Schramm-Loewner Evolution(SLE)

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Pre-requisites : 1. Advanced Probability, 2. Complex Analysis
Co-requisites : Stochastic calculus

(17th January, Thursday)

1 Introduction

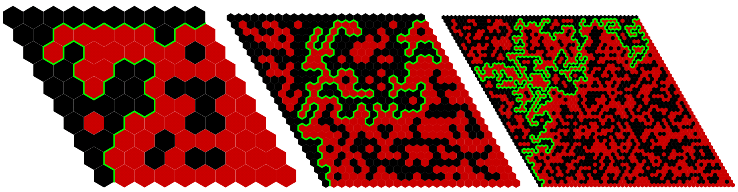
SLE is the random, fractal curve which lives in a domain $D \subset \mathbb{C}$. It was introduced in 1999 by Schramm in order to describe the scaling limits of different discrete models. This was a transformative idea in probability because it has led to new hits between probabilistic models and different areas of mathematics.

Examples

1. **Loop-erased random walk on** $\mathbb{Z}^2$: Consider a regular random walk $X(n)$ on $\mathbb{Z}^2$, randomly moving particle on the vertices of $\mathbb{Z}^2$ which in each time step goes up/down/left/right with probability $1/4$. It's loop-erasure is obtained by erasing the loops that $X(n)$ makes chronologically. The loop-erased random wak is important in probability, in that it builds uniform spanning tree (UST) with Wilson's algorithm.

   On the other hand, *Donsker's invariance principle* tells us that $X(n)/\sqrt{n} \to$two-dimensional BM. The question is, "What is the scaling limit of the loop-erased of X"? This is an SLE_2 (proved by Lawler-Schramm-Werner).

2. **Percolation on hexagoanl lattice** : Color each hexagon black or white independently

   each with probability $1/2$.

   Q. What is the large-scale behavior of the cluster boundaries?

   A. (Smirnov) They are SLE_6. (the proof is not very long, and not very technical)

   <u>Famous open question</u> : prove the same thing on any other planar lattice.

3. Suppose $X = B_1 + iB_2$, a complex Brownian motion with $B_1, B_2 \sim BM(\mathbb{R})$ independent. The *outer boundary* of $X[0,1]$ is the boundary of the unbounded complement of $\mathbb{C}\backslash X[0,1]$.

   Mandelbrot's conjecture : the "dimension" of the outer boundary of Brownian motion is 4/3.

   This was proved by Lawler-Schramm-Werner using SLE.

1.1 Sturcture

1. Will introduce background on complex analysis and define SLE.
2. Analyze basic properties of SLE, after seeing Ito's formula in Stochastic calculus.
3. Have a look on more recent devolopments of the subject.

2 Conformal Mappings

Suppose that $U, V \subset \mathbb{C}$ are domains, $f : U \to V$ is a map.

- Then f is **holomorphic** if $f'(z) = \lim_{w\to z}(f(w)-f(z))/(w-z)$ exists for all $z \in U$. A map is a **conformal transformation (or conformal equivalences, or just conformal)** if it is holomorphic and a bijection.
- $U \subset \mathbb{C}$ is **simply connected** if $\mathbb{C}\backslash U$ is connected.
- Denote unit disk $\mathbb{D} = \{|z| < 1\}$, the half plane $\mathbb{H} = \{im(z) > 0\}$.

Theorem) *(Riemann mapping theorem)* Suppose $U \subset \mathbb{C}$ is simply connected, $U \neq \mathbb{C}$, $z \in U$. Then there exists a unique conformal transformation $f : \mathbb{D} \to U$ with $f(0) = z$, $f'(0) \in \mathbb{R}_{>0}$.

proof) You can find the proof of this theorem in any standard complex analysis text. Or, see the extra problem provided at the first example sheet of this course.

Corollary) If $U, V \subset \mathbb{C}$ are simply connected, $U, V \neq \mathbb{C}$, $z \in U$, $w \in V$, then there exists a unique conformal transformation $f : U \to V$ with $f(z) = w$, $f'(0) \in \mathbb{R}_{>0}$.

proof) Map U, V to $\mathbb{D}$ using Riemann mapping theorem. *(End of proof)* $\square$

Important Examples :

1. Conformal tranformations of $\mathbb{D} \to \mathbb{D}$, $U, \mathbb{D}$, $z \in \mathbb{D}$, $f : \mathbb{D} \to \mathbb{D}$ given by

$$f(w) = \frac{w + z}{1 + \overline{z}w}$$

   is the unique conformal transformation $\mathbb{D} \to \mathbb{D}$, $0 \mapsto z$, $f'(0) \in \mathbb{R}_{>0}$.

   More generally, every conformal transformation $f : \mathbb{D} \to \mathbb{D}$ takes the form

$$f(w) = \lambda \frac{w - z}{\overline{z}w - 1}, \quad \lambda \in \partial\mathbb{D}, z \in \mathbb{D}$$

   Note that there are three real parameter family : λ has one parameter, z has two parameters.

2. The map $\mathbb{H} \to \mathbb{D}$ given by
$$f(z) = \frac{z-i}{z+i}$$
is a conformal transformation, "Layley transform".

3. The conformal transformations $\mathbb{H} \to \mathbb{H}$ are given by
$$f(z) = \frac{az+b}{cz+d}, \quad a,b,c,d \in \mathbb{R},\ ad-bc=1.$$
Note this also has three real parameter family. More generally, there is always a 3-real parameter family of conformal transformations $U \to V$, for $U, V \subset \mathbb{C}$ simply connected domains.

(diagram 2) $g_t(z) = \sqrt{z^2+4t}$ maps $\mathbb{H}\backslash[0, 2\sqrt{t}i] \to \mathbb{H}$. Note
$$\partial_t g_t(z) = \frac{4}{2\sqrt{z^2+4t}} = \frac{2}{g_t(z)}$$
so (g_t) is described by the ODE : $\partial_t g_t(z) = 2/g_t(z)$, $g_0(z) = z$.

Loewner's theorem : Suppose γ is a curve in $\mathbb{H}$ with $\gamma(0) = 0$ and is simple. Suppose there is a uniuqe conformal transformation $g_t : \mathbb{H}\backslash\gamma[0,t] \to \mathbb{H}$ with $g_t(z) - z \to 0$ as $z \to \infty$. Then
$$\partial_t g_t(z) = \frac{2}{g_t(z) - u_t}$$
where u_t is a continuous $\mathbb{R}$-valued function.

$SLE_t \leftrightarrow u_t = \sqrt{\kappa} B_t$, $t > 0$, B is a standard BM.

(22nd January, Tuesday)

Last time : conformal transformations, Reimann mapping theorem.

Important example : For each $t \geq 0$, let $\mathbb{H}_t = \mathbb{H}\backslash[0, 2i\sqrt{t}]$. Let $g_t : \mathbb{H}_t \to \mathbb{H}$ be the map $z \mapsto \sqrt{z^2+4t}$. Then g_t has the following properties.

1. g_t has the special property that $|g_t(z) - z| \to 0$ as $|z| \to \infty$. That is "g_t looks like the identity map at ∞".

2. $\partial_t g_t(z) = \frac{2}{g_t(z)}$, $g_0(z) = z$. So $(g_t(z))$ solves the ODE
$$\partial_t = \frac{2}{g_t(z)}, \quad g_0(z) = z \quad \cdots\cdots\cdots (\star)$$
For each $z \in \mathbb{H}$, $(\star)$ has a unique solution up until $\tau(z) = \inf\{t \geq 0 : |g_t(z)| = 0\}$. The maps (g_t) are characterized by $(\star)$.

   Idea : The ODE $(\star)$ encods the curve $\gamma(t) = 2\sqrt{t}i$. (Conversely, given the curve, this parametrization is the one that makes the characterizing ODE $(\star)$ nice)

Preview : Suppose γ is a simple curve in $\mathbb{H}$ with $\gamma(0) = 0$. For each $t \geq 0$, let g_t be the unique conformal transformation $\mathbb{H}_t \to \mathbb{H}$ with $|g_t(z) - z| \to 0$ as $z \to \infty$. *[Will prove existence/uniqueness of such a map later]* Then there exists continuous $\mathbb{R}$-values function w such that

$$\partial_t g_t(z) = \frac{2}{g_t(z) - w_t}, \quad g_0(z) = z$$

This is called the **Chordal Loewner equation**. We also have a correspondence

$$\text{Curves } \gamma \quad \leftrightarrow \quad \mathbb{R} - \text{valued, continuous functions } w$$

As noted earlier, the example above corresponds to $w \equiv 0$ and SLE_κ corresponds to $w_z = \sqrt{\kappa} B_t$ for $\kappa \geq 0$, B a standard Brownian motion.

Brownian Motion, Conformal Maps, Harmonic Functions

Definition) A **complex Brownian motion** B_t started from $z \in \mathbb{C}$ is a process of the form $B_t = B_t^1 + iB_t^2$ where B^1, B^2 are independent standard Brownian motions with $B_0 = z$.

These are objects intimately connected with harmonis functions and conformal maps.

Recall : $f = u + iv : \mathbb{C}_{x,y} \to \mathbb{C}_{u,v}$ is holomorphic *iff* u, v solve the *Cauchy-Riemann equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Consequence : f is holomorphic, then u, v are harmoic, *i.e.* $\triangle u = \triangle v = 0$.

In *Advanced* $\mathbb{P}$, we saw that harmonic functions and Brownian motions are closely related.

Theorem) Let u be a harmonic function on a bounded domain $D \subset \mathbb{C}$ which is continuous on $\overline{D}$. Let $\mathbb{P}_z$ be the law of a complex Brownian motion starting from $z \in D$ and $\tau = \inf\{t \geq 0 : B_t \notin D\}$. Then :

$$u(z) = \mathbb{E}_z[u(B_\tau)]$$

proof) Saw in *Advnced probability.* Will see another proof in *Stochastic Calculus* using Itô's formula.

The following two theorems are a good characterization of harmonic functions.(These properties are also true in higher dimensions, but we do not need those results for this course.)

Theorem) *(Mean-value property)* In the setting of the pervious theorem, if $z \in D$, $r > 0$, so that $B(z, r) \subset D$, then

$$u(z) = \frac{1}{2\pi} \int_0^\pi u(z + re^{i\theta}) d\theta$$

proof) Apply the previous theorem with $D = B(z, r)$ and u restrited to $B(z, r)$. Use that the first exit distribution of a complex BM from $B(z, r)$ starting from z is uniform on the circle $\partial B(z, r)$.

(End of proof) $\square$

Theorem) *(Maximum principle)* Suppose that u is harmonic on a domain $D \subset \mathbb{C}$. If u attains its maximum at an inteiror point z_0 then u is constant.

> **proof)** Let $D_0 = \{z \in D : u(z) = u(z_0)\}$. Then $D_0 \neq \phi$ as $z_0 \in D$. Since u is continuous, D_0 is closed. By the mean-value property, D_0 is open. Therefore, $D = D_0$.
>
> *(End of proof)* $\square$

As a consequence, we obtain a similar principle for holomorphic functions.

Theorem) *(Maximum modulus principle)* Let $D \subset \mathbb{C}$ be a domain, $f : D \to \mathbb{C}$ be holomorphic. If $|f|$ attains its maximum at an interior point, then f is constant.

> **proof)** Let $K \subset D$ be compact. By replacing f with $f + M$ for $M > 0$ sufficiently large, we can assume wlog that $|f| \neq 0$ on K. Note that $\log |f| = \Re(\log f)$ is harmonic on K. If $|f|$ attians its maximum on K, then so does $\log |f|$. Hence $\log |f|$ is constant on K and therefore so is $|f|$. Since $K \subset D$ was arbitrary, if $|f|$ has an interior maximum then $|f|$ is constant on D. So area$(f(D))$ =0.
>
> But If $\exists s z \in D$ with $f'(z) \neq 0$, then $f'(w) \neq 0$ for w in a neighbourhood of z. So
>
> $$\int |f'(w)|^2 > 0 \quad \Rightarrow \quad \text{area}(f(D)) > 0,$$
>
> a contardiction.
>
> *(End of proof)* $\square$

Let us conclude our review of complex analysis with an important lemma.

Schwarz lemma : Suppose $f : \mathbb{D} \to \mathbb{D}$ is holomorphic with $f(0) = 0$. Then $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$. If $|f(z)| = |z|$ for some $z \in \mathbb{D}\backslash\{0\}$, then $f(w) = e^{i\theta} w$ for some $\theta \in \mathbb{R}$. *i.e.* f is just a rotation.

> **proof)** Let
>
> $$g(w) = \begin{cases} f(w)/w & \text{if } w \neq 0 \\ f'(0) & \text{if } w = 0 \end{cases}$$
>
> Then g is holomorphic, $|g(w)| \leq 1$ for all $w \in \mathbb{D}$. By the maximum modulus principle, $|f(w)| \leq |w|$. If $|f(z)| = |z|$ for some $z \in \mathbb{D}\backslash\{0\}$, then g is constant, and therefore $f(w) = e^{i\theta} w$ for some $\theta \in \mathbb{R}$.
>
> *(End of proof)* $\square$

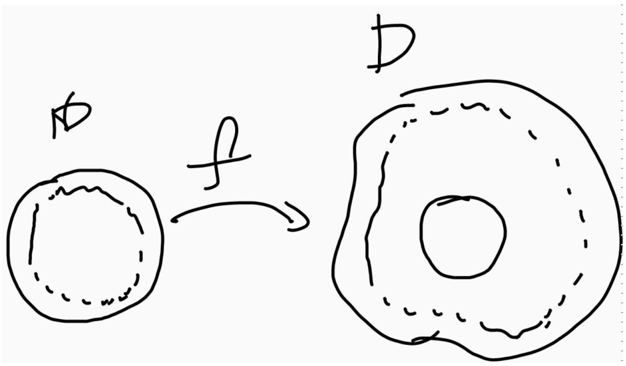
(24th January, Thursday)

Distortion estimates for conformal maps

Let $U = \{\text{conformal transformations } f : \mathbb{D} \to D\}$ where $D \subset \mathbb{C}$, $D \neq \mathbb{C}$, $0 \in D$, $f(0) = 0$, $f'(0) = 1$.

Theorem) *(Koebe-1/4)* If $f \in U$ and $0 < r \leq 1$, then $B(0, r/4) \subset f(r\mathbb{D})$.

If $f \in U$, since $f(0) = 0$, $f'(0) = 1$, we have $f(z) = \sum_{n=0}^{\infty} a_n z^n = z + \sum_{n=2}^{\infty} a_n z^n$.



Proposition 1) If $f \in U$, then $|a_2| \le 2$.

This is related to Bieberbach conjecture : $|a_n| \le n$ for all n. This is a famous complex analysis conjecture made in 1916. Loewner developed *Loewner flows* while working on it. The conjecture was proved by de Branges in 1985.

We will first prove the theorem assuming the propostion.

proof of Koebe-1/4) Suppose $f : \mathbb{D} \to D$ with $f \in U$. Fix $z_0 \notin D$. The goal is to show that $|z_0| \ge 1/4$.

Set $\tilde{f}(z) = \frac{z_0 f(z)}{z_0 - f(z)}$. Since $f \in U$, we have that

$$\tilde{f}(0) = \frac{z_0 f(0)}{z_0 - f(0)} = 0, \quad \tilde{f}'(0) = \frac{z_0^2 f'(0)}{z_0^2} = 1$$

Moreover, $\tilde{f}$ is a conformal transformation as it is a composition of conformal transformations. Hence, we have $\tilde{f} \in U$.

Write $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, then $\tilde{f}(z) = z + (a_2 + \frac{1}{z_0})z^2 + \cdots$. By the proposition, we have $|a_2| \le 2$ and $|a_2 + \frac{1}{z_0}| \le 2$. So we have that

$$2 \ge \left| a_2 + \frac{1}{z_0} \right| \ge \left| \frac{1}{z_0} \right| - |a_2| \ge \left| \frac{1}{z_0} \right| - 2$$

Re-arranging, $\left| \frac{1}{z_0} \right| \le 4$ and therfore $|z_0| \ge 1/4$. This gives the theorem for the case $r = 1$.

For general values of $r \in (0, 1]$, the theorem follows by replacing f with $f(rz)/r$.

(End of proof) $\square$

This theorem is useful in SLE, becuase it allows bound the derivative just using the geometry of the domain and the image. Conversely, one can bound the domain or the image just using the information of the derivative - often, dealing with derivative is much natural then dealing with the domain and the image. The following corollary captures this feature of the theorem very well.

Corollary) Suppose $D, \tilde{D} \subset \mathbb{C}$ are domains, $z \in D$, $\tilde{z} \in \tilde{D}$, $f : D \to \tilde{D}$ is a conformal transformaiton with $f(z) = \tilde{z}$. Then :

$$\frac{\tilde{d}}{4d} \le |f'(z)| \le \frac{4\tilde{d}}{d}$$

where $d = \mathrm{dist}(z, \partial D)$ and $\tilde{d} = \mathrm{dist}(f(z), \partial f(D))$.

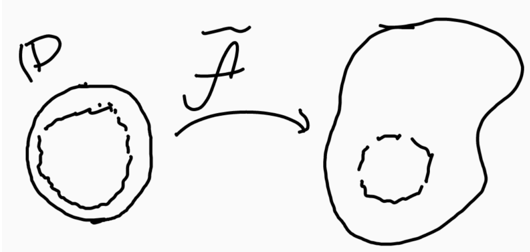


Figure 1: in proof of corollary

proof) By translation, can assume that $z = \tilde{z} = 0$. Let $\tilde{f}(w) = \frac{f(wd)}{f'(0)d}$. Then $\tilde{f} \in U$. So *Koebe-1/4 theorem* gives $B(0, r/4) \subset f(r\mathbb{D})$ for all $r \in [0,1]$.

For all $\epsilon > 0$, there eixsts $\delta > 0$ so that $\forall w \in \mathbb{D} \backslash (1-\delta)\mathbb{D}$ so that

$$\frac{\tilde{d}}{d|f'(0)|} \geq \left|\frac{f(wd)}{f'(0)d}\right| = |\tilde{f}(w)| \geq \frac{1}{4} - \epsilon$$

By re-arranging, get $\tilde{d} \geq (\frac{1}{4} - \epsilon)d|f'(0)|$. Send $\epsilon \to 0$, then we obtain the upper bound.

To obtain the lower bound, use the same arguement with f^{-1} in place of f and not that $|(f^{-1})'(f(z))| = |f'(z)|^{-1}$.

(End of proof) $\square$

Now we are just left to prove the propostion. However, the proposition is more trickier to prove than it looks. The proposition is proved using a series of methodology called *area theorm*. The next propostion is the first version of *area theorem*.

Proposition 2) If $f \in U$, then $\text{area}(f(\mathbb{D})) = \pi \sum_{n=1}^{\infty} n|a_n|^2$.

proof) Fix $r \in (0,1)$, let $\gamma(0) = f(re^{i\theta})$, $\theta \in [0, 2\pi]$. Then

$$\begin{aligned} \frac{1}{2i}\int_\gamma \overline{z}dz &= \frac{1}{2i}\int_\gamma (x-iy)(dx+idy) \\ &= \frac{1}{2i}\int_\gamma \Big[(x-iy)dx + (ix+y)dy\Big] \\ &= \frac{1}{2i}\int_{f(r\mathbb{D})} 2idxdy = \text{area}(f(r\mathbb{D})) \quad \textit{[Green's formula]} \end{aligned}$$

On the other hand,

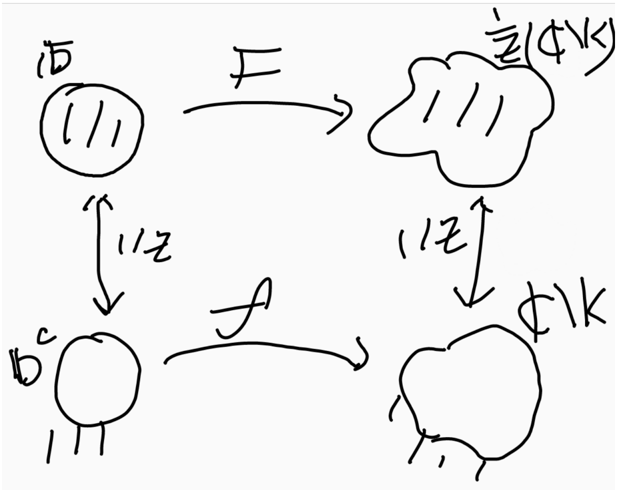
$$\begin{aligned} \frac{1}{2i}\int_\gamma \overline{z}dz &= \frac{1}{2i}\int_0^{2\pi} \overline{f(re^{i\theta})} f'(re^{i\theta}) ire^{i\theta} d\theta \\ &= \frac{1}{2i}\int_0^{2\pi} \sum_{n=1}^{\infty} \overline{a}_n r^n e^{-i\theta n} \sum_{n=1}^{\infty} a_n n r^{n-1} e^{i\theta(n-1)} \cdot ire^{i\theta} d\theta \\ &= \pi \sum_{n=0}^{\infty} r^{2n} |a_n|^2 \cdot n \end{aligned}$$

Send $r \to 1$, then we are done.

(End of proof) $\square$

Definition) Say that $K \subset \mathbb{C}$ compact is a **compact hull** if $\mathbb{C}\backslash K$ is connected and K consists of more than a single point. Let $\mathscr{H} = \{\text{compact hulls}\}$.

If $K \in \mathscr{H}$, *Riemann mapping theorem* implies that there exists a unique conformal transformation $f : \mathbb{C}\backslash\overline{\mathbb{D}} \to \mathbb{C}\backslash K$ which fixes ∞ and has positive derivative at ∞, *i.e.* $\lim_{z\to\infty} f(z)/z > 0$.



If F is the unique conformal transformation $\mathbb{D} \to \frac{1}{z}(\mathbb{C}\backslash K)$ (the map $z \mapsto 1/z$ applied to $\mathbb{D}\backslash K$) with $F(0) = 0, F'(0) > 0$, then

$$f(z) = \frac{1}{F(1/z)}, \quad f(z)/z = \frac{1}{zF(1/z)} \xrightarrow{z\to\infty} \frac{1}{F'(0)}$$

The following is an another version of the *area theorem.*

Proposition 3) Suppose $K \in \mathscr{H}$ and $f(z) = z + b_0 + \sum_{n=1}^{\infty} \frac{b_n}{z^n}$ be such that $f(\mathbb{D}) = K$. Then

$$\text{area}(K) = \pi\left[1 - \sum_{n=1}^{\infty} |b_n|^2 n\right]$$

In particular, $\sum_{i=1}^{\infty} n|b_n|^2 \leq 1$.

proof) Fix $r > 1$ and let $K_r = f(r\mathbb{D})$ and $\gamma = f(r\partial\mathbb{D})$. Then doing a bit of algebra,

$$\begin{aligned} \text{area}(K_r) =& \frac{1}{2i}\int_\gamma \overline{z}dz = \frac{1}{2i}\int_0^{2\pi} \overline{f(re^{i\theta})}f'(re^{i\theta}) \\ =& \left[r^2 - \sum_{n=1}^{\infty} n|b_n|^2 r^{-2n}\right] \\ \xrightarrow{r\to 1}& \text{ area}(K) = \pi\left[1 - \sum_{n=1}^{\infty} n|b_n|^2\right] \end{aligned}$$

(End of proof) $\square$

Next time, we are going to finish proving $|a_2| \leq 2$ and start deriving Loewner's equation.

(29th January, Tuesday)

Recap : $U = \{\text{conformal transforamtions } f : \mathbb{D} \to D, f(0) = 0, f'(0) = 1\}$.

Lemma) If $f \in U$, then there exists $h \in U$ odd such that $h(z)^2 = f(z^2)$

proof) Let

$$\tilde{f}(z) = \begin{cases} f(z)/z & z \neq 0 \\ f'(0) & z = 0 \end{cases}$$

Then $\tilde{f}$ is a non-zero holomorphic map on $\mathbb{D}$, so there exists a holomoprhic square-root of $\tilde{f}$, $g(z)^2 = \tilde{f}(z)$. Set $h(z) = zg(z^2)$. Then h is odd and

$$(h(z))^2 = z^2(g(z^2))^2 = z^2 \cdot \tilde{f}(z^2) = f(z^2)$$

Also, $h(0) = 0$, $h'(0) = 1$. We need to show h is injective.

: Suppose $h(z_1) = h(z_2)$, then $z_1 g(z_1^2) = z_2 g(z_2^2)$, so $z_1^2(g(z_1^2))^2 = z_2^2(g(z_2^2))^2$, and $f(z_1^2) = f(z_2^2)$, which implies $z_1^2 = z_2^2$ and so $g(z_1^2) = g(z_2^2)$. Recall $z_1 g(z_1^2) = z_2 g(z_2^2)$, so we have $z_1 = z_2$.

Having this, we see that $h \in U$.

(End of proof) □

proof of Proposition 1) Suppose that $f \in U$. Suppose that $h \in U$ as in the lemma. since h is odd, its series expansion only has odd powers of z :

$$h(z) = z + c_3 z^3 + c_5 z^5 + \cdots$$

Also

$$\begin{aligned} f(z^2) &= z^2 + a_2 z^4 + a_3 z^6 + \cdots = (h(z))^2 \\ &= (z + c_3 z^3 + c_5 z^5 + \cdots)^2 = z^2 + 2c_3 z^4 + \cdots \end{aligned}$$

so $c_3 = a_2/2$. Let $g(z) = 1/h\left(\frac{1}{z}\right)$. Then

$$\begin{aligned} g(z) &= \frac{1}{z^{-1} + (a_2/2)z^{-3} + \cdots} = \frac{z}{1 + (a_2/2)z^{-2} + \cdots} \\ &= z(1 - \frac{a_2}{2}z^{-2} + \cdots) = z - \frac{a_2}{2}z^{-1} + \cdots \end{aligned}$$

so $|a_2/2|^2 \leq 1$ by **Proposition 3**. We conclude $|a_2| \leq 2$

(End of proof) □

Half-plane capacity

Definition) Call $A \subset \mathbb{H}$ a **compact $\mathbb{H}$-hull** if $A = \overline{H} \cap \mathbb{H}$, $\mathbb{H} \backslash A$ are simply connected. Denote $\mathcal{Q} = \{$compact $\mathbb{H}$-hulls$\}$.

We are going to be interested in (1) "correct" notion of size of $A \in \mathcal{Q}$, (2) "correct" conformal transformation $\mathbb{H} \backslash A \to \mathbb{H}$.

Proposition 1) For each $A \in \mathcal{Q}$, there is a unique conformal transformation $g_A : \mathbb{H} \backslash A \to \mathbb{H}$ with $|g_A(z) - z| \to 0$ as $z \to \infty$ (that is, g_A looks like the identity at ∞).

Proposition 2) *(Schwarz reflection)* Let $D \subset \mathbb{H}$ be simply connected and $\phi : D \to \mathbb{H}$ be a conformal transformation which is bounded on bounded sets. Then ϕ extends by reflection to a conformal transformation on $D^* = D \cup \{\overline{z} : z \in D\} \cup \{x \in \partial\mathbb{H} : D$ is a neighbourhood of x in $\mathbb{H}\}$ by setting $\phi(\overline{z}) = \overline{\phi(z)}$.

proof) Can find a proof in any complex analysis text. Also will be practicing a harmnoic function version of the theorem in the example sheet.

proof of Propisition 1) By the *Riemann mapping theorem*, there exists a conformal transformation $g : \mathbb{H}\backslash A \to \mathbb{H}$. by post-composing g with a conformal transformation $\mathbb{H} \to \mathbb{H}$, we may assume that g fixes ∞. By Schwarz reflection, we can extend g to a conformal transformation $\mathbb{C}\backslash(\overline{A} \cup \{\overline{z} : z \in A\})$ by seetting $g(\overline{z}) = \overline{g(z)}$.

By performing a series expansion for $1/(g(1/z))$, we see that

$$g(z) = b_{-1}z + b_0 + \sum_{n=1}^{\infty} b_n/z^n$$

If $z \in \mathbb{R}\backslash\overline{A}$, $z = \overline{z}$ and $g(z) = g(\overline{z}) = \overline{g(z)}$, so g maps $\mathbb{R}\backslash\overline{A}$ into $\mathbb{R}$, so

$$b_{-1}z + b_0 + b_1 z^{-1} + \cdots = \overline{b_{-1}}z + \overline{b_0} + \overline{b_1}z^{-1} + \cdots$$

Set $g_A(w) = \frac{g(w)-b_0}{b_{-1}}$, then we have $|g_A(z) - z| \to 0$ as $z \to \infty$.

We are just left to show the uniqueness of such map. Suppose $\tilde{g}_A$ is another such conformal transformation. Then $\tilde{g}_A \circ g_A^{-1}$ is a conformal transforamtion $\mathbb{H} \to \mathbb{H}$. So by a standard propoerty of conformal map $\mathbb{H} \to \mathbb{H}$, we can find $a, b, c, d \in \mathbb{R}$ with $ad - bc = 1$ and

$$\tilde{g}_A \circ g_A^{-1}(z) = \frac{az+b}{cz+d}$$

Since $|\tilde{g}_A \circ g_A^{-1}(z) - z| \to 0$ as $z \to \infty$, we should have $\tilde{g}_A \circ g_A^{-1}(z) = z$, and hence $\tilde{g}_A = g_A$.

(End of proof) $\square$

Definition) If $A \in \mathcal{Q}$, the **half-plane capacity** of A is

$$\mathrm{hcap}(A) = \lim_{z\to\infty} z(g_A(z) - z)$$

Equivalently, $g_A(z) = z + \frac{\mathrm{hcap}(A)}{z} + \cdots$. hcap is a notion of "size" of A.

We will prove several properties of hcap that helps us interpret it as a "size function".

Properties : If $r > 0$, $x \in \mathbb{R}$, $A \in \mathcal{Q}$, then (i) $g_{rA} = rg_A(z/r)$, (ii) $g_{A+x} = g_A(z-x) + x$.

Indeed, $rg_A(z/r)$ and $g_A(z-x) + x$ are the unique conformal transformations $\mathbb{H}\backslash(rA) \to \mathbb{H}$ and $\mathbb{H}\backslash(A + x) \to \mathbb{H}$, respectively, which look like the identity at ∞.

As $rg_A(z/r) = z + \frac{r^2\mathrm{hcap}(A)}{z} + \cdots$, so

(i) $\mathrm{hcap}(rA) = r^2\mathrm{hcap}(A)$, *[scaling]*

(ii) $\mathrm{hcap}(A + x) = \mathrm{hcap}(A)$, *[translation invariant]*

If $A, \tilde{A} \in \mathcal{A}$, $A \subset \tilde{A}$, then

$$g_{\tilde{A}} = g_{g_A(\tilde{A}\backslash A)} \circ g_A$$

as the RHS is the unique confromal transformation $\mathbb{H}\backslash\tilde{A} \to \mathbb{H}$ which looks like the identity at ∞.

(31st January, Thursday)

Recap : $\text{hcap} = \lim_{z\to\infty} z(g_A(z)-z)$, *i.e.* $g_A(z) = z + \frac{\text{hcap}(A)}{z} + \cdots$.

If $A, \tilde{A} \in \mathcal{A}$, $A \subset \tilde{A}$, then

$$g_{\tilde{A}} = g_{g_A(\tilde{A}\backslash A)} \circ g_A$$

as the RHS is the unique confromal transformation $\mathbb{H}\backslash\tilde{A} \to \mathbb{H}$ which looks like the identity at ∞, so writing

$$\begin{aligned} g_A(z) &= z + \frac{\text{hcap}(A)}{z} + \cdots \\ g_{g_A(\tilde{A}\backslash A)}(z) &= z + \frac{\text{hcap}(g_A(\tilde{A}\backslash A))}{z} + \cdots \end{aligned}$$

we have

$$g_{\tilde{A}}(z) = z + \frac{\text{hcap}(\tilde{A})}{z} + \cdots = z + \frac{\text{hcap}(A) + \text{hcap}(g_A(\tilde{A}\backslash A))}{z} + \cdots$$

hence $\text{hcap}(\tilde{A}) = \text{hcap}(A) + \text{hcap}(g_A(\tilde{A}\backslash A))$. We will show in **Proposition 3** that $\text{hcap} \geq 0$, so this shows hcap is monotone.

Examples :

1. $z \mapsto \sqrt{z^2+4t}$ is the unique conformal transformation $\mathbb{H}\backslash[0, 2i\sqrt{t}] \to \mathbb{H}$ with $|\sqrt{z^2+4t} - z| \to 0$ as $z \to \infty$. Then

$$\sqrt{z^2+4t} = z + \frac{2t}{z} + \cdots$$

so $\text{hcap}([0, 2i\sqrt{t}]) = 2t$.

2. The map $z \mapsto z + \frac{1}{z}$ maps $\mathbb{H}\backslash\overline{\mathbb{D}} \to \mathbb{H}$ with $|z + \frac{1}{z} - z| \to 0$ as $z \to \infty$. So $\text{hcap}(\overline{\mathbb{D}} \cap \mathbb{H}) = 1$.

3. If $A \in Q$ with $A \subset r(\overline{\mathbb{D}} \cap \mathbb{H})$ for $r > 0$, has

$$\text{hcap}(A) \leq \text{hcap}(r(\overline{\mathbb{D}} \cap \mathbb{H})) = r^2 \text{hcap}(\mathbb{D} \cap \mathbb{H}) = r^2$$

Proposition 3) Suppose $A \in \mathcal{Q}$, B a complex Brownian motion, and $\tau = \inf\{t \geq 0 : B_t \notin \mathbb{H}\backslash A\}$. Then for all $z \in \mathbb{H}\backslash A$,

(i) $\text{Im}(z - g_A(z)) = \mathbb{E}_z[\text{Im}(B_\tau)]$.

(ii) $\text{hcap}(A) = \lim_{y\to\infty} y\mathbb{E}_{iy}[\text{Im}(B_\tau)]$. In particular, $\text{hcap}(A) \geq 0$.

(iii) $\text{hcap}(A) = \frac{2}{\pi}\int_0^\pi \mathbb{E}_{e^{i\theta}}[\text{Im}(B_\tau)] \sin\theta d\theta$, provided $A \subset \overline{\mathbb{D}} \cap \mathbb{H}$.

proof)

(i) Note $\phi(z) = \text{Im}(z - g_A(z))$ is harmonic on $\mathbb{H}\backslash A$ as $z - g_A(z)$ is holomorphic. As $g_A(z) = z + \frac{\text{hcap}(A)}{z} + \cdots$, it follows that $\phi(z)$ is bounded. Since $\text{Im}(g_A(z)) = 0$ when $z \in \partial(\mathbb{H}\backslash A)$, *(i)* follows from the representatioin of harmonic functions using Brownian motion.

(ii)

$$\begin{aligned}\mathrm{hcap}(A) &= \lim_{z\to\infty} z(g_A(z)-z) \\ &= \lim_{y\to\infty} iy(g_A(iy)-iy) \quad (\in \mathbb{R}) \\ &= \lim_{y\to\infty} y\mathrm{Im}(y-g_A(iy)) \quad \text{(took real part)} \\ &= \lim_{y\to\infty} y\mathbb{E}_{iy}[\mathrm{Im}(B_\tau)] \quad \text{(by part } \textit{(i)}\text{)}\end{aligned}$$

(iii) See Example Sheets.

Interlude

Conformal invariance of Brownian motion : up to a random time-change, the conformal image of a Brownian motion is a Brownian motion.

Theorem) Let $D, \tilde{D} \subset \mathbb{C}$ be domains and $f : D \to \tilde{D}$ be a conformal transformation. Let $B, \tilde{B}$ be complex Brownian motions starting from $z \in D$, $\tilde{z} = f(z) \in \tilde{D}$, respectively. Let

$$\tau = \inf\{t \geq 0 : B_t \notin D\}, \quad \tilde{\tau} = \inf\{t \geq 0 : \tilde{B}_t \notin \tilde{D}\}$$

Let $\tau' = \int_0^\tau |f'(B_s)|^2 ds$ and

$$\begin{aligned}\sigma(t) &= \inf\{s \geq 0 : \int_0^s |f'(B_r)|^2 dr = t\} \\ B'_t &= f(B_{\sigma(t)})\end{aligned}$$

Then $(\tau', B'_t : t < \tau') \overset{\mathrm{d}}{=} (\tilde{\tau}, \tilde{B}_t : t < \tilde{\tau})$.

proof) Use Stochastic Calculus, Itô's formula.

(End of proof) $\square$

Using the conformal invariance of Brownian motion, one can deduce that the first exit distribution of a Brownian motion is (see *Example Sheet*)

- from $\mathbb{D}$ starting from $z \in \mathbb{D}$ is $\frac{1}{2\pi}\left[\frac{1-|z|^2}{|e^{i\theta}-z|^2}\right]$ at $e^{i\theta}$. (Recall, this is just the Poisson kernel)
- from $\mathbb{H}$ starting at $z = x+iy$ is $\frac{1}{\pi}\left[\frac{y}{(x-u)^2+y^2}\right]$ at $u \in \partial\mathbb{H}$.

Proposition) Suppose that $A \in \mathfrak{Q}$, $\mathrm{rad}(A) = \sup\{|z| : z \in A\}$. If $x > \mathrm{rad}(A)$, $g_A(x) = \lim_{y\to\infty} \pi y\Big[\frac{1}{2} - \mathbb{P}_{iy}\Big[B_\tau \in (x,\infty)\Big]\Big]$.

If $x < -\mathrm{rad}(A)$, $g_A(x) = \lim_{y\to\infty} \pi y\Big[\frac{1}{2} - \mathbb{P}_{iy}\Big[B_\tau \in (-\infty,x)\Big]\Big]$.

proof) Suppose $A = \phi$. Then

$$\lim_{y\to\infty} \pi y\Big[\frac{1}{2} - \mathbb{P}_{iy}\Big[B_\tau \in (x,\infty)\Big]\Big] = \lim_{y\to\infty} \pi y\mathbb{P}_{iy}\Big[B_\tau \in [0,x]\Big]$$
$$= \lim_{y\to\infty} \pi y\int_0^x \frac{y}{\pi(u^2+y^2)}du = \lim_{y\to\infty} \pi y\int_0^{x/y} \frac{1}{\pi(1+u^2)}du = x = g_{\phi(x)}$$

Now suppose $A \in \mathcal{Q}$ with $A \neq \phi$. Write $g_A(z) = u_A(z) + iv_A(z)$ and $\sigma = \inf\{B_t \notin \mathbb{H}\}$. By the conformal invariance of Brownian motion,

$$\begin{aligned}\mathbb{P}_{iy}[B_\tau \in (x, \infty)] &= \mathbb{P}_{g_A(iy)}[B_\sigma \in (g_A(x), \infty)] \\ &= \mathbb{P}_{iv_A(iy)}[B_\sigma \in (g_A(x) - u_A(iy), \infty)] \quad \text{(translate by } -u_A(iy))\end{aligned}$$

As $y \to \infty$, $v_A(iy)/y \to 1$ and $yu_A(iy) \to 0$, as $g_A(iy) = u_A(iy) + iv_A(iy) = iy + \frac{\mathrm{hcap}(A)}{iy} + \cdots$. By computing this probabilty in the limit $y \to \infty$ as before, the proposition follows.

(End of proof) □

(5th February, Tuesday)

Corollary) If $A \in \mathcal{Q}$, $\mathrm{rad}(A) \leq 2$, then

$$\begin{aligned} x \leq &g_A(x) \leq x + \frac{1}{x} \quad \text{if } x > 1 \\ x + \frac{1}{x} \leq &g_A(x) \leq x \quad \text{if } x < -1 \end{aligned}$$

proof) See Example Sheet #1.

Lemma) Let $p(z, e^{i\theta})$ be the exit distribution of a Brownian motion started from $z \in \mathbb{H}\backslash\mathbb{D}$ at point $e^{i,\theta}$ for $\theta \in [0, \pi]$. Then it satisfies

$$p(z, e^{i\theta}) = \frac{2}{\pi}\frac{\mathrm{Im}(z)}{|z|^2}\sin\theta(1 + O(1/|z|))$$

proof) See *Example Sheet #1.*

Proposition) There exists $c > 0$ so that for all $A \in \mathcal{Q}$, $|z| \geq \mathrm{rad}(A)$, have that

$$\left| g_A(z) - z - \frac{\mathrm{hcap}(A)}{z} \right| \leq c \cdot \frac{\mathrm{rad}(A)\mathrm{hcap}(A)}{|z|^2}$$

[This tells us how good $z + hcap(A)/z$ is as an estimate of $g_A(z)$. Also later, we will see that this is what we all need to prove Loewner's theorem.]

proof) By scaling, we can assume that $\mathrm{rad}(A) = 1$. Let $h(z) = z + \frac{\mathrm{hcap}(A)}{z} - g_A(z)$. Let

$$v(z) = \mathrm{Im}(h(z)) = \mathrm{Im}(z - g_A(z)) - \frac{\mathrm{hcap}(A)\mathrm{Im}(z)}{|z|^2}$$

Let B be a complex Brownian motion and let $\sigma = \inf\{t \geq 0 : B_t \notin \mathbb{H}\backslash\overline{\mathbb{D}}\}$. For $\theta \in [0, \pi]$, let $p(z, e^{i\theta})$ be the density with respect to Lebesgue mesure of B_σ at $e^{i\theta}$ of Brownian motion started at z. Then

$$\mathrm{Im}(z - g_A(z)) = \int_0^\pi \mathbb{E}_{e^{i\theta}}[\mathrm{Im}(B_\tau)]p(z, e^{i\theta})d\theta$$

where $\tau = \inf\{t \geq 0 : B_t \notin \mathbb{H}\backslash A\}$ by the strong Markov property of B.

Recall by the previous lemma, $p(z, e^{i\theta}) = \frac{2}{\pi}\frac{\text{Im}(z)}{|z|^2}\sin\theta(1 + O(1/|z|))$ and by a previous result, $\text{hcap}(A) = \frac{2}{\pi}\int \mathbb{E}_{e^{i\theta}}[\text{Im}(B_\tau)]\sin\theta d\theta$. So

$$\begin{aligned}
|v(z)| &= \left|\text{Im}(z - g_A(z)) - \frac{\text{Im}(z)}{|z|^2}\text{hcap}(A)\right| \\
&= \Big|\int_0^\pi \mathbb{E}_{e^{i\theta}}[\text{Im}(B_\tau)]p(z, e^{i\theta})d\theta - \frac{2}{\pi}\int_0^\pi \mathbb{E}_{e^{i\theta}}[\text{Im}(B_\tau)]\sin\theta d\theta \cdot \frac{\text{Im}(z)}{|z|^2}\Big| \\
&\le c \cdot \frac{\text{hcap}(A)\text{Im}(z)}{|z|^3} \quad \text{by previous lemma}
\end{aligned}$$

where $c > 0$ is a constant. Since v is harmonic, by Example Sheet #1, Problem #8, can bound

$$|\partial_x v(z)| \le c\frac{\text{hcap}(A)}{|z|^3}, \quad |\partial_y v(z)| \le c\frac{\text{hcap}(A)}{|z|^3} \quad \cdots\cdots\cdots \ (\star)$$

Note that $h(iy) \to 0$ as $y \to \infty$, so

$$\begin{aligned}
|h(iy)| &= \Big|\int_y^\infty h'(is)ds\Big| \le \int_y^\infty |h'(is)|ds \\
&\le \int_y^\infty c\frac{\text{hcap}(A)}{s^3}ds \quad \text{by } (\star) \text{ and the Cauchy-Riemann equation} \\
&\le \tilde{c} \cdot \frac{\text{hcap}(A)}{y^2} \quad \text{for a constant } \tilde{c} > 0
\end{aligned}$$

By a similar argument, except integrating along a semicircle of radius $r \ge 2$, have that

$$\begin{aligned}
|h(re^{i\theta})| \le& \int_{\pi/2}^\theta |h'(re^{i\theta})|rd\theta \\
\le& c \cdot \frac{\text{hcap}(A)}{r^2} + |h(ir)| \\
\le& c' \cdot \frac{\text{hcap}(A)}{r^2}
\end{aligned}$$

(End of proof) □

Chordal Loewner equation

Theorem) *(Beurling estimate)* There exists a constant $c > 0$ so that the following is true. Let B be a complex Brownian motion $A \subset \overline{\mathbb{D}}$, with $0 \in A$, $A \cap \partial\overline{\mathbb{D}} \neq \phi$, connected. Then

$$\mathbb{P}_z[B([0,\tau]) \cap A = \phi] \le c|z|^{1/2}, \quad z \in \mathbb{D}$$

where $\tau = \inf\{t \ge 0 : B_t \notin \mathbb{D}\}$ and $B([0,\tau]) = \{B_t : t \in [0,\tau]\}$. *[Note that the constant c does not depend on the set A - this makes this estimate useful.]*

The worst case behavior is attined for $A = [0,1]$. To see this, consider a conformal map $\mathbb{D}\backslash[0,1] \to \mathbb{H}$ which fixes 0 behaves like the map $z \mapsto \sqrt{z}$ near 0. (needs a diagram)

Theorem is quite tricky to prove, so will skip it here. However, it is not difficult to show that the theorem holds with $\frac{1}{2}$ replaced by some constant $\alpha > 0$ (which is not obtained explicit, but does not depend on A).

Idea : fix $r > 0$, and let $C_r = B(0,r)\backslash\overline{B(0,r/2)}$. Complex Brownian motion starting from $-\frac{3}{4}ir$ has a positive chance p_0 of disconnecting 0 from ∞ before leaving C_r (that is, going around the origin not leaving C_r). Moreover, p_0 does not depend on r. [can deduce using the first exit disribution of Brownian motion from a disk when started at its center is uniform.]

Then partition the annulus $B(0,1)\backslash B(0,|z|)$ in to annuli

$$C_k = B(0,\partial^k|z|)\backslash\overline{B(0,2^{k-1}|z|)}$$

for $1 \le k \le \lceil \log_2 1/|z| \rceil$. For B to make it to $\partial\mathbb{D}$ without hitting A, it must have been that it did not disconnect 0 from ∞ in the time interval between when it first hits $\partial B(0,\frac{3}{4}2^{k-1}|z|)$ and when it subseqeuntly exits C_k. Therefore the probabiilty that B does not hit A is

$$\le (1-p_0)^{\lceil \log_2(1/|z|)\rceil} = |z|^\alpha \quad \text{for } \alpha = \alpha(p_0)$$

(7th February, Thursday)

(Announcement : Class on February 12 (next Tuesday) to be mdae up a later point in the term)

Today : We need one more estimate. Then prove Loewner's theorem, derive SLE.

Proposition) There exists a constant $C > 0$ so that the following holds. Suppose $A, \tilde{A} \in \mathcal{Q}$ with $A \subset \tilde{A}$, $\tilde{A}\backslash A$ connected. Then

$$\operatorname{diam}(g_A(\tilde{A}\backslash A)) \le C \begin{cases} (dr)^{1/2} & d \le r \\ \operatorname{rad}(\tilde{A}) & d > r \end{cases}$$

where $d = \operatorname{diam}(\tilde{A}\backslash A)$, $r = \sup\{\operatorname{Im}(z) : z \in \tilde{A}\}$

proof) By scaling, can assume that $r = 1$.

If $d > 1$, the bound follows since : $|g_A(z) - z| \le 3\operatorname{rad}(A)$ (*Example Sheet #1*). So

$$\operatorname{diam}(g_A(\tilde{A}\backslash A)) \le \operatorname{diam}(\tilde{A}) + 6\operatorname{rad}(A) \le 8\operatorname{rad}(\tilde{A})$$

Next assume that $d < 1$. Let B be a complex Brownian motion starting from iy, $y \ge 2$. Let $U = B(z,d)$ be chosen so that $U \supset \tilde{A}\backslash A$. Let $\tau = \inf\{t \ge 0 : B_t \notin \mathbb{H}\backslash A\}$. For $B[0,\tau]$ to hit U, it must

(1) Reach $B(z,1)$ before leaving $\mathbb{H}\backslash A$. This happens with probability $\le c/y$, $c > 0$ constant,

(2) Given (1) has happend, must reach U before leaving $\mathbb{H}\backslash A$. Beurling estimte says that this happens with probability $\le cd^{1/2}$, $c > 0$ constant.

Therefore, $\limsup_{y\to\infty} y\mathbb{P}_{iy}[B[0,\tau] \cap U \ne \phi] \le cd^{1/2}$. By the conformal invariance of Brownian motion, with $\sigma = \inf\{t \ge 0 : B_t \notin \mathbb{H}\}$, has

$$\limsup_{y\to\infty} y\mathbb{P}_{iy}[B[0,\sigma] \cap g_A(\tilde{A}\backslash A) \ne \phi] \le cd^{1/2}$$

Since $g_A(\tilde{A}\backslash A)$ is connected, by Example Sheet #1, has $\operatorname{diam}(g_A(\tilde{A}\backslash A)) \le cd^{1/2}$.

(End of proof) $\square$

Definition) Now suppose (A_t) is a family of compact $\mathbb{H}$-hulls with $A_0 = \phi$. Say that (A_t) is

(i) **non-decreasing** if $s \le t$ implies $A_s \subset A_t$.

(ii) **locally growing** if $\forall T > 0$, $\epsilon > 0$, $\exists \delta > 0$ so that $0 \le s \le t \le s + \delta T$ implies $\mathrm{diam}(g_s(A_t \backslash A_s)) \le \epsilon$.

(iii) **parameterized by capacity** if $\mathrm{hcap}(A_t) = 2t$ for all $t \ge 0$.

Let

$$\mathcal{A} = \{\text{families of compact } \mathbb{H}\text{-hulls satisfying (i),(ii),(iii)}\}$$
$$\mathcal{A}_T = \{\text{families of compact } \mathbb{H}\text{-hulls satisfying (i),(ii),(iii) defined on } [0,T]\}$$

Surely, $\mathcal{A} = \mathcal{A}_\infty$.

Example : Proposition implies that if γ is a simple curve in $\mathbb{H}$ starting from 0, then $A_t = \gamma[0,t]$ gives a family in $\mathcal{A}$. Also in Example sheet #1, Problem #11, will see that the can be parameterized by capacity.

Proposition) Suppose $(A_t) \in \mathcal{A}$, $g_t = g_{A_t}$. Then there exists $U : [0,\infty) \to \mathbb{R}$ continuous so that

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}, \quad g_0(z) = z$$

proof) First note that $\bigcap_{s>t} \overline{g_t(A_s)}$ consists of a single point since (A_t) is locally growing (hence $\overline{g_t(A_s)}$ are bounded, and therefore forms a nested family of compact sets). Set U_t to be this point. It is easy to see that U_t is continuous since (A_t) is locally growing.

Recall if $B \in \mathcal{Q}$, then $g_B(z) = z + \frac{\mathrm{hcap}(B)}{z} + O\Big(\frac{\mathrm{hcap}(B)\mathrm{rad}(A)}{|z|^2}\Big)$ and that if $x \in \mathbb{R}$, then $g_{B+x}(z) - x = g_B(z-x)$. So

$$g_B(z) = g_{B+x}(z+x) - x = z + \frac{\mathrm{hcap}(B)}{z+x} + \mathrm{hcap}(B)\mathrm{rad}(B+x)O\Big(\frac{1}{|z+x|^2}\Big) \quad \cdots\cdots\cdots \ (\star)$$

Fix $\epsilon > 0$. For $0 \le s \le t$, let $g_{s,t} = g_t \circ g_s^{-1}$. Note that since (A_t) is parametrized by capacity and $\mathrm{hcap}(A_{t+\epsilon}) = \mathrm{hcap}(A_t) + \mathrm{hcap}(g_t(A_{t+\epsilon} \backslash A_t))$, we have $\mathrm{hcap}(A_{t+\epsilon} \backslash A_t)) = 2\epsilon$. Apply $(\star)$ with $B = g_t(A_{t+\epsilon} \backslash A_t)$, $x = -U_t$, and use that $\mathrm{rad}(B - U_t) \le \mathrm{diam}(B)$ to see that

$$g_{t,t+\epsilon}(z) = z + \frac{2\epsilon}{z - U_t} + 2\epsilon \cdot \mathrm{diam}(B) O\Big(\frac{1}{|z - U_t|^2}\Big)$$

so

$$\begin{aligned} g_{t+\epsilon}(z) - g_t(z) &= (g_{t,t+\epsilon} - id) \circ g_t(z) \\ &= \frac{2\epsilon}{g_t(z) - U_t} + 2\epsilon \mathrm{diam}(g_t(A_{t+\epsilon} \backslash A_t)) O\Big(\frac{1}{|g_t(z) - U_t|^2}\Big) \end{aligned}$$

Divide both sides by ϵ, send $\epsilon \to 0$, get that $\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}$.

(End of proof) $\square$

Conversely, if $U : [0,\infty) \to \mathbb{R}$ is continuous, then we can let $(g_t)_{t \ge 0}$ to solve

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}, \quad g_0(z) = z$$

Then $A_t = \mathbb{H} \backslash \mathrm{Image}(g_t)$ is a family in $\mathcal{A}$. [*Example Sheet #1*]. Here, U is called the **"Loewner driving function"**. (However, the corresponding g is not always a curve)

Derivation of SLE

Definition) Suppose that (A_t) is a random family in $\mathcal{A}$ encoded by U. Let $\mathcal{F}_t = \sigma(U_s : s \leq t)$. We say that (A_t) satisfies the **conformal Markov property** if

(i) **("Markov")** Given $\mathcal{F}_t$, $(g_t(A_{t+s} \backslash A_t) - U_t)_{s \geq 0} \stackrel{\mathrm{d}}{=} (A_s)_{s \geq 0}$.

(ii) **("Conformal")** Is scale invariant in that $(rA_{t/r^2})_{t \geq 0} \stackrel{\mathrm{d}}{=} (A_t)$ for all $r > 0$. *[called "conformal" since the only conformal transformations $\mathbb{H} \to \mathbb{H}$ which fix $0, \infty$ are rescaling.]*

Theorem) *(Schramm)* If (A_t) satisfies the *conformal Markov property*, then there exists $\kappa > 0$ so that $U = \sqrt{\kappa} B$, where B is a standard Brownian motion.

Definition) The family (g_t) with $U = \sqrt{\kappa} B$ is called SLE$_\kappa$.
Later : We will see properties of g_t depending on the values of κ.

(14th February, Thursday)

(example class 1 at Feb 21 in MR15, 2pm or 4pm. Submit solutions to question 9 and 10 to pigeon hole of jason miller, unitl 5pm Feb 19)

Theorem) *(Schramm)* If (A_t) satisfies the *conformal Markov property*, then there exists $\kappa > 0$ so that $U = \sqrt{\kappa} B$, where B is a standard Brownian motion.

proof) Condition (i) of conformal Markov property is

$$\begin{aligned} & (U_{t+s} - U_t)_{s \geq 0} \stackrel{\mathrm{d}}{=} (U_s)_{s \geq 0} \quad \text{given } \mathcal{F}_t \\ \Leftrightarrow \quad & (U_t) \text{ has stationary independent incremetns} \\ \Rightarrow \quad & \exists \kappa \geq 0, a \in \mathbb{R} \text{ so that } U_t = \sqrt{\kappa} B_t + at \text{ where B is a standard BM} \end{aligned}$$

Condition (ii) of conformal Markov property is saying that

$$(rA_{t/r^2}) \stackrel{\mathrm{d}}{=} (A_t) \quad \Leftrightarrow \quad (rU_{t/r^2}) \stackrel{\mathrm{d}}{=} (U_t) \quad (U \text{ satisfies Brownian scaling})$$

so

$$\sqrt{\kappa} \tilde{B}_t + at \stackrel{\mathrm{d}}{=} rU_{t/r^2} = \sqrt{\kappa} r B_{t/r^2} + ar \cdot \frac{t}{r^2} \stackrel{\mathrm{d}}{=} \sqrt{\kappa} \tilde{B}_t + \frac{a}{r} t \quad (\tilde{B} \text{ a standard BM})$$

where the first equality makes use of property (ii). So $a = 0$.

(End of proof) $\square$

For $\kappa \geq 0$, SLE_κ is the random family of hulls encoded by $U_t = \sqrt{\kappa} B_t$, B a standard BM. Also, SLE_0 has $U(t) = 2\sqrt{t} i$.

Remarks :

(1) SLE_κ is generated by a continuous curves. This measn that for all $t \geq 0$, $\mathbb{H} \backslash A_t =$ unbounded compnent of $\mathbb{H} \backslash \gamma[0, t]$. This was proves by Rohde-Scharamm, and we will take this as an ssumption in this course.

(2) Behaviour of SLE_κ depend strongly on κ. We will show that SLE_κ is

(i) *Simple* if $\kappa \in (0, 4]$.

(ii) *Self-intersecting* if $\kappa \in (4, 8)$ - In this case, A_t corresponds to $\gamma[0, t]$ with holes filled in.

(iii) *Space-filling* if $\kappa \geq 8$.

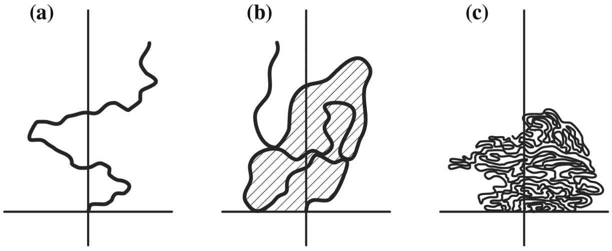


Fig. 2.7 Schematic pictures of $\mathrm{SLE}^{(D)}$ paths in **a** Phase 1 ($D \geq D_c = 2 \Leftrightarrow 0 < \kappa \leq \kappa_c = 4$), **b** Phase 2 ($\overline{D}_c = 3/2 < D < D_c = 2 \Leftrightarrow \kappa_c = 4 < \kappa < \overline{\kappa}_c = 8$), and **c** Phase 3 ($1 < D \leq \overline{D}_c = 3/2 \Leftrightarrow \kappa \geq \overline{\kappa}_c = 8$)

Figure 2: reference : M. Katori, "Bessel Processes, Schramm-Loewner Evolution and the Dyson Model" (2015)

(3) SLE_κ is singled out by the conformal Markov property. This comes from conjecture from the physics literature, about scaling limits of discrte models, *e.g. percolation*.

(4) Important tool : Stochastic calculus.

Stochastic Calculus Review

(Yippi!!!)
Basic object of study is continuous semi-martingale,

$$X_t = M_t + A_t$$

where M is a continuous local martingale and A a bounded variation process.

Important concepts :

(1) Stochastic integral

(2) Quadratic variation

(3) Itô formula

(4) Levy characterization.

(5) SDE's

General Setting : One has probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $(\mathcal{F}_t)$ satisfying the **"usual conditions"**.

(i) $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets.

(ii) $\mathcal{F}_t$ is right-continuous, *i.e.* $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$.

Stochastic Integral : $X_t = M_t + A_t$ continuous semi-martingale, H_t a prvisible process. Set

$$\int_0^t H_s dX_s = \int_0^t X_s dM + \int_0^t H_s dA_s$$

where the first integral is Itô integral for continuous local martingales and the second one is Lebesgue-Stieljes integral continuous with bounded variation. We have defined and constructed the stochastic integral in the spirit of the Riemann integral - it is not a measure. It is the extra cancellation in the definition of the Itô intgral which makes it converge.

Quadratic variation of a continous local martingale M_t is

$$[M]_t = \lim_{n\to\infty} \sum_{k=1}^{\lceil 2^n t\rceil - 1} (M_{(k+1)2^{-n}} - M_{(k)2^{-n}})^2$$

This is characterized by the property that it is the unique continuous process of bounded variation so that $M^2 - [M]$ is a continuous local martingale.

Quadratic variation of a continuous finite variation process vanishes. So

$$[X]_t = [M + A]_t = [M]_t$$

Also,

$$\left[\int_0^\cdot H_s dM_s\right]_t = \int_0^t H_s^2 d[M]_s$$

which is just a Lebesgue-Stieltjes integral.

Itô's formula : This is just a stochastic analogue of the *Fundamental Theorem of Calculus*. If $f \in C^2$, then

$$f(t) = f(0) + \sum_{k=1}^{n} (f(t_k) - f(t_{k-1}))$$

where $0 = t_0 < \cdots < t_n = t$ is a partition of $[0, t]$. By Taylor's formula,

$$\begin{aligned} f(t) =& f(0) + \sum_{k=1}^{n} \Big(f'(t_{k-1})(t_k - t_{k-1}) + o(t_k - t_{k-1})\Big) \\ \to& f(0) + \int_0^t f'(s)ds \quad \text{as } \max|t_k - t_{k-1}| \to 0 \end{aligned}$$

In a similar context, suppose B is a Brownian motion with $B_0 = 0$ and make Taylor expension with point B_t :

$$\begin{aligned} f(B_t) =& f(B_0) + \sum_{k=1}^{n} \Big(f'(B_{t_{k-1}}))(B_{t_k} - B_{t_{k-1}}) + \frac{1}{2} f''(B_{t_{k-1}})(B_{t_k} - B_{t_{k-1}})^2 + o((B_{t_k} - B_{t_{k-1}})^2)\Big) \\ \to& f(0) + \int_0^t f'(B_s)dB_s + \frac{1}{2}\int_0^t f''(B_s)ds \end{aligned}$$

since the summation over the quadratic terms is not negligible anymore in this setting.

We can make the general version accrodingly. If $X_t = M_t + A_t$ is a contiuous semi-martingale, with $f \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R})$ then

$$\begin{aligned} f(t, X_t) &= f(0, X_0) + \int_0^t \partial_s f(s, X_s)ds + \int_0^t \partial_x f(s, X_s)dX_s + \frac{1}{2}\int_0^t \partial_x^2 f(s, X_s)d[M]_s \\ &= f(0, X_0) + \left[\int_0^t \partial_s f(s, X_s)ds + \int_0^t \partial_x f(s, X_s)dA_s + \frac{1}{2}\int_0^t \partial_x^2 f(s, X_s)d[M]_s\right] + \int_0^t \partial_x f(s, X_s)dMS \end{aligned}$$

Example : For a standard Brownian motion B_t, $B_t^2 = B_0^2 + \int_0^t 2B_s dB_s + t$, so $B_t^2 - t = 2\int_0^t B_s dB_s$ is a martingale.

Lévy Characterization : If M is a continous local martingale, then M is a standard Brownian motion *iff* $[M]_t = t$.

> **proof)** Use Itô's formula with $e^{i\theta M_t + \frac{1}{2}\theta^2 [M]_t}$ - this is called an *exponential martingale.* So one can calculate the characteristic function of M_t from this.
>
> *(End of proof)* $\square$

Stochastic differential equations : $(\Omega, \mathcal{F}, \mathbb{P})$, $(\mathcal{F}_t)$ be satisfying the usual conditions. Suppose B is a standard Brownian motion adapted to $(\mathcal{F}_t)$. For function b, σ, we say a process (X_t) solves the SDE

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t$$

if and only if

$$X_t = X_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dB_s \quad \forall t \geq 0$$

One can prove existence/uniqueness when b, σ are Lipschitz functions. *[This will be done in the Stochastic Calculus course.]*

(19th Tuesday)

(Example class #1 at 1:30 Thursday 21st MR15 or 4:00 pm Friday 22nd MR5)

Goal : Estblish phases of SLE$_\kappa$.

Bessel Processes

Suppose that $X = (B^1, \cdots, B^d)$ is a d-dimensional Brownian motion. Let $Z_t = \left\|X_t\right\| = \left((B_t^1)^2 + \cdots + (B_t^d)^2\right)^{1/2}$. Itô's formula implies

$$\begin{aligned} Z_t^2 =& (B_t^2) + \cdots + (B_t^d)^2 \\ =& Z_0^2 + 2\int_0^t B_s^1 dB_s^1 + \cdots + 2\int_0^t B_s^t dB_s^d + td \end{aligned}$$

Set $Y_t = \int_0^t \frac{B_s^1 dB_s^1 + \cdots + B_s^d dB_s^d}{Z_s}$, then

$$Z_t^2 = Z_0^2 + 2\int_0^t Z_s dY_s + td$$

Note $[Y]_t = \int_0^t \frac{(B_s^1)^2 + \cdots + (B_s^d)^2}{Z_s^2} ds = t$ and also Y_t is a continuous local martingale. Therefore, the *Lévy characterization* implies Y_t is a standard Brownianm motion. So letting $\tilde{B}_t = Y_t$,

$$\begin{aligned} & Z_t^2 = Z_0^2 + 2\int_0^t Z_s d\tilde{B}_s + td \\ \Leftrightarrow \quad & dZ_t^2 = 2Z_s d\tilde{B}_s + dt \cdot d \end{aligned}$$

This is referred as **squared Besesl SDE with dimension d** and Z_t^2 is a **squared Bessel process with dimension** d and denoted BESQ^d.

Apply Itô's formula with $f(x)=\sqrt{x}$ and insert the expression for $d(Z_t^2)$ above to see that

$$\begin{aligned}
Z_t =& Z_0 + \frac{1}{2}\int_0^t Z_s^{-1} d(Z_s^2) - \frac{1}{8}\int_0^t Z_s^{-3} d[Z^2]_s \\
=& Z_0 + \tilde{B}_t + \frac{d}{2}\int_0^t Z_s^{-1} ds - \frac{1}{2}\int_0^t Z_s^{-1} ds \\
=& Z_0 + \tilde{B}_t + \frac{d-1}{2}\int_0^t Z_s^{-1} ds \\
\Leftrightarrow \quad dZ_t =& \frac{d-1}{2} Z_t^{-1} dt + d\tilde{B}_t
\end{aligned}$$

This is called **Bessel SDE of dimension** d and Z_t is a **Bessel process of dimension** d and denoted BES^d. For $d \in \mathbb{N}$, a BES^d corresponds to the modulus of a d-dimensional Brownian motion. However, the BES^d SDE has a solution for all $d \in \mathbb{R}$, up until hitting 0.

♡ **Claim :** If $d \in \mathbb{R}$, Z_t is a BES^d, then

- $d < 2$ then Z_t hits 0 almost surely.
- $d \geq 2$ then Z_t almost surely does not reach 0.

[If d = 2, the Bessel process gets arbitrarily close to 0 infinitely often without hitting 0 as a 2-diemnsional Brownian motion does not hit 0, but gets arbitrarily close]

proof) Consider Z_t^{2-d}. By Itô's formula and expression for dZ_t,

$$\begin{aligned}
Z_t^{2-d} =& Z_0^{2-d} + \int_0^t (2-d) Z_t^{1-d} dZ_t + \frac{1}{2}\int_0^t (2-d)(1-d) Z_t^{-d} d[Z]_t \\
=& Z_0^{2-d} + \int_0^t (2-d) Z_t^{1-d} d\tilde{B}_t + \int_0^t \frac{(2-d)(d-1)}{2} Z_t^{-1} dt + \frac{1}{2}\int_0^t (2-d)(1-d) Z_t^{-1} dt \\
=& Z_0^{2-d} + \int_0^t (2-d) Z_t^{1-d} d\tilde{B}_t
\end{aligned}$$

So Z_t^{2-d} is a continuous local martingale. For $a \in \mathbb{R}$, let $\tau_a = \inf\{t \geq 0 : Z_t = a\}$. Fix $0 < a < Z_0 < b$. Then $Z_{t\wedge\tau_a\wedge\tau_b}^{2-d}$ is a bounded martingale. We may apply *Optional Stopping Theorem* to see

$$Z_0^{2-d} = \mathbb{E}\Big[Z_{\tau_a\wedge\tau_b}^{2-d}\Big] = a^{2-d}\mathbb{P}[\tau_a < \tau_b] + b^{2-d}\mathbb{P}[\tau_b < \tau_a]$$

If $d < 2$, take a limit as $a \to 0$ to see that $Z_0^{2-d} = b^{2-d}\mathbb{P}[\tau_b < \tau_0]$. So

$$\Big(\frac{Z_0}{b}\Big)^{2-d} = \mathbb{P}[\tau_b < \tau_a] \to 0 \quad \text{as } b \to \infty$$

If $d > 2$, $\mathbb{P}[\tau_a < \tau_b] = \left(\frac{Z_0}{a}\right)^{2-d} - \left(\frac{b}{a}\right)^{2-d}\mathbb{P}[\tau_b < \tau_a] \to 0$ as $a \to 0$ for any b.

For the case $d = 2$, similar argument works with $\log(x)$ in place of x^{2-d}.

(End of proof) □

Back to SLE : Let $U_t = \sqrt{\kappa} B_t$, where B_t is a standrad Brownian motion and g solve the equation

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}, \quad g_0(z) = z$$

For each $x \in \mathbb{R}$, let $V_t^x = g_t(x) - U_t$. Let

$$\tau_x = \inf\{t \geq 0 : V_t^x = 0\} = \text{first time that } x \text{ is cut off from } \infty \text{ by } \gamma.$$

Then

$$\begin{aligned} dV_t^x =& d(g_t(x) - U_t) = \frac{2dt}{g_t(x) - U_t} - \sqrt{\kappa} dB_t \\ =& \frac{2dt}{V_t^x} - \sqrt{\kappa} dB_t \\ \Leftrightarrow \quad & d(V_t^x/\sqrt{\kappa}) = \frac{2/\kappa}{V_t^x/\sqrt{\kappa}} dt + d\tilde{B}_t, \quad (\tilde{B}_t = -B_t) \end{aligned}$$

So $(V_t^x/\sqrt{\kappa}) \sim \mathrm{BES}^d$ where $\frac{d-1}{2} = \frac{2}{\kappa}$ or equivalently $d = 1 + \frac{4}{\kappa}$. Note $d \geq 2$ *iff* $\kappa \leq 4$.

Proposition) SLE is simple for $\kappa \leq 4$, self-intersecting for $\kappa > 4$.

proof) Fix $t > 0$. Then $s \mapsto g_t(\gamma(t+s)) - U_t$ in an SLE_κ. This process intersects $\partial\mathbb{H}$ when $\kappa > 4$ and does not when $\kappa \leq 4$.

Suppose γ intersects itself at the time s. We can map map back at time between the two times that γ hits $\gamma(s)$. Then self-intersection time becomes a boundary hiting time. Therefore whether SLE_κ is self-intersecting is equivalent to whether it is boundary intersecting.

(End of proof) □

(21st February, Thursday)

Last time : We have seen that SLE_κ simple for $\kappa \leq 4$, self-intersecting for $\kappa > 4$. Furthremore, you will see that, in *Example sheet #2*, SLE_κ is space-filling for $\kappa \geq 8$.

Today, we will show how SLE_κ cuts regions off from ∞, i.e. its complememt is not connected for $\kappa \in (4, 8)$.

Recall our contructions, $\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}$, $g_0(z) = z$ where $U_t = \sqrt{\kappa} B_t$, $B \sim$ BM. We have also seen that if we let

$$V_t^x = g_t(x) - U_t$$

for $x \in \mathbb{R}$ then this is just a constant multiple of a Bessel process, $V_t^x/\sqrt{\kappa} \sim \mathrm{BES}^d$ with $d = 1 + \frac{4}{\kappa}$.

In order to characterize how an SLE_κ cuts regions from infinity, define a stopping time

$$\tau_x = \inf\{t \geq 0 : V_t^x = 0\} = \text{ first time that } x \text{ is cut off from infinity by the } \mathrm{SLE}_\kappa.$$

Fix $0 < x < y \in \mathbb{R}$. Let

$$g(x, y) = \mathbb{P}[\tau_x = \tau_y] = \text{probability that } x, y \text{ separated from } \infty \text{ simultaneously}$$

Our goal for now is to show that $g(x, y) > 0$ for $\kappa \in (4, 8)$ and $g(x, y) = 0$ for $\kappa \geq 8$. (In fact, SLE_κ for $\kappa \geq 8$ visits every point on $\partial\mathbb{H}$).

Observations :

1. $g(x, y) = g(1, y/x)$ as SLE_κ is scale-invariant. *[The scale invariance is just a general conseqeunce of being a conformal Markov property.]*

2. $g(1, r) \to 0$ as $r \to \infty$. This is because $\mathbb{P}[\tau_1 < t] \to 1$ as $t \to \infty$ and $\mathbb{P}[\tau_r < t] \to 0$ as $r \to \infty$ with t fixed.

Definition) We say that events A, B are **equivalent** if

$$\mathbb{P}[A \backslash B] = \mathbb{P}[B \backslash A] = 0,$$

i.e. A, B differ by an event with probability 0.

Proposition) $\{\tau_1 = \tau_r\}$, $r > 1$, is equivalent to the event

$$\daleth := \Big\{ \sup_{t<\tau_1} \frac{V_t^r - V_t^1}{V_t^1} < \infty \Big\}$$

proof) Indeed, $\{\tau_1 < \tau_r\}$ occurs, then the denominator hits 0 before the numerator so we can not have $\daleth$. So we see that $\{\tau_r = \tau_1\} \supset \daleth$.

On the other hand, for $m > 0$,

$$\mathbb{P}\Big[\tau_1 = \tau_r | \sup_{t<\tau_1} \frac{V_t^r - V_t^1}{V_t^1} \geq m\Big] = \mathbb{P}[\tau_1 = \tau_r | \sigma_m < \tau_1]$$

where $\sigma_m = \inf\{t \geq 0 : \frac{V_t^r - V_t^1}{V_t^1} \geq m\}$. While

$$\mathbb{P}[\tau_1 = \tau_r | \sigma_m < \tau_1] \leq g(1, m + 1)$$

by the Markov property and the scale-invariance of SLE_κ. But this converges to 0 as $r \to \infty$, so

$$\mathbb{P}\Big[\tau_1 = \tau_r | \sup_{t<\tau_1} \frac{V_t^r - V_t^1}{V_t^1} = \infty\Big] = 0$$

So $\{\tau_1 = \tau_r\} \subset \daleth \cup$ (a null set), which implies the claim.

(End of proof) □

Our problem is reformulated to showing $\mathbb{P}\Big[\sup_{t<\tau_1} \frac{V_t^r - V_t^1}{V_t^1} < \infty\Big]$ is 0 when $\kappa \in (4, 8)$ and 0 when $\kappa \geq 8$. Let

$$Z_t = \log\Big(\frac{V_t^r - V_t^1}{V_t^1}\Big)$$

and $d = 1 + 4/\kappa$. Note $d \leq 3/2$ for $\kappa \geq 8$ and $d > 3/2$ for $\kappa \in (4, 8)$. (Below this line, all V_t^s should be fixed to $V_t^s/\sqrt{\kappa}$) The *Itô derivative* of Z_t is

$$dZ_t = \Big[\Big(\frac{3}{2} - d\Big)\frac{1}{(V_t^1)^2} + \frac{d-1}{2}\frac{V_t^r - V_t^1}{(V_t^1)^2 V_t^r}\Big]dt - \frac{1}{V_t^1}dB_t$$

and $Z_0 = \log(r-1)$. To make the formula simple, we make time-change $\sigma(t) = \inf\{u \geq 0 : \int_0^u \frac{1}{(V_s^1)^2} ds = t\}$. then

$$t = \int_0^{\sigma(t)} \frac{1}{(V_s^1)^2} ds \quad \Leftrightarrow \quad dt = \frac{d\sigma(t)}{(V_{\sigma(t)}^1)^2}$$

If we define $\tilde{B}_t := -\int_0^{\sigma(t)} \frac{1}{V_s^1} dB_s$, it is a continuous local martingale with

$$[\tilde{B}]_t = \Big[-\int_0^{\sigma(\cdot)} \frac{1}{V_s^1} dB_s \Big]_t = \int_0^{\sigma(t)} \frac{1}{(V_s^1)^2} ds = t$$

so by *Lévy characterization of Brownian motions*, $\tilde{B}_t$ is a Brownian motion. Let $\tilde{Z}_t = Z_{\sigma(t)}$, then we have that

$$d\tilde{Z}_t = \Big[\Big(\frac{3}{2} - d \Big) + \frac{d-1}{2} \frac{V_{\sigma(t)^r} - V_{\sigma(t)^1}}{V_{\sigma(t)}^1} \Big] dt + d\tilde{B}_t$$

so

$$\tilde{Z}_t = \tilde{Z}_0 + \tilde{B}_t + \Big(\frac{3}{2} - d \Big) + \frac{d-1}{2} \int_0^t \frac{V_{\sigma(s)}^r - V_{\sigma(s)}^1}{V_{\sigma(s)}^r} ds \geq \tilde{Z}_t + \tilde{B}_t + \Big(\frac{3}{2} - d \Big) t$$

If $\kappa \geq 8$, then $d = 1 + \frac{4}{\kappa} \leq \frac{3}{2}$, so $\tilde{Z}_t \geq \tilde{Z}_0 + \tilde{B}_t$, and

$$\sup_t \tilde{Z}_t \geq \tilde{Z}_0 + \sup_t \tilde{B}_t = \infty \quad \text{a.s.}$$

and hence $\sup_{t<\tau_1} e^{Z_t} = \sup_{t<\tau_1} \frac{V_t^r - V_t^1}{V_t^1} = \infty$ a.s. Therefore we see that $g(x,y) = 0$ for all $0 < x < y$ if $\kappa \geq 8$.

Now suppose $\kappa \in (4,8)$. (We can in fact the exact value of $g(x,y)$, but we will make the calculation crude and just show that it is positive) Fix $\epsilon > 0$, and assume that $r = 1 + \frac{\epsilon}{2}$. Then $\tilde{Z}_0 = \log(r-1) = \log(\epsilon/2)$. Let $\tau = \inf\{t \geq 0 : \tilde{Z}_t = \log \epsilon\}$. Then

$$\begin{aligned}
\tilde{Z}_{t\wedge\tau} =& \tilde{Z}_0 + \tilde{B}_{t\wedge\tau} + \Big(\frac{3}{2} - d \Big) t \wedge \tau + \frac{d-1}{2} \int_0^{t\wedge\tau} \frac{V_{\sigma(s)}^r - V_{\sigma(s)}^1}{V_{\sigma(s)}^r} ds \\
\leq& \tilde{Z}_0 + \tilde{B}_{t\wedge\tau} + \Big(3/2 - d \Big) t \wedge \tau + \frac{d-1}{2} \int_0^{t\wedge\tau} e^{\tilde{Z}_s} ds \quad (\text{used } V_{\sigma(s)}^r \geq V_{\sigma(s)}^1) \\
\leq& \tilde{Z}_0 + \tilde{B}_{t\wedge\tau} + \Big(3/2 - d \Big) t \wedge \tau + \frac{d-1}{2} \epsilon \cdot t \wedge \tau \\
=& \tilde{Z}_0 + \tilde{B}_{t\wedge\tau} + \Big(\frac{3}{2} - d + \frac{d-1}{2} \epsilon \Big) t \wedge \tau
\end{aligned}$$

Let $Z_t^* = \tilde{Z}_0 + \tilde{B}_t + (\frac{3}{2} - d + \frac{d-1}{2}\epsilon)t$. Then above calculation shows $Z_{t\wedge\tau}^* \geq \tilde{Z}_{t\wedge\tau}$. Assume $\epsilon > 0$ is small so that

$$\frac{3}{2} - d + \frac{d-1}{2} \epsilon < 0$$

(such choice exists because $d > 3/2$ whenever $\kappa \in (4,8)$). Then Z_t^* is a Brownian motion with negative drift starting from $\log(\epsilon/2)$, so

$$\begin{aligned}
&\mathbb{P}[\sup_{t\geq 0} Z_t^* < \log \epsilon] > 0 \\
\Rightarrow &\mathbb{P}[\sup_{t\geq 0} \tilde{Z}_t < \log \epsilon] > 0 \\
\Rightarrow &\mathbb{P}[\sup_{t<\tau_1} e^{Z_t} < \epsilon] > 0 \\
\Rightarrow &g(1, 1 + \epsilon/2) > 0
\end{aligned}$$

But we wanted $g(x,y) > 0$ for all $0 < x < y$. Will finish this proof in Example sheet 2 (use the Markov property of SLE_κ to deduce $g(x,y) > 0$ for all $0 < x < y$ from $g(1, 1 + \frac{\epsilon}{2}) > 0$)

(26th February, Tuesday)

SLE on a simply connected domain

So far, we have only defined SLE_κ in $\mathbb{H}$ from 0 to ∞. If $D \subset \mathbb{C}$ is a simply connected domain, $x, y \in \partial D$ distinct, then there exitsst a conformal transformation $\phi : \mathbb{H} \to D$ with $\phi(0) = x$, $\phi(\infty) = y$. An SLE_κ in D from x to y is defined by taking it to be $\gamma = \phi(\tilde{\gamma})$ where $\tilde{\gamma}$ is a SLE_κ in $\mathbb{H}$ from 0 to ∞. *[See Example Sheet #2 to see why this is well-defined].*

Which SLE_κ should correspond ot the scaling limit of percolation?

Let $D \subset \mathbb{C}$ be simply connected, $x, y \in \partial D$ distinct. Consider $p = \frac{1}{2}$ percolation on the hexagonal lattic in D with hexagons of size $\epsilon > 0$. Color the hexagons on the clockwise/counterclockwise arc of ∂D from x to y and black/white. Then there exists a unique interface γ_ϵ from x to y with black/white on its left/right sides. *[I think this part needs more explanation]*

It was conjectured that the limit γ of γ_ϵ as $\epsilon \to 0$ exists in distribution and conformally invariant. This was proved, in the case of hexagonal lattice, by Smirnov in 2006. (The argument is completely elementary, so you might want to have a look on it). This means that if $\tilde{D}$ is another simply connected domian, $\tilde{x}$, $\tilde{y} \in \partial\tilde{D}$ are distinct, $\psi : D \to \tilde{D}$ a conformal transformation with $\psi(x) = \tilde{x}$, $\psi(y) = \tilde{y}$, then $\psi(\gamma) \overset{\mathrm{d}}{=}$ scaling limit of percolation on $\tilde{D}$.

Moreover, percolation satisfies a natural Markov property, the conditional law of γ_ϵ given $\gamma_\epsilon|_{[0,t]}$ is the same as that of percolation in the remaining domain. That is, we only need to see the hexagons adjacent to γ_ϵ to generate γ_ϵ.

These two properties say that γ should satisfy *Schramm's conformal Markov characterization* of SLE_κ. So γ is an SLE_κ for *some* κ. So the question is, which value of κ fits in? We will now go back to the continuum case and see that $\kappa = 6$ is the only possible choice that makes sense. (So we are not proving the actual convergence, but just seeing that the if the convergence is made, then the convergence is made to $\kappa = 6$)

A special property of percolation is **"locality"**. That is,

- Suppose $D \subset \mathbb{H}$ is simply connected, and $0 \in \partial D$.
- A percolation exploration in D with black/white boundary conditions on $\mathbb{R}_- / \mathbb{R}_+$ up until hitting $\partial D \backslash \partial \mathbb{H} \overset{\mathrm{d}}{=}$ percolation exploartion in $\mathbb{H}$ with the same boundary condtions and stopped at the same time.

Which SLE_κ has the analogous property? Suppose γ is an SLE_κ in $\mathbb{H}$ from 0 to ∞ stopped upon hitting $\partial D \backslash \partial \mathbb{H}$. We want γ to have same distribution as SLE_κ in D stopped at the same time. Equivalently if $\psi : D \to \mathbb{H}$ is a conformal transforamtion with $\psi(0) = 0$, $\psi(y) = \infty$, we want $\psi(\gamma)$ to be an SLE_κ in $\mathbb{H}$ from 0 to ∞ upon hitting $\psi(\partial D \backslash \partial \mathbb{H})$.

Suppose D and ψ are as before. Suppose that (A_t) is a locally growing family of compact $\mathbb{H}$ hulls with $A_0 = \phi$ and $\text{hcap}(A_t) = 2t$ for all $t \geq 0$. Let $\tilde{A}_t = \psi(A_t)$. Then $(\tilde{A}_t)$ is a locally growing family of compact $\mathbb{H}$ hulls with $\tilde{A}_0 = \phi$. Let $\tilde{a}(t) = \text{hcap}(\tilde{A}_t)$. In *Example Sheet #2, Problem #3*, you will see that

"If $\tilde{g}_t = g_{\tilde{A}_t}$ is the unique conformal transforamtion $\mathbb{H}\backslash\tilde{A}_t \to \mathbb{H}$ with $\tilde{g}_t(z) - z \to 0$ as $z \to \infty$, then

$$\partial_t \tilde{g}_t(z) = \frac{\partial_t \tilde{a}(t)}{\tilde{g}_t(z) - \tilde{u}(t)}, \quad \tilde{g}_0(z) = z$$

where $\tilde{u}_t = \psi_t(u_t)$, $\psi_t = \tilde{g}_t \circ \psi \circ g_t^{-1}$, u_t the Loewner driving function for (A_t) and $g_t = g_{A_t}$. Also, $\tilde{a}(t) = \int_0^t 2(\psi_s'(u_s))^2 ds$."

Proposition) The maps (ψ_t) satisfy

$$\partial_t \psi_t(z) = 2\Big[\frac{(\psi_t'(u_t))^2}{\psi_t(z) - \psi_t(u_t)} - \psi_t'(u_t)\frac{1}{z - u_t}\Big]$$

At $z = u_t$, $\partial_t \psi_t(u_t) = \lim_{z \to u_t} \psi_t(u_t) = -3\psi_t''(u_t)$.

proof) We have that

$$\begin{aligned}\partial_t \psi_t(z) =& \partial_t\Big(\tilde{g}_t(\psi(g_t(z)))\Big) = (\partial_t \tilde{g}_t)(\psi(g_t^{-1}(z))) + \tilde{g}_t'(\psi(g_t(z)))\psi(g_t^{-1}(z)) \cdot \partial_t g_t^{-1}(z) \\ =& \frac{2(\psi_t'(u_t))^2}{\psi_t(z) - \psi_t(u_t)} - \psi_t'(z)\frac{2}{z - u_t}\end{aligned}$$

Note,

$$\partial_t(g_t^{-1}(g_t(z))) = \partial_t(id(z)) = 0 = \partial_t g_t^{-1}(g_t(z)) + (g_t^{-1})'\frac{2}{g_t(z) - u_t}$$

and this gives formula for $\partial_t g_t^{-1}$, whenever $z \neq u_t$.

On Example Sheet #2, will check the result for $z = u_t$.

(End of proof) $\square$

Next time, we will show that $\tilde{u}_t$ is a martingael *iff* $u_t = \sqrt{6}B_t$ for a Brownian motion B.
The next topic will be to argue that self-avoding walk *iff* $\text{SLE}_{8/3}$.

(28th February, Thursday)

Goal : show that SLE_6 is singles out by the property that if $\gamma \sim \text{SLE}_\kappa$ in $\mathbb{H}$ from 0 to ∞, then $\psi(\gamma)$ is an SLE_6 in $\mathbb{H}$ from 0 to ∞, up until hitting $\psi(\partial D\backslash\partial\mathbb{H})$.
Recall our notations $(A_t) \in \mathcal{A}$, $\tilde{A}_t = \psi(A_t)$, $\tilde{g}_t = g_{\tilde{A}_t}$. In *Example sheet #2*, will show that

$$\partial_t \tilde{g}_t(z) = \frac{\partial_t \tilde{a}(t)}{\tilde{g}_t(z) - \tilde{u}_t}, \quad \tilde{g}_0(z) = z$$

where u_t is the Loewner driving function for (A_t), $\tilde{u}_t = \psi_t(u_t)$, $\tilde{a}(t) = \int_0^t 2(\psi_s'(u_s))^2 ds$. Also recall that we have seen that

Proposition)

$$\begin{aligned}\psi_t =& \tilde{g}_t \circ \psi \circ g_t^{-1}, \\ \partial_t \psi_t(u_t) =& \lim_{z \to u_t} \partial_t \psi_t(z) = -3\psi_t''(u_t).\end{aligned}$$

Now by Itô's formula,

$$\begin{aligned}
d\tilde{u}_t =& d\psi_t(u_t) \\
=& (\partial_t\psi_t(u_t) + \frac{\kappa}{2}\psi_t''(u_t))dt + \sqrt{\kappa}\psi_t'(u_t)dB_t \\
=& \frac{\kappa-6}{2}\psi_t''(u_t)dt + \sqrt{\kappa}\psi_t'(u_t)dB_t \quad \text{(by Proposition)}
\end{aligned}$$

Make time change $\sigma(t) = \inf\{u \geq 0 : \int_0^u (\psi_s'(u_s))^2 ds = t\}$. Then $\mathrm{hcap}(\tilde{A}_{\sigma(t)}) = 2t$ and

$$\partial_t \tilde{g}_{\sigma(t)}(z) = \frac{2}{\tilde{g}_{\sigma(t)} - \tilde{u}_{\sigma(t)}}, \quad \tilde{g}_0(z) = z$$

Also

$$d\tilde{u}_{\sigma(t)} = \frac{\kappa-6}{2} \cdot \frac{\psi''_{\sigma(t)}(u_{\sigma(t)})}{\psi'_{\sigma(t)}(u_{\sigma(t)})}dt + \sqrt{\kappa}d\tilde{B}_t$$

where $\tilde{B}_t = \int_0^{\sigma(t)} \psi_s'(u_s)dB_s$ is a stamdard Brownian motion by the definiiton of $\sigma(t)$ and the *Lévy characterization*. This process is an SLE_κ *iff* $\kappa = 6$. So we have proved :

Theorem) SLE_κ satisfies the locality property *iff* $\kappa = 6$.

Therefore SLE_6 is the only possible scaling limit for critical percolation.

Restriction property

The goal in this section is to show that $\mathrm{SLE}_{8/3}$ is the only possible SLE_κ which can arise as the scaling limit of *self-avoiding walks (SAW)*.

Self-avoiding walk (SAW) : Consider a graph $G = (V, E)$, $x \in V$, $n \in \mathbb{N}$. The SAW on G starting from x of length n is the uniform measure on simple paths in G starting from x of length n.

SAW was introduced in 1953 by P. Flory as a model for a polymer. (Flory was a Nobel prize winning chemist).

- SAW on $\mathbb{Z}^d$ for $d \geq 5$ converges to Brownian motion after rescaling by $n^{-1/2}$ (proved by Hara and Slade). This is because a simple random walk does not alreday self-intersect that much.
- It was conjecture that the same will happen for $\mathbb{Z}^4$, but with an extra logarithmic correction in the scaling.
- In $\mathbb{Z}^3$, there is no conjecture for the scaling limit or what the scaling factor sholud be.
- In $\mathbb{Z}^2$, it is conjectured to converge to $\mathrm{SLE}_{8/3}$ with scaling factor $n^{-4/3}$ (conjectured by Lawler-Schramm-Werner)

The story would be similar to that of critical percolation. We will check that the only reasonable scaling limit of SAW in form of SLE_κ is $\mathrm{SLE}_{8/3}$ with aid of a special property. The special property of SAW is the **restriction property**. That is, if $G' = (V', E)$ is a subgraph of G with $x \in V'$, then the SAW conditioned to stay in G' is a SAW in G'. *[Note that uniform measure restricted to a smaller set is the uniform measure.]* To derive the $\mathrm{SLE}_{8/3} \leftrightarrow$ SAW conjecture, we will show that the only SLE_κ which satisfies a continuum version of restriction is $\mathrm{SLE}_{8/3}$.

Assume that $\kappa \leq 4$ so that SLE$_\kappa$ is simple and let

$$\mathcal{Q}_+ = \{A \in \mathcal{Q} : \overline{A} \cap (-\infty, 0] = \phi\}$$
$$\mathcal{Q}_- = \{A \in \mathcal{Q} : \overline{A} \cap [0, \infty) = \phi\}$$

Suppose $A \in \mathcal{Q}_{+/-} = \mathcal{Q}_+ \cup \mathcal{Q}_-$ (so that $0 \in \mathbb{H}\backslash A$) and let $\psi_A = g_A - g_A(0)$. Then $\psi_A : \mathbb{H}\backslash A \to \mathbb{H}$ is the uique conformal transformation with $\psi_A(0) = 0$, $\psi_A(z)/z \to 1$ as $z \to \infty$.

Fact : SLE$_\kappa$ is "transient". That is, if $\gamma \sim$ SLE$_\kappa$ in $\mathbb{H}$ from 0 to ∞, then $\lim_{t\to\infty} \gamma(t) = \infty$ a.s.

Since SLE$_\kappa$ is simple for $\kappa \leq 4$ and transient, we have that

$$\mathbb{P}[V_A] := \mathbb{P}[\gamma([0, \infty)) \cap A = \phi] \in (0, 1)$$

Moreover, the law of γ is determined by the probabilities $\mathbb{P}[V_A]$ when $A \in \mathcal{Q}_{+/-}$ (why?).

Definition) We sasy that SLE$_\kappa$ satisfies the **restriction property** if for all $a \in \mathcal{Q}_{+/-}$, the conditional law of γ given V_A is an SLE$_\kappa$ in $\mathbb{H}\backslash A$. Equivalently, the conditional law of $\psi_A(\gamma)$ given V_A is an SLE$_\kappa$ in $\mathbb{H}$ from 0 to ∞.

Lemma) Suppose that there exists $\alpha > 0$ so that $\mathbb{P}[V_A] = (\psi_A'(0))^\alpha$ for all $A \in \mathcal{Q}_{+/-}$. Then SLE$_\kappa$ satisfies *restriction property.*

proof) Assume that $\mathbb{P}[V_A] = (\psi_A'(0))^\alpha$ for all $A \in \mathcal{Q}_{+/-}$. For $A, B \in \mathcal{Q}_{+/-}$, we have that

$$\begin{aligned}
\mathbb{P}\Big[\psi_A \circ \gamma([0,\infty)) \cap B = \phi | V_A\Big] &= \mathbb{P}\Big[\psi_A \circ \gamma([0,\infty)) \cap B = \phi, \gamma([0,\infty)) \cap A = \phi\Big] / \mathbb{P}\Big[V_A\Big] \\
&= \mathbb{P}\Big[\gamma([0,\infty)) \cap (A \cup \psi_A^{-1}(B)) = \phi\Big] / \mathbb{P}[V_A] \\
&= (\psi'_{\psi_A^{-1}(B)\cup A}(0))^\alpha / (\psi_A'(0))^\alpha \quad \text{(by assumptoin)} \\
&= (\psi_B'(0))^\alpha (\psi_A'(0))^\alpha / (\psi_A'(0))^\alpha \quad \text{(chain rule)} \\
&= (\psi_B'(0))^\alpha \\
&= \mathbb{P}[V_B]
\end{aligned}$$

where we have used $\psi_{\psi_A^{-1}(B)\cup A} = \psi_B \circ \psi_A$. So the law of $\psi_A \circ \gamma$ given V_A is exactly the law of γ.

(End of proof) $\square$

(5th March, Tuesday)

Brownian excursions

- A Brownian excursion is a Borwnian motion in a simple connected domain $D \subset \mathbb{C}$ starting from $x \in \overline{D}$ and conditioned to leave at $y \in \partial D$, with x, y are distinct.
- We are conditioning on a zero probability event, so we need to make this sense precise. We will first define it in the case that $D = \mathbb{H}$. For other domians, define it by applying a conformal mapping.

Definition) *(Rigoruous construction for $D = \mathbb{H}$)* Let $B = (B^1, B^2)$ be a complex Brownian motion with $B^1_0 = 0$, $B^2_0 = \epsilon > 0$. Condition B on the event that B^2 hits $R > 0$ (is a height, not a radius) very large before hitting 0. Take a limit as $R \to \infty$. To define it starting from $\partial\mathbb{H}$, take another limit as $\epsilon \to 0$.

As the result, we get **Brownian excursion** $\hat{B} = (\hat{B}^1, \hat{B}^2)$ in $\mathbb{H}$ from 0 to ∞, with $\hat{B}^1$, $\hat{B}^2$ independent, $\hat{B}^1$ a standard Brownian motion, $\hat{B}^2 \sim \mathrm{BES}^3$.

[On the Example Sheet #2, will check that these limits work and the details of the computation.]

Proposition) Suppose $A \in \mathcal{Q}_{+/-}$, and g_A is as usual. Then

$$\mathbb{P}_0[\hat{B}[0,\infty) \cap A = \phi] = g_A'(0) \ (= \psi_A'(0))$$

[Note, in the setting of an SLE, this equals $\psi_t'(U_t)$]

proof) Let $\mathcal{I}_R = \{z \in \mathbb{H} : \mathrm{Im}(z) = R\}$. Recall that $|g_A(z) - z| \leq 3\mathrm{rad}(A)$, $\forall z \in \mathbb{H}\backslash A$ *(Example Sheet #1)*. So

$$g_A(\mathcal{I}_R) \subset \{z \in \mathbb{H} : R - 3\mathrm{rad}(A) \leq \mathrm{Im}(z) \leq R + 3\mathrm{rad}(A)\}$$

Fix $z \in \mathbb{H}$, and let

$$\begin{aligned} \sigma_R &= \inf\{t \geq 0 : \mathrm{Im}(B_t) = R\} \quad B \text{ a complex BM} \\ \hat{\sigma}_R &= \inf\{t \geq 0 : \mathrm{Im}(\hat{B}_t) = R\} \quad \hat{B} \text{ a Brownian excursion} \end{aligned}$$

Then

$$\begin{aligned} \mathbb{P}_z[\hat{B}[0,\infty) \cap A = \phi] &= \lim_{R\to\infty} \mathbb{P}_z[\hat{B}[0,\hat{\sigma}_R] \cap A = \phi] \\ &= \lim_{R\to\infty} \frac{\mathbb{P}_z[B[0,\sigma_R] \cap (A \cup \mathbb{R}) = \phi]}{\mathbb{P}_z[B[0,\sigma_R] \cap \mathbb{R} = \phi]} = \lim_{R\to\infty} \frac{(\text{Numerator})}{(\text{Denominator})} \end{aligned}$$

Since the denominator part is asking for the probability of a Gambler's ruin for Brownian motion, we have

$$(\text{Denominator}) = \frac{\mathrm{Im}(z)}{R}$$

Also, by conformal invariance of Brownian motions,

$$\mathbb{P}_{g_A(z)}[B[0,\sigma_{R+3\mathrm{rad}(A)}] \cap \mathbb{R} = \phi] \leq \mathbb{P}_z[B[0,\sigma_R] \cap (A \cup \mathbb{R}) = \phi] \leq \mathbb{P}_{g_A(z)}[B[0,\sigma_{R-3\mathrm{rad}(A)}] \cap \mathbb{R} = \phi]$$

and therefore

$$\frac{\mathrm{Im}(g_A(z))}{R + 3\mathrm{rad}(A)} \leq (\text{Numerator}) \leq \frac{\mathrm{Im}(g_A(z))}{R - 3\mathrm{rad}(A)}.$$

Combining these and taking limit $R \to \infty$ gives

$$\mathbb{P}_z[\hat{B}[0,\infty) \cap A = \phi] = \frac{\mathrm{Im}(g_A(z))}{\mathrm{Im}(z)}$$

the result follows by taking a limit as $\mathrm{Im}(z) \to 0$.

(End of proof) $\square$

So indeed by the last lemma of the last lecture, we see that Brownian excursion satisfies a restriction property. People knew about such properties before SLE was invented, so it is bit more(???) natural to think of restriction property as the characteristic feature of $\mathrm{SLE}_{8/3}$.

Restriction Theorem for SLE$_{8/3}$

Let us go back to prove the restriction property for SLE$_{8/3}$. Recall,

$$V_A = \{\gamma[0,\infty) \cap A = \phi\}, \quad A \in \mathcal{Q}_{+/-}, \quad \gamma \sim \mathrm{SLE}_\kappa$$

Let $\mathcal{F}_t = \sigma(U_s : s \leq t)$ be the filtration generated by $U_t = \sqrt{\kappa} B_t$ for some Brownian motion B_t. Let $\tilde{M}_t = \mathbb{P}[V_A|\mathcal{F}_t]$. Then $\tilde{M}_t$ is a bounded martingale, $\tilde{M}_0 = \mathbb{P}[V_A]$ and $\tilde{M}_t \to \mathbf{1}_{V_A}$ by the martingale convergence theorem.

Let $\tau = \inf\{t \geq 0 : \gamma(t) \in A\}$. Then

$$\begin{aligned}
\tilde{M}_t = \mathbb{P}[V_A|\mathcal{F}_t] &= \mathbb{P}[V_A|\mathcal{F}_t](\mathbf{1}_{\{\tau>t\}} + \mathbf{1}_{\{\tau\leq t\}}) \\
&= \mathbb{P}[V_A \mathbf{1}_{\{\tau>t\}}|\mathcal{F}_t] = \mathbb{P}[V_A|\mathcal{F}_t]\mathbf{1}_{\{\tau>t\}} \quad \text{(as } \tau \text{ is a stopping time)} \\
&= \mathbb{P}[V_{g_t(A)-g_t(0)}]\mathbf{1}_{\{\tau>t\}} \quad \text{(by the confromal Markov property)} \\
&= \mathbb{P}[V_{g_t(A)-U_t}]\mathbf{1}_{\{\tau>t\}}
\end{aligned}$$

Observe : If M_t is another $\mathcal{F}_t$-martingale with $M_t \to \mathbf{1}_{V_A}$ as $t \to \infty$, then $M_t = \tilde{M}_t$ for all $t \geq 0$. So if we guess M_t and show that it coverges to $\mathbf{1}_{V_A}$, then we can deduce the form of $\tilde{M}_t$ from this.

Try $M_t = (\psi'_{g_t(A)-g_t(0)}(0))^\alpha \mathbf{1}_{\{\tau>t\}}$, for α a parameter to be chosen later. (Recall that ψ_B was the unique confromal map $\mathbb{H}\backslash B \to \mathbb{H}$ with $\psi_B(0)$ and $\psi_B(z)/z \to 1$ as $z \to \infty$ for $B \in \mathcal{Q}_{+/-}$.) We may also write

$$M_t = (\psi'_t(U_t))^\alpha \mathbf{1}_{\{\tau>t\}}$$

where $\psi_t = \tilde{g}_t \circ \psi_A \circ g_t^{-1}$ and $\tilde{g}_t = g_{\psi_A(\gamma[0,t])}$. The goal is to show that M_t is a martingale for $\kappa = 8/3$ with the correct limit in $t \to \infty$ then it would be as desired.

Claim : M_t is a continuous martingale.

proof) In the *Example Sheet #2*, will see that

$$\partial_t \psi'_t(U_t) = \frac{(\psi''_t(U_t))^2}{2\psi'_t(U_t)} - \frac{4}{3}\psi'''_t(U_t)$$

Conditioned on $\tau > t$, *Ito's formula* will give

$$dM_t = \alpha M_t \Big[\frac{(\alpha-1)\kappa+1}{2}\Big(\frac{\psi''_t(U_t)}{\psi'_t(U_t)}\Big)^2 + \Big(\frac{\kappa}{2} - \frac{4}{3}\Big)\frac{\psi'''_t(U_t)}{\psi'_t(U_t)}\Big]dt + \alpha M_t \frac{\psi''_t(U_t)}{\psi'_t(U_t)}\sqrt{\kappa}dB_t$$

[Note that, for the case $\tau \leq t$, we have to first prove that $M_t \to 0$ as $t \to \tau^-$ so that M is continuous at $t = \tau$. This will be done in the proof of the theorem below.] If $\kappa = 8/3$ and $\alpha = 5/8$, then

$$dM_t = \alpha M_t \frac{\psi''_t(U_t)}{\psi'_t(U_t)}\sqrt{\kappa}dB_t$$

so M_t is a continuous local martingale.

Moreover, $\psi'_t(U_t)$ is the probability that a Brownian excursion in $\mathbb{H}\backslash\gamma[0,t]$ from $\gamma(t)$ to ∞ does not hit A (by the last proposition), so this implies $0 \leq M_t \leq 1$. In particular, M_t is a bounded continuous local martingale, hence is just a continuous martingale.

(End of proof) $\square$

Theorem) $\text{SLE}_{8/3}$ satisfies the restriction property. Moreover,

$$\mathbb{P}[\gamma[0,\infty) \cap A = \phi] = (\psi_A'(0))^{5/8} \quad \forall A \in \mathcal{Q}_{+/-}$$

Remark : $\gamma_1, \cdots, \gamma_8$ be independent $\text{SLE}_{8/3}$. Then

$$\mathbb{P}[\gamma_j[0,\infty) \cap A = \phi, \ \forall 1 \le j \le 8] = (\psi_A'(0))^5$$

Also, let $\hat{B}^1$, $\hat{B}^2, \cdots$, $\hat{B}^5$ be independent *Brownian excursions*, then

$$\mathbb{P}[\hat{B}^j[0,\infty) \cap A = \phi, \ \forall 1 \le j \le 5] = (\psi_A'(0))^5$$

So the hull of $\gamma_1, \cdots, \gamma_8 \overset{\mathrm{d}}{=}$ hull of $\hat{B}^1, \cdots, \hat{B}^5$, where the hull of $\gamma_1, \cdots, \gamma_8$ is the bounded component of $\mathbb{H} \backslash \cup_{j=1}^8 \gamma_j[0,\infty)$ and the hull of $\hat{B}^1, \cdots, \hat{B}^5$ is defined accordingly. Although we can not prove any interesting properties from this result, this suggests that the Brownian excursions and the $\text{SLE}_{8/3}$ curves share certain geometry properties.

(7th March, Thursday)

Recall, $\psi_A = g_A - g_A(0)$, $V_A = \{\gamma[0,\infty) \cap A = \phi\}$, $\psi_t = \tilde{g}_t \circ \psi_A \circ g_t^{-1}$, $\tilde{g}_t = g_{\psi_A(\gamma[0,t])}$. We were proving :

Theorem) $\text{SLE}_{8/3}$ satisfies the restriction property. Moreover, if $\gamma \sim \text{SLE}_{8/3}$, then $\mathbb{P}[\gamma[0,\infty) \cap = \phi] = (\psi'(0))^{5/8}$, for all $A \in \mathcal{Q}_{+/-}$.

proof) It will be sufficient to just prove for $A \in \mathcal{Q}_+$.

Let $M_t = \mathbf{1}_{\{\tau > t\}}(\psi_t'(U_t))^\alpha$, $\alpha = 5/8$, with $\tau = \inf\{t \ge 0 : \gamma(t) \in A\}$. Here, $\psi_t'(U_t)$ is the probability that a Brownian excursion in $\mathbb{H} \backslash \gamma[0,t]$ from $\gamma(t)$ to ∞ does not hit A. We have already seen that M_t is a martingale for $t < \tau$, so $M_{t\wedge\tau}^- := \lim_{\epsilon \to 0^+} M_{(t\wedge\tau)-\epsilon}$ is always a martingale with $0 \le M_{t\wedge\tau}^- \le 1$. So we would have $M_{t\wedge\tau}^- \to M_\infty$ a.s. as $t \to \infty$ with $0 \le M_\infty \le 1$. As indicated earlier, our goal is to prove that $M_\infty = \tilde{M}_\infty = \mathbf{1}_{V_A}$. (From now on, we will just write $M_{t\wedge\tau}$ in place of $M_{t\wedge\tau}^-$). We need to show that

1. $M_{t\wedge\tau} \to 1$ on V_A as $t \to \infty$.
2. $M_{t\wedge\tau} \to 0$ on V_A^c as $t \to \infty$. *[Of course, this is clear from the definition of M, but what we really get to prove is that $M_{t\wedge\tau}^- \to 0$ as $t \to \infty$. That is, in fact $M_t \to 0$ as $t \to \tau^-$.]*

By scaling, we can assume that $\sup\{\text{Im}(\omega) : \omega \in A\} = 1$. For each $r > 0$, let $\sigma_r = \inf\{t \ge 0 : \text{Im}(\gamma(t)) = r\}$. Note $\sigma_r < \infty$ a.s. for all $r > 0$ since $\text{SLE}_{8/3}$ is "transient".

♠ **Claim 1 :** $M_{t\wedge\tau} \to 1$ on V_A as $t \to \infty$.

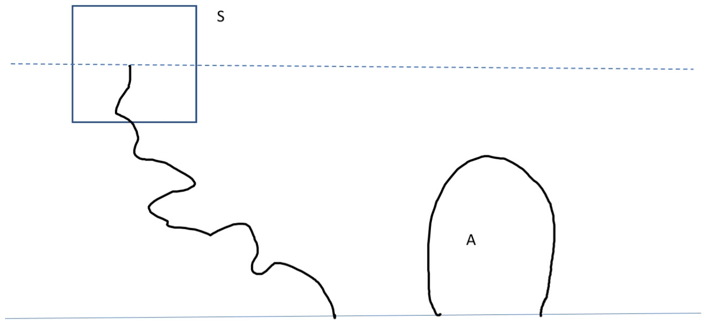
: Let $\hat{B}$ be a Brownian excursion in $\mathbb{H} \backslash \gamma[0,\sigma_r]$ from $\gamma(\sigma_r)$ to ∞. Probability that $\hat{B}$ hits A is $1 - \psi_{\sigma_r}'(U_{\sigma_r})$ (by a propsition from last lecture), and

$$1 - \psi_{\sigma_r}'(U_{\sigma_r}) = \lim_{\epsilon \to 0} \lim_{R \to \infty} \frac{\mathbb{P}_z\Big[B[0,\tau_R] \subset \mathbb{H} \backslash \gamma[0,\sigma_r], B[0,\tau_R] \cap A \ne \phi\Big]}{\mathbb{P}_z\Big[B[0,\tau_R] \subset \mathbb{H} \backslash \gamma[0,\sigma_r]\Big]}$$

where B is a complex Brownian motion, $\tau_R = \inf\{t \ge 0 : \text{Im}(B_t) = R\}$ and $z = \gamma(\sigma_r) + i\epsilon$ (by definition of a Brownian excursion).

Sub-Claim: The limit $\le Cr^{-1/2}$ for some $C > 0$ a constant.

: We will prove a lower bound for the denominator and an upper bound for the numerator. Let $S = [-1,1]^2 + \gamma(\sigma_r)$ be the square with side length 2 centred at $\gamma(\sigma_r)$. Let $l = [-1,1] \times \{1\} + \gamma(\sigma_r)$ be the top of S, B be a complex BM starting from $z = \gamma(\sigma_r) + i\epsilon$ and let $\eta = \inf\{t \geq 0 : B(t) \notin S\}$.



Then by a simple(?) symmetry argument,

$$\mathbb{P}_z[B(\eta) \in l] > 1/4,$$
$$\mathbb{P}_w[B(\eta) \in l] \leq 1/4, \qquad \forall w \in S,\ \mathrm{Im}(w) \leq r$$

So

$$\begin{aligned}
\frac{1}{4} &< \mathbb{P}_z[B(\eta) \in l] \\
&= \mathbb{P}_z[B(\eta) \in l, B[0,\eta] \cap \gamma[0,\sigma_r] = \phi] + \mathbb{P}_z[B(\eta) \in l, B[0,\eta] \cap \gamma[0,\sigma_r] \neq \phi] \\
&= \mathbb{P}_z[B(\eta) \in l \,|B[0,\eta] \cap \gamma[0,\sigma_r] = \phi]\mathbb{P}_z[B[0,\eta] \cap \gamma[0,\sigma_r] = \phi] \\
&\quad + \mathbb{P}_z[B(\eta) \in l \,|B[0,\eta] \cap \gamma[0,\sigma_r] \neq \phi]\mathbb{P}_z[B[0,\eta] \cap \gamma[0,\sigma_r] \neq \phi]
\end{aligned}$$

Observe that the $\mathbb{P}_z[B(\eta) \in l \,|B[0,\eta] \cap \gamma[0,\sigma_r] \neq \phi] \leq 1/4$, by the *Strong Markov Property* of BM applied upon B hits $\gamma[0,\sigma_r]$. Hence

$$\begin{aligned}
&\mathbb{P}_z[B(\eta) \in l \,|B[0,\eta] \cap \gamma[0,\sigma_r] = \phi] > \frac{1}{4} \\
\Rightarrow \quad &(\text{Denominator}) \geq \frac{1}{4}\mathbb{P}_z[B[0,\eta] \cap \gamma[0,\sigma_r] = \phi] \times \frac{1}{R-r+1}
\end{aligned}$$

where the factor $\frac{1}{R-r+1}$ comes from $\mathbb{P}[\inf\{t \geq \eta : \mathrm{Im}(B_t) = R\} \leq \inf\{t \geq \eta : \mathrm{Im}(B_t) = r\}|B_\eta \in l]$ and is computed using the probability of a *Gambler's Ruin.*

Also, the *Strong Markov Property* for B at time η and the *Beurling estimate* implies (because we have assumed the hieght of A is ≤ 1, B has to escape $D(B_\eta, r)$ before hitting A)

$$\mathbb{P}_z[B \text{ hits } A \text{ before hitting } \mathbb{R} \cup \gamma[0,\sigma_r]] \leq Cr^{-1/2}\mathbb{P}_z[B[0,\eta] \cap \gamma[0,\sigma_r] = \phi]$$

So (Numerator) $\leq \frac{1}{R}Cr^{-1/2}$, with the factor $\frac{1}{R}$ comes from the probability that B reaches hieght R before hitting $\mathbb{R}$, and is again computed using the probability of a *Gambler's Ruin.* Putting the two estimates for the denominator and the numerator, we have the desired result.

This will finish claim 1 as

$$\begin{aligned}
\psi'_{\sigma_r}(U_{\sigma_r}) &= \text{Probability that a Brownian excursion } \mathbb{H} \text{ from } U_{\sigma_r} \text{ to } \infty \text{ does not hit } g_{\sigma_r}(A) \\
&= \text{Probaibilty that a Brownian excursion in } \mathbb{H}\backslash\gamma[0,\sigma_r] \text{ from } \gamma(\sigma_r) \text{ to } \infty \text{ does not hit } A.
\end{aligned}$$

So, we see that

$$1 \geq \psi'_{\sigma_r}(U_{\sigma_r}) \geq 1 - Cr^{-1/2} \to 1 \quad \text{as } r \to \infty$$

This implies $M_{\sigma_r \wedge \tau} \to 1$ as $r \to \infty$ on V_A, and so $M_{t\wedge\tau} \to 1$ as $t \to 1$ on V_A (as both limits have to be the same).

♠ **Claim 2:** $M_{t\wedge\tau} \to 0$ as $t \to \infty$ on V_A^c.

: Assume that A is bounded by a smooth, simple curve $\beta : (0,1) \to \mathbb{H}$ that is sufficiently close to A. (Will be justifying on *Example Sheet #2*). Note, because we are working on the event V_A^c, we have $\gamma(\tau) = \beta(s)$ for some $s \in (0,1)$. Since β is a smooth, simple curve, there exists $\delta > 0$ so that $l = [\beta(s), \beta(s) + \delta \underline{n}] \in$, where $\underline{n}$ is the inward pointing normal at $\beta(s)$.

Let $l_t = g_t(l) - g_t(0)$, and $t_m = \inf\{t \geq 0 : |\gamma(t) - \beta(s)| \leq \frac{1}{m}\}$ for each $m \in \mathbb{N}$. We want to show that as $m \to \infty$, the probability that a Brownian excursion in $\mathbb{H}\backslash\gamma[0,t_m]$ from $\gamma(t_m)$ to ∞ hits $A \to 1$.

Note, that the chance for a Brownian motion starting on any point on l and hitting $\mathbb{R} \cup \gamma[0,t_m]$ on either its left or right sides is bounded below by a strictly positive value. So

$$l_{t_m} \subset \{w \in \mathbb{H} : \mathrm{Im}(\omega) \geq a|\mathrm{Re}(\omega)|\}$$

for some $a > 0$.

In *Example Sheet #2*, will be showing that a Brownian excursion from 0 to ∞ in $\mathbb{H}$ hits l_{t_m} with probability $\to 1$ as $m \to \infty$. So this implies that $\psi'_t(U_t) \to 0$ as $t \to \tau^-$.

(End of proof) □

In next two lectures, we are going to talk about more recent works on SLEs, including Gaussian free field(="Random Mountain"). SLE$_4$ curves describe the topographical map of the GFF.

(12th March, Tuesday)

The Gaussian free field (GFF)

We have defined a Brownian motion $B : [0,\infty) \to \mathbb{R}$ to be a "random curve" indexed by time. We will define *Gaussion free field (GFF)* $h : D \to \mathbb{R}$ as a "random filed" indexed by $D \subset \mathbb{C}$. This can be seen as a "two-time dimensional analogue of Brownian motion". The construction would be based on the theory Hilbert spaces.

Definition) Let

$$\begin{aligned} C^\infty &= \{\text{functions on } \mathbb{C}\text{which are infinitely differentiable}\} \\ C_0^\infty &= \{f \in C^\infty \text{ with compact support}\} \\ C_0^\infty(D) &= \{f \in C_0^\infty \text{ with support } \subset D\} \end{aligned}$$

Suppose $f, g \in C_0^\infty$. Then the **Dirichlet inner product** of f, g is

$$(f,g)_\nabla = \frac{1}{2\pi} \int \nabla f(x) \cdot \nabla(g)(x) dx$$

[Poincaré's inequality ensures this is an inner product.] Suppose that $D \subset \mathbb{C}$ is simply connected, but $D \neq \mathbb{C}$. Let $H_0^1(D)$=Hilbert space completion of $C_0^\infty(D)$ with respect to $(\cdot,\cdot)_\nabla$. Let $\|f\|_\nabla^2 = (f,f)_\nabla$ be the norm on $H_0^1(D)$.

Propeerties of $H_0^1(D)$ and $(\cdot,\cdot)_\nabla$

(1) **Conformal invariance :** Suppose $\varphi : D \to \tilde{D}$ is a conformal transformation, $f, g \in C_0^\infty(D)$. Then

$$(f,g)_\nabla = (f \circ \varphi^{-1}, g \circ \varphi^{-1})_\nabla$$

I other words, the Dirichlet inner product is conformally invariant. (The proof is on *Ex Sheet #2.*)

This tells us that $H_0^1(D) \to H_0^1(\tilde{D})$ is given by $f \mapsto f \circ \varphi^{-1}$ is an isomrphism of Hilbert spaces.

(2) **Inclusion :** Suppose that $U \subset D$ is opne. If $f \in C_0^\infty(U)$, then $f \in C_0^\infty(D)$. Therefore we have a well-defined inclusion map $H_0^1(U) \hookrightarrow H_0^1(D)$. Therefore, $H_0^1(U)$ is a subspace of $H_0^1(D)$

(3) **Orthogonal decomposition :** For $U \subset D$ open, let $H_{\text{supp}}(U) = H_0^1(U) = H_0^1(U) \subset H_0^1(D)$ and $H_{\text{harm}}(U) = \{f \in H_0^1(D) : f \text{ is harmonic on } U\}$. Then $H_0^1(D) = H_{\text{harm}}(U) \oplus H_{\text{supp}}(U)$ is an orthogonal decomposition.

proof) To check orthogonality, suppose $f \in H_{\text{supp}}(U)$ and $g \in H_{\text{harm}}(U)$. Then

$$(f,g)_\nabla = \frac{1}{2\pi}\int \nabla f \cdot \nabla g = \frac{-1}{2\pi}\int f \triangle g = 0$$

To see that they span $H_0^1(D)$, fix $f \in H_0^1(D)$. Let f_0 be the orthogonal projection f onto $H_{\text{supp}}(U)$ and $g_0 = f - f_0$. We want g_0 to be harmonic on U. We will instead check that g is 'weakly harmonic'. Suppose $\varphi \in C_0^\infty(U)$. Then

$$0 = (\varphi, g_0)_\nabla = \frac{1}{2\pi}\int \nabla\varphi \cdot \nabla g_0 = \frac{-1}{2\pi}\int (\triangle\varphi) g_0$$

In *Example Sheet #2*, will check that this implies g_0 is harmonic on U (and hence also C^∞ on U).

(End of proof) □

Definition of Gaussian free field

Aside about normal random variables on $\mathbb{R}^n$:
Let $h = \alpha_1 e_1 + \cdots + \alpha_n e_n$ where $\alpha_1, \cdots, \alpha_n$ are iid $N(0,1)$ and $e_1, \cdots, e_n$ are standard basis on $\mathbb{R}^n$. Then h is a standard Gaussian random variable on $\mathbb{R}^n$.

- If $x \in \mathbb{R}^n$, then $(h, x) = \sum_{j=1}^n \alpha_j x_j \sim N(0, \|x\|^2)$.
- If $x, y \in \mathbb{R}^n$, then (h, x), (h, y) are jointly Gaussian with

$$\text{Cov}((h,x),(h,y)) = \sum_{j=1}^n x_j y_j = (x, y)$$

So associated with h is a *family* of Gaussian random variables, (h, x) indexed by $x \in \mathbb{R}^n$ with covariance given by the standard inner product on $\mathbb{R}^n$. This is an example of a *Gaussian Hilbert space.*

Definition) Let (f_n) be an orthonormal basis (ONB) of $H_0^1(D)$ and (α_n) be iid $N(0,1)$. Then h on D is given by

$$h = \sum_{n=1}^{\infty} \alpha_n f_n$$

If $f = \sum_{n=1}^{\infty} \beta_n f_n \in H_0^1(D)$ for $\beta_n \in \mathbb{R}$, then

$$(h, f)_\nabla = \sum_{n=1}^{\infty} \alpha_n \beta_n \sim N(0, \|f\|_\nabla^2)$$

If $g = \sum_{n=1}^{\infty} \gamma_n f_n \in H_0^1(D)$, then $(h, f)_\nabla$, $(h, g)_\nabla$ are jointly Gaussian and $\mathrm{Cov}((h, f)_\nabla, (h, g)_\nabla) = \sum_{n=1}^{\infty} \beta_n \gamma_n = (f, g)_\nabla$. The **Gaussian free field (GFF)** is a family of Gaussian random variables $(h, f)_\nabla$ indexed by $f \in H_0^1(D)$ with covariance given by $(\cdot, \cdot)_\nabla$.
[The expression $h = \sum_n \alpha_n f_n$ makes it seem like as if h random varaible. Hoever, h is not a function on D as $\mathbb{E}[\|h\|_D^2] = \mathbb{E}[\sum_{n=1}^{\infty} \alpha_n^2] = \infty$ and does not take finite value. It should rather be thought of as a collection of random variables.]

Properties of GFF

(1) **Conformally invariant :** if $\varphi : D \to \tilde{D}$ is a conformal transformation, $h = \sum_n \alpha_n f_n$ is a GFF on D, then $h \circ \varphi^{-1} = \sum_{n=1}^{\infty} \alpha_n f_n \circ \varphi^{-1}$ is a GFF on $\tilde{D}$.

(2) **Markov property :** if $U \subset D$ is open and h is a GFF on D, then we can write $h = h_1 + h_2$ where h_1 is a GFF on U, h_2 is harmonic on U and h_1, h_2 are independent. That is, if we are conditioned with the value of GFF outside of U, then the rest is just a GFF on U.

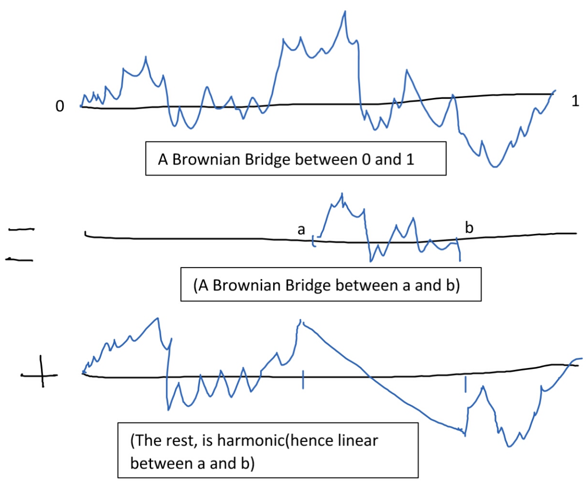
proof) Choose the ONB (f_n) for $H_0^1(D)$ to consist of an ONB (f_n^1) of $H_{\mathrm{supp}}(U)$ and (f_n^2) of $H_{\mathrm{harm}}(U)$. Then

$$h = \sum_n \alpha_n f_n = \sum_n \alpha_n^1 f_n^1 + \sum_n \alpha_n^2 f_n^2$$

but $h_1 := \sum_n \alpha_n^1 f_n^1$ is a GFF in U and $h_2 := \sum_n \alpha_n^2 f_n^2$ is harmonic in U.

(End of proof) $\square$

Analogue for Browanian Bridge



(14th March, Thursday)

L^2 inner product of a C_0^∞ function and the GFF

Idea : if h were a function then, we could have defined $(f, h) = -\frac{1}{2\pi}(h, \triangle f)$. We would like to do something similar with distribution h.

If $\varphi \in C_0^\infty(D)$, then we have

$$-2\pi\triangle^{-1}\varphi(x) = \int G(x,y)\varphi(y)dy$$

where $G(x,y) = G_D(x,y) = -\log|x-y| - \tilde{G}_x(y)$ where $\tilde{G}_x(y)$ is harmonic in D with boundary conditions $y \mapsto -\log|x-y|$ (x fixed). $G = G_D$ is the *Dirichlet Green's function* for $\triangle$ on D.

Example : When $D = \mathbb{H}$, has $G_{\mathbb{H}}(x,y) = -\log|x-y| + \log|x-\overline{y}|$.

To define the L^2 inner product of $\varphi \in C_0^\infty$, we let

$$(h,\varphi) := (h, -2\pi\triangle^{-1}\varphi))_\nabla$$

Thus (h,φ) is a mean-zero normal random variable with variance

$$\begin{aligned}
\mathrm{Var}((h,\varphi)) &= \mathrm{Var}((h, -2\pi\triangle^{-1}\varphi)_\nabla) \\
&= (2\pi)^2\|\triangle^{-1}\varphi\|_\nabla^2 = (2\pi)^2(\triangle^{-1}\varphi, \triangle^{-1}\varphi)_\nabla \\
&= -2\pi(\triangle^{-1}\varphi, \triangle(\triangle^{-1}\varphi)) \quad \text{(integration by parts)} \\
&= \iint \varphi(x)G(x,y)\varphi(y)dxdx
\end{aligned}$$

Similarly

$$\mathrm{Cov}((h,\varphi),(h,\psi)) = \iint \varphi(x)G(x,y)\psi(y)dxdy$$

Proposition) *(Conformal invariance of the Grenns' fucntions)* If $D, \tilde{D} \subset \mathbb{C}$ are domains in $\mathbb{C}, \varphi : D \to \tilde{D}$ is a conformal transforamtion, then $G_D(x,y) = G_{\tilde{D}}(\varphi(x), \varphi(y))$.

> **proof)** We have that, since confromal transformation preserves harmonicity, $G_D(x,y) - G_{\tilde{D}}(\varphi(x),\varphi(y))$ is harmonic on D. Also, it is clear from their deifinitions that $G_D(x,y) - G_{\tilde{D}}(\varphi(x),\varphi(y)) \equiv 0$ on the boundary, hence identically 0.
>
> *(End of proof)* $\square$

The GFF is not a function. It is a generalized function (a.k.a. a distribution). But speaking in the language of functions, we are going to see that '0-level sets' of the GFF $\{x : h(x) = 0\}$ are described by SLE_4 (Schramm, Sheffield).

Theorem) Let $\lambda = \pi/2$. Let γ be an SLE_4 in $\mathbb{H}$ from 0 to ∞, (g_t) be its Loewner evolution, with $U_t = \sqrt{\kappa}B_t = 2B_t$ and $f_t = g_t - U_t$. Fix $U \subset \mathbb{H}$ open and let $\tau = \inf\{t \geq 0 : \gamma(t) \in U\}$. Let h be a GFF in $\mathbb{H}$, and $\mathfrak{h} = \lambda - \frac{2\lambda}{\pi}\arg(\cdot)$. Then

$$h \circ f_{t\wedge\tau} + \mathfrak{h} \circ f_{t\wedge\tau} \overset{\mathrm{d}}{=} h + \mathfrak{h}$$

where the left/right sides are restricted to U (only integrate with respect to test functions supported in U).

The theorem suggsets two different methods for sampliing a GFF.

Let us think about the statement for a while. We will have

- $\mathfrak{h}$ = harmonic in $\mathbb{H}$ with boundary condition $-\lambda$ on $\mathbb{R}_-$, $+\lambda$ on $_+$.
- $\mathfrak{h}\circ f_{t\wedge\tau}$=harmonic in $\mathbb{H}\backslash\gamma[0,t\wedge\tau]$ with boundary condtion $-\lambda$ on $\mathbb{R}_-$ and left side of $\gamma[0,t\wedge\tau]$, $+\lambda$ on $\mathbb{R}_+$ and the right side of $\gamma[0,t\wedge\tau]$.
- γ is a "level ridge" of h (remind that h is not a function)

This version of the theorem is a simplified from the original one. This theorem is not dealing with what happens after γ hits U.

proof) Suppose $\phi\in C_0^\infty(U)$. We want to prove :

$$(h\circ f_{t\wedge\tau}+\mathfrak{h}\circ f_{t\wedge\tau},\phi)\overset{\mathrm{d}}{=}(h+\mathfrak{h},\phi)$$

i.e. is $\sim N(m_0(\phi),\sigma_0^2(\phi))$ where $m_0(\phi)=(\mathfrak{h},\phi)$, $\sigma_0^2(\phi)=\iint\phi(x)G_{\mathbb{H}}(x,y)\phi(y)dxdy$. This is equivalent to showing

$$\mathbb{E}\Big[\exp[i\sigma(h\circ f_{t\wedge\tau}+\mathfrak{h}\circ f_{t\wedge\tau}),\phi\Big]=\exp\left[i\theta m_0(\phi)-\frac{\theta^2}{2}\sigma_0^2(\phi)\right]$$

Let $\mathcal{F}_t=\sigma(U_s:s\le t)$. Given $\mathcal{F}_{t\wedge\tau}$, $h\circ f_{t\wedge\tau}$ is a GFF in $\mathbb{H}\backslash\gamma[0,t\wedge\tau]$, so

$$\begin{aligned}&\mathbb{E}[\exp(i\theta(h\circ f_{t\wedge\tau}+\mathfrak{h}\circ f_{t\wedge\tau},\phi))|\mathcal{F}_{t\wedge\tau}]\\&=\mathbb{E}\Big[\exp(i\theta(h\circ f_{t\wedge\tau,\phi}))|\mathcal{F}_{t\wedge\tau}\Big]\exp\Big[i\theta m_{t\wedge\tau}(\phi)\Big]\\&=\exp\left(i\theta m_{t\wedge\tau}(\phi)-\frac{\theta^2}{2}\sigma^2_{t\wedge\tau}(\phi)\right)\end{aligned}$$

where $\sigma_t^2(\phi)=\iint\phi(x)G_t(x,y)\phi(y)dxdy$ with $G_t(x,y)=G_{\mathbb{H}}(f_t(x),f_t(y))$ is the Greens' function on $\mathbb{H}\backslash\gamma[0,t]$ and $m_t(\phi):=(\mathfrak{h}\circ f_t,\phi)$.

We want to show that $\exp(i\theta m_t(\phi)-\frac{\theta^2}{2}\sigma_t^2(\phi))$ is a martingale. This is in the form of an exponential martingale, so showing $m_t(\phi)$ is a martingale with $[m_\cdot(\phi)]_t=\sigma_0^2(\phi)-\sigma_t^2(\phi)$ would be sufficient for this.

- Check that $m_t(\phi)$ is a martingale (recall $m_t(\phi)=(\mathfrak{h}\circ f_t,\phi)$):

$$\begin{aligned}\mathfrak{h}\circ f_t(z)&=\lambda-\frac{2\lambda}{\pi}\arg(f_t(z))\\&=\lambda-\frac{2\lambda}{\pi}\mathrm{Im}(\log(g_t(z)-U_t))=\lambda-\mathrm{Im}(\log(g_t(z)-U_t))\end{aligned}$$

We want to see that $\log(g_t(z)-U_t)$ is a martingale. To see this,

$$\begin{aligned}d\log(g_t(z)-U_t)&=\frac{1}{g_t(z)-U_t}\cdot\frac{2}{g_t(z)-U_t}dt-\frac{1}{g_t(z)-U_t}dU_t-\frac{\kappa/2}{(g_t(z)-U_t)^2}dt\\&=\frac{2-\kappa/2}{(g_t(z)-U_t)^2}dt-\frac{2}{g_t(z)-U_t}dB_t=\frac{-2}{g_t(z)-U_t}dB_t\end{aligned}$$

since $\kappa=4$. So $m_t(\phi)$ is a martingale.

- The quadratic variation is, by *Itô isometry*,

$$d[m_\cdot(\phi)]_t=\iint\phi(z)\mathrm{Im}(\frac{2}{g_t(z)-U_t})\mathrm{Im}(\frac{2}{g_t(w)-U_t})\phi(w)dzdw$$

It will be enougth to show that $d\sigma_t^2(\phi)$ has the same form. To see this, observe

$$G_{\mathbb{H}}(z,w) = -\mathrm{Re}(\log(z-w) - \log(z-\overline{w}))$$
$$\log(f_t(z) - f_t(w)) = \log(g_t(z) - g_t(w)) \quad (\text{used } f_t = g_t - U_t)$$

so

$$\begin{aligned} d\log(g_t(z) - g_t(w)) &= \frac{1}{g_t(z) - g_t(w)} \cdot \Big(\frac{2}{g_t(z) - U_t} - \frac{2}{g_t(w) - U_t}\Big) dt \\ &= \frac{-2dt}{(g_t(z) - U_t)(g_t(w) - U_t)} \end{aligned}$$

and similarly

$$d\log(g_t(z) - \overline{g_t(w)}) = \frac{-2dt}{(g_t(z) - U_t)(\overline{g_t(w)} - U_t)}$$

Hence, using conformal invariance of Green's function,

$$\begin{aligned} dG_t(z,w) &= -\mathrm{Im}\Big(\frac{2}{g_t(z) - U_t}\Big)\mathrm{Im}\Big(\frac{2}{g_t(w) - U_t}\Big) \\ \Rightarrow \quad d\sigma_t^2(\phi) &= \Big[-\iint \phi(z)\mathrm{Im}\Big(\frac{2}{g_t(z) - U_t}\Big)\mathrm{Im}\Big(\frac{2}{g_t(w) - U_t}\Big)\phi(w)dzdw\Big]dt \end{aligned}$$

after some computation. so indeed $d[m.(\phi)]_t = d\sigma_t^2(\phi)$.

These complete the proof.

(End of proof) $\square$