Advanced Probability

-Martingales
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(15th October 2018, Monday)

Chapter 2. Martingales in Discrete Time

2.1. Definitions.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

- A **Filtration** for $(\Omega, \mathcal{F}, \mathbb{P})$ is a sequence $(\mathcal{F}_n)_{n\geq 0}$ of σ-algebras s.t. for all $n \geq 0$, we have

$$\mathcal{F}_n \subset \mathcal{F}_{n+1} \subset \mathcal{F}$$

  Set $F_\infty = \sigma(\mathcal{F}_n : n \geq 0)$ then $\mathcal{F}_\infty \subset \mathcal{F}$. We allow $\mathcal{F}_\infty \neq \mathcal{F}$. We interpret n as times and $\mathcal{F}_n$ as the extent of knowledge at time n.

- A **Random process(in discrete time)** is a sequence of random variables $(X_n)_{n\geq 0}$. It has a natural filtration $(F_n^X)_{n\geq 0}$ given by

$$\mathcal{F}_n^X = \sigma(X_0, \cdots, X_n)$$

  That is, the knowledge obtained from X_n by time n. We say $(X_n)_{n\geq 0}$ is **adapted to** $(\mathcal{F}_n)_{n\geq 0}$ if X_n is $\mathcal{F}_n$-measurable for all $n \geq 0$. This is equivalent to having $\mathcal{F}_n^X \subset \mathcal{F}_n$, for all $n \geq 0$. (Here, X_n are real-valued)

- We would say $(X_n)_{n\geq 0}$ is **integrable** if X_n is integrable for all $n \geq 0$.

- A **martingale** is an *adapted, integrable random process* $(X_n)_{n\geq 0}$ s.t. for all $n \geq 0$,

$$\mathbb{E}[X_{n+1}|\mathcal{F}_n] = X_n \quad \text{a.s.}$$

  In the case $\mathbb{E}[X_{n+1}|\mathcal{F}_n] \leq X_n$ a.s., $(X_n)_n$ is called a **super-martingale** and in the case $\mathbb{E}[X_{n+1}|\mathcal{F}_n] \geq X_n$ a.s., $(X_n)_n$ is called a **sub-martingale**.

Optional Stopping

- A random variable $T : \Omega \to \{0, 1, 2, \cdots\} \cup \{\infty\}$ is a **stopping time** if $\{T \leq n\} \in \mathcal{F}_n$ for all $n \geq 0$.

- For a stopping time T, we set $\mathcal{F}_T = \{A \in \mathcal{F}_\infty : A \cap \{T \leq n\} \in \mathcal{F}_n \text{ for all } n \geq 0\}$. It is easy to check $\mathcal{F}_T$ is indeed a σ-algebra and that if $T(\omega) = n$ for all $\omega \in \Omega$, then T is a stopping time and $\mathcal{F}_T = \mathcal{F}_n$.

- Given X, define $X_T(\omega) = X_{T(\omega)}(\omega)$ whenever $T(\omega) < \infty$ and define the **stopped process** X^T by

$$X_n^T(\omega) = X_{T(\omega)\wedge n}(\omega) \quad \text{for } n \geq 0$$

Proposition 2.2.1.) Let X be an adapted process. Let S, T be stopping times for X. Then

(a) $S \wedge T$ is a stopping time for X.

(b) $\mathcal{F}_T$ is a σ-algebra.

(c) If $S \le T$ then $\mathcal{F}_S \subset \mathcal{F}_T$.

(d) $X_T 1_{T<\infty}$ is an $\mathcal{F}_T$-measurable random variable.

(e) X^T is adapted.

(f) If X is integrable, then X^T is also integrable.

proof)

(a) $\{S \wedge T \le n\} = \{S \le n\} \cup \{T \le n\} \in \mathcal{F}_n$ for all $n \ge 0$, so $S \wedge T$ is a stopping times

(b) Directly from the definition, we see that $\phi \mathcal{F}_T$. Also, given $A \in \mathcal{F}_T$ and a sequence $(A_m)_m \subset \mathcal{F}_T$, we have

$$
\begin{aligned}
A^c \cap \{T \le n\} &= \{T \le n\} - A \cap \{T \le n\} \in \mathcal{F}_n \quad &\Rightarrow A^c \in \mathcal{F}_T \\
(\cup_m A_m) \cap \{T \le n\} &= \cup_m (A_m \cap \{T \le n\}) \in \mathcal{F}_n \quad &\Rightarrow \cup_m A_m \in \mathcal{F}_T
\end{aligned}
$$

hence $\mathcal{F}_T$ is a σ-algebra.

(c) Let $A \in \mathcal{F}_S$. Then $A \cap \{T \le n\} = A \cap \{S \le n\} \cap \{T \le n\} \in \mathcal{F}_n$, hence $A \in \mathcal{F}_T$.

(d) For each $t \in \mathbb{R}$, we have $\{X_T 1_T > t\} = \cup_m \{X_m > t, T = n\}$ so for any $n \ge 0$,

$$\{X_T 1_T > t\} \cap \{T \le n\} = \cup_{m=1}^n \{X_m > t, T = n\} \in \mathcal{F}_n$$

and so $X_T 1_T$ is $\mathcal{F}_T$-measurable.

(e) By definition of being a stopping time, for any $t \in \mathbb{R}$,

$$\{(X^T)_n > t\} = \{T > n, X_n > t\} \cup \Big(\cup_{m=0}^n \{T = m, X_m > t\} \Big) \in \mathcal{F}_n$$

so X^T is adapted.

(f) First consider the case where X is non-negative integrable. Then

$$\mathbb{E}(X_n^T) = \mathbb{E}(\mathbb{E}(X_n^T | T)) = \sum_{m \ge n} \mathbb{P}(T = m)\mathbb{E}(X_m) + \mathbb{P}(T > n)\mathbb{E}(X_n) < \infty$$

for any n, so we have the result for non-negative X.

For the general case, divide X into a non-negative and a negative part.

(End of proof) $\square$

Theorem 2.2.2) *(Optional stopping theorem)* Let X be a super-martingale and let S, T be *bounded* stopping times with $S \le T$ a.s. Then

$$\mathbb{E}[X_T] \le \mathbb{E}[X_S]$$

proof) Fix $n \ge 0$ such that $T \le n$ a.s. Then

$$
\begin{aligned}
X_T &= X_S + \sum_{S \le k < T} X_{k+1} - X_k \\
&= X_S + \sum_{k=0}^{n} (X_{k+1} - X_k) 1_{S \le k < T}
\end{aligned}
$$

Now $\{S \le k\}$ is in $\mathcal{F}_k$ and $\{T > k\}$ is in $\mathcal{F}_k$, so

$$
\begin{aligned}
\mathbb{E}[(X_{k+1} - X_k) 1_{S \le k < T}] &= \mathbb{E}[\mathbb{E}[(X_{k+1} - X_k) 1_{S \le k < T} | \mathcal{F}_k]] \\
&= \mathbb{E}[\mathbb{E}[X_{k+1} - X_k | \mathcal{F}_k] 1_{S \le k < T}]
\end{aligned}
$$

but since (X_n) was a super-martingale, $\mathbb{E}[X_{k+1} - X_k | \mathcal{F}_k] \le 0$ a.s. and therefore $\mathbb{E}[(X_{k+1} - X_k) 1_{S \le k < T}] \le 0$ a.s. Hence $\mathbb{E}(X_T) \le \mathbb{E}(X_S)$.

(End of proof) $\square$

⋆ Note that X is a sub-martingale *if and only if* $(-X)$ is a super-martingale, and that X is a martingale *if and only if* X and $(-X)$ are super-martingales. Hence, we obtain sub-martingale and martingale versions of the theorem :

$$\begin{aligned}&\text{If } (X_n) \text{ is a sub-martingale, } \mathbb{E}[X_T] \geq \mathbb{E}[X_S]\\&\text{If } (X_n) \text{ is a martingale, } \mathbb{E}[X_T] = \mathbb{E}[X_S]\end{aligned}$$

Theorem 2.2.3.) Let X be an adapted integrable process. Then the followings are equivalent.

(a) X is a super-martingale.

(b) for all bounded stopping times T and stopping time S,

$$\mathbb{E}(X_T|\mathcal{F}_S) \leq X_{S\wedge T} \quad \text{a.s.},$$

(c) for all stopping times T, the stopped process X^T is a super-martingale,

(d) for all bounded stopping times T and all stopping times S with $S \leq T$ a.s,

$$\mathbb{E}(X_T) \leq \mathbb{E}(X_S)$$

⋆ The theorem gives an inverse statement of the optional stopping theorem.

proof)

(a) ⇒ (b) Suppose X is a super-martingale and S, T are stopping times. Let $T \leq n$, for some $n < \infty$. Then

$$X_T = X_{S\wedge T} + \sum_{k=0}^{T}(X_{k+1} - X_k)1_{S\leq k<T} \cdots\cdots (*)$$

Let $A \in \mathcal{F}_S$. Then $A \cap \{S \leq k\} \in \mathcal{F}_k$ and $\{T > k\} \in \mathcal{F}_k$ so

$$\mathbb{E}[(X_{k+1} - X_k)1_{S\leq k<T}1_A] = \mathbb{E}[\mathbb{E}[(X_{k+1} - X_k)1_{S\leq k<T}1_A|\mathcal{F}_k]] \leq 0$$

and

$$\begin{aligned}\mathbb{E}[(X_T - X_{S\wedge T})1_A] =&\mathbb{E}[\sum_{n=0}^{T}(X_{k+1} - X_k)1_{S\leq k<T}1_A] \leq 0\\ \Rightarrow& \quad \mathbb{E}[X_T 1_A] \leq \mathbb{E}[X_{S\wedge T}1_A]\end{aligned}$$

But since this inequality is true for any $A \in \mathcal{F}_S$ and noting that $X_{S\wedge T} \in \mathcal{F}_S$), we see

$$\mathbb{E}[X_T|\mathcal{F}_S] \leq X_{S\wedge T} \quad \text{a.s.}$$

The implications (b)⇒(c) and (c)⇒(d) are obvious.

(d) ⇒ (a) Let $m \leq n$ and $A \in \mathcal{F}_n$. Set $T = m1_A + n1_{A^c}$. Then T is a stopping with $T \leq n$. Then

$$\mathbb{E}(X_n 1_A - X_m 1_A) = \mathbb{E}(X_n) - \mathbb{E}(X_T) \leq 0$$

(note, if $\omega \in A$ then $(X_n 1_A - X_m 1_A)(\omega) = X_n(\omega) - X_m(\omega)$ and 0 otherwise) so

$$\mathbb{E}[X_n|\mathcal{F}_m] \leq X_m$$

(End of proof) □

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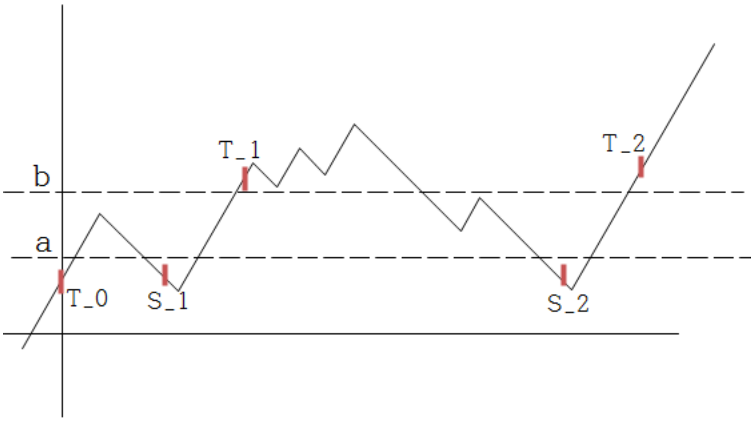
(17th October, Wednesday)

2.3. Doob's upcrossing inequality

- Let X be a random process and let $a, b \in \mathbb{R}$ s.t. $a < b$. Fix $\omega \in \Omega$. By an **upcrossing** of $[a, b]$ by $X(\omega)$, we mean an interval of times $\{j, j+1, \cdots, k\}$ s.t. $X_j(\omega) < a$, $X_k(\omega) > b$.
- Write $U_n[a, b](\omega)$ for the number of disjoint upcrossings contained in $\{0, 1, \cdots, n\}$, and $U_n[a, b] \nearrow U[a, b]$ as $n \to \infty$.



Theorem 2.3.1.) (Doob's upcrossing inequality) Let X be a *super-martingale*. Then

$$(b-a)\mathbb{E}[U[a, b]] \leq \sup_{n \geq 0} \mathbb{E}[(X_n - a)^-]$$

(Recall, $x^- = (-x) \vee 0$)

In fact, in this theorem, we prove $(b-a)\mathbb{E}[U_n[a, b]] \leq \mathbb{E}[(X_n - a)^-]$.

proof) Set $T_0 = 0$ and define recursively for $k \geq 0$,

$$S_{k+1} = \inf\{m \geq T_k : X_m < a\}, \quad T_{k+1} = \sup\{m \geq S_{k+1} : X_m > b\}$$

Note that if $T_k < \infty$, then $\{S_k, S_k + 1, \cdots, T_k\}$ is an upcrossing of $[a, b]$ by X, and T_k is the time of completion of the $k - th$ upcrossing. Also note that $U_n[a, b] \leq n$. For $m \leq n$, we have

$$\{U_n[a, b] = m\} = \{T_m \leq n < T_{m+1}\}$$

On this event,

$$X_{T_k \wedge n} - X_{S_k \wedge n} = \begin{cases} X_{T_k} - X_{S_k} \geq b - a & \text{if } k \leq m \\ X_n - X_{S_k} \geq X_n - a & \text{if } k \geq m+1, S_{m+1} \leq n \\ 0 \quad \text{otherwise} \end{cases}$$

Hence

$$\begin{aligned} \sum_{k=1}^{n} (X_{T_k \wedge n} - X_{S_k \wedge n}) &\geq (b-a)U_n[a, b] + X_n - a \\ &\geq (b-a)U_n[a, b] - (X_n - a)^- \end{aligned}$$

Since X is a super-martingale and $T_k \wedge n$ and $S_k \wedge n$ are *bounded stopping times* with $S_k \leq T_k$, by optional stopping theorem, we have

$$\mathbb{E}(X_{T_k \wedge n}) \leq \mathbb{E}(X_{S_k \wedge n})$$

By $\mathbb{E}(\sum_{k=1}^{n} (X_{T_k \wedge n} - X_{S_k \wedge n})) \leq 0$ we get

$$(b-a)\mathbb{E}(U_n[a, b]) \leq \mathbb{E}[(X_n - a)^-]$$

Apply monotone convergence, with $n \to \infty$, then we are done.

(End of proof) $\square$

This theorem does not seem to have any significance at the moment, but it will turn out to be important later on.

2.4. Doob's maximal inequalities.

Define $X_n^* = \sum_{k\geq n} |X_k|$

In the next two theorems, we see that the martingale(or sub-martingale) property allows us to obtain estimates on this X_n^* in terms of expectations for X_n.

Theorem 2.4.1) (Doob's maximal inequality) Let X be a *martingale or a non-negative sub-martingale.* Then for all $\lambda \geq 0$,

$$\lambda \mathbb{P}(X_n^* \geq \lambda) \leq \mathbb{E}(|X_n| 1_{\{X_n^* \geq \lambda\}}) \leq \mathbb{E}(|X_n|)$$

proof) If X is a martingale, then $|X|$ is a non-negative sub-martingale. It suffices to consider the case where X is a non-negative sub-martingale.

Set $T = \inf\{k \geq 0 : X_k \geq \lambda\} \wedge n$. Then T is a stopping time and $T \leq n$, so by optional stopping, has

$$\begin{aligned} \mathbb{E}(X_n) \geq \mathbb{E}(X_T) &= \mathbb{E}(X_T 1_{X_n^* \geq \lambda}) + \mathbb{E}(X_T 1_{X_n^* < \lambda}) \\ &= \mathbb{E}(\lambda 1_{X_n^* \geq \lambda}) + \mathbb{E}(X_n 1_{X_n^* < \lambda}) \end{aligned}$$

and

$$\mathbb{E}(X_n 1_{X^* \geq \lambda}) \geq \lambda \mathbb{P}(X_n^* \geq \lambda)$$

(End of proof) $\square$

Theorem 2.4.2) (Doob's L^p-inequality) Let X be a *martingale or a non-negative sub-martingale.* Then, for all $p > 1$ and $q = p/(p-1)$, we have

$$\| X_n^* \|_p \leq q \| X_n \|_q$$

proof) Again, it suffices to consider when X is a non-negative sub-martingale. Fix $k < \infty$. Then

$$\begin{aligned} \mathbb{E}[(X_n^* \wedge k)^p] &= \mathbb{E} \int_0^k p\lambda^{p-1} 1_{\{x_n^* \lambda\}} d\lambda \quad \text{(integration by parts)} \\ &= \int_0^k p\lambda^{p-1} \mathbb{P}(X_n^* \geq \lambda) d\lambda \quad \text{(Fubini)} \\ &\leq \int +0^k p\lambda^{p-2} \mathbb{E}(X_n 1_{X_n^* \geq \lambda}) d\lambda \quad \text{(Doob's maximal inequaltiy)} \\ &= \frac{p}{p-1} \mathbb{E}(X_n (X_n^* \wedge k)^{p-1}) \\ &\leq q \| X_n \|_p \| X_n^* \wedge k \|_p^{p-1} \quad \text{(Hölder's inequality)} \end{aligned}$$

Hence , $\| X_n^* \wedge k \|_p \leq q \| X_n \|_p$. Apply monotone convergence theorem with $k \to \infty$, then we have the desired result.

(End of proof) $\square$

Doob's maximal and L^p inequalities have different versions which apply under the same hypothesis to

$$X^* = \sum_{n\geq 0} |X_n|$$

since $X_n^* \nearrow X^*$. Letting $n \to \infty$ in Doob's maximal inequality gives

$$\lambda \mathbb{P}(X^* \geq \lambda) \lim_{n\to\infty} \lambda \mathbb{P}(X_n^* \geq \lambda) \leq \sup_{n\geq 0} \mathbb{E}(|X_n|)$$

We can then replace $\lambda \mathbb{P}(X^* > \lambda)$ by $\lambda \mathbb{P}(X^* \geq \lambda)$ by taking limits from the right in λ.

Similarly, for $p \in (1, \infty)$ by monotone convergence,

$$\| X^* \|_p \leq q \sup_{n\geq 0} \| X_n \|_p$$

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(19th October, Friday)

2.5. Doob's martingale convergence theorems

We are going to study three different martingale convergence theorems. They are all important.

- We say that a random process X is **L^p-bounded** if $\sum_{n\geq 0} \| X_n \|_p < \infty$.
- We say that X is **uniformly integrable** if

$$\sup_{n\geq 0} \mathbb{E}(|X_n| 1_{|X_n|>\lambda}) \to 0 \quad \text{as } \lambda \to \infty$$

- If X is L^p bounded for some $p > 1$, then this implies that X is uniformly integrable. This again implies that X is L^1 bounded. The first implication follows from Hölder inequality. The second implication is true because $\mathbb{E}(|X_n|) = \mathbb{E}(|X_n| 1_{|X_n|\leq\lambda}) + \mathbb{E}(|X_n| 1_{|X_n|>\lambda}) \leq \lambda + \mathbb{E}(|X_n| 1_{|X_n|>\lambda})$.

$$\boxed{X \text{ is } L^p\text{-bounded, } p > 1}$$
$$\Downarrow$$
$$\boxed{X \text{ is Uniformly integrable}}$$
$$\Downarrow$$
$$\boxed{X \text{ is } L^1\text{-bounded}}$$

Theorem 2.5.1) *(Almost sure martingale convergence theorem)* Let X be an *L^1-bounded super-martingale.* Then there exists an integrable and $\mathcal{F}_\infty$-measurable random variable X_∞ such that

$$X_n \to X \quad a.s. \text{ as } n \to \infty$$

proof) For a sequence of real numbers $(x_n)_{n\geq 0}$, as $n \to \infty$, $(x_n)_n$ either converges *or* $|x_n| \to \infty$, *or* $\liminf_n x_n < \limsup_n x_n$. In the last case, since the rationals are dense in $\mathbb{R}$, there exist $a, b \in \mathbb{Q}$ such that $\liminf x_n < a < b \limsup x_n$.

Set $\Omega_0 = \Omega_\infty \cap (\bigcap_{a,b\in\mathbb{Q}, a<b} \Omega_{a,b})$ where $\Omega_\infty = \{\liminf |X_n| < \infty\}$, $\Omega_{a,b} = \{U[a,b] < \infty\}$ (Recall that $U[a,b]$ is the number of upcrossings). Then $X_n(\omega)$ converges for all $\omega \in \Omega_0$. By Fatous' lemma,

$$\mathbb{E}(\liminf |X_n|) \leq \liminf \mathbb{E}|X_n| < \infty$$

so this implies $\mathbb{P}(\Omega_\infty) = 1$. By Doob's inequality, for $a < b$, has

$$(b-a)\mathbb{E}(U[a,b]) \leq |a| + \sup_{n\geq 0} \mathbb{E}|X_n| < \infty$$

and therefore $\mathbb{P}(\Omega_{a,b}) = 1$. Putting this together, we deduce that $\mathbb{P}(\Omega_0) = 1$, and we can find a random variable X_∞ defined by

$$X_\infty = \lim_{n\to\infty} X_n 1_{\Omega_0}$$

Then $X_n \to X_\infty$ a.s. Also X_∞ is $\mathcal{F}_\infty$-measurable and $|X_\infty| \leq \liminf |X_n|$ so $\mathbb{E}(|X_\infty|) < \infty$. Hence X_∞ is integrable.

(End of proof) □

Remark : Every non-negative integrable super-martingale is L^1-bounded, hence it converges a.s.

Theorem 2.5.2) *(L^1 martingale convergence theorem)* Let $(X_n)_{n\geq 0}$ be a *uniformly integrable martingale.* Then there exists a random variable $X_\infty \in L^1(\mathcal{F}_\infty)$ such that

$$X_n \xrightarrow{n\to\infty} X_\infty \quad \text{a.s. and in } L^1$$

Moreover, $X_n = \mathbb{E}(X_\infty|\mathcal{F}_n)$ a.s. for all $n \geq 0$.

Conversely, for all $Y \in L^1(\mathcal{F}_\infty)$, on choosing version X_n of $\mathbb{E}(Y|\mathcal{F}_n)$ for all n, we obtain a uniformly integrable martingale $(X_n)_{n\geq 0}$ such that

$$X_n \xrightarrow{n\to\infty} Y \quad \text{a.s. and in } L^1$$

We can think of this theorem as establishing the bijection unif. integrable martingale/a.s. $\leftrightarrow$ $L^1(\mathcal{F}_\infty)$

proof) Let $(X_n)_{n\geq 0}$ be a uniformly integrable martingale. By the almost sure martingale convergence theorem, there exists $X_\infty \in L^1(\mathcal{F}_\infty)$ s.t. $X_n \to X_\infty$ a.s. Since X is uniformly integrable, it also follows that $X_n \to X_\infty$ in L^1.(see PM, Thm 2.5.1. and 6.2.3.)

Next, for $m \geq n$,

$$\begin{aligned}\| X_n - \mathbb{E}(X_\infty|\mathcal{F}_n) \|_1 &= \| \mathbb{E}(X_m - X_\infty|\mathcal{F}_n) \|_1 \\ &= \| X_m - X_\infty \|_1 \to 0 \quad \text{as } m \to \infty\end{aligned}$$

Hence $X_n = \mathbb{E}(X_\infty|\mathcal{F}_n)$ a.s.

For the converse statement, suppose $Y \in L^1(\mathcal{F}_\infty)$ and let X_n be a version of $\mathbb{E}(Y|\mathcal{F}_n)$ for all n. Then $(X_n)_{n\geq 0}$ is a martingale by the tower property, and is uniformly integrable by **Lemma 1.5.1.** Hence there exists $X_\infty \in L^1(\mathcal{F}_\infty)$ such that $X_n \to X_\infty$ a.s. and in L^1. For all $n \geq 0$ and all $A \in \mathcal{F}_n$, we have

$$\mathbb{E}(X_\infty 1_A) = \lim_{m\to\infty} \mathbb{E}(X_m 1_A) = \lim_{n\leq m\to\infty} \mathbb{E}(\mathbb{E}(Y 1_A|\mathcal{F}_m)) = \mathbb{E}(Y 1_A)$$

where the second equality follows because $\mathbb{E}(X_m|\mathcal{F}_n) = \mathbb{E}(Y|\mathcal{F}_n)$. Now X_∞, $Y \in L^1(\mathcal{F}_\infty)$ and $\cup_n \mathcal{F}_n$ is a π-system generating $\mathcal{F}_\infty$. Hence, by Dynkin's lemma,

$$X_\infty = Y \quad \text{a.s.}$$

(End of proof) □

Theorem 2.5.3) *(L^p-martingale convergence theorem)* Let $p \in (1, \infty)$. Let $(X_n)_{n\geq 0}$ be an L^p-bounded martingale. Then there exists a random variable $X_\infty \in L^p(\mathcal{F}_\infty)$ s.t.

$$X_n \to X_\infty \quad \text{a.s. and in } L^p$$

Moreover, $X_n = \mathbb{E}(X_\infty|\mathcal{F}_n)$ a.s. for all $n \geq 0$.

Conversely, for all $Y \in L^p(\mathcal{F}_\infty)$, on choosing a version X_n of $\mathbb{E}(Y|\mathcal{F}_n)$ for all n, we obtain an L^p-bounded martingale such that $X_n \to Y$ a.s. and in L^p.

This is very similar to the statement of L^1-martingale convergence theorem. Indeed, the proof is also very similar.

proof) Let (X_n) be an L^p-bounded martingale. By *a.s. martingale convergence theorem*, there exists $X_\infty \in L^1(\mathcal{F}_\infty)$, $X_n \to X_\infty$ a.s.

By *Doob's L^p-inequality*, $\| X^* \|_p \leq q \sup_{n\geq 0} \| X_n \|_p < \infty$, where $X^* = \sup_{n\geq 0} |X_n|$. Also, since $|X_n - X_\infty|^p \leq (2X^*)^p$ for all n, we may apply dominated convergence theorem to deduce that $X_n \to X_\infty$ in L^p. Then $X_n = \mathbb{E}(X_\infty|\mathcal{F}_n)$ a.s. for all n, as in the L^1-convergence.

For the converse statement, suppose $Y \in L^p(\mathcal{F}_\infty)$ and let X_n be a version of $\mathbb{E}(Y|\mathcal{F}_n)$. Then $(X_n)_{n\geq 0}$ is a martingale by the tower property and by Jensen inequality,

$$\| X_n \|_p = \| \mathbb{E}(Y|\mathcal{F}_n) \|_p \leq \| Y \|_p$$

Let $X_n \to X_\infty$ a.s. and in L^P for $X_\infty \in L^p(\mathcal{F}_\infty)$, using the previous part. Then proceed as in the proof of L^1-convergence to prove that in fact $Y = X_\infty$ a.s.

(End of proof) □

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(22nd October, Monday)

Recall that, for a stopping time T and a random process X, X_T has been defined only on $\{T < \infty\}$. Given an almost sure limit X_∞ for X, we define $X_T = X_\infty$ on $\{T = \infty\}$. Then the optional stopping theorem extends to all stopping times for uniformly integrable martingales.

Theorem 2.5.5.) Let X be a uniformly integrable martingale and let T be any stopping time. Then $\mathbb{E}(X_T) = \mathbb{E}(X_0)$. Moreover, for all stopping time S and T, we have

$$\mathbb{E}(X_T|\mathcal{F}_S) = X_{S\wedge T} \quad \text{a.s.}$$

This theorem is an extension of Optional stopping theorem, **Theorem 2.2.2** and **Theorem 2.2.3**.

proof) By the optional stopping time theorem and **2.2.3**, when applied to the bounded stopping time $T \wedge n$, we have

$$\begin{aligned}\mathbb{E}(X_{T\wedge n}) &= \mathbb{E}(X_0)\\ \mathbb{E}(X_{T\wedge n}|\mathcal{F}_S) &= X_{S\wedge T\wedge n}\end{aligned}$$

In order to get the claim by letting $n \to \infty$, we need to prove $X_{T\wedge n} \to X_T$ *a.s.* and in L^1. This will imply that

$$\mathbb{E}(X_{T\wedge n}|\mathcal{F}_S) \to \mathbb{E}(X_T|\mathcal{F}_S) \quad \text{in } L^1$$

Claim : $X_{T\wedge n} \to X_T$ *a.s.* and in L^1
proof) By the L^1 martingale convergence theorem, there exists $X_\infty \in L^1(\mathcal{F}_\infty)$ s.t. $X_n \to X_\infty$ a.s. and in L^1 and $X_n = \mathbb{E}(X_\infty|\mathcal{F}_n)$. This implies $X_{T\wedge n} \to X_T$ *a.s.* as $n \to \infty$.(if $T < \infty$, the convergence trivial, and in the case $T = \infty$, the convergence justified the previous statement). Since $F_{T\wedge n} \subset F_n$, by **Theorem 2.2.3.** and the tower property we have

$$X_{T\wedge n} = \mathbb{E}(X_n|\mathcal{F}_{T\wedge n}) = \mathbb{E}(X_\infty|\mathcal{F}_{T\wedge n})$$

By **Lemma 1.5.1**, $(X_{T\wedge n})_{n\geq 0}$ is uniformly integrable. Hence

$$X_{T\wedge n} \to X_T \quad \text{in } L^1$$

(End of proof) □

Backward martingale

- A **backward filtration** $(\hat{\mathcal{F}}_n)_{n\geq 0}$ is a sequence of σ-algebras such that $\mathcal{F} \supset \hat{\mathcal{F}}_n \supset \hat{\mathcal{F}}_{n+1}$.
- This also defines $\hat{\mathcal{F}}_\infty = \bigcap_{n\geq 0} \hat{\mathcal{F}}_n$

Theorem 2.5.4.) *(Backward martingale convergence theorem)* For all $Y \in L^1(\mathcal{F})$, we have

$$\mathbb{E}(Y|\hat{\mathcal{F}}_n) \to \mathbb{E}(Y|\hat{\mathcal{F}}_\infty) \quad \text{a.s. and in } L^1 \quad \text{as } n \to \infty$$

Note that we do not need a uniformly integrability condition, because our assumption of backward filtration already implies uniform convergences.

proof) Write $X_n = \mathbb{E}(Y|\hat{\mathcal{F}}_n)$ for all $n \geq 0$. Fix $n \geq 0$, by the Tower property, $(X_{n-k})_{0\leq k\leq n}$ is a martingale for the filtration $(\hat{\mathcal{F}}_{n-k})_{0\leq k\leq n}$. For $a < b$, the number $U_n[0,\infty]$ of upcrossings of $[a,b]$ by $(X_k)_{0\leq k\leq n}$ equals the number of upcrossings of $[-b,-a]$ by the process $(-X_{n-k})_{0\leq k\leq n}$. Hence by (the note on) **Theorem 2.3.1**,

$$(b-a)\mathbb{E}(U_n[a,b]) \leq \mathbb{E}((X_0 - b)^+)$$

and so by monotone convergence,

$$(b-a)\mathbb{E}(U[a,b]) \leq \mathbb{E}((X_0 - b)^+) \leq \mathbb{E}(|X|) + |b| \leq \mathbb{E}(|Y|) + |b| < \infty$$

where the third inequality follows because of Jensen's inequality. Also,

$$\mathbb{E}(\liminf |X_n|) \leq \liminf \mathbb{E}|X_n| \leq \mathbb{E}|Y| < \infty$$

With these properties in hand, we can apply the same proof used to prove almost sure martingale convergence theorem to show that $\mathbb{P}(\hat{\Omega}_0) = 1$, where $\hat{\Omega}_0 = \{X_n$ converges as $n \to \infty\}$ - observe that $\hat{\Omega}_0 = \{\liminf_n |X_n| < \infty\} \cap (\bigcap_{a,b\in\mathbb{Q},a<b}\{U[a,b] < \infty\})$ and we see that each set in the intersection has measure 1, and therefore $\mathbb{P}(\hat{\Omega}_0) = 1$.

Set $X_\infty = 1_{\hat{\Omega}_0} \lim_{n\to\infty} X_n$. Then $X_\infty \in L^1(\hat{\mathcal{F}}_\infty)$ and $X_n \to X_\infty$ a.s. Now $(X_n)_{n\geq 0}$ is uniformly integrable (by **Lemma 1.5.1**), so $X_n \xrightarrow{L^1} X_\infty$. Finally, for all $A \in \hat{F}_\infty$, we have

$$\mathbb{E}((X_\infty - \mathbb{E}(Y|\hat{\mathcal{F}}_\infty))1_A) = \lim_{n\to\infty} \mathbb{E}((X_n - Y)1_A) = 0$$

This implies $X_\infty = \mathbb{E}(Y|\hat{\mathcal{F}}_\infty)$ a.s.

(End of proof) □

3. Applications of martingale theory

Sums of independent random variables

Let $S_n = X_1 + \cdots + X_n$, where $(X_n)_{n\geq 0}$ is a sequence of independent random variables.

Theorem 3.1.1) *(Strong Law of Large Numbers)* Let $(X_n)_{n\geq 0}$ be a sequence of independent identically distributed (*i.i.d*) integrable random variables. Set $\mu = \mathbb{E}(X_1)$. Then

$$S_n/n \to \mu \quad \text{a.s. and in } L^1$$

proof) Define $\hat{\mathcal{F}}_n = \sigma(S_m : m \geq n)$, $\mathcal{T}_n = \sigma(X_m : m \geq n+1)$ and $\mathcal{T} = \cap_{n\geq 1}\mathcal{T}_n$. Then $\hat{\mathcal{F}}_n = \sigma(S_n, \mathcal{T}_n)$ and $(\hat{\mathcal{F}}_n)_{n\geq 1}$ is a backward filtration. Since $\sigma(X_1, S_n)$ is independent of $\mathcal{T}_n$, we have

$$\mathbb{E}(X_1|\hat{\mathcal{F}}_n) = \mathbb{E}(X_1|S_n) \quad \text{a.s.}$$

For $k \leq n$ and all Borel sets B, we have

$$\mathbb{E}(X_k 1_{\{S_n\in B\}}) = \mathbb{E}(X_1 1_{\{S_n\in B\}})$$

by symmetry $(X_k, S_n) \overset{\mathrm{d}}{=} (X_1, S_n)$ in distribution, so $\mathbb{E}(X_k|S_n) = \mathbb{E}(X_1|S_n)$ a.s. But

$$\mathbb{E}(X_1|S_n) + \cdots + \mathbb{E}(X_n|S_n) = \mathbb{E}(S_n|S_n) = S_n \quad \text{a.s.}$$

so $\mathbb{E}(X_1|\hat{\mathcal{F}}_n) = S_n/n$ almost surely. Then by backward martingale convergence theorem, has $S_n/n \to Y$ a.s. and in L^1 for some random variable Y. Then $Y \in \mathcal{T}$. By Kolmogorov's 0-1 law [PM **Theorem 2.6.1**], Y is almost surely a constant. Hence

$$Y = \mathbb{E}(Y) = \lim \mathbb{E}(S_n/n) = \mu \quad \text{a.s.}$$

where the second equality follows from L^1 convergence $S_n/n \to Y$.

(End of proof) □

Since a.s. convergence implies convergence in probability, we have the following corollary.

Corollary 3.1.2) *(Weak law of large numbers)* Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. integrable r.v.. Set $\mu = \mathbb{E}(X_1)$. Then

$$\mathbb{P}(|\frac{S_n}{n} - \mu| > \epsilon) \to 0 \quad \text{as } n \to \infty \quad \forall \epsilon > 0$$

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(24th October, Wednesday)

3.2. Non-negative martingale and change of measure

- Given a random variable X, $\mathcal{F}$-measurable with $X \geq 0$ and $\mathbb{E}(X) = 1$, we can define a new probability measure for $\tilde{\mathbb{P}}$ on $\mathcal{F}$ by

$$\tilde{\mathbb{P}}(A) = \mathbb{E}(X 1_A) \quad \forall A \in \mathcal{F}$$

  Moreover, by [PM, Prop 3.1.4], given $\tilde{\mathbb{P}}$, this equation determines X uniquely, up to a.s. modification. We say $\tilde{\mathbb{P}}$ **has a density w.r.t.** $\mathbb{P}$ and X is a version of the density.

- Let $(\mathcal{F}_n)_{n\geq 0}$ be a filtration in $\mathcal{F}$ and assume $\mathcal{F} = \mathcal{F}_\infty$. Let $(X_n)_{n\geq 0}$ be an adapted random process, with $X_n \geq 0$ and $\mathbb{E}(X_n) = 1$ for all n. We can define, for each n, a new probability measure $\tilde{\mathbb{P}}_n$ on $\mathcal{F}_n$ by

$$\tilde{\mathbb{P}}_n(A) = \mathbb{E}(X_n 1_A) \quad \forall A \in \mathcal{F}_n$$

  Since we require each X_n to be $\mathcal{F}_n$-measurable, this equation determines X_n uniquely up to a.s. modification.

Proposition 3.2.1.) The measures $\tilde{\mathbb{P}}_n$ are consistent. That is

$$\tilde{\mathbb{P}}_{n+1}|\mathcal{F}_n = \tilde{\mathbb{P}}_n \quad \forall n \quad \textit{iff} \quad (X_n)_{n\geq 0} \quad \text{is a martingale}$$

Moreover, there is a measure $\tilde{\mathbb{P}}$ on $\mathcal{F}$, which has a density w.r.t $\mathbb{P}$ such that

$$\tilde{\mathbb{P}}|\mathcal{F}_n = \tilde{\mathbb{P}}_n \quad \forall n \quad \textit{iff} \quad (X_n)_n \quad \text{is a uniformly integrable martingale}$$

proof) (The proof was an exercise.) For the first point,

$$\begin{aligned} \tilde{\mathbb{P}}_n(A) = \tilde{\mathbb{P}}_{n+1}(A|\mathcal{F}_n) = \mathbb{E}(X_{n+1} 1_A|\mathcal{F}_n) = \mathbb{E}(X_{n+1}|\mathcal{F}_n) 1_A \quad \forall A \in \mathcal{F}_n \\ \Leftrightarrow \quad \mathbb{E}(X_{n+1}|\mathcal{F}_n) = X_n \quad \text{a.s.} \quad \Leftrightarrow \quad (X_n) \text{ is a martingale} \end{aligned}$$

For the second point, suppose $\tilde{\mathbb{P}}|\mathcal{F}_n = \tilde{\mathbb{P}}_n \quad \forall n$. Then $\tilde{\mathbb{P}}_n|\mathcal{F}_m = (\tilde{\mathbb{P}}|\mathcal{F}_n)|\mathcal{F}_m = \tilde{\mathbb{P}}_m$ whenever $n \geq m$, so we find that $(X_n)_n$ is a martingale. Since we assumed that $\mathbb{E}(X_n) = 1$ for all n, by almost everywhere martingale convergence theorem, we find a random variable X such that $X_n \to X$ a.s. Now for any $A \in \mathcal{F}$, we may find $N \geq 0$ such that $A \in \mathcal{F}_k$ for all $k \geq N$, so

$$\tilde{\mathbb{P}}(A) = \tilde{\mathbb{P}}(A|\mathcal{F}_k) = \tilde{\mathbb{P}}_k(A) = \mathbb{E}(X_k 1_A) \xrightarrow{k\to\infty} \mathbb{E}(X 1_A)$$

and therefore $\tilde{\mathbb{P}}(A) = \mathbb{E}(X 1_A)$. Hence for all $n \geq 0$, we have

$$\mathbb{E}(X 1_A|\mathcal{F}_n) = \tilde{\mathbb{P}}(A|\mathcal{F}_n) = \mathbb{E}(X_n 1_A) \quad \forall A \in \mathcal{F}_n$$

and therefore $X_n = \mathbb{E}(X|\mathcal{F}_n)$. This shows that $(X_n)_n$ is uniformly integrable martingale.

For the converse direction, assume that $(X_n)_n$ is uniformly integrable. Then by L^1-martingale convergence theorem, we may find $X \in \mathcal{F}_\infty$ such that $X_n \to X$ in L^1 and a.s. Define $\tilde{\mathbb{P}}(A) = \mathbb{E}(X 1_A)$. Then $\tilde{\mathbb{P}}(A|\mathcal{F}_n) = \mathbb{E}(\mathbb{E}(X 1_A|F_n)) = \mathbb{E}(X_n 1_A)$ for any $n \geq 0$ and $A \in \mathcal{F}_n$, and therefore $\tilde{\mathbb{P}}|\mathcal{F}_n = \tilde{\mathbb{P}}_n$.

(End of proof) □

Theorem 3.2.3) *(Radon-Nikodym theorem)* Let μ and ν be σ-finite measures on a measurable space $(E, \mathcal{E})$. Then the followings are equivalent :

(a) $\nu(A) = 0$ for all $A \in \mathcal{E}$ such that $\mu(A) = 0$, i.e. ν is **absolutely continuous** with respect to μ.

(b) There exists a measurable function f on E such that $f \geq 0$ and $\nu(A) = \mu(f 1_A)$ for all $A \in \mathcal{E}$.

The function f which is unique up to modification μ-a.e. is called (a version of) the **Radon-Nikodym derivative** of ν with respect to μ. We write $f = d\nu/d\mu$ almost surely.

We will give a proof for the case *where $\mathcal{E}$ is countably generated*. We assume there is a sequence $(G_n : n \in \mathbb{N})$ of subsets of E which generates $\mathcal{E}$. This holds, for example, whenever $\mathcal{E}$ is the Borel σ-algebra. of a topology with countable basis. A further martingale argument is required to prove the general case, but we omit it.

proof) The direction (b) $\Rightarrow$ (a) is obvious. So we aim to prove (a) $\Rightarrow$ (b)

By assumption, there is a countable partition of E by measurable sets on which both μ and ν are finite. (since μ, ν are σ-finite.) It suffices to show (b) holds on each of these sets, so we can reduce to the case where μ, ν are finite.

The case $\nu(E) = 0$ is clear, as we can just take $f \equiv 0$. So assume $\nu(E) > 0$. Then $\mu(E) > 0$ by (a). Write $\Omega = E$ and $\mathcal{F} = \mathcal{E}$ and consider the probability measures

$$\mathbb{P} = \mu/\mu(E) \quad \text{and} \quad \tilde{\mathbb{P}} = \nu/\nu(E) \quad \text{on } (\Omega, \mathcal{F})$$

It will suffice to show that there is a random variable $X \geq 0$ such that $\tilde{\mathbb{P}}(A) = \mathbb{E}(X1_A)$ for all $A \in \mathcal{F}$.

Set $\mathcal{F}_n = \sigma(G_k : k \leq n)$. There exists $m \in \mathbb{N}$ and a partition of Ω by events $A_1, \cdots, A_m$ such that $\mathcal{F}_n = \sigma(A_1, \cdots, A_m)$ (e.g. choose $A_1 = G_1$, $A_2 = G_2 \backslash G_1$, $A_3 = G_3 \backslash (G_1 \cup G_2)$ and so on). Set

$$X_n = \sum_{j=1}^{m} a_j 1_{A_j}$$

where $a_j = \tilde{\mathbb{P}}(A_j)/\mathbb{P}(A_j)$ if $\mathbb{P}(A_j) > 0$ and $a_j = 0$ otherwise. Then $X_n \geq 0$, $X_n \in \mathcal{F}_n$.

Observe that $(\mathcal{F}_n)_{n\geq 0}$ is a filtration and $(X_n)_{n\geq 0}$ is a non-negative martingale adapted to $(\mathcal{F}_n)_{n\geq 0}$(has to check this). We will show that $(X_n)_{n\geq 0}$ is *uniformly integrable*. Once shown this, by the L^1-martingale convergence theorem, there exists $X \geq 0$ such that $\mathbb{E}(X1_A) = \mathbb{E}(X_n 1_A)$ for all $A \in \mathcal{F}_n$. Define a probability measure $\mathbb{Q}$ on $\mathcal{F}$ by

$$\mathbb{Q}(A) = \mathbb{E}(X1_A) \quad \forall A \in \mathcal{F}$$

Then $\mathbb{Q} = \tilde{\mathbb{P}}$ on $\cup_n \mathcal{F}_n$ which is a π-system generating $\mathcal{F}$. Hence $\mathbb{Q} = \tilde{\mathbb{P}}$ on $\mathcal{F}$, by uniqueness of extension.[PM, Thm 1.7.1], which implies (b).

It remains to show that $(X_n)_n$ is uniformly integrable. Given $\epsilon > 0$, we can find $\delta > 0$ such that $\tilde{\mathbb{P}}(B) < \epsilon$ for all $B \in \mathcal{F}$ with $\mathbb{P}(B) < \delta$.(If not, there would be a sequence of sets $(B_n)_n \subset \mathcal{F}$ with $\mathbb{P}(B_n) < 2^{-n}$ and $\tilde{\mathbb{P}}(B) \leq \epsilon$ for all n. Then by Borel-Cantielli lemma, $\mathbb{P}(\limsup B_n) = 0$, but $\tilde{\mathbb{P}}(\limsup B_n) > \epsilon$, which contradicts (a)). Set $\lambda = 1/\delta$. Then by Markov inequality,

$$\mathbb{P}(X_n > \lambda) \leq \frac{\mathbb{E}(X_n)}{\lambda} = \frac{1}{\lambda} = \delta \quad \forall n$$

so $\mathbb{E}(X_n 1_{X_n > \lambda}) = \tilde{\mathbb{P}}(X_n > \lambda) < \epsilon$ for all n. Hence $(X_n)_n$ is uniformly integrable by its definition

(End of proof) $\square$

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(26th October, Friday)

3.3. Markov Chains

- Let E be a *countable set*. We identify each measure μ on E with $(\mu_x : x \in E)$ where $\mu_x = \mu(\{x\})$. Then for each function f on E write

$$\mu(f) = \mu f = \sum_{x \in E} \mu_x f_x \quad \text{(vector product)}$$

  where $f_x = f(x)$.

- A **transition matrix** on E is a matrix $P = (p_{xy} : x, y \in E)$ such that each row $(p_{xy} : y \in E)$ is a probability measure.

- Given a filtration $(\mathcal{F}_n)_{n\geq 0}$ and $(X_n)_{n\geq 0}$, and adapted process with values in E, we say that $(X_n)_{n\geq 0}$ is a **Markov chain with transition matrix** P if, for all $n \geq 0$, all $x, y \in E$ and all $A \in \mathcal{F}_n$ with $A \subset \{x_n = x\}$ and $\mathbb{P}(A) > 0$,

$$\mathbb{P}(X_{n+1} = y | A) = p_{xy}$$

Our notion of Markov chain depends on the choice of $(\mathcal{F}_n)_n$. The following results show that our definition agrees with the usual one with the choice of the natural filtration of $(X_n)_n$.

Proposition 3.3.1) Let $(X_n)_{n\geq 0}$ be a random process in E and take $\mathcal{F}_n = \sigma(X_k : k \geq n)$. Then the following are equivalent :

(a) $(X_n)_{n\geq 0}$ is a Markov chain with initial distribution μ and transition matrix P.

(b) For all n and all $x_0, x_1, \cdots, x_n \in E$,

$$\mathbb{P}(X_0 = x_0, X_1 = x_1, \cdots, X_n = x_n) = \mu_{x_0} p_{x_0 x_1} \cdots p_{x_{n-1} x_n}$$

Proposition 3.3.2) Let E^* denote the set of sequence $x = (x_n : n \geq 0)$ taking values in E and define $X_n : E^* \to E$ by $X_n(x) = x_n$. Set $\mathcal{E} = \sigma(X_k : k \geq 0)$. Let P be a transition matrix on E. Then, for each $y \in E$, there is a unique probability measure $\mathbb{P}_y$ on $(E^*, \mathcal{E}^*)$ such that $(X_n)_{n\geq 0}$ is a Markov chain with transition matrix P and starting from y.

proof) The choice of probability measure should be obvious from the transition matrix P. To show uniqueness, use Dynkin's lemma.

An example of a Markov chain in $\mathbb{Z}^d$ is the simple symmetric random walk with transition matrix

$$p_{xy} = \begin{cases} 1/2d & \text{if } |x-y| = 1 \\ 0 & \text{otherwise} \end{cases}$$

The following result shows a simple instance of a general relationship between Markov processes and martingale.

Proposition 3.3.3) Let $(X_n)_{n\geq 0}$ be an adapted process in E. TFAE :

(a) $(X_n)_{n\geq 0}$ is a Markov chain with transition matrix P.

(b) For all bounded functions f on E, the following process is a *martingale*

$$M_n^f = f(X_n) - f(X_0) - \sum_{k=0}^{n-1} (P-I) f(X_k)$$

proof) (exercise) (Be careful that $(P-I)f(X_n)$ is not $P-I$ applied to $f(X_n)$ but $(P-I)f$ applied to X_n.) Suppose that $(X_n)_n$ is a Markov chain. Then

$$\mathbb{E}(f(X_{n+1})|\mathcal{F}_n) = \mathbb{E}(\sum_{y\in E} f(X_{n+1}) 1_{X_{n+1}=y} |\mathcal{F}_n) = \sum_{y\in E} f(y) \mathbb{E}(1_{X_{n+1}=y}|\mathcal{F}_n)$$

Claim : $\mathbb{E}(1(X_{n+1} = y)|\mathcal{F}_n) = \sum_{x\in X} p_{xy} 1(X_n = x)$

proof) Observe that $\mathbb{E}(1_{X_{n+1}=y}|\mathcal{F}_n) = \mathbb{E}\big(\mathbb{E}(1(X_{n+1} = y)|X_n)|\mathcal{F}_n\big)$, so it is sufficient to prove that $\mathbb{E}(1(X_{n+1} = y)|X_n) = \sum_{x\in X} p_{xy} 1(X_n = x)$. The expression on the right hand side is clearly $\sigma(X_n)$-measurable. Also, for any $A = \{X_n = w\} \in \sigma(X_n)$

$$\mathbb{E}(\mathbb{E}(1_{X_{n+1}=y}|X_n))1_A) = \mathbb{E}(1_{X_{n+1}=y} 1_A) = \mathbb{P}(X_{n+1} = y, X_n = w) = p_{wy} \mathbb{P}(X_n = w)$$

and

$$\mathbb{E}(\sum_{x\in X} p_{xy} 1(X_n = x) 1_A) = \sum_{x\in X} p_{xy} \mathbb{P}(X_n = x, 1_A) = p_{wy} \mathbb{P}(X_n = w)$$

Since $\{\{X_n = w\} : w \in E\}$ generates $\sigma(X_n)$, we have the result.

Therefore, $\mathbb{E}(f(X_{n+1})|\mathcal{F}_n) = \sum_{x,y\in E} f(y) p_{xy} 1_{X_n=x} = P(f)(X_n)$ and therefore

$$\mathbb{E}(f(X_n+1) - f(X_0) - \sum_{k=0}^{n} (P-I) f(X_k)|\mathcal{F}_n) = f(X_n) - f(X_0) - \sum_{k=0}^{n-1} (P-I) f(X_k)$$

Now if $(M_n^f)_n$ is a martingale for any bounded function, then it follows that $\mathbb{E}(f(X_n+1)|X_n) = P(f)(X_n)$ for any bounded f and n, and therefore X_n is a Markov chain.

- A bounded function f on E is said to be **harmonic** (for the transition matrix P) if

$$P(f) = f \quad \text{i.e.} \sum_{y \in E} p_{xy} f_y = f_x \quad \forall x \in E$$

- If f is a *bounded harmonic function*, then $(f(X_n))_{n \geq 0}$ is a *bounded martingale*. Then by Doob's convergence theorem, $f(X_n)$ converges a.s. and in L^p for all $p < \infty$.

- More generally, for $D \subset E$, a *bounded function* f on E is **harmonic on** D if

$$\sum_{y \in E} p_{xy} f_y = f_x \quad \forall x \in D$$

- Let $\partial D = E \backslash D$ and fix a bounded function f on ∂D. Set $T = \inf\{n \geq 0 : X_n \in \partial D\}$ and define a function u on E by

$$u(x) = \mathbb{E}_x(f(X_T) 1_{T<\infty})$$

  where E_x is the unique probability measure of a Markov chain starting at $x \in E$, as defined in **Prop 3.3.2**.

Theorem 3.3.4) The function u is bounded, harmonic in D, and $u = f$ on ∂D. Moreover, if $\mathbb{P}_x(T < \infty) = 1$ for all $x \in D$, then u is the unique bounded extension of f which is harmonic in D.

proof) It is clear that u is bounded and $u = f$ on ∂D. For all $x, y \in E$ with $p_{xy} > 0$ under $\mathbb{P}_x$, conditional on $\{X_1 = y\}$, $(X_{n+1})_{n \geq 0}$ has distribution $\mathbb{P}_y$. So for $x \in D$, $u(x) = \sum_{y \in E} p_{xy} u(y)$ showing u is harmonic in D.

On the other hand, suppose that g is a bounded function harmonic in D such that $g = f$ on ∂D. Then $M = M^g$(where M is as defined in **Prop 3.3.3**) is a martingale and T is a stopping time, so M^T is also a martingale by optional stopping theorem. But $M_{T \wedge n} = g(X_{T \wedge n})$ so if $\mathbb{P}_x(T < \infty) = 1$ for all $x \in D$, then

$$M_{T \wedge n} \to g(X_T) = f(X_T) \quad \text{a.s.}$$

So by bounded convergence, for all $x \in D$,

$$g(x) = \mathbb{E}_x(M_0) = \mathbb{E}_x(M_{T \wedge n}) \to \mathbb{E}_x(f(X_T)) = u(x)$$

therefore $g(x) = u(x)$ for all $x \in E$.

(End of proof) □