

# Analysis of PDEs

Claude Warnick(cmw50)  
www.dpmms.cam.ac.uk/ cmw50  
Example Classes : Dr. Ivan Moyano

Texts : (1)Evans. PDEs, (2)Rauch, PDEs, (3)F.John, PDEs, (4)Gilberg + Raudinger, Elliptic PDE, (5) Ladyzhenskaya, The Boundary Value Problems of Mathematical Physics.

=====

(5th October 2018, Friday)

## Introduction

Suppose  $U \subset \mathbb{R}^n$  is open. A *partial differential equation* of order  $k$  is an expression of the following form:

$$F(x, u(x), Du(x), \dots, D^{(k)}u(x)) = 0 \quad (1)$$

Here,  $F : U \times \mathbb{R} \times \mathbb{R}^n \times \dots \times \mathbb{R}^{n^k} \rightarrow \mathbb{R}$  is a given function and  $u : U \rightarrow \mathbb{R}$  is the 'unknown'. We say  $u \in C^k(U)$  is a classical solution of 1 if 1 is satisfied on  $U$  when we substitute  $u$  into the expression.

We could also consider the case where  $u : U \rightarrow \mathbb{R}^p$  and  $F$  takes values in  $\mathbb{R}^q$ , then we speak of a *system of PDE's*.

### Examples)

1. The Transport Equation: Suppose  $V : \mathbb{R}^{n+2} \rightarrow \mathbb{R}^n$  is given.

$$\frac{\partial u}{\partial t}(x, t) + V(x, t, u(t, x)) \cdot D_x u(x, t) = f(x, t) \quad \text{for } x \in \mathbb{R}^n$$

is a PDE for  $u : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ . This describes evolution of some chemical produced at rate  $f(x, t)$  and being advected by a flow of velocity  $V(x, t, u(t, x))$ .

2. The Laplace and Poisson Equations:

$$\Delta u(x) = \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i \partial x_j}(x) = 0 \quad (\text{Laplace Equation})$$

This describes:

- + Electrostatic potential in empty space
- + Static distribution of heat in a solid body
- + Applications to steady flows in 2D
- + Connections to complex analysis

$$\Delta u(x) = f(x) \quad \text{some given } f : \mathbb{R}^n \rightarrow \mathbb{R} \quad (\text{Poisson's Equation})$$

This describes:

- + Electric field produced by charge distribution  $f$
- + Gravitational field in Newton's Theory ( $f$  is mass density)

3. Heat/Diffusion Equation:

$$\frac{\partial u}{\partial t} = \Delta u$$

This describes evolution of temperature in a solid homogeneous body.

4. Wave Equation:

$$-\frac{\partial^2 u}{\partial t^2} + \Delta u = 0$$

This describe:

- + Displacement of a stretched string (dimension=1)
- + Ripples on surface of water (dimension=2)
- + Density of air in a sound wave (dimension=3)

5. Maxwell's Equations: With  $E, B : \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{R}^3$ ,

$$\begin{aligned} \nabla \cdot E &= \rho & \nabla \cdot B &= 0 \\ \nabla \times E + \frac{\partial B}{\partial t} &= 0 & \nabla \times B - \frac{\partial E}{\partial t} &= J \end{aligned}$$

$\rho, J$  are charge density/current respectively, are given.

6. Ricci Flow:

$$\partial_t g_{ij} = -2R_{ij}$$

where  $g_{ij}$  is a Riemannian metric,  $R_{ij}$  is its Ricci curvature.

7. Minimal Surface Equation: For  $u : \mathbb{R}^2 \rightarrow \mathbb{R}$ ,

$$\operatorname{div}\left(\frac{Du}{\sqrt{1-|Du|^2}}\right) = 0$$

Condition for the graph  $\{(x, y, u(x, y))\}$  to locally extremise area.

8. Eikonal Equation: for  $U \subset \mathbb{R}^3$  and  $u : U \rightarrow \mathbb{R}$

$$|Du| = 1$$

Level sets parametrise a wave-front moving according to the ray theory of light.

9. Schrödinger's Equation: For  $u : \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{C} \equiv \mathbb{R}^2$ ,

$$i\frac{\partial u}{\partial t} + \Delta u - Vu = 0$$

for  $V : \mathbb{R}^3 \rightarrow \mathbb{R}$  given.  $u$  is the wavefunction of a quantum mechanical particle moving in a potential  $V$ .

10. Einstein's Equations for General Relativity:

$$R_{\mu\nu}[g] = 0$$

where  $g$  is Lorentzian metric.  $R_{\mu\nu}$  is Ricci tensor. This describes gravitational field in vacuum.

-. There are Many more examples.

## Data and Well-Posedness

In all examples, there is extra information required beyond the equation. We call this the *data*. An important question is what data is appropriate. We typically ask of a PDE problem that:

- a) A solution exists,
- b) for given data the solution is unique,
- c) the solution depends on the data continuously.

If these hold, we say the problem is 'well-posed'. To make these precise, we have to (usually) specify function spaces for the data and solution to belong to.

=====

8th October, Monday

Let  $U \subset \mathbb{R}^n$ ,  $u : U \rightarrow \mathbb{R}$  be unknown. Then our system of interest will be

$$F(x; u, Du, \dots, D^k u) = 0 \quad (2)$$

**Notations)** Let  $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$  be a multi-index (where  $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ ). Then we let:

- $D^\alpha f(x) = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$  where  $|\alpha| = \alpha_1 + \dots + \alpha_n$  is the order of  $\alpha$ .
- For  $x \in \mathbb{R}^n$ ,  $x^\alpha = x_1^{\alpha_1} \times \dots \times x_n^{\alpha_n}$
- $\alpha! = \alpha_1! \dots \alpha_n!$ .
- For  $\beta = (\beta_1, \dots, \beta_n)$ ,  $\beta \leq \alpha$  is equivalent to having  $\beta_k \leq \alpha_k$  for all  $k$ .

## Classifying PDEs

- We say (2) is **linear** if  $F$  is a linear function of  $u$  and its derivatives. We can write (2) as

$$\sum_{|\alpha| \leq k} a_\alpha(x) D^\alpha u(x) = f(x)$$

- We say (2) is **semi-linear** if it is of the form

$$\sum_{|\alpha| \leq k} a_\alpha(x) D^\alpha u(x) + a_0(x; u(x), \dots, D^{k-1} u(x)) = 0$$

|

- We say (2) is **quasi-linear** if it is of the form

$$\sum_{|\alpha| \leq k} a_\alpha(x; u(x), \dots, D^{k-1} u(x)) D^\alpha u(x) + a_0(x; u(x), \dots, D^{k-1} u(x)) = 0$$

|

- We say (2) is **fully non-linear** if its not linear, semi-linear, nor quasi-linear

## Examples)

- $\Delta u = f$  is linear
- $\Delta u = u^3$  is semi-linear
- $uu_{xx} + u_x u_{yy} = f$  is quasi-linear
- $u_{xx} u_{yy} - u_{xy}^2 = f$  is fully non-linear.

## Cauchy-Kovalevskaya Theorem

For motivation, we recall some ODE theory. Fix  $U \subset \mathbb{R}^n$ , and assume  $f : U \rightarrow \mathbb{R}^n$  is given. Consider the ODE

$$\dot{u}(t) = f(u(t)), u(0) = u_0 \in U \quad (3)$$

with  $u : I \subset \mathbb{R} \rightarrow U$ .

**Theorem** (Picard-Lindelöf) Suppose there exist  $r, K > 0$  s.t.  $B_r(u_0) = \{w \in \mathbb{R}^n : |w - u_0| < r\}$  and  $|f(x) - f(y)| \leq K|x - y|$  for all  $x, y \in B_r(u_0)$ . Then there exists  $\epsilon > 0$  (depending in  $r$  and  $K$ ) and a unique  $C^1$ -function  $u : (-\epsilon, \epsilon) \rightarrow U$  solving (3).

**proof**) Use  $U$  solves (3), then

$$u(t) = u_0 + \int_0^t f(u(s))ds \quad (4)$$

and conversely, if  $U$  is  $C^0$  and solves (4), then in fact  $U$  is  $C^1$  by FTC, and  $u$  solves (3). (in context of PDEs, this is called *weak formulation*)

Then our solution, if exists, is a fixed point of the map  $B : w \mapsto u_0 + \int_0^t f(w(s))ds$ . (use Banach fixed point theorem)

### Observations:

- We start by reformulating the problem in a weak form and find a unique  $C^0$  solution. Then  $C^1$  the regularity follows a posteriori.
- to construct the fixed point map, we solve the linear problem  $\dot{w}(t) = f(w(t))$ .

Lets consider an alternative approach to solving (3). Assuming  $f$  is differentiable, we have

$$\begin{aligned} u^{(1)}(t) &= f(u(t)) \\ u^{(2)}(t) &= f'(u(t))\dot{u}(t) \\ u^{(3)}(t) &= f''(u(t))(\dot{u}(t))^2 + f'(u(t))\ddot{u}(t) \\ &\vdots \\ u^{(k)}(t) &= f_k(u(t), \dot{u}(t), \dots, u^{(k-1)}(t)) \end{aligned}$$

So in principle, given  $u(0) = u_0$ , we can determine  $u_k = u^{(k)}(0)$  for all  $k \geq 0$ . *Formally* at least, we can write

$$u(t) = \sum_{k=0}^{\infty} u_k t^k / k! \quad (5)$$

ignoring the issues of convergence. Call this a **formal power series solution**. When will this agree with the Picard-Lindelöf solution we have constructed?

**Theorem** (Cauchy-Kovalevskaya, for the case of ODEs) The series in (5) converges to a solution of (3) in a neighbourhood of  $t = 0$  if  $f$  is real analytic at  $u_0$ .

-This will follow from a more general result later.

**Definition**) Let  $U \subset \mathbb{R}^n$  be open and suppose  $f : U \rightarrow \mathbb{R}^n$ .  $f$  is called **real analytic** near  $x_0 \in U$  if  $\exists r > 0$  and constants  $f_\alpha$  ( $\alpha$  are multi-indices) such that

$$f(x) = \sum_{\alpha} f_{\alpha} (x - x_0)^{\alpha} \quad \text{for } x \in B_r(x_0)$$

**Note:** if  $f$  is real analytic, then it is  $C^\infty$ . Furthermore, the constants  $f_\alpha$  are given by  $f_\alpha = D^\alpha f(x_0) / \alpha!$ . Thus  $f$  equals its Taylor expansion about  $x_0$ , in a neighbourhood of  $x_0$ .

$$f(x) = \sum_{\alpha} \frac{D^\alpha f(x_0)}{\alpha!} (x - x_0)^{\alpha} \quad \text{for } x \in B_r(x_0)$$

By translation, we usually assume  $x_0 = 0$

=====

(10th October, Wednesday)

- Last lecture :  $U \subset \mathbb{R}^n$  open,  $f : U \rightarrow \mathbb{R}$  is real analytic at  $x_0 \in U$  if  $\exists f_\alpha \in \mathbb{R}, r > 0$  s.t.

$$f(x) = \sum_{\alpha} f_{\alpha} (x - x_0)^{\alpha} \quad \forall |x - x_0| < r$$

### Properties of real analytic functions

- $f$  is real analytic at  $x_0$  if and only if  $\exists s > 0$  and  $C, \rho > 0$  such that:

$$\sup_{|x - x_0| < s} |D^{\alpha} f(x)| \leq C \frac{|\alpha|!}{\rho^{|\alpha|}}$$

- If  $f$  is RA(real analytic) at  $x_0$ , it is RA for all  $x$  close enough to  $x_0$ .
- If  $f : U \rightarrow \mathbb{R}$  is real analytic everywhere on a connected set  $U$ , then  $f$  is determined by its values on any open subset of  $U$ . (Or by its Taylor expansion at a single point.)

**Example :** If  $r > 0$  set

$$f(x) = \frac{r}{r - (x_1 + \dots + x_n)} \quad \text{for } |x| < r/\sqrt{n}$$

Then for  $|x| < r/\sqrt{n}$ ,

$$\begin{aligned} f(x) &= \frac{1}{1 - (x_1 + \dots + x_n)/r} = \sum_{k=0}^{\infty} \left( \frac{x_1 + \dots + x_n}{r} \right)^k = \sum_{k=0}^{\infty} \frac{1}{r^k} \sum_{|\alpha|=k} \binom{|\alpha|}{\alpha} x^{\alpha} \\ &= \sum_{\alpha} \frac{|\alpha|!}{r^{|\alpha|} \alpha!} x^{\alpha} \end{aligned}$$

by multinomial theorem. This is valid for  $|x_1 + \dots + x_n|/r < 1$ , which holds for  $|x| < r/\sqrt{n}$ . In fact, on this domain, the series converges absolutely. Indeed :

$$\sum_{\alpha} \frac{|\alpha|!}{r^{|\alpha|} \alpha!} |x|^{\alpha} = \sum_{k=0}^{\infty} \left( \frac{|x_1| + \dots + |x_n|}{r} \right)^k < \infty$$

since  $|x_1| + \dots + |x_n| \leq |x| \sqrt{n} < r$ .

**Definition)** Let  $f = \sum_{\alpha} f_{\alpha} x^{\alpha}$ ,  $g = \sum_{\alpha} g_{\alpha} x^{\alpha}$  be two formal power series. We say  $g$  **majorises**  $f$ , written  $g \gg f$  if

$$|f_{\alpha}| \leq g_{\alpha}$$

for all  $\alpha$ , and say that  $g$  is a **majorant** of  $f$ .

### Lemma)

- If  $g \gg f$  and  $g$  converges for  $|x| < r$  then  $f$  also converges (absolutely) for  $|x| < r$ .
- If  $f$  converges for  $|x| < r$ , then for any  $s \in (0, r/\sqrt{n})$ ,  $f$  has a majorant that converges for  $|x| < s/\sqrt{n}$ . ( $n$  is the dimension of the space)

**proof)**

- We note that

$$\begin{aligned} \sum_{\alpha} |f_{\alpha} x^{\alpha}| &\leq \sum_{\alpha} |f_{\alpha}| |x_1|^{\alpha_1} \dots |x_n|^{\alpha_n} \\ &\leq \sum_{\alpha} g_{\alpha} \tilde{x}^{\alpha} \end{aligned}$$

where  $\tilde{x} = (|x_1|, \dots, |x_n|)$ . Now  $|\tilde{x}| = |x| < r$  so  $\sum_{\alpha} g_{\alpha} \tilde{x}^{\alpha}$  converges, hence  $\sum_{\alpha} |f_{\alpha} x^{\alpha}|$  converges. Hence  $f$  converges on  $|x| < r$  absolutely.

- (ii) Pick  $s$  s.t.  $0 < s\sqrt{n} < r$ , and set  $y = s(1, \dots, 1)$ . Then  $|y| = s\sqrt{n} < r$ . Hence  $\sum_{\alpha} f_{\alpha} y^{\alpha}$  converges. A convergent series has bounded terms,  $\exists C > 0$  s.t.  $|f_{\alpha} y^{\alpha}| \leq C$  for all  $\alpha$ , and therefore

$$|f_{\alpha}| \leq \frac{C}{y_1^{\alpha_1} \dots y_n^{\alpha_n}} = \frac{C}{s^{|\alpha|}} \leq \frac{C|\alpha|!}{s^{\alpha} \alpha!}$$

But then  $g(x)$  defined by

$$g(x) = \frac{Cs}{s - (x_1 + \dots + x_n)} = C \sum_{\alpha} \frac{|\alpha|!}{s^{\alpha} \alpha!} x^{\alpha}$$

majorises  $f$  and converges for  $|x| < s/\sqrt{n} < r/n$ .

(End of proof)  $\square$

**Remark :** If  $f = (f^1, \dots, f^m)$  and  $g = (g^1, \dots, g^m)$  are formal power series, then we say

$$g \gg f \quad \text{if} \quad g^i \gg f^i \quad i = 1, \dots, m$$

## Cauchy-Kovalevskaya for First Order Systems

We will study a problem that generalises the Cauchy problem for ODEs we have already discussed.

As coordinates on  $\mathbb{R}^n$  we take  $(x', t) = x$  where

$$x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}, \quad t = x^n \in \mathbb{R}$$

Set

$$B_r^n = \{t^2 + |x'|^2 < r^2\}, \quad B_r^{n-1} = \{|x'| < r, t = 0\}$$

We consider a system of equations for unknown  $\underline{u}(x) \in \mathbb{R}^m$ . More concretely, we seek a solution to

$$\begin{aligned} \underline{u}_t &= \sum_{j=1}^{n-1} \underline{B}_j(\underline{u}, x') \cdot \underline{u}_{x_j} + \underline{c}(\underline{u}, x') \quad \text{on } B_r^n \\ \underline{u} &= 0 \quad \text{on } B_r^{n-1} \end{aligned} \tag{6}$$

where  $\underline{u}_{x_j} = \partial u / \partial x_j$  etc. We assume that we are given the real analytic functions

$$\begin{aligned} \underline{B}_j : \mathbb{R}^m \times \mathbb{R}^{n-1} &\rightarrow \text{Mat}(m \times m) \\ \underline{c} : \mathbb{R}^m \times \mathbb{R}^{n-1} &\rightarrow \mathbb{R}^m \end{aligned}$$

(these functions do not have to be defined on the entire space, but just have to be defined on  $\mathbb{R}^n \times B_r^{n-1}$ )

Note we assume  $\underline{B}_j$  and  $\underline{u}$  do not depend explicitly on  $t$ . We can always introduce  $u^{m+1}$  satisfying  $\partial_t u^{m+1} = 1$ ,  $u^{m+1} = 0$  on  $B_r^{n-1}$  and extending the system.

We will write  $\underline{B}_j = ((b_j^{kl}))$  and  $\underline{c} = (c^1, \dots, c^m)^T$ . Then in components (6) reads:

$$u_t^k = \sum_{j=1}^{n-1} \sum_{l=1}^m b_j^{kl}(\underline{u}, x') u_{x_j}^l + c^k(\underline{u}, x') \quad k = 1, \dots, m$$

**Examples :** Take  $m = 2$ , write  $\underline{u} = (f, g)^T$ .

(a)

$$\begin{cases} f_t = g_x + F \\ g_t = f_x \end{cases}$$

together imply  $f_{tt} - f_{xx} = F_t$

(b)

$$\begin{cases} f_t = -g_x + F \\ g_t = f_x \end{cases}$$

together imply  $f_{tt} + f_{xx} = F_t$ . (Note  $F = 0$  gives Cauchy-Riemann equation)

**Theorem)** (Cauchy-Kovalevskaya) Assume  $\{\underline{B}_j\}_{j=1}^{n-1}$  and  $\underline{c}$  are real analytic. Then for sufficiently small  $r > 0$  there exists a unique real analytic function  $\underline{u} : B_r^n \rightarrow \mathbb{R}^m$  solving the problem (6).

=====

(12th October, Friday)

**Theorem)** (Cauchy-Kovalevskaya) Assume  $\{\underline{B}_j\}_{j=1}^{n-1}$  and  $\underline{c}$  are real analytic. Then for sufficiently small  $r > 0$  there exists a unique real analytic function  $\underline{u} : B_r^n \rightarrow \mathbb{R}^m$  solving the problem (6).

**proof)**

1. The strategy will be to write

$$\underline{u}(x) = \sum_{\alpha} \underline{u}_{\alpha} x^{\alpha} \quad (7)$$

and compute coefficients

$$\underline{u}_{\alpha} = \frac{D^{\alpha} \underline{u}(0)}{\alpha!}$$

in terms of  $\underline{B}_j$ ,  $\underline{c}$  and show that the series (7) converges on  $B_r^n$  for  $r$  small enough.

2. As  $\underline{B}_j$  and  $\underline{c}$  are real analytic, we can write

$$\begin{aligned} \underline{B}_j(z, x') &= \sum_{\gamma, \delta} \underline{B}_{j, \gamma, \delta} z^{\gamma} (x')^{\delta} \quad \gamma \in \mathbb{N}^m, \delta \in \mathbb{N}^{n-1} \text{ multiindices} \\ \underline{c}(z, x') &= \sum_{\gamma, \delta} \underline{c}_{\gamma, \delta} z^{\gamma} (x')^{\delta} \end{aligned}$$

where these power series converge for  $|z|^2 + |x'|^2 < s^2$ , wlog  $s > r$ . Thus:

$$\begin{aligned} \underline{B}_{j, \gamma, \delta} &= \frac{D_z^{\delta} D_{x'}^{\gamma} \underline{B}_j(0, 0)}{\gamma! \delta!} \\ \underline{c}_{\gamma, \delta} &= \frac{D_z^{\delta} D_{x'}^{\gamma} \underline{c}(0, 0)}{\gamma! \delta!} \end{aligned} \quad (8)$$

3. Since  $\underline{u} \equiv 0$  on  $\{t = x^n = 0\}$ , we have

$$\underline{u}_{\alpha} = \frac{D^{\alpha} \underline{u}(0)}{\alpha!} = 0$$

for all multi-indices  $\alpha$  with  $\alpha_n = 0$ .

Now, we use the evolution equation (6) to deduce

$$\underline{u}_{x_n}(0) = \underline{u}_t(0) = \sum_{j=1}^{n-1} \underline{B}_j(\underline{u}(0), 0) \underline{u}_{x_j}(0) + \underline{c}(\underline{u}(0), 0) = \underline{c}(0, 0)$$

Fix  $i \in \{1, 2, \dots, n-1\}$ , differentiate (6) with respect to  $x^i$  :

$$\begin{aligned} \underline{u}_{tx_i} &= \sum_{j=1}^{n-1} \left[ \partial_{x_i} \underline{B}_j(\underline{u}, x') \underline{u}_{x_j} + \left( \sum_{i=1}^m \partial_{z_i} \underline{B}_j(\underline{u}, x') \frac{\partial u^i}{\partial x^j} \underline{u}_{x_j} \right) + \underline{B}_j(\underline{u}, x') \underline{u}_{x_i x_j} \right] \\ &\quad + \partial_{x_i} \underline{c}(\underline{u}, x') + \sum_{i=1}^m \partial_{z_i} \underline{c}(\underline{u}, x') \frac{\partial u^i}{\partial x^i} \\ \underline{u}_{tx_i}(0) &= \partial_{x_i} \underline{c}(0, 0) \end{aligned}$$

Iterating this, we deduce  $D^{\alpha} \underline{u}(0) = D^{\delta} \underline{c}(0, 0)$  where  $\alpha = (\delta, 1)$ .

4. Now, suppose  $\alpha = (\delta, 2)$ , for  $\delta \in \mathbb{N}^{n-1}$ . Then

$$\begin{aligned} D^\alpha u^k &= D^\delta (u_{x_n x_n}^k) = D^\delta (u_t^k)_t \\ &= D^\delta \left( \sum_{j=1}^{n-1} \sum_{l=1}^m b_j^{kl} u_{x_j}^l + c^k \right)_t \\ &= D^\delta \left( \sum_{j=1}^{n-1} \sum_{i=1}^m \left[ b_j^{kl} u_{x_j t}^l + \sum_{p=1}^m (b_j^{kl})_{z_p} u_{x_j}^l u_t^p \right] + \sum_{p=1}^m c_{z_p}^k u_t^p \right) \end{aligned}$$

so

$$D^\alpha u^k(0) = D^\alpha \left( \sum_{j=1}^{n-1} \sum_{i=1}^m b_j^{kl} u_{x_j t}^l + \sum_{p=1}^m c_{z_p}^k u_t^p \right) \Big|_{x=0, \underline{u}=0}$$

Now crucially, the expression on the right can be expanded to produce a polynomial with non-negative coefficients involving derivative of  $\underline{B}_j$  and  $\underline{c}$ , and derivatives  $D^\beta \underline{u}$  where  $\beta_n \leq 1$ . More generally, for each multi-index  $\alpha$  and each  $k \in \{1, \dots, n\}$ , we can compute

$$D^\alpha u^k(0) = p_\alpha^k \left( D_z^\alpha D_{x'}^\delta \underline{B}_j, D_z^\alpha D_{x'}^\delta \underline{c}, D^\beta \underline{u} \right) \Big|_{x=0, \underline{u}=0}$$

where  $\beta_n \leq \alpha_n - 1$  and  $p_\alpha^k$  is some polynomial in its arguments with non-negative coefficients. Equivalently, for each  $\alpha, k$

$$u_\alpha^k = q_\alpha^k(\underline{B}_{j, \alpha, \delta}, \underline{c}_{\gamma, \delta}, u_\beta)$$

where  $q_\alpha^k$  is a polynomial with non-negative coefficients, with  $\beta_n \leq \alpha_n - 1$ .

5. We have shown that if a solution exists, we can compute all derivatives at 0 in terms of known quantities. We will construct a series which majorises the formal sum  $\sum_\alpha u_\alpha x^\alpha$ .

First suppose

$$\underline{B}_j^* \gg \underline{B}_j \quad \underline{c}^* \gg \underline{c}$$

where

$$\begin{aligned} \underline{B}_j^* &= \sum_{\gamma, \delta} \underline{B}_{j, \gamma, \delta}^* z^\gamma (x')^\delta \\ \underline{c}^* &= \sum_{\gamma, \delta} \underline{c}_{\gamma, \delta}^* z^\gamma (x')^\delta \end{aligned}$$

Assume these converge for  $|z|^2 + |x'|^2 < s^2$  (decrease  $s$  if necessary). For all  $j, \gamma, \delta, k, l$ ,

$$0 \leq |B_{j, \gamma, \delta}^{kl}| \leq (B^*)_{j, \gamma, \delta}^{kl}, \quad 0 \leq |c_{\gamma, \delta}^k| \leq (c^*)_{\gamma, \delta}^k$$

We consider the modified problem:

$$\begin{aligned} \underline{u}_t^* &= \sum_{j=1}^{n-1} \underline{B}_j^*(\underline{u}^*, x') \underline{u}_{x_j}^* + \underline{c}^*(\underline{u}^*, x') \quad \text{for } |x| < r \\ \underline{u}^* &= \underline{0} \quad \text{on } B_r^{n-1} \end{aligned}$$

As above, seek a real analytic solution

$$\underline{u}^* = \sum_\alpha \underline{u}_\alpha^* x^\alpha \quad \text{where } \underline{u}_\alpha^* = \frac{D^\alpha \underline{u}(0)}{\alpha!}$$

6. We claim  $0 \leq |u_\alpha^k| \leq (u^*)_\alpha^k$  for all  $\alpha \in \mathbb{N}^n$ .

We do this by proof by induction on  $\alpha_n$ .

For  $\alpha_n = 0$ ,  $u_\alpha^* = u_\alpha = 0$



For the induction step: (for  $\beta_\alpha \leq \alpha_n - 1$ )

$$\begin{aligned} |u_\alpha^k| &= |q_\alpha^k(\underline{B}_{j,\gamma,\delta}, \underline{c}_{\gamma,\delta}, \underline{u}_\beta)| \\ &\leq q_\alpha^k(|B_{j,\gamma,\delta}^{kl}|, |C_{\gamma,\delta}^k|, |u_\beta^k|) \\ &\leq q_\alpha^k((B^*)_{j,\gamma,\delta}^{kl}, (c^*)_{\gamma,\delta}^k, (u^*)_\beta^k) \\ &= (u^*)_\alpha^k \end{aligned}$$

Using positivity of coefficients of  $q$  and induction assumption. Thus  $\underline{u}^* \gg \underline{u}$ . Remains to show we can find  $\underline{B}_j^*, \underline{c}^*$  s.t. a solution  $\underline{u}^*$  exists and converges near 0.

=====

(15th October, Monday)

Last lecture :

- a formal power series solution  $\underline{u} = \sum_\alpha \underline{u}_\alpha x^\alpha$  exists.
- If  $\underline{B}_j^* \gg \underline{B}_j, \underline{c}^* \gg \underline{c}$  and  $\underline{u}^*$  satisfies

$$\begin{aligned} \underline{u}_t^* &= \sum_{j=1}^{n-1} \underline{B}_j^*(\underline{u}^*, x') \underline{u}_{x_j}^* + \underline{c}^*(\underline{u}^*, x') \quad \text{for } |x| < r \\ \underline{u}^* &= \underline{0} \quad \text{on } B_r^{n-1} \end{aligned}$$

then the power series for  $\underline{u}^* = \sum_\alpha \underline{u}_\alpha^* x^\alpha$ .

**proof, continued)** To complete the proof, it suffices to show that for any  $\underline{B}_j, \underline{c}$ , we can find  $\underline{B}_j^*, \underline{c}_j^*$  such that the corresponding  $\underline{u}_j^*$  is a convergent series.

We make a particular choice for  $\underline{B}_j^*, \underline{c}^*$ . For this we recall from an earlier lemma that

$$\begin{aligned} \underline{B}_j^* &= \frac{Cr}{r - (x_1 + \dots + x_{n-1}) - (z_1 + \dots + z_m)} \begin{pmatrix} 1 & 1 & \dots & 1 \\ \vdots & & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix} \\ \underline{c}^* &= \frac{Cr}{r - (x_1 + \dots + x_{n-1}) - (z_1 + \dots + z_m)} (1, \dots, 1)^T \end{aligned}$$

will majorise  $\underline{B}_j, \underline{c}$ , provided  $C$  is large enough,  $r$  is small enough and  $\underline{B}_j^*, \underline{c}^*$  are given by convergent series for  $|x'|^2 + |z|^2 < r^2$ . With these choices of majorants, the modified equation takes the form :

$$\begin{aligned} (u^*)_t^k &= \frac{Cr}{r - (x_1 + \dots + x_{n-1}) - ((u^*)^1 + \dots + (u^*)^m)} \left( \sum_{j,l} (u^*)_{x_j}^l + 1 \right) \quad \text{for } |x'|^2 + t^2 < r^2 \\ u^* &= 0 \quad \text{for } t = 0, |x'| < r \end{aligned}$$

This problem has an explicit solution.

$$\underline{u}^* = v^*(1, \dots, 1)^T$$

where

$$v^* = \frac{1}{mn} \left( r - (x_1 + \dots + x_{n-1}) - \sqrt{(r - (x_1 + \dots + x_{n-1}))^2 - 2nmCrt} \right)$$

(Check this is indeed the solution!!)  $v^*$  is real analytic for  $|x'|^2 + t^2 < r^2$ , provided  $r$  is small enough. Hence  $\underline{u}^*$  is given by a convergent series since  $\underline{u}^* \gg \underline{u}$ . Our formal power series for  $\underline{u}$  converges.

Initial condition hold for  $\underline{u}$  since

$$\underline{u}_\alpha = \underline{0} \quad \text{if } \alpha_n = 0$$

Moreover, the functions  $\underline{u}_t$  and  $\sum_{j=1}^{n-1} \underline{B}_j(\underline{u}, x') \underline{u}_{x_j} + \underline{c}(\underline{u}, x')$  are both real analytic near 0 and by construction, have the same Taylor expansion. Hence they must agree on a neighbourhood of 0, so the equation holds in some ball about 0.

(End of proof)  $\square$

## Reduction to a First Order System

### Example)

Consider the PDE problem for  $u : \mathbb{R}^3 \rightarrow \mathbb{R}$

$$\begin{aligned} u_{tt} &= uu_{xy} - u_{xx} + u_t \\ u|_{t=0} &= u_0 \\ u_t|_{t=0} &= u_1 \end{aligned} \tag{9}$$

where  $u_0, u_1 : \mathbb{R}^2 \rightarrow \mathbb{R}$  are given real analytic functions (near 0).

First note that  $f = u_0 + tu_1$  is analytic in a neighbourhood of  $0 \in \mathbb{R}^3$  and  $f|_{t=0} = u_0$ ,  $f_t|_{t=0} = u_1$ .

Set  $w = u - f$ , then

$$\begin{aligned} w_{tt} &= ww_{xy} - w_{xx} + w_t + fw_{xy} + f_{xy}w + F \\ w|_{t=0} &= w_t|_{t=0} = 0 \end{aligned}$$

where  $F = ff_{xy} - f_{xx} + f_t - f_{tt}$ .

Let  $(x, y, t) = (x^1, x^2, x^3)$  and set  $\underline{u} = (w, w_x, w_y, w_t) = (u^1, u^2, u^3, u^4)$ . Then

$$\begin{aligned} u_{x^3}^1 &= w_t = u^4 \\ u_{x^3}^2 &= w_{xt} = u_{x^1}^4 \\ u_{x^3}^3 &= w_{yt} = u_{x^2}^4 \\ u_{x^3}^4 &= w_{tt} = u^1 u_{x^2}^2 - u_{x^1}^2 + u^4 + f u_{x^2}^2 + f_{xy} u^1 + F \end{aligned}$$

Now, defining:

$$\underline{\underline{B}}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \underline{\underline{B}}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ u_1 + f & 0 & 0 & 0 \end{pmatrix}$$

$$\underline{c} = (u^4, 0, 0, u^4 + f_{xy}u^1 + F)^T$$

The system of equations is in the form

$$\underline{u}_{x^2} = \sum_{j=1}^4 \underline{\underline{B}}_j \underline{u}_{x^j} + \underline{c}$$

where  $\underline{\underline{B}}_j$ ,  $\underline{c}$  are real analytic near 0. By Cauchy-Kovalevskaya, a real analytic solution to (9) exists near 0.

**Note :** this procedure relied on

- (a) being able to solve for  $u_{tt}$ ,
- (b)  $u_{tt}$  depending on at most two derivatives of  $u$  (in a quasilinear fashion)

More generally, suppose we wish to solve the quasilinear problem :

$$\begin{aligned} \sum_{|\alpha|=k} a_\alpha(D^{k-1}u, \dots, u, x) D^\alpha u + a_0(D^{k-1}u, \dots, u, x) &= 0 \quad \text{for } |x| < r \\ u = \frac{\partial u}{\partial x_n} = \dots = \frac{\partial^{k-1} u}{\partial x_n^{k-1}} &= 0 \quad \text{for } |x'| < r, x_n = 0 \end{aligned}$$

called a **Cauchy problem**.

We introduce

$$\underline{u} = (u, \frac{\partial u}{\partial x_n}, \dots, D^\alpha u, \dots)_{|\alpha| \leq k-1} = (u^1, \dots, u^m)$$

$\underline{u}$  contains all derivative of  $u$  up to order  $k-1$ . Wlog, (by changing the order if necessary) put  $u^m = \partial^{k-1}u/\partial x_n^{k-1}$ . For  $j < m$ , we can compute  $\partial u^j/\partial x^n$  in terms of  $\partial u^l/\partial x^p$  for some  $l \in \{1, \dots, m\}$  and  $p < n$ .

To compute  $\partial u^m/\partial x_n$  we need to use the equation. Suppose that

$$a_{(0, \dots, 0, k)}(0, \dots, 0) \neq 0$$

Then we can write the equation as :

$$\frac{\partial^k u}{\partial x_n^k} = \frac{-1}{a_{(0, \dots, k)}(D^{k-1}u, \dots, u, x)} \left[ \sum_{|\alpha|=k, \alpha_n < k} a_\alpha D^\alpha u + a_0 \right]$$

Assuming  $a_\alpha$  are real analytic, the denominator will be non-zero near the origin. The RHS can be written in terms of  $\frac{\partial u^l}{\partial x^p}$  for  $p < n$  and  $\underline{u}$ . We see we can write the equation as a first ordered system for  $\underline{u}$ , *provided* (this condition is important! would come back to this later)

$$a_{(0, \dots, k)}(0, \dots, 0) \neq 0 \quad (\text{non-characteristic condition})$$

In this case we can apply Cauchy-Kovalevskaya.

=====  
(17th October, Wednesday)

(Problem sheet 1 handed out. Example classes sign-up. First example class (probably) at Thur/Fri next week)

## Cauchy Problems for Quasilinear Equations with Data on a Surface

We say  $\Sigma \subset \mathbb{R}^n$  is a real analytic **hypersurface** near  $x \in \Sigma$  if there exists  $\epsilon > 0$  and a real analytic map  $\Phi : B_\epsilon(x) \rightarrow U \subset \mathbb{R}^n$  where  $U = \Phi(B_\epsilon(x))$  such that

- $\Phi$  is bijective, and the inverse  $\Phi^{-1} : U \rightarrow B_\epsilon(x)$  is real analytic.
- $\Phi(\Sigma \cap B_\epsilon(x)) = \{x_n = 0\} \cap U$ .

We think of  $\Phi$  as 'straightening out the boundary'.

There are many examples, e.g.  $\{|x| = 1\}$ .

Let  $\gamma$  be the unit normal to  $\Sigma$  and suppose  $u$  solves

$$\sum_{|\alpha|=k} a_\alpha (D^{k-1}u, \dots, u, x) D^\alpha u + a_0 (D^{n-1}u, \dots, u, x) = 0 \quad \text{in } B_\epsilon(x) \quad (10)$$

$$u = \gamma^i \partial_i u = \dots = (\gamma^i \partial_i)^{k-1} u = 0 \quad \text{on } \Sigma$$

(note that the boundary condition is equivalent to having  $D^\alpha u = 0$  for all  $|\alpha| < k$  on  $\Sigma$ .)

Define  $v(y) = u(\Phi^{-1}(y)) \Leftrightarrow u(x) = v(\Phi(x))$ . Note

$$\frac{\partial u}{\partial x^i} = \frac{\partial v}{\partial y^j} \frac{\partial \Phi^j}{\partial x^i}$$

$$\frac{\partial^2 u}{\partial x^i \partial x^k} = \frac{\partial v^2}{\partial y^j \partial y^i} \frac{\partial \Phi^j}{\partial x^i} \frac{\partial \Phi^l}{\partial x^k} + \frac{\partial v}{\partial y^j} \frac{\partial^2 \Phi^j}{\partial x^i \partial x^k} \quad \text{etc.}$$

So we can compute  $D^\alpha u$  as a linear combination of  $D^\beta v$  for  $|\beta| \leq |\alpha|$ , with coefficients depending on  $\Phi$ . So if  $u$  solves (10), then  $v$  will solve

$$\sum_{|\alpha|=k} b_\alpha (D^{k-1}v, \dots, v, x) D^\alpha v + b_0 (D^{k-1}v, \dots, v, x) = 0$$

Moreover,

$$v|_{x_n=0} = u|_\Sigma = 0$$

$$\partial_i v|_{x_n=0} = (D\Phi^{-1})_{ij} \partial_j u|_\Sigma = 0$$

and proceeding similarly for  $\partial^{k-1}v/(\partial x^n)^{k-1}$ , we have each  $D^\beta v$  for  $|\beta| < k$  as a linear combination of  $D^\alpha u$  ( $|\alpha| < k$ ) and hence  $D^\beta v = 0$  for each  $|\alpha| < k$ . Hence, we have (check that this is an equivalent condition)

$$v = \frac{\partial v}{\partial x^n} = \dots = \partial^{k-1}v/(\partial x^n)^{k-1} = 0 \quad \text{on } \{x_n = 0\}$$

We can solve this, provided

$$b_{(0,\dots,0,k)}(0,0,\dots,0,y) \neq 0 \quad \text{on } \{x_n = 0\}$$

Note if  $|\alpha| = k$ ,

$$D^\alpha u = \frac{\partial^k v}{\partial y_n^k} (D\Phi^n)^\alpha + (\text{terms not involving } \frac{\partial^k v}{\partial y_n^k})$$

So the coefficient of  $\partial^k v / \partial y_n^k$  in

$$\sum_{|\alpha|=k} a_\alpha (D^{k-1}u, \dots, u, x) D^\alpha u + a_0 (D^{n-1}u, \dots, u, x) = 0$$

is

$$b_{(0,\dots,k)} = a_\alpha (D\Phi^n)^\alpha$$

But  $\Sigma = \{\Phi^n = 0\}$  so  $D\Phi^n \propto \gamma$ . Therefore,

$$b_{(0,\dots,k)} \neq 0 \quad \Leftrightarrow \quad \sum_{|\alpha|=k} a_\alpha (D\Phi^n)^\alpha \neq 0 \quad \Leftrightarrow \quad \sum_{|\alpha|=k} a_\alpha \gamma^\alpha \neq 0$$

**Definition)**  $\Sigma$  is a non-characteristic at  $x \in \Sigma$  for the problem (10) provided

$$\sum_{|\alpha|=k} a_\alpha (0, \dots, 0, x) \gamma^\alpha(x) \neq 0$$

Finally, we have a more general version of Cauchy-Kovalevskaya.

**Theorem)** (Cauchy-Kovalevskaya Redux) Suppose  $\Sigma \subset \mathbb{R}^n$  is a real analytic hypersurface. If  $\Sigma$  is non-characteristic for (10) at  $x \in \Sigma$ , there exists a unique real analytic solution to (10) in a neighbourhood of  $x$ .

**proof)** We have already seen that we can solve the problem for  $v$  uniquely, then  $u(x) = v(\Phi(x))$  is the unique solution for (10)

(End of proof)  $\square$

## Characteristic Surfaces for 2nd Order Linear PDE

Consider the linear operator

$$Lu = \sum_{i,j=1}^n a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i \frac{\partial u}{\partial x_i} + cu$$

with  $a_{ij}, b_i, c : \mathbb{R}^n \rightarrow \mathbb{R}$ .

Consider the Cauchy problem

$$\begin{aligned} Lu &= f \\ u &= \sum_{i=1}^n \xi^i \frac{\partial u}{\partial x^i} = 0 \quad \text{on } \Pi_\xi = \{\xi \cdot x = 0\} \end{aligned}$$

$\Pi_\xi$  is characteristic at  $x \in \mathbb{R}^n$  if :

$$\sigma_p(\xi, x) = \sum_{i,j=1}^n a_{ij} \xi^i \xi^j = 0$$

$\sigma_p$  is the **principal symbol** of  $L$ .

- If  $\sigma_p(\xi, x) > 0$  for all  $x, \xi \neq 0$ , then no plane is characteristic, and such operations are called **elliptic**.

Let us restrict to the case where  $a_{ij}, b_i, c$  are constants. Suppose  $b_i = c = 0$  and  $\Pi_\xi$  is characteristic. Then

$$u(x) = e^{i\lambda\xi \cdot x}$$

solve  $Lu = 0$  for any  $\lambda$ . By taking  $\lambda$  large, we can construct solutions to  $Lu = 0$  whose derivative (in the  $\xi$  direction) is as large as we like. In particular,  $Lu$  is very regular, but  $u$  need not be. In the elliptic setting, this cannot happen.

=====

(19th October, Friday)

## Criticisms/Shortcomings of Cauchy-Kovalevskaya

1. Real analyticity is (sometimes) too strong a condition. For example, if solutions of Maxwell's equations were required to be real analytic. We'd know electro-magnetic field everywhere if we could measure it in some small set. This is absurd
2. We don't necessarily get continuous dependence on data in the form we would like.

**Example :** Consider Laplace's equation on  $\mathbb{R}^2$ .  $u_{xx} + u_{yy} = 0$ , with Cauchy data  $u(x, 0) = \cos(kx)$ ,  $u_y(x, 0) = 0$ . This has a real analytic solution

$$u(x, y) = \cos(kx) \cosh(ky)$$

We can check that  $\sup_{x \in \mathbb{R}} |u(x, 0)| \leq 1$  but  $\sup_{x \in \mathbb{R}} |u(x, \epsilon)| \rightarrow \infty$  as  $k \rightarrow \infty$  for all  $\epsilon > 0$ .

In fact, we can require as many derivatives of  $u$  on  $\{y = 0\}$  to be bounded and we can still find solutions which are arbitrarily large at  $y = \epsilon$ . This Cauchy problem is *not* well posed in  $C^k$ , as there is no continuous dependence on data.

This suggests the Cauchy problem for Laplace's equation is not the natural one to consider.

## Elliptic Boundary Value Problems

A more natural problem arising in physics is the **Dirichlet Problem** :

$$\begin{aligned} \Delta u &= 0 & \text{in } U \subset \mathbb{R}^n, & \quad U \text{ open, bounded} \\ u &= g & \text{on } \partial U \end{aligned}$$

e.g.  $u$  is electrostatic potential in a cavity whose walls are held at voltage  $g$ .

We shall develop methods to solve such problems. First we develop some technology.

## Hölder and Sobolev Spaces

We need to discuss various function spaces in which to seek solutions our PDEs.

### Hölder spaces

Suppose  $U \subset \mathbb{R}^n$  is open. We write  $u \in C^k(U)$  if  $u : U \rightarrow \mathbb{R}$  is  $k$ -times differentiable at each  $x \in U$  and  $D^\alpha u$  is continuous on  $U$  for all  $|\alpha| \leq k$ . This is not a Banach space, so we would like to restrict to a smaller complete space with a norm.

We say  $u \in C^k(\bar{U})$  if  $u \in C^k(U)$  and  $D^\alpha u$  is uniformly continuous and bounded on  $U$ . We introduce a norm :

$$\|u\|_{C^k(\bar{U})} = \sum_{|\alpha| \leq k} \sup_{x \in U} |D^\alpha u(x)|$$

With this norm,  $C^k(\bar{U})$  is a Banach space. (\*Be aware, that  $C^k(\bar{U})$  is constructed from the closure of  $U$ , but this is not true. It is constructed from  $U$  and just depends on  $U$ . This matter when  $U$  does not have a nice boundary e.g. if  $U$  is a complement of the Cantor set  $\cap [0, 1]$ , then  $C^k(\bar{U}) \neq C^k \cap [0, 1]$ ).

For  $0 < \gamma \leq 1$ , we say that  $u$  is **Hölder continuous with exponent**  $\gamma$  if there exists a constant  $C \geq 0$  s.t.

$$|u(x) - u(y)| \leq C|x - y|^\gamma \quad \forall x, y \in U$$

We define  $\gamma^{\text{th}}$  **Hölder seminorm** by

$$[u]_{C^{0,\gamma}(\bar{U})} = \inf_{x,y \in U} \frac{|u(x) - u(y)|}{|x - y|^\gamma}$$

We say  $u \in C^{k,\gamma}(\bar{U})$  if  $u \in C^k(\bar{U})$  and  $D^\alpha u$  is Hölder continuous, with exponent  $\gamma$  for all  $|\alpha| = k$ . We define a norm :

$$\|u\|_{C^{k,\gamma}(\bar{U})} = \|u\|_{C^k(\bar{U})} + \sum_{|\alpha|=k} [D^\alpha u]_{C^{0,\gamma}(\bar{U})}$$

This is again a Banach space.

### The Spaces $L^p(U)$ , $L^p_{\text{loc}}(U)$

For  $U \subset \mathbb{R}^n$  open, suppose  $1 \leq p < \infty$ . We define the space  $L^p(U)$  by

$$L^p(U) = \{u : U \rightarrow \mathbb{R} \text{ measurable} \mid \|u\|_{L^p(U)} < \infty\} / \sim$$

where

$$\|u\|_{L^p(U)} = \begin{cases} \left( \int_U |u(x)|^p dx \right)^{1/p} & \text{for } 1 \leq p < \infty \\ \text{ess sup}_x u(x) = \inf \{C \geq 0 : |u(x)| \leq C \text{ for almost every } x\} & \text{for } p = \infty \end{cases}$$

and the  $\sim$  is an equivalence relation defined by  $u_1 \sim u_2$  if and only if  $u_1 = u_2$  almost everywhere.

$L^p(U)$  is a Banach space with norm  $\|\cdot\|_{L^p(U)}$ . Completeness follows from dominated convergence theorem.

We define a local versions of  $L^p(u)$  : we say  $u \in L^p_{\text{loc}}(U)$  if  $u \in L^p(V)$  for every  $V \subset\subset U$  (or  $V \Subset U$ ) should be read ' **$V$  is compactly contained in  $U$** ', meaning there exists a compact  $K$  s.t.  $V \subset K \subset U$ . Note  $L^p_{\text{loc}}(U)$  is *not* a Banach space.(it is a Fréchet space)

### Weak Derivatives

We would like a notion of differentiability for  $L^p$  functions. Since  $L^p$  functions like to be integrated, it makes sense to seek a definition involving integration.

**Definition)** Suppose  $u, v \in L^1_{\text{loc}}(U)$  and  $\alpha$  is a multi-index. We say  $v$  is a  $\alpha^{\text{th}}$  **weak derivative** of  $u$  if

$$(-1)^{|\alpha|} \int_U u D^\alpha \phi dx = \int_U v \phi dx \quad \forall \phi \in C_c^\infty(U)$$

In other words,  $u, v$  obey the correct integration of parts formula, when integrated against a test function  $\phi \in C_c^\infty(U)$ .

-check that if  $D^\alpha u = v$ , then  $v$  is indeed also a weak derivative of  $u$ .