Analysis of PDEs

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Example Classes : Dr. Ivan Moyano

Texts : (1)Evans. PDEs, (2)Rauch, PDEs, (3)F.John, PDEs, (4)Gilberg + Raudinger, Elliptic PDE, (5) Ladyzhenskay, The Boundary Value Problems of Mathematical Physics.

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(5th October 2018, Friday)

Introduction

Suppose $U \subset \mathbb{R}^n$ is open. A *partial differential equation* of order k is an expression of the following form:

$$F(x, u(x), Du(x), \cdots, D^{(k)}u(x)) = 0 \tag{1}$$

Here, $F : U \times \mathbb{R} \times \mathbb{R}^n \times \cdots \times \mathbb{R}^{n^k} \to \mathbb{R}$ is a given function and $u : U \to \mathbb{R}$ is the 'unknown'. We say $u \in C^k(U)$ is a classical solution of 1 if 1 is satisfied on U when we substitute u into the expression.

We could also consider the case where $u : U \to \mathbb{R}^p$ and F takes values in $\mathbb{R}^q$, then we speak of a *system of PDE's*.

Examples)

1. The Transport Equation: Suppose $V : \mathbb{R}^{n+2} \to \mathbb{R}^n$ is given.

$$\frac{\partial u}{\partial t}(x,t) + V(x,t,u(t,x)) \cdot D_x u(x,t) = f(x,t) \quad \text{for } x \in \mathbb{R}^n$$

   is a PDE for $u : \mathbb{R}^{n+1} \to \mathbb{R}$. This describes evolution of some chemical produced at rate $f(x,t)$ and being advected by a flow of velocity $V(x,t,u(t,x))$.

2. The Laplace and Poisson Equations:

$$\Delta u(x) = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i \partial x_j}(x) = 0 \quad \text{(Laplace Equation)}$$

   This describes:

   + Electrostatic potential in empty space
   + Static distribution of heat in a solid body
   + Applications to steady flows in 2D
   + Connections to complex analysis

$$\Delta u(x) = f(x) \quad \text{some given } f : \mathbb{R}^n \to \mathbb{R} \quad \text{(Poisson's Equation)}$$

   This describes:

+ Electric field produced by charge distribution f
+ Gravitational field in Newton's Theory(f is mass density)

3. Heat/Diffusion Equation:

$$\frac{\partial u}{\partial t} = \Delta u$$

This describes evolution of temperature in a solid homogeneous body.

4. Wave Equation:

$$-\frac{\partial^2 u}{\partial t^2} + \Delta u = 0$$

This describe:

+ Displacement of a stretched string (dimension=1)
+ Ripples on surface of water (dimension=2)
+ Density of air in a sound wave (dimension=3)

5. Maxwell's Equations: With $E, B : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$,

$$\nabla \cdot E = \rho \quad \nabla \cdot B = 0$$
$$\nabla \times E + \frac{\partial B}{\partial t} = 0 \quad \nabla \times B - \frac{\partial E}{\partial t} = J$$

ρ, J are charge density/current respectively, are given.

6. Ricci Flow:

$$\partial_t g_{ij} = -2R_{ij}$$

where g_{ij} is a Riemannian metric, R_{ij} is its Ricci curvature.

7. Minimal Surface Equation: For $u : \mathbb{R}^2 \to \mathbb{R}$,

$$\operatorname{div}\left(\frac{Du}{\sqrt{1 - |Du|^2}}\right) = 0$$

Condition for the graph $\{(x, y, u(x, y))\}$ to locally extremise area.

8. Eikonal Equation: for $U \subset \mathbb{R}^3$ and $u : U \to \mathbb{R}$

$$|Du| = 1$$

Level sets parametrise a wave-front moving according to the ray theory of light.

9. Schrödinger's Equation: For $u : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{C} \equiv \mathbb{R}^2$,

$$i\frac{\partial u}{\partial t} + \Delta u - Vu = 0$$

for $V : \mathbb{R}^3 \to \mathbb{R}$ given. u is the wavefunction of a quantum mechanical particle moving in a potential V.

10. Einstein's Equations for General Relativity:

$$R_{\mu\nu}[g] = 0$$

where g is Lorentzian metric. $R_{\mu\nu}$ is Ricci tensor. This describes gravitational field in vacuum.

-. There are Many more examples.

Data and Well-Posedness

In all examples, there is extra information required beyond the equation. We call this the *data*. An important question is what data is appropriate. We typically ask of a PDE problem that:

a) A solution exists,

b) for given data the solution is unique,

c) the solution depends on the data continuously.

If these hold, we say the problem is 'well-posed'. To make these precise, we have to (usually) specify function spaces for the data and solution to belong to.

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8th October, Monday

Let $U \subset \mathbb{R}^n$, $u : U \to \mathbb{R}$ be unknown. Then our system of interest will be

$$F(x; u, Du, \cdots, D^k u) = 0 \tag{2}$$

Notations) Let $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$ be a multi-index(where $\mathbb{N} = \{0, 1, 2, 3, \cdots\}$). Then we let:

- $D^\alpha f(x) = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$ where $|\alpha| = \alpha_1 + \cdots + \alpha_n$ is the order of α.
- For $x \in \mathbb{R}^n$, $x^\alpha = x_1^{\alpha_1} \times \cdots \times x_n^{\alpha_n}$
- $\alpha! = \alpha_1! \cdots \alpha_n!$.
- For $\beta = (\beta_1, \cdots, \beta_n)$, $\beta \leq \alpha$ is equivalent to having $\beta_k \leq \alpha_k$ for all k.

Classifying PDEs

- We say (2) is **linear** if F is a linear function of u and its derivatives. We can write (2) as

$$\sum_{|\alpha \leq k} a_\alpha(x) D^\alpha u(x) = f(x)$$

- We say (2) is **semi-linear** if it is of the form

$$\sum_{|\alpha \leq k} a_\alpha(x) D^\alpha u(x) + a_0(x; u(x), \cdots, D^{k-1} u(x)) = 0$$

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- We say (2) is **quasi-linear** if it is of the form

$$\sum_{|\alpha \leq k} a_\alpha\big(x; u(x), \cdots, D^{k-1} u(x)\big) D^\alpha u(x) + a_0(x; u(x), \cdots, D^{k-1} u(x)) = 0$$

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- We say (2) is **fully non-linear** if its not linear, semi-linear, nor quasi-linear

Examples)

- $\Delta u = f$ is linear
- $\Delta u = u^3$ is semi-linear
- $u u_{xx} + u_x u_{yy} = f$ is quasi-linear
- $u_{xx} u_{yy} - u_{xy}^2 = f$ is fully non-linear.

Cauchy-Kovalevskaya Theorem

For motivation, we recall some ODE theory. Fix $U \subset \mathbb{R}^n$, and assume $f : U \to \mathbb{R}^n$ is given. Consider the ODE

$$\dot{u}(t) = f(u(t)), u(0) = u_0 \in U \tag{3}$$

with $u : I \subset \mathbb{R} \to U$.

Theorem) (Picard-Lindelöf) Suppose there exist $r, K > 0$ s.t. $B_r(u_0) = \{w \in \mathbb{R}^n : |w - u_0| < r\}$ and $|f(x) - f(y) \leq K|x - y|$ for all $x, y \in B_r(x_0)$. Then there exists $\epsilon > 0$(depending in r and K) and a unique C^1-function $u : (-\epsilon, \epsilon) \to U$ solving (3).
proof) Use U solves (3), then

$$u(t) = u_0 + \int_0^t f(u(s))ds \tag{4}$$

and conversely, if U is C^0 and solves (4), then in fact U is C^1 by FTC, and u solves (3).(in context of PDEs, this is called *weak formulation*)
Then our solution, if exists, is a fixed point of the map $B : w \mapsto u_0 + \int_0^t f(w(s))ds$.(use Banach fixed point theorem)

Observations:

- We start by reformulating the problem in a weak form and find a unique C^0 solution. Then C^1 the regularity follows a posteriori.
- to construct the fixed point map, we solve the linear problem $\dot{w}(t) = f(w(t))$.

Lets consider an alternative approach to solving (3). Assuming f is differentiable, we have

$$\begin{aligned}
u^{(1)}(t) &= f(u(t)) \\
u^{(2)}(t) &= f'(u(t))\dot{u}(t) \\
u^{(3)}(t) &= f''(u(t))(\dot{u}(t))^2 + f'(u(t))\ddot{u}(t) \\
&\vdots \\
u^{(k)}(t) &= f_k(u(t), \dot{u}(t), \cdots, u^{(k-1)})(t))
\end{aligned}$$

So in principle, given $u(0) = u_0$, we can determine $u_k = u^{(k)}(0)$ for all $k \geq 0$. *Formally* at least, we can write

$$u(t) = \sum_{k=0}^{\infty} u_k t^k / k! \tag{5}$$

ignoring the issues of convergence. Call this a **formal power series solution**. When will this agree with the Picard-Lindelöf solution we have constructed?

Theorem) (Cauchy-Kovalevskaya, for the case of ODEs) The series in (5) converges to a solution of (3) in a neighbourhood of $t = 0$ if f is real analytic at u_0.
-This will follow from a more general result later.

Definition) Let $U \subset \mathbb{R}^n$ be open and suppose $f : U \to \mathbb{R}$. f is called **real analytic** near $x_0 \in U$ if $\exists r > 0$ and constants f_α(α are multi-indices) such that

$$f(x) = \sum_\alpha f_\alpha (x - x_0)^\alpha \quad \text{for } x \in B_r(x_0)$$

Note: if f is real analytic, then it is C^∞. Furthermore, the constants f_α are given by $f_\alpha = D^\alpha f(x_0)/\alpha!$. Thus f equals its Taylor expansion about x_0, in a neighbourhood of x_0.

$$f(x) = \sum_\alpha \frac{D^\alpha f(x_0)}{\alpha!}(x - x_0)^\alpha \quad \text{for } x \in B_r(x_0)$$

By translation, we usually assume $x_0 = 0$

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(10th October, Wednesday)

• Last lecture : $U \subset \mathbb{R}^n$ open, $f : U \to \mathbb{R}$ is real analytic at $x_0 \in U$ if $\exists f_\alpha \in \mathbb{R}$, $r > 0$ s.t.

$$f(x) = \sum_\alpha f_\alpha (x - x_0)^\alpha \quad \forall |x - x_0| < r$$

Properties of real analytic functions

- f is real analytic at x_0 if and only if $\exists s > 0$ and $C, \rho > 0$ such that:

$$\sup_{|x-x_0|<s} \left| D^\alpha f(x) \right| \le C \frac{|\alpha|!}{\rho^{|\alpha|}}$$

- If f is RA(real analytic) at x_0, it is RA for all x close enough to x_0.
- If $f : U \to \mathbb{R}$ is real analytic everywhere on a connected set U, then f is determined by its values on any open subset of U. (Or by its Taylor expansion at a single point.)

Example : If $r > 0$ set

$$f(x) = \frac{r}{r - (x_1 + \cdots + x_n)} \quad \text{for } |x| < r/\sqrt{n}$$

Then for $|x| < r/\sqrt{n}$,

$$\begin{aligned} f(x) &= \frac{1}{1 - (x_1 + \cdots + x_n)/r} = \sum_{k=0}^\infty \left(\frac{x_1 + \cdots + x_n}{r} \right)^k = \sum_{k=0}^\infty \frac{1}{r^k} \sum_{|\alpha|=k} \binom{|\alpha|}{\alpha} x^\alpha \\ &= \sum_\alpha \frac{|\alpha|!}{r^{|\alpha|}\alpha!} x^\alpha \end{aligned}$$

by multinomial theorem. This is valid for $|x_1 + \cdots + x_n|/r < 1$, which holds for $|x| < r/\sqrt{n}$. In fact, on this domain, the series converges absolutely. Indeed :

$$\sum_\alpha \frac{|\alpha|!}{r^{|\alpha|}\alpha!} |x|^\alpha = \sum_{k=0}^\infty \left(\frac{|x_1| + \cdots + |x_n|}{r} \right)^k < \infty$$

since $|x_1| + \cdots + |x_n| \le |x|\sqrt{n} < r$.

Definition) Let $f = \sum_\alpha f_\alpha x^\alpha$, $g = \sum_\alpha g_\alpha x^\alpha$ be two formal power series. We say g **majorises** f, written $g \gg f$ if

$$\left| f_\alpha \right| \le g_\alpha$$

for all α, and say that g is a **majorant** of f.

Lemma)

(i) If $g \gg f$ and g converges for $|x| < r$ then f also converges (absolutely) for $|x| < r$.

(ii) If f converges for $|x| < r$, then for any $s \in (0, r/\sqrt{n})$, f has a majorant that converges for $|x| < s/\sqrt{n}$.(n is the dimension of the space)

proof)

(i) We note that

$$\begin{aligned} \sum_\alpha \left| f_\alpha x^\alpha \right| &\le \sum_\alpha \left| f_\alpha \right| |x_1|^{\alpha_1} \cdots |x_n|^{\alpha_n} \\ &\le \sum_\alpha g_\alpha \tilde{x}^\alpha \end{aligned}$$

where $\tilde{x} = (|x_1|, \cdots, |x_n|)$. Now $|\tilde{x}| = |x| < r$ so $\sum_\alpha g_\alpha \tilde{x}^\alpha$ converges, hence $\sum_\alpha \left| f_\alpha x^\alpha \right|$ converges. Hence f converges on $|x| < r$ absolutely.

(ii) Pick s s.t. $0 < s\sqrt{n} < r$, and set $y = s(1, \cdots, 1)$. Then $|y| = s\sqrt{n} < r$. Hence $\sum_\alpha f_\alpha y^\alpha$ converges. A convergent series has bounded terms, $\exists C > 0$ s.t. $\left|f_\alpha y^\alpha\right| \leq C$ for all α, and therefore

$$\left|f_\alpha\right| \leq \frac{C}{y_1^{\alpha_1} \cdots y_n^{\alpha_n}} = \frac{C}{s^{|\alpha|}} \leq \frac{C|\alpha|!}{s^\alpha \alpha!}$$

But then $g(x)$ defined by

$$g(x) = \frac{Cs}{s - (x_1 + \cdots + x_n)} = C \sum_\alpha \frac{|\alpha|!}{s^\alpha \alpha!} x^\alpha$$

majorises f and converges for $|x| < s/\sqrt{n} < r/n$.

(End of proof) □

Remark : If $f = (f^1, \cdots, f^m)$ and $g = (g^1, \cdots, g^m)$ are formal power series, then we say

$$g \gg f \quad \text{if} \quad g^i \gg f^i \quad i = 1, \cdots, m$$

Cauchy-Kovalevskaya for First Order Systems

We will study a problem that generalises the Cauchy problem for ODEs we have already discussed.

As coordinates on $\mathbb{R}^n$ we take $(x', t) = x$ where

$$x' = (x_1, \cdots, x_{n-1}) \in \mathbb{R}^{n-1}, \quad t = x^n \in \mathbb{R}$$

Set

$$B_r^n = \{t^2 + |x'|^2 < r^2\}, \quad B_r^{n-1} = \{|x'| < r, t = 0\}$$

We consider a system of equations for unknown $\underline{u}(x) \in \mathbb{R}^m$. More concretely, we seek a solution to

$$\begin{aligned} \underline{u}_t &= \sum_{j=1}^{n-1} \underline{\underline{B}}_j(\underline{u}, x') \cdot \underline{u}_{x_j} + \underline{c}(\underline{u}, x') \quad \text{on } B_r^n \\ \underline{u} &= 0 \quad \text{on } B_r^{n-1} \end{aligned} \tag{6}$$

where $\underline{u}_{x_j} = \partial u/\partial x_j$ etc. We assume that we are given the real analytic functions

$$\begin{aligned} \underline{\underline{B}}_j &: \mathbb{R}^m \times \mathbb{R}^{n-1} \to \mathrm{Mat}(m \times m) \\ \underline{c} &: \mathbb{R}^m \times \mathbb{R}^{n-1} \to \mathbb{R}^m \end{aligned}$$

(these functions do not have to defined on the entire space, but just have to be defined on $\mathbb{R}^n \times B_r^{n-1}$)
Note we assume $\underline{\underline{B}}_j$ and $\underline{u}$ do not depend explicitly on t. We can always introduce u^{m+1} satisfying $\partial_t u^{m+1} = 1$, $u^{m+1} = 0$ on B_r^{n-1} and extending the system.

We will write $\underline{\underline{B}}_j = ((b_j^{kl}))$ and $\underline{c} = (c^1, \cdots, c^m)^T$. Then in components (6) reads:

$$u_t^k = \sum_{j=1}^{n-1} \sum_{l=1}^{m} b_j^{kl}(\underline{u}, x') u_{x_j}^l + c^k(\underline{u}, x') \quad k = 1, \cdots, m$$

Examples : Take $m = 2$, write $\underline{u} = (f, g)^T$.

(a)

$$\begin{cases} f_t = g_x + F \\ g_t = f_x \end{cases}$$

together imply $f_{tt} - f_{xx} = F_t$

(b)

$$\begin{cases} f_t = -g_x + F \\ g_t = f_x \end{cases}$$

together imply $f_{tt} + f_{xx} = F_t$. (Note $F = 0$ gives Cauchy-Riemann equation)

Theorem) (Cauchy-Kovalevskaya) Assume $\{\underline{\underline{B}}_j\}_{j=1}^{n-1}$ and $\underline{c}$ are real analytic. Then for sufficiently small $r > 0$ there exists a unique real analytic function $\underline{u} : B_r^n \to \mathbb{R}^m$ solving the problem (6).

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(12th October, Friday)

Theorem) (Cauchy-Kovalevskaya) Assume $\{\underline{\underline{B}}_j\}_{j=1}^{n-1}$ and $\underline{c}$ are real analytic. Then for sufficiently small $r > 0$ there exists a unique real analytic function $\underline{u} : B_r^n \to \mathbb{R}^m$ solving the problem (6).

proof)

1. The strategy will be to write

$$\underline{u}(x) = \sum_{\alpha} \underline{u}_\alpha x^\alpha \tag{7}$$

and compute coefficients

$$\underline{u}_\alpha = \frac{D^\alpha \underline{u}(0)}{\alpha!}$$

in terms of $\underline{\underline{B}}_j$, $\underline{c}$ and show that the series (7) converges on B_r^n for r small enough.

2. As $\underline{\underline{B}}_j$ and $\underline{c}$ are real analytic, we can write

$$\begin{aligned} \underline{\underline{B}}_j(z, x') &= \sum_{\gamma,\delta} \underline{\underline{B}}_{j,\gamma,\delta} z^\gamma (x')^\delta \quad \gamma \in \mathbb{N}^m, \delta \in \mathbb{N}^{n-1} \text{ multiindices} \\ \underline{c}(z, x') &= \sum_{\gamma,\delta} \underline{c}_{\gamma,\delta} z^\gamma (x')^\delta \end{aligned}$$

where these power series converge for $|z|^2 + |x'|^2 < s^2$, wlog $s > r$. Thus:

$$\begin{aligned} \underline{\underline{B}}_{j,\gamma,\delta} &= \frac{D_z^\delta D_{x'}^\delta \underline{\underline{B}}_j(0,0)}{\gamma!\delta!} \\ \underline{c}_{\gamma,\delta} &= \frac{D_z^\delta D_{x'}^\delta \underline{c}(0,0)}{\gamma!\delta!} \end{aligned} \tag{8}$$

3. Since $\underline{u} \equiv 0$ on $\{t = x^n = 0\}$, we have

$$\underline{u}_\alpha = \frac{D^\alpha \underline{u}(0)}{\alpha!} = 0$$

for all multi-indices α with $\alpha_n = 0$.
Now, we use the evolution equation (6) to deduce

$$\underline{u}_{x_n}(0) = \underline{u}_t(0) = \sum_{j=1}^{n-1} \underline{\underline{B}}_j(\underline{u}(0), 0)\underline{u}_{x_j}(0) + \underline{c}(\underline{u}(0), 0) = \underline{c}(0, 0)$$

Fix $i \in \{1, 2, \cdots, n-1\}$, differentiate (6) with respect to x^i :

$$\begin{aligned} \underline{u}_{tx_i} = \sum_{j=1}^{n-1} \Big[\partial_{x_i} \underline{\underline{B}}_j(\underline{u}, x')\underline{u}_{x_j} + \big(\sum_{i=1}^{m} \partial_{z_i} \underline{\underline{B}}_j(\underline{u}, x') \frac{\partial u^i}{\partial x^j} \underline{u}_{x_j} \big) + \underline{\underline{B}}_j(\underline{u}, x')\underline{u}_{x_i x_j} \Big] \\ + \partial_{x_i} \underline{c}(\underline{u}, x') + \sum_{i=1}^{m} \partial_{z_l} \underline{c}(\underline{u}, x') \frac{\partial u^l}{\partial x^i} \end{aligned}$$

$$\underline{u}_{tx_i}(0) = \partial_{x_i} \underline{c}(0, 0)$$

Iterating this, we deduce $D^\alpha \underline{u}(0) = D^\delta \underline{c}(\underline{0}, 0)$ where $\alpha = (\delta, 1)$.

4. Now, suppose $\alpha = (\delta, 2)$, for $\delta \in \mathbb{N}^{n-1}$. Then

$$\begin{aligned} D^\alpha u^k &= D^\delta(u^k_{x_n x_n}) = D^\delta(u^k_t)_t \\ &= D^\delta\Big(\sum_{j=1}^{n-1}\sum_{l=1}^{m} b_j^{kl} u^l_{x_j} + c^k\Big)_t \\ &= D^\delta\Big(\sum_{j=1}^{n-1}\sum_{i=1}^{m}\Big[b_j^{kl} u^l_{x_j t} + \sum_{p=1}^{m}(b_j^{kl})_{z_p} u^l_{x_j} u^p_t\Big] + \sum_{p=1}^{m} c^k_{z_p} u^p_t\Big) \end{aligned}$$

so

$$D^\alpha u^k(0) = D^\alpha\Big(\sum_{j=1}^{n-1}\sum_{i=1}^{m} b_j^{kl} u^l_{x_j t} + \sum_{p=1}^{m} c^k_{z_p} u^p_t\Big)\Big|_{x=0,\underline{u}=0}$$

Now crucially, the expression on the right can be expanded to produce a polynomial with non-negative coefficients involving derivative of $\underline{\underline{B}}_j$ and $\underline{c}$, and derivatives $D^\beta \underline{u}$ where $\beta_n \leq 1$. More generally, for each multi-index α and each $k \in \{1, \cdots, n\}$, we can compute

$$D^\alpha u^k(0) = p^k_\alpha\Big(D^\alpha_z D^\delta_{x'} \underline{\underline{B}}_j, D^\alpha_z D^\delta_{x'} \underline{c}, D^\beta \underline{u}\Big)\big|_{x=0,\underline{u}=0}$$

where $\beta_n \leq \alpha_n - 1$ and p^k_α is some polynomial in its arguments with non-negative coefficients. Equivalently, for each α, k

$$u^k_\alpha = q^k_\alpha(\underline{\underline{B}}_{j,\alpha,\delta}, \underline{c}_{\gamma,\delta}, u_\beta)$$

where q^k_α is a polynomial with non-negative coefficients, with $\beta_n \leq \alpha_n - 1$.

5. We have shown that if a solution exists, we can compute all derivatives at 0 in terms of known quantities. We will construct a series which majorises the formal sum $\sum_\alpha u_\alpha x^\alpha$.

   First suppose

$$\underline{\underline{B}}^*_j \gg \underline{\underline{B}}_j \quad \underline{c}^* \gg \underline{c}$$

where

$$\begin{aligned} \underline{\underline{B}}^*_j &= \sum_{\gamma,\delta} \underline{\underline{B}}^*_{j,\gamma,\delta} z^\gamma (x')^\delta \\ \underline{c}^* &= \sum_{\gamma,\delta} \underline{c}^*_{\gamma,\delta} z^\gamma (x')^\delta \end{aligned}$$

Assume these converge for $|z|^2 + |x'|^2 < s^2$ (decrease s if necessary). For all j, γ, δ, k, l,

$$0 \leq |B^{kl}_{j,\gamma,\delta}| \leq (B^*)^{kl}_{j,\gamma,\delta}, \quad 0 \leq |c^k_{\gamma,\delta}| \leq (c^*)^{kl}_{\gamma,\delta}$$

We consider the modified problem:

$$\begin{aligned} &\underline{u}^*_t = \sum_{j=1}^{n-1} \underline{\underline{B}}^*_j(\underline{u}^*, x')\underline{u}^*_{x_j} + \underline{c}^*(\underline{u}^*, x') \quad \text{for } |x| < r \\ &\underline{u}^* = \underline{0} \quad \text{on } B^{n-1}_r \end{aligned}$$

As above, seek a real analytic solution

$$\underline{u}^* = \sum_\alpha \underline{u}^*_\alpha x^\alpha \quad \text{where } \underline{u}^*_\alpha = \frac{D^\alpha \underline{u}(0)}{\alpha!}$$

6. We claim $0 \leq |u^k_\alpha| \leq (u^*)^k_\alpha$ for all $\alpha \in \mathbb{N}^n$.

   We do this by proof by induction on α_n.

   For $\alpha_n = 0$, $u^*_\alpha = u_\alpha = 0$

For the induction step: (for $\beta_\alpha \leq \alpha_n - 1$)

$$\begin{aligned} |u_\alpha^k| &= |q_\alpha^k(\underline{\underline{B}}_{j,\gamma,\delta}, \underline{c}_{\gamma,\delta}, \underline{u}_\beta)| \\ &\leq q_\alpha^k(|B_{j,\gamma,\delta}^{kl}|, |C_{\gamma,\delta}^k|, |u_\beta^k|) \\ &\leq q_\alpha^k((B^*)_{j,\gamma,\delta}^{kl}, (c^*)_{\gamma,\delta}^k, (u^*)_\beta^k) \\ &= (u*)_\alpha^k \end{aligned}$$

Using positivity of coefficients of q and induction assumption. Thus $\underline{u}^* \gg \underline{u}$. Remains to show we can find $\underline{\underline{B}}_j^*$, $\underline{c}^*$ s.t. a solution $\underline{u}^*$ exists and converges near 0.

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(15th October, Monday)

Last lecture :

- a formal power series solution $\underline{u} = \sum_\alpha \underline{u}_\alpha x^\alpha$ exists.
- If $\underline{\underline{B}}_j^* \gg \underline{\underline{B}}_j$, $\underline{c}^* \gg \underline{c}$ and $\underline{u}^*$ satisfies

$$\begin{aligned} \underline{u}_t^* &= \sum_{j=1}^{n-1} \underline{\underline{B}}_j^*(\underline{u}^*, x')\underline{u}_{x_j}^* + \underline{c}^*(\underline{u}^*, x') \quad \text{for } |x| < r \\ \underline{u}^* &= \underline{0} \quad \text{on } B_r^{n-1} \end{aligned}$$

then the power series for $\underline{u}^* = \sum_\alpha \underline{u}_\alpha^* x^\alpha$.

proof, continued) To complete the proof, it suffices to show that for any $\underline{\underline{B}}_j$, $\underline{c}$, we can find $\underline{\underline{B}}_j^*$, $\underline{c}_j^*$ such that the corresponding $\underline{u}_j^*$ is a convergent series.

We make a particular choice for $\underline{\underline{B}}_j^*$, $\underline{c}^*$. For this we recall from an earlier lemma that

$$\begin{aligned} \underline{\underline{B}}_j^* &= \frac{Cr}{r - (x_1 + \cdots + x_{n-1}) - (z_1 + \cdots + z_m)} \begin{pmatrix} 1 & 1 & \cdots 1 \\ \vdots & & \ddots \vdots \\ 1 & 1 & \cdots 1 \end{pmatrix} \\ \underline{c}^* &= \frac{Cr}{r - (x_1 + \cdots + x_{n-1}) - (z_1 + \cdots + z_m)} (1, \cdots, 1)^T \end{aligned}$$

will majorise $\underline{\underline{B}}_j$, $\underline{c}$, provided C is large enough, r is small enough and $\underline{\underline{B}}_j^*$, $\underline{c}^*$ are given by convergent series for $|x'|^2 + |z|^2 < r^2$. With these choices of majorants, the modified equation takes the form :

$$\begin{aligned} (u^*)_t^k &= \frac{Cr}{r - (x_1 + \cdots + x_{n-1}) - ((u^*)^1 + \cdots + (u^*)^m)} \big(\sum_{j,l} \big(u^*\big)_{x_j}^l + 1\big) \quad \text{for } |x'|^2 + t^2 < r^2 \\ u^* &= 0 \qquad\qquad \text{for } t = 0, |x'| < r \end{aligned}$$

This problem has an explicit solution.

$$\underline{u}^* = v^*(1, \cdots, 1)^T$$

where

$$v^* = \frac{1}{mn}\Big(r - (x_1 + \cdots + x_{n-1}) - \sqrt{(r - (x_1 + \cdots + x_{n-1}))^2 - 2nmCrt}\Big)$$

(*Check this is indeed the solution!!*) v^* is real analytic for $|x'|^2 + t^2 < r^2$, provided r is small enough. Hence $\underline{u}^*$ is given by a convergent series since $\underline{u}^* \gg \underline{u}$. Our formal power series for $\underline{u}$ converges.

Initial condition hold for $\underline{u}$ since

$$\underline{u}_\alpha = \underline{0} \quad \text{if} \quad \alpha_n = 0$$

Moreover, the functions $\underline{u}_t$ and $\sum_{j=1}^{n-1} \underline{\underline{B}}_j(\underline{u}, x')\underline{u}_{x_j} + \underline{c}(\underline{u}, x')$ are both real analytic near 0 and by construction, have the same Taylor expansion. Hence they must agree on a neighbourhood of 0, so the equation holds in some ball about 0.

(End of proof) □

Reduction to a First Order System

Example)

Consider the PDE problem for $u : \mathbb{R}^3 \to \mathbb{R}$

$$\begin{aligned} u_{tt} &= u u_{xy} - u_{xx} + u_t \\ u\big|_{t=0} &= u_0 \\ u_t\big|_{t=0} &= u_1 \end{aligned} \tag{9}$$

where $u_0, u_1 : \mathbb{R}^2 \to \mathbb{R}$ are given real analytic functions (near 0).
First note that $f = u_0 + tu_1$ is analytic in a neighbourhood of $0 \in \mathbb{R}^3$ and $f\big|_{t=0} = u_0$, $f_t\big|_{t=0} = u_1$.
Set $w = u - f$, then

$$\begin{aligned} w_{tt} &= w w_{xy} - w_{xx} + w_t + f w_{xy} + f_{xy} w + F \\ w\big|_{t=0} &= w_t\big|_{t=0} = 0 \end{aligned}$$

where $F = f f_{xy} - f_{xx} + f_t - f_{tt}$.
Let $(x, y, t) = (x^1, x^2, x^3)$ and set $\underline{u} = (w, w_x, w_y, w_t) = (u^1, u^2, u^3, u^4)$. Then

$$\begin{aligned} u^1_{x^3} &= w_t = u^4 \\ u^2_{x^3} &= w_{xt} = u^4_{x_1} \\ u^3_{x^3} &= w_{yt} = u^4_{x_2} \\ u^4_{x^3} &= w_{tt} = u^1 u^2_{x^2} - u^2_{x^1} + u^4 + f u^2_{x^2} + f_{xy} u^1 + F \end{aligned}$$

Now, defining:

$$\underline{\underline{B}}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \underline{\underline{B}}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ u_1 + f & 0 & 0 & 0 \end{pmatrix}$$

$$\underline{c} = (u^4, 0, 0, u^4 + f_{xy} u^1 + F)^T$$

The system of equations is in the form

$$\underline{u}_{x_2} = \sum_{j=1} 2 \underline{\underline{B}}_j \underline{u}_{x_j} + \underline{c}$$

where $\underline{\underline{B}}_j$, $\underline{c}$ are real analytic near 0. By Cauchy-Kovalevskaya, a real analytic solution to (9) exists near 0.

Note : this procedure relied on

(a) being able to solve for u_{tt},

(b) u_{tt} depending on at most two derivatives of u (in a quasilinear fashion)

More generally, suppose we wish to solve the quasilinear problem :

$$\begin{aligned} &\sum_{|\alpha|=k} a_\alpha(D^{k-1}u, \cdots, u, x) D^\alpha u + a_0(D^{k-1}u, \cdots, u, x) = 0 \quad \text{for } |x| < r \\ &u = \frac{\partial u}{\partial x_n} = \cdots = \frac{\partial^{k-1} u}{\partial x_n^{k-1}} = 0 \qquad \text{for } |x'| < r,\ x_n = 0 \end{aligned}$$

called a **Cauchy problem**.
We introduce

$$\underline{u} = (u, \frac{\partial u}{\partial x_n}, \cdots, D^\alpha u, \cdots)_{|\alpha| \le k-1} = (u^1, \cdots, u^m)$$

$\underline{u}$ contains all derivative of u up to order $k-1$. Wlog, (by changing the order if necessary) put $u^m = \partial^{k-1}u/\partial x_n^{k-1}$. For $j < m$, we can compute $\partial u^j/\partial x^n$ in terms of $\partial u^l/\partial x^p$ for some $l \in \{1, \cdots, m\}$ and $p < n$.

To computed $\partial u^m/\partial x_n$ we need to use the equation. Suppose that

$$a_{(0,\cdots,0,k)}(0,\cdots,0) \neq 0$$

Then we can write the equation as :

$$\frac{\partial^k u}{\partial x_n^k} = \frac{-1}{a_{(0,\cdots,k)}(D^{k-1}u,\cdots,u,x)}\Big[\sum_{|\alpha|=k,\alpha_n<k} a_\alpha D^\alpha u + a_0\Big]$$

Assuming a_α are real analytic, the denominator will be non-zero near the origin. The RHS can be written in terms of $\frac{\partial u^l}{\partial x^p}$ for $p < n$ and $\underline{u}$. We see we can write the equation as a first ordered system for $\underline{u}$, *provided*(this condition is important! would come back to this later)

$$a_{(0,\cdots,k)}(0,\cdots,0) \neq 0 \quad \text{(non-characteristic condition)}$$

In this case we can apply Cauchy-Kovalevskaya.

==
(17th October, Wednesday)

(Problem sheet 1 handed out. Example classes sign-up. First example class (probably) at Thur/Fri next week)

Cauchy Problems for Quasilinear Equations with Data on a Surface

We say $\Sigma \subset \mathbb{R}^n$ is a real analytic **hypersurface** near $x \in \Sigma$ if there exists $\epsilon > 0$ and a real analytic map $\Phi : B_\epsilon(x) \to U \subset \mathbb{R}^n$ where $U = \Phi(B_\epsilon(x))$ such that

- Φ is bijective, and the inverse $\Phi^{-1} : U \to B_\epsilon(x)$ is real analytic.
- $\Phi(\Sigma \cap B_\epsilon(x)) = \{x_n = 0\} \cap U$.

We think of Φ as 'straightening out the boundary'.

There are many examples, e.g. $\{|x| = 1\}$.

Let γ be the unit normal to Σ and suppose u solves

$$\sum_{|\alpha|=k} a_\alpha(D^{k-1}u,\cdots,u,x)D^\alpha u + a_0(D^{k-1}u,\cdots,u,x) = 0 \quad \text{in } B_\epsilon(x) \tag{10}$$
$$u = \gamma^i\partial_i u = \cdots = (\gamma^i\partial_i)^{k-1} = 0 \quad \text{on } \Sigma$$

(note that the boundary condition is equivalent to having $D^\alpha u = 0$ for all $|\alpha| < k$ on Σ.)

Define $v(y) = u(\Phi(y)) \quad \Leftrightarrow \quad u(x) = v(\Phi^{-1}(x))$. Note

$$\frac{\partial v}{\partial x^i} = \frac{\partial u}{\partial y^j}\frac{\partial \Phi^j}{\partial x^i}$$
$$\frac{\partial^2 v}{\partial x^i \partial x^k} = \frac{\partial u^2}{\partial y^j \partial y^i}\frac{\partial \Phi^j}{\partial x^i}\frac{\partial \Phi^l}{\partial x^k} + \frac{\partial u}{\partial y^j}\frac{\partial^2 \Phi^j}{\partial^i \partial x^k} \quad \text{etc.}$$

So we can compute $D^\alpha u$ as a linear combination of $D^\beta v$ for $|\beta| \le |\alpha|$, with coefficients depending on Φ. So if u solves (10), then v will solve

$$\sum_{|\alpha|=k} b_\alpha(D^{k-1}v,\cdots,v,x)D^\alpha v + b_0(D^{k-1}v,\cdots,v,x) = 0$$

Moreover,

$$v\big|_{x_n=0} = u\big|_\Sigma = 0$$
$$\partial_i v\big|_{x_n=0} = (D\Phi)_{ij}\partial_j u\big|_\Sigma = 0$$

and proceeding similarly for $\partial^{k-1}v/(\partial x^n)^{k-1}$, we have each $D^\beta v$ for $|\beta| < k$ as a linear combination of $D^\alpha u (|\alpha| < k)$ and hence $D^\beta v = 0$ for each $|\alpha| < k$. Hence, we have (check that this is an equivalent condition)

$$v = \frac{\partial v}{\partial x^n} = \cdots = \partial^{k-1}v/(\partial x^n)^{k-1} = 0 \quad \text{on } \{x_n = 0\}$$

We can solve this, provided

$$b_{(0,\cdots,0,k)}(0,0,\cdots,0,y) \neq 0 \quad \text{on } \{x_n = 0\}$$

Note if $|\alpha| = k$,

$$D^\alpha u = \frac{\partial^k v}{\partial y_n^k}(D\Phi^n)^\alpha + (\text{terms not involving } \frac{\partial^k v}{\partial y_n^k})$$

So the coefficient of $\partial^k v/\partial y_n^k$ in

$$\sum_{|\alpha|=k} a_\alpha(D^{k-1}u,\cdots,u,x)D^\alpha u + a_0(D^{n-1}u,\cdots,u,x) = 0$$

is

$$b_{(0,\cdots,k)} = a_\alpha(D\Phi^n)^\alpha$$

But $\Sigma = \{\Phi^n = 0\}$ so $D\Phi^n \propto \gamma$. Therefore,

$$b_{(0,\cdots,k)} \neq 0 \quad \Leftrightarrow \quad \sum_{|\alpha|=k} a_\alpha(D\Phi^n)^\alpha \neq 0 \quad \Leftrightarrow \quad \sum_{|\alpha|=k} a_\alpha \gamma^\alpha \neq 0$$

Definition) Σ is a **non-characteristic** at $x \in \Sigma$ for the problem (10) provided

$$\sum_{|\alpha|=k} a_\alpha(0,\cdots,0,x)\gamma^\alpha(x) \neq 0$$

Finally, we have a more general version of Cauchy-Kovalevskaya.

Theorem) (Cauchy-Kovalevskaya Redux) Suppose $\Sigma \subset \mathbb{R}^n$ is a real analytic hypersurface. If Σ is non-characteristic for (10) at $x \in \Sigma$, there exists a unique real analytic solution to (10) in a neighbourhood of x.

> **proof)** We have already seen that we can solve the problem for v uniquely, then $u(x) = v(\Phi(x))$ is the unique solution for (10)
>
> *(End of proof)* □

Characteristic Surfaces for 2nd Order Linear PDE

Consider the linear operator

$$Lu = \sum_{i,j=1}^n a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i \frac{\partial u}{\partial x^i} + cu$$

with $a_{ij}, b_i, c : \mathbb{R}^n \to \mathbb{R}$.

Consider the Cauchy problem

$$\begin{aligned} &Lu = f \\ &u = \sum_{i=1}^n \xi^i \frac{\partial u}{\partial x^i} = 0 \quad \text{on } \Pi_\xi = \{\xi \cdot x = 0\} \end{aligned}$$

Π_ξ is characteristic at $x \in \mathbb{R}^n$ if :

$$\sigma_p(\xi, x) = \sum_{i,j=1}^n a_{ij}\xi^i\xi^j = 0$$

σ_p is the **principal symbol** of L.

- If $\sigma_p(\xi, x) > 0$ for all x, $\xi \neq 0$, then no plane is characteristic, and such operations are called **elliptic**.

Let us restrict to the case where a_{ij}, b_i, c are constants. Suppose $b_i = c = 0$ and Π_ξ is characteristic. Then

$$u(x) = e^{i\lambda\xi\cdot x}$$

solve $Lu = 0$ for any λ. By taking λ large, we can construct solutions to $Lu = 0$ whose derivative (in the ξ direction) is a large as we like. In particular, Lu is very regular, but u need not be. In the elliptic setting, this cannot happen.

==
(19th October, Friday)

Criticisms/Shortcomings of Cauchy-Kovalevskaya

1. Real analyticity is (sometimes) too strong a condition. For example, if solutions of Maxwell's equations were required to be real analytic. We'd know electro-magnetic field everywhere if we could measure it in some small set. This is absurd.

2. We don't necessarily get continuous dependence on data in the form we would like.

   Example : Consider Laplace's equation on $\mathbb{R}^2$. $u_{xx} + u_{yy} = 0$, with Cauchy data $u(x, 0) = \cos(kx)$, $u_y(x, 0) = 0$. This has a real analytic solution

   $$u(x, y) = \cos(kx)\cosh(ky)$$

   We can check that $\sup_{x\in\mathbb{R}} |u(x, 0) \leq 1|$ but $\sup_{x\in\mathbb{R}} |u(x, \epsilon)| \to \infty$ as $k \to \infty$ for all $\epsilon > 0$.

   In fact, we can require as many derivatives of u on $\{y = 0\}$ to be bounded and we can still find solutions which are arbitrarily large at $y = \epsilon$. This Cauchy problem is *not* well posed in C^k, as there is no continuous dependence on data.

These suggest the Cauchy problem for Laplace's equation is not the natural one to consider.

Elliptic Boundary Value Problems

A more natural problem arising in physics is the **Dirichlet Problem** :

$$\begin{aligned} \Delta u &= 0 \quad \text{in } U \subset \mathbb{R}^n, \quad U \text{ open, bounded} \\ u &= g \quad \text{on } \partial U \end{aligned}$$

e.g. u is electrostatic potential in a cavity whose walls are held at voltage g.

We shall develop methods to solve such problems. First we develop some technology.

Hölder and Sobolev Spaces

We need to discuss various function spaces in which to seek solutions our PDEs.

Hölder spaces

Suppose $U \subset \mathbb{R}^n$ is open. We write $u \in C^k(U)$ if $u : U \to \mathbb{R}$ is k-times differentiable at each $x \in U$ and D^α is continuous on U for all $|\alpha| \leq k$. This is not a Banach space, so we would like to restrict to a smaller complete space with a norm.

We say $u \in C^k(\overline{U})$ if $u \in C^k(U)$ and $D^\alpha u$ is uniformly continuous and bounded on U for each $|\alpha| \leq k$. We introduce a norm :

$$\| u \|_{C^k(\overline{U})} = \sum_{|\alpha|\leq k} \sup_{x\in U} |D^\alpha u(x)|$$

With this norm, $C^k(\overline{U})$ is a Banach space. (*Be aware, that $C^k(\overline{U})$ seems to be constructed from the closure $\overline{U}$, but this is not true. It is constructed from U and just depends on U. This matters when U does not have a nice boundary e.g. if U is a complement of the Cantor set$\cap[0, 1]$, then $C^k(\overline{U}) \neq C^k \cap [0, 1]$).

For $0 < \gamma \leq 1$, we say that u is **Hölder continuous with exponent** γ if there exists a constant $C \geq 0$ s.t.

$$|u(x) - u(y)| \leq C|x-y|^\gamma \quad \forall x, y \in U$$

We define γ^{th} **Hölder seminorm** by

$$[u]_{C^{0,\gamma}(\overline{U})} = \inf_{x,y \in U} \frac{|u(x)-u(y)|}{|x-y|^\gamma}$$

We say $u \in C^{k,\gamma}(\overline{U})$ if $u \in C^k(\overline{U})$ and $D^\alpha u$ is Hölder continuous, with exponent γ for all $|\alpha| = k$. We define a norm :

$$\| u \|_{C^{k,\gamma}(\overline{U})} = \| u \|_{C^k(\overline{U})} + \sum_{|\alpha|=k} [D^\alpha k]_{C^{0,\gamma}(\overline{U})}$$

This is again a Banach space.

The Spaces $L^p(U)$, $L^p_{\text{loc}}(U)$

For $U \subset \mathbb{R}^n$ open, suppose $1 \leq p < \infty$. We define the space $L^p(U)$ by

$$L^p(U) = \{u : U \to \mathbb{R} \text{ measurable} | \; \| u \|_{L^p(U)} < \infty\} / \sim$$

where

$$\| u \|_{L^p(U)} = \begin{cases} \left(\int_U |u(x)|^p dx \right)^{1/p} & \text{for } 1 \leq p < \infty \\ \\ \text{assup}_x u(x) = \inf\{C \geq 0 : |u(x)| \leq C \text{ for almost every } x\} \end{cases}$$

and the $\sim$ is an equivalence relation defined by $u_1 \sim u_2$ *if and only if* $u_1 = u_2$ almost everywhere.
$L^p(U)$ is a Banach space with norm $\| \cdot \|_{L^p(U)}$. Completeness follows from dominated convergence theorem.

We define a local versions of $L^p(u)$: we say $u \in L^p_{\text{loc}}(U)$ if $u \in L^p(V)$ for every $V \subset\subset U$ should be read **'V is compactly contained in U'**, meaning there exists a compact K s.t. $V \subset K \subset U$. Note $L^p_{\text{loc}}(U)$ is *not* a Banach space.(it is a Fréchet space)

Weak Derivatives

We would like a notion of differentiability for L^p functions. Since L^p functions like to be integrated, it makes sense to seek a definition involving integration.

Definition) Suppose $u, v \in L^1_{\text{loc}}(U)$ and α is a multi-index. We say v is a α^{th} **weak derivative** of u if

$$(-1)^{|\alpha|} \int_U u D^\alpha \phi dx = \int_U v \phi dx \quad \forall \phi \in C_c^\infty(U)$$

In other words, u, v obey the correct integration of parts formula, when integrated against a test function $\phi \in C_c^\infty(U)$.

$\star$ Check that if $D^\alpha u = v$, then v is indeed also a weak derivative of u.

==
(22nd October, Monday)

(Example Classes : Group A - Thurs 2:00 - 3:30 pm, Group B - Thurs 4:00 - 5:30 pm, MR5)
(I am Group B)
(Submission in the problem class)
(Also a handout distributed - I'll try to add them in my notes!)

Last lecture : Suppose $u, v \in L^1_{\text{loc}}(U)$ and α is a multi-index. We say v is a α^{th} **weak derivative** of u if

$$(-1)^{|\alpha|} \int_U u D^\alpha \phi dx = \int_U v \phi dx \quad \forall \phi \in C_c^\infty(U)$$

In other words, u, v obey the correct integration of parts formula, when integrated against a test function $\phi \in C_c^\infty(U) = \{u \in C^k(U) \;\; \forall k = 1, 2, \cdots | \text{supp}(U) \subset\subset U\}$.

If $u \in C^k(U)<$ then $D^\alpha u$ is a weak α-derivative of u for all $|\alpha| \leq k$ (use integration by part for proof)

Lemma) Suppose $v, \tilde{v} \in L^1_{\text{loc}}(U)$ are both weak α-derivatives of $u \in L^1_{\text{loc}}(U)$. Then $v = \tilde{v}$ almost everywhere.

proof)

$$(-1)^{|\alpha|} \int_U u D^\alpha \phi dx = \int_U v \phi dx = \int_U \tilde{v} \phi dx$$

by the definition of v and $\tilde{v}$ being α-derivatives. Then

$$\int_U (v - \tilde{v}) \phi dx = 0 \quad \forall \phi \in C^\infty_c(Y) \quad \Rightarrow \quad v = \tilde{v} \quad \text{a.e.}$$

(End of proof) $\square$

Since weak derivative is unique, we denote the α^{th} weak derivative of u by $D^\alpha u$.

Definition)

- We say $u \in L^1_{\text{loc}}(U)$ belongs to the **Sobolev space** $W^{k,p}(U)$ (k is the number of derivatives we want and p is the exponent of L^p space we are working on) if $u \in L^p(U)$ and the weak derivative $D^\alpha u$ exist for all $|\alpha| \leq k$ and $D^\alpha \in L^p(U)$.
- If $p = 2$ we write $H^k(U) = W^{k,2}(U)$.
- We define the $W^{k,p}$ norm by

$$\| u \|_{W^{k,p}(U)} = \begin{cases} \left(\sum_{|\alpha| \leq k} \int_U |D^\alpha u|^p dx \right)^{1/p} & 1 \leq p < \infty \\ \sum_{|\alpha| \leq k} \| D^\alpha u \|_{L^\infty(U)} & p = \infty \end{cases}$$

  (there are various equivalent ways of defining the norm)
- We denote by $W^{k,p}_0(U)$ the completion of $C^\infty_c(U)$ in the $W^{k,p}(U)$-norm.

We will find out that these spaces will be useful in fining solutions of PDEs. In particular, the H^k spaces will be useful.

Example : Let $U = B_1(0) = \{|x| < 1\} \subset \mathbb{R}^n$. Set $u(x) = |x|^{-\lambda}$ for $x \in U \backslash \{0\}$ and $\lambda > 0$. This diverges at $x = 0$, so this is not a $C^k(U)$ function.

Note for $x \neq 0$, $D_i u = -\lambda x_i / |x|^{\lambda+2}$. By considering test functions $\phi \in C^\infty_c(U \backslash \{0\})$, if u is weakly differentiable, then the weak derivative must agree with this for $x \neq 0$. We can check $U \in L^1(U)$ if

$$\infty > \int_U |u| dx = \int_{B_1(0)} |x|^{-\lambda} dx = \omega_{n-1} \int_0^1 r^{-\lambda} r^{n-1} dr$$

where ω_{n-1} is the area of S^{n-1}. The integral is finite if $n - 1 - \lambda > -1$, or equivalently λn.

A similar computation gives $-\lambda x_i / |x|^{\lambda+2} \in L^1(U)$, and equivalently $\lambda + 1 < n$.

Now we take $\lambda < n - 1$. Now, suppose $\phi \in C^\infty_c(U)$.

$$-\int_{U \backslash B_\epsilon(0)} u \phi_{x_i} dx = \int_{U \backslash B_\epsilon(0)} u_{x_i} \phi dx - \int_{\partial B_\epsilon(0)} u \phi \nu^i dS$$

where $\underline{\nu} = (\nu^1, \cdots, \nu^n)$ is the inward normal to $\partial B_\epsilon(0)$. Then integration by parts is justified since u is smooth on $U \backslash B_\epsilon(0)$. We estimate

$$\begin{aligned} &\Big| \int_{\partial B_\epsilon(0)} u \phi \nu^i dS \Big| \leq \int_{\partial B_\epsilon(0)} |u \phi| dS \\ \leq& \| \phi \|_{L^\infty(U)} \int_{\partial B_\epsilon(0)} |u| dS = \| \phi \|_{L^\infty(U)} \int_{\partial B_\epsilon(0)} \epsilon^{-d} dS \\ =& \omega_{n-1} \| \phi \|_{L^\infty(U)} \epsilon^{n+1-\lambda} \xrightarrow{\epsilon \to 0} 0 \quad \text{for } \lambda < n - 1 \end{aligned}$$

Thus if $\lambda < n-1$, $|u|^\lambda$ has a i^{th} weak derivative, equal to $-\lambda x_i/|x|^{\lambda+1}$. Moreover, $|Du| = -\lambda/|x|^{\lambda+1} \in L^p(U)$ *if and only if* $p(\lambda+1) < n$. Also, $u \in L^p(U)$ *if and only if* $p\lambda < n$.
$\therefore u \in W^{1,p}(U) \quad \Leftrightarrow \quad \lambda < \frac{n-p}{p}$.

Notice that if $p > n$, we don't have an example of with $\lambda > 0$ (look back once we've done some Sobolev embeddings).

Theorem) For each $k = 1, 2, \cdots$ and $1 \le p \le \infty$. Then the space $W^{k,p}(U)$ is a Banach space.

We need some nice properties of weak derivatives, e.g. linearity, but we differ them to the example sheets.

proof) We just prove the $p < \infty$ case here.

1. Homogeneity and positivity of $\| \cdot \|_{W^{k,p}(U)}$ are obvious. To prove triangular inequality, recall Minkowski's inequality :

$$\Big(\sum_{i=1}^n |a_i + b_i|^p\Big)^{1/p} \le \Big(\sum_{i=1}^n |a_i|^p\Big)^{1/p} + \Big(\sum_{i=1}^n |b_i|^p\Big)^{1/p}$$

We compute

$$\begin{aligned}
\| u+v \|_{W^{k,p}(U)} &= \Big(\sum_{|\alpha| \le k} \| D^\alpha u + D^\alpha v \|^p_{L^p(U)} \Big)^{1/p} \\
&\le \Big(\sum_{|\alpha| \le k} \big(\| D^\alpha u \|_{L^p(U)} + \| D^\alpha v \|_{L^p(U)} \big)^p \Big)^{1/p} \\
&\le \Big(\sum_{|\alpha| \le k} \| D^\alpha u \|^p_{L^p(U)} \Big)^{1/p} + \Big(\sum_{|\alpha| \le ks} \| D^\alpha v \|^p_{L^p(U)} \Big)^{1/p} \\
&= \| u \|_{W^{k,p}(U)} + \| v \|_{W^{k,p}(U)}
\end{aligned}$$

2. For completeness, we note :

$$\| D^\alpha u \|_{L^p(U)} \le \| u \|_{W^{k,p}(U)} \qquad |\alpha| \le k$$

If $(u_l)_l$ is a Cauchy sequence in $W^{k,p}(U)$, then

$$\| D^\alpha(u_l - u_m) \|_{L^p(U)} \le \| u_l - u_m \|_{W^{k,p}(U)}$$

and hence $(D^\alpha u_l)_l$ is a Cauchy sequence in $L^p(U)$ for all $|\alpha| \le k$. By completeness of $L^p(U)$, we may find $u^\alpha \in L^p(U)$ such that $D^\alpha u_l \to u^\alpha$ in $L^p(U)$ for each $|\alpha| \le k$. In particular, let $u = u^{(0,\cdots,0)}$

Claim : $u^\alpha = D^\alpha u$ for each $|\alpha| \le k$.
proof) To see this, let $\phi \in C_c^\infty(U)$. Then

$$(-1)^{|\alpha|} \int_U u_l D^\alpha \phi dx = \int_U D^\alpha u_l \phi dx$$

Sending $l \to \infty$, we have $D^\alpha u_l \to u^\alpha$, $u_l \to u$ in $L^p(U)$. Therefore,

$$(-1) \int_U u D^\alpha \phi dx = \int_U u^\alpha \phi dx$$

and $D^\alpha u = u^\alpha \in L^p(U)$ so $u \in W^{k,p}(U)$ and $u_l \to u$ in $W^{k,p}(U)$

(End of proof) □