

# **Elliptic Partial Differential Equations**

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# 1 Introduction

In this course we focus on the study of 2nd order elliptic equations on a domain in  $\mathbb{R}^n$ . A primary scenario of elliptic PDEs are *variational equations*. These lead to nonlinear elliptic PDEs. So first we need knowledge of the linear theory to study the non-linear problems.

A general set up is the following:

Consider  $F : \underbrace{\Omega \times \mathbb{R} \times \mathbb{R}^n}_{\text{coordinates } (x,z,p)} \rightarrow \mathbb{R}$ , with  $\Omega \subset \mathbb{R}^n$  open and bounded. Then consider the functional:

$$\mathcal{F}(u) = \int_{\Omega} F(x, u(x), Du(x)) \, dx.$$

Assume  $F$  is sufficiently regular (i.e. differentiable). Let  $u \in \mathcal{S}$ , for  $\mathcal{S}$  a suitable **linear** space of functions (so that we can add elements and not leave  $\mathcal{S}$ ). Here,  $u : \Omega \rightarrow \mathbb{R}$ , and we consider cases such as  $\mathcal{S} = W^{1,2}(\Omega)$ , or Hölder spaces such as  $C^{1,\alpha}(\overline{\Omega})$ ,  $C^{2,\alpha}(\overline{\Omega})$ , etc.

Suppose we have a function  $u \in \mathcal{S}$  which minimises  $\mathcal{F}$  subject to some further constraint, such as fixed boundary values, i.e. we require  $u|_{\partial\Omega} = g$ , where  $g : \partial\Omega \rightarrow \mathbb{R}$  is given. This tends to be needed or else we can get trivial minimisers, such as  $u \equiv 0$ . Then for any  $\varphi \in \mathcal{S}$ , we have:

$$\mathcal{F}(u + t\varphi) \geq \mathcal{F}(u) \quad \text{for all } t \in \mathbb{R}.$$

So with enough regularity, this means we need:  $\frac{d}{dt}\big|_{t=0} \mathcal{F}(u + t\varphi) = 0$ , i.e. the directional derivative along  $\varphi$  is 0. This means:

$$\frac{d}{dt}\bigg|_{t=0} \int_{\Omega} F(x, u + t\varphi, Du + tD\varphi) dx = 0.$$

Then from this we get a differential equation, which will be elliptic if there are certain bounds on the integrand.

So assuming appropriate regularity to exchange the integral and derivative, this becomes:

$$\int_{\Omega} D_z F(x, u, Du) \varphi + D_{p_j} F(x, u, Du) D_j \varphi \, dx = 0 \quad (\text{summation convention})$$

for all  $\varphi \in \mathcal{S}$  with  $\varphi|_{\partial\Omega} = 0$  (to ensure that  $u + t\varphi$  has the correct boundary condition). Then using the divergence theorem, we get

$$\int_{\Omega} [\nabla \cdot D_p F(x, u, Du) - D_z F(x, u, Du)] \varphi \, dx = 0$$

for all such  $\varphi$ , and so hence:

$$\nabla \cdot D_p F(x, u, Du) - D_z F(x, u, Du) = 0 \quad \text{in } \Omega.$$

This is the **Euler-Lagrange Equation** associated with  $\mathcal{F}$ . This then becomes

$$\sum_{i,j=1}^n D_{p_i p_j} F(x, u, Du) D_{ij} u - D_z F(x, u, Du) = 0$$

a quasi-linear 2nd order PDE. We can write this in the general form:

$$a^{ij}(x, u, Du) D_{ij} u - b(x, u, Du) = 0.$$

This type of equation is **elliptic** if  $(a^{ij}(x, u, Du))_{ij}$  is a positive definite matrix in  $\Omega$ . But then noting that the  $a^{ij}$  were the 2nd derivatives of  $F$ , this is the same as saying that  $F$  is convex in the  $p$ -variable.

**Example 1.1 (Dirichlet Energy)** Consider the case  $F(x, z, p) = |p|^2$ . Then repeating the calculation leads to an Euler-Lagrange equation of:  $\Delta u = 0$ . So the minimisers (or critical points) of the energy functional are harmonic functions.

□

**Example 1.2 (Minimal Surfaces)** Here we take  $F(x, z, p) = \sqrt{1 + |p|^2}$ , so that

$$\mathcal{F}(u) = \int_{\Omega} \sqrt{1 + |Du|^2}$$

is the  $n$ -dimensional area of the surface determined by  $u$ . So the Euler-Lagrange equations here become:

$$\nabla \cdot \left( \frac{Du}{\sqrt{1 + |Du|^2}} \right) = 0 \quad \text{in } \Omega,$$

which is the minimal surface equation.

□

**Remark:** The existence theory of solutions to Laplace's equation, subject to fixed boundary data, is 'trivial'. However the existence theory of solutions to the minimal surface equation is highly non-trivial. In fact, existence to the Dirichlet problem (i.e. solutions subject to a fixed boundary) fails unless  $\Omega$  has a suitable convexity property (known as *mean convex*).

**Remark:** Local properties of solutions to Laplace's equation and the minimal surface equation have much in common. This is intuitively seen from the equations, since locally,  $|Du| \sim \text{constant}$ , and so the denominator in the minimal surface equation is approximately constant, and so pulling this out gives  $\sim C \nabla \cdot Du = C \Delta u$ , which has the same form as Laplace's equation.

However for entire solutions, i.e. those defined on all of  $\mathbb{R}^n$ , the global behaviour is very different. For example, compare the following results.

**Theorem 1.1 (Liouville's Theorem for Harmonic Functions)** If  $u : \mathbb{R}^n \rightarrow \mathbb{R}$  is  $C^2$ , bounded and satisfies  $\Delta u = 0$ , then  $u \equiv \text{constant}$ .

**Theorem 1.2** The only entire solutions to the minimal surface equation in  $\mathbb{R}^n$  are affine solutions (i.e. planar)  $\iff n \leq 7$ .

**Note:** In the former result, there is an assumption at  $\infty$ , namely that  $u$  is bounded. In the latter, no assumption at  $\infty$  is needed. So note that the global behaviour is very different (e.g. there is a dependence on dimension in the latter but not the former).

## 2 Harmonic Functions

### 2.1 Basic Properties

Let  $\Omega \subset \mathbb{R}^n$  be a domain throughout (i.e. open and connected). As usual we set  $\Delta = \sum_i \frac{\partial^2}{\partial x_i^2}$ .

**Definition 2.1** A function  $u \in C^2(\Omega)$  is  $\left\{ \begin{array}{c} \text{harmonic} \\ \text{subharmonic} \\ \text{superharmonic} \end{array} \right\}$  in  $\Omega$  if  $\left\{ \begin{array}{c} \Delta u = 0 \\ \Delta u \geq 0 \\ \Delta u \leq 0 \end{array} \right\}$  in  $\Omega$ .

The first theorem we prove about harmonic functions is the *mean value property* (MVP). It is of critical importance, and simply tells us that harmonic functions are equal to their averages. We write  $B_\rho(y) := \{x : |x - y| < \rho\}$ .

**Theorem 2.1 (Mean Value Property (MVP))** If  $u \in C^2(\Omega)$  is subharmonic and  $\overline{B_r(y)} \subset \Omega$ , then:

$$u(y) \leq \frac{1}{\omega_n r^n} \int_{B_r(y)} u \quad (1)$$

and

$$u(y) \leq \frac{1}{n\omega_n r^{n-1}} \int_{\partial B_r(y)} u \quad (2)$$

where  $\omega_n = \text{volume of } B_1(0) \subset \mathbb{R}^n$ .

Moreover if  $u$  is superharmonic, then the inequalities reverse. If  $u$  is harmonic, then both are equalities.

**Proof.** We have

$$\begin{aligned} 0 &\leq \int_{B_\rho(y)} \Delta u = \int_{B_\rho(y)} \nabla \cdot Du \stackrel{\text{divergence theorem}}{=} \int_{\partial B_\rho(y)} Du(x) \cdot \underbrace{\frac{x-y}{\rho}}_{\text{outward normal}} dx \\ &\stackrel{\rho w = x-y}{=} \rho^{n-1} \int_{S^{n-1}} w Du(y + \rho w) dw \stackrel{\text{chain rule}}{=} \rho^{n-1} \int_{S^{n-1}} \frac{\partial}{\partial \rho} (u(y + \rho w)) dw \end{aligned}$$

and this is true for all  $\rho \leq r$ . So hence we see, for all  $\rho \leq r$ ,

$$0 \leq \int_{S^{n-1}} \frac{\partial}{\partial \rho} (u(y + \rho w)) dw = \frac{\partial}{\partial \rho} \int_{S^{n-1}} u(y + \rho w) dw.$$

Hence  $\int_{S^{n-1}} u(y + \rho w) dw$  is increasing with  $\rho$ , and so hence we see that

$$\int_{S^{n-1}} u(y + \rho w) dw \leq \int_{S^{n-1}} u(y + r w) dw$$

for all  $0 < \rho \leq r$ . So letting  $\rho \rightarrow 0$  in this, we get (by continuity of  $u$ )

$$n\omega_n u(y) \leq \frac{1}{r^{n-1}} \int_{\partial B_r(y)} u(x) dx$$

(after changing variables in the integral on the RHS), thus we get (2).

To see (1), note that this gives:  $n\omega_n \rho^{n-1} \leq \int_{\partial B_\rho(y)} u$  for all  $\rho \leq r$ , and so integrating this inequality over  $\rho \in (0, r]$  gives:

$$\omega_n r^n u(y) \leq \int_{B_r(y)} u$$

which is (1).

This proves the subharmonic case. To see the superharmonic case, just note that then  $-u$  is subharmonic and apply that case. For the harmonic case, note that harmonic functions are both superharmonic and subharmonic, and so combine those results to get equalities.

□

**Remark:** In fact the MVP characterises harmonic functions - see Example Sheet 1.

**Theorem 2.2 (The Strong Maximum Principle (SMP))**

*Suppose  $u \in C^2(\Omega)$  is subharmonic in  $\Omega$ . Suppose  $\exists y_0 \in \Omega$  with  $u(y_0) = \sup_\Omega u$ . Then  $u = \text{constant in } \Omega$ .*

*If instead  $u$  is superharmonic, then the same result holds if  $u$  attains its infimum.*

*If  $u$  is instead harmonic, the same conclusion holds if  $u$  attains either its supremum or infimum.*

**Proof.** It suffices to prove the subharmonic case, as the others follow just as before from the subharmonic case.

So let  $\Sigma = \{y \in \Omega : u(y) = M\}$  where we have let  $M = \sup_\Omega u$  ( $< \infty$  by assumption). Then clearly by assumption  $M \neq \emptyset$ , and is closed (as  $u$  is continuous). So by connectedness, it suffices to show that  $\Sigma$  is open.

So pick  $y \in \Sigma$ . Then by the MVP we have, for any  $\rho > 0$  such that  $\overline{B_\rho(y)} \subset \Omega$ ,

$$M = u(y) \leq \frac{1}{n\omega_n \rho^n} \int_{B_\rho(y)} u \implies \int_{B_\rho(y)} (M - u) \leq 0.$$

But then  $M - u \geq 0$  and is continuous, and so it follows that we must have  $u \equiv M$  on  $B_\rho(y)$ , i.e.  $B_\rho(y) \subset \Sigma$ . So hence  $\Sigma$  is open, and so we are done.

□

Here the SMP is easy to prove, given the MVP. However in general PDE's it is not simple to prove, and a weaker statement is proven first. We now deduce it from the SMP for this harmonic function case.

**Theorem 2.3 (The Weak Maximum Principle (WMP))**

Suppose  $\Omega \subset \mathbb{R}^n$  is a **bounded** domain, and that  $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ . Then, if  $u$  is subharmonic in  $\Omega$ , we have  $\sup_{\Omega} u = \sup_{\partial\Omega} u$ .

If instead  $u$  were superharmonic, the same result holds but with  $\inf$ 's instead.

If instead  $u$  is harmonic, both hold.

**Proof.** This is immediate from the SMP, as  $\sup_{\Omega} u$  and  $\inf_{\Omega} u$  are attained in  $\overline{\Omega}$  (as  $\Omega$  is bounded, so nothing escapes to infinity), but by the SMP, they are not attained in  $\text{Int}(\Omega)$  (unless  $u$  is constant, but then this is trivial). □

The MVP indicates that  $u$  is always a average of itself. This suggests that  $u$  can never vary too much relative to itself, and thus we should be able to related the sup and inf of  $u$  to one another. The Harnack inequality makes this precise. However note that if  $u$  ever changed sign, then there would be no hope of bounding  $\sup u$  by  $\inf u$ , as they would be of different signs. So to get around this technicality, we assume  $u$  is (wlog) non-negative.

**Theorem 2.4 (The Harnack Inequality)** Suppose  $u \in C^2(\Omega)$  is **non-negative** and harmonic in  $\Omega$ . Then if  $\Omega' \subset \subset \Omega$  is any bounded subdomain, we have

$$\sup_{\Omega} u \leq C \inf_{\Omega} u$$

for some  $C = C(n, \Omega', \Omega)$ .

**Proof.** First, choose  $y \in \Omega$  and  $\rho > 0$  such that  $\overline{B_{4\rho}(y)} \subset \Omega$ . Then pick  $x_1, x_2 \in B_{\rho}(y)$ . Then the MVP gives:

$$\begin{aligned} u(x_1) &= \frac{1}{\omega_n \rho^n} \int_{B_{\rho}(x_1)} u \leq \frac{1}{\omega_n \rho^n} \int_{B_{2\rho}(y)} u && \text{as } B_{\rho}(x_1) \subset B_{2\rho}(y), \\ \text{and } u(x_2) &= \frac{1}{\omega_n (3\rho)^n} \int_{B_{3\rho}(x_2)} u \geq \frac{1}{\omega_n (3\rho)^n} \int_{B_{2\rho}(y)} u && \text{as } B_{2\rho}(y) \subset B_{3\rho}(x_2). \end{aligned}$$

Hence combining these inequalities we see that  $u(x_1) \leq 3^n u(x_2)$  for all  $x_1, x_2 \in B_{\rho}(y)$ . So we get such an inequality locally in balls, with constant independent of  $\rho, y$ , so long as  $\overline{B_{4\rho}(y)} \subset \Omega$ .

So now choose  $x_1, x_2 \in \overline{\Omega'} \subset \Omega$  such that  $u(x_1) = \sup_{\Omega'} u$  and  $u(x_2) = \inf_{\Omega'} u$ . Then by (path) connectedness of  $\Omega$ ,  $\exists$  a continuous path  $\gamma \subset \overline{\Omega'}$  joining  $x_1$  and  $x_2$ .

Then as  $\gamma$  is compact in  $\Omega$ , and  $\Omega$  is open, we can choose  $\rho > 0$  such that  $4\rho < \text{dist}(\gamma, \partial\Omega)$ . Then choose  $N = N(\Omega', \Omega)^*$  such that we can cover  $\gamma$  by  $N$  balls of radius  $\rho$ , so  $\gamma \subset \bigcup_{i=1}^N B_{\rho}(y_i)$  for some  $y_i \in \Omega'$ .

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\*We need to be careful here: this  $N$  depends on  $\gamma$ , which depends on  $u$  via its endpoints, so it is not immediately true that  $N$  is independent of  $u$ . What we need to do is as follows: find a finite cover of  $\overline{\Omega'}$  by such balls of radius  $\rho$ , where the above inequality holds. Say there are  $N$  such balls. Then take a path from  $x_1$  to  $x_2$  going through each of these balls at most once (which we can do as the balls cover, e.g. could take a piecewise linear path). Then we get that this argument is true with  $C = 3^{nN}$ , independent of  $u$ .

Then apply the local result along  $\gamma$  in each ball to get:

$$u(x_1) \leq \underbrace{3^n \cdots 3^n}_{N \text{ times}} u(x_2) = 3^{nN} u(x_2)$$

and so we are done, by the choice of  $x_1, x_2$ , with  $C = 3^{nN}$ .

□

**Theorem 2.5 (Derivative Estimates)** *Suppose  $u \in C^2(\Omega)$  is harmonic in  $\Omega$ . Then if  $B_\rho(y) \subset \Omega$ , then*

$$|Du(y)| \leq \frac{C}{\rho} \sup_{\partial B_\rho(y)} |u|$$

where  $C = C(n)$ .

**Proof.**  $\Delta u = 0 \implies 0 = D_i(\Delta u) = \Delta(D_i u)$  in  $\Omega$ . So  $D_i u$  is harmonic. So by the MVP applied to  $D_i u$ ,

$$\begin{aligned} D_i u(y) &= \frac{1}{\omega_n \rho^n} \int_{B_\rho(y)} D_i u = \frac{1}{\omega_n \rho^n} \int_{B_\rho(y)} \nabla \cdot (0, \dots, 0, \underbrace{u}_{i\text{th}}, 0, \dots, 0) \\ &= \frac{1}{\omega_n \rho^n} \int_{B_\rho(y)} u(x) v_i(x) dx \quad (\text{by the divergence theorem}) \end{aligned}$$

where  $v_i(x) = \frac{x_i - y_i}{\rho}$  is a component of the outward normal. Thus as  $|v_i(x)| \leq 1$ , we have

$$|D_i u(y)| \leq \frac{1}{\omega_n \rho^n} \cdot \sup_{\partial B_\rho(y)} |u| \cdot \int_{\partial B_\rho(y)} dx = \frac{n}{\rho} \sup_{\partial B_\rho(y)} |u|,$$

and thus

$$|Du(x)| \leq \frac{C}{\rho} \sup_{\partial B_\rho(y)} |u|$$

for some  $C = C(n)$ .

□

**Remark:** Applying this result iteratively, we get that for any  $\Omega' \subset\subset \Omega$ , and for any multi-index  $\alpha$ , we have

$$\sup_{\Omega'} |D^\alpha u| \leq C \sup_{\Omega} |u|$$

for some  $C = C(n, \alpha, \Omega, \Omega')$ .

It is now very easy to prove the uniqueness of solutions to the Dirichlet problem.

**Theorem 2.6 (Uniqueness of Solutions to the Dirichlet Problem)** *Suppose  $\Omega$  is a **bounded** domain, and  $u_1, u_2 \in C^2(\Omega) \cap C^0(\overline{\Omega})$  satisfy*

$$\Delta u_1 = \Delta u_2 \quad \text{in } \Omega \quad \text{and} \quad u_1 = u_2 \quad \text{on } \partial\Omega.$$

*Then  $u_1 = u_2$  on  $\overline{\Omega}$ .*



**Proof.** Set  $w = u_1 - u_2$ . Then  $\Delta w = 0$  in  $\Omega$ , and  $w = 0$  on  $\partial\Omega$ . Then by applying the WMP to  $w$  we get  $w = 0$  in  $\overline{\Omega}$ . □

So we know that if  $u \in C^2(\Omega)$  and  $\Delta u = 0$  in  $\Omega$ , then  $\sup_{\Omega'} |D^\alpha u| \leq C \sup_{\Omega'} |u|$  for some  $C = C(n, \alpha, \Omega, \Omega')$ . Also, in view of the MVP, we have

$$\sup_{\Omega'} |D^\alpha u| \leq C \int_{\Omega} |u|,$$

where again  $C = C(n, \alpha, \Omega, \Omega')$ . So in particular we can prove Liouville's theorem from before:

**Corollary 2.1 (Liouville's Theorem for Harmonic Functions)** *If  $u \in C^\infty(\mathbb{R}^n)$  is harmonic in  $\mathbb{R}^n$ , and if  $u$  grows **sublinearly at  $\infty$** , then  $u = \text{constant}$ .*

**Note:** By growing sublinearly at  $\infty$ , we mean that  $|u(x)| \leq C(1 + |x|^\alpha)$ , for some  $\alpha \in (0, 1)$ , with  $C$  some fixed constant, for all  $x \in \mathbb{R}^n$ .

**Proof.** From Theorem 2.5, we know that

$$\sup_{\partial B_{\rho/2}(y)} |Du| \leq \frac{C}{\rho} \sup_{B_\rho(y)} |u|$$

where  $C = C(n)$ . So fix any  $y \in \mathbb{R}^n$ . Thus combining this inequality with the growth hypothesis gives

$$|Du(y)| \leq \frac{C}{\rho} \cdot \sup_{B_\rho(y)} |u| \leq \frac{C}{\rho} \cdot (1 + (\rho + |y|)^\alpha).$$

Then letting  $\rho \rightarrow \infty$  in this to see that  $Du(y) = 0$ . So hence as  $y$  was arbitrary, we see  $Du \equiv 0$ , and so  $u = \text{constant}$ . □

**Note:** We needed to use this derivative bound, because we needed to know the  $\rho$  dependence on the RHS in order to take the limit.

**Example 2.1** *Consider the minimal surface equation:*

$$\nabla \cdot \left( \frac{Du}{\sqrt{1 + |Du|^2}} \right) = 0.$$

*Then this is:*

$$\sum_{i,j=1}^n \left( \delta_{ij} - \frac{D_i u D_j u}{1 + |Du|^2} \right) D_{ij} u = 0$$

*i.e.  $\sum_{i,j} a^{ij} D_{ij} u = 0$ , where  $a^{ij}(x) = \delta_{ij} - \frac{D_i u D_j u}{1 + |Du|^2}$ . This equation is always elliptic, but not uniformly elliptic, as the smallest eigenvalue  $\rightarrow 0$  as  $|x| \rightarrow \infty$ .* □

## 2.2 Existence Theory for Harmonic Functions

*Typical Problem:* Solve the Dirichlet problem for the Laplacian.

By this we mean, given  $\Omega \subset \mathbb{R}^n$  a bounded domain, and a function  $\varphi : \overline{\Omega} \rightarrow \mathbb{R}$  continuous, then find  $u \in C^\infty(\Omega) \cap C^0(\overline{\Omega})$  such that

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega. \end{cases}$$

We will assume for simplicity that  $\partial\Omega$  is smooth, and that  $\varphi \in C^\infty(\overline{\Omega}; \mathbb{R})$ . There are at least 3 different methods of solving this problem:

### 1. Hilbert Space Method

Here the idea is to use the Riesz representation theorem, which depends on the linearity of the equation, to get:

- Existence of a solution in a Sobolev space  $W^{1,2}(\Omega)$ ,
- Then deal with the regularity separately.

### 2. Direct Method of the Calculus of Variations

The idea here is to prove the existence of solutions in  $W^{1,2}(\Omega)$  based on the fact that Laplace's equation is variational, i.e. it is the Euler-Lagrange equation for the Dirichlet energy.

### 3. Perron's Method

Here we use the fact that the solvability of the Dirichlet problem in balls implies the solvability of the Dirichlet problem in more general domains. This idea is based on the maximum principles. (We will look at this idea later in more general.)

Note that in all of these methods, we first prove the existence of solutions in some larger space of functions, and then prove the regularity theory afterward to show that it is in the correct space.

We now look at the direct method (Method 2. in the above) in more detail.

### The Direct Method for the Laplace Equation

Define  $\mathcal{S} = \{w \in W^{1,2}(\Omega) : w - \varphi \in W_0^{1,2}(\Omega)\}$ , i.e. the functions in  $W^{1,2}(\Omega)$  which agree with  $\varphi$  locally about  $\partial\Omega$ . Then  $\mathcal{S} \neq \emptyset$  since  $\varphi \in \mathcal{S}$ .

So set  $\beta = \inf_{w \in \mathcal{S}} \mathcal{E}(w) \in \mathbb{R}$ , where  $\mathcal{E}$  is the Dirichlet energy. Note  $\beta \geq 0$  clearly. So hence as this is an infimum, we get that  $\exists$  a sequence  $(w_j)_j \subset \mathcal{S}$  such that  $\mathcal{E}(w_j) \rightarrow \beta$ . We now want to extract a convergent subsequence, and show that the limit is a solution.

So we see that  $\int_\Omega |Dw_j|^2 \leq \beta + 1$  for all  $j$  sufficiently large. So to hence to bound  $\|w_j\|_{W^{1,2}(\Omega)}$ , we just need to bound  $\|w_j\|_{L^2(\Omega)}$ .

But as  $w_j - \varphi \in W_0^{1,2}(\Omega)$ , we have by the Poincaré inequality,

$$\int_{\Omega} |w_j - \varphi|^2 \leq C \int_{\Omega} |D(w_j - \varphi)|^2$$

for some  $C = C(n, \Omega)$  which implies that

$$\int_{\Omega} |w_j|^2 \leq \tilde{C}(C, \beta) < \infty$$

as  $\int_{\Omega} |\varphi|^2 < \infty$ . So hence we have control on  $\|w_j\|_{W^{1,2}(\Omega)}$  for  $j$  large. Hence by the Rellich Compactness Theorem,  $\exists$  a subsequence  $(j')$  and  $w \in W^{1,2}(\Omega)$  such that

$$\begin{aligned} w_{j'} &\rightharpoonup w && \text{in } W^{1,2}(\Omega) \\ w_{j'} &\rightarrow w && \text{in } L^2(\Omega). \end{aligned}$$

So in particular,

$$\int_{\Omega} Dw_{j'} \cdot Dv \rightarrow \int_{\Omega} Dw \cdot Dv \quad \text{for all } v \in W^{1,2}(\Omega)$$

as the map  $w' \mapsto \int_{\Omega} Dw' \cdot Dv$  is a linear functional on  $W^{1,2}(\Omega)$ .

Also we must have  $w_{j'} - \varphi \rightharpoonup w - \varphi$  in  $W^{1,2}(\Omega)$ . But  $w_{j'} - \varphi \in W_0^{1,2}(\Omega)$ , which is a norm-closed subspace, and hence is weakly closed (since in a Banach space, a convex subset is norm-closed  $\iff$  it is weakly closed).

So hence we must have  $w - \varphi \in W_0^{1,2}(\Omega)$ , i.e.  $w \in \mathcal{S}$ . So in particular,  $\mathcal{E}(w) \geq \beta$  by definition of  $\beta$ .

Finally, noting that  $\mathcal{E}(\cdot)$  is lower semi-continuous<sup>†</sup> with respect to weak convergence in  $W^{1,2}(\Omega)$ , we have

$$\mathcal{E}(w) \leq \liminf_{j' \rightarrow \infty} \mathcal{E}(w_{j'}) = \beta$$

and so hence we have  $\mathcal{E}(w) = \beta = \inf_{w' \in \mathcal{S}} \mathcal{E}(w')$ .

So hence we see that this implies

$$\int_{\Omega} Dw \cdot Dv = 0 \quad \text{for all } v \in W_0^{1,2}(\Omega),$$

which can be proved via using integration by parts and approximation, by approximating  $v$  by smooth compactly supported functions. This is the Euler-Lagrange equation in weak-form. But then this is just the weak form of Laplace's equation, and so  $\Delta w = 0$  weakly, and so we have existence. □

To prove the regularity theory, we will show that weak solutions of Laplace's equation are in fact actual solutions to Laplace's equation (this is Weyl's lemma).

So we have shown the existence of a weak solution to the Dirichlet problem.

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<sup>†</sup>To see this, note that by weak convergence,  $\int_{\Omega} Dw_{j'} \cdot Dv \rightarrow \int_{\Omega} Dw \cdot Dv$  for all  $v \in W^{1,2}(\Omega)$ , and so apply this with  $v = w$  and use Holder's inequality.

### 2.3 Interior Regularity

Next we need to prove the regularity of the weak solution. Indeed, note that we have shown that  $\exists u \in L^1(\Omega)$  which satisfies

$$\int_{\Omega} u \cdot \Delta v = 0 \quad \text{for all } v \in C_c^\infty(\Omega).$$

The key result is Weyl's Lemma, which tells us that weakly harmonic functions are in fact smooth harmonic.

**Theorem 2.7 (Weyl's Lemma)** *Suppose  $\Omega \subset \mathbb{R}^n$  is open and  $u \in L_{loc}^1(\Omega)$ . Then if*

$$\int_{\Omega} u \cdot \Delta v = 0 \quad \text{for all } v \in C_c^\infty(\Omega),$$

*then  $u \in C^\infty(\Omega)$  and  $\Delta u = 0$  in  $\Omega$ .*

**Proof.** First mollify  $u$ : take  $\varphi \in C^\infty(\mathbb{R}^n)$  with  $\varphi \equiv 0$  in  $\mathbb{R}^n \setminus B_1(0)$ ,  $\varphi \geq 0$  in  $\mathbb{R}^n$  and  $\int_{\mathbb{R}^n} \varphi = 1$ . Wlog assume that  $\varphi$  is radially symmetric.

Then for  $\sigma > 0$ , let

$$\varphi_\sigma(x) = \frac{1}{\sigma^n} \varphi\left(\frac{x}{\sigma}\right).$$

Then  $\varphi \in C_c^\infty(B_\sigma(0))$  is  $\geq$  with  $\int_{\mathbb{R}^n} \varphi_\sigma = 1$ . The let  $u_\sigma = \varphi_\sigma * u$  be this convolution. This is well-defined and finite for  $x \in \Omega_\sigma := \{x \in \Omega : \text{dist}(x, \partial\Omega) > \sigma\}$ .

Then we know that  $u_\sigma$  is smooth in  $\Omega_\sigma$  (via differentiation under the integral sign), and also that  $u_\sigma \rightarrow u$  in  $L_{loc}^1(\Omega)$ . Then we can see that  $\Delta u_\sigma = 0$  in  $\Omega_\sigma$ : indeed,

$$D_{x_i} u_\sigma(x) = \int_{\Omega} u(y) \cdot D_{x_i} (\varphi_\sigma(x - y)) dy = - \int_{\Omega} u(y) \cdot D_{y_i} (\varphi_\sigma(x - y)) dy$$

and so hence we see:

$$\Delta_x u_\sigma(x) = \int_{\Omega} u(y) \cdot \Delta_y (\varphi_\sigma(x - y)) dy = 0$$

where we have used the definition of weakly harmonic in the last equality. So hence  $u_\sigma$  is harmonic in  $\Omega_\sigma$ .

Thus by the a priori derivative estimates we proved for smooth harmonic functions, we see that for any multi-index  $\alpha$ ,

$$\sup_{\Omega'} |D^\alpha u_\sigma| \leq C \int_{\Omega_{\sigma_1}} |u_\sigma|$$

for some fixed  $\sigma_1$  small enough, depending on  $\Omega'$ , and where  $C = C(\Omega', \sigma, u)$ , and this is true for all  $\sigma < \sigma_1$ . But then as  $u_\sigma \rightarrow u$  in  $L_{loc}^1(\Omega)$ , for small enough  $\sigma_1$  we can bound the RHS of this by  $C \left( \int_{\Omega_{\sigma_1}} |u| + 1 \right)$ , and thus we get a uniform bound in  $\sigma$  on  $\sup_{\Omega'} |D^\alpha u_\sigma|$ .

So hence by the Arzela-Ascoli theorem (as bounded derivatives implies equicontinuous), we can find a sequence  $(\sigma_j)_{j=1}^\infty$  with  $\sigma_j \downarrow 0$ , and  $\tilde{u} \in C^\infty(\Omega)$  s.t.  $u_{\sigma_j} \rightarrow \tilde{u}$  in  $C^k(\Omega')$ , for any  $k \in \mathbb{N}$  and any  $\Omega' \subset \subset \Omega$  (this involves a diagonal argument).

Hence we see that  $\Delta \tilde{u} = \lim_j \Delta u_{\sigma_j} = 0$  in  $\Omega$  by continuity of these derivatives. Then since  $u_{\sigma_j} \rightarrow u$  a.e. in  $\Omega$  (from above), we must have that  $u = \tilde{u}$  a.e. in  $\Omega$ , and so we are done.  $\square$

The results for boundary regularity we shall omit here - they should be in the Analysis of PDEs Part III course.

**Theorem 2.8 (Exist. and Uniq. of Solutions to the DP with Contin. Boundary Data)**  
*Suppose  $\Omega$  is a bounded domain with sufficiently smooth boundary<sup>‡</sup>. Then for any  $\varphi \in C^0(\partial\Omega)$  **continuous**,  $\exists!$   $u \in C^\infty(\Omega) \cap C^0(\overline{\Omega})$  solving the Dirichlet problem*

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega. \end{cases}$$

**Warning:** We might not have  $\int_\Omega |Du|^2 < \infty$  for this solution  $u$  in the theorem!

**Proof.** Choose a sequence  $(\varphi_n)_n \in C^\infty(\mathbb{R}^n)$  s.t.  $\varphi_n \rightarrow \varphi$  uniformly on  $\partial\Omega$ . Then by the boundary assumption, we know that we can solve the Dirichlet problem with boundary data  $\varphi_n$  for each  $n$ , i.e.  $\exists u_n \in C^\infty(\Omega) \cap C^0(\overline{\Omega})$  such that  $\Delta u_n = 0$  in  $\Omega$ , and  $u_n = \varphi_n$  on  $\partial\Omega$ .

Then for each  $n, m$  we have  $\Delta(u_n - u_m) = 0$  in  $\Omega$  and  $u_n - u_m = \varphi_n - \varphi_m$  on  $\partial\Omega$ . By the WMP, we have

$$\sup_{\overline{\Omega}} |u_n - u_m| \leq \sup_{\partial\Omega} |u_n - u_m| = \sup_{\partial\Omega} |\varphi_n - \varphi_m| \rightarrow 0$$

as  $n, m \rightarrow \infty$ . Hence  $(u_n)_n$  is Cauchy in  $C^0(\overline{\Omega})$ . So hence as this space is complete, we get that  $\exists u \in C^0(\overline{\Omega})$  such that  $u_n \rightarrow u$  uniformly on  $\overline{\Omega}$ . In particular,  $u = \varphi$  on  $\partial\Omega$ .

Then by the interior derivative estimates for  $u_n$ , we see that  $(u_n)_n$  also converges in  $C^k(\Omega')$  for any  $\Omega' \subset\subset \Omega$ . Hence we have  $u \in C^\infty(\Omega)$ , and so by taking limits of second order derivatives,  $\Delta u = 0$  in  $\Omega$ . So done.  $\square$

### Remarks:

1. If  $\Omega$  is a  $C^2$ -domain, then the requirement of the theorem is satisfied.

More generally, if  $\Omega$  satisfies the exterior sphere condition, then it is also satisfied (i.e.  $\forall z \in \partial\Omega, \exists B_\rho(y)$  such that  $\overline{B_\rho(y)} \cap \overline{\Omega} = \{y\}$ .)

2. There are bounded domains for which the conclusion of the theorem does not hold  $\rightarrow$  see “Lebesgue’s Example [Dirichlet Energy]” (e.g. if there is a cusp in  $\Omega$ ).

Now we move away from harmonic functions and consider more general 2<sup>nd</sup> order elliptic operators.

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<sup>‡</sup>Precisely,  $\partial\Omega$  should be smooth enough so that the Dirichlet problem:  $\Delta u = 0$  in  $\Omega$ , with  $u = \varphi$  on  $\partial\Omega$ , has a solution  $u \in C^\infty(\Omega) \cap C^0(\overline{\Omega})$  for any given  $\varphi \in C^\infty(\overline{\Omega})$ , i.e.  $\varphi = \tilde{\varphi}|_{\overline{\Omega}}$  for some  $\tilde{\varphi} \in C^\infty(\Omega')$ , where  $\Omega' \supset \overline{\Omega}$  is open.

### 3 General 2nd Order Elliptic Operators

From now on, we shall write

$$Lu := a^{ij} D_{ij}u + b^i D_i u + cu$$

where we have used summation convention. We work on  $\Omega \subset \mathbb{R}^n$  open, and here  $u \in C^2(\Omega)$ , with  $a^{ij}, b^i, c : \Omega \rightarrow \mathbb{R}$ .

We then look at solving the Dirichlet problem (DP),

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$$

for given  $f : \Omega \rightarrow \mathbb{R}$  and  $\varphi : \partial\Omega \rightarrow \mathbb{R}$ .

If we can write  $L$  in **divergence form**, i.e.  $Lu = D_i(a^{ij} D_j u) + b^i D_i u + cu$ , then we can use Hilbert space methods (see Analysis of PDEs Part III course). However if it cannot, then we need a different method. In this course, we look at **Schauder estimates**: this does not involve Sobolev spaces, and the philosophy is to deform the elliptic operator  $L$  into the Laplacian  $\Delta$  by several rescalings, as we shall see.

Due to  $u \in C^2(\Omega)$ , by the symmetry of mixed partial derivatives we can wlog assume that  $a^{ij} = a^{ji}$  (since if not simply replace  $a^{ij}$  with  $\tilde{a}^{ij} = \frac{1}{2}(a^{ij} + a^{ji})$ ). Then as  $D_{ij}u = D_{ji}u$ , we have  $a^{ij} D_{ij}u = \tilde{a}^{ij} D_{ij}u$  (sum over  $i, j$ ).

**Remark:** Summation Convention will always apply here (indices from 1 to  $n$ ).

**Definition 3.1** We make the following definitions for the  $L$  above.

1.  $L$  is **elliptic** in  $\Omega$  if the matrix  $(a^{ij}(x))_{ij}$  is positive definite  $\forall x \in \Omega$ . Equivalently,

$$0 < \lambda(x) |\xi|^2 \leq a^{ij}(x) \xi^i \xi^j \leq \Lambda(x) |\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^n \setminus \{0\}$$

where  $\lambda$  is the **smallest eigenvalue** of this matrix, and  $\Lambda$  the largest.

2.  $L$  is **strictly elliptic** in  $\Omega$  if  $\exists$  a constant  $\lambda_0 > 0$  s.t.  $\lambda_0 < \lambda(x) \forall x \in \Omega$ .
3.  $L$  is **uniformly elliptic** in  $\Omega$  if  $L$  is elliptic and  $\Lambda/\lambda$  is uniformly bounded.

**Note:** It is not true in general that uniformly elliptic implies strict ellipticity.

The goal of this section is to develop an existence theory for solutions  $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$  of the DP above, for bounded domains  $\Omega$  and where  $f, \varphi$  are continuous.

In fact, we will assume, out of necessity, that  $f, a^{ij}, b^i, c$  are in  $C^{0,\alpha}(\overline{\Omega})$ , and prove the existence of  $u \in C^{2,\alpha}(\Omega) \cap C^0(\overline{\Omega})$ . We will also need further assumptions, such as  $c \leq 0$ , but we give these when they are needed.

Since we will only assume  $C^{0,\alpha}$  regularity of the  $a^{ij}$ , the operator  $L$  cannot be written in divergence form (where energy methods and Euler-Lagrange methods are used). Hence we will not use any Sobolev theory for our approach.

This method will be based on a priori estimates for the  $C^{2,\alpha}$  norm of the solutions (interior and boundary estimates). The starting points of these estimates is the **weak maximum principle (WMP)**.

### 3.1 Basic Properties

#### Theorem 3.1 (Weak Maximum Principle (WMP))

Suppose that  $L$  is elliptic and that  $\sup_{\Omega} |\frac{b^i}{\lambda}| < \infty$ . Suppose that  $\Omega$  is bounded, open, and that  $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$  satisfies  $Lu \geq 0$  (i.e. a **subsolution** of  $Lu = 0$ ). Then:

- (i) If  $c = 0$ , then  $\sup_{\Omega} u = \sup_{\partial\Omega} u$ .
- (ii) If  $c \leq 0$ , then  $\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+$ , where  $u^+ = \max(u, 0)$ .

**Remark:** in (ii), we cannot drop the assumption that  $c \leq 0$  in  $\Omega$ . For example, consider the Dirichlet eigenfunctions of the Laplacian on bounded domains. Then for example:

- If  $n = 1$ , consider  $u(x) = \sin(x)$  for  $x \in (0, \pi)$ . Then,  $u'' + u = 0$ . For this equation, we have  $c \equiv 1 > 0$ , but  $\sup_{\Omega} u = 1$  whilst  $\sup_{\partial\Omega} u^+ = \max(u(0), u(\pi)) = 0$ , a contradiction.
- If  $n = 2$ , then consider  $u(x) = \sin(x) \sin(y)$  on  $\Omega = (0, \pi) \times (0, \pi)$ . Then  $\Delta u + 2u = 0$  in  $\Omega$ , but  $u \equiv 0$  on  $\partial\Omega$ , again a contradiction.

**Proof. (i):** If  $Lu > 0$  in  $\Delta$ , then the conclusion follows easily: in fact, in this case there cannot be a local maximum in  $\text{Int}(\Omega)$  (i.e. the strong maximum principle also holds).

Indeed, to see this, note that if  $x_0 \in \text{Int}(\Omega)$  is a local maximum, then  $Du(x_0) = 0$ , and  $D^2u(x_0) = [D_{ij}u(x_0)]_{ij}$  is non-positive definite. Thus as  $A = [a^{ij}(x_0)]_{ij}$  is positive definite (by ellipticity assumption), we see that

$$a^{ij} D_{ij}u(x_0) = \text{Trace}(A \cdot D^2u(x_0)) \leq 0,$$

as then  $A \cdot D^2u(x_0)$  is non-positive (as negative  $\times$  positive = negative). Hence,

$$0 < Lu(x_0) = a^{ij} D_{ij}u(x_0) + \underbrace{b^i D_i u(x_0)}_{=0} \leq 0,$$

a contradiction (remember that  $c = 0$  here). So we cannot have an interior maximum in this case, so done.

In the general case where  $Lu \geq 0$ , then consider the function  $v(x) = e^{\gamma x_1}$ , for some  $\gamma > 0$  is a constant we will choose. So then we can directly compute:

$$\begin{aligned} D_1 v &= \gamma e^{\gamma x_1}, \quad \text{and} \quad D_i v = 0 \quad \text{if } i > 1 \\ D_{11} v &= \gamma^2 e^{\gamma x_1}, \quad \text{and} \quad D_{ij} v = 0 \quad \text{for } (i, j) \neq (1, 1). \end{aligned}$$

So then

$$Lv = e^{\gamma x_1} (\gamma^2 a^{11} + b^1 \gamma) \stackrel{\text{by ellipticity}}{\geq} e^{\gamma x_1} (\lambda \gamma^2 + b^1 \gamma) = e^{\gamma x_1} \lambda (\gamma^2 + \frac{b^1}{\lambda} \cdot \gamma) > 0$$

in  $\Omega$ , if we choose  $\gamma$  sufficiently large depending on  $\sup_{\Omega} \left| \frac{b^1}{\lambda} \right| < \infty$ .

So hence,  $Lu \geq 0 \implies L(u + \epsilon v) > 0$  for any  $\epsilon > 0$  (where  $\gamma$  is now fixed). So applying the above case gives:

$$u(x) \stackrel{\text{as } v \geq 0}{\leq} \sup_{\Omega} (u + \epsilon v) \stackrel{\text{by above case}}{\leq} \sup_{\partial\Omega} (u + \epsilon v) \leq \sup_{\Omega} u + \epsilon \underbrace{\sup_{\Omega} v}_{\text{bounded}}.$$

So hence take  $\epsilon \downarrow 0$  to get that  $u(x) \leq \sup_{\partial\Omega} u$  for all  $x \in \Omega$ , i.e.  $\sup_{\Omega} u \leq \sup_{\partial\Omega} u$ . But the other inequality is trivial (by inclusion essentially), and thus we are done.

(ii): Define  $L_0$  by just the 1st and 2nd order terms of  $L$ , i.e.

$$L_0 u = a^{ij} D_{ij} u + b^i D_i u.$$

So, in the set  $\Omega^+ = \{x \in \Omega : u(x) > 0\}$ , we have that, since  $cu \leq 0$  here,

$$L_0 u = Lu - cu \geq -cu \geq 0$$

Note that if  $\Omega^+ = \emptyset$ , then the conclusion is trivially true, as then  $u^+ = 0$  and so the RHS of the conclusion is 0.

So we can wlog assume that  $\Omega^+ \neq \emptyset$ . But then we must have  $\partial\Omega^+ \cap \partial\Omega \neq \emptyset$ , **and** there must exist  $x_0 \in \partial\Omega^+ \cap \partial\Omega$  such that  $u(x_0) > 0$  (since if not, then  $u \leq 0$  on  $\partial\Omega^+$  (since then  $\partial\Omega^+ \cap \partial\Omega = \emptyset$ , and so  $\partial\Omega^+ \subset \text{Int}(\Omega)$ , and so  $\partial\Omega^+ \subset \Omega \setminus \Omega^+$  (as  $\Omega^+$  is open), and so hence  $u \leq 0$  here as not in  $\Omega^+$ ). But then this contradicts (i) with  $L \rightarrow L_0$  and  $\Omega \rightarrow \Omega^+$ ).

So hence

$$\sup_{\Omega} u \stackrel{\sup_{\Omega} u > 0}{=} \sup_{\Omega^+ \neq \emptyset} u \stackrel{\text{by (i)}}{=} \sup_{\partial\Omega^+} u \stackrel{(\star)}{\leq} \sup_{\partial\Omega} u \stackrel{\text{as } u \leq u^+}{=} \sup_{\partial\Omega} u^+$$

where inequality  $(\star)$  is true because  $\partial\Omega^+$  is either in  $\text{Int}(\Omega)$  (in which case  $u = 0$  here, as  $\Omega = \Omega^+ \cup \{x \in \Omega : u(x) \leq 0\}$ ) or on  $\partial\Omega$ , where we could have  $u > 0$ . So hence the sup is equal to just that over the  $\partial\Omega^+ \cap \partial\Omega$ , which is  $\leq$  that over  $\partial\Omega$ .

So hence we are done. □

We now move to proving the Strong maximum principle (SMP), using the WMP. Let  $M = \sup_{\Omega} u$ , and consider  $\Sigma = \{x \in \Omega : u(x) = M\}$ . We first deduce some corollaries of the WMP.

**Corollary 3.1** *Let  $\Omega$  be a bounded, open set, and  $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ . Suppose that  $L$  is elliptic, with  $\sup_{\Omega} |b|/\lambda < \infty$ , where  $b = (b_1, \dots, b_n)$ , and  $c \leq 0$  in  $\Omega$ . Then:*

(i) *If  $Lu \leq 0$  in  $\Omega$ , then  $\inf_{\Omega} u \geq \inf_{\partial\Omega} u^-$ , where  $u^- = \min(u, 0)$ .*

(ii) *If  $Lu = 0$ , then  $\sup_{\Omega} |u| = \sup_{\partial\Omega} |u|$ .*

**Proof.** (i): Apply the WMP to  $-u$ .

(ii): Combine (i) and the usual WMP, as here both  $Lu \geq 0$  and  $Lu \leq 0$  hold.



□

**Corollary 3.2** *Let  $\Omega$  and  $L$  be as above. Suppose that  $u, v, w \in C^2(\Omega) \cap C^0(\overline{\Omega})$  satisfy in  $\Omega$ ,*

$$Lu \geq 0, \quad Lv = 0, \quad Lw \leq 0.$$

*Then,*

1. *If  $u \leq v$  on  $\partial\Omega$ , then  $u \leq v$  in  $\overline{\Omega}$ .*
2. *If  $v \leq w$  on  $\partial\Omega$ , then  $v \leq w$  in  $\Omega$ .*

**Proof.** (1): Then  $L(u - v) \geq 0$ , and so by the WMP,  $u - v \leq \sup_{\partial\Omega}(u - v) \leq 0$  on  $\Omega$ .

(2): The same (or apply (1) to  $-v, -w$ ).

□

**Remark:** Hence we see the reason for the terminology:

1. If  $u$  satisfies  $Lu \leq 0$  then it is called a **subsolution** of the equation  $Lu = 0$  (as such functions are below all solutions of the DP, by the above).
2. If  $u$  satisfies  $Lu \geq 0$  then it is called a **supersolution** of the equation  $Lu = 0$  (as such functions are above all solutions of the DP, by the above).

Now we work toward the SMP. We need the following lemma, to get the sign of terms.

**Theorem 3.2 (The Hopf Boundary Point Lemma)** *Let  $\Omega \subset \mathbb{R}^n$  be open. Suppose  $y \in \partial\Omega$ , and suppose that  $\Omega$  satisfies the **interior sphere condition** at  $y$  (i.e.  $\exists R > 0, z \in \Omega$  such that  $B_R(z) \subset \Omega$  and  $y \in \partial B_R(z)$ ).*

*Let  $Lu = a^{ij}D_{ij}u + b^iD_iu + cu$  is uniformly elliptic in  $\Omega$ , and suppose  $\sup_{\Omega}|b|/\lambda + \sup_{\Omega}|c|/\lambda < \infty$ . Suppose that  $u \in C^2(\Omega) \cap C^0(\{y\} \cup \Omega)$  satisfies  $u(y) > u(x) \forall x \in \Omega$ , and  $Lu \geq 0$  in  $\Omega$ . Suppose also that one of the following holds:*

- (i)  $c \equiv 0$  in  $\Omega$ ,
- (ii)  $c \leq 0$  in  $\Omega$  and  $u(y) \geq 0$ ,
- (iii)  $u(y_0) = 0$  (no hypothesis on the sign of  $c$ ).

**Then,**  $\frac{\partial u}{\partial \nu}(y) > 0$  if the derivative exists, where  $\nu$  is the outward pointing unit normal to  $\partial B_R(y)$  to  $y$  (and  $\frac{\partial u}{\partial \nu}(y) = \lim_{t \uparrow 0} \frac{u(y+t\nu) - u(y)}{t}$ ).

**Remark:** The significance of this theorem is in the **strict** inequality of the conclusion ( $\frac{\partial u}{\partial \nu}(y) \geq 0$  is trivially true, since  $u(x) < u(y)$  for all  $x \in \Omega$ ).

**Proof.** Let  $A$  be the annulus  $B_R(z) \setminus B_r(z)$ , for some  $0 < r < R$  (e.g.  $r = R/2$ ).

Cases (i)+(ii): Consider the function on  $A$  defined by

$$v(x) = e^{-\alpha|x-z|^2} - e^{-\alpha R^2}.$$

Then, we clearly have  $D_i v(x) = -2\alpha(x_i - z_i) \cdot e^{-\alpha|x-z|^2}$  and

$$D_{ij} v(x) = -2\alpha e^{-\alpha|x-z|^2} \delta_{ij} + 4\alpha^2 (x_i - z_i)(x_j - z_j) e^{-\alpha|x-z|^2}.$$

So hence:

$$\begin{aligned} Lv &= e^{-\alpha|x-z|^2} \left( \underbrace{\sum_{i,j=1}^n a^{ij} (x_i - z_i)(x_j - z_j)}_{\text{bound by ellipticity}} - 2\alpha \underbrace{\sum_{i=1}^n a^{ii}}_{=\text{Trace}(A)=\sum_i \lambda_i \leq n\Lambda} - 2\alpha \sum_{i=1}^n b^i (x_i - z_i) + c \right) - \underbrace{ce^{-\alpha R^2}}_{\geq 0 \text{ by condition on } c} \\ &\geq e^{\alpha|x-z|^2} \left( 4\alpha^2 \lambda(x) \underbrace{|x-z|^2}_{\geq (R/2)^2 \text{ on } A} - 2\alpha n \Lambda(x) - 2\alpha |b| |x-z| - |c| \right) \\ &\geq e^{-\alpha|x-z|^2} \lambda(x) \left( \alpha^2 R^2 - 2\alpha n \cdot \sup_{\Omega} \left( \frac{\Lambda}{\lambda} \right) - \alpha R \cdot \sup_{\Omega} \frac{|b|}{\lambda} - \sup_{\Omega} \frac{|c|}{\lambda} \right) \\ &> 0 \quad \text{for } \alpha \text{ sufficiently large.} \end{aligned}$$

Fix such an  $\alpha$  so that this is  $> 0$ . Then let  $w(x) = u(x) - u(y) + \epsilon v(x)$ , where  $\epsilon > 0$  is to be chosen. Then,  $Lw = Lu + \epsilon Lv - cu(y) \geq 0$  in  $A$ , by the above. Note also that as  $v = 0$  on  $\partial B_R(z)$  and  $u(x) \leq u(y)$  on  $\bar{\Omega}$ , we have that  $w|_{\partial B_R(z)} \leq 0$ . Also, since  $u(x) < u(y)$  on  $\partial B_{R/2}(z)$  (which is compact and so  $u - u(y)$ ,  $v$  attain their minima, which will be  $< 0$  and  $> 0$  respectively), we can choose  $\epsilon > 0$  small enough such that  $w|_{\partial B_{R/2}(z)} < 0$ . Fix this  $\epsilon > 0$ .

So then  $w|_{\partial A} \leq 0$ . So by applying the WMP to  $w$  in  $A$ , we get that  $u(x) - u(y) + \epsilon v(x) \leq 0$  in  $A$ . So hence for  $t < 0$  with  $|t|$  small, we have (as  $v(y) = 0$ , since  $y \in \partial B_R(z)$ ),

$$\frac{u(y+tv) - u(y)}{t} \geq -\epsilon \cdot \frac{v(y+tv) - v(y)}{t}.$$

(The inequality flips as  $t < 0$ .) So hence letting  $t \uparrow 0$ , we get that:

$$\frac{\partial u}{\partial \nu}(y) \geq -\epsilon \cdot \frac{\partial v}{\partial \nu}(y) = -\epsilon Dv(y) \cdot \frac{y-x}{R} = 2\epsilon \alpha R e^{-\alpha R^2} > 0$$

and so done.

**Case (iii):** Consider the operator  $\tilde{L} := L - c^+$ , so that

$$\tilde{L}u = a^{ij} D_{ij} u + b^i D_i u + \underbrace{(c - c^+)}_{\leq 0} u.$$

Thus,  $\tilde{L}u = \underbrace{Lu}_{\geq 0 \text{ by assumption}} - \underbrace{c^+ u}_{\leq c^+ u(y)=0 \text{ here}} \geq 0$ . Then simply apply case (ii) to  $\tilde{L}$  to finish the proof.

□

From the Hopf boundary point lemma, the SMP is now an easy corollary.

**Theorem 3.3 (The Strong Maximum Principle (SMP))** *Suppose  $\Omega \subset \mathbb{R}^n$  is a domain (not necessarily bounded). Suppose that  $L$  is uniformly elliptic, with  $\sup_{\Omega} (|b|/\lambda + |c|/\lambda) < \infty$ . Suppose also that  $u \in C^2(\Omega)$ ,  $Lu \geq 0$  in  $\Omega$ , and that  $M := \sup_{\Omega} u < \infty$ . Then:*

- (i) *If  $c = 0$  and  $u(y) = M$  for some  $y \in \Omega$ , then  $u \equiv M$  in  $\Omega$ .*
- (ii) *If  $c \leq 0$ ,  $M \geq 0$  and  $u(y) = M$  for some  $y \in \Omega$ , then  $u \equiv M$  in  $\Omega$ .*
- (iii) *If  $M = 0$  any  $u(y) = M = 0$  for some  $y \in \Omega$ , then  $u \equiv 0$  in  $\Omega$ .*

**Proof.** Let  $\Sigma = \{x \in \Omega : u(x) = M\}$ . Then by continuity of  $u$ ,  $\Sigma$  is (relatively) closed in  $\Omega$ . So by connectedness, it suffices to show that  $\Sigma$  is (relatively) open (as by assumption  $\Sigma \neq \emptyset$ ).

So pick  $z \in \Omega \cap \Sigma$  s.t.  $\text{dist}(z, \partial\Omega) > \text{dist}(z, \partial\Sigma)$  (i.e. first pick  $z_1 \in \partial\Sigma \cap \Omega$ , which we can do as  $\Omega$  is connected. Then choose  $\rho_1 > 0$  such that  $B_{\rho_1}(z_1) \subset \Omega$ . Then by the triangle inequality, choose our  $z$  to be any  $z \in B_{\rho_1/2}(z_1) \setminus \Sigma$ ).

Then let  $R = \sup\{\rho : B_{\rho}(y) \subset \Omega \subset \Sigma\}$  (the balls of increasing radii will intersect  $\Sigma$  first rather than  $\partial\Omega$  by how  $z$  is chosen). Then  $\exists y \in \partial B_R(z) \cap \Sigma \cap \Omega$ . Then since  $Du(y) = 0$  (as it is a maximum), we get a direct contradiction to the Hopf lemma (as this should be  $> 0$ ). So we have a contradiction, and must have  $u$  constant in this ball, equal to  $M$ . But then this shows that  $\partial\Sigma$  is open, and so  $\Sigma$  is open, and so done.

□

Now we can deduce some corollaries, including uniqueness of the solutions for the Neumann problem.

**Corollary 3.3** *Let  $L = a^{ij}D_{ij} + b^iD_i + c$  is uniformly elliptic in a domain  $\Omega \subset \mathbb{R}^n$ . with  $\sup_{\Omega} (|b|/\lambda + |c|/\lambda) < \infty$ . Suppose that  $u, v \in C^2(\Omega)$  satisfy:*

$$Lu \geq Lv \quad \text{and} \quad u \leq v \quad \text{in } \Omega.$$

*Then,  $u = v$  on  $\Omega$ , or  $u < v$  on all of  $\Omega$ .*

**Proof.** Then we have  $L(u - v) \geq 0$  in  $\Omega$  and  $u - v \leq 0$  in  $\Omega$ . Then if  $\exists x_0 \in \Omega$  with  $u(x_0) = v(x_0)$ , then by Case (iii) of the SMP, we have  $u \equiv v$  in  $\Omega$ . Otherwise we have  $u \neq v$  in  $\Omega$ , and so by have  $u < v$  in  $\Omega$  (as we have by assumption  $u \leq v$ ).

□

**Corollary 3.4 (Uniqueness of Solutions for the Neumann Problem (NP))** *Suppose  $\Omega \subset \mathbb{R}^n$  is a bounded domain and that  $\partial\Omega$  satisfies the **interior sphere condition** at each point. Suppose  $L$  is uniformly elliptic in  $\Omega$ , with  $\sup_{\Omega} (|b|/\lambda + |c|/\lambda) < \infty$ , and  $c \leq 0$ . Then if  $u_1, u_2 \in C^2(\Omega) \cap C^0(\overline{\Omega})$  satisfy :*

$$\begin{cases} Lu_i = f & \text{in } \Omega \\ \frac{\partial u_i}{\partial \nu} = g & \text{on } \partial\Omega \end{cases}$$

for each  $i$ , for some functions  $f : \Omega \rightarrow \mathbb{R}$ ,  $g : \partial\Omega \rightarrow \mathbb{R}$  (i.e. the Neumann problem), then  $u_1 = u_2 + c$ , for some constant  $c$ .

**Proof.** By considering  $u = u_1 - u_2$ , it suffices to show that if

$$\begin{cases} Lu = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

then  $u = c$  is a constant.

So let  $M = \sup_{\overline{\Omega}} u$ . Assume wlog  $M \geq 0$  (or else consider  $-u$ ). Then by the SMP, if  $u \not\equiv M$  on  $\overline{\Omega}$ , then  $\exists y \in \partial\Omega$  such that  $u(y) = M$  **and**  $u(x) < u(y)$  for all  $x \in \Omega$  (as  $c \leq 0$ ).

Then by the Hopf boundary point lemma, we get that  $\frac{\partial u}{\partial \nu}(y) \neq 0$ , which is a contradiction as we have  $\frac{\partial u}{\partial \nu} \equiv 0$  on  $\partial\Omega$ . So hence we must have  $u \equiv M$  is constant in  $\overline{\Omega}$ .

□

**Note:** As a consequence of this, we see that the solution to the NP with zero data must be a constant, and so we must have  $L(M) = 0$ , where  $M$  is this constant. Hence as the derivatives of a constant vanish, this says we must have  $Mc(x) = 0$  for all  $x \in \Omega$ . So hence if  $c(x) \neq 0$  for any  $x \in \Omega$ , we must have  $M = 0$ , and thus the solutions are unique. So it is only when  $c \equiv 0$  that the solutions differ by a constant (which makes sense as then constants will always vanish, as then both  $L$  and the boundary data depend all on derivatives).

**Theorem 3.4 (IMPORTANT: Maximum Principle a priori Estimate (MPAPE))**

Suppose  $\Omega \subset \mathbb{R}^n$  is a bounded domain, with  $L$  elliptic,  $c \leq 0$ , and  $\beta := \sup_{\Omega} |b|/\lambda < \infty$ . Let  $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$  and  $f : \Omega \rightarrow \mathbb{R}$ . Then:

(i) If  $Lu \geq f$  in  $\Omega$ , then  $\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ + C \cdot \sup_{\Omega} \left( \frac{|f|}{\lambda} \right)$ .

(ii) If  $Lu = f$  in  $\Omega$ , then  $\sup_{\Omega} |u| \leq \sup_{\partial\Omega} |u| + C \cdot \sup_{\Omega} \left( \frac{|f|}{\lambda} \right)$

where in both (i) and (ii),  $C = C(\beta, \text{diam}(\Omega))$ .

**Proof.** Let  $d = \text{diam}(\Omega)$ . Then as  $\Omega$  is bounded, we have  $\Omega \subset \{x : a \leq x_1 \leq a + d\}$  for some  $a \in \mathbb{R}$ . We can assume wlog that  $a = 0$  (or else just consider  $\tilde{u}(x) = u(\tilde{x})$ ,  $\tilde{f}(x) = f(\tilde{x})$ , where  $\tilde{x} = (x_1 + a_1, x_2, \dots, x_n)$ , for  $x \in \tilde{\Omega} := \Omega - (a, 0, \dots, 0)$ ).

So  $\Omega \subset \{x : 0 \leq x_1 \leq d\}$  wlog. Then let

$$v(x) = \sup_{\partial\Omega} u^+ + (e^{\alpha d} - e^{\alpha x_1}) \cdot \sup_{\Omega} \frac{|f|}{\lambda}$$

(this expression can be deduced by letting  $v(x) = A + Be^{\alpha x_1}$  and working the calculation through to find what constants  $A$  and  $B$  we need). Then we can just compute:

$$(a^{ij} D_{ij} + bD_i) e^{\alpha x_1} = e^{\alpha x_1} (a^{11} \alpha^2 + b^1 \alpha) \stackrel{a^{11} \geq \lambda \text{ by ellipticity}}{\geq} e^{\alpha x_1} \lambda \left( \alpha^2 + \frac{b^1}{\lambda} \cdot \alpha \right)$$

$$\underset{\text{as } b^1/\lambda \geq -\beta}{\geq} e^{\alpha x_1} \lambda (\alpha^2 - \beta \alpha) \geq \lambda$$

where the last inequality is true **if**  $\alpha = \beta + 1$ , since  $e^{\alpha x_1} \geq 1$  as  $\alpha, x_1 \geq 0$  here.

So then,

$$Lv \leq cv - \lambda \cdot \sup_{\Omega} \frac{|f|}{\lambda} \leq -\lambda \cdot \sup_{\Omega} \frac{|f|}{\lambda}$$

since  $c \leq 0, v \geq 0$ .

(i): Then

$$L(u - v) \geq f + \lambda \cdot \sup_{\Omega} \frac{|f|}{\lambda} = \lambda \left( \frac{f}{\lambda} + \sup_{\Omega} \frac{|f|}{\lambda} \right) \geq 0$$

in  $\Omega$ . Then since  $u \leq u^+$ , we see that  $u(x) \leq v(x)$  for  $x \in \partial\Omega$ . So by the WMP,  $u \leq v$  in  $\Omega$ . So:

$$\sup_{\Omega} u \leq \sup_{\Omega} v \leq \sup_{\partial\Omega} u^+ + C \cdot \sup_{\Omega} \frac{|f|}{\lambda}$$

where  $C = \sup_{x \in \Omega} (e^{(\beta+1)d} - e^{(\beta+1)x_1}) = C(\beta, d)$ . So done with this case.

(ii): Apply case (i) with  $-u$  in place of  $u$  and then combine the results.

□

## 3.2 Hölder Spaces

Let  $\Omega \subset \mathbb{R}^n$  be open. Let  $\alpha \in (0, 1]$ . Fix these throughout.

**Definition 3.2** We say that  $u : \Omega \rightarrow \mathbb{R}$  is **uniformly Hölder continuous with exponent  $\alpha$**  if:

$$[u]_{\alpha; \Omega} := \sup_{x, y \in \Omega} \frac{|u(x) - u(y)|}{|x - y|^\alpha} < \infty.$$

This quantity is known as the **Hölder semi-norm**.

If  $\alpha = 1$ , then we say that  $u$  is **uniformly Lipschitz**.

**Note:** If this were to hold with  $\alpha > 1$ , then  $u$  would be constant (easily seen by considering the derivative of  $u$ ).

**Definition 3.3** We say that  $u$  is **locally Hölder continuous with exponent  $\alpha$**  in  $\Omega$ , if  $u|_K : K \rightarrow \mathbb{R}$  is uniformly Hölder continuous with exponent  $\alpha$  (in  $K$ ), for every compact  $K \subset \Omega$ .

Now let  $k \in \mathbb{N} \cup \{\infty\} = \{0, 1, \dots\} \cup \{\infty\}$ . Then recall that we define:

$$C^k(\Omega) = \left\{ u : \Omega \rightarrow \mathbb{R} \left| D^\beta u \text{ exists and is continuous in } \Omega \text{ for any multi-index } \beta \text{ with } |\beta| = \sum_i \beta_i \leq k \right. \right\}.$$

Then we define the **Hölder spaces** by:

$$C^{k, \alpha}(\Omega) = \{ u \in C^k(\Omega) \mid D^\beta u \text{ is locally Hölder continuous with exponent } \alpha \text{ for every } \beta \text{ with } |\beta| = k \}$$

and

$C^{k,\alpha}(\overline{\Omega}) = \{u \in C^k(\Omega) \mid D^\beta u \text{ is uniformly Hölder continuous in } \Omega \text{ with exponent } \alpha \text{ for every } \beta \text{ with } |\beta| = k\}.$

If  $\alpha \in (0, 1)$ , we write  $C^\alpha(\Omega)$ ,  $C^\alpha(\overline{\Omega})$  to denote  $C^{0,\alpha}(\Omega)$ ,  $C^{0,\alpha}(\overline{\Omega})$  respectively. Also also write  $C^{k,\alpha}(\Omega) := C^k(\Omega)$ , and  $C^{k,0}(\overline{\Omega}) = C^k(\overline{\Omega})$ , where here  $k \in \mathbb{N} \cup \{\infty\}$ .

This then defines  $C^{k,\alpha}(\Omega)$ ,  $C^{k,\alpha}(\overline{\Omega})$  for any  $k \in \mathbb{N} \cup \{\infty\}$ ,  $\alpha \in [0, 1]$ .

**Note:** We do not have  $C^{k+1}(\Omega) = C^{k,1}(\Omega)$ , e.g.  $C^1 \neq C^{0,1}$  as Lipschitz functions are not continuously differentiable (e.g.  $|x|$ ).

We then also define:

$$C_0^{k,\alpha}(\Omega) \equiv C_c^{k,\alpha}(\Omega) := \{u \in C^{k,\alpha}(\Omega) : \text{supp}(u) \subset \Omega \text{ is compact (in } \Omega)\},$$

where  $\text{supp}(u) = \overline{\{x \in \Omega : u(x) \neq 0\}}$  is the support of  $u$ .

## Norms and Semi-Norms

For  $k \in \mathbb{N}$ , id  $u \in C^k(\overline{\Omega})$  we define:

$$[u]_{k;\Omega} \equiv |D^k u|_{0;\Omega} := \sup_{|\beta|=k} |D^\beta u|_{0;\Omega} = \sup_{|\beta|=k} \left( \sup_{x \in \Omega} |D^\beta u(x)| \right).$$

For  $u \in C^{k,\alpha}(\overline{\Omega})$ , we set

$$[u]_{k,\alpha;\Omega} \equiv [D^k u]_{\alpha;\Omega} := \sup_{|\beta|=k} [D^\beta u]_{\alpha;\Omega}.$$

Note that these are only semi-norms on the respective spaces (as they vanish for constants). There to get norms on each space, we set:

$$\|u\|_{C^k(\overline{\Omega})} \equiv |u|_{k;\Omega} \equiv |u|_{k,0;\Omega} := \sum_{j=0}^k |D^j u|_{0;\Omega}$$

and

$$\|u\|_{C^{k,\alpha}(\overline{\Omega})} \equiv |u|_{k,\alpha;\Omega} := |u|_{k;\Omega} + [D^k u]_{\alpha;\Omega}.$$

One can also check that these spaces, equipped with the relevant norm, are Banach spaces.

### 3.2.1 Compactness Properties

With all normed spaces, it is of critical importance to understand the compactness properties of the space, and when we can extract convergent subsequences. Of particular importance here, is the Arzela-Ascoli theorem.

**Theorem 3.5 (Arzela-Ascoli)** Let  $\Omega \subset \mathbb{R}^n$  be open,  $k \in \mathbb{N}$ , and  $\alpha \in (0, 1]$ . Then, if  $(u_j)_j \subset C^{k,\alpha}(\Omega)$  are such that:

$$\sup_j |u_j|_{k,\alpha;\Omega'} < \infty \quad \text{for all } \Omega' \subset\subset \Omega,$$

then  $\exists u \in C^{k,\alpha}\Omega$  and a subsequence  $(u_{j'})_{j'}$  such that  $u_{j'} \rightarrow u$  in  $C^k(\overline{\Omega'})$ , for all  $\Omega' \subset\subset \Omega$ .

**Note:** There is no conclusion/claim on the convergence in  $C^{k,\alpha}(\overline{\Omega'})$ !

**Proof.** Exercise (see example sheet 2 - it is good for the soul).

□

Next we prove an interpolation inequality. This is an instance of a result, where if we have Banach spaces  $X \subset\subset Y \subset Z$  each with their own norms, we can bound the norm in  $Y$  by that in  $Z$  and  $X$ , but the norm in  $X$  can be made smaller by making the constant in front of the one in  $Z$  larger, etc. In this case, we have  $C^{k,\alpha}(\overline{\Omega}) \subset\subset C^k(\overline{\Omega}) \subset C^0(\overline{\Omega})$ . So we have:

**Theorem 3.6 (Interpolation Inequality for Hölder Spaces)** Let  $\epsilon > 0$ ,  $l \in \mathbb{Z}^+$ , and  $\alpha \in (0, 1]$ . Then,  $\exists C = C(n, l, \alpha, \epsilon) \in (0, \infty)$  such that, if  $u \in C^{l,\alpha}(\overline{B_R(x_0)})$ , then:

$$R^k |D^k u|_{0,B_R(x_0)} \leq \epsilon R^{l+\alpha} [D^l u]_{\alpha;B_R(x_0)} + C \cdot |u|_{0;B_R(x_0)},$$

for all  $k \in \{0, 1, \dots, l\}$ .

**Proof.** The idea of the proof is as follows - full details can be found on Example Sheet 1.

First, by rescaling and so considering  $v(x) = u(x_0 + Rx)$ , it suffices to prove this lemma when  $R = 1$ ,  $x_0 = 0$ , since we have

$$\begin{aligned} |D^k v|_{0;B_1(0)} &= R^k |D^k u|_{0;B_R(x_0)} \\ |D^l v|_{\alpha;B_1(0)} &= R^{l+\alpha} [D^l u]_{\alpha;B_R(x_0)}. \end{aligned}$$

Then one can argue by contradiction using Arzela-Ascoli.

□

We will also need the following abstract lemma before starting the Schauder theory.

**Theorem 3.7 (Absorbing Lemma)** Let  $\lambda \in [0, \infty)$  and  $\theta \in (0, 1)$ . Then,  $\exists \delta = \delta(n, \lambda, \theta) \in (0, 1)$  such that the following is true:

Let  $B_R(x)$  be a fixed ball in  $\mathbb{R}^n$ . Let  $S$  be a non-negative, subadditive function on the collection of sub-balls of  $B_R(x)$ , i.e. if  $B_\rho(y) \subset \bigcup_{j=1}^N B_{\rho_j}(y_j) \subset B_R(x)$ , then

$$S(B_\rho(y)) \leq \sum_{j=1}^N S(B_{\rho_j}(y_j)).$$

Suppose that for all balls  $B_\rho(y) \subset B_R(x)$ , we have  $\rho^\lambda S(B_{\theta\rho}(y)) \leq \delta \rho^\lambda S(B_\rho(y)) + \gamma$ , for some fixed  $\gamma$ . Then,

$$R^\lambda S(B_{\theta R}(x)) \leq C\gamma, \quad \text{where } C = C(n, \theta, \lambda).$$

**Note:** This is essentially saying if we have some local bound on  $S$ , then we can ‘absorb’ the term dependent on  $S$  on the RHS to get a global bound.

**Proof.** Let

$$Q = \sup_{B_\rho(y) \subset B_R(x)} \rho^\lambda S(B_{\theta\rho}(y)).$$

Note that by sub-additivity of  $S$  we have  $Q \leq R^\lambda S(B_R(x)) < \infty$ , and so  $Q$  is finite.

Iterating our bound, with  $\theta\rho$  in place of  $\rho$ , we have that whenever  $B_\rho(y) \subset B_R(x)$ ,

$$\begin{aligned} (\theta\rho)^\lambda S(B_{\theta^2\rho}(y)) &\leq \delta(\theta\rho)^\lambda S(B_{\theta\rho}(y)) + \gamma \\ &\leq \delta\theta^\lambda Q + \gamma. \end{aligned}$$

So fix any  $B_\rho(y) \subset B_R(x)$ . Then cover  $B_{\theta\rho}(y)$  by balls  $\{B_{(1-\theta)\theta^2\rho}(z_j)\}_{j=1}^N$ , with  $N \leq C(\theta, n)$ . To do this, choose a maximal, pairwise disjoint, collection of balls  $\{B_{\frac{(1-\theta)\theta^2\rho}{2}}(z_j)\}_{j=1}^N$ , subject to the condition that  $z_j \in B_{\theta\rho}(y)$ . Then these  $z_j$  work, since if not, then  $\exists z \in B_{\theta\rho}(y) \setminus \bigcup_{j=1}^N B_{(1-\theta)\theta^2\rho}(z_j)$ . But then we would have  $z$  is at least  $(1-\theta)\theta^2\rho$  away from each  $z_j$ , and so  $B_{\frac{(1-\theta)\theta^2\rho}{2}}(z) \cap B_{\frac{(1-\theta)\theta^2\rho}{2}}(z_j) = \emptyset$ , which is a contradiction to maximality.

Then to see that bound on  $N$ , note that we also have

$$\bigcup_{j=1}^N B_{\frac{(1-\theta)\theta^2\rho}{2}}(z_j) \subset B_{\frac{(1-\theta)\theta^2\rho}{2} + \theta\rho}(y)$$

(just from considering radii), which gives us a volume bound. Indeed, as the balls on the LHS are disjoint of the same volume, we get:

$$\begin{aligned} N\omega_n \cdot \left(\frac{(1-\theta)\theta^2\rho}{2}\right)^n &\leq \omega_n \left(\frac{(1-\theta)\theta^2\rho}{2} + \theta\rho\right)^n \\ \implies N &\leq C(\theta, n). \end{aligned}$$

So hence we get, by sub-additivity,

$$S(B_{\theta\rho}(y)) \leq \sum_{j=1}^N S(B_{(1-\theta)\theta^2\rho}(z_j)),$$

and so

$$\begin{aligned} \rho^\lambda S(B_{\theta\rho}(y)) &\leq \rho^\lambda \sum_{j=1}^N S(B_{(1-\theta)\theta^2\rho}(z_j)) \\ &\stackrel{\text{assumption with } \rho \rightarrow (1-\theta)\rho}{\leq} ((1-\theta)\theta)^{-\lambda} \cdot \sum_{j=1}^N \left( \underbrace{\delta((1-\theta)\theta\rho)^\lambda S(B_{(1-\theta)\theta^2\rho}(z_j))}_{\leq Q} + \gamma \right) \end{aligned}$$



$$\leq \delta((1-\theta)\theta)^{-\lambda}NQ + N\gamma((1-\theta)\theta)^{-\lambda},$$

where this term is  $\leq Q$  since  $z_j \in B_{\theta\rho}(y)$ , and so  $B_{(1-\theta)\theta\rho}(z_j) \subset B_\rho(y) \subset B_R(x)$ .

So taking the *sup* over all  $B_\rho(y) \subset B_R(x)$ , we get (as the RHS is independent of  $\rho, y$ )

$$Q \leq \delta CQ + C_1\gamma$$

where  $C, C_1$  only depend on  $n, \theta, \lambda$ . So choose  $\delta$  s.t.  $\delta C = 1/2$ . Then we get  $Q \leq 2C_1\gamma$ , and so in particular, we get the result we are after. □

## 4 Schauder Theory

### 4.1 Interior Schauder Estimates

We start with proving the interior Schauder estimates. We prove the results in the unit ball, and extend them to more general domains afterward. The point is if the coefficients of  $L$  are  $\alpha$ -Hölder continuous, then any  $C^{2,\alpha}$  solution of  $Lu = f$  can have its  $C^{2,\alpha}$ -norm **on a smaller ball** bounded by its supremum and a Hölder norm of  $f$ .

#### Theorem 4.1 (Unit Scale Interior Schauder Estimate)

Let  $\alpha \in (0, 1)$ , and  $\beta > 0$ . Suppose  $a^{ij}, b^i, c \in C^{0,\alpha}(\overline{B_1(0)})$ , with  $|a^{ij}|_{0,\alpha;B_1(0)} + |b^i|_{0,\alpha;B_1(0)} + |c|_{0,\alpha;B_1(0)} \leq \beta$ . Suppose also that  $L$  is strictly elliptic, so  $\exists \lambda > 0$  fixed such that  $a^{ij}(x)\xi^i\xi^j \geq \lambda|\xi|^2$ , for all  $x \in B_1(0)$  and  $\xi \in \mathbb{R}^n$ .

Then if  $u \in C^{2,\alpha}(B_1(0)) \cap C^0(\overline{B_1(0)})$  and  $f \in C^{0,\alpha}(\overline{B_1(0)})$  satisfy  $Lu = f$  in  $B_1(0)$ , then:

$$|u|_{2,\alpha;B_{1/2}(0)} \leq C(|u|_{0;B_1(0)} + |f|_{0,\alpha;B_1(0)})$$

for some constant  $C = C(n, \alpha, \beta, \lambda) < \infty$ .

#### Note:

- We can never take  $\alpha = 0$  or  $1$  in these Schauder estimates - see Example Sheet 1 for details of this.
- The strict ellipticity gives a lower bound on  $\lambda$ , the smallest eigenvalue. The bound of  $|a^{ij}|_{0,\alpha;B_1(0)}$  gives us an upper bound on  $\Lambda$ , the largest eigenvalue. Combining these shows that  $\Lambda/\lambda$  is bounded, and so under these assumptions, strict ellipticity  $\implies$  uniform ellipticity here.

**Remark:** This is a remarkable result for many reasons: we get control of  $u$  and its derivatives (up to 2nd order) just from  $\sup_{B_1(0)} |u|$  and  $f!$  In fact:

1. We will, using this, strengthn this conclusion to:

$$|u|_{2,\alpha;B_\theta(0)} \leq C (|u|_{0;B_1(0)} + |f|_{0,\alpha;B_1(0)})$$

for any  $\theta \in (0, 1)$ , provided  $C$  can also depend on  $\theta$ .

2. This estimate gives a **compactness property for the space of solutions to  $Lu = f$** , for given  $L, f$ .

Indeed, if  $(u_k)_k \subset C^{2,\alpha}(B_1(0)) \cap C^0(\overline{B_1(0)})$  solve  $Lu = f$  in  $B_1(0)$ , and if  $\gamma := \sup_k \left( \sup_{B_1(0)} |u_k| \right) < \infty$ , then the estimates give for each  $\theta \in (0, 1)$ ,

$$|u_k|_{2,\alpha;B_\theta(0)} \leq C(\gamma, n, \theta, \beta, \lambda, f).$$

So by Arzela-Ascoli,  $\exists$  a subsequence  $(k')$  and  $u \in C^{2,\alpha}(B_1(0))$  such that  $u_{k'} \rightarrow u$  in  $C^2(B_\theta(0))$ , for any  $\theta \in (0, 1)$  (this requires a diagonal argument: we get a subsequence for each  $\theta$ , and then can take a suitable diagonal subsequence).

Then we can pass to the limit pointwise in  $Lu_k = f$  to see that  $Lu = f$  as well.

3. This also assumes nothing about the  $C^2$  norm of  $u$  near the boundary, and makes no “up-to-the-boundary” conclusion.
4. The theorem is false if  $\alpha = 0$  or  $\alpha = 1$ , as mentioned above.

**Proof.** We shall write  $B_r := B_r(0)$  throughout for ease of notation.

By working in a slightly smaller ball than  $B_1$ , we can assume wlog that  $|u|_{2;B_1} < \infty$ . The proof will consist of several steps: first a reduction step, then a contradiction step, before lastly finding a PDE satisfied by a certain function, which will be easier to work with. We first claim that:

**Step 1:** Reduction step.

**Claim:** It suffices to prove the following: For any given  $\delta \in (0, 1)$ ,  $\exists C$  such that

$$[D^2u]_{\alpha;B_{1/2}} \leq \delta [D^2u]_{\alpha;B_1} + C(|u|_{2;B_1} + |f|_{0,\alpha;B_1}). \quad (3)$$

**Proof of Claim.** Suppose (3) were true. Then by the interpolation inequality, we have

$$[D^2u]_{\alpha;B_{1/2}} \leq 2\delta [D^2u]_{\alpha;B_1} + C(|u|_{0;B_1} + |f|_{0,\alpha;B_1}). \quad (4)$$

Then by the absorbing lemma, choosing  $\delta = \delta(n, \alpha)$  small enough, we get that the following is true:

*Given any sub-ball  $B_\rho(y) \subset B_1(0)$ , let  $\tilde{u}(x) = u(y + \rho x)$ . Then  $\tilde{u}$  satisfies in  $B_1(0)$  the equation*

$$\tilde{a}^{ij} D_{ij} \tilde{u} + \tilde{b}^i D_i \tilde{u} + \tilde{c} \tilde{u} = \tilde{f}$$

*where  $\tilde{a}^{ij}(x) = a^{ij}(y + \rho x)$ ,  $\tilde{b}^i(x) = \rho b^i(y + \rho x)$ ,  $\tilde{c}(x) = \rho^2 c(y + \rho x)$ , and  $\tilde{f}(x) = \rho^2 f(y + \rho x)$ . Also,*

$$|\tilde{a}^{ij}|_{0,\alpha;B_1} + |\tilde{b}^i|_{0,\alpha;B_1} + |\tilde{c}|_{0,\alpha;B_1} \leq |a^{ij}|_{0;B_\rho(y)} + \rho^\alpha [a^{ij}]_{\alpha;B_\rho(y)} + \rho |b^i|_{0;B_\rho(y)} + \rho^{1+\alpha} [b^i]_{\alpha;B_\rho(y)} + \dots$$

$$\leq \beta.$$

Also, clearly we have  $\tilde{a}^{ij}(x)\xi^i\xi^j \geq \lambda|\xi|^2$ , so this PDE is still strictly elliptic.

So we can apply (4) with  $\tilde{u}$  in place of  $u$  and  $\tilde{f}$  in place of  $f$  to get (after changing variables back to  $u$ ):

$$\begin{aligned} \rho^{2+\alpha}[D^2u]_{\alpha;B_{\rho/2}(y)} &\leq 2\delta\rho^{2+\alpha}[D^2u]_{\alpha;B_\rho(y)} + C(|u|_{0;B_\rho(y)} + \rho^2|f|_{0;B_\rho(y)} + \rho^{2+\alpha}[f]_{\alpha;B_\rho(y)}) \\ &\leq 2\delta\rho^{2+\alpha}[D^2u]_{\alpha;B_\rho(y)} + \underbrace{C(|u|_{0;B_1} + |f|_{0,\alpha;B_1})}_{=:\gamma, \text{ independent of } B_\rho(y) \subset B_1(0)} \end{aligned}$$

where for the last inequality we have used that  $B_\rho(y) \subset B_1(0)$ . So by the absorbing lemma, we get (after choosing  $\delta$ ),

$$[D^2u]_{\alpha;B_{1/2}} \leq C \underbrace{(|u|_{0;B_1} + |f|_{0,\alpha;B_1})}_{=:\gamma \text{ from above}}$$

where  $C = C(n, \alpha, \lambda, \beta)$ , which (after using the interpolation inequality again) is the conclusion of the theorem. So indeed, we have shown that it suffices to prove (3). ■

**Step 2:** Argue by contradiction and use Arzela-Ascoli.

If (3) does not hold for any  $C > 0$ , then  $\exists \delta$  such that  $\forall k \in \mathbb{N}$ , we can find  $a_k^{ij}, b_k^i, c_k$  with  $|a_k^{ij}|_{0,\alpha;B_1} + |b_k^i|_{0,\alpha;B_1} + |c_k|_{0,\alpha;B_1} \leq \beta$  (the same  $\beta$  for each  $k$ ), with  $a_k^{ij}\xi^i\xi^j \geq \lambda|\xi|^2$ , and  $u_k \in C^{2,\alpha}(\overline{B_1}) \cap C^0(\overline{B_1})$ , solving  $L_k u_k = f_k$ , for some  $f_k \in C^{0,\alpha}(\overline{B_1})$ , but that

$$[D^2u_k]_{\alpha;B_{1/2}} > \delta[D^2u_k]_{\alpha;B_1} + k(|u_k|_{2;B_1} + |f_k|_{0,\alpha;B_1}). \quad (5)$$

Then, by passing to a subsequence of  $(k)$ , by definition of  $[D^2u_k]_{\alpha;B_{1/2}}$ , we can wlog assume  $\exists x_k, y_k \in B_{1/2}$  and fixed  $l, m \in \{1, \dots, n\}$  (i.e. the same for the entire subsequence) such that

$$\frac{|D_{lm}u_k(x_k) - D_{lm}u_k(y_k)|}{|x_k - y_k|^\alpha} > \frac{1}{2}[D^2u_k]_{\alpha;B_{1/2}} \quad \forall k.$$

[This is simply because for each  $k$ , we can find some  $l, m$  and  $x_k, y_k$  such that this is true. So, as there are only finitely many choices of  $l, m$ , we can find some subsequence where the same  $l, m$  are used: so restrict to that subsequence.]

Then let  $\rho_k = |x_k - y_k|$ . Then, by the triangle inequality,

$$\frac{1}{2}[D^2u_k]_{\alpha;B_{1/2}} < \frac{|D_{lm}u_k(x_k)| + |D_{lm}u_k(y_k)|}{\rho_k^\alpha} \leq \frac{2|u_k|_{\alpha;B_{1/2}}}{\rho_k^\alpha} \stackrel{\text{by (5)}}{\leq} \frac{2[D^2u_k]_{\alpha;B_{1/2}}}{k\rho_k^\alpha}$$

which in particular shows that  $\rho_k^\alpha < 4/k$ , and so hence  $\rho_k \rightarrow 0$ .

The process from here on is simple: we rescale and subtract the 2-jet from to get a bounded function which vanishes at 0. These functions will then have some limit, which by all the assumptions will satisfy a Laplace-like equation, which we can work with.

So rescale as follows: set

$$\tilde{u}(x) := \frac{u_k(x_k + \rho_k x) - q_k(x)}{\rho_k^{2+\alpha} [D^2 u_k]_{\alpha; B_1}},$$

where  $q_k(x) = u_k(x_k) + \rho_k x \cdot Du_k(x_k) + \frac{\rho_k^2}{2} \sum_{i,j} D_{ij} u_k(x_k) x^i x^j$  is the **2-jet** of  $u_k$  from the origin (i.e. up to 2nd order Taylor expansion). The  $\rho_k^{2+\alpha}$  factor in the definition of  $\tilde{u}_k$  is a scaling factor from differentiation  $\tilde{u}$ . Also, note that here  $x = (x^1, \dots, x^n)$  is a variable, which is completely different to  $x_k$ , which are chosen points.

So clearly by definition of  $q_k$  (or just by direct calculation) we have

$$\tilde{u}_k(0) = 0, \quad D\tilde{u}_k(0) = 0, \quad D^2\tilde{u}_k(0) = 0,$$

and also by direct calculation, we see that

$$[D^2\tilde{u}_k]_{\alpha; B_{1/2\rho_k}(0)} \leq 1$$

noting that  $\tilde{u}_k$  is defined on  $B_{1/\rho_k}(-\frac{x_k}{\rho_k}) \supset B_{1/2\rho_k}(0)$ .

Hence for any  $R > 1$ , we have

$$|\tilde{u}_k|_{2,\alpha; B_R(0)} \leq CR^{2+\alpha}$$

(since the  $\rho_k$  converge and so are bounded). So hence by Arzela-Ascoli, by passing to a subsequence, we get that  $\exists v \in C^{2,\alpha}(\mathbb{R}^n)$  such that  $\tilde{u}_k \rightarrow v$  in  $C^2(B_R(0))$  for all  $R > 0$ , and with  $[D^2 v]_{\alpha; \mathbb{R}^n} \leq 1$  (by taking a limit in these Hölder norm expressions). [Note that this is the same  $v$  for every  $R$ ].

**Step 3:** Find the PDE satisfied by  $v$ .

Note first that  $w_k(x) = u_k(x_k + \rho_k x)$  satisfies

$$\tilde{L}_k w_k = \tilde{f}_k \quad \text{in } B_{1/2\rho_k}(0)$$

where

$$\tilde{L}_k = \tilde{a}_k^{ij} D_{ij} + \tilde{b}_k^i D_i + \tilde{c}_k$$

where the summation is over  $i, j$  and not  $k$ , where  $\tilde{a}_k^{ij}$ , etc, are as before. Then since we have

$$w_k(x) = \rho_k^{2+\alpha} [D^2 u_k]_{\alpha; B_1} \tilde{u}_k(x) + q_k(x),$$

we see that  $\tilde{u}_k$  satisfies

$$\tilde{L}_k \tilde{u}_k = g_k$$

where

$$g_k := \frac{\tilde{f}_k - \tilde{L}_k q_k}{\rho_k^{2+\alpha} [D^2 u_k]_{\alpha; B_1}}.$$

Now we shall see that all of these coefficients in  $\tilde{L}$  converge uniformly to constants, and thus when we pass to the limit,  $v$  will satisfy a PDE with constant coefficients and constant inhomogeneous term.

So note that

$$[\tilde{a}_k^{ij}]_{\alpha; B_{1/2\rho_k}(0)} \stackrel{\text{containment} + \text{scaling}}{\leq} \rho_k^\alpha [a_k^{ij}]_{\alpha; B_1} \leq \rho_k^\alpha \beta \rightarrow 0$$

as  $k \rightarrow \infty$ , since we know  $\rho_k \rightarrow 0$ . Moreover we need  $\alpha > 0$  for this. If  $\alpha = 0$ , then instead we just have  $|\tilde{a}_k^{ij}|_{0; B_{1/2\rho_k}(0)} \leq \beta$ , which tells us that  $|\tilde{a}_k^{ij}|_{0; \alpha; B_R(0)} \leq C(R)$  for all  $R > 0$ , for  $k$  sufficiently large. Hence we can use Arzela-Ascoli on the  $\tilde{a}_k^{ij}$ . Hence we get that  $\tilde{a}_k^{ij} \rightarrow \tilde{a}^{ij}$  as  $k \rightarrow \infty$  for some  $\tilde{a}^{ij}$ , and the limit has Hölder semi-norm 0,  $\forall \alpha > 0$  (by the above computation), and thus is constant (the convergence is locally uniformly).

Also,  $|\tilde{b}_k^i|_{0; B_{1/2\rho_k}(0)} \leq \beta \rho_k \rightarrow 0$ , and  $|\tilde{c}_k|_{0; B_{1/2\rho_k}(0)} \leq \beta \rho_k^2 \rightarrow 0$ , and so hence we have  $\tilde{b}_k^i \rightarrow 0$  and  $\tilde{c}_k \rightarrow 0$  locally uniformly on  $\mathbb{R}^n$ .

**Claim:**  $g_k \rightarrow 0$  locally uniformly.

**Proof of Claim:** Note that we have (slightly disgusting expression - there might be a nicer way of doing this?)

$$g_k(x) = \frac{\rho_k^2 f_k(x'_k) - (\rho_k^2 a^{ij}(x'_k) D_{ij} u_k(x_k) + \rho_k^2 b_k^i(x'_k) (D_i u_k(x_k) + D_{is} u_k(x_k) x^s) + \rho_k^2 c_k(x'_k) q_k(x))}{\rho_k^{2+\alpha} [D^2 u_k]_{\alpha; B_1}}$$

where  $x'_k = x_k + \rho_k x$ , and so summation is over all indices but  $k$ . So hence, using the fact that  $f_k(x_k) = a^{ij}(x_k) D_{ij} u_k(x_k) + b_k^i(x_k) D_i u_k(x_k) + c_k(x_k) u_k(x_k)$ , adding and subtracting this in the numerator, using the triangle inequality, and then bounding the terms involving  $u$  gives

$$|g_k(x)| \leq \frac{\rho_k^2 (|f_k(x'_k) - f_k(x_k)| + |(a^{ij}(x'_k) - a^{ij}(x_k)) \cdots| \cdot |u_k|_{2; B_1})}{\rho_k^{2+\alpha} [D^2 u_k]_{\alpha; B_1}}.$$

Then using the bounds on the Hölder norms of the coefficients, this gives

$$\begin{aligned} |g_k(x)| &\leq \frac{\rho_k^\alpha \left( |f_k|_{\alpha; B_1} + ([a_k^{ij}]_{\alpha; B_1} + [b_k^i]_{\alpha; B_1} + [c_k]_{\alpha; B_1}) |u_k|_{2; B_1} \right)}{\rho_k^\alpha [D^2 u_k]_{\alpha; B_1}} \\ &\leq \frac{|f_k|_{\alpha; B_1} + \beta |u_k|_{2; B_1}}{[D^2 u_k]_{\alpha; B_1}} \stackrel{\text{by (5)}}{<} \frac{1 + \beta}{k} \rightarrow 0. \end{aligned}$$

So hence  $g_k \rightarrow 0$  locally uniformly. ■

With all of this, we can pass to the limit in  $\tilde{L}_k \tilde{u}_k = g_k$  to get

$$\tilde{a}^{ij} D_{ij} v = 0 \quad \text{in } \mathbb{R}^n,$$

where the  $a^{ij}$  are constants. Also, by passing to the limit in the ellipticity condition (thus it is crucial that this is **strictly** elliptic), we see that  $\tilde{a}^{ij} \xi^i \xi^j \geq \lambda |\xi|^2$  as well. So the eigenvalues of this

matrix are positive, and so we can diagonalise  $A = (\tilde{a}^{ij})_{ij}$ . So write

$$PAP^T = Q = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

where  $\lambda_i \geq \lambda > 0$  for all  $i$ . So now let  $w(x) = v(P^T x)$ , i.e.  $v(x) = w(Px)$ . Then since  $D^2 v(x) = P^T D^2 w(Px) P$ , our equation becomes:

$$0 = \text{trace}(A \cdot D^2 v) = \text{trace}(P^T Q P P^T \cdot D^2 w(Px) \cdot P) = \text{trace}(Q \cdot D^2 w(Px)) = \sum_{i=1}^n \lambda_i D_{ii} w(Px).$$

So as  $P$  gives a linear isomorphism  $\mathbb{R}^n \rightarrow \mathbb{R}^n$ , this is the same as

$$\sum_{i=1}^n \lambda_i D_{ii} w(x) = 0 \quad \text{for all } x \in \mathbb{R}^n.$$

So if we rescale via  $\tilde{w}(x) = w(\sqrt{\lambda_1} x_1, \dots, \sqrt{\lambda_n} x_n)$ , then we have  $\Delta \tilde{w} = 0$  in  $\mathbb{R}^n$ . We also have  $[D^2 \tilde{w}]_{\alpha; \mathbb{R}^n} < \infty$ , and so hence we have  $\tilde{w} \in C^\infty(\mathbb{R}^n)$ . So hence for any  $i, j \in \{1, \dots, n\}$ , we have  $\nabla(D_{ij} \tilde{w}) = 0$  in  $\mathbb{R}^n$ .

But by the Hölder continuity condition, we know that

$$|D_{ij} \tilde{w}(x)| \leq |D_{ij} \tilde{w}(0)| + [D^2 \tilde{w}]_{\alpha; \mathbb{R}^n} \cdot |x|^\alpha$$

i.e.  $D_{ij} \tilde{w}$  grows sub-linearly, as  $\alpha < 1$ . So hence Liouville's theorem for harmonic functions implies that  $D_{ij} \tilde{w} = \text{constant}$ , for all  $i, j$ . Hence  $D_{ij} v = \text{constant}$  as well, for each  $i, j$ . But we know  $D^2 v(0) = 0$  from taking the limit in  $D^2 \tilde{u}_k(0) = 0$ , and thus we have  $D^2 v = 0$  identically.

But on the other hand, consider  $\zeta = \frac{y_k - x_k}{\rho_k}$ . Then  $|\zeta_k| = 1$ , and  $u_k(x_k + \rho_k \zeta_k) = u_k(y_k)$ . So hence:

$$|D_{lm} \tilde{u}_k(\zeta_k)| = \left| \frac{\rho_k^2 (D_{lm} u_k(y_k) - D_{lm} u_k(x_k))}{\rho_k^{2+\alpha} [D^2 u_k]_{\alpha; B_1}} \right| \stackrel{\text{by choice of } x_k, y_k}{\geq} \frac{1}{2} \cdot \frac{[D^2 u_k]_{\alpha; B_{1/2}}}{[D^2 u_k]_{\alpha; B_1}} \stackrel{\text{by (5)}}{>} \frac{\delta}{2}.$$

So hence as  $S^{n-1}$  is compact, and  $\zeta_n \in S^{n-1}$  for every  $k$ , we can pass to an appropriate subsequence where we have a limit  $\zeta$ , and then hence taking the limit in this inequality above gives

$$|D_{lm} v(\zeta)| \geq \frac{\delta}{2}$$

but then this directly contradicts  $D_{lm} v$  being identically 0, so we have our contradiction, so done. □

Now we give some corollaries of this result.

**Corollary 4.1 (Scale Invariant Interior Schauder Estimate)** *Suppose  $B_R(x_0) \subset \mathbb{R}^n$ , and  $a^{ij}, b^i, c \in C^{0,\alpha}(\overline{B_R(x_0)})$  are such that  $a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2$  for some fixed  $\lambda > 0$ , for all  $x \in B_R(x_0), \xi \in \mathbb{R}^n$ . Suppose also that*

$$|a^{ij}|_{0;B_R(x_0)} + R^\alpha[a^{ij}]_{\alpha;B_R(x_0)} + R(|b^i|_{0;B_R(x_0)} + R^\alpha[b^i]_{\alpha;B_R(x_0)}) + R^2(|c|_{0;B_R(x_0)} + R^\alpha[c]_{\alpha;B_R(x_0)}) \leq \beta$$

*for some  $\beta$ . Suppose also that  $u \in C^{2,\alpha}(B_R(x_0)) \cap C^0(\overline{B_R(x_0)})$  satisfies  $Lu = f$  in  $B_R(x_0)$  for some  $f \in C^{0,\alpha}(\overline{B_R(x_0)})$ .*

*Then, we have*

$$|u|'_{2,\alpha;B_R(x_0)} \leq C(|u|_{0;B_R(x_0)} + R^2(|f|_{0;B_R(x_0)} + R^\alpha[f]_{\alpha;B_R(x_0)})),$$

*where  $C = C(n, \lambda, \alpha, \beta)$  is independent of  $u, R$ , where*

$$|u|_{k,\alpha;B_\rho(y)} := \sum_{j=0}^k \rho^j |D^j u|_{0;B_\rho(y)} + \rho^{k+\alpha} [D^k u]_{\alpha;B_\rho(y)}.$$

**Proof:** Apply Theorem 4.1 (Interior Schauder estimates) with  $\tilde{a}^{ij}, \tilde{b}^i, \tilde{c}, \tilde{u}, \tilde{f}$  in place of the relevant quantities, where for a function  $g$  we set  $\tilde{g}(x) := g(x_0 + Rx)$ . □

Now we can further extend these interior estimates to arbitrary domains, instead of just balls.

**Corollary 4.2 (Interior Schauder Estimates in General Domains)** *Let  $\alpha \in (0, 1)$ , and let  $\Omega \subset \mathbb{R}^n$  be an open, bounded set. Suppose that  $a^{ij}, b^i, c \in C^{0,\alpha}(\overline{\Omega})$ , where*

$$|a^{ij}|_{0,\alpha;\Omega} + |b^i|_{0,\alpha;\Omega} + |c|_{0,\alpha;\Omega} \leq \beta,$$

*with  $a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2$  for some  $\lambda > 0$ , for all  $x \in \Omega, \xi \in \mathbb{R}^n$ . Suppose also that  $u \in C^{2,\alpha}(\Omega) \cap C^0(\overline{\Omega})$  satisfies  $Lu = f$  in  $\Omega$ , for some  $f \in C^{0,\alpha}(\overline{\Omega})$ .*

*Then  $\forall \tilde{\Omega} \subset\subset \Omega$ , we have that*

$$|u|_{2,\alpha;\tilde{\Omega}} \leq C(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega}),$$

*where  $C = C(n, \alpha, \beta, \lambda, \text{dist}(\tilde{\Omega}, \partial\Omega))$ .*

**Proof.** Let  $d = \text{dist}(\tilde{\Omega}, \partial\Omega) := \sup\{r > 0 : (\tilde{\Omega})_r \subset \Omega\}$ , where  $(\tilde{\Omega})_r = \bigcup_{x \in \tilde{\Omega}} B_r(x)$  is the  $r$ -neighbourhood of  $\tilde{\Omega}$ .

Then for any  $x \in \tilde{\Omega}$ , we have  $B_d(x) \subset \Omega$ , and

$$|a^{ij}|'_{0,\alpha;B_d(x)} + d|b^i|'_{0,\alpha;B_d(x)} + d^2|c|_{0,\alpha;B_d(x)} \leq C(d)\beta$$

for some constant  $C$ . Then by Corollary 4.1, we get:

$$\begin{aligned} |u|_{0;B_{d/2}(x)} + d|Du|_{0;B_{d/2}(x)} + d^2|D^2u|_{0;B_{d/2}(x)} &\leq C(|u|_{0;B_d(x)} + d^2|f|_{0;B_d(x)} + d^{2+\alpha}[f]_{\alpha;B_d(x)}) \\ &\leq C(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega}) \end{aligned} \quad (6)$$

where  $C = C(n, \lambda, \alpha, \beta, d)$ . In particular, we have

$$|u(x)| + |Du(x)| + |D^2u(x)| \leq C(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega})$$

for all  $x \in \tilde{\Omega}$ . So, hence this says that we have

$$|u|_{2;\tilde{\Omega}} \leq C(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega}).$$

Then by (6), we also have that

$$\sup_{x,y \in \tilde{\Omega} : x \neq y, |x-y| < d/2} \frac{|D_{ij}u(x) - D_{ij}u(y)|}{|x-y|^\alpha} \leq C(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega}).$$

But also, if  $x, y \in \tilde{\Omega}$  are such that  $|x-y| \geq d/2$ , then

$$\begin{aligned} \frac{|D_{ij}u(x) - D_{ij}u(y)|}{|x-y|^\alpha} &\leq \left(\frac{d}{2}\right)^{-\alpha} \cdot (|D_{ij}u(x)| + |D_{ij}u(y)|) \leq \left(\frac{d}{2}\right)^{-\alpha} |u|_{2;\tilde{\Omega}} \\ &\leq C(|u|_{0;\Omega} + |f|_{0,\alpha;\tilde{\Omega}}), \end{aligned}$$

where  $C = C(n, \lambda, \alpha, \beta, d)$ . So hence combining these we see that

$$[D^2u]_{\alpha;\tilde{\Omega}} \leq C(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega}).$$

So hence combining we get the result. □

## 4.2 Boundary Schauder Estimates

Now we aim to prove boundary Schauder estimates. First we set up some notation.

**Notation:** We shall write

$$\mathbb{R}_+^n := \{(x', x^n) : x' \in \mathbb{R}^{n-1}, x^n > 0\}.$$

$$\mathbb{R}_-^n := \{(x', x^n) : x' \in \mathbb{R}^{n-1}, x^n < 0\}.$$

$$B_R^\pm(y) := B_R(y) \cap \mathbb{R}_\pm^n.$$

$$B_R^\pm := B_R^\pm(0).$$

$$S_R(y) = B_R(y) \cap \{x_n = 0\}.$$

$$S_R = S_R(0).$$



**Theorem 4.2 (Boundary Schauder Estimate in the Unit Ball)** Suppose  $0 < \alpha < 1$ ,  $a^{ij}, b^i, c \in C^{0,\alpha}(\overline{B_1^+})$ , and  $|a^{ij}|_{0,\alpha;B_1^+} + |b^i|_{0,\alpha;B_1^+} + |c|_{0,\alpha;B_1^+} \leq \beta$ . Suppose also that  $a^{ij}(x)\xi^i\xi^j \leq \lambda|\xi|^2$  for some fixed  $\lambda > 0$ , for all  $x \in B_1^+$ ,  $\xi \in \mathbb{R}^n$ . Suppose  $u \in C^{2,\alpha}(\overline{B_1^+})$  satisfies

$$\begin{cases} Lu = f & \text{in } B_1^+ \\ u = \varphi & \text{on } S_1, \end{cases}$$

for some  $f \in C^{0,\alpha}(\overline{B_1^+})$ ,  $\varphi \in C^{2,\alpha}(\overline{B_1^+})$ .

Then, we have

$$|u|_{2,\alpha;B_{1/2}^+} \leq C \left( |u|_{0;B_1^+} + |f|_{0,\alpha;B_1^+} + |\varphi|_{2,\alpha;B_1^+} \right)$$

where  $C = C(n, \lambda, \alpha, \beta)$ .

**Proof:** By considering  $v = u - \varphi$ , it suffices to prove the theorem with  $\varphi \equiv 0$  (as the coefficients for the equation for  $v$  are still of this form, and the bound for  $v$  implies it for  $u$ ).

So first we will prove the following:

**Claim:**  $\forall \delta > 0, \exists C = C(n, \alpha, \lambda, \beta, \delta)$  such that if the hypotheses are as in the theorem, then

$$[D^2u]_{\alpha;B_{1/2}^+} \leq \delta[D^2u]_{\alpha;B_1^+} + C \left( |u|_{2;B_1^+} + |f|_{0,\alpha;B_1^+} \right).$$

**Proof of Claim:** To prove this, we argue by contradiction. Suppose it was not true. Then by passing to a subsequence, as before (in the interior case proof) we get that  $\exists x_k, y_k \in B_{1/2}^+$  and  $u_k \in C^{2,\alpha}(B_1^+)$  such that the hypotheses are satisfied with  $u_k$  in place of  $u$ , and with some  $a_k^{ij}, b_k^i, c_k, f_k \in C^{0,\alpha}(\overline{B_1^+})$  in place of  $a^{ij}, b^i, c, f$ , and with, for some fixed  $l, m \in \{1, \dots, n\}$ ,

$$\frac{|D_{lm}u_k(x_k) - D_{lm}u_k(y_k)|}{|x - y|^\alpha} > \frac{1}{2}[D^2u_k]_{\alpha;B_{1/2}^+},$$

and,

$$[D^2u_k]_{\alpha;B_{1/2}^+} > \delta[D^2u_k]_{\alpha;B_1^+} + k \left( |u_k|_{2;B_1^+} + |f_k|_{0,\alpha;B_1^+} \right).$$

Then as before, we see that  $\rho_k := |x_k - y_k| \rightarrow 0$  and  $k \rightarrow \infty$ . So we have two cases:

**Either:** (1)  $\limsup_{k \rightarrow \infty} \frac{\text{dist}(x_k, S_1)}{\rho_k} = \infty$

**or:** (2)  $\limsup_{k \rightarrow \infty} \frac{\text{dist}(x_k, S_1)}{\rho_k} = \mu < \infty$ .

We consider each case separately.

**Case (1):** for any  $R$  and sufficiently large  $k$ , we have  $\text{dist}(x_k, S_1) \geq R\rho_k$ , and so hence (as  $x_k \in B_{1/2}^+$ ) we then have  $B_{R\rho_k}(x_k) \subset B_1^+$ .

So set

$$\tilde{u}_k(x) = \frac{u_k(x_k + \rho_k x) - q_k(x)}{\rho_k^{2+\alpha} [D^2 u_k]_{\alpha; B_1^+}}$$

where

$$q_k(x) := u_k(x_k) + \rho_k Du_k(x_k) \cdot x + \frac{\rho_k^2}{2} \cdot D_{ij} u_k(x_k) x^i x^j$$

where we are implicitly summing over  $i, j$ .

Note that then  $\tilde{u}_k$  is defined in  $B_R(0)$ , and satisfies  $|u_k|_{2, \alpha; B_R(0)} \leq C(R)$  for sufficiently large  $k$ . Then using the same argument as before, in Theorem 4.1, this case is done.

**Case (2):** Let  $z_k = \text{proj}_{\{x^n=0\}}(x_k)$  be the projection of  $x_k$  onto the  $x^n$ -axis. So,  $x_k = (x_k^1, \dots, x_k^{n-1}, 0)$ . Then subtract off the 2nd order terms as usual (to get the  $C^{2, \alpha}$  bounds), so set

$$\tilde{u}_k(x) = \frac{u_k(z_k + \rho_k x) - q_k(x)}{\rho_k^{2+\alpha} [D^2 u_k]_{\alpha; B_1^+(0)}}$$

where  $q_k(x) = u_k(z_k) + \rho_k Du_k(z_k) \cdot x + \frac{\rho_k^2}{2} \sum_{i,j=1}^n D_{ij} u_k(z_k) x^i x^j$ . So in particular,  $[D^2 \tilde{u}_k]_{\alpha; B_R^+(0)} \leq 1$ .

Then as before, for any  $R > 0$ , we have  $|\tilde{u}_k|_{2, \alpha; B_R^+(0)} \leq C(R)$  for sufficiently large  $k$ , and  $\tilde{u}_k|_{S_R} = 0$ , since in fact we have

$$q_k(x) = \rho_k D_n u_k(z_k) x^n + \rho_k^2 \sum_{j=1}^n D_{nj} u_k(z_k) x^n x^j$$

since  $u|_{S_R} = 0$ , so the only non-zero derivatives must have a derivative in the  $x^n$ -direction.

Also, if we set  $\xi_k = \frac{x_k - z_k}{\rho_k}$ ,  $\eta_k = \frac{y_k - z_k}{\rho_k}$ , then we have  $|\xi_k| \leq 2\mu$  and  $|\eta_k| \leq \frac{|x_k - y_k| + |x_k - z_k|}{\rho_k} = 1 + \frac{|x_k - z_k|}{\rho_k} \leq 1 + 2\mu$ . So hence these sequences are bounded, and lie in compact subsets of  $\mathbb{R}^n$ , and so we can find convergent subsequences for both (which have the same index, wlog). So passing to this subsequence, we can wlog assume that  $\xi_k \rightarrow \xi$  and  $\eta_k \rightarrow \eta$ . Then we clearly have (just by plugging into the expressions) that

$$D_{lm} \tilde{u}_k(\xi_k) = \frac{D_{lm} u_k(x_k) - D_{lm} u_k(z_k)}{\rho_k^\alpha [D^2 u_k]_{\alpha; B_1^+(0)}}$$

and

$$D_{lm} \tilde{u}_k(\eta_k) = \frac{D_{lm} u_k(y_k) - D_{lm} u_k(z_k)}{\rho_k^\alpha [D^2 u_k]_{\alpha; B_1^+(0)}}.$$

So hence,

$$|D_{lm} \tilde{u}_k(\xi_k) - D_{lm} \tilde{u}_k(\eta_k)| = \frac{|D_{lm} u_k(x_k) - D_{lm} u_k(y_k)|}{\rho_k^\alpha [D^2 u_k]_{\alpha; B_1^+(0)}} \geq \frac{\frac{1}{2} [D^2 u_k]_{\alpha; B_{1/2}(0)^+}}{[D^2 u_k]_{\alpha; B_1(0)^+}} \geq \delta/2. \quad (7)$$

Then by Arzela-Ascoli, we get that  $\exists v \in C^{2,\alpha}(\mathbb{R}_+^n \cup \{x^n = 0\})$  such that, by passing to an appropriate subsequence, we have  $\tilde{u}_k \rightarrow v$  in  $C^2$  on compact subsets of  $\mathbb{R}_+^n \cup \{x^n = 0\}$ . As before, we find that  $v$  satisfies an elliptic equation of the form:  $\tilde{a}^{ij} D_{ij} v = 0$ , where the  $\tilde{a}^{ij}$  are constants in  $\mathbb{R}_+^n$ . Also,  $v = 0$  on  $\{x^n = 0\}$ .

Then as again, by rotation and scaling of constants, we get that we have  $w \in C^2(\overline{H})$  with

$$\begin{cases} \Delta w = 0 & \text{in a half-space } H \\ w|_{\partial H} = 0. \end{cases}$$

So  $w$  is harmonic. Then by making an odd reflection on  $\partial H$  for  $w$ , we can extend  $w$  to a harmonic function  $\tilde{w}$  on all of  $\mathbb{R}^n$  (see below for details of this). Also,  $[D^2 \tilde{w}]_{\alpha; \mathbb{R}^n} < \infty$ .

But then this implies (as  $D^2 \tilde{w}$  is the harmonic in  $\mathbb{R}^n$ ) and sublinear by the above) by Liouville's theorem that  $D^2 \tilde{w} = \text{constant}$ . This then implies that  $D^2 v$  is constant. But this then contradicts (7), since that shows  $D_{lm} v(\xi) \neq D_{lm} v(\eta)$  (after taking the limit to extend it to  $v$  instead of  $\tilde{u}_k$ ). So hence we have a contradiction, so hence we are done with proving this claim. ■

So to now finish the proof, we shall use this claim we have just established. By the interpolation inequality and scaling, this claim then gives (just as before in Theorem 4.1), that for any ball  $B_\rho(y) \subset B_1(0)$ , with  $y \in \{x^n = 0\}$ , we have

$$\rho^{2+\alpha} [D^2 u]_{\alpha; B_{\rho/2}(y)^+} \leq \delta \rho^{2+\alpha} [D^2 u]_{\alpha; B_\rho(y)^+} + C \left( |u|_{0; B_1^+} + |f|_{0, \alpha; B_1^+} \right).$$

Also, by the interior Schauder estimate, for any ball  $\overline{B_\rho(y)} \subset B_1^+$ , we have

$$\rho^{2+\alpha} [D^2 u]_{\alpha; B_{\rho/2}(y)} \leq C \left( |u|_{0; B_1^+} + |f|_{0, \alpha; B_1^+} \right).$$

Then the conclusion of the theorem then follows by applying the following variant of the absorbing lemma. □

Now we quickly state formally the other results we used in this prove, which will be checked on Example Sheet 2.

**Proposition 4.1 (Reflection Principle for Harmonic Functions)** *Let  $\Omega^+$  be an open subset of  $\mathbb{R}_+^n$ , and let  $T = \partial\Omega^+ \cap \{x^n = 0\}$ . Let  $\Omega^-$  be the reflection of  $\Omega^+$  in  $\{x_n = 0\}$ , i.e.*

$$\Omega^- = \{(x^1, \dots, -x^n : (x^1, \dots, x^n) \in \Omega^+\}.$$

*Let  $v \in C^2(\Omega^+) \cap C^0(\Omega^+ \cup T)$ . Then let  $\bar{v}$  be the odd reflection of  $v$  in  $T$ , i.e.  $\bar{v} : \Omega^+ \cup T \cup \Omega^- \rightarrow \mathbb{R}$  is:*

$$\bar{v}(x^1, \dots, x^n) = \begin{cases} v(x^1, \dots, x^n) & \text{if } (x^1, \dots, x^n) \in \Omega^+ \cup T \\ -v(x^1, \dots, -x^n) & \text{if } (x^1, \dots, x^n) \in \Omega^-. \end{cases}$$

*Then if  $\Delta v = 0$  in  $\Omega^+$  with  $v|_T = 0$ , then we have  $\bar{v} \in C^2(\text{Int}(\Omega^+ \cup T \cup \Omega^-))$ , and  $\Delta \bar{v} = 0$  in  $\text{Int}(\Omega^+ \cup T \cup \Omega^-)$ .*

**Proof.** See example sheet 2 (just use the mean-value property characterisation of harmonic functions to get that  $\bar{v}$  is harmonic here, which implies the result).

□

**Note:** This result is trivial if  $T = \emptyset$ , since then  $\Omega^+$  and  $\Omega^-$  are disjoint. The important part of this result is that it is  $C^2$  across the boundary  $T$ .

**Lemma 4.1 (Absorbing Lemma, Boundary Version)** *Given  $\theta \in (0, 1)$ ,  $\mu \in \mathbb{R}$ , then  $\exists \delta = \delta(n, \theta, \mu) \in (0, 1)$  and  $C = C(n, \theta, \mu) > 0$  such that, if  $R > 0$ ,  $\mathcal{B} = \{B_\rho(y) : B_\rho(y) \subset B_R^+(0)\}$  and  $\mathcal{B}^+ = \{B_\rho^+(y) : y^n = 0, B_\rho^+(y) \subset B_R(0)^+\}$  and  $S : \mathcal{B} \cup \mathcal{B}^+ \rightarrow \mathbb{R}$  is a sub-additive, non-negative function, satisfying*

$$\rho^\mu S(B_{\theta\rho}^+(y)) \leq \delta \rho^\mu S(B_\rho^+(y)) + \gamma \quad \text{if } B_\rho^+(y) \in \mathcal{B}^+$$

and

$$\rho^\mu S(B_{\theta\rho}(y)) \leq \gamma \quad \text{if } B_\rho(y) \in \mathcal{B}.$$

Then we have  $R^\mu S(B_{\theta R}^+(0)) \leq C\gamma$ .

**Proof.** See example sheet 2 (the idea is to first take a maximal pairwise disjoint collection of balls covering strips/boundary, and then double the radii to cover the domain  $\mathcal{D}$  and then cover the rest).

□

So we have proven the boundary Schauder estimate on balls. So all that is left is to extend this to general domains.

Because of how often we use this hypothesis, we denote the following hypotheses by (H):

*Suppose  $\Omega \subset \mathbb{R}^n$  is a bounded domain, with  $0 < \alpha < 1$ . Suppose  $a^{ij}, b^i, c \in C^{0,\alpha}(\bar{\Omega})$  are such that*

$$|a^{ij}|_{0,\alpha;\Omega} + |b^i|_{0,\alpha;\Omega} + |c|_{0,\alpha;\Omega} \leq \beta.$$

*Suppose also that  $\exists \lambda > 0$  constant such that:  $a^{ij}(x)\xi^i\xi^j \geq \lambda|\xi|^2 \forall x \in \Omega, \xi \in \mathbb{R}^n$ . Also let  $Lu = a^{ij}D_{ij}u + b^iD_iu + cu$ .*

Then we have the following boundary Schauder estimate, as a Corollary of Theorem 4.2.

**Corollary 4.3** *Suppose hypotheses (H) hold, and  $\Omega$  is a  $C^{2,\alpha}$  domain. Then,  $\exists \epsilon = \epsilon(\Omega) > 0$  such that, if  $u \in C^{2,\alpha}(\bar{\Omega})$ ,  $f \in C^{0,\alpha}(\bar{\Omega})$ ,  $\varphi \in C^{2,\alpha}(\bar{\Omega})$  satisfy*

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega. \end{cases}$$

Then,  $\forall x \in \partial\Omega$ , we have

$$|u|_{2,\alpha;B_\epsilon(x) \cap \Omega} \leq C(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega} + |\varphi|_{2,\alpha;\Omega}).$$

**Remark:** Clearly we need  $\Omega$  to be a  $C^{2,\alpha}$  domain for us to have any chance of  $u$  being  $C^{2,\alpha}$  on  $\partial\Omega$ .

**Proof.** For any  $z \in \partial\Omega$ , by definition of  $\Omega$  being a  $C^{2,\alpha}$ -domain, we know that  $\exists \psi : B_R(z) \rightarrow \mathcal{D}$  a  $C^{2,\alpha}$ -diffeomorphism, for some  $R > 0$ , such that

$$\psi(B_R(z) \cap \Omega) \subset \{y^n > 0\}, \quad \text{and} \quad \psi(B_R(z) \cap \partial\Omega) \subset \{y^n = 0\},$$

i.e.  $\psi$  provides a  $C^{2,\alpha}$ -straightening out of the boundary near  $z$ . Let  $x = (x^1, \dots, x^n)$  be coordinates for  $\Omega$  and let  $y = (y^1, \dots, y^n)$  be coordinates of  $\mathcal{D}$ . So let

$$\tilde{u}(y) = u(\psi^{-1}(y)).$$

Then, we have  $\tilde{u}|_{\{y^n=0\} \cap \mathcal{D}} = (\varphi \circ \psi^{-1})|_{\{y^n=0\} \cap \mathcal{D}} =: \tilde{\varphi}$ . So to apply Theorem 4.2, we need to find the PDE satisfied by  $\tilde{u}$  and show that it is strictly elliptic, etc.

So note,  $u(x) = \tilde{u}(\psi(x))$ . So,  $D_{x_i}u = D_{y_k}\tilde{u}(\psi(x)) \cdot \frac{\partial \psi^k}{\partial x^i}$  (sum over  $k$ ), and hence

$$D_{x_i x_j}u = \sum_{l,k=1}^n D_{y_l y_k} \tilde{u}(\psi(x)) \cdot \frac{\partial \psi^k}{\partial x^i} \cdot \frac{\partial \psi^l}{\partial x_j} + \sum_{k=1}^n D_{y_k} \tilde{u}(\psi(x)) \cdot \frac{\partial^2 \psi^k}{\partial x^i \partial x^j}.$$

So using this, we can check that  $\tilde{u}$  satisfies the equation:

$$\begin{cases} A^{lk} D_{y_l y_k} \tilde{u} + B^l D_{y_l} \tilde{u} + C \tilde{u} = \tilde{f} & \text{on } \mathcal{D} \\ \tilde{u} = \tilde{\varphi} & \text{on } \mathcal{D} \cap \{y^n = 0\} \end{cases}$$

where we can find the coefficients explicitly from the above, e.g.  $A^{lk} = \sum_{i,j} a^{ij} \frac{\partial \psi^k}{\partial x^i} \cdot \frac{\partial \psi^l}{\partial x^j}$ . Also,  $\tilde{f} := f \circ \psi^{-1}$ .

Now we want to rescale: choose  $\sigma > 0$  such that  $B_\sigma(\psi(z)) \subset \mathcal{D}$  (can do as this is open). Then, provided the assumptions hold, apply Theorem 4.2 with  $\bar{u}(y) := \tilde{u}(\psi(z) + \sigma y)$  in place of  $u$  to see:

$$|\tilde{u}'|_{2,\alpha;B_{\sigma/2}(\psi(z))} \leq C \left( |\tilde{u}|_{0;B_\sigma(\psi(z))} + \sigma^2 |\tilde{f}|_{0,\alpha;B_\sigma(\psi(z))} + |\tilde{\varphi}'|_{2,\alpha;B_\sigma(\psi(z))} \right), \quad (8)$$

where  $C = C(n, \lambda, \beta, \alpha, \psi)$ . So to apply this Theorem 4.2, we need to check that:

$$|A^{lk}|'_{0,\alpha;B_\sigma(\psi(z))} + |B^l|'_{0,\alpha;B_\sigma(\psi(z))} + |c|_{0,\alpha;B_\sigma(\psi(z))} \leq \mu \beta$$

for some  $\mu = \mu(\psi)$ , and we also need to check the ellipticity condition of the  $(A^{lk})_{lk}$ . For the latter, we know the form of  $A^{lk}$  (by the above), and so note

$$\sum_{l,k=1}^n A^{lk}(y) \xi^l \xi^k = \sum_{i,j=1}^n a^{ij} D_i(\xi \cdot \psi) D_j(\xi \cdot \psi) \geq \lambda |D(\xi \cdot \psi)|^2|_{\psi^{-1}(y)} \geq C(\psi) \lambda |\xi|^2$$

for some  $C(\psi) > 0^\dagger$  (where we have also used the ellipticity condition on the  $(a^{ij})_{ij}$ ). The other condition is checked similarly, and so the application of Theorem 4.2 here is justified.

---

<sup>†</sup>Indeed, we have  $\xi \cdot y = \xi \cdot \psi(\psi^{-1}(y)) \implies \xi = D(\xi \cdot \psi)|_{\psi^{-1}(y)} \cdot D\psi^{-1}(y)$ , which then implies  $|\xi| \leq |D(\xi \cdot \psi)|_{\psi^{-1}(y)} \cdot \|D\psi(y)^{-1}\|$ . So hence noting that  $\|D\psi(y)^{-1}\| = c(\psi)$  is a  $> 0$  constant depending only on  $\psi$ , we get this.

So hence transforming back  $\tilde{f} \rightarrow f$ , etc, we get from (8) that

$$|\tilde{u}|_{2,\alpha;B_{\sigma/2}(\psi(z))} \leq C (|u|_{0;\Omega} + |f|_{0,\alpha;\Omega} + |\psi|_{2,\alpha;\Omega}),$$

where  $C = C(n, \lambda, \beta, \alpha, \psi, \sigma)$ .

Now pick  $r > 0$   $r = r(z)$  such that  $B_r(z) \subset \psi^{-1}(B_{\sigma/2}(\psi(z)))$ . Then by the above, we get

$$|u|_{2,\alpha;B_r(z)} \leq C (|u|_{0;\Omega} + |f|_{0,\alpha;\Omega} + |\varphi|_{2,\alpha;\Omega}),$$

where  $C = C(n, \lambda, \beta, \alpha, \psi, \sigma)$ . Do denote the dependence on the boundary point  $z$ , we write that this was all done for some  $\psi = \psi_z$ ,  $\sigma = \sigma_z$ . So here,  $C = C_z$  as well.

This is the result we want for a ball. So we finish the proof by a compactness argument. Clearly  $\partial\Omega \subset \bigcup_{x \in \partial\Omega} B_{r(z)/2}(z)$ . Then by compactness of  $\partial\Omega$ , we get that there is a finite subcover, and so have  $\partial\Omega \subset \bigcup_{j=1}^N B_{r(z_j)/2}(z_j)$ . Then let

$$\epsilon = \min_j \left\{ \frac{r(z_j)}{2} : j = 1, \dots, N \right\}$$

$$C = \max_j \{C_{z_j} : j = 1, \dots, N\}.$$

Then for any  $x \in \partial\Omega$ , we have by the above and our choices of  $\epsilon$ ,  $C$ , that

$$|u|_{2,\alpha;B_\epsilon(x)} \leq C (|u|_{0;\Omega} + |f|_{0,\alpha;\Omega} + |\varphi|_{2,\alpha;\Omega})$$

and so we are done. □

### 4.3 Global Schauder Estimates

Now we finally combine everything we have done to get global Schauder estimates.

**Theorem 4.3 (Global Schauder Estimates)** *Suppose hypotheses (H) hold. Suppose also that  $\Omega$  is a  $C^{2,\alpha}$ -domain. Then, if  $u \in C^{2,\alpha}(\Omega)$ ,  $f \in C^{0,\alpha}(\overline{\Omega})$ ,  $\varphi \in C^{2,\alpha}(\overline{\Omega})$ , where*

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

*then in fact*

$$|u|_{2,\alpha;\Omega} \leq C (|u|_{0;\Omega} + |f|_{0,\alpha;\Omega} + |\varphi|_{2,\alpha;\Omega}),$$

*where  $C = C(n, \lambda, \alpha, \beta, \Omega)$ .*

**Proof.** Let  $\epsilon = \epsilon(\Omega)$  be as in Corollary 4.3. Then let

$$\Omega_\epsilon := \left\{ x \in \Omega : \text{dist}(x, \partial\Omega) \geq \frac{\epsilon}{4} \right\}.$$

Then by the interior Schauder estimates (Theorem 4.1) we have

$$|u|_{2,\alpha;\Omega_\epsilon} \leq C(|u|_{0,\Omega} + |f|_{0,\alpha;\Omega})$$

where  $C = C(n, \alpha, \lambda, \beta, \Omega)$ .

Then note that  $\Omega \setminus \Omega_\epsilon \subset \bigcup_{x \in \partial\Omega} B_{\epsilon/4}(x)$ . Therefore, for any  $x \in \Omega$ , we have either:

- If  $x \in \Omega_\epsilon$ , then

$$|u(x)| + |Du(x)| + |D^2u(x)| \leq C(|u|_{0,\Omega} + |f|_{0,\alpha;\Omega}),$$

by the above interior estimate.

- If  $x \in \Omega \setminus \Omega_\epsilon$ , then  $x \in B_{\epsilon/4}(y)$  for some  $y \in \partial\Omega$ , and so by Corollary 4.3, we have

$$|u(x)| + |Du(x)| + |D^2u(x)| \leq |u|_{2,\alpha;B_{\epsilon/4}(y)} \leq C(|u|_{0,\Omega} + |f|_{0,\alpha;\Omega} + |\varphi|_{2,\alpha;\Omega})$$

where the first inequality is by definition of the norm, and the second is by Corollary 4.3.

Hence combining these, we see that we get

$$|u|_{2;\Omega} \leq C(|u|_{0,\Omega} + |f|_{0,\alpha;\Omega} + |\varphi|_{2,\alpha;\Omega}).$$

So to finish, we just need to bound  $[D^2u]_{\alpha;\Omega}$  (since  $|u|_{2,\alpha;\Omega} = |u|_{2;\Omega} + [D^2u]_{\alpha;\Omega}$ ). So let  $x, y \in \Omega$ . Then we have several cases.

- Suppose they are such that  $|x - y| < \frac{\epsilon}{4}$ . Then we have two cases.

- If either of  $\text{dist}(x, \partial\Omega) < \frac{\epsilon}{4}$  or  $\text{dist}(y, \partial\Omega) < \frac{\epsilon}{4}$  hold, then we have  $x, y \in B_{\epsilon/2}(z)$ , for some  $z \in \partial\Omega$ . So then by Corollary 4.3 we get

$$\frac{|D_{ij}u(x) - D_{ij}u(y)|}{|x - y|^\alpha} \leq C(|u|_{0,\Omega} + |f|_{0,\alpha;\Omega} + |\varphi|_{2,\alpha;\Omega})$$

simply from the bound on the norm. So done in this case (as it incorporates this Hölder coefficient).

- Otherwise we have both  $x, y \in \Omega_\epsilon$ , and so again we are done from the norm bound in the interior estimate.

- Otherwise we have  $|x - y| \geq \frac{\epsilon}{4}$ . So in this case we clearly have:

$$\frac{|D_{ij}u(x) - D_{ij}u(y)|}{|x - y|^\alpha} \leq \frac{1}{(\epsilon/4)^\alpha} \cdot 2 \cdot \overbrace{|u|_{2;\Omega}}^{\text{bounded by above}} \leq C(|u|_{0,\Omega} + |f|_{0,\alpha;\Omega} + |\varphi|_{2,\alpha;\Omega}).$$

So hence in each case we get this bound for some  $C$ , and so we are done (by taking the maximum  $C$  over the 3 cases).

□

Now with all of our Schauder estimates, we prove some applications. We start with the Dirichlet problem.

## 5 Solvability of the Dirichlet Problem

We now have a new set of hypotheses which we will assume in this section. We denote the following by (H):

Let  $\Omega \subset \mathbb{R}^n$  be a bounded domain. Suppose that  $a^{ij}, b^i, c \in C^{0,\alpha}(\overline{\Omega})$ , and that  $\exists \lambda > 0$  such that  $a^{ij}(x)\xi^i\xi^j \geq \lambda|\xi|^2$  for all  $x \in \Omega$ ,  $\xi \in \mathbb{R}^n$  (i.e. strict ellipticity). Let  $Lu = a^{ij}D_{ij}u + b^iD_iu + cu$ .

With the above set up, we shall prove that if additionally  $c \leq 0$  in  $\Omega$ , then under some mild regularity assumption on  $\partial\Omega$  (namely,  $\Omega$  will need to satisfy the exterior sphere condition), the Dirichlet problem

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$$

is solvable, for a unique solution  $u \in C^{2,\alpha}(\Omega) \cap C(\overline{\Omega})$ , for any given  $f \in C^{0,\alpha}(\overline{\Omega})$ .

**Recall:** Remember that  $\alpha \in (0,1)$  always here: so this does not hold for general continuous coefficient equations.

The first step to doing this is proving the following theorem, which reduces the existence of a solution to this problem to the existence of a solution to an easier Dirichlet problem (namely, Poisson's equation).

**Theorem 5.1** *Let hypotheses (H) above hold. Assume further that  $\Omega$  is a  $C^{2,\alpha}$ -domain, and  $c \leq 0$  in  $\Omega$ . Then:*

*The Dirichlet problem  $\begin{cases} Lu = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$  has a solution  $u \in C^{2,\alpha}(\overline{\Omega})$ ,  $\forall f \in C^{0,\alpha}(\overline{\Omega})$ ,  $\varphi \in C^{2,\alpha}(\overline{\Omega})$*

*$\iff$  The Dirichlet problem  $\begin{cases} \Delta u = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$  has a solution  $u \in C^{2,\alpha}(\overline{\Omega})$ , for all such  $f, \varphi$ .*

**Remark:** Since  $c \leq 0$ , the solution (for either problem), if it exists, is unique.

**Remark:** Note that there is no assumption on the coefficients of  $L$ , as this is an equivalence of existence statements, not a solvability statement.

**Proof.** By consider  $u - \varphi$  it suffices (since  $\varphi \in C^{2,\alpha}(\overline{\Omega})$ ) to establish the equivalence in the special case.  $\varphi \equiv 0$  (indeed, this may change  $f$ , but that does not matter).

So consider  $C_0^{2,\alpha}(\overline{\Omega}) := \{u \in C^{2,\alpha}(\overline{\Omega}) : u \equiv 0 \text{ on } \partial\Omega\}$ . This is clearly a Banach space with respect to the usual  $C^{2,\alpha}$ -norm.

Then consider for  $t \in [0,1]$ , the 1-parameter family of operators  $L_t : C_0^{2,\alpha}(\overline{\Omega}) \rightarrow C^{0,\alpha}(\overline{\Omega})$ , where

$$\begin{aligned} L_t u &:= t \cdot Lu + (1-t) \cdot \Delta u \\ &= a_t^{ij} D_{ij}u + b_t^i D_i u + c_t u \end{aligned}$$



where  $a_t^{ij} = ta^{ij} + (1-t)\delta_{ij}$ ,  $b_t^i = tb^i$ ,  $c_t = tc$ . Note that clearly  $L_0 = \Delta$ , and  $L_1 = L$ . So this is a linear interpolation between these operators.

Clearly we have

$$|a_t^{ij}|_{0,\alpha;\Omega} + |b_t^i|_{0,\alpha;\Omega} + |c_t|_{0,\alpha;\Omega} \leq \max(1, \beta)$$

where  $\beta = |a^{ij}|_{0,\alpha;\Omega} + |b^i|_{0,\alpha;\Omega} + |c|_{0,\alpha;\Omega}$ . Also, clearly we have  $a_t^{ij}(x)\xi^i\xi^j \geq \min(1, \lambda)|\xi|^2$  for all  $t \in [0, 1]$ . So hence we can apply the global Schauder estimate to get:

$$|u|_{2,\alpha;\Omega} \leq C_1 (|u|_{0,\Omega} + |L_t u|_{0,\alpha;\Omega})$$

for all  $u \in C_0^{2,\alpha}(\overline{\Omega})$ , where  $C_1 = C_1(n, \alpha, \lambda, \beta, \Omega)$ .

Then by the maximum principle a priori estimate (MPAPE), we always have that (as  $u$  in this set is 0 on the boundary)  $|u|_{0,\Omega} \leq C_2 |L_t u|_{0,\alpha;\Omega}$ , where  $C_2 = C_2(n, \lambda, \Omega)$ . So hence, combining these two inequalities we have that

$$|u|_{2,\alpha;\Omega} \leq C |L_t u|_{0,\alpha;\Omega} \quad (9)$$

is always true, for all  $t \in [0, 1]$  and all  $u \in C_0^{2,\alpha}(\overline{\Omega})$ , where  $C = C(n, \alpha, \lambda, \beta, \Omega)$ . In particular, this tells us that  $L_t$  is always injective.

However the solvability of  $L_t u = f$  in  $C_0^{2,\alpha}(\overline{\Omega})$  (i.e. to ensure correct boundary conditions) for arbitrary  $f \in C^{0,\alpha}(\overline{\Omega})$  is asking for surjectivity of  $L_t$ . (Or equivalently, bijectivity of  $L_t$  since we know it is always injective).

So now fix  $f \in C^{0,\alpha}(\overline{\Omega})$ . We will show that if  $L_t$  is surjective at some  $t$ , then it is surjective at all  $t \in [0, 1]$  (this is a connectivity argument). So suppose  $L_s$  is surjective (and hence bijective) for some  $s \in [0, 1]$ . Fix this  $s$ . Then by (9),

$$|L_s^{-1}g|_{2,\alpha;\Omega} \leq C |g|_{0,\alpha;\Omega} \quad (10)$$

for all  $g \in C^{0,\alpha}(\overline{\Omega})$ . Then, for any  $t \in [0, 1]$ ,

$$L_t u = f \iff L_s u + (L_t - L_s)u = f \iff u = L_s^{-1}(f) + L_s^{-1}((L_s - L_t)u)$$

where we have used linearity of  $L_s^{-1}$  (which send  $f$  to the solution). So hence to find such a solution for  $L_t$ , we are looking for a fixed point of a certain map, determined by the above.

So let  $T : C_0^{2,\alpha}(\overline{\Omega}) \rightarrow C_0^{2,\alpha}(\overline{\Omega})$  be the map above, i.e.

$$T(u) := L_s^{-1}(f) + L_s^{-1}((L_s - L_t)u).$$

So we have seen that solvability of  $L_t u = f$  is equivalent to  $T$  having a fixed point. We will show that  $T$  always has a fixed point by using the contraction mapping theorem (and thus  $L_t u = f$  will always have a solution in this case).

**Claim:**  $T$  is a contraction map for  $t$  sufficiently close to  $s$ .

**Proof of Claim:** For any  $u, v \in C_0^{2,\alpha}(\overline{\Omega})$ , we have

$$T(u) - T(v) = L_s^{-1}((L_s - L_t)(u - v)) = (s - t)L_s^{-1}((L - \Delta)(u - v))$$

where we have used the form of  $L_t$ . So hence using (10), we have

$$|T(u) - T(v)|_{2,\alpha;\Omega} \leq |s - t| \cdot C|(L - \Delta)(u - v)|_{0,\alpha;\Omega} \leq |s - t| \cdot \tilde{C}|u - v|_{2,\alpha;\Omega},$$

where in the second inequality we have used the fact that both  $L, \Delta$  are bounded operators here (as they only go up to 2nd order derivatives, which are included in the  $C^{2,\alpha}$ -norm), where  $\tilde{C} = \tilde{C}(n, \alpha, \lambda, \beta, \Omega)$ .

So hence if  $|s - t| < \frac{1}{2\tilde{C}}$ , then  $T$  is a contraction map for such  $t$ . ■

Hence for  $t$  sufficiently close to  $s$ , we have that  $T$  is a contraction map. Hence by the contraction mapping theorem,  $T$  has a fixed point for such  $t$ , and so hence  $L_t u = f$  is solvable.

Note that the width of  $t$  to which we could do this,  $1/2\tilde{C}$ , did not depend on  $s$ . So by breaking up the interval  $[0, 1]$  into subintervals of length  $1/2\tilde{C}$ , we can apply this argument to each one (propagating the result onto the next interval) to get that  $L_t u = f$  is solvable for all  $t \in [0, 1]$  if it is solvable for some  $s \in [0, 1]$  (i.e. apply this argument again to  $s \pm \frac{1}{2\tilde{C}}$ , etc, until we cover  $[0, 1]$ ).

Hence the result follows by assuming we can solve the equation  $L_t u = f$  at either  $t = 0$  or  $t = 1$ . □

**Note:** Indeed, we get the equivalence if any  $L_t u = f$  is solvable, not just for  $L = \Delta$ .

**Remark:** This method is called the **continuity method**: we “continuously deform our operator into a nicer one and relate the two”. The above argument is essentially showing that the set  $\{t \in [0, 1] : L_t \text{ is a bijection}\}$  is open, but that around each point, we can find a ball about that point lying in this set which has radius independent of the point. This then clearly implies  $L$  must be the whole of  $[0, 1]$  if it is non-empty.

**Recall:**  $\Omega \subset \mathbb{R}^n$  is a bounded domain, and  $\alpha \in (0, 1)$ . Suppose  $a^{ij}, b^i, c \in C^{0,\alpha}(\overline{\Omega})$ ,  $c \leq 0$  and let  $L$  be strictly elliptic in  $\Omega$  (these were our hypotheses, (H)).

So we now know from Theorem 5.1, we have reduced the existence (in  $C^{2,\alpha}(\overline{\Omega})$ ) of a solution to the Dirichlet problem

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$$

to the existence in  $C^{2,\alpha}(\overline{\Omega})$  of a solution to the problem:

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

for any given  $f \in C^{0,\alpha}(\overline{\Omega})$ ,  $\varphi \in C^{2,\alpha}(\overline{\Omega})$  (this is useful as not only is  $\Delta$  a simpler operator, but it is also in divergence form).

The idea is to then prove the existence of such solutions to this equations in balls, and then to extend to more general domains using Perron's method.

This then allows us, in the first instance, to prove the existence of a solution to  $Lu = f$  in  $\Omega$ , with  $u|_{\partial\Omega} = \varphi$ , when the domain is a ball.

**Proposition 5.1** *Let  $B = B_R(y) \subset \mathbb{R}^n$  be any ball. Then, if  $f \in C^\infty(\overline{B})$  and  $\varphi \in C^\infty(\overline{B})$ , then  $\exists! u \in C^\infty(\overline{B})$  such that:*

$$\begin{cases} \Delta u = f & \text{in } B \\ u = \varphi & \text{on } \partial B. \end{cases}$$

**Proof.** There are several ways of proving this, e.g. using the Riesz representation theorem (taking advantage of the divergence structure of  $\Delta$ ) - See the Analysis of PDE Part III course for details.

□

Next we can extend this from balls to more general domains:

**Proposition 5.2** *Suppose that  $B$  is a ball. Then if  $f \in C^{0,\alpha}(\overline{B})$  and  $\varphi \in C^0(\overline{B})$ , then  $\exists! u \in C^{2,\alpha}(B) \cap B^0(\overline{B})$  such that:*

$$\begin{cases} \Delta u = f & \text{in } B \\ u = \varphi & \text{on } \partial B. \end{cases}$$

Moreover, if we have  $\varphi \in C^{2,\alpha}(\overline{B})$ , then this solution is in fact  $u \in C^{2,\alpha}(\overline{B})$ .

**Proof.** Extend  $f$  to  $\tilde{f} \in C_c^{0,\alpha}(\mathbb{R}^n)$ , and  $\varphi$  to  $\tilde{\varphi} \in C_c^0(\mathbb{R}^n)$ . Then mollify both  $\tilde{f}$  and  $\tilde{\varphi}$ . Then choose  $\eta \in C_c^\infty(\mathbb{R}^n)$  such that  $\eta|_{\mathbb{R}^n \setminus B_1(0)} = 0$ ,  $\eta \geq 0$ , and  $\int_{\mathbb{R}^n} \eta = 1$ . Let  $\eta_\sigma(x) = \sigma^{-n} \eta\left(\frac{x}{\sigma}\right)$ .

Then choose a sequence  $(\sigma_k)_k$  with  $\sigma_k \downarrow 0$ , and let

$$f_k(x) := \int_{\mathbb{R}^n} \tilde{f}(y) \eta_{\sigma_k}(x - y) \, dy \equiv \int_{\mathbb{R}^n} \tilde{f}(x - y) \eta_{\sigma_k}(y) \, dy$$

and

$$\varphi_k(x) := \int_{\mathbb{R}^n} \tilde{\varphi}(y) \eta_{\sigma_k}(x - y) \, dy = \int_{\mathbb{R}^n} \tilde{\varphi}(x - y) \eta_{\sigma_k}(y) \, dy$$

(i.e. convolutions). Then the  $f_k$ ,  $\varphi_k$  are smooth. So by Proposition 5.1,  $\exists$  a smooth  $u_k$  such that:

$$\begin{cases} \Delta u_k = f_k & \text{in } B \\ u_k = \varphi_k & \text{on } \partial B. \end{cases}$$

Moreover, we know  $f_k \rightarrow f$  uniformly on  $\overline{B}$ , and  $\varphi_k \rightarrow \varphi$  uniformly on  $\partial B$ . We also have  $\sup_B |f_k| \leq \sup |\tilde{f}|$  for all  $k$  (this is just the usual convolution bound), and so:

$$\begin{aligned} |f_k(x) - f_k(z)| &\leq \int_{\mathbb{R}^n} |\tilde{f}(x - y) - \tilde{f}(z - y)| \eta_{\sigma_k}(y) \, dy \leq [\tilde{f}]_{0,\alpha} \cdot |x - z|^\alpha \\ &\implies |f_k|_{0,\alpha} \leq [\tilde{f}]_{0,\alpha}. \end{aligned}$$

Also we have (again as this is a convolution)  $\sup |\varphi_k| \leq \sup |\tilde{\varphi}|$ .

Then as  $\Delta(u_k - u_l) = f_k - f_l$ , and  $(u_k - u_l)|_{\partial B} = \varphi_k - \varphi_l$ , by the maximum principle a priori estimate (Theorem 3.4), we have:

$$\sup_{\overline{B}} |u_k - u_l| \leq \sup_{\partial B} |\varphi_k - \varphi_l| + C \sup_{\overline{B}} |f_k - f_l|.$$

So hence  $(u_k)_k$  is uniformly Cauchy in  $\overline{B}$ , and so  $\exists u \in C^0(\overline{B})$  such that  $u_k \rightarrow u$  uniformly in  $\overline{B}$  (as continuity is preserved under uniform limits). Moreover, by taking limits we see  $u = \varphi$  on  $\partial B$ .

Now we shall upgrade the regularity of  $u$  via the interior Schauder estimates.

By the interior Schauder estimate, for any sub-ball  $\tilde{B} \subset\subset B$ , we have:

$$|u_k|_{2,\alpha,\tilde{B}} \leq C \left( \sup_{\tilde{B}} |u_k| + |f_k|_{0,\alpha,\tilde{B}} \right) \leq C \left( \sup_{\partial B} |\varphi_k| + |\tilde{f}|_{0,\alpha,\mathbb{R}^n} \right) \leq C \left( \sup_{\partial B} |\varphi| + |\tilde{f}|_{0,\alpha,\mathbb{R}^n} \right)$$

where  $C = C(\tilde{B}, B)$ .

So this is a bound independent of  $k$ , and so by Arzela-Ascoli, (a subsequence of)  $(u_k)_k$  converges in  $C_{\text{loc}}^2(B)$  to some limit in  $C^2(B)$ , which must be  $u$  (by uniqueness of limits).

So hence we get that  $u \in C^2(B)$  and (by now passing to the limit in  $\Delta u_k = f_k$ ), we get  $\Delta u = f$  in  $B$  (we get the Hölder condition on  $u$  by taking limits).

For the “Moreover” part, if  $\varphi \in C^{2,\alpha}(\overline{B})$ , then use the same argument but this time we can use the global Schauder estimates in place of the interior Schauder estimates to get the result.

□

**Remark:** By combining Theorem 5.1 and the argument of Proposition 5.2, we get that if hypothesis (H) hold in a ball  $B$ , then the Dirichlet problem  $Lu = f$  in  $B$ ,  $u = \varphi$  on  $\partial B$ , has a unique solution  $u \in C^{2,\alpha} \cap C^0(\overline{B})$ , for any given  $f \in C^{0,\alpha}(\overline{B})$ ,  $\varphi \in C^0(\partial B)$ .

Now we extend the solvability of the Dirichlet problem associated with  $L$  from balls to more general bounded domains.

## 5.1 Perron’s Method

The idea of Perron’s method is that solvability in balls implies solvability in general domains. It uses the notion of subsolutions, extended to continuous functions. It is based on the maximum principle.

We first make two observations.

Observation 1: Fix  $f \in C^{0,\alpha}(\overline{\Omega})$ . Let  $u \in C^2(\Omega)$ . Then, we have that:  $u$  is a subsolution to  $Lu = f$  (i.e.  $Lu \geq f$  in  $\Omega$ )  $\Leftrightarrow$  for every ball  $B \subset\subset \Omega$ , we have  $u \leq u_B$  in  $B$ , where  $u_B \in C^2(B) \cap C^0(\overline{B})$

is the (unique) solution to the Dirichlet problem:

$$\begin{cases} Lu_B = f & \text{in } B \\ u_B = u & \text{on } \partial B. \end{cases}$$

[**Exercise** to check this equivalence.]

Note that such a  $u_B$  exists by the previous discussion. So note that this equivalent notion of subsolution can be extended to include continuous functions (i.e. we can use it to define when  $u$  is a subsolution, even if  $u$  is just continuous, whereas we cannot always substitute a continuous function into  $L$ ).

Observation 2: Let  $f \in C^{0,\alpha}(\overline{\Omega})$  and  $\varphi \in C^0(\partial\Omega)$ . Then set:

$$S_\varphi := \{v \in C^2(\Omega) \cap C^0(\overline{\Omega}) : Lv \geq f \text{ in } \Omega \text{ and } v \leq \varphi \text{ on } \partial\Omega\}.$$

Then if  $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$  solves:

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$$

then in fact we have  $u(x) = \sup_{v \in S_\varphi} v(x)$  [**Exercise** to check].

[i.e. subsolutions give us a representation of the solution in this way.]

Now modify our assumptions (H) to include  $c \leq 0$  as an assumption.

**Definition 5.1** Let  $f \in C^{0,\alpha}(\overline{\Omega})$ . Then a function  $u \in C^0(\Omega)$  is said to be a **sub (super) solution** to  $Lu = f$  in  $\Omega$  if, for all open balls  $B$  with  $\overline{B} \subset \Omega$ , we have that  $u \leq u_B$  ( $u \geq u_B$ ) in  $B$ , where  $u_B$  is the unique function in  $C^{2,\alpha}(B) \cap C^0(\overline{B})$  satisfying:

$$\begin{cases} Lu_B = f & \text{in } B \\ u_B = u & \text{on } \partial B. \end{cases}$$

We tend to check that things are subsolutions, etc, by using the maximal principle and its corollaries.

**Note:** The point is that as sub/super solutions need only be continuous, if we take the maximum/minimum of sub/super solutions, we will get another sub/super solution. This would not be true for  $C^2$  functions, as taking max/min might destroy the regularity.

**Lemma 5.1** (i) Let  $u, v \in C^0(\overline{\Omega})$  be sub and super solutions respectively to  $L\tilde{u} = f$  in  $\Omega$ . Then if  $u \leq v$  on  $\partial\Omega$ , then we have  $u \leq v$  on  $\overline{\Omega}$ .

(ii) If  $u_1, u_2 \in C^0(\Omega)$  are both subsolutions to  $Lu = f$  in  $\Omega$ , then  $v := \max\{u_1, u_2\}$  is again a subsolution.

(iii) Let  $u \in C^0(\Omega)$  be a subsolution to  $Lu = f$  in  $\Omega$ . Let  $B$  be a ball with  $\overline{B} \subset \Omega$ , and let  $u_B \in C^{2,\alpha}(B) \cap C^0(\overline{B})$  be the unique solution to:

$$\begin{cases} Lu_B = f & \text{in } B \\ u_B = u & \text{on } \partial B. \end{cases}$$

Then define:

$$U := \begin{cases} u_B & \text{on } B \\ u & \text{on } \Omega \setminus B \end{cases}$$

which is called the **L-lift** of  $u$ . Then,  $U \in C^0(\Omega)$ , and  $U$  is a subsolution to  $Lu = f$  in  $\Omega$ .

**Proof.** See Example Sheet 3. □

**Note:** The latter two properties are the usefulness in the generalisation of this notion of subsolution/supersolution.

Now fix  $\varphi \in C^0(\partial\Omega)$ . Then, let

$$\mathcal{S}_\varphi := \{v \in C^0(\overline{\Omega}) : v \text{ is a subsolution to } Lu = f \text{ in } \Omega \text{ such that } v \leq \varphi \text{ on } \partial\Omega\}.$$

**Theorem 5.2** Let hypothesis (H) holds, with  $c \leq 0$ . Suppose that  $f \in C^{0,\alpha}(\overline{\Omega})$  and  $\varphi \in C^0(\partial\Omega)$ , and define  $\mathcal{S}_\varphi$  as above. Then set:

$$u = \sup_{v \in \mathcal{S}_\varphi} v.$$

Then,  $u \in C^{2,\alpha}(\Omega)$ , and  $Lu = f$  in  $\Omega$ .

**Proof.** We first show that  $\mathcal{S}_\varphi$  is non-empty, so that this is well-defined.

**Claim:**  $\mathcal{S}_\varphi \neq \emptyset$ .

**Proof of Claim.** In fact, we see that if  $y = (y_1, \dots, y_n) \in \mathbb{R}^n$  is such that  $\Omega \subset \{x \in \mathbb{R}^n : x_1 \geq y_1\}$ , and if  $\varphi(y) := \sup_{x \in \Omega} |x - y|$ , then,

$$s(x) := -\sup_{\partial\Omega} |\varphi| - (e^{\gamma d} - e^{\gamma(x_1 - y_1)}) \sup_{\Omega} |f|$$

for  $\gamma$  sufficiently large, is a subsolution to  $Lu = f$  (and of course  $s \leq \varphi$  on  $\partial\Omega$ ).

Given this explicit expression, the fact that  $s \in \mathcal{S}_\varphi$  is easily checked by direction calculation of  $L(s)$ . ■.

Also, the function  $s_1 = -s$  satisfies  $LS_1 \leq f$  in  $\Omega$ , and  $s_1 \geq \varphi$  on  $\partial\Omega$ .

So by Lemma 5.1, we have that  $v \leq s_1$  for all  $v \in \mathcal{S}_\varphi$ , and so in particular,  $u \leq s_1$  in  $\Omega$ . So thus  $u$  is well-defined.

Now fix  $z \in \Omega$ , and fix a ball  $B_R(z)$  with  $\overline{B_R(z)} \subset \Omega$ . Then by definition of  $u$ ,  $\exists$  a sequence  $v_j \subset \Omega$  such that  $v_j(z) \rightarrow u(z)$ . Then by replacing  $v_j$  with  $\max\{v_j, s\}$ , we may assume wlog that  $\sup_{\Omega} |v_j| \leq \mu$ , for some  $\mu \in \mathbb{R}$ , for all  $j$ .

Then let  $V_j$  be the  $L$ -lift of  $v_j$  in  $B = B_R(z)$ . Then, we know that  $V_j \in C^{2,\alpha}(B) \cap C^0(\overline{B})$ , with  $LV_j = f$  in  $B$ ,  $V_j = v_j$  on  $\partial B$ , and  $V_j = v_j$  in  $\Omega \setminus B$ .

So hence by Lemma 5.1 (iii),  $V_j \in \mathcal{S}_{\varphi}$ .

SO,  $V_j \leq u$  in  $\Omega$ . Then let  $w_j = V_j|_{B_R(z)}$ . Then by the interior Schauder estimate,

$$\begin{aligned} |w_j|_{2,\alpha;B_{R/2}(z)} &\leq C(|w_j|_{0;B_R(z)} + |f|_{0,\alpha;B_R(z)}) \\ &\leq C(|v_j|_{0;\partial B_R(z)} + C_1|f|_{0;\Omega} + |f|_{0,\alpha;\Omega}) \\ &\leq C(\mu + C_1|f|_{0,\alpha;\Omega}) \end{aligned}$$

where on the second line we have used that  $w_j = v_j$  on  $\partial B_R(z)$  and the maximum principle a priori estimate.

So hence by Arzela-Ascoli, by passing to a subsequence,  $\exists V \in C^{2,\alpha}(\overline{B_{R/2}(z)})$  such that  $V_j \rightarrow V$  in  $C^2(\overline{B_{R/2}(z)})$ , and  $LV = f$  in  $B_{R/2}(z)$ .

Then we know that  $V \leq u$  in  $B_{R/2}(z)$  (as each  $V_j \leq u$ ). But then as the  $v_j$  are subsolution and  $V_j$  solves the problem, we have that  $v_j(x) \leq V_j(x)$  in  $B_R(z)$ . So hence taking  $j \rightarrow \infty$ , we get that  $u(z) = V(z)$ .

**Claim:**  $u = V$  in  $B_{R/16}(z)$ .

[Note that since  $z \in \Omega$  was arbitrary, and  $V$  is  $C^{2,\alpha}$  here, with  $LV = f$ , it then follows (by covering  $\Omega$  with balls) that  $u \in C^{2,\alpha}(\Omega)$  and  $Lu = f$  in  $\Omega$ .]

**Proof of Claim:** Suppose not. Then,  $\exists z_1 \in B_{R/16}(z)$  such that  $V(z_1) < u(z_1)$  (as we know in general that  $V \leq u$  in  $B_{R/2}(z)$ ).

So then as  $u$  is defined as a supremum, we get that  $\exists \tilde{w} \in \mathcal{S}_{\varphi}$  such that  $V(z_1) < \tilde{w}(z_1) \leq u(z_1)$ . So then let  $\tilde{v}_j = \max\{\tilde{w}, V_j\}$ . Then note that  $\tilde{v}_j \in \mathcal{S}_{\varphi}$ , and let  $\tilde{V}_j$  be the  $L$ -lift of  $\tilde{v}_j$  in  $B_{R/4}(z_1)$ .

Then as before, by the interior Schauder estimates,  $\exists \tilde{V} \in C^{2,\alpha}(B_{R/8}(z_1))$  such that  $\tilde{V}_j \rightarrow \tilde{V}$  in  $C^2(B_{R/4}(z_1))$  (by passing to a subsequence).

So hence as:  $V(z_1) < \tilde{w}(z_1) \leq \tilde{v}_j(z_1) \leq \tilde{V}_j(z_1)$ , we have  $V(z_1) < \tilde{V}(z_1)$ .

But then as  $V_j \leq \tilde{V}_j$  in  $B_{R/4}(z_1) \Rightarrow V \leq \tilde{V}$  in  $B_{R/8}(z)$ . But then as  $V(z) = u(z)$ , we know  $V(z) = \tilde{V}(z)$ .

So since  $LV = f$ ,  $L\tilde{V} = f$  in  $B_{R/8}(z_1)$ , we have  $L(V - \tilde{V}) = 0$  in  $B_{R/8}(z_1)$ .

So hence by the SMP (as they agree at a point), we get that  $V \equiv \tilde{V}$ , which contradicts  $V(z_1) < \tilde{V}(z_1)$ .

So hence we must have  $u = V$  on  $B_{R/16}(z)$ .

■

So hence as  $V$  solves the equation, so does  $u$ . But then as this ball was arbitrary, we get that  $u$  satisfies the equation on all of  $\Omega$ , and that  $u \in C^{2,\alpha}(\Omega)$  as required.

□

Now we want to look at the boundary behaviour of the Perron solution  $u$ , since the above only gives us that  $u$  satisfies the equation in  $\Omega$  with no information on its boundary values. We will see that under mild assumptions on  $\partial\Omega$  (i.e. if  $\partial\Omega$  satisfies the exterior sphere condition at every point of  $\partial\Omega$ ), then the Perron solution about satisfies  $u \in C^0(\overline{\Omega})$ , with  $u = \varphi$  on  $\partial\Omega$ .

To do this, we will need to important concept of **barriers**.

**Definition 5.2** Assume the standard hypotheses hold. Let  $f \in C^{0,\alpha}(\overline{\Omega})$  and  $\varphi \in C^0(\partial\Omega)$ , and fix  $x_0 \in \partial\Omega$ . Then:

1. A sequence of functions  $w_i^+ : \overline{\Omega} \rightarrow \mathbb{R}$  is said to be an **upper barrier at  $x_0$**  to  $L, f, \varphi$  if:
  - $w_i^+ \in C^0(\overline{\Omega})$  and  $w_i^+$  is a supersolution to  $Lu = f$  in  $\Omega$ , with  $w_i^+ \geq \varphi$  on  $\partial\Omega$ .
  - $w_i^+(x_0) \rightarrow \varphi(x_0)$  as  $i \rightarrow \infty$ .
2. Similarly, a sequence  $w_i^- : \overline{\Omega} \rightarrow \mathbb{R}$  is a **lower barrier at  $x_0$**  relative to  $L, f, \varphi$  if:
  - $w_i^- \in C^0(\overline{\Omega})$  and  $w_i^-$  is a subsolution to  $Lu = f$  in  $\Omega$ , and  $w_i^- \leq \varphi$  on  $\partial\Omega$ .
  - $w_i^-(x_0) \rightarrow \varphi(x_0)$  as  $i \rightarrow \infty$ .

**Proposition 5.3** Suppose hypothesis (H) hold, and that  $f \in C^{0,\alpha}(\overline{\Omega})$ ,  $\varphi \in C^0(\partial\Omega)$  and  $x_0 \in \partial\Omega$ . Suppose  $\exists$  upper and lower barriers at  $x_0$  relative to  $L, f, \varphi$ . Then, the Perron solution (given by Theorem 5.2) satisfies  $u(x) \rightarrow \varphi(x_0)$  as  $x \rightarrow x_0$  with  $x \in \Omega$

i.e if the boundary obeys this barrier condition, the Perron solution agrees on the boundary.

**Proof.** From the defining expression for the Perron solution  $u$ , we clearly have  $u(x) \geq w_i^-(x)$  for all  $x \in \Omega$  and  $\forall i$ , where the  $w_i^-$  are the lower barriers.

For the other inequality, note that for any  $v \in S_\varphi$ , we have  $w_i^+(x) \geq v(x)$  for all  $x \in \Omega$  (as supersolutions lie above subsolutions if they are so at the boundary). So hence we get

$$u(x) := \sup_{v \in S_\varphi} v(x) \leq w_i^+(x) \quad \text{for all } i, x.$$

So hence  $w_i^-(x) \leq u(x) \leq w_i^+(x)$  for all  $x \in \Omega$  and  $i$ .

So now given  $\epsilon > 0$ , we can choose  $i$  such that  $0 \leq w_i^+(x_0) - \varphi(x_0) \leq \epsilon/2$  and  $0 \leq \varphi(x_0) - w_i^-(x) \leq \epsilon/2$  (from the convergence in the definition of barriers).

Then we can choose by continuity,  $\delta > 0$  such that  $|w_i^\pm(x) - w_i^\pm(x_0)| < \epsilon/2$  for all  $x \in B_\delta(x_0) \cap \overline{\Omega}$ . Then it is easy to see by the triangle inequality that  $|u(x) - \varphi(x)| \leq \epsilon$  if  $x \in B_\delta(x_0) \cap \Omega$ , and so hence we are done.

□



**Proposition 5.4** *Suppose hypotheses (H) hold. Let  $f \in C^{0,\alpha}(\overline{\Omega})$  and  $\varphi \in C^0(\partial\Omega)$ . Let  $x_0 \in \partial\Omega$ . Then, if  $\exists w \in C^2(\Omega) \cap C^0(\overline{\Omega})$  such that:*

- (i)  $Lw \leq -1$  in  $\Omega$
- (ii)  $w(x_0) = 0$  and  $w(x) > 0$  for all  $x \in \partial\Omega \setminus \{x_0\}$

*then  $\exists$  lower and upper barriers at  $x_0$  relative to  $L, f, \varphi$ .*

*In fact, for any sequence  $\epsilon \downarrow 0$ ,  $\exists$  constants  $k_i > 0$  such that:*

$$w_i^\pm = \varphi(x_0) \pm \epsilon_i \pm k_i w(x)$$

*are upper and lower barriers at  $x_0$ .*

**Proof.** For any  $\epsilon > 0$ , we can choose  $\delta > 0$  such that  $\varphi(x) \leq \varphi(x_0) + \epsilon$  for all  $x \in \partial\Omega \cap B_\delta(x_0)$ , by continuity of  $\varphi$ . So as  $\varphi$  is bounded on  $\partial\Omega$ , and  $w$  is positive and continuous on the compact set  $\partial\Omega \setminus B_\delta(x_0)$  (so attains a positive minima), we can choose  $l_\epsilon > 0$  constant such that  $l_\epsilon w(x) \geq \varphi(x) - \varphi(x_0) - \epsilon$  for all  $x \in \partial\Omega \setminus B_\delta(x_0)$ .

So hence,  $\varphi(x_0) + \epsilon + l_\epsilon w(x) \geq \varphi(x)$  for all  $x \in \partial\Omega$  (since on  $B_\delta(x_0)$  this is true by choice of  $\delta/\epsilon$ ).

So let  $k_\epsilon = \max\{l_\epsilon, \sup_{x \in \partial\Omega} |f(x) - c(x)\varphi(x_0)|\}$  (i.e. make the constant  $l_\epsilon$  larger if necessary).

Then we have  $Lw_\epsilon \leq f$  in  $\Omega$ , where  $w_\epsilon = \varphi(x_0) + \epsilon + k_\epsilon w$ . So hence taking a sequence  $\epsilon_i \downarrow 0$ , we get that  $w_{\epsilon_i}^+ := w_{\epsilon_i}$  is an upper barrier. Similarly,  $w_i^- = \varphi(x_0) - \epsilon_i - k_{\epsilon_i} w(x)$  is a lower barrier, so done. □

So the question now becomes, when does such a  $w$  exist? Well, it turns out that it suffices for  $\partial\Omega$  to satisfy the exterior sphere condition.

**Proposition 5.5** *If  $\Omega$  satisfies the exterior sphere condition at  $x_0 \in \partial\Omega$ , then upper and lower barriers exist at  $x_0$  relative to  $L, f, \varphi$ .*

*In particular, if  $\partial\Omega$  is  $C^2$ , then the Perron solution agrees with  $\varphi$  on  $\partial\Omega$  (and so is a solution).*

**Proof.** We know by assumption that  $\exists R > 0$  such that  $\overline{B_R(y)} \cap \overline{\Omega} = \{x_0\}$ . So for some constants  $m, \mu, \sigma > 0$ , set:

$$w(x) := \mu (R^{-\sigma} - |x - y|^{-\sigma})$$

for  $x \in \Omega$ . Then by direct calculation, we can check that for  $\mu, \sigma > 0$  sufficiently larger (depending on  $\text{diam}(\Omega)$ ) we have  $Lw \leq -1$  on  $\Omega$ .

Then clearly  $w(x_0) = 0$ , and  $w(x) > 0$  for all  $x \in \partial\Omega \setminus \{x_0\}$ . So hence the claim follows from Proposition 5.4. □

Now putting together all of Propositions 5.3-5.5 with Theorem 5.2, we get our existence theorem:

**Theorem 5.3** *Let hypotheses (H) hold. Let  $f \in C^{0,\alpha}(\overline{\Omega})$  and  $\varphi \in C^0(\partial\Omega)$ . Then,  $\exists! u \in C^{2,\alpha}(\Omega) \cap C^0(\overline{\Omega})$  solving:*

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$$

*provided that  $\Omega$  satisfies the exterior sphere condition at every point on  $\partial\Omega$  (e.g. if  $\Omega$  is a  $C^2$  domain).*

**Proof.** Just combine Theorem 5.2 with Propositions 5.3-5.5. □

Finally, we prove that if  $\varphi$  is  $C^{2,\alpha}$  as well, then  $u$  is  $C^{2,\alpha}$  up to and including the boundary as well.

**Theorem 5.4** *Suppose hypothesis (H) hold, and also suppose  $\Omega$  is a bounded  $C^{2,\alpha}$  domain. Then, for any  $f \in C^{0,\alpha}(\overline{\Omega})$  and  $\varphi \in C^{2,\alpha}(\overline{\Omega})$ ,  $\exists! u \in C^{2,\alpha}(\overline{\Omega})$  such that*

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega. \end{cases}$$

**Proof.** Let  $u \in C^{2,\alpha}(\Omega) \cap C^0(\overline{\Omega})$  be the unique solution to this problem, given by Theorem 5.3. Note that we know this is the unique such solution, so we need to show that it is more regular on  $\partial\Omega$  in this case.

By definition of  $\Omega$  being  $C^{2,\alpha}$ , we know  $\exists \psi_1 : B_R(x_0) \rightarrow \tilde{U}$  which is a  $C^{2,\alpha}$ -map, where  $\tilde{U} \subset \mathbb{R}^n$ , and  $\psi_1(B_R(x_0) \cap \Omega) = U \cap \{x_n > 0\}$ .

Then further compose  $\psi_1$  with a map  $\psi_2$  such that  $\psi_2$  pushes  $\{x_n = 0\}$  to touch a ball  $B$ . Let  $\psi$  be the composition of these.

Hence we get that  $\exists \psi : B_R(x_0) \rightarrow U$  a  $C^{2,\alpha}$  diffeomorphism such that  $\exists$  a ball  $B$  with  $\overline{B} \subset \psi(\overline{\Omega} \cap B_R(x_0))$ , such that:  $T := \psi(\partial\Omega \cap B_\rho(x_0)) \subset \partial B$ , for some  $\rho > 0$ .

Letting  $y := \psi(x)$  be the coordinates on  $U$ , the equation  $Lu(x) = f(x)$  transforms to a strictly elliptic equation  $\tilde{L}\tilde{u}(y) = \tilde{f}(y)$  in  $U$ , with  $\tilde{u}(y) = \tilde{\varphi}(y)$  on  $T$ .

So now solve the Dirichlet problem:

$$\begin{cases} \tilde{L}v = \tilde{f} & \text{in } B \\ v = \tilde{u} & \text{on } \partial B \end{cases}$$

(which we know we can do as this is a ball). Then we know that  $v \in C^{2,\alpha}(B) \cap C^0(\overline{B})$ . Moreover (**key**) we know  $v \in C^{2,\alpha}(B \cup T)$  (since  $\tilde{u} = \tilde{\varphi}$  is  $C^{2,\alpha}$  on  $T$ ) (this is by the same mollification and compactness argument used before in a previous theorem).

Now we will show via the maximum principle that  $v = \tilde{u}$ , and so hence  $\tilde{u}$  is  $C^{2,\alpha}$  on  $T$ , and so  $u$  is  $C^{2,\alpha}$  on some neighbourhood of  $x$  on  $\partial\Omega$ , which completes the proof.

On the other hand, since  $\tilde{L}(v - \tilde{u}) = 0$  in  $B$  and  $v - \tilde{u} = 0$  on  $\partial B$ , and  $v - \tilde{u} \in C^0(\overline{B})$ , it follows from the maximum principle that  $v = \tilde{u}$  in  $\overline{B}$ . So hence  $\tilde{u} \in C^{2,\alpha}(B \cup T)$ . So transforming back, we have  $u \in C^{2,\alpha}(\overline{B} \cap B_\rho(x_0))$ . So since  $x_0$  was arbitrary, we get  $u \in C^{2,\alpha}(\overline{\Omega})$ .

□

## 5.2 Fredholm Alternative for Elliptic PDEs

Let  $V$  be a normed space, and let  $T : V \rightarrow V$  be a bounded, linear compact map (note that  $T$  compact means that if  $(x_j)_j$  is a bounded sequence in  $V$ , then  $(T(x_j))_j$  has a convergent subsequence). Then the general Fredholm alternative says that, either:

- (i) The equation  $x + T(x) = 0$  has a non-zero solution, or
- (ii) For any given  $y \in V$ , the equation  $x + T(x) = y$  has a unique solution  $x \in V$ .

[See Gilbarg + Trudinger, Chapter 5, for a proof.]

We can apply this to get a Fredholm alternative for our setting:

**Theorem 5.5 (Fredholm Alternative)** *Let  $\Omega$  be a bounded,  $C^{2,\alpha}$  domain in  $\mathbb{R}^n$  and  $\alpha \in (0, 1)$ . Let  $a^{ij}, b^i, c \in C^{0,\alpha}(\overline{\Omega})$ , and  $L$  be a strictly elliptic operator in  $\Omega$ . Then either:*

- (i) *The homogeneous problem  $Lu = 0$  in  $\Omega$ ,  $u = 0$  on  $\partial\Omega$ , has a non-zero solution  $u \in C^{2,\alpha}(\overline{\Omega})$*
- (ii) *or for any given  $f \in C^{0,\alpha}(\overline{\Omega})$  and  $\varphi \in C^{2,\alpha}(\overline{\Omega})$ , the Dirichlet problem  $Lu = f$  in  $\Omega$ ,  $u = \varphi$  on  $\partial\Omega$ , has a unique solution  $u \in C^{2,\alpha}(\overline{\Omega})$ .*

i.e. so what we really need to uniquely solve the Dirichlet problem is the fact that the homogeneous problem has only the zero solution, not that  $c \leq 0$ .

[The way to think about this Fredholm alternative is that if uniqueness holds, then existence also holds, i.e. if both  $u_1, u_2$  satisfy  $Lu = f$  in  $\Omega$  and  $u = \varphi$  on  $\partial\Omega$ , then we know  $w = u_1 - u_2$  solves the homogeneous problem, and so we want  $w = 0$ , etc.]

**Proof.** It suffices to establish the dichotomy in the special case  $\varphi = 0$ , since:

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases} \text{ has a unique solution} \Leftrightarrow \begin{cases} Lv = f - L\varphi & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases} \text{ has a unique solution.}$$

So let  $\sigma$  be a constant such that  $\sigma \geq \sup_{\overline{\Omega}} c$ . Then consider the map  $L_\sigma : C_0^{2,\alpha}(\overline{\Omega}) \rightarrow C^{0,\alpha}(\overline{\Omega})$  given by  $L_\sigma u := Lu - \sigma u$  (note we work in  $C_0^{2,\alpha}$  as our boundary condition is 0).

Then by Theorem 5.4, we know  $L_\sigma$  is a bijection. Then by the global Schauder estimates and the MPAPE, we have:

$$|u|_{2,\alpha;\overline{\Omega}} \leq C |L_\sigma u|_{0,\alpha;\Omega}$$

for all  $u \in C_0^{2,\alpha}(\overline{\Omega})$ , and some fixed constant  $C$ . Equivalently, this says:

$$|L_\sigma^{-1}(f)|_{2,\alpha;\overline{\Omega}} \leq C |f|_{0,\alpha;\Omega} \quad \forall f \in C^{0,\alpha}(\overline{\Omega})$$

i.e.  $L_\sigma^{-1}$  is a bounded linear map.

The inclusion  $I : C^{2,\alpha}(\overline{\Omega}) \hookrightarrow C^{0,\alpha}(\overline{\Omega})$  is a compact map (by Arzela-Ascoli). So hence  $T_\sigma = I \circ L_\sigma^{-1} : C^{0,\alpha}(\overline{\Omega}) \rightarrow C^{0,\alpha}(\overline{\Omega})$  is a compact linear map, and so hence so is  $\sigma T_\sigma$ .

So applying the abstract Fredholm alternative to  $\sigma T_\sigma$ , we see that either:

- (i)  $u + \sigma T_\sigma(u) = 0$  has a non-zero solution  $u \in C^{0,\alpha}(\overline{\Omega})$
- (ii) or for any  $f \in C^{0,\alpha}(\overline{\Omega})$ ,  $\exists! u \in C^{0,\alpha}(\overline{\Omega})$  such tha  $u + \sigma T_\sigma(u) = L_\sigma^{-1}(f)$ .

Note that in either case, we know that since  $u, f \in C^{0,\alpha}(\overline{\Omega})$ , that  $L_\sigma^{-1}(u), L_\sigma^{-1}(f) \in C_0^{2,\alpha}(\overline{\Omega})$  (by definition of  $L_\sigma$ ). But then in each case,  $u$  is expressible in terms of these, and so hence we have that  $u \in C_0^{2,\alpha}(\overline{\Omega})$ .

So then we can apply  $L_\sigma$  to both sides (which we can do now as we know  $u$  is in the correct space), and then we get that either (as  $L = L_\sigma + \sigma$ ):

- (i)  $Lu = 0$  has a non-zero solution  $u \in C_0^{2,\alpha}(\overline{\Omega})$
- (ii) or for any  $f \in C^{0,\alpha}(\overline{\Omega})$ ,  $\exists! u \in C_0^{2,\alpha}(\overline{\Omega})$  such that  $Lu = f$  in  $\Omega$ .

So hence as we have 0 boundary conditions, this implies the result. □

**Remark:** Clearly  $K = \{u \in C_0^{2,\alpha}(\overline{\Omega}) : Lu = 0 \text{ in } \Omega\}$ , which is a linear subspace, of  $C_0^{2,\alpha}(\overline{\Omega})$ , is always a finite dimensional subspace of  $C^{2,\alpha}(\overline{\Omega})$  (with the  $C^{0,\alpha}(\overline{\Omega})$  norm). This is because  $Lu = 0 \Leftrightarrow u = -\sigma L_\sigma^{-1}(u)$ , and by the estimate in the proof, we have  $|u|_{2,\alpha;\overline{\Omega}} = |\sigma| \cdot |L_\sigma^{-1}(u)| \leq C|u|_{0,\alpha;\overline{\Omega}}$ .

i.e. if we take the unit ball in the  $C^{0,\alpha}$ -norm, this gives rise to a uniform bound on the  $|u|_{2,\alpha;\overline{\Omega}}$  norms, and so by Arzela-Ascoli we get a convergent subsequence, which shows that the closed unit ball in the space  $(K, \|\cdot\|_{0,\alpha;\overline{\Omega}})$  is compact, and so hence  $K$  is finite dimensional (as compactness of the unit ball characterises finite-dimensionality).

Now we finish by looking at the higher regularity theory.

### 5.3 Higher Regularity of Solutions

**Lemma 5.2** *Let  $\Omega \subset \mathbb{R}^n$  be an open set, and  $\alpha \in (0, 1)$ . Let  $a^{ij}, b^i, c \in C^{0,\alpha}(\Omega)$ . Suppose that  $L$  is strictly elliptic in  $\Omega$ . Then suppose  $f \in C^{0,\alpha}(\Omega)$  and  $u \in C^2(\Omega)$  solves  $Lu = f$  in  $\Omega$ . Then in fact  $u \in C^{2,\alpha}(\Omega)$ , and:*

$$|u|_{2,\alpha;\tilde{\Omega}} \leq C(|u|_{0,\Omega'} + |f|_{0,\alpha;\Omega'})$$

for all  $\tilde{\Omega} \subset \subset \Omega' \subset \subset \Omega$ , where  $C = C(n, \alpha, \lambda, \text{dist}(\tilde{\Omega}, \partial\Omega'))$ .

**Note:** The estimate just comes from that of the interior Schauder theory. The interesting part is the fact that we automatically have  $C^{2,\alpha}$  regularity.

**Proof.** Define  $L_1 u := a^{ij} D_{ij} u + b^i D_i u = f_1$ , where  $f_1 = f - cu \in C^{0,\alpha}(\Omega)$ .

So fix  $B = B_\rho(y) \subset\subset \Omega$ . Then by Theorem 5.3,  $\exists v \in C^{2,\alpha}(B) \cap B^0(\overline{B})$  such that  $L_1 v = f_1$  in  $B$ , and  $v = u$  on  $\partial B$ .

But then  $L_1(v - u) = 0$  in  $B$ , and  $v - u = 0$  on  $\partial B$ , and  $v - u \in C^0(\overline{B})$ . So hence by the weak maximum principle, we get that  $v = u$  in  $B$ , and so  $u \in C^{2,\alpha}(\Omega)$ .

□

With this we can now prove higher regularity results.

**Theorem 5.6 (Higher Interior Regularity)** *Suppose  $\Omega \subset \mathbb{R}^n$  is open and  $\alpha \in (0, 1)$ . Let  $a^{ij}, b^i, c \in C^{k,\alpha}(\Omega)$  for some  $k \geq 0$ . Suppose  $L$  is strictly elliptic in  $\Omega$ , and  $f \in C^{k,\alpha}(\Omega)$ . Suppose that  $u \in C^2(\Omega)$  solves  $Lu = f$  in  $\Omega$ .*

*Then,  $u \in C^{k+2,\alpha}(\Omega)$ , and:*

$$|u|_{k+2,\alpha;\Omega'} \leq C(|u|_{0;\Omega_1} + |f|_{k,\alpha;\Omega_1})$$

*for all  $\Omega' \subset\subset \Omega_1 \subset\subset \Omega$ , where  $C = C(n, k, \alpha, \lambda, L, \text{dist}(\Omega', \partial\Omega_1))$ .*

**Proof.** We prove this by induction on  $k$  (we have already done the  $k = 0$  case, as this is just Lemma 5.2 above).

**Step 1:** Establish the  $k = 1$  case, i.e. if  $a^{ij}, b^i, c, f \in C^{1,\alpha}(\Omega)$ , and  $u \in C^{2,\alpha}(\Omega)$ , then  $u \in C^{3,\alpha}(\Omega)$  and

$$|u|_{3,\alpha;\Omega'} \leq C(|u|_{0;\Omega_1} + |f|_{1,\alpha;\Omega_1}).$$

[Note that we can wlog assume that  $u \in C^{2,\alpha}$  by the  $k = 0$  case.]

To see this, let  $\Omega'' \subset\subset \Omega'$  and fix a direction  $l \in \{1, \dots, n\}$ . Then for  $0 < |h| < \text{dist}(\Omega'', \Omega_1)$ , let  $\delta_{l,h}$  be the difference quotient operator, i.e.

$$\delta_{l,h}g(x) := \frac{g(x + he_l) - g(x)}{h}$$

for  $g : \Omega_1 \rightarrow \mathbb{R}$ . Then, we have

$$\begin{aligned} L(\delta_{l,h}u)(x) &= \frac{1}{h} \left[ L(u(x + he_l)) - \overbrace{Lu(x)}^{=f(x)} \right] \\ &= \frac{1}{h} [a^{ij}(x)D_{ij}u(x + he_l) + b^i(x)D_iu(x + he_l) + c(x)u(x + he_l) - f(x)] \\ &= \frac{1}{h} [(a^{ij}(x) - a^{ij}(x + he_l))D_{ij}u(x + he_l) + (b^i(x) - b^i(x + he_l))D_iu(x + he_l) \\ &\quad + (c(x) - c(x + he_l))u(x + he_l) + f(x + he_l) - f(x)] \\ &= -(\delta_{l,h}a^{ij})(x)D_{ij}u(x + he_l) - (\delta_{l,h}b^i)(x)D_iu(x + he_l) - (\delta_{l,h}c)(x)u(x + he_l) + \delta_{h,l}f(x) \\ &=: g_{l,h}(x) \end{aligned}$$

where in the long expression we have added and subtracted  $f(x + he_l) \equiv Lu(x + he_l)$ .

Now, as  $f$  is differentiable, we have

$$(\delta_{l,h}f)(x) = \frac{1}{h}(f(x + he_l) - f(x)) = \frac{1}{h} \int_0^1 \frac{d}{dt} f(x + the_l) dt = \int_0^1 (D_l f)(x + the_l) dt.$$

So, we have  $\sup_{\Omega''} |\delta_{l,h}f| \leq \sup_{\Omega'} |D_l f|$ . Moreover, for  $x \neq y$ , we have

$$(\delta_{l,h}f)(x) - (\delta_{l,h}f)(y) = \int_0^1 (D_l f)(x + the + l) - (D_l f)(y + the_l) dt$$

which implies  $[\delta_{l,h}f]_{\alpha;\Omega''} \leq [D_l f]_{\alpha;\Omega'}$ . Similarly, we get that same for the other coefficients, e.g.  $[\delta_{l,h}a^{ij}]_{\alpha;\Omega''} \leq [D_l a^{ij}]_{\alpha;\Omega'}$ .

Hence by the interior Schauder estimates, we get:

$$|\delta_{l,h}u|_{2,\alpha;\Omega''} \leq C(|g_{l,h}|_{0,\alpha;\Omega_1} + |\delta_{l,h}u|_{0;\Omega_1}) \leq C(|u|_{2,\alpha;\Omega_1} + |f|_{1,\alpha;\Omega_1}) \quad (11)$$

where the second inequality comes from our bounds above. Hence we have a uniform bound on this, independent on  $h$ . Hence by Arzela-Ascoli, we can find  $(h_j)_j \downarrow 0$  such that  $\delta_{l,h_j} \rightarrow w$ , with the convergence in  $C^2(\overline{\Omega''})$ , where  $w \in C^{2,\alpha}(\overline{\Omega''})$ . But we know that  $\delta_{l,h_j}u \rightarrow D_l u$ , and so hence we have  $D_l u = w \in C^{2,\alpha}(\Omega'')$ , i.e.  $u \in C^{3,\alpha}(\Omega'')$ . So hence by (11) and the previous Lemma, this proves the  $k = 1$  case.

**Step 2:** Prove the  $k \geq 2$  case.

Now we want to prove that if  $a^{ij}, b^i, c, f \in C^{k,\alpha}(\Omega)$  and  $u \in C^{k+1,\alpha}(\Omega)$ , then in fact  $u \in C^{k+2,\alpha}(\Omega)$ , with the required estimate.

[Note by the induction hypothesis, we can assume that  $u \in C^{k+1,\alpha}$ .]

To see this, let  $\gamma$  be any multi-index with  $|\gamma| = k - 1$ . Then we know from then product rule of derivatives,

$$L(D^\gamma u) = D^\gamma f - \sum_{\beta < \gamma} \frac{\gamma!}{\beta!(\gamma - \beta)!} \left( D^{\gamma - \beta} a^{ij} \cdot D^\beta D_{ij} u + D^{\gamma - \beta} b^i \cdot D^\beta D_i u + D^{\gamma - \beta} c \cdot D^\beta u \right)$$

where here  $\beta < \gamma$  means that  $\beta_i \leq \gamma_i$  for all  $i$  and  $\beta_i < \gamma_i$  for some  $i$ .

Then one can check to see that the regularity assumptions on this hold (i.e. the RHS is in  $C^{1,\alpha}(\Omega)$ ), and so we can apply Step 1/the  $k = 1$  case to get that  $D^\gamma u \in C^{3,\alpha}(\Omega)$ , and so hence  $u \in C^{k+2,\alpha}(\Omega)$ , and the estimate holds.

**Step 3:** Prove the theorem with the estimate.

This just follows by induction on  $k$ , from Step 2. So done. □

There is also a higher global regularity result as well (i.e. up to the boundary - see Gilbarg and Trudinger, Chapter 6).

**Note:** We can apply Theorem 5.6 to certain non-linear problems, e.g. the minimal surface equation.

**Example:** Suppose  $u \in C^2(\Omega)$  solves the minimal surface equation,

$$D_i \left( \frac{D_i u}{\sqrt{1 + |Du|^2}} \right) = 0$$

in  $\Omega$ . Then, in fact we have  $u \in C^\infty(\Omega)$ .

**Proof.** The equation in non-divergence form is:

$$\underbrace{\left( \delta_{ij} - \frac{D_i u D_j u}{1 + |Du|^2} \right)}_{=a^{ij}} D_{ij} u = 0.$$

So hence if  $u \in C^2(\Omega)$ , then the lemma gives  $u \in C^{2,\alpha}(\Omega)$ . Then we know  $D_i u \in C^{1,\alpha}(\Omega)$  for all  $i$ , and so hence in fact  $a^{ij} \in C^{1,\alpha}(\Omega)$ . So hence by Theorem 5.6, we get that  $u \in C^{3,\alpha}(\Omega)$ . But then in the same way, we now have  $a^{ij} \in C^{2,\alpha}$ , and so by induction we get  $u \in C^{k,\alpha}(\Omega)$  for all  $k$ , i.e.  $u \in C^\infty(\Omega)$ .

□

**End of lecture course.**