# Advanced Probability

-Martingales
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(15th October 2018, Monday)

## Chapter 2. Martingales in Discrete Time

### 2.1. Definitions.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

- A **Filtration** for $(\Omega, \mathcal{F}, \mathbb{P})$ is a sequence $(\mathcal{F}_n)_{n\geq 0}$ of $\sigma$-algebras s.t. for all $n \geq 0$, we have

$$\mathcal{F}_n \subset \mathcal{F}_{n+1} \subset \mathcal{F}$$

  Set $F_\infty = \sigma(\mathcal{F}_n : n \geq 0)$ then $\mathcal{F}_\infty \subset \mathcal{F}$. We allow $\mathcal{F}_\infty \neq \mathcal{F}$. We interpret $n$ as times and $\mathcal{F}_n$ as the extent of knowledge at time $n$.

- A **Random process(in discrete time)** is a sequence of random variables $(X_n)_{n\geq 0}$. It has a natural filtration $(F_n^X)_{n\geq 0}$ given by

$$\mathcal{F}_n^X = \sigma(X_0, \cdots, X_n)$$

  That is, the knowledge obtained from $X_n$ by time $n$. We say $(X_n)_{n\geq 0}$ is **adapted to** $(\mathcal{F}_n)_{n\geq 0}$ if $X_n$ is $\mathcal{F}_n$-measurable for all $n \geq 0$. This is equivalent to having $\mathcal{F}_n^X \subset \mathcal{F}_n$, for all $n \geq 0$. (Here, $X_n$ are real-valued)

- We would say $(X_n)_{n\geq 0}$ is **integrable** if $X_n$ is integrable for all $n \geq 0$.

- A **martingale** is an *adapted, integrable random process* $(X_n)_{n\geq 0}$ s.t. for all $n \geq 0$,

$$\mathbb{E}[X_{n+1}|\mathcal{F}_n] = X_n \quad \text{a.s.}$$

  In the case $\mathbb{E}[X_{n+1}|\mathcal{F}_n] \leq X_n$ a.s., $(X_n)_n$ is called a **super-martingale** and in the case $\mathbb{E}[X_{n+1}|\mathcal{F}_n] \geq X_n$ a.s., $(X_n)_n$ is called a **sub-martingale**.

### Optional Stopping

- A random variable $T : \Omega \to \{0, 1, 2, \cdots\} \cup \{\infty\}$ is a **stopping time** if $\{T \leq n\} \in \mathcal{F}_n$ for all $n \geq 0$.

- For a stopping time $T$, we set $\mathcal{F}_T = \{A \in \mathcal{F}_\infty : A \cap \{T \leq n\} \in \mathcal{F}_n \text{ for all } n \geq 0\}$. It is easy to check $\mathcal{F}_T$ is indeed a $\sigma$-algebra and that if $T(\omega) = n$ for all $\omega \in \Omega$, then $T$ is a stopping time and $\mathcal{F}_T = \mathcal{F}_n$.

- Given $X$, define $X_T(\omega) = X_{T(\omega)}(\omega)$ whenever $T(\omega) < \infty$ and define the **stopped process** $X^T$ by

$$X_n^T(\omega) = X_{T(\omega)\wedge n}(\omega) \quad \text{for } n \geq 0$$

**Proposition 2.2.1.)** Let $X$ be an adapted process. Let $S$, $T$ be stopping times for $X$. Then

(a) $S \wedge T$ is a stopping time for $X$.

(b) $\mathcal{F}_T$ is a $\sigma$-algebra.

(c) If $S \leq T$ then $\mathcal{F}_S \subset \mathcal{F}_T$.

(d) $X_T 1_{T<\infty}$ is an $\mathcal{F}_T$-measurable random variable.

(e) $X^T$ is adapted.

(f) If $X$ is integrable, then $X^T$ is also integrable.

**proof)**

(a) $\{S \wedge T \leq n\} = \{S \leq n\} \cup \{T \leq n\} \in \mathcal{F}_n$ for all $n \geq 0$, so $S \wedge T$ is a stopping times

(b) Directly from the definition, we see that $\phi \mathcal{F}_T$. Also, given $A \in \mathcal{F}_T$ and a sequence $(A_m)_m \subset \mathcal{F}_T$, we have

$$\begin{aligned} A^c \cap \{T \leq n\} &= \{T \leq n\} - A \cap \{T \leq n\} \in \mathcal{F}_n \quad &\Rightarrow A^c \in \mathcal{F}_T \\ (\cup_m A_m) \cap \{T \leq n\} &= \cup_m (A_m \cap \{T \leq n\}) \in \mathcal{F}_n \quad &\Rightarrow \cup_m A_m \in \mathcal{F}_T \end{aligned}$$

hence $\mathcal{F}_T$ is a $\sigma$-algebra.

(c) Let $A \in \mathcal{F}_S$. Then $A \cap \{T \leq n\} = A \cap \{S \leq n\} \cap \{T \leq n\} \in \mathcal{F}_n$, hence $A \in \mathcal{F}_T$.

(d) For each $t \in \mathbb{R}$, we have $\{X_T 1_T > t\} = \cup_m \{X_m > t, T = n\}$ so for any $n \geq 0$,

$$\{X_T 1_T > t\} \cap \{T \leq n\} = \cup_{m=1}^n \{X_m > t, T = n\} \in \mathcal{F}_n$$

and so $X_T 1_T$ is $\mathcal{F}_T$-measurable.

(e) By definition of being a stopping time, for any $t \in \mathbb{R}$,

$$\{(X^T)_n > t\} = \{T > n, X_n > t\} \cup \Big( \cup_{m=0}^n \{T = m, X_m > t\} \Big) \in \mathcal{F}_n$$

so $X^T$ is adapted.

(f) First consider the case where $X$ is non-negative integrable. Then

$$\mathbb{E}(X_n^T) = \mathbb{E}(\mathbb{E}(X_n^T | T)) = \sum_{m \geq n} \mathbb{P}(T = m)\mathbb{E}(X_m) + \mathbb{P}(T > n)\mathbb{E}(X_n) < \infty$$

for any $n$, so we have the result for non-negative $X$.

For the general case, divide $X$ into a non-negative and a negative part.

*(End of proof)* □

**Theorem 2.2.2)** *(Optional stopping theorem)* Let $X$ be a super-martingale and let $S, T$ be *bounded* stopping times with $S \leq T$ a.s. Then

$$\mathbb{E}[X_T] \leq \mathbb{E}[X_S]$$

**proof)** Fix $n \geq 0$ such that $T \leq n$ a.s. Then

$$\begin{aligned} X_T &= X_S + \sum_{S \leq k < T} X_{k+1} - X_k \\ &= X_S + \sum_{k=0}^n (X_{k+1} - X_k) 1_{S \leq k < T} \end{aligned}$$

Now $\{S \leq k\}$ is in $\mathcal{F}_k$ and $\{T > k\}$ is in $\mathcal{F}_k$, so

$$\begin{aligned} \mathbb{E}[(X_{k+1} - X_k) 1_{S \leq k < T}] &= \mathbb{E}[\mathbb{E}[(X_{k+1} - X_k) 1_{S \leq k < T} | \mathcal{F}_k]] \\ &= \mathbb{E}[\mathbb{E}[X_{k+1} - X_k | \mathcal{F}_k] 1_{S \leq k < T}] \end{aligned}$$

but since $(X_n)$ was a super-martingale, $\mathbb{E}[X_{k+1} - X_k | \mathcal{F}_k] \leq 0$ a.s. and therefore $\mathbb{E}[(X_{k+1} - X_k) 1_{S \leq k < T}] \leq 0$ a.s. Hence $\mathbb{E}(X_T) \leq \mathbb{E}(X_S)$.

*(End of proof)* □

•Note that $X$ is a sub-martingale *if and only if* $(-X)$ is a super-martingale, and that $X$ is a martingale *if and only if* $X$ and $(-X)$ are super-martingales. Hence, we obtain sub-martingale and martingale versions of the theorem :

$$\begin{aligned}&\text{If } (X_n) \text{ is a sub-martingale, } \mathbb{E}[X_T] \geq \mathbb{E}[X_S]\\ &\text{If } (X_n) \text{ is a martingale, } \mathbb{E}[X_T] = \mathbb{E}[X_S]\end{aligned}$$

**Theorem 2.2.3.)** Let $X$ be an adapted integrable process. Then the followings are equivalent.

(a) $X$ is a super-martingale.

(b) for all bounded stopping times $T$ and stopping time $S$,

$$\mathbb{E}(X_T|\mathcal{F}_S) \leq X_{S\wedge T} \quad \text{a.s.},$$

(c) for all stopping times $T$, $X_T$ is a super-martingale,

(d) for all bounded stopping times $T$ and all stopping times $S$ with $S \leq T$ a.s,

$$\mathbb{E}(X_T) \leq \mathbb{E}(X_S)$$

⋆ The theorem gives an inverse statement of the optional stopping theorem.

**proof)**

(a) ⇒ (b) Suppose $X$ is a super-martingale and $S, T$ are stopping times. Let $T \leq n$, for some $n < \infty$. Then

$$X_T = X_{S\wedge T} + \sum_{k=0}^{T}(X_{k+1} - X_k)1_{S\leq k<T} \cdots\cdots (*)$$

Let $A \in \mathcal{F}_S$. Then $A \cap \{S \leq k\} \in \mathcal{F}_k$ and $\{T > k\} \in \mathcal{F}_k$ so

$$\mathbb{E}[(X_{k+1} - X_k)1_{S\leq k<T}1_A] = \mathbb{E}[\mathbb{E}[(X_{k+1} - X_k)1_{S\leq k<T}1_A|\mathcal{F}_k]] \leq 0$$

and

$$\begin{aligned}\mathbb{E}[(X_T - X_{S\wedge T})1_A] =&\mathbb{E}[\sum_{n=0}^{T}(X_{k+1} - X_k)1_{S\leq k<T}1_A] \leq 0\\ \Rightarrow &\quad \mathbb{E}[X_T 1_A] \leq \mathbb{E}[X_{S\wedge T}1_A]\end{aligned}$$

But since this inequality is true for any $A \in \mathcal{F}_S$ and noting that $X_{S\wedge T} \in \mathcal{F}_S$), we see

$$\mathbb{E}[X_T|\mathcal{F}_S] \leq X_{S\wedge T} \quad \text{a.s.}$$

The inclusions (b)⇒(c) and (c)⇒(d) are obvious.

(d) ⇒ (a) Let $m \leq n$ and $A \in \mathcal{F}_n$. Set $T = m1_A + n1_{A^c}$. Then $T$ is a stopping with $T \leq n$. Then

$$\mathbb{E}(X_n 1_A - X_m 1_A) = \mathbb{E}(X_n) - \mathbb{E}(X_T) \leq 0$$

(note, if $\omega \in A$ then $(X_n 1_A - X_m 1_A)(\omega) = X_n(\omega) - X_m(\omega)$ and 0 otherwise) so

$$\mathbb{E}[X_n|\mathcal{F}_m] \leq X_m$$

*(End of proof)* □

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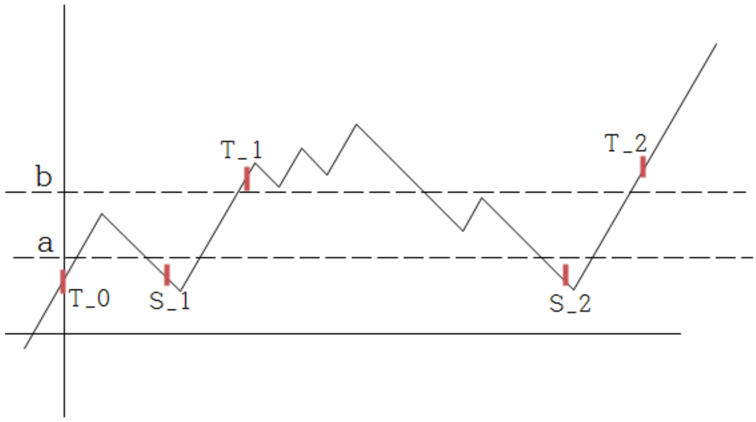
(17th October, Wednesday)




## 2.3. Doob's upcrossing inequality

- Let $X$ be a random process and let $a, b \in \mathbb{R}$ s.t. $a < b$. Fix $\omega \in \Omega$. By an **upcrossing** of $[a, b]$ by $X(\omega)$, we mean an interval of times $\{j, j+1, \cdots, k\}$ s.t. $X_j(\omega) < a$, $X_k(\omega) > b$.
- Write $U_n[a, b](\omega)$ for the number of disjoint upcrossings contained in $\{0, 1, \cdots, n\}$, and $U_n[a, b] \nearrow U[a, b]$ as $n \to \infty$.

**Theorem 2.3.1.)** (Doob's upcrossing inequality) Let $X$ be a *super-martingale*. Then

$$(b-a)\mathbb{E}[U[a,b]] \leq \sup_{n \geq 0} \mathbb{E}[(X_n - a)^-]$$

(Recall, $x^- = (-x) \vee 0$)

**proof)** Set $T_0 = 0$ and define recursively for $k \geq 0$,

$$S_{k+1} = \inf\{m \geq T_k : X_m < a\}, \quad T_{k+1} = \sup\{m \geq S_{k+1} : X_m > b\}$$

Note that if $T_k < \infty$, then $\{S_k, S_k + 1, T_k\}$ is an upcrossing of $[a, b]$ by $X$, and $T_k$ is the time of completion of the $k-th$ upcrossing. Also note that $U_n[a, b] \leq n$. For $m \leq n$, we have

$$\{U_n[a,b] = m\} = \{T_m \leq n < T_{m+1}\}$$

On this event,

$$X_{T_k \wedge n} - X_{S_k \wedge n} = \begin{cases} X_{T_k} - X_{S_k} \geq b - a & \text{if } k \leq m \\ X_n - X_{S_k} \geq X_n - a & \text{if } k = m1, S_{m+1} \leq n \\ 0 \quad \text{otherwise} \end{cases}$$

Hence

$$\begin{aligned} \sum_{k=1}^{n} (X_{T_k \wedge n} - X_{S_k \wedge n}) &\geq (b-a)U_n[a,b] + X_n - a \\ &\geq (b-a)U_n[a,b] - (X_n - a)^- \end{aligned}$$

Since $X$ is a super-martingale and $T_k \wedge n$ and $S_k \wedge n$ are *bounded stopping times* with $S_k \leq T_k$, by optional stopping theorem, we have

$$\mathbb{E}(X_{T_k \wedge n}) \leq \mathbb{E}(X_{S_k \wedge n})$$

By $\mathbb{E}(\sum_{k=1}^{n}(X_{T_k \wedge n} - X_{S_k \wedge n}))$ we get

$$(b-a)\mathbb{E}(U_n[a,b]) \leq \sum_{n \geq 0} \mathbb{E}[(X_n - a)^-]$$

Apply monotone convergence, with $n \to \infty$, then we are done.

*(End of proof)* □

This theorem does not seem to have any significance at the moment, but it will turn out to be important later on.

## 2.4. Doob's maximal inequalities.

Define $X_n^* = \sum_{k \geq n} |X_k|$

In the next two theorems, we see that the martingale(or sub-martingale) property allows us to obtain estimates on this $X_n^*$ in terms of expectations for $X_n$.

**Theorem 2.4.1)** (Doob's maximal inequality) Let $X$ be a *martingale or a non-negative sub-martingale.* Then for all $\lambda \geq 0$,

$$\lambda \mathbb{P}(X_n^* \geq \lambda) \leq \mathbb{E}(|X_n| 1_{\{X_n^* \geq \lambda\}}) \leq \mathbb{E}(|X_n|)$$

**proof)** If $X$ is a martingale, then $|X|$ is a non-negative sub-martingale. It suffices to consider the case where $X$ is a non-negative sub-martingale.

Set $T = \inf\{k \geq 0 : X_k \geq \lambda\} \wedge n$. Then $T$ is a stopping time and $T \leq n$, so by optional stopping, has

$$\begin{aligned} \mathbb{E}(X_n) \geq \mathbb{E}(X_T) &= \mathbb{E}(X_T 1_{X_n^* \geq \lambda}) + \mathbb{E}(X_T 1_{X_n^* < \lambda}) \\ &= \mathbb{E}(\lambda 1_{X_n^* \geq \lambda}) + \mathbb{E}(X_n 1_{X_n^* < \lambda}) \end{aligned}$$

and

$$\mathbb{E}(X_n 1_{X^* \geq \lambda}) \geq \lambda \mathbb{P}(X_n^* \geq \lambda)$$

*(End of proof)* □

**Theorem 2.4.2)** (Doob's $L^p$-inequality) Let $X$ be a *martingale or a non-negative sub-martingale.* Then, for all $p > 1$ and $q = p/(p-1)$, we have

$$\| X_n^* \|_p \leq q \| X_n \|_q$$

**proof)** Again, it suffices to consider when $X$ is a non-negative sub-martingale. Fix $k < \infty$. Then

$$\begin{aligned} \mathbb{E}[(X_n^* \wedge k)^p] &= \mathbb{E} \int_0^k p\lambda^{p-1} 1_{\{x_n^* \lambda\}} d\lambda \quad \text{(integration by parts)} \\ &= \int_0^k p\lambda^{p-1} \mathbb{P}(X_n^* \geq \lambda) d\lambda \quad \text{(Fubini)} \\ &\leq \int +0^k p\lambda^{p-2} \mathbb{E}(X_n 1_{X_n^* \geq \lambda}) d\lambda \quad \text{(Doob's maximal inequaltiy)} \\ &= \frac{p}{p-1} \mathbb{E}(X_n (X_n^* \wedge k)^{p-1}) \\ &\leq q \| X_n \|_p \| X_n^* \wedge k \|_p^{p-1} \quad \text{(Hölder's inequality)} \end{aligned}$$

Hence , $\| X_n^* \wedge k \|_p \leq q \| X_n \|_p$. Apply monotone convergence theorem with $k \to \infty$, then we have the desired result.

*(End of proof)* □

Doob's maximal and $L^p$ inequalities have different versions which apply under the same hypothesis to

$$X^* = \sum_{n \geq 0} |X_n|$$

since $X_n^* \nearrow X^*$. Letting $n \to \infty$ in Doob's maximal inequality gives

$$\lambda \mathbb{P}(X^* \geq \lambda) \lim_{n \to \infty} \lambda \mathbb{P}(X_n^* \geq \lambda) \leq \sup_{n \geq 0} \mathbb{E}(|X_n|)$$

We can then replace $\lambda \mathbb{P}(X^* > \lambda)$ by $\lambda \mathbb{P}(X^* \geq \lambda)$ by taking limits from the right in $\lambda$.

Similarly, for $p \in (1, \infty)$ by monotone convergence,

$$\| X^* \|_p \leq q \sup_{n \geq 0} \| X_n \|_p$$

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(19th October, Friday)

## 2.5. Doob's martingale convergence theorems

We are going to study three different martingale convergence theorems. They are all important.

- We say that a random process $X$ is **$L^p$-bounded** if $\sum_{n\geq 0} \| X_n \|_p < \infty$.
- We say that $X$ is **uniformly integrable** if

$$\sup_{n\geq 0} \mathbb{E}(|X_n| 1_{|X_n|>\lambda}) \to 0 \quad \text{as } \lambda \to \infty$$

- If $X$ is $L^p$ bounded for some $p > 1$, then this implies that $X$ is uniformly integrable. This again implies that $X$ is $L^1$ bounded. The first implication follows from Hölder inequality. The second implication is true because $\mathbb{E}(|X_n|) = \mathbb{E}(|X_n| 1_{|X_n|\leq\lambda}) + \mathbb{E}(|X_n| 1_{|X_n|>\lambda}) \leq \lambda + \mathbb{E}(|X_n| 1_{|X_n|>\lambda})$.

$X$ is $L^p$-bounded, $p > 1$

$\Downarrow$

$X$ is Uniformly integrable

$\Downarrow$

$X$ is $L^1$-bounded

**Theorem 2.5.1)** *(Almost sure martingale convergence theorem)* Let $X$ be an *$L^1$-bounded super-martingale.* Then there exists an integrable and $\mathcal{F}_\infty$-measurable random variable $X_\infty$ such that

$$X_n \to X \quad a.s. \text{ as } n \to \infty$$

**proof)** For a sequence of real numbers $(x_n)_{n\geq 0}$, as $n \to \infty$, $(x_n)_n$ either converges *or* $|x_n| \to \infty$, *or* $\liminf_n x_n < \limsup_n x_n$. In the last case, since the rationals are dense in $\mathbb{R}$, there exist $a, b \in \mathbb{Q}$ such that $\liminf x_n < a < b \limsup x_n$.

Set $\Omega_0 = \Omega_\infty \cap (\bigcap_{a,b\in\mathbb{Q}, a<b} \Omega_{a,b})$ where $\Omega_\infty = \{\liminf |X_n| < \infty\}$, $\Omega_{a,b} = \{U[a, b] < \infty\}$ (Recall that $U[a, b]$ is the number of upcrossings). Then $X_n(\omega)$ converges for all $\omega \in \Omega_0$. By Fatous' lemma,

$$\mathbb{E}(\liminf |X_n|) \leq \liminf \mathbb{E}|X_n| < \infty$$

so this implies $\mathbb{P}(\Omega_\infty) = 1$. By Doob's inequality, for $a < b$, has

$$(b - a)\mathbb{E}(U[a, b]) \leq |a| + \sup_{n\geq 0} \mathbb{E}|X_n| < \infty$$

and therefore $\mathbb{P}(\Omega_{a,b}) = 1$. Putting this together, we deduce that $\mathbb{P}(\Omega_0) = 1$, and we can find a random variable $X_\infty$ defined by

$$X_\infty = \lim_{n\to\infty} X_n 1_{\Omega_0}$$

Then $X_n \to X_\infty$ a.s. Also $X_\infty$ is $\mathcal{F}_\infty$-measurable and $|X_\infty| \leq \liminf |X_n|$ so $\mathbb{E}(|X_\infty|) < \infty$. Hence $X_\infty$ is integrable.

*(End of proof)* $\square$

**Remark :** Every non-negative integrable super-martingale is $L^1$-bounded, hence it converges a.s.

**Theorem 2.5.2)** *($L^1$ martingale convergence theorem)* Let $(X_n)_{n\geq 0}$ be a *uniformly integrable martingale.* Then there exists a random variable $X_\infty \in L^1(\mathcal{F}_\infty)$ such that

$$X_n \xrightarrow{n\to\infty} X_\infty \quad \text{a.s. and in } L^1$$

Moreover, $X_n = \mathbb{E}(X_\infty|\mathcal{F}_n)$ a.s. for all $n \geq 0$.

Conversely, for all $Y \in L^1(\mathcal{F}_\infty)$, on choosing version $X_n$ of $\mathbb{E}(Y|\mathcal{F}_n)$ for all $n$, we obtain a uniformly integrable martingale $(X_n)_{n\geq 0}$ such that

$$X_n \xrightarrow{n\to\infty} Y \quad \text{a.s. and in } L^1$$

We can think of this theorem as establishing the bijection unif. integrable martingale/a.s. $\leftrightarrow$ $L^1(\mathcal{F}_\infty)$

**proof)** Let $(X_n)_{n\geq 0}$ be a uniformly integrable martingale. By the almost sure martingale convergence theorem, there exists $X_\infty \in L^1(\mathcal{F}_\infty)$ s.t. $X_n \to X_\infty$ a.s. Since $X$ is uniformly integrable, it also follows that $X_n \to X_\infty$ in $L^1$.(see PM, Thm 2.5.1. and 6.2.3.)

Next, for $m \geq n$,

$$\begin{aligned}\| X_n - \mathbb{E}(X_\infty|\mathcal{F}_n) \|_1 &= \| \mathbb{E}(X_m - X_\infty|\mathcal{F}_n) \|_1 \\ &= \| X_m - X_\infty \|_1 \to 0 \quad \text{as } m \to \infty\end{aligned}$$

Hence $X_n = \mathbb{E}(X_\infty|\mathcal{F}_n)$ a.s.

For the converse statement, suppose $Y \in L^1(\mathcal{F}_\infty)$ and let $X_n$ be a version of $\mathbb{E}(Y|\mathcal{F}_n)$ for all $n$. Then $(X_n)_{n\geq 0}$ is a martingale by the tower property, and is uniformly integrable by **Lemma 1.5.1.** Hence there exists $X_\infty \in L^1(\mathcal{F}_\infty)$ such that $X_n \to X_\infty$ a.s. and in $L^1$. For all $n \geq 0$ and all $A \in \mathcal{F}_n$, we have

$$\mathbb{E}(X_\infty 1_A) = \lim_{m\to\infty} \mathbb{E}(X_m 1_A) = \lim_{n\leq m\to\infty} \mathbb{E}(\mathbb{E}(Y 1_A|\mathcal{F}_m)) = \mathbb{E}(Y 1_A)$$

where the second equality follows because $\mathbb{E}(X_m|\mathcal{F}_n) = \mathbb{E}(Y|\mathcal{F}_n)$. Now $X_\infty$, $Y \in L^1(\mathcal{F}_\infty)$ and $\cup_n \mathcal{F}_n$ is a $\pi$-system generating $\mathcal{F}_\infty$. Hence, by Dynkin's lemma,

$$X_\infty = Y \quad \text{a.s.}$$

*(End of proof)* $\square$

**Theorem 2.5.3)** *($L^p$-martingale convergence theorem)* Let $p \in (1,\infty)$. Let $(X_n)_{n\geq 0}$ be an $L^p$-bounded martingale. Then there exists a random variable $X_\infty \in L^p(\mathcal{F}_\infty)$ s.t.

$$X_n \to X_\infty \quad \text{a.s. and in } L^p$$

Moreover, $X_n = \mathbb{E}(X_\infty|\mathcal{F}_n)$ a.s. for all $n \geq 0$.
Conversely, for all $Y \in L^p(\mathcal{F}_\infty)$, on choosing a version $X_n$ of $\mathbb{E}(Y|\mathcal{F}_n)$ for all $n$, we obtain an $L^p$-bounded martingale such that $X_n \to Y$ a.s. and in $L^p$.

This is very similar to the statement of $L^1$-martingale convergence theorem. Indeed, the proof is also very similar.

**proof)** Let $(X_n)$ be an $L^p$-bounded martingale. By *a.s. martingale convergence theorem*, there exists $X_\infty \in L^1(\mathcal{F}_\infty)$, $X_n \to X_\infty$ a.s.

By *Doob's $L^p$-inequality*, $\| X^* \|_p \leq q \sup_{n\geq 0} \| X_n \|_p < \infty$, where $X^* = \sup_{n\geq 0} |X_n|$. Also, since $|X_n - X_\infty|^p \leq (2X^*)^p$ for all $n$, we may apply dominated convergence theorem to deduce that $X_n \to X_\infty$ in $L^p$. Then $X_n = \mathbb{E}(X_\infty|\mathcal{F}_n)$ a.s. for all $n$, as in the $L^1$-convergence.

For the converse statement, suppose $Y \in L^p(\mathcal{F}_\infty)$ and let $X_n$ be a version of $\mathbb{E}(Y|\mathcal{F}_n)$. Then $(X_n)_{n\geq 0}$ is a martingale by the tower property and by Jensen inequality,

$$\| X_n \|_p = \| \mathbb{E}(Y|\mathcal{F}_n) \|_p \leq \| Y \|_p$$

Let $X_n \to X_\infty$ a.s. and in $L^P$ for $X_\infty \in L^p(\mathcal{F}_\infty)$, using the previous part. Then proceed as in the proof of $L^1$-convergence to prove that in fact $Y = X_\infty$ a.s.

*(End of proof)* $\square$

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(22nd October, Monday)

Recall that, for a stopping time $T$ and a random process $X$, $X_T$ has been defined only on $\{T < \infty\}$. Given an almost sure limit $X_\infty$ for $X$, we define $X_T = X_\infty$ on $\{T = \infty\}$. Then the optional stopping theorem extends to all stopping times for uniformly integrable martingales.

**Theorem 2.5.5.)** Let $X$ be a uniformly integrable martingale and let $T$ be any stopping time. Then $\mathbb{E}(X_T) = \mathbb{E}(X_0)$. Moreover, for all stopping time $S$ and $T$, we have

$$\mathbb{E}(X_T|\mathcal{F}_S) = X_{S\wedge T} \quad \text{a.s.}$$

This theorem is an extension of Optional stopping theorem, **Theorem 2.2.2** and **Theorem 2.2.3**.

**proof)** By the optional stopping time theorem and **2.2.3**, when applied to the bounded stopping time $T \wedge n$, we have

$$\mathbb{E}(X_{T\wedge n}) = \mathbb{E}(X_0)$$
$$\mathbb{E}(X_{T\wedge n}|\mathcal{F}_S) = X_{S\wedge T\wedge n}$$

In order to get the claim by letting $n \to \infty$, we need to prove $X_{T\wedge n} \to X_T$ *a.s.* and in $L^1$. This will imply that

$$\mathbb{E}(X_{T\wedge n}|\mathcal{F}_S) \to \mathbb{E}(X_T|\mathcal{F}_S) \quad \text{in } L^1$$

**Claim :** $X_{T\wedge n} \to X_T$ *a.s.* and in $L^1$

**proof)** By the $L^1$ martingale convergence theorem, there exists $X_\infty \in L^1(\mathcal{F}_\infty)$ s.t. $X_n \to X_\infty$ a.s. and in $L^1$ and in $L^1$ and $X_n = \mathbb{E}(X_\infty|\mathcal{F}_n)$. This implies $X_{T\wedge n} \to X_T$ *a.s.* as $\to \infty$.(if $T < \infty$, the convergence trivial, and in the case $T = \infty$, the convergence justified the previous statement). Since $F_{T\wedge n} \subset F_n$, by **Theorem 2.2.3.** and the tower property we have

$$X_{T\wedge n} = \mathbb{E}(X_n|\mathcal{F}_{T\wedge n}) = \mathbb{E}(X_\infty|\mathcal{F}_{T\wedge n})$$

By **Lemma 1.5.1**, $(X_{T\wedge n})_{n\geq 0}$ is uniformly integrable. Hence

$$X_{T\wedge n} \to X_T \quad \text{in } L^1$$

*(End of proof)* □

**Backward martingale**

- A **backward filtration** $(\hat{\mathcal{F}}_n)_{n\geq 0}$ is a sequence of $\sigma$-algebras such that $\mathcal{F} \supset \hat{\mathcal{F}}_n \supset \hat{\mathcal{F}}_{n+1}$.
- This also defines $\hat{\mathcal{F}}_\infty = \bigcap_{n\geq 0} \hat{\mathcal{F}}_n$

**Theorem 2.5.4.)** *(Backward martingale convergence theorem)* For all $Y \in L^1(\mathcal{F})$, we have

$$\mathbb{E}(Y|\hat{\mathcal{F}}_n) \to \mathbb{E}(Y|\hat{\mathcal{F}}_\infty) \quad \text{a.s. and in } L^1 \quad \text{as } n \to \infty$$

Note that we do not need a uniformly integrability condition, because our assumption of backward filtration already implies uniform convergences.

**proof)** Write $X_n = \mathbb{E}(Y|\hat{\mathcal{F}}_n)$ for all $n \geq 0$. Fix $n \geq 0$, by the Tower property, $(X_{n-k})_{0\leq k\leq n}$ is a martingale for the filtration $(\hat{\mathcal{F}}_{n-k})_{0\leq k\leq n}$. For $a < b$, the number $U_n[0,\infty]$ of upcrossings of $[a,b]$ by $(X_k)_{0\leq k\leq n}$ equals the number of upcrossings of $[-b,-a]$ by the process $(-X_{n-k})_{0\leq k\leq n}$. Hence by **Theorem 2.3.1**,

$$(b-a)\mathbb{E}(U_n[a,b]) \leq \mathbb{E}((X_0 - b)^+)$$

and so by monotone convergence,

$$(b-a)\mathbb{E}(U[a,b]) \leq \mathbb{E}((X_0 - b)^+) \leq \mathbb{E}(|Y|) + |b| < \infty$$

Also,

$$\mathbb{E}(\liminf |X_n|) \leq \liminf \mathbb{E}|X_n| \leq \mathbb{E}|Y| < \infty$$

The only used(???) in the proof of the almost sure martingale convergence theorem applies to show that $\mathbb{P}(\hat{\Omega}_0) = 1$. where $\hat{\Omega}_0 = \{X_n \text{ converges as } n \to \infty\}$

Set $X_\infty 1_{\hat{\Omega}_0} \lim_{n\to\infty} X_n$. Then $X_\infty \in L^1(\hat{\mathfrak{F}}_\infty)$ and $X_n \to X_\infty$ a.s. Now $(X_n)_{n\geq 0}$ is uniformly integrable (by **Lemma 1.5.1**), so $X_n \xrightarrow{L^1} X_\infty$. Finally, for all $A \in \hat{F}_\infty$, we have

$$\mathbb{E}((X_\infty - \mathbb{E}(Y|\hat{\mathfrak{F}}_\infty))1_A) = \lim_{n\to\infty} \mathbb{E}((X_n - Y)1_A) = 0$$

This implies $X_\infty = \mathbb{E}(Y|\hat{\mathfrak{F}}_\infty)$ a.s.

*(End of proof)* □

# 3. Applications of martingale theory

## Sums of independent random variables

Let $S_n = X_1 + \cdots + X_n$, where $(X_n)_{n\geq 0}$ is a sequence of independent random variables.

**Theorem 3.1.1)** *(Strong Law of Large Numbers)* Let $(X_n)_{n\geq 0}$ be a sequence of independent identically distributed ($i.i.d$) integrable random variables. Set $\mu = \mathbb{E}(X_1)$. Then

$$S_n/n \to \mu \quad \text{a.s. and in } L^1$$

**proof)** Define $\hat{\mathfrak{F}}_n = \sigma(S_m : m \geq n)$, $\mathfrak{T}_n = \sigma(X_m : m \geq n+1)$ and $\mathfrak{T} = \cap_{n\geq 1}\mathfrak{T}_n$. Then $\hat{\mathfrak{F}}_n = \sigma(S_n, \mathfrak{T}_n)$ and $(\hat{\mathfrak{F}}_n)_{n\geq 1}$ is a backward filtration. Since $\sigma(X_1, S_n)$ is independent of $\mathfrak{T}_n$, we have

$$\mathbb{E}(X_1|\hat{\mathfrak{F}}_n) = \mathbb{E}(X_1|S_n) \quad \text{a.s.}$$

For $k \leq n$ and all Borel sets $B$, we have

$$\mathbb{E}(X_k 1_{\{S_n\in B\}}) = \mathbb{E}(X_1 1_{\{S_n \in B\}})$$

by symmetry $(X_k, S_n) \overset{\text{d}}{=} (X_1, S_n)$ in distribution, so $\mathbb{E}(X_k|S_n) = \mathbb{E}(X_1|S_n)$ a.s. But

$$\mathbb{E}(X_1|S_n) + \cdots + \mathbb{E}(X_n|S_n) = \mathbb{E}(S_n|S_n) = S_n \quad \text{a.s.}$$

so $\mathbb{E}(X_1|\hat{\mathfrak{F}}_n) = S_n/n$ almost surely. Then by backward martingale convergence theorem, has $S_n/n \to Y$ a.s. and in $L^1$ for some random variable $Y$. Then $Y \in \mathfrak{T}$. By Kolmogorov's 0-1 law [PM **Theorem 2.6.1**], $Y$ is almost surely a constant. Hence

$$Y = \mathbb{E}(Y) = \lim \mathbb{E}(S_n/n) = \mu \quad \text{a.s.}$$

where the second equality follows from $L^1$ convergence $S_n/n \to Y$.

*(End of proof)* □

Since a.s. convergence implies convergence in probability, we have the following corollary.

**Corollary 3.1.2)** *(Weak law of large numbers)* Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. integrable r.v.. Set $\mu = \mathbb{E}(X_1)$. Then

$$\mathbb{P}(|\frac{S_n}{n} - \mu| > \epsilon) \to 0 \quad \text{as } n \to \infty \quad \forall \epsilon > 0$$