true if  $K$  is locally bounded]

Then  $\int_0^t H_s^n dX_s \xrightarrow{ucp} \int_0^t H_s dX_s$  as  $n \rightarrow \infty$ .

**Corollary)** Let  $X$  be a continuous semimartingale, and let  $H$  be a locally bounded adapted left-continuous process. Then for any subdivision  $0 = t_0^{(m)} < \dots < t_{n_m}^{(m)} = t$  of  $[0, t]$  with  $\max_i |t_i^{(m)} - t_{i-1}^{(m)}| \rightarrow 0$  as  $m \rightarrow \infty$ , has :

$$\lim_{m \rightarrow \infty} \sum_{i=1}^{n_m} H_{t_{i-1}^{(m)}} (X_{t_i^{(m)}} - X_{t_{i-1}^{(m)}}) = \int_0^t H_s dX_s$$

where the convergence is made ucp. [Taking the left end-point  $t_{i-1}^{(m)}$  is important, and is consistent with the choice of Itô integral. Different choice corresponds to what the integral means.]

**Remark :** Suppose  $H$  is continuous. Unlike the case that  $X$  is of finite variation, it is essential here that  $H$  is evaluated at the left end point. - state why

### 3.5 Itô formula

**Theorem)** (*Integration by parts*) Let  $X$  and  $Y$  be continuous semimartingales. Then a.s.,

$$X_t Y_t - X_0 Y_0 = \int_0^t X_s dY_s + \int_0^t Y_s dX_s + \langle X, Y \rangle_t$$

The last term  $\langle X, Y \rangle$  is called the **Itô correction**.

**Theorem)** (*Itô formula*) Let  $X^1, \dots, X^p$  be continuous semimartingales, and let  $f \in C^2(\mathbb{R}^p; \mathbb{R})$ . Then, writing  $X = (X^1, \dots, X^p)$ , a.s.,

$$f(X_t) = f(X_0) + \sum_{i=1}^p \int_0^t \frac{\partial f}{\partial x^i}(X_s) dX_s^i + \frac{1}{2} \sum_{i,j=1}^p \int_0^t \frac{\partial^2 f}{\partial x^i \partial x^j}(X_s) d\langle X^i, X^j \rangle_s \quad \dots\dots\dots (*)$$

Informally, we may write

$$df(X_t) = \sum_{i=1}^p \frac{\partial f}{\partial x^i}(X_t) dX_t^i + \frac{1}{2} \sum_{i,j=1}^p \frac{\partial^2 f}{\partial x^i \partial x^j}(X_t) d\langle X^i, X^j \rangle_t$$

**Summary of calculation rules for the Itô integral :**

Let us adopt the notations

$$\begin{aligned} Z_t - Z_0 &= \int_0^t H_s dX_s & \Leftrightarrow & dZ_t = H_t dX_t \\ Z_t - Z_0 &= \langle X, Y \rangle_t & \Leftrightarrow & dZ_t = dX_t dY_t \end{aligned}$$

Then, :

<b>“Associativity”</b>	$H_t(K_t dX_t) = (H_t K_t) dX_t, \quad (i.e. \ H \cdot (K \cdot X) = (HK) \cdot X)$
<b>“Kunita-Watanabe equality”</b>	$H_t dX_t dY_t = (H_t dX_t) dY_t, \quad (i.e. \ H \cdot \langle X, Y \rangle = \langle H \cdot X, Y \rangle)$
<b>“Itô formula”</b>	$df(X_t) = \sum_i \frac{\partial f}{\partial x^i} dX_t^i + \frac{1}{2} \sum_{i,j} \frac{\partial^2 f}{\partial x^i \partial x^j}(X_t) dX_t^i dX_t^j$

## 4 Applications to Brownian Motion and Martingales

### 4.1 Lévy's characterisation of Brownian motion

**Theorem)** Let  $X = (X^1, \dots, X^p)$  be continuous local martingales. Suppose  $X_0 = 0$  and that  $\langle X^i, X^j \rangle_t = \delta_{ij}t$  for all  $t \geq 0$ . Then  $X$  is a standard  $p$ -dimensional Brownian motion. That is, *the covariation singles out the Brownian motion*.

**proof)** It suffices to show that for all  $0 \leq s < t$ , (i)  $X_t - X_s$  is independent of  $\mathcal{F}_s$  and (ii)  $X_t - X_s \sim N(0, (t-s)id_{p \times p})$ .

Both properties follow from the following claim :

$$\mathbb{E}(e^{i\theta \cdot (X_t - X_s)} | \mathcal{F}_s) = e^{-\frac{1}{2}|\theta|^2(t-s)} \quad \text{for all } \theta \in \mathbb{R}^p, s < t \quad \dots\dots\dots (\diamond)$$

Indeed,  $(\diamond)$  implies  $\mathbb{E}(e^{i\theta \cdot (X_t - X_s)}) = \exp(-\frac{1}{2}|\theta|^2(t-s))$  thus  $X_t - X_s \sim N(0, (t-s)id)$ . To show independence, consider  $A \in \mathcal{F}_s$  with  $\mathbb{P}(A) > 0$ . Then  $X_t - X_s \sim N(0, (t-s)id)$  under  $\mathbb{P}(\cdot|A) = \mathbb{P}(\cdot \cap A)/\mathbb{P}(A)$ . Hence

$$\mathbb{E}(1_A f(X_t - X_s)) = \mathbb{P}(A) \mathbb{E}(f(X_t - X_s)) \quad \forall f \text{ measurable,}$$

i.e.  $A$  is independent of  $X_t - X_s$ . The same is trivial if  $\mathbb{P}(A) = 0$ . Thus,  $X_t - X_s$  is independent of  $\mathcal{F}_s$ .

To show  $(\diamond)$ , fix  $\theta \in \mathbb{R}^p$  and set  $Y = \theta \cdot X_t = \sum_{i=1}^p \theta^i X_t^i$ . Then

$$\langle Y \rangle_t = \langle Y, Y \rangle_t = \sum_{i,j=1}^p \theta^i \theta^j \langle X^i, X^j \rangle_t = \sum_{i=1}^p (\theta^i)^2 t = |\theta|^2 t$$

where the third equality follows from the assumption in the statement of the theorem. Let  $Z_t = e^{iY_t + \frac{1}{2}\langle Y \rangle_t} = e^{iY_t + \frac{1}{2}|\theta|^2 t}$ . By Itô formula with  $f(x) = e^x$  and  $\bar{X} = iY + \frac{1}{2}\langle Y \rangle$ ,

$$\begin{aligned} dZ_t &= df(X_t) = Z_t(d\bar{X}_t + \frac{1}{2}d\langle \bar{X} \rangle_t) \\ &= Z_t(i \cdot dY_t + \frac{1}{2}d\langle Y \rangle_t - \frac{1}{2}d\langle Y \rangle_t) = iZ_t dY_t \end{aligned}$$

(verify these calculations - might be faulty.) In particular,  $Z$  is a continuous local martingale. Since  $Z$  is bounded on every bounded interval,  $Z$  is in fact a martingale (when we are checking martingale property, we only do computations on bounded times, so it suffices to have absolute bound on each bounded interval). So  $\mathbb{E}(Z_t | \mathcal{F}_s) = Z_s$  and hence

$$\mathbb{E}(e^{i\theta \cdot (X_t - X_s)} | \mathcal{F}_s) = e^{-\frac{1}{2}|\theta|^2(t-s)}$$

which was the claim.

(End of proof)  $\square$

This theorem can be of intrinsic interest, but is also interesting in terms of applications. We often construct a stochastic process not directly from a Brownian motion, but in different ways, for example by Wiener measure. Then this characterization verifies if this process is a Brownian motion or not.

### 4.2 Dubins-Schwarz Theorem

**Theorem)** Let  $M$  be a continuous local martingale with  $M_0 = 0$  and  $\langle M \rangle_\infty = \infty$  a.s. Let  $T_s = \inf\{t \geq 0 : \langle M \rangle_t > s\}$  be the right-continuous inverse of  $\langle M \rangle$ .

$$B_s = M_{T_s}, \quad \mathcal{G}_s = \mathcal{F}_{T_s}$$

Then  $T_s$  is an  $(\mathcal{F}_t)$  stopping times,  $\langle M \rangle_{T_s}$  for all  $s \geq 0$ ,  $B$  is a  $(\mathcal{G}_s)_{s \geq 0}$ -Brownian motion and

$$M_t = B_{\langle M \rangle_t},$$

i.e.  $M$  is a random time change of a Brownian motion.

(20th February, Wednesday)

Before we prove the theorem, we need a lemma.

**Lemma)** Let  $M$  be a continuous local martingale. Almost surely for all  $u < v$ ,  $M$  is constant on  $[u, v]$  iff  $\langle M \rangle$  is constant on  $[u, v]$ .

**proof)** By continuity, it suffices to prove that for any fixed  $u, v$ , a.s.

$$\{M_t = M_u \ \forall t \in [u, v]\} = \{\langle M \rangle_u = \langle M \rangle_v\}$$

Let  $N_t = M_t - M_{t \wedge u} = \int_{t \wedge u}^t dM_s$ . Then  $\langle N \rangle_t = \langle M \rangle_t - \langle M \rangle_{t \wedge u}$  (by *Kunita-Watanabe* formula). For any  $\epsilon$ , let  $T_\epsilon = \inf\{t \geq 0 : \langle N \rangle_t > \epsilon\}$ . Then  $N^{T_\epsilon} \in M_c^2$  since  $\langle N^{T_\epsilon} \rangle \leq \epsilon$  and  $\mathbb{E}((N^{T_\epsilon})^2) = \mathbb{E}(\langle N^{T_\epsilon} \rangle_t) \leq \epsilon$ . So

$$\mathbb{E}(N_t^2 \mathbf{1}_{\{\langle M \rangle_v = \langle M \rangle_u\}}) = \mathbb{E}(\mathbf{1}_{\langle N \rangle_v = 0} N_{t \wedge T_\epsilon}^2) \leq \mathbb{E}(N_{t \wedge T_\epsilon}^2) \leq \epsilon$$

so  $N_t = 0$  a.s. on  $\{\langle M \rangle_v = \langle M \rangle_u\}$  for any  $t \in [u, v]$ . Hence we have shown that a.s.  $\langle M \rangle_u = \langle M \rangle_v$  implies  $M$  is constant on  $[u, v]$ .

The other direction is implied by the approximation formula for the quadratic variation. Without loss of generality, put  $u = 0 < v$  and let  $\langle M \rangle_v^{(m)} = \sum_{i=1}^{2^m} (M_{i \cdot 2^{-m}v} - M_{(i-1) \cdot 2^{-m}v})^2$ , then  $\langle M \rangle_v^{(m)} = 0$  whenever  $M \equiv M_0$  on  $[0, v]$ . So for any  $\epsilon > 0$ ,

$$\begin{aligned} & \mathbb{P}(|\langle M \rangle_v - \langle M \rangle_v^{(m)}| \geq \epsilon, M_t = M_0 \ \forall t \in [0, v]) \\ &= \mathbb{P}(|\langle M \rangle_v| \geq \epsilon, M_t = M_0 \ \forall t \in [0, v]) \rightarrow 0 \quad \text{as } m \rightarrow \infty \end{aligned}$$

so in fact  $\mathbb{P}(|\langle M \rangle_v| \geq \epsilon, M_t = M_0 \ \forall t \in [0, v]) = 0$  for any  $\epsilon > 0$ , which concludes the proof.

[Note that we will never use the last implication]

(End of proof)  $\square$

**Theorem)** (*Dubins-Schwarz*) Let  $M$  be a continuous local martingale with  $M_0 = 0$  and  $\langle M \rangle_\infty = \infty$  a.s. Let  $T_s = \inf\{t \geq 0 : \langle M \rangle_t > s\}$  be the right-continuous inverse of  $\langle M \rangle$ . Let

$$B_s = M_{T_s}, \quad \mathcal{G}_s = \mathcal{F}_{T_s}$$

Then  $T_s$  is an  $(\mathcal{F}_t)$  stopping time,  $\langle M \rangle_{T_s} = s$  for all  $s \geq 0$ ,  $B$  is a  $(\mathcal{G}_s)_{s \geq 0}$ -Brownian motion and

$$M_t = B_{\langle M \rangle_t}$$

**proof)** Since  $\langle M \rangle$  is continuous and adapted, the  $T_s$  are stopping times and  $T_s < \infty$  a.s. Redefine  $T_s = 0$  if  $\langle M \rangle_\infty < \infty$ . Note that  $T_s$  is still a stopping times since  $(\mathcal{F}_s)$  is complete, hence  $\mathcal{F}_0$  includes all events of probability 0.

♡ **Claim :**  $(\mathcal{G}_s)$  is a filtration obeying the *usual conditions*. [note, we have constructed our theory on the implicit assumption that each filtration obeys the usual condition. So should  $(\mathcal{G}_s)$ ]

: For  $A \in \mathcal{G}_s = \mathcal{F}_{T_s}$  and  $s < t$ ,

$$A \cap \{T_t \leq u\} = A \cap \{T_s \leq u\} \cap \{T_t \leq u\} \in \mathcal{F}_u$$

so  $A \in \mathcal{A}_{T_t} = \mathcal{G}_t$ . Hence  $(\mathcal{G}_s)$  is a filtration.

Right-continuity and completeness are descended from the properties  $(\mathcal{F}_t)$  and right-continuity of  $t \mapsto T_t$ .

♡ **Claim :**  $B$  is adapted to  $(\mathcal{G}_s)$ .

: Recall, from *Advanced Probability*, that if  $X$  is càdlàg and  $T$  is a stopping time then  $X_T \mathbf{1}_{T < \infty} \in \mathcal{F}_T$ . Apply this with  $X = M$ ,  $T = T_s$  and  $\mathcal{F}_T = \mathcal{G}_s$ . This gives  $B_t \in \mathcal{G}_s$ .

♡ **Claim :**  $B$  is continuous.

:  $T_s$  is càdlàg in  $s$ , so  $B_s = M_{T_s}$  is càdlàg and thus right-continuous.

Also to check  $B$  is left continuous, observe  $B$  is left-continuous at  $s \Leftrightarrow B_s = B_{s-} \Leftrightarrow M_{T_s} = M_{T_{s-}}$  where  $T_{s-} = \inf\{t \geq 0 : \langle M \rangle_t = s\}$ . We divide into two cases,



- if  $T_s = T_{s-}$  then  $B$  is left-continuous.
- if  $T_s > T_{s-}$  then  $\langle M \rangle$  is constant on  $[T_{s-}, T_s]$ . Hence  $M_{T_s} = M_{T_{s-}}$  holds by the previous lemma, *i.e.*  $B$  is left-continuous.

By these two cases, we see that  $B$  is left-continuous

♡ **Claim :**  $B$  is a continuous martingale with respect to  $(\mathcal{G}_s)$  and  $\langle B \rangle_s = s$  for all  $s \geq 0$ .

: Let  $0 \leq r < s$ . Then  $\langle M^{T_s} \rangle_\infty = \langle M \rangle_{T_s} = s$  (here we used the fact that  $\langle M^{T_s} \rangle_t$  tends to  $\infty$  a.s.). So  $M^{T_s} \in M_c^2$ , and so  $(M^2 - \langle M \rangle)^{T_s}$  is a uniformly integrable martingale. Now *Optional Stopping Theorem* implies

$$\begin{aligned}\mathbb{E}(B_s | \mathcal{G}_r) &= \mathbb{E}(M_\infty^{T_s} | \mathcal{F}_{T_r}) = M_{T_r} = B_r \\ \mathbb{E}(B_s^2 - s | \mathcal{G}_r) &= \mathbb{E}((M^2 - \langle M \rangle)_\infty^{T_s} | \mathcal{F}_{T_r}) = M_{T_r}^2 - \langle M \rangle_{T_r} = B_r^2 - r\end{aligned}$$

By the claim and the *Lévy's characterisation* of Brownian motions, it follows that  $B$  is a Brownian motion.

(End of proof) □

*Caution!* : Schwarz in *Dubins-Schwarz* does not have a 't' in his name.

### 4.3 Girsanov's Theorem

This is a next application of Itô's formula.

**Example :** Let  $X \sim N(0, C)$ . be an  $n$ -dimensional Gaussian vector with positive definite covariance matrix  $C = (C_{ij})_{i,j=1}^n$  and mean 0. Then

$$\mathbb{E}(f(X)) = \left(\det \frac{M}{2\pi}\right)^{1/2} \int_{\mathbb{R}^n} f(x) e^{-\frac{1}{2}x \cdot Mx} dx, \quad (M = C^{-1})$$

Let  $a \in \mathbb{R}^n$ . Then

$$\begin{aligned}\mathbb{E}(f(X + a)) &= \left(\det \frac{M}{2\pi}\right)^{1/2} \int_{\mathbb{R}^n} f(x) e^{-\frac{1}{2}(x-a) \cdot M(x-a)} dx \\ &= \left(\det \frac{M}{2\pi}\right)^{1/2} \int_{\mathbb{R}^n} f(x) e^{-\frac{1}{2}x \cdot Mx - \frac{1}{2}a \cdot Ma + x \cdot Ma} dx =: \left(\det \frac{M}{2\pi}\right)^{1/2} \int_{\mathbb{R}^n} f(x) e^{-\frac{1}{2}x \cdot Mx} Z dx\end{aligned}$$

where  $Z := \exp(\frac{1}{2}a \cdot Ma + x \cdot Ma)$ , so  $\mathbb{E}(f(X + a)) = \mathbb{E}(f(X)Z)$ . Thus if  $\mathbb{P}$  denotes the distribution of  $X$ , then the measure  $\mathbb{Q}$  with

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = Z$$

is that of a  $N(0, C)$  Gaussian vector, *i.e.* of  $X + a$ .

**Example :** Let  $B$  be a standard Brownian motion with  $B_0 = 0$ . Fix finitely many times  $0 = t_0 = t_1 < \dots < t_n$ . Then  $(B_{t_i})_{i=1}^n$  is a centred Gaussian vector with

$$\mathbb{E}(f((B_{t_i})_{i=1}^n)) = \text{const.} \times \int_{\mathbb{R}^n} f(x) e^{-\frac{1}{2} \sum_{i=1}^n \frac{(x_i - x_{i-1})^2}{t_i - t_{i-1}}} dx_1 \dots dx_n$$

Let  $h : \mathbb{R}_+ \rightarrow \mathbb{R}$  be a deterministic function. Then

$$\mathbb{E}(f((B + h)_{t_i})) = \mathbb{E}(Z f(B_{t_i}))$$

with  $Z = \exp\left(-\frac{1}{2} \sum_{i=1}^n \frac{(h_{t_i} - h_{t_{i-1}})^2}{t_i - t_{i-1}} + \sum_{i=1}^n \frac{(h_{t_i} - h_{t_{i-1}})(B_{t_i} - B_{t_{i-1}})}{t_i - t_{i-1}}\right)$ .

The Girsanov's Theorem is essentially a generalized version of these with the summation replaced by integration with more careful treatment.

**Definition)** Let  $L$  be a continuous local martingale with  $L_0 = 0$ . Then the **stochastic exponential** of  $L$  is

$$\mathcal{E}(L)_t = e^{L_t - \frac{1}{2}\langle L \rangle_t}$$

**Fact :**  $Z = \mathcal{E}(L)$  is a continuous local martingale and it satisfies

$$dZ_t = Z_t dL_t$$

i.e.  $Z_t = 1 + \int_0^t Z_s dL_s$ .

**proof)** By Itô's formula applied to  $X = L - \frac{1}{2}\langle L \rangle$  and  $f(x) = e^x$ ,

$$dZ_t = df(X_t) = Z_t(dL_t - \frac{1}{2}d\langle L \rangle_t + \frac{1}{2}d\langle L \rangle_t) = Z_t dL_t$$

Since  $L$  is a continuous local martingale, so is  $Z \cdot L$  since  $Z = 1 + Z \cdot L$ , hence  $Z$  is a continuous local martingale.

(End of proof)  $\square$

**Theorem)** (*Girsanov*) Let  $L$  be a continuous local martingale with  $L_0 = 0$ . Suppose that  $\mathcal{E}(L)$  is a *UI*(uniformly integrable) martingale. Define a probability measure  $\mathbb{Q}$  by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathcal{E}(L)_\infty$$

If  $M$  is a continuous local martingale with respect to  $\mathbb{P}$ , then  $\tilde{M} = M - \langle M, L \rangle$  is a continuous local martingale with respect to  $\mathbb{Q}$ .

*Remark :* The quadratic variation does not change,  $\langle M \rangle = \langle \tilde{M} \rangle$ . This follows, for example, from

$$\langle M \rangle_t = \lim_{n \rightarrow \infty} \sum_{i=1}^{\lfloor 2^m t \rfloor} (M_{2^{-m}i} - M_{2^{-m}(i-1)})^2 \quad \mathbb{P}\text{-a.s. along a subsequence}$$

and  $\mathbb{Q}$ -null sets are identical to  $\mathbb{P}$ -null sets, so the limit is also true in  $\mathbb{Q}$ -a.s. sense.

**proof)** Let  $T_n = \inf\{t \geq 0 : |\tilde{M}_t| > n\}$ . Then  $T_n$  is a stopping time, and  $\mathbb{Q}(T_n \nearrow \infty) = \mathbb{P}(T_n \nearrow \infty) = 1$  by continuity of  $\tilde{M}$  and that  $\mathbb{P} \sim \mathbb{Q}$ . Thus it suffices to show that  $\tilde{M}^{T_n}$  is a continuous local martingale with respect to  $\mathbb{Q}$  for all  $n$ . Let  $Y_t = M_t^{T_n} - \langle M^{T_n}, L \rangle_t$  and  $Z_t = \mathcal{E}(L)_t$ .

♥ **Claim :**  $(ZY)_t$  is a continuous local martingale with respect to  $\mathbb{P}$ .

: we just check using Itô formula, that

$$\begin{aligned} d(ZY) &= Y_t dZ_t + Z_t dY_t + d\langle Z, Y \rangle_t \\ &= (M_t^{T_n} - \langle M^{T_n}, L \rangle_t) Z_t dL_t + Z_t (dM_t^{T_n} - d\langle M^{T_n}, L \rangle_t) + Z_t d\langle L, M^{T_n} \rangle_t \\ &= (M_t^{T_n} - \langle M^{T_n}, L \rangle_t) Z_t dL_t + Z_t dM_t^{T_n} \end{aligned}$$

where we used  $d\langle Z, Y \rangle = d\langle Z, M^{T_n} \rangle = Z d\langle L, M^{T_n} \rangle$  since  $dZ = Z dL$  and  $\langle Z \cdot L, M^{T_n} \rangle = Z \cdot \langle L, M^{T_n} \rangle$ . Thus  $d(ZY)$  is a sum of stochastic differentials with respect to local martingales. Hence  $ZY$  is a continuous local martingale.

♥ **Claim :**  $ZY$  is a uniformly integrable martingale with respect to  $\mathbb{P}$ .

: this follows from the fact that  $Z = \mathcal{E}(L)$  is by assumption a uniformly integrable martingale, together with the fact that  $Y$  is bounded (recall, by definition that  $|Y| \leq n$ ) (but be careful that if  $Z$  was just a uniformly integrable local martingale, then this is not necessarily true so we need a bit more). Indeed, a local martingale  $M$  is a martingale *iff*

$$\forall t, \quad \mathcal{X}_t = \{M_T : T \text{ is a stopping time with } T \leq t\} \text{ is UI}$$

: (verifying this statement was an Exercise. Or see online lecture notes p21-22) Forward implication follows from the fact that, for any  $t > 0$ ,  $\{\mathbb{E}[X_t | \mathcal{G}] : \mathcal{G} \subset \mathcal{F}_t \text{ a sub-}\sigma\text{-algebra}\}$  is uniformly integrable (see *Advanced Probability* or *Example Sheet #1*).

To see the backward implication, first note that for any bounded stopping time  $T$ , say  $T \leq t_0$  a.s., the process  $(X_t^T)_t$  is uniformly integrable and therefore  $\mathbb{E}[X_t^T | \mathcal{F}_s] = X_s^T$  whenever  $s \leq t \leq t_0$ . Then the result follows from a part of the *Optional Stopping Theorem* stating that  $X$  is a martingale *iff* the process  $X^T$  is a martingale for any bounded stopping time  $T$ .

Since being UI is preserved under multiplication by bounded random variables, this implies the claim.

♡ **Claim :**  $Y$  is a martingale with respect to  $\mathbb{Q}$  (recall  $\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathcal{E}(L)_\infty = Z_\infty$ ).

: First observe that

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}(Y_t - Y_s | \mathcal{F}_s) &= \mathbb{E}^{\mathbb{P}}(Z_\infty Y_t - Z_\infty Y_s | \mathcal{F}_s) \\ &= \mathbb{E}^{\mathbb{P}}(\mathbb{E}^{\mathbb{P}}(Z_\infty Y_t - Z_\infty Y_s | \mathcal{F}_t) | \mathcal{F}_s) = \mathbb{E}^{\mathbb{P}}(Z_t Y_t - Z_s Y_s | \mathcal{F}_s) = 0\end{aligned}$$

Since  $Y = (M - \langle M, L \rangle)^{T_n}$  is a  $\mathbb{Q}$ -martingale and  $T_n \nearrow \infty$  a.s., thus  $M - \langle M, L \rangle$  is a  $\mathbb{Q}$ -local martingale.

Having these claims, the proof is complete.

(End of proof)  $\square$

To apply the theorem, we need to verify that the exponential martingale is uniformly integrable. One criterion for doing this is the following, yet not the most general one.

**Proposition)** Suppose that  $\langle L \rangle$  is bounded, say  $\langle L \rangle_\infty \leq C$ . Then  $\mathcal{E}(L)$  is a UI martingale.

**proof)** It suffices to show that  $\sup_t L_t$  has Gaussian tail :

$$\mathbb{P}\left(\sup_{t \geq 0} L_t \geq a\right) \leq e^{-a^2/2C}$$

Indeed, then

$$\begin{aligned}\mathbb{E}\left(\sup_t \mathcal{E}(L)_t\right) &\leq \mathbb{E}\left(\exp(\sup_t L_t)\right) \quad (\text{used } \mathcal{E}(L) = e^{L - \frac{1}{2}\langle L \rangle} \leq e^L) \\ &= \int_0^\infty \mathbb{P}\left(\exp(\sup_t L_t) \geq \lambda\right) d\lambda \\ &= \int_0^\infty \mathbb{P}\left(\sup_t L_t \geq \log \lambda\right) d\lambda \\ &\leq 1 + \int_1^\infty e^{-\frac{(\log \lambda)^2}{2C}} d\lambda < \infty\end{aligned}$$

so  $\mathcal{E}(L)_t$  is bounded by the random variable  $\sup_t \mathcal{E}(L)_t \in L^1$  and thus  $\mathcal{E}(L)$  is uniformly integrable.

(Checking the tail bound is an exercise, on the Example Sheet)

♡ **Claim :** Let  $M$  be a continuous local martingale with  $M_0 = 0$ . Then for any  $a, b > 0$ ,

$$\mathbb{P}\left(\sup_{t \geq 0} M_t \geq a, \langle M \rangle_\infty \leq b\right) \leq \exp\left(-\frac{a^2}{2b}\right)$$

: Let  $T = \inf\{t \geq 0 : M_t = a\}$ , then  $\{M_\infty^T = a\} \supset \{\sup_t M_t > a\}$ . Fix  $\lambda \in \mathbb{R}_{>0}$ , and let

$$Z_t = \exp\left(\lambda M_t^T - \frac{1}{2}\lambda^2 \langle M \rangle_t^T\right)$$

By *Ito's formula*,  $Z$  is a continuous local martingale and  $Z_t \leq e^{\lambda a}$  for all  $t \geq 0$ , so is bounded. Hence,  $Z$  is in fact a true martingale. Now

$$\begin{aligned}1 = Z_0 = \mathbb{E}Z_\infty &\geq \mathbb{E}(Z_\infty \mathbf{1}_{\sup_t M_t > a, \langle M \rangle_\infty \leq b}) \\ &\geq \mathbb{E}(\exp(\lambda a - \frac{1}{2}\lambda^2 b) \mathbf{1}_{\sup_t M_t > a, \langle M \rangle_\infty \leq b})\end{aligned}$$

since upon the event  $\{\sup_t M_t > a\}$ , we always have  $M_\infty^T = a$ . Hence

$$\mathbb{P}(\sup_t M_t > a, \langle M \rangle_\infty \leq b) \leq e^{-\lambda a + \frac{1}{2}\lambda^2 b}$$

Optimizing over  $\lambda$ , we get

$$\mathbb{P}(\sup_t M_t > a, \langle M \rangle_\infty \leq b) \leq e^{-\frac{a^2}{2b}}$$

and

$$\mathbb{P}(\sup_t M_t \geq a, \langle M \rangle_\infty \leq b) \leq \inf\{\mathbb{P}(\sup_t M_t > a', \langle M \rangle_\infty \leq b); a' < a\} \leq e^{-a^2/2b}$$

as desired.

(End of proof)  $\square$

In the proof, we had to check tedious details using techniques of measure theory and martingale theory, but the real core of the proof of the theorem in fact comes from the examples displayed at the end of the last lecture. We just need to see the finite dimensional cases to see how the proof in general works.

(25th February, Monday)

(A comment from last lecture) **Proposition** Let  $M$  be a continuous local martingale with  $M_0 = 0$ . Then  $M \in M_c^2$  iff  $\mathbb{E}\langle M \rangle_\infty < \infty$  and then  $M^2 - \langle M \rangle$  is a UI martingale and  $\|M\|_{M^2} = (\mathbb{E}\langle M \rangle_\infty)^{1/2}$ .

**proof**) See *Example Sheet #1*.

There is a more general criterion for  $\mathcal{E}(M)$  to be uniformly integrable.

**Theorem** (*Novikov*) Let  $M$  be a continuous local martingale with  $M_0 = 0$ . Then  $\mathbb{E}(e^{\frac{1}{2}\langle M \rangle_\infty}) < \infty$  implies that  $\mathcal{E}(M)$  is a UI martingale.

**Corollary** (*corollary of Girsanov's Theorem*) Let  $B$  be a standard Brownian motion (under  $\mathbb{P}$ ) and let  $L$  be a continuous local martingale with  $L_0 = 0$  such that  $\mathcal{E}(L)$  is a UI martingale. Then  $\tilde{B} = B - \langle B, L \rangle$  is a standard Brownian motion under the measure  $\mathbb{Q}$  where

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathcal{E}(L)_\infty$$

**proof**) By *Girsanov's Theorem*,  $\tilde{B}$  is a continuous local martingale. Moreover,  $\langle \tilde{B} \rangle_t = \langle B \rangle_t = t$ . By *Lévy's characterisation*,  $\tilde{B}$  is a standard Brownian motion.

(End of proof)  $\square$

Why is this useful? Consider the following (informal) example.

**Example :** Consider the SDE (this is yet to be defined), for a fixed time  $T < \infty$ ,

$$dX_t = b(X_t)dt + dB_t, \quad t \leq T$$

We can construct a solution as follows. Let  $X$  be a standard Brownian motion under  $\mathbb{P}$ . Let

$$L_t = \int_0^{t \wedge T} b(X_s) dX_s$$

Assume that  $\mathcal{E}(L)$  is a UI martingale. Then

$$X_t - \langle X, L \rangle_t = X_t - \int_0^{t \wedge T} b(X_s) d\langle X \rangle_s = X_t - \int_0^{t \wedge T} b(X_s) ds$$

is a standard Brownian motion under  $\mathbb{Q}$  given by  $d\mathbb{Q}/d\mathbb{P} = \mathcal{E}(L)_\infty$ . Thus if we call this Brownian motion  $\tilde{B}$  then the last equation is written by

$$X_t - \int_0^{t \wedge T} b(X_s) ds = \tilde{B}_t$$

So instead of solving the equation  $dX_t = b(X_t)dt + dB_t$ , we can just start with a Brownian motion in the changed measure  $\mathbb{Q}$  and change the measure back to  $\mathbb{P}$  to see the distribution of  $X$ .

When is  $\mathcal{E}(L)$  is a UI martingale? We have

$$\langle L \rangle_\infty = \int_0^T b(X_s)^2 ds$$

So it is sufficient if  $b$  is a bounded function.

## 4.4 The Cameron-Martin formula

This section is about a different application of Girsanov's Theorem.

**Definition** The (classical/canonical) **Wiener space**  $(W, \mathcal{W}, P)$  is given by  $W = C(\mathbb{R}, \mathbb{R})$ ,  $\mathcal{W} = \sigma(X_t : t \geq 0)$  where  $X_t : W \rightarrow \mathbb{R}$  is given by  $X_t(w) = w(t)$  for  $w \in W$  and  $P$  is the unique probability measure on  $(W, \mathcal{W})$  such that  $(X_t)$  is a standard Brownian motion.  $X$  is also called the **canonical version of Brownian motion**. (This is a Banach space)

**Definition** The **Cameron-Martin space** is

$$\mathcal{H} = \{h \in W : h(t) = \int_0^t g(s)ds \text{ for some } g \in L^2(\mathbb{R}_+)\}$$

For  $h \in \mathcal{H}$ , the function  $\dot{h} = g$  is the weak derivative of  $g$ .

*Exercise* :  $\mathcal{H}$  is a Hilbert space with inner product

$$(h, f)_{\mathcal{H}} = \int_0^\infty \dot{h}(s)\dot{f}(s)ds$$

The dual space of  $\mathcal{H}$  can be identified with

$$\mathcal{H}^* = \{\mu \in \mathcal{M}(\mathbb{R}_+) : \int_0^\infty (s \wedge t)\mu(ds)\mu(dt) = (\mu, \mu)_{\mathcal{H}^*} < \infty, \mu(\{0\}) = 0\}$$

in the sense that for any  $l : \mathcal{H} \rightarrow \mathbb{R}$  bounded and linear, there is  $\mu \in \mathcal{H}^*$  such that  $l(h) = \int_0^\infty h(t)\mu(dt)$  and vice-versa.

*Remark* : We would like to think of a Brownian motion as the standard Gaussian measure on  $\mathcal{H}$ . This measure does not exist. But the next theorem shows it almost does.

**Theorem** (*Cameron-Martin*) Let  $h \in \mathcal{H}$  and define  $P^h$  by ( $P^h$  is going to be a canonical measure on the Wiener space)

$$P^h(A) = P(\{w \in W : w + h \in A\})$$

for  $A \in \mathcal{W}$ . Then the measure  $P^h$  is absolutely continuous with respect to the Wiener measure  $P$  and

$$\frac{dP^h}{dP} = \exp\left(\int_0^\infty \dot{h}(s)dX_s - \frac{1}{2}\int_0^\infty \dot{h}(s)^2 ds\right)$$

**proof**) Apply *Girsanov's Theorem* with  $L_t = \int_0^t \dot{h}(s)dX_s$ . Since  $\langle L \rangle_\infty = \int_0^\infty \dot{h}^2 ds = \|h\|_{\mathcal{H}}^2 < \infty$ .  $\mathcal{E}(L)$  is a UI martingale. Then

$$\mathcal{E}(L)_\infty = \exp\left(\int_0^\infty \dot{h}(s)dX_s - \frac{1}{2}\int_0^\infty \dot{h}(s)^2 ds\right)$$

so the rest is as in pervious examples.

(End of proof)  $\square$

(See a refernce book for deeper analytical viewpoint)

## 5 Stochastic Differential Equations

### 5.1 Notions of Solutions

In *Section 1*, we considered the SDE(Stochastic Differntial Equation)  $\dot{x}(t) = F(x(t)) + \eta(t)$ , where  $\eta$  is a white noise. Since the integral of white noise should be interpreted as a Brownian motion, it is reasonable to interpret this SDE as

$$X_t - X_0 = \int_0^t F(X_s)ds + B_t \quad \Leftrightarrow \quad dX_t = F(X_t)dt + dB_t$$

Note that  $\Leftrightarrow$  holds because it is defined as a notaion.

**Definition)** Let  $d, m \in \mathbb{N}$ ,  $B : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ ,  $\sigma : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times m}$  be locally bounded Borel functions. The **stochastic differential equation (SDE)**  $E(\sigma, b)$  is

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t$$

The SDE  $E(\sigma, b)$  together with the initial condition  $X_0 = x \in \mathbb{R}^d$  is denoted  $E_x(\sigma, b)$ .

**Definition)** A **(weak) solution** to the SDE  $E(\sigma, b)$  consists of

- a filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$  obeying the *usual conditions*;
- an  $m$ -dimensional  $(\mathcal{F}_t)$ -Brownian motion  $B$ ;
- an  $(\mathcal{F}_t)$ -adapted continuous process  $X$  with values in  $\mathbb{R}^d$  such that

$$X_t = X_0 + \int_0^t \sigma(s, X_s)dB_s + \int_0^t b(s, X_s)ds.$$

**Definition)** For a **strong solution** to  $E(\sigma, b)$ , we specify the probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  and a Brownian motion  $B$ , and choose  $(\mathcal{F}_t)_t$  to be the *completed filtration* induced by  $B$ . A strong solution is an  $(\mathcal{F}_t)$ -adapted continuous process  $X$  as in the definition of weak solution. That is, it satisfies

$$X_t = X_0 + \int_0^t \sigma(s, X_s)dB_s + \int_0^t b(s, X_s)ds.$$

This means one can think of a strong solution as a function of the Brownian motion.

The main difference in strong solution from weak solution is that the filtration and the Brownian motion is given *a priori*. As the names suggest, weak solutions are easier to find but strong solutions have better properties. They are both useful.

**Definition)** For the SDE  $E(\sigma, b)$  we say that there is

- **weak uniqueness** or **uniqueness in law** if all solutions to  $E_x(\sigma, b)$  has the same law.
- **pathwise uniqueness** if, for  $(\Omega, \mathcal{F}, \mathcal{F}_t, \Omega, \mathbb{P})$  and  $B$  fixed, all solutions with the same initial conditions are indistinguishable. [*This does not mean that the filtration and  $B$  are given as in the definition of a strong solution. This just means that if two weak solutions have the same attached filtration and Brownian motion, then they are indistinguishable.*]

Pathwise uniqueness implies weak uniqueness, but this is not obvious (called Yamada-Watanabe theorem, see below).

**Example :** (*Tanaka*) The SDE

$$dX_t = \text{sign}(X_t)dB_t, \quad X_0 = x \quad \dots\dots\dots (\text{TK})$$

where  $\text{sign}(x) = 1$  if  $x > 0$ ,  $\text{sign}(x) = -1$  if  $x \leq 0$ , has a weak solution that is unique in law, but it is pathwise uniqueness does not hold.

**proof)** Let  $X$  be a one-dimensional Brownian motion with  $X_0 = x$ . Set  $\tilde{B}_t = \int_0^t \text{sign}(X_s)dX_s$  then

$$x + \int_0^t \text{sign}(X_s)d\tilde{B}_s = x + \int_0^t \text{sign}(X_s)^2 dX_s = x + (X_t - X_0) = X_t$$

so  $dX_t = \text{sign}(X_t)d\tilde{B}_t$  with  $X_0 = x$ .

Moreover,  $\tilde{B}$  is a Brownian motion since it is a continuous local martingale with

$$\langle \tilde{B} \rangle_t = \int_0^t \text{sign}(X_s)^2 d\langle X \rangle_s = \langle X \rangle_t = t,$$

so  $\tilde{B}$  and  $X$  are both standard Brownian motion by *Lévy's characterisation*.

By the same argument, in fact, any solution is a standard Brownian motion. Therefore weak uniqueness holds.

To show that pathwise uniqueness fails, at least when  $x = 0$ , we will show that if  $X$  is a solution with  $X_0 = 0$  then  $-X$  is a solution with the same Brownian motion  $B$ . Indeed,

$$-X_t = -\int_0^t \text{sign}(X_s)dB_s = \int_0^t \text{sign}(-X_s)dB_s + 2\int_0^t 1_{X_s=0}dB_s$$

Let  $N_t = 2\int_0^t 1_{X_s=0}dB_s$ . It will be sufficient to prove that  $N = 0$  to see that  $-X_t$  is also a solution.

♡ **Claim :**  $N$  is indistinguishable from 0.

: Clearly,  $N$  is a continuous local martingale and  $\langle N \rangle_t = 4\int_0^t 1_{X_s=0}ds = 0$  a.s. since the zero set of Brownian motion has Lebesgue measure 0 a.s. (why?). So  $N = 0$  up to indistinguishability.

So  $-X$  also solves (TK).

(End of proof) □

*Remark :*  $X$  above is not a strong solution.

**Theorem** (*Pathwise uniqueness for SDEs with Lipschitz coefficients*) Suppose that  $b$  and  $\sigma$  are locally Lipschitz (in space variable), i.e., for each  $n > 0$ , there exists  $K_n > 0$  such that for all  $|x|, |y| \leq n$ ,  $t \geq 0$ , has

$$\begin{aligned} |b(t, x) - b(t, y)| &\leq K_n |x - y| \quad \text{and} \\ |\sigma(t, x) - \sigma(t, y)| &\leq K_n |x - y|. \end{aligned}$$

Then pathwise uniqueness holds for  $E(\sigma, b)$ .

**proof**) Let  $X$  and  $X'$  be two solutions to  $E(\sigma, b)$  defined on the same probability space such that  $X_0 = X'_0$ . Let  $T_n = \inf\{t \geq 0 : |X_t| > n \text{ or } |X'_t| > n\}$  and

$$f_n(t) = \mathbb{E}(|X_{t \wedge T_n} - X'_{t \wedge T_n}|^2)$$

By continuity of  $X$  and  $X'$ , it suffices to show that for all  $n$  and all  $t$ , one has  $f_n(t) = 0$ .

♡ **Claim :**  $f_n(t) = 0$  for all  $n, t > 0$ .

: By Itô's formula,

$$\begin{aligned} |X_{t \wedge T_n} - X'_{t \wedge T_n}|^2 &= \int_0^{t \wedge T_n} 2(X_s - X'_s) \cdot (b(X_s) - b(X'_s))ds \\ &\quad + \int_0^{t \wedge T_n} 2(X_s - X'_s) \cdot (\sigma(X_s) - \sigma(X'_s))dB_s + \int_0^{t \wedge T_n} |\sigma(X_s) - \sigma(X'_s)|^2 ds \end{aligned}$$

Since  $(X_s - X'_s)(\sigma(X_s) - \sigma(X'_s))$  is bounded on  $s < T_n$ , the second term is a martingale with expectation 0, so

$$\begin{aligned} \mathbb{E}(|X_{t \wedge T_n} - X'_{t \wedge T_n}|^2) &\leq (2K_n + K_n^2) \int_0^t \mathbb{E}(|X_{s \wedge T_n} - X'_{s \wedge T_n}|^2) ds \\ \Rightarrow f_n(t) &\leq (2K_n + K_n^2) \int_0^t f_n(s) ds \end{aligned}$$

By Gronwall's inequality (see below), we have  $f_n(t) \leq f_n(0)e^{(2K_n + K_n^2)t} = 0$ .

(End of proof) □

(1st March, Friday)

**Gronwall's Lemma** (on Example Sheet #3) Let  $T > 0$  and let  $f : [0, T] \rightarrow \mathbb{R}$  be non-negative bounded Borel function. Assume  $f(t) \leq a + b \int_0^t f(s)ds$  for all  $t \leq T$ . Then

$$f(t) \leq ae^{bt} \quad \text{for all } t \leq T$$

*Remark :* The proof of the uniqueness proves (for locally Lipschitz coefficients) processes defined up to the time  $T$  must agree up to time  $T$ .

## 5.2 Strong existence for Lipschitz coefficients

Recall, we denote  $E(\sigma, b)$  for  $dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t$ .

**Theorem)** Assume  $b$  and  $\sigma$  are globally Lipschitz, *i.e.* there is  $K > 0$  such that for all  $x, y \in \mathbb{R}^d$ ,  $t \geq 0$ ,

$$|b(t, x) - b(t, y)| \leq K|x - y|, \quad |\sigma(t, x) - \sigma(t, y)| \leq K|x - y|$$

For any  $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$  (obeying usual condition), any  $(\mathcal{F}_t)$ -Brownian motion  $B$ , any  $x \in \mathbb{R}$ , there is a unique strong solution to  $E_x(\sigma, b)$ .

**proof)** To simplify notation, we assume  $d = m = 1$ . Define

$$F(X)_t = x + \int_0^t \sigma(s, X_s)dB_s + \int_0^t b(s, X_s)ds$$

Then  $X$  is a strong solution to  $E_x(\sigma, b)$  if  $F(X) = X$ . To find such a fixed point, we use *Picard iteration*. Fix  $T > 0$ . For  $X$  continuous, adapted process, set

$$\|X\|_T = \mathbb{E} \left( \sup_{t \in [0, T]} |X_t|^2 \right)^{1/2}$$

Then  $B = \{X : \Omega \times [0, T] \rightarrow \mathbb{R}, \|X\|_T < \infty\}$  is a Banach space.

♣ **Claim :**  $\|F(X) - F(Y)\|_T^2 \leq (2T + 8)K^2 \int_0^T \|X - Y\|_t^2 dt$ .

: Just estimate  $\|F(X) - F(Y)\|_T$ ,

$$\begin{aligned} \|F(X) - F(Y)\|_T^2 &\leq 2\mathbb{E} \left( \sup_{t \leq T} \left| \int_0^t (b(s, X_s) - b(s, Y_s))ds \right|^2 \right) \\ &\quad + 2\mathbb{E} \left( \sup_{t \leq T} \left| \int_0^t (\sigma(s, X_s) - \sigma(s, Y_s))dB_s \right|^2 \right) =: (A) + (B) \end{aligned}$$

Then

$$\begin{aligned} (A) &\leq 2T\mathbb{E} \left( \sup_{t \leq T} \int_0^t |b(s, X_s) - b(s, Y_s)|^2 ds \right) \quad (\text{Cauchy-Schwarz}) \\ &\leq 2TK^2 \int_0^T \|X - Y\|_t^2 dt \end{aligned}$$

and

$$\begin{aligned} (B) &\leq 8\mathbb{E} \left( \int_0^T |\sigma(s, X_s) - \sigma(s, Y_s)|^2 ds \right) \quad (\text{Doob's } L^2) \\ &\leq 8K^2 \int_0^T \|X - Y\|_t^2 dt \end{aligned}$$

♣ **Claim :**  $\|F(0)\|_T < \infty$ .

: the argument is the same as putting 0 for  $F(Y)$  above,

$$\begin{aligned} F(0)_t &= x + \int_0^t b(s, 0)ds + \int_0^t \sigma(s, 0)dB_s \\ \Rightarrow \|F(0)\|_T^2 &\leq 3 \left( |x|^2 + \left\| \int_0^t b(s, 0)ds \right\|_T^2 + \left\| \int_0^t \sigma(s, 0)dB_s \right\|_T^2 \right) \\ &\leq 3 \left( |x|^2 + T \int_0^T b(s, 0)^2 ds + 4 \int_0^T |\sigma(s, 0)|^2 ds \right) < +\infty \end{aligned}$$

Now use *Picard iteration* : Let  $X_t^0 = 0$  for all  $t$  and set  $X^{i+1} = F(X^i)$  so by the first claim, for some  $C > 0$ ,

$$\begin{aligned} \|X^{i+1} - X^i\|_T^2 &\leq CT \int_0^T \|X^i - X^{i-1}\|_t^2 dt \leq (CT)^2 \int_0^T \int_0^t \|X^{i-1} - X^{i-2}\|_s^2 ds \\ &\leq \dots \leq \frac{(CT)^i}{i!} \|X^1 - X^0\|_T = \frac{(CT)^i}{i!} \|F(0)\|_T \end{aligned}$$



Therefore,

$$\sum_{i=1}^{\infty} \|X^i - X^{i-1}\|_T^2 < \infty \quad \forall T > 0$$

and hence  $X^i$  converges uniformly on  $[0, T]$  a.s. for all  $T > 0$ . Say  $X$  is the limit. Then  $F(X) = X$ .

By uniqueness, solutions up to different times  $T$  must agree when both are defined. Hence we can extend them to all of  $[0, \infty)$ .

(End of proof)  $\square$

The following proposition provides a (rough) estimate on the dependence of the solution on the initial condition.

**Proposition)** Under the assumptions of the theorem, let  $X^x$  be the solution with initial condition  $X_0^x = x$ . Then for any  $p \geq 2$ ,

$$\mathbb{E}\left(\sup_{s \leq t} |X_t^x - X_s^y|^p\right) \leq C_p |x - y|^p e^{C_p(t \vee 1)^p t}$$

(4th March, Monday)

To prove the proposition, we need :

**Lemma)** (*Burkholder-Davis-Gundy (BDG) inequality*) For every real  $p > 0$ , there exists  $C_p > 0$  depending only on  $p$  such that, for every continuous local martingale  $M$  with  $M_0 = 0$  and every stopping time  $T$ ,

$$\mathbb{E}\left[\sup_{0 \leq s \leq T} (M_s)^p\right] \leq C_p \mathbb{E}[(M)_T^{p/2}]$$

For proof for the case  $p \geq 2$ , see *Example sheet #2*. For the case  $p < 2$ , a good reference would be “Brownian motions, Martingales, and Stochastic Calculus” by Jean-Francois Le Gall.

Note that we will only use the  $p \geq 2$  case.

**Proposition)** Under the assumptions as in the theorem, let  $X^x$  be the solution with initial condition  $X_0^x = x \in \mathbb{R}^d$ . That is,

$$X_t^x = x + \int_0^t \sigma(r, X_r^x) dB_r + \int_0^t b(r, X_r^x) dr.$$

Then for  $p \geq 2$ ,

$$\mathbb{E} \sup_{s \leq t} |X_s^x - X_s^y|^p \leq C_p |x - y|^p \exp(C_p(t \vee 1)^p t)$$

**proof)** For simplicity, assume  $d = m = 1$ . Fix  $x, y \in \mathbb{R}^d$  and let  $T_n = \inf\{t \geq 0 : |X_t^x| > n \text{ or } |X_t^y| > n\}$ . Since  $|a + b + c|^p \leq 3^{p-1}(|a|^p + |b|^p + |c|^p)$ ,

$$\begin{aligned} \mathbb{E}\left(\sup_{s \leq t} |X_{s \wedge T_n}^x - X_{s \wedge T_n}^y|^p\right) &\leq 3^{p-1} \left[ |x - y|^p + \mathbb{E}\left(\sup_{s \leq t} \left| \int_0^{s \wedge T_n} (\sigma(r, X_r^x) - \sigma(r, X_r^y)) dB_r \right|^p\right) \right. \\ &\quad \left. + \mathbb{E}\left(\sup_{s \leq t} \left| \int_0^{s \wedge T_n} (b(r, X_r^x) - b(r, X_r^y)) dr \right|^p\right) \right] \\ &=: 3^{p-1} (|x - y|^p + (A) + (B)) \end{aligned}$$

By *BDG inequality*,

$$\begin{aligned} (A) &\leq C_p \mathbb{E}\left(\int_0^{t \wedge T_n} (\sigma(r, X_r^x) - \sigma(r, X_r^y))^2 dr\right)^{p/2} \\ &\leq C_p t^{\frac{p}{2}-1} \mathbb{E}\left(\int_0^{t \wedge T_n} |\sigma(r, X_r^x) - \sigma(r, X_r^y)|^p dr\right) \quad (\text{H\"older}) \\ &\leq C_p t^{\frac{p}{2}-1} \int_0^t \mathbb{E}|\sigma(t \wedge T_n, X_{r \wedge T_n}^x) - \sigma(t \wedge T_n, X_{r \wedge T_n}^y)|^p dr \\ &\leq C_p t^{\frac{p}{2}-1} \int_0^t \mathbb{E}(K |X_{r \wedge T_n}^x - X_{r \wedge T_n}^y|^p) dr \end{aligned}$$

Also, only using Hölder,

$$\begin{aligned} (B) &\leq t^{p-1} \mathbb{E} \left( \int_0^t |b(r \wedge T_n, X_{r \wedge T_n}^x) - b(r \wedge T_n, X_{r \wedge T_n}^y)|^p dr \right) \\ &\leq t^{p-1} \mathbb{E} \int_0^t K |X_{r \wedge T_n}^x - X_{r \wedge T_n}^y|^p dr \end{aligned}$$

In summary,

$$f_n(t) := \mathbb{E}(\sup_{s \leq t} |X_{s \wedge T_n}^x - X_{s \wedge T_n}^y|^p) \leq 3^{p-1} |x - y|^p + \tilde{C}_p (t \vee 1)^{p/2} t^{\frac{p}{2}} \int_0^t \mathbb{E}(\sup_{s \leq t} |X_{s \wedge T_n}^x - X_{s \wedge T_n}^y|^p) dr$$

Note that  $f_n$  is bounded on any interval  $[0, T]$  for  $n < \infty$ . By *Grönwall's inequality*,

$$f_n(t) \leq 3^{p-1} |x - y|^p \exp(\tilde{C}_p (t \vee 1)^{p/2} t^{\frac{p}{2} + 1})$$

By Fatou, taking  $n \rightarrow \infty$ , we obtain the claimed inequality.

(End of proof)  $\square$

Strong solution can be considered functions of Brownian motion in the following sense. Recall the (*d-dimensional*) *Wiener space*  $(W^d, \mathcal{W}^d, P^d)$  where

$$W^d = C(\mathbb{R}_+, \mathbb{R}^d), \quad \mathcal{W} = \sigma(X_t^i : i \in \mathbb{R}_+, i = 1, \dots, d), \quad \text{where } X_t(w) = w(t) \text{ for } w \in W^d$$

and  $P^d$  is the probability measure on  $(W^d, \mathcal{W}^d)$  such that  $(X_t)_{t \geq 0}$  is a standard Brownian motion with  $X_0 = 0$ .

The space  $C(\mathbb{R}_+, \mathbb{R}^d)$  can be given the topology of uniform convergence on compact intervals. This topology is induced by the metric

$$d(w, \tilde{w}) = \sum_{k=1}^{\infty} \alpha_k (\|w - \tilde{w}\|_{L^\infty([0, t]; \mathbb{R}^d)} \wedge 1)$$

for any sequence  $(\alpha_k) \subset \mathbb{R}_+$  with  $\sum_{k=1}^{\infty} \alpha_k = 1$ .

*Remark :* This metric makes  $C(\mathbb{R}_+, \mathbb{R}^d)$  a complete separable metric space (a so called *Polish space*).

**Theorem)** Under the assumptions of the last theorem (strong solution for Lipschitz coefficients), for  $x \in \mathbb{R}^d$ , there exists maps

$$F_x : W^m = C(\mathbb{R}_+, \mathbb{R}^m) \rightarrow W^d = C(\mathbb{R}_+, \mathbb{R}^d)$$

measurable with respect to the completion of  $\mathcal{W}^m$  on  $W^m$  and w.r.t.  $\mathcal{W}^d$  on  $W^d$  such that

- (i)  $\forall t \geq 0$ ,  $F_x(w)_t$  is a measurable function of  $\sigma(w(s) : s \leq t)$  for  $P^d$ -a.s.  $w \in W^m$ .
  - (ii)  $\forall w \in C(\mathbb{R}_+, \mathbb{R}^m) : x \in \mathbb{R}^d \mapsto F_x(w) \in C(\mathbb{R}_+, \mathbb{R}^d)$  is continuous.
  - (iii)  $\forall x \in \mathbb{R}^d$ ,  $\forall (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  satisfying the usual conditions, every  $(\mathcal{F}_t)$ -Brownian motion  $\hat{B}$  with  $\hat{B}_0 = 0$ , the unique solution to  $E_x(\sigma, b)$  is  $\hat{X}_t = F_x(\hat{B})_t$ .
  - (iv) In the set-up of (iii), if  $U$  is  $\mathcal{F}_0$ -measurable, then  $F_U(\hat{B})_t$  is the unique solution to  $E(\sigma, b)$  with  $X_0 = U$ .
- (Such  $F$  is called the **Itô map**.)

**proof)** For simplicity, assume  $d = m = 1$  in the notation. Let

$$\mathcal{G}_t = \sigma(w(s) : 0 \leq s \leq t) \vee \mathcal{N}, \quad \mathcal{G} = \mathcal{G}_\infty$$

where  $\mathcal{N}$  are the  $P$ -null sets. Then by the existence theorem applied to  $(W^n, \mathcal{G}, (\mathcal{G}_t), P^d)$ ,  $B_t(w) = w(t)$ , there is a unique solution  $X^x$  to  $E_x(\sigma, b)$ .

Let  $p \geq 2$ , that is to be specified later. By the last proposition and  $d(w, \tilde{w}) = \sum_k \alpha_k (\sup_{s \leq k} |w(s) - \tilde{w}(s)| \wedge 1)$  with  $(\alpha_k)$  to be chosen,

$$\begin{aligned} \mathbb{E}(d(X^x, X^y)^p) &\leq \mathbb{E} \left( \left( \sum_k \alpha_k \sup_{s \leq k} |X_s^x - X_s^y| \right)^p \right) \\ &\leq \sum_{k=1}^{\infty} \alpha_k \mathbb{E} \left( \sup_{s \leq k} |X_s^x - X_s^y|^p \right) \quad (\text{Jensen}) \\ &\leq C_p |x - y|^p \sum_k \alpha_k \exp(C_p k^{p+1}) \\ &\leq \tilde{C}_p |x - y|^p \quad \dots \dots \dots (\dagger) \end{aligned}$$

where the last inequality follows by choosing  $(\alpha_k)$  appropriately. A version of *Kolmogorov's continuity criterion* applies to processes in a complete metric space indexed by  $\mathbb{R}^d$  if  $(\dagger)$  holds with  $p > d$ . Applying this to  $(X^x, x \in \mathbb{R}^d)$ , there is a modification  $(\tilde{X}^x, x \in \mathbb{R}^d)$  that is continuous in  $x \in \mathbb{R}^d$ . We set  $F_x(w) = \tilde{X}^x(w) = (\tilde{X}_t^x(w))_{t \geq 0}$ .

(6th March, Wednesday)

Last time :  $X^x$  was strong solution with respect to filtration induced by  $B$  on the canonical space and  $\tilde{X}^x$  was a continuous (in  $x$ ) modification by Komogorov continuity theorem, and let  $F_x(w) = \tilde{X}^x(w)$  for  $w \in C(\mathbb{R}_+, \mathbb{R}^m)$ .

**proof continued)** The construction from last lecture proves point (ii).

For (i), we observe  $w \mapsto F_x(w)_t = \tilde{X}_t^x(w)$ . We have  $\tilde{X}_t^x(w) = X_t^x(x)$  for  $P$ -a.e.  $w$ . But  $X_t^x$  measurable with respect to  $\sigma(w(s) : s \leq t)$  completed by null sets. Hence  $\tilde{X}_t^x$  also is  $\mathcal{G}_t$ -measurable (recall,  $\mathcal{G}_t$  was the completed filtration of the Brownian motion).

To show (iii), fix  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$  and  $\hat{B}$ , we set

$$\hat{X} = F_x(\hat{B})_t$$

Since  $F_x$  maps into  $C(\mathbb{R}_+, \mathbb{R}^d)$ ,  $\hat{X}$  is continuous in  $t$ . Since  $F_x(\hat{B})_t$  coincides a.s. with a measurable function of  $(\hat{B}_t : 0 \leq s \leq t)$  and  $(\mathcal{F}_t)$  in complete it follows that  $\hat{X}$  is adapted. By definition,

$$\begin{aligned} \tilde{X}_t &= x + \int_0^t \sigma(s, \tilde{X}_s) dB_s + \int_0^t b(s, \tilde{X}_s) ds \\ &= x + \lim_{m \rightarrow \infty} \sum_{i=1}^{\lfloor 2^m t \rfloor} \sigma(s, \tilde{X}_{(i-1)2^{-m}})(B_{i2^{-m}} - B_{(i-1)2^{-m}}) + \int_0^t b(s, \tilde{X}_s) ds \end{aligned}$$

Since the limit converges in  $P^m$ -probability, we also have convergence  $P^m$ -a.s. along a subsequence. Since  $F_x(w)_t = \tilde{X}_t^x(w)$  along this subsequence,

$$\begin{aligned} F_x(w)_t &= x + \lim_{m \rightarrow \infty} \sum_{i=1}^{\lfloor 2^m t \rfloor} \sigma(s, F_x(w)_{(i-1)2^{-m}})(w(i2^{-m}) - w((i-1)2^{-m})) \\ &\quad + \int_0^t b(s, F_x(w)_s) ds \quad \text{for } P^m\text{-a.s. } w \in W^m \end{aligned}$$

Since  $\hat{B}$  has law  $P^m$  on  $W^m$ , we can substitute  $w = \hat{B}$  and then revert the approximation of the stochastic integral to get

$$\hat{X}_t = x + \int_0^t \sigma(s, \hat{X}_s) d\hat{B}_s + \int_0^t b(s, \hat{X}_s) ds$$

as desired. Refer to a reference for the proof of point (iv).

(End of proof)  $\square$

[Note that, this technical procedure is necessary because we are changing from one probability space to the other. Since stochastic integrals not only refers to a path of a Brownian motion but also refers to a larger part of the probability space (as we consider convergence in the construction of integrals), we have to be careful when we are changing the probability space.]

**Corollary)** The solutions to  $E_x(\sigma, b)$  can be constructed for all  $x \in \mathbb{R}^d$  simultaneously such that a.s.  $X^x$  is continuous in the initial condtion.

**proof)** Direct from the theorem.

### 5.3 Some examples of SDEs

#### Geometric Brownian motion

For  $w \in C(\mathbb{R}_+, \mathbb{R})$ , define  $F_x(w)$  by

$$[F_x(w)](t) = x \exp\left(\sigma(w(t)) + \left(\mu - \frac{\sigma^2}{2}\right)t\right)$$

If  $B$  is a standard Brownian motion with  $B_0 = 0$ , then  $X_t = F_x(B)_t$  satisfies

$$dX_t = \sigma X_t dB_t + \mu X_t dt, \quad X_0 = x \quad \dots\dots\dots (*)$$

On the other hand, if we choose  $w$  to be any smooth path, then  $x_t = F_x(w)_t$  satisfies the ODE

$$dx_t = \sigma x_t dw_t + x_t \left(\mu - \frac{\sigma^2}{2}\right) dt, \quad \text{where } x_0 = x$$

Thus the *Itô map*  $F$  satisfies the ‘wrong’ equation on smooth paths! The process solving  $(*)$  is called *Geometric Brownian motion*.

#### The Ornstein-Uhlenbeck process

Let  $\lambda > 0$ . The *Ornstein-Uhlenbeck process* is the (unique) solution to

$$dX_t = -\lambda X_t dt + dB_t$$

To solve this SDE, apply *Itô’s formula* to  $e^{\lambda t} X_t$ :

$$\begin{aligned} d(e^{\lambda t} X_t) &= e^{\lambda t} dX_t + \lambda e^{\lambda t} X_t dt = e^{\lambda t} dB_t \\ \Leftrightarrow e^{\lambda t} X_t - X_0 &= \int_0^t e^{\lambda s} dB_s \\ \Leftrightarrow X_t &= e^{-\lambda t} X_0 + \int_0^t e^{-\lambda(t-s)} dB_s \end{aligned}$$

The last term  $\int_0^t e^{-\lambda(t-s)} dB_s$  integrates a deterministic function in the Brownian motion, so it can be thought as a Wiener integral.

**Fact :** If  $X_0 = x$  is fixed (or if  $X_0$  is Gaussian), then  $(X_t)$  is a *Gaussian process*, i.e.  $(X_{t_i})_{i=1}^n$  is jointly Gaussian, for  $0 = t_0 < t_1 < \dots < t_n$ .

**proof)** Exercise - use of Wiener integral simplifies the proof.

A Gaussian process is determined by its mean and its covariance.

**Fact :** If  $X_0 = x$ , then  $\mathbb{E}(X_t) = e^{-\lambda t} x$ ,  $\text{Cov}(X_t, X_s) = \frac{1}{2\lambda}(e^{-\lambda|t-s|} - e^{-\lambda(t+s)})$ .

**proof)** Clearly,  $\mathbb{E}X_t = e^{-\lambda t} \mathbb{E}X_0 + \mathbb{E} \int_0^t e^{-\lambda(t-s)} dB_s = e^{-\lambda t} \mathbb{E}X_0$ . Also, by *Itô isometry*,

$$\begin{aligned} \text{Cov}(X_t, X_s) &= \mathbb{E}((X_t - \mathbb{E}X_t)(X_s - \mathbb{E}X_s)) \\ &= \mathbb{E}\left(\int_0^t e^{-\lambda(t-u)} dB_u \int_0^s e^{-\lambda(s-u)} dB_u\right) \\ &= \int_0^\infty \mathbf{1}_{u < t} e^{-\lambda(t-u)} \mathbf{1}_{u < s} e^{-\lambda(s-u)} du \\ &= e^{-\lambda(t+s)} \int_0^{t+s} e^{2\lambda u} du = \frac{1}{2\lambda} e^{-\lambda(t+s)} (e^{2\lambda(s \wedge t)} - 1) \end{aligned}$$

(End of proof)  $\square$

**Corollary)**  $X_t \sim N(e^{-\lambda t} x, \frac{1-e^{-2\lambda t}}{2\lambda})$

**Fact :** If  $X_0 \sim N(0, \frac{1}{\lambda})$ , then  $X_t \sim N(0, \frac{1}{2\lambda})$  for all  $t > 0$ , and  $X_t$  is a *stationary* Gaussian process with  $\text{Cov}(X_s, X_t) = \frac{1}{2\lambda} e^{-\lambda|t-s|}$

## 5.4 Local Solutions

**Proposition** (*Local Itô formula*) Let  $X = (X^1, \dots, X^d)$  be semimartingales. Let  $U \subset \mathbb{R}^d$  be open, and let  $f : U \rightarrow \mathbb{R}^d$  be  $C^2$ . Set  $T = \inf\{t \geq 0 : X_t \notin U\}$ . Then for all  $t < T$ ,

$$f(X_t) = f(X_0) + \sum_{i=1}^d \frac{\partial f}{\partial x^i}(X_s) dX_s^i + \frac{1}{2} \frac{\partial^2 f}{\partial x^i \partial x^j}(X_s) d\langle X^i, X^j \rangle_s$$

**proof**) Apply Itô's formula to  $X^{T_n}$  where  $T_n = \inf\{t \geq 0 : \text{dist}(X_t, U^c) \leq \frac{1}{n}\}$ . Observe that  $T_n \nearrow T$  as  $n \rightarrow \infty$ .

(End of proof)  $\square$

**Example :** Let  $B$  be a standard Brownian motion with  $B_0 = 1$  (in dimension 1). Taking  $U = (0, \infty)$ ,  $f(x) = \sqrt{x}$  gives

$$\sqrt{B_t} = 1 + \frac{1}{2} \int_0^t B_s^{-1/2} dB_s - \frac{1}{8} \int_0^t B_s^{-3/2} ds$$

for  $t < T = \inf\{t \geq 0 : B_t = 0\}$ .

**Theorem**) Let  $U \subset \mathbb{R}^d$  be open and  $b : \mathbb{R}_+ \times U \rightarrow \mathbb{R}^d$  and  $\sigma : \mathbb{R}_+ \times U \rightarrow \mathbb{R}^{d \times m}$  be *locally Lipschitz continuous*. Then for every  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$ , a Brownian motion  $B$  adapted to this filtration, and every  $x \in U$ , there exists a stopping time  $T$  such that, for  $t < T$ ,

$$X_t = x + \int_0^t b(s, X_s) ds + \int_0^t \sigma(s, X_s) dB_s$$

where  $T$  is such that for all  $K \subset U$  compact, we have  $\sup\{t < T : X_t \in K\} < T$ .

Such  $T$  is called the **explosion time**.

**proof**) Fix  $K_n \subset K$  compact such that  $K_{n+1} \supset K_n$  and  $\bigcup_n K_n = U$ . One can find  $b_n$  and  $\sigma_n$  defined on all of  $\mathbb{R}^d$  such that

$$b_n|_{K_n} = b|_{K_n}, \quad \text{and} \quad \sigma_n|_{K_n} = \sigma|_{K_n}$$

and such that  $b_n$  and  $\sigma_n$  are *globally Lipschitz continuous*. Hence there is a unique solution  $X^n$  to  $E_x(\sigma_n, b_n)$  for each  $n$ . Let  $T_n = \inf\{t \geq 0 : X_t^n \notin K_n\}$ . By uniqueness,  $X^{n+1}$  also solves  $E_x(\sigma_n, b_n)$  up to time  $T_n$ . Thus  $X_t^{n+1} = X_t^n$  for  $t < T_n$  and we can define  $X_t$  for  $t < T = \sup_n T_n$  by requiring that  $X_t = X_t^n$  for  $t < T_n$ .

♠ **Claim :** Let  $K$  be compact. Then on  $\{T < \infty\}$ ,

$$\text{a.s.} \quad \sup\{t < T : X_t \in K\} < T$$

: Let  $L$  be another compact set such that  $K \subset \text{int}(L) \subset L \subset U$ . Let  $\varphi : U \rightarrow \mathbb{R}$  be  $C^\infty$  such that  $\varphi|_K = 1$  and  $\varphi|_{\text{int}(L)^c} = 0$ . Let

$$\begin{aligned} R_1 &= \inf\{t \geq 0 : X_t \notin L\}, \quad S_n = \inf\{t \geq R_n : X_t \in K\} \\ R_n &= \inf\{t \geq S_{n-1} : X_t \notin L\} \end{aligned}$$

Let  $N$  be the number of crossings of  $X$  from  $\text{int}(L)^c$  to  $K$ . Then on  $\{T \leq t, N \geq n\}$ ,

$$\begin{aligned} n &= \sum_{k=1}^n (1 - 0) = \sum_{k=1}^n (\varphi(X_{S_k}) - \varphi(X_{R_k})) \\ &= \int_0^t \sum_{k=1}^n \mathbf{1}_{(R_k, S_k]}(s) (D\varphi(X_s) \cdot dX_s + \frac{1}{2} D^2\varphi(X_s) d\langle X \rangle_s) \\ &= \int_0^t (H_s^n dB_s + \tilde{H}_s^n ds) =: Z_t^n \end{aligned}$$

with  $H^n$  and  $\tilde{H}^n$  are predictable and *bounded* independently of  $n$ . So

$$\begin{aligned} n^2 \mathbf{1}_{\{T \leq t, N \geq n\}} &= (Z_t^n)^2 \\ \Rightarrow \mathbb{P}(T \leq t, N \geq n) &= \frac{\mathbb{E}(Z_t^n)^2}{n^2} \leq \frac{C(t)}{n^2} \end{aligned}$$

and hence  $\mathbb{P}(T \leq t, N = \infty) = 0$ , and in particular  $\mathbb{P}(T < \infty, N = \infty) = 0$ , which implies the claim.

(End of proof)  $\square$

**Example :** Consider the SDEs

$$dX_t^i = -\nabla_i H(X_t)dt + dB_t^i, \quad X_0 = x$$

Assume that there are  $a \geq 0, b \geq 0$  such that

$$x \cdot \nabla H(x) \geq -a|x|^2 - b$$

Then, the SDE has a global solution, *i.e.*  $T = \infty$  a.s.

**proof)** Let  $T_n = \inf\{t \geq 0 : |X_t|^2 > n\}$ . Then by Itô's formula to  $X^{T_n}$ ,

$$\begin{aligned} \mathbb{E}|X_{t \wedge T_n}|^2 &= \mathbb{E}|X_0|^2 - \mathbb{E}\left(2 \int_0^{t \wedge T_n} X_s \cdot \nabla H(X_s)ds - t \wedge T_n\right) \\ &\leq \mathbb{E}|X_0|^2 + 2a\mathbb{E}\left(\int_0^{t \wedge T_n} |X_s|^2 ds\right) + (1 + 2b)\mathbb{E}(t \wedge T_n) \\ &\leq \mathbb{E}|X_0|^2 + (1 + 2b)t + 2a \int_0^t \mathbb{E}|X_{s \wedge T_n}|^2 ds \end{aligned}$$

By Gronwall's lemma,

$$\mathbb{E}|X_{t \wedge T_n}|^2 \leq (\mathbb{E}|X_0|^2 + (1 + 2b)t)e^{2at}$$

If  $\mathbb{P}(T < \infty) > 0$ , then for sufficiently large  $t$ ,  $|X_{t \wedge T_n}|^2 \rightarrow \infty$  as  $n \rightarrow \infty$  with positive probability, so it follows that  $\mathbb{P}(T < \infty) = 0$ .

(End of proof)  $\square$

(11th March, Monday)

## 6 Applications to PDEs and Markov Processes

### 6.1 Probabilistic representations of solutions to PDEs

**Exercise :** Let  $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$  and  $\sigma : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times m}$  be (locally) bounded Borel functions and let  $x \in \mathbb{R}^d$ . Assume that  $X$  is a solution to  $E_x(\sigma, b)$ . Then for every  $f \in C^1(\mathbb{R}_+) \otimes C^2(\mathbb{R}^d)$ ,

$$M_t^f = f(t, X_t) - f(0, X_0) - \int_0^t \left( \frac{\partial}{\partial s} + L \right) f(s, X_s) ds$$

is a continuous local martingale where

$$Lf(y) = \frac{1}{2} \sum_{i,j=1}^d a_{ij}(y) \frac{\partial^2 f}{\partial y^i \partial y^j} + \sum_{i=1}^d b_i(y) \frac{\partial f}{\partial y^i}$$

where  $a(y) = \sigma(y)\sigma(y)^T \in \mathbb{R}^{d \times d}$ .

**Definition)** The  $L$  is called the **(infinitesimal) generator** of  $X$ .

**Example :**

- $dX = dB$ , a Brownian motion has  $L = \frac{1}{2} \Delta$ .
- $dX = -Xdt + dB$ , an Ornstein-Uhlenbeck process has  $L = \frac{1}{2} \Delta - x \cdot \nabla$ .

### Dirichlet-Poisson problem

Let  $U \subset \mathbb{R}^d$ ,  $U \neq \emptyset$  be open and bounded. Given  $f \in C(\overline{U})$  and  $g \in C(\partial U)$ , a (DP) asks to find  $u \in C^2(\overline{U}) = C^2(U) \cap C(\overline{U})$  such that

$$\begin{cases} -Lu(x) = f(x) & \text{for } x \in U \\ u(x) = g(x) & \text{for } x \in \partial U \end{cases} \dots\dots\dots (\text{DP})$$

This is called a **Poisson problem** if  $f = 0$  and called a **Dirichlet Problem** if  $g = 0$ .

**Definition)**  $a : \overline{U} \rightarrow \mathbb{R}^{d \times d}$  is **uniformly elliptic** if there is  $c > 0$  such that

$$\xi^T a(x) \xi \geq c |\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^d, x \in \overline{U}$$

**Theorem)** Assume that  $U$  has a smooth boundary, that  $a$  and  $b$  are Hölder continuous functions, and that  $a$  is uniformly elliptic. Then for every Hölder continuous  $f : \overline{U} \rightarrow \mathbb{R}$  and every continuous  $g : \partial U \rightarrow \mathbb{R}$ , (DP) has a solution.

[See PDE textbooks. Can also use probabilistic method to prove this.]

**Theorem)** Let  $U \subset \mathbb{R}^d$  be open, bounded and non-empty. Let  $b$  and  $\sigma$  be bounded measurable, assume  $a = \sigma \sigma^T$  is uniformly elliptic and let  $u$  be the solution of (DP) with coefficients  $\sigma$  and  $b$ . Let  $x \in U$ , let  $X$  be a solution to  $E_x(\sigma, b)$ . Let  $T_U = \inf\{t \geq 0 : X_t \notin U\}$ . Then  $\mathbb{E}[T_U] < \infty$  and

$$u(x) = \mathbb{E}_x \left( u(X_{T_U}) - \int_0^{T_U} Lu(X_s) ds \right) = \mathbb{E}_x \left( g(X_{T_U}) + \int_0^{T_U} f(X_s) ds \right)$$

**proof)** Let  $U_n = \{x \in U : \text{dist}(x, \partial U) > \frac{1}{n}\}$ ,  $T_n = \inf\{t \geq 0 : X_t \notin U_n\}$ . There are  $u_n \in C_b^2(\mathbb{R}^d)$  such that  $u|_{U_n} = u_n|_{U_n}$ . Then

$$M_t^n = (M^{U_n})_t^{T_n} = u_n(X_{t \wedge T_n}) - u_n(X_0) - \int_0^{t \wedge T_n} Lu_n(X_s) ds$$

is a continuous local martingale, bounded for  $t \leq t_0$  for any  $t_0 > 0$ , so a martingale. Hence  $u(x) = u_n(x)$  for  $x \in U$ ,  $n$  large enough so that  $x \in U_n$  and

$$\begin{aligned} u(x) = u_n(x) &= \mathbb{E} \left( u_n(X_{t \wedge T_n}) - \int_0^{t \wedge T_n} Lu_n(X_s) ds \right) \\ &= \mathbb{E} \left( u(X_{t \wedge T_n}) + \int_0^{t \wedge T_n} f(X_s) ds \right) \end{aligned}$$

To take the limit  $t \wedge T_n \rightarrow T_U$  we will need  $\mathbb{E}T_U < \infty$ . To see this, let  $v$  be a solution to (DP) with  $f(x)1$  and  $g(x) = 0$  for all  $x$ . Then

$$\mathbb{E}(t \wedge T_n) = \mathbb{E} \left( \int_0^{t \wedge T_n} 1 ds \right) = v(x) - \mathbb{E}(v(X_{t \wedge T_n})) \leq 2\|v\|_\infty < \infty$$

By monotone convergence and since  $T_n \wedge t \nearrow T_U$ , has

$$\mathbb{E}(T_U) = \lim_{t \rightarrow \infty} \lim_{n \rightarrow \infty} \mathbb{E}(t \wedge T_n) \leq 2\|v\|_\infty < \infty$$

♡ **Claim:**  $u(x) = \mathbb{E}(u(X_{T_U}) + \int_0^{T_U} f(X_s) ds)$

: Since  $t \wedge T_n \nearrow T_U$  as  $n \rightarrow \infty$  and  $t \rightarrow \infty$ , and since

$$\mathbb{E} \left( \int_0^{T_U} |f(X_s)| ds \right) \leq \|f\|_\infty \mathbb{E}[T_U] \leq C < \infty.$$

The *Dominated convergence theorem* implies

$$\mathbb{E} \left( \int_0^{t \wedge T_n} f(X_s) ds \right) \rightarrow \mathbb{E} \left( \int_0^{T_U} f(X_s) ds \right).$$

Since  $u$  is continuous on  $\overline{U}$ , also by DCT,

$$\mathbb{E}(u(X_{t \wedge T_n})) \rightarrow \mathbb{E}(u(X_{T_U}))$$

This completes the proof.

(End of proof)  $\square$

A similar method can also be used not only to prove the existence of solution but to find the solution. (not going to do this here).

### Cauchy Problem

Given  $f \in C_b^2(\mathbb{R}^d)$ , find  $u \in C(\mathbb{R}_+) \otimes C^2(\mathbb{R}^d)$  such that

$$\begin{cases} \frac{\partial u}{\partial t} = Lu & \text{on } (0, \infty) \times \mathbb{R}^d \\ u(0, \cdot) = f & \text{on } \mathbb{R}^d \end{cases} \dots\dots\dots (\text{CP})$$

where  $L$  is given as above.

**Theorem** For  $f \in C_b^2(\mathbb{R}^d)$ , there exists a solution to (CP).  
[Again, refer to a standard PDE texts, such as Evans.]

**Theorem** Let  $u$  be a (bounded) solution to (CP). Let  $x \in \mathbb{R}^d$ , let  $X$  be any solution to  $E_x(\sigma, b)$ ,  $0 \leq s \leq t$ , then

$$\mathbb{E}(f(X_t)|\mathcal{F}_s) = u(t-s, X_s)$$

In particular,

$$\mathbb{E}_x f(X_t) = u(t, x)$$

**proof** Let  $g(s, x) = u(t-s, x)$  (time runs backward). Then

$$\left(\frac{\partial}{\partial s} + L\right)g(s, x) = 0$$

so  $M^g = g(s, X_s) - g(s, x)$  is a (true) martingale, so

$$u(t-s, X_s) = g(s, X_s) = \mathbb{E}(g(t, X_t)|\mathcal{F}_s) = \mathbb{E}(u(0, X_t)|\mathcal{F}_s) = \mathbb{E}(f(X_t)|\mathcal{F}_s)$$

(End of proof)  $\square$

(13th March, Wednesday)

**Theorem** (Feynman-Kac formula) Let  $L, b, \sigma$  as before. Let  $f \in C_b^2(\mathbb{R}^d)$ ,  $V \in C_b(\mathbb{R}^d)$  and suppose that  $u : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}$  satisfies

$$\begin{cases} \frac{\partial u}{\partial t} = Lu + Vu & \text{on } \mathbb{R}_+ \times \mathbb{R}^d \\ u(0, \cdot) = f & \text{on } \mathbb{R}^d \end{cases}$$

( $Vu$  here is just a pointwise multiplication). Let  $X$  be a solution to  $E_x(\sigma, b)$  for some  $x \in \mathbb{R}^d$ . Then for all  $t \geq 0$ ,

$$u(t, x) = \mathbb{E}_x \left[ f(X_t) \exp \left( \int_0^t V(X_s) ds \right) \right]$$

[This result resembles the form of path integral solution of Schrödinger's equation. This is the reason why this is called the 'Feynman'-Kac formula. However, proving the same with Schrödinger's equation (where there is  $-i$  in front of  $\frac{\partial u}{\partial t}$ ) is much more difficult as we lose positivity of  $L$ .]

**proof** Let  $E_t = \exp(\int_0^t V(X_s) ds)$ . For  $s < t$ , set  $M_s = u(t-s, X_s)E_s$ , then

$$\begin{aligned} dM_s &= -\frac{\partial}{\partial t} u(t-s, X_s)E_s ds + \nabla u(t-s, X_s)E_s \sigma_s dB_s + Lu(t-s, X_s)E_s ds + u(t-s, X_s)V(X_s)E_s ds \\ &= \left( -\frac{\partial}{\partial t} + L + V(X_s) \right) u(t-s, X_s)E_s ds + d(\text{martingale}) \\ &= d(\text{martingale}) \end{aligned}$$



Thus  $M$  is a continuous local martingale on  $[0, t]$ . by assumption,  $M$  is also bounded, so a martingale. Hence

$$u(t, x) = M_0 = \mathbb{E}_x M_t = \mathbb{E}_x u(0, X_t) E_t = \mathbb{E}(f(X_t) E_t)$$

(End of proof)  $\square$

## 6.2 Markov property

Let  $B(\mathbb{R}^d)$  be the Banach space of **bounded Borel functions** on  $\mathbb{R}^d$ , with  $\|f\| = \sup_{x \in \mathbb{R}^d} |f(x)|$  for  $f \in B(\mathbb{R}^d)$ .

### Definition)

- (i) A collection of bounded linear operators  $Q_t$  on  $B(\mathbb{R}^d)$  is a **transition semigroup** if  $Q_t f \geq 0$  if  $f \geq 0$  (pointwise),  $Q_t \mathbf{1} = \mathbf{1}$ ,  $\|Q_t\| \leq 1$ , and

$$Q_{t+s} = Q_t Q_s \quad \forall t, s \geq 0$$

- (ii) An  $(\mathcal{F}_t)$ -adapted process  $X$  is a **Markov process** with transition semigroup  $(Q_t)_t$  if

$$\mathbb{E}(f(X_{s+t}) | \mathcal{F}_s) = Q_t f(X_s) \quad \forall s, t \geq 0, f \in B(\mathbb{R}^d)$$

**Theorem)** Let  $b : \mathbb{R}^d \rightarrow \mathbb{R}$ ,  $\sigma : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times m}$  be *Lipschitz* (this can be weakened). Assume  $X$  is a solution to  $E(\sigma, b)$  on some  $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$  and  $B$ . Then  $X = (X_t)_{t \geq 0}$  is a Markov process with semigroup

$$Q_t f(x) = \mathbb{E}(f(X_t^x)) = \int f(F_x(w)_t) P^m(dw)$$

where  $X_t^x$  is an arbitrary solution to  $E_x(\sigma, b)$ , and  $F_x$  is the *Itô solution map*,  $P^m$  is the Wiener measure.

**proof)** Let  $X$  be a solution to  $E(\sigma, b)$ .

♡ **Claim :**  $\mathbb{E}(f(X_{t+s}) | \mathcal{F}_s) = Q_t f(X_s)$

: By definition of  $X$ ,

$$\begin{aligned} X_t &= X_0 + \int_0^t \sigma(X_u) dB_u + \int_0^t b(X_u) du \quad \forall t \geq 0 \\ \Rightarrow X_{t+s} &= X_s + \int_s^{t+s} \sigma(X_u) dB_u + \int_s^{t+s} b(X_u) du \quad \forall s, t \geq 0 \end{aligned}$$

Set  $X'_t = X_{s+t}$ ,  $\mathcal{F}'_t = \mathcal{F}_{s+t}$ ,  $B'_t = B_{s+t} - B_s$ . Then  $(\Omega, \mathcal{F}, (\mathcal{F}'), P)$  is another filtered probability space obeying the usual conditions, and  $B'$  is a  $(\mathcal{F}'_t)$ -Brownian motion with  $B'_0 = 0$ . Then

$$X'_t = X'_0 + \int_0^t \sigma(X'_u) dB'_u + \int_0^t b(X'_u) du$$

(Justification of  $\int_s^{s+t} \sigma(X_u) dB_u = \int_0^t \sigma(X'_u) dB'_u$  uses approximation of integrals - this should not be treated naively!). Thus  $X'$  solves  $E(\sigma, b)$  with  $X'_0 = X_s$ . By the theorem about the solution map, we have  $X' = F_{X_s}(B')$  a.s. So

$$\mathbb{E}(f(X_{s+t}) | \mathcal{F}_s) = \mathbb{E}(f(X'_t) | \mathcal{F}_s) = \mathbb{E}(f(F_{X_s}(B')_t) | \mathcal{F}_s) = \int f(F_{X_s}(w)_t) P^m(dw) = Q_t f(X_s)$$

Also,

$$Q_{t+s} f(x) = \mathbb{E}(f(X_{t+s}^x)) = \mathbb{E}\left(\mathbb{E}(f(X_{t+s}^x) | \mathcal{F}_s)\right) = \mathbb{E}(Q_t f(X_s^x)) = Q_s Q_t f(x)$$

so  $(Q_t)_{t \geq 0}$  indeed forms a transition semigroup.

(End of proof)  $\square$

**Definition)** Let  $Q_t$  be the transition semigroup.

(i) A probability measure  $\mu$  on  $\mathbb{R}^d$  is **invariant** under  $(Q_t)$  if

$$\int Q_t f(x) \mu(dx) = \int f(x) d\mu(x) \quad \forall f \in B(\mathbb{R}^d)$$

(ii) A probability measure  $\mu$  is **reversible** with respect to  $(Q_t)$  if

$$\int g(x) Q_t f(x) \mu(dx) = \int f(x) Q_t g(x) \mu(dx)$$

(check this.)

**Fact :** Reversibility of  $\mu$  implies it is invariant. (Take  $g = 1$  and use  $Q_t 1 = 1$ .)

**Example :** Consider the transition semigroup associated to the SDE, with suitable on  $H$ ,

$$dX_t = -\frac{1}{2} \nabla H(X_t) dt + dB_t$$

(Note that, if taking  $H(x) = \lambda|x|^2$ , this gives an Orstein-Uhlenbeck process.) Then the measure  $\mu(dx) = \frac{1}{Z} e^{-H(x)} dx$ , where  $Z = \int e^{-H(x)} dx$  is reversible for  $(*)$ .

An application, if you want to sample from the measure  $\mu(dx) = \frac{1}{Z} e^{-H(x)} dx$ , then we can simulate the SDE for a suitably long time and how the distribution is made after long time. (called Markov chain Monte-Carlo simulation)

**Lemma)** Assume that the explosion time for  $(*)$  is infinite. Then for  $f : C([0, T], \mathbb{R}^d) \rightarrow \mathbb{R}$ ,

$$\mathbb{E}\left(f(X|_{[0, T]})\right) = \mathbb{E}^{\text{BM}}\left[f(X|_{[0, T]}) \exp\left(\frac{1}{2}H(X_0) - \frac{1}{2}H(X_T) - \int_0^T \left(\frac{1}{8}|\nabla H|^2 - \frac{1}{4}\Delta H\right)(X_s) ds\right)\right]$$

(where  $\mathbb{E}^{\text{BM}}$  takes average over law under which  $X$  is a Brownian motion with same initial condition.) (An interpretation(??) of Girsanov's theorem.)