# Advanced Probability

-Martingales
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(15th October 2018, Monday)

## Chapter 2. Martingales in Discrete Time

### 2.1. Definitions.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

- A **Filtration** for $(\Omega, \mathcal{F}, \mathbb{P})$ is a sequence $(\mathcal{F}_n)_{n\geq 0}$ of $\sigma$-algebras s.t. for all $n \geq 0$, we have
$$\mathcal{F}_n \subset \mathcal{F}_{n+1} \subset \mathcal{F}$$
Set $F_\infty = \sigma(\mathcal{F}_n : n \geq 0)$ then $\mathcal{F}_\infty \subset \mathcal{F}$. We allow $\mathcal{F}_\infty \neq \mathcal{F}$. We interpret $n$ as times and $\mathcal{F}_n$ as the extent of knowledge at time $n$.

- A **Random process(in discrete time)** is a sequence of random variables $(X_n)_{n\geq 0}$. It has a natural filtration $(F_n^X)_{n\geq 0}$ given by
$$\mathcal{F}_n^X = \sigma(X_0, \cdots, X_n)$$
That is, the knowledge obtained from $X_n$ by time $n$. We say $(X_n)_{n\geq 0}$ is **adapted to** $(\mathcal{F}_n)_{n\geq 0}$ if $X_n$ is $\mathcal{F}_n$-measurable for all $n \geq 0$. This is equivalent to having $\mathcal{F}_n^X \subset \mathcal{F}_n$, for all $n \geq 0$. (Here, $X_n$ are real-valued)

- We would say $(X_n)_{n\geq 0}$ is **integrable** if $X_n$ is integrable for all $n \geq 0$.

- A **martingale** is an *adapted, integrable random process* $(X_n)_{n\geq 0}$ s.t. for all $n \geq 0$,
$$\mathbb{E}[X_{n+1}|\mathcal{F}_n] = X_n \quad \text{a.s.}$$
In the case $\mathbb{E}[X_{n+1}|\mathcal{F}_n] \leq X_n$ a.s., $(X_n)_n$ is called a **super-martingale** and in the case $\mathbb{E}[X_{n+1}|\mathcal{F}_n] \geq X_n$ a.s., $(X_n)_n$ is called a **sub-martingale**.

### Optional Stopping

- A random variable $T : \Omega \to \{0, 1, 2, \cdots\} \cup \{\infty\}$ is a **stopping time** if $\{T \leq n\} \in \mathcal{F}_n$ for all $n \geq 0$.

- For a stopping time $T$, we set $\mathcal{F}_T = \{A \in \mathcal{F}_\infty : A \cap \{T \leq n\} \in \mathcal{F}_n \text{ for all } n \geq 0\}$. It is easy to check $\mathcal{F}_T$ is indeed a $\sigma$-algebra and that if $T(\omega) = n$ for all $\omega \in \Omega$, then $T$ is a stopping time and $\mathcal{F}_T = \mathcal{F}_n$.

- Given $X$, define $X_T(\omega) = X_{T(\omega)}(\omega)$ whenever $T(\omega) < \infty$ and define the **stopped process** $X^T$ by
$$X_n^T(\omega) = X_{T(\omega)\wedge n}(\omega) \quad \text{for } n \geq 0$$

**Proposition 2.2.1.)** Let $X$ be an adapted process. Let $S$, $T$ be stopping times for $X$. Then

(a) $S \wedge T$ is a stopping time for $X$.

(b) $\mathcal{F}_T$ is a $\sigma$-algebra.

(c) If $S \le T$ then $\mathcal{F}_S \subset \mathcal{F}_T$.

(d) $X_T 1_{T<\infty}$ is an $\mathcal{F}_T$-measurable random variable.

(e) $X^T$ is adapted.

(f) If $X$ is integrable, then $X^T$ is also integrable.

**proof)**

(a) $\{S \wedge T \le n\} = \{S \le n\} \cup \{T \le n\} \in \mathcal{F}_n$ for all $n \ge 0$, so $S \wedge T$ is a stopping times

(b) Directly from the definition, we see that $\phi \mathcal{F}_T$. Also, given $A \in \mathcal{F}_T$ and a sequence $(A_m)_m \subset \mathcal{F}_T$, we have

$$A^c \cap \{T \le n\} = \{T \le n\} - A \cap \{T \le n\} \in \mathcal{F}_n \quad \Rightarrow A^c \in \mathcal{F}_T$$
$$(\cup_m A_m) \cap \{T \le n\} = \cup_m (A_m \cap \{T \le n\}) \in \mathcal{F}_n \quad \Rightarrow \cup_m A_m \in \mathcal{F}_T$$

hence $\mathcal{F}_T$ is a $\sigma$-algebra.

(c) Let $A \in \mathcal{F}_S$. Then $A \cap \{T \le n\} = A \cap \{S \le n\} \cap \{T \le n\} \in \mathcal{F}_n$, hence $A \in \mathcal{F}_T$.

(d) For each $t \in \mathbb{R}$, we have $\{X_T 1_T > t\} = \cup_m \{X_m > t, T = n\}$ so for any $n \ge 0$,

$$\{X_T 1_T > t\} \cap \{T \le n\} = \cup_{m=1}^n \{X_m > t, T = n\} \in \mathcal{F}_n$$

and so $X_T 1_T$ is $\mathcal{F}_T$-measurable.

(e) By definition of being a stopping time, for any $t \in \mathbb{R}$,

$$\{(X^T)_n > t\} = \{T > n, X_n > t\} \cup \Big( \cup_{m=0}^n \{T = m, X_m > t\} \Big) \in \mathcal{F}_n$$

so $X^T$ is adapted.

(f) First consider the case where $X$ is non-negative integrable. Then

$$\mathbb{E}(X_n^T) = \mathbb{E}(\mathbb{E}(X_n^T | T)) = \sum_{m \ge n} \mathbb{P}(T = m)\mathbb{E}(X_m) + \mathbb{P}(T > n)\mathbb{E}(X_n) < \infty$$

for any $n$, so we have the result for non-negative $X$.

For the general case, divide $X$ into a non-negative and a negative part.

*(End of proof)* $\square$

**Theorem 2.2.2)** *(Optional stopping theorem)* Let $X$ be a super-martingale and let $S, T$ be *bounded* stopping times with $S \le T$ a.s. Then

$$\mathbb{E}[X_T] \le \mathbb{E}[X_S]$$

**proof)** Fix $n \ge 0$ such that $T \le n$ a.s. Then

$$\begin{aligned} X_T &= X_S + \sum_{S \le k < T} X_{k+1} - X_k \\ &= X_S + \sum_{k=0}^n (X_{k+1} - X_k) 1_{S \le k < T} \end{aligned}$$

Now $\{S \le k\}$ is in $\mathcal{F}_k$ and $\{T > k\}$ is in $\mathcal{F}_k$, so

$$\begin{aligned} \mathbb{E}[(X_{k+1} - X_k) 1_{S \le k < T}] &= \mathbb{E}[\mathbb{E}[(X_{k+1} - X_k) 1_{S \le k < T} | \mathcal{F}_k]] \\ &= \mathbb{E}[\mathbb{E}[X_{k+1} - X_k | \mathcal{F}_k] 1_{S \le k < T}] \end{aligned}$$

but since $(X_n)$ was a super-martingale, $\mathbb{E}[X_{k+1} - X_k | \mathcal{F}_k] \le 0$ a.s. and therefore $\mathbb{E}[(X_{k+1} - X_k) 1_{S \le k < T}] \le 0$ a.s. Hence $\mathbb{E}(X_T) \le \mathbb{E}(X_S)$.

*(End of proof)* $\square$

•Note that $X$ is a sub-martingale *if and only if* $(-X)$ is a super-martingale, and that $X$ is a martingale *if and only if* $X$ and $(-X)$ are super-martingales. Hence, we obtain sub-martingale and martingale versions of the theorem :

$$\text{If } (X_n) \text{ is a sub-martingale, } \mathbb{E}[X_T] \geq \mathbb{E}[X_S]$$
$$\text{If } (X_n) \text{ is a martingale, } \mathbb{E}[X_T] = \mathbb{E}[X_S]$$

**Theorem 2.2.3.)** Let $X$ be an adapted integrable process. Then the followings are equivalent.

(a) $X$ is a super-martingale.

(b) for all bounded stopping times $T$ and stopping time $S$,

$$\mathbb{E}(X_T|\mathcal{F}_S) \leq X_{S\wedge T} \quad \text{a.s.,}$$

(c) for all stopping times $T$, $X_T$ is a super-martingale,

(d) for all bounded stopping times $T$ and all stopping times $S$ with $S \leq T$ a.s,

$$\mathbb{E}(X_T) \leq \mathbb{E}(X_S)$$

⋆ The theorem gives an inverse statement of the optional stopping theorem.

**proof)**

(a) ⇒ (b) Suppose $X$ is a super-martingale and $S, T$ are stopping times. Let $T \leq n$, for some $n < \infty$. Then

$$X_T = X_{S\wedge T} + \sum_{k=0}^{T}(X_{k+1} - X_k)1_{S\leq k<T} \cdots\cdots (*)$$

Let $A \in \mathcal{F}_S$. Then $A \cap \{S \leq k\} \in \mathcal{F}_k$ and $\{T > k\} \in \mathcal{F}_k$ so

$$\mathbb{E}[(X_{k+1} - X_k)1_{S\leq k<T}1_A] = \mathbb{E}[\mathbb{E}[(X_{k+1} - X_k)1_{S\leq k<T}1_A|\mathcal{F}_k]] \leq 0$$

and

$$\begin{aligned}\mathbb{E}[(X_T - X_{S\wedge T})1_A] =& \mathbb{E}[\sum_{n=0}^{T}(X_{k+1} - X_k)1_{S\leq k<T}1_A] \leq 0 \\ \Rightarrow& \quad \mathbb{E}[X_T 1_A] \leq \mathbb{E}[X_{S\wedge T}1_A]\end{aligned}$$

But since this inequality is true for any $A \in \mathcal{F}_S$ and noting that $X_{S\wedge T} \in \mathcal{F}_S$), we see

$$\mathbb{E}[X_T|\mathcal{F}_S] \leq X_{S\wedge T} \quad \text{a.s.}$$

The inclusions (b)⇒(c) and (c)⇒(d) are obvious.

(d) ⇒ (a) Let $m \leq n$ and $A \in \mathcal{F}_n$. Set $T = m1_A + n1_{A^c}$. Then $T$ is a stopping with $T \leq n$. Then

$$\mathbb{E}(X_n 1_A - X_m 1_A) = \mathbb{E}(X_n) - \mathbb{E}(X_T) \leq 0$$

(note, if $\omega \in A$ then $(X_n 1_A - X_m 1_A)(\omega) = X_n(\omega) - X_m(\omega)$ and 0 otherwise) so

$$\mathbb{E}[X_n|\mathcal{F}_m] \leq X_m$$

*(End of proof)* □

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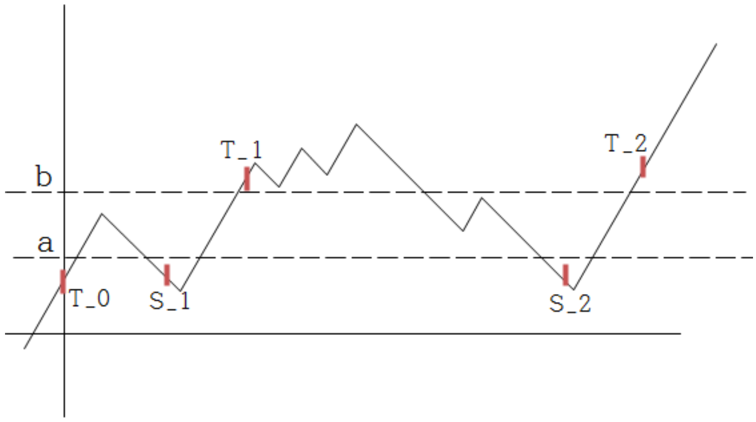
(17th October, Wednesday)




## 2.3. Doob's upcrossing inequality

- Let $X$ be a random process and let $a, b \in \mathbb{R}$ s.t. $a < b$. Fix $\omega \in \Omega$. By an **upcrossing** of $[a, b]$ by $X(\omega)$, we mean an interval of times $\{j, j+1, \cdots, k\}$ s.t. $X_j(\omega) < a$, $X_k(\omega) > b$.
- Write $U_n[a, b](\omega)$ for the number of disjoint upcrossings contained in $\{0, 1, \cdots, n\}$, and $U_n[a, b] \nearrow U[a, b]$ as $n \to \infty$.

**Theorem 2.3.1.)** (Doob's upcrossing inequality) Let $X$ be a *super-martingale*. Then

$$(b-a)\mathbb{E}[U[a, b]] \leq \sup_{n \geq 0} \mathbb{E}[(X_n - a)^-]$$

(Recall, $x^- = (-x) \vee 0$)

Set $T_0 = 0$ and define recursively for $k \geq 0$,

$$S_{k+1} = \inf\{m \geq T_k : X_m < a\}, \quad T_{k+1} = \sup\{m \geq S_{k+1} : X_m > b\}$$

Note that if $T_k < \infty$, then $\{S_k, S_k + 1, T_k\}$ is an upcrossing of $[a, b]$ by $X$, and $T_k$ is the time of completion of the $k-th$ upcrossing. Also note that $U_n[a, b] \leq n$. For $m \leq n$, we have

$$\{U_n[a, b] = m\} = \{T_m \leq n < T_{m+1}\}$$

On this event,

$$X_{T_k \wedge n} - X_{S_k \wedge n} = \begin{cases} X_{T_k} - X_{S_k} \geq b - a & \text{if } k \leq m \\ X_n - X_{S_k} \geq X_n - a & \text{if } k = m1, S_{m+1} \leq n \\ 0 \quad \text{otherwise} \end{cases}$$

Hence

$$\begin{aligned} \sum_{k=1}^{n} (X_{T_k \wedge n} - X_{S_k \wedge n}) &\geq (b-a)U_n[a, b] + X_n - a \\ &\geq (b-a)U_n[a, b] - (X_n - a)^- \end{aligned}$$

Since $X$ is a super-martingale and $T_k \wedge n$ and $S_k \wedge n$ are *bounded stopping times* with $S_k \leq T_k$, by optional stopping theorem, we have

$$\mathbb{E}(X_{T_k \wedge n}) \leq \mathbb{E}(X_{S_k \wedge n})$$

By $\mathbb{E}(\sum_{k=1}^{n} (X_{T_k \wedge n} - X_{S_k \wedge n}))$ we get

$$(b-a)\mathbb{E}(U_n[a, b]) \leq \sum_{n \geq 0} \mathbb{E}[(X_n - a)^-]$$

Apply monotone convergence, with $n \to \infty$, then we are done.

*(End of proof)* □

This theorem does not seem to have any significance at the moment, but it will turn out to be important later on.

### 2.4. Doob's maximal inequalities.

Define $X_n^= \sum_{k\geq n} |X_k|$

In the next two theorems, we see that the martingale(or sub-martingale) property allows us to obtain estimates on this $X_n^*$ in terms of expectations for $X_n$.

**Theorem 2.4.1)** (Doob's maximal inequality) Let $X$ be a *martingale or a non-negative sub-martingale.* Then for all $\lambda \geq 0$,

$$\lambda \mathbb{P}(X_n^* \geq \lambda) \leq \mathbb{E}(|X_n| 1_{\{X_n^* \geq \lambda\}}) \leq \mathbb{E}(|X_n|)$$

**proof)** If $X$ is a martingale, then $|X|$ is a non-negative sub-martingale. It suffices to consider the case where $X$ is a non-negative sub-martingale.

Set $T = \inf\{k \geq 0 : X_k \geq \lambda\} \wedge n$. Then $T$ is a stopping time and $T \leq n$, so by optional stopping, has

$$\begin{aligned}\mathbb{E}(X_n) \geq \mathbb{E}(X_T) &= \mathbb{E}(X_T 1_{X_n^* \geq \lambda}) + \mathbb{E}(X_T 1_{X_n^* < \lambda}) \\ &= \mathbb{E}(\lambda 1_{X_n^* \geq \lambda}) + \mathbb{E}(X_n 1_{X_n^* < \lambda})\end{aligned}$$

and

$$\mathbb{E}(X_n 1_{X^* \geq \lambda}) \geq \lambda \mathbb{P}(X_n^* \geq \lambda)$$

*(End of proof)* □

**Theorem 2.4.2)** (Doob's $L^p$-inequality) Let $X$ be a *martingale or a non-negative sub-martingale.* Then, for all $p > 1$ and $q = p/(p-1)$, we have

$$\| X_n^* \|_p \leq q \| X_n \|_q$$

**proof)** Again, it suffices to consider when $X$ is a non-negative sub-martingale. Fix $k < \infty$. Then

$$\begin{aligned}\mathbb{E}[(X_n^* \wedge k)^p] &= \mathbb{E} \int_0^k p\lambda^{p-1} 1_{\{x_n^* \lambda\}} d\lambda \quad \text{(integration by parts)} \\ &= \int_0^k p\lambda^{p-1} \mathbb{P}(X_n^* \geq \lambda) d\lambda \quad \text{(Fubini)} \\ &\leq \int +0^k p\lambda^{p-2} \mathbb{E}(X_n 1_{X_n^* \geq \lambda}) d\lambda \quad \text{(Doob's maximal inequaltiy)} \\ &= \frac{p}{p-1} \mathbb{E}(X_n (X_n^* \wedge k)^{p-1}) \\ &\leq q \| X_n \|_p \| X_n^* \wedge k \|_p^{p-1} \quad \text{(Hölder's inequality)}\end{aligned}$$

Hence , $\| X_n^* \wedge k \|_p \leq q \| X_n \|_p$. Apply monotone convergence theorem with $k \to \infty$, then we have the desired result.

*(End of proof)* □

Doob's maximal and $L^p$ inequalities have different versions which apply under the same hypothesis to

$$X^* = \sum_{n \geq 0} |X_n|$$

since $X_n^* \nearrow X^*$. Letting $n \to \infty$ in Doob's maximal inequality gives

$$\lambda \mathbb{P}(X^* \geq \lambda) \lim_{n \to \infty} \lambda \mathbb{P}(X_n^* \geq \lambda) \leq \sup_{n \geq 0} \mathbb{E}(|X_n|)$$

We can then replace $\lambda \mathbb{P}(X^* > \lambda)$ by $\lambda \mathbb{P}(X^* \geq \lambda)$ by taking limits from the right in $\lambda$.

Similarly, for $p \in (1, \infty)$ by monotone convergence,

$$\| X^* \|_p \leq q \sup_{n \geq 0} \| X_n \|_p$$