Analysis of Partial Differential Equations
Exercise sheet IV (Chapter 4)

1. (Cole-Hopf transformation) Consider the following viscous Burgers' equation:

$$\begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} + u_\varepsilon \dfrac{\partial u_\varepsilon}{\partial x} - \varepsilon \dfrac{\partial^2 u_\varepsilon}{\partial x^2} = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}, \\ u_\varepsilon(0,x) = u^{in}(x) & x \in \mathbb{R}, \end{cases} \tag{1}$$

where $\varepsilon > 0$ is a viscosity constant. Let us make the following change of unknowns:

$$\varphi_\varepsilon(t,x) = \exp\left(-\frac{1}{2\varepsilon} \int_{-\infty}^{x} u_\varepsilon(t,y)\mathrm{d}y \right).$$

Show that the new unknown φ_ε satisfies the following Heat equation:

$$\frac{\partial \varphi_\varepsilon}{\partial t} - \varepsilon \frac{\partial^2 \varphi_\varepsilon}{\partial x^2} = 0 \quad \text{in } \mathbb{R}_+ \times \mathbb{R}.$$

Admitting the existence and uniqueness of a solution $\varphi_\varepsilon \in C^1(\mathbb{R}_+; C^2(\mathbb{R}))$ to the above heat equation, compute the solution to the initial value problem (1).

2. Let $\{u_\varepsilon\}$ be a sequence of solutions associated with (1) for different values of ε.
- Prove that the sequence $\{u_\varepsilon\}$ converges almost everywhere to a limit u when $\varepsilon \to 0$.
- Suppose that $u^{in} \in C_b(\mathbb{R})$ i.e., continuous and bounded. Show that the limit function u solves the following Burgers' equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} + u \dfrac{\partial u}{\partial x} = 0 & \text{in } (0,T] \times \mathbb{R}, \\ u(0,x) = u^{in}(x) & x \in \mathbb{R}. \end{cases} \tag{2}$$

- Suppose further that

$$\exists\ x_0 \in \mathbb{R} \text{ such that } \left.\frac{\partial u^{in}}{\partial x}\right|_{x=x_0} < 0.$$

Using the method of characteristics, show that the global C^1 solutions to (2) seize to exist i.e., there exists a $T^* > 0$ such that $\dfrac{\partial u}{\partial x}(T^*, x)$ blows up and the blow-up time is given by

$$T^* = \min_{x\in\mathbb{R},\, \frac{\partial u^{in}}{\partial x} < 0} \left\{ \frac{-1}{\partial u^{in}/\partial x} \right\}.$$

3. Compute explicitly the unique entropy solution $u \in C(\mathbb{R}_+; L^1_{loc}(\mathbb{R}))$ of the following Burger's equation:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0 \quad \text{in } \mathbb{R}_+ \times \mathbb{R}$$

with initial data

$$u(0,x) = u^{in}(x) = \begin{cases} 1 & \text{for } x < -1 \\ 0 & \text{for } -1 < x < 0 \\ 2 & \text{for } 0 < x < 1 \\ 0 & \text{for } x > 1. \end{cases}$$

Give a graphical representation of the solution thus constructed.

4. (Finite speed of propagation) Consider a scalar conservation law with flux, $f \in C^2$:

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial f(u)}{\partial x} = 0 \quad \text{in } \mathbb{R}^+ \times \mathbb{R}, \\ u(0,x) = u^{in}(x). \end{cases}$$

Show that if $\mathrm{supp}(u^{in}) \subset [-B, +B]$ for some $B > 0$, then

$$\mathrm{supp}(u(t,\cdot)) \subset [-B + t \min_{x\in\mathbb{R}} f'(u^{in}(x)), B + t \max_{x\in\mathbb{R}} f'(u^{in}(x))].$$

5. (Kružkov's uniqueness result) Let $f \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $u^{in} \in L^\infty(\mathbb{R})$. Consider the following nonlinear transport equation:

$$\begin{aligned} &\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}[f(u)] = 0 \qquad \text{for } (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \\ &u(0,x) = u^{in}(x) \qquad\quad x \in \mathbb{R}. \end{aligned} \tag{3}$$

- Recall the definitions of the *weak* and *entropic* solution to (3).
- Show that any entropic solution to (3) is a weak solution to (3).
- By the *doubling of variables method* (consult notes if necessary) show that for any two entropic solutions u, v of (3) with initial data $u^{in}, v^{in} \in L^\infty \cap L^1(\mathbb{R})$, we have the following contraction inequality:

$$\forall t > 0, \qquad \int_{\mathbb{R}} |u(t,x) - v(t,x)| \mathrm{d}x \le \int_{\mathbb{R}} |u^{in}(x) - v^{in}(x)| \mathrm{d}x.$$

- Using the above contraction inequality, deduce the uniqueness result of Kružkov in $u \in C(\mathbb{R}_+; L^1_{loc}(\mathbb{R}))$.

6. (D'Alembert's formula: $d = 1$) Consider the following factorization of the wave operator in space dimension 1:

$$\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} = \Big(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\Big)\Big(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\Big).$$

Then, deduce that the solution $u(t,x)$ for the following wave equation

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial^2 u}{\partial x^2} = 0 \qquad \text{in } (0,\infty) \times \mathbb{R}, \\ u = g, \quad \dfrac{\partial u}{\partial t} = h \qquad \text{on } \{t = 0\} \times \mathbb{R}, \end{cases} \tag{4}$$

is given explicitly as

$$u(t,x) = \frac{1}{2}\Big[g(x+t) + g(x-t)\Big] + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,\mathrm{d}y. \tag{5}$$

7. (Equipartition of Energy) Let $u \in C^2(\mathbb{R}_+ \times \mathbb{R})$ solve the initial boundary value problem for the wave equation in one dimension:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial^2 u}{\partial x^2} = 0 \qquad \text{in } (0,\infty) \times \mathbb{R}, \\ u = g, \quad \dfrac{\partial u}{\partial t} = h \qquad \text{on } \{t = 0\} \times \mathbb{R}. \end{cases} \tag{6}$$

Suppose g, h have compact support. Denote

$$\begin{cases} \text{Kinetic Energy } k(t) = \dfrac{1}{2}\displaystyle\int_{\mathbb{R}} \Big(\frac{\partial u}{\partial t}(t,x)\Big)^2 \mathrm{d}x \\ \text{Potential Energy } p(t) = \dfrac{1}{2}\displaystyle\int_{\mathbb{R}} \Big(\frac{\partial u}{\partial x}(t,x)\Big)^2 \mathrm{d}x \end{cases}$$

• Prove that the total energy $E(t) = k(t) + p(t)$ is constant in time.
• Using D'Alembert's formula (5) for the solution $u(t, x)$ of the wave equation in one dimension, prove that $k(t) = p(t)$ for $t \geq T$ with T large enough.
• Can the above equipartition of energy be true for wave equation in higher dimensions?

8. (Telegraph equation) Show that there exists at most one solution $u \in L^2((0,T) \times (0,1))$ such that $\dfrac{\partial u}{\partial t} \in L^2((0,T) \times (0,1))$ and $\dfrac{\partial u}{\partial x} \in L^2((0,T) \times (0,1))$ to the following initial-boundary value problem:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} + d\dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} = f & \text{in } (0,1) \times (0,T), \\ u = 0 & \text{on } (\{0\} \times (0,T)) \cup (\{1\} \times (0,T)), \\ u = g, \quad \dfrac{\partial u}{\partial t} = h & \text{on } \{t = 0\} \times (0,1). \end{cases}$$

where $d \in \mathbb{R}$ is a constant and f, g, h are smooth and have compact support.
Hint: Derive Energy estimates.

9. (Beam equation) Show that there exists at most one solution $u \in L^2((0,T) \times (0,1))$ such that $\dfrac{\partial u}{\partial t} \in L^2((0,T) \times (0,1))$ and $\dfrac{\partial^2 u}{\partial x^2} \in L^2((0,T) \times (0,1))$ to the following initial-boundary value problem:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} + \dfrac{\partial^4 u}{\partial x^4} = 0 & \text{in } (0,1) \times (0,T), \\ u = \dfrac{\partial u}{\partial x} = 0 & \text{on } (\{0\} \times [0,T]) \cup (\{1\} \times [0,T]), \\ u = g, \quad \dfrac{\partial u}{\partial t} = h & \text{on } \{t = 0\} \times [0,1]. \end{cases}$$

Hint: Derive Energy estimates.