Analysis of Partial Differential Equations
Exercise sheet I (Chapter 1)

1. (The Picard-Lindelöf / Cauchy-Lipschitz theorem) Consider $\mathbf{F} : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ continuous and locally Lipschitz in the second variable i.e., for each $\bar{\mathbf{u}} \in \mathbb{R}^d$, there exist constants $\delta > 0$ and $L > 0$ such that

$$|\mathbf{u} - \bar{\mathbf{u}}| < \delta \implies |\mathbf{F}(\cdot, \mathbf{u}) - \mathbf{F}(\cdot, \bar{\mathbf{u}})| \leq L|\mathbf{u} - \bar{\mathbf{u}}|.$$

Consider the following ODE:

$$\mathbf{u}'(t) = \mathbf{F}(t, \mathbf{u}(t)), \quad \mathbf{u}(0) = \mathbf{u}_0 \in \mathbb{R}^d.$$

(a) Prove the existence and uniqueness of a *maximal* C^1 solution on $(-T_c^-, T_c^+)$ with $T_c^-, T_c^+ > 0$ i.e., the solution cannot be continued for times beyond $-T_c^-$ and T_c^+.
(b) Prove moreover that if $T_c^+ < +\infty$ then $\mathbf{u}(t)$ is unbounded as $t \to T_c^+$.

2. Show that the following ODEs have infinitely many solutions:

$$\begin{cases} u'(t) \;=\; \sqrt{u(t)}, \quad t \in (0, a), \\ u(0) \;=\; 0. \end{cases} \tag{1}$$

Show further that the solutions to (1) split into TWO different types.

$$\begin{cases} u'(t) \;=\; \dfrac{4\,t\,u(t)}{u(t)^2 \,+\, t^2}, \quad t \in (0, a), \\ u(0) \;=\; 0. \end{cases} \tag{2}$$

Show further that the solutions to (2) split into FIVE different types.

3. Show that the solution to the following ODE blows up in finite time:

$$\begin{cases} u'(t) \;=\; u(t)^2, \quad t \in (0, t_c), \\ u(0) \;=\; u_0 > 0. \end{cases} \tag{3}$$

(a) What is the value of the critical time t_c?
(b) Does the blow-up behaviour of $u(t)$ still holds when $u(t)^2$ is replaced by $-u(t)^2$ in (3)?

4. Show that the solution $u : \mathbb{R}_+ \to \mathbb{R}$ to the following second order ODE is global:

$$\begin{cases} u''(t) \,+\, \sin(u(t)^2) \,=\, 0, \qquad t \in \mathbb{R}_+, \\ (u(0), u'(0)) \,=\, (u_0, u_1) \in \mathbb{R}^2. \end{cases} \tag{4}$$

5. (The Gronwall lemma) For some $T > 0, C \geq 0$, let $u, v \in C^1([0, T); \mathbb{R}_+)$ be such that:

$$\forall\, t \in [0, T), \quad u(t) \leq C + \int_0^t v(s)u(s)\,\mathrm{d}s.$$

Show that u satisfies

$$\forall\, t \in [0, T), \quad u(t) \leq C \exp\left(\int_0^T v(s)\,\mathrm{d}s\right).$$

Hint: Set $w(t) := C + \int_0^t v(s)u(s)\,\mathrm{d}s$ and check $w'(t) - v(t)w(t) \leq 0$ for all $t \in [0, T)$.

6. (Approximation of solutions to ODE) Let $F \in C^1(\mathbb{R};\mathbb{R})$. First let $u, v \in C^1([0,T];\mathbb{R})$ solutions to

$$\begin{aligned} u'(t) &= F(u(t)) \text{ for } t \in \mathbb{R}_+ \text{ and } \ u(0) = u_0 \in \mathbb{R}, \\ v'(t) &= F(v(t)) \text{ for } t \in \mathbb{R}_+ \text{ and } \ v(0) = v_0 \in \mathbb{R}. \end{aligned}$$

(a) Prove that for any $T > 0$,

$$\forall\, t \in [0,T], \quad |u(t) - v(t)| \leq |u_0 - v_0| e^{C_T t} \tag{5}$$

(b) Identity the constant C_T in (5).
Assume second that $u, v \in C^1([0,T];\mathbb{R})$ are solutions to

$$\begin{aligned} |u'(t) - F(u(t))| &\leq \varepsilon_1 \text{ for } t \in \mathbb{R}_+ \text{ and } \ u(0) = u_0 \in \mathbb{R}, \\ |v'(t) - F(v(t))| &\leq \varepsilon_2 \text{ for } t \in \mathbb{R}_+ \text{ and } \ v(0) = v_0 \in \mathbb{R}, \end{aligned}$$

for some $\varepsilon_1, \varepsilon_2 > 0$.
(c) Prove that for any $T > 0$,

$$\forall\, t \in [0,T], \quad |u(t) - v(t)| \leq |u_0 - v_0| e^{C_T t} + (\varepsilon_1 + \varepsilon_2) \frac{e^{C_T t} - 1}{C_T}.$$

7. (Osgood uniqueness Theorem) Let I be an interval of $\mathbb{R}$, and $\mathbf{F} : I \times \mathbb{R}^d \to \mathbb{R}^d$ a continuous function. Let Ω be an open subset of $\mathbb{R}^d$ endowed with the euclidean norm, $t_0 \in I$, $\mathbf{u}_0 \in \Omega$. We suppose that

$$\forall\, (t, \mathbf{y}_1, \mathbf{y}_2) \in I \times \Omega \times \Omega, \quad |\mathbf{F}(t, \mathbf{y}_1) - \mathbf{F}(t, \mathbf{y}_2)| \leq \omega(|\mathbf{y}_1 - \mathbf{y}_2|)$$

where $\omega \in C(\mathbb{R}_+, \mathbb{R}_+)$ an increasing function which satisfies

$$\omega(0) = 0, \quad \forall\, \sigma > 0, \quad \omega(\sigma) > 0, \quad \text{and} \quad \forall\, \alpha > 0, \quad \int_0^\alpha \frac{1}{\omega(\sigma)}\,\mathrm{d}\sigma = +\infty. \tag{6}$$

Let $\mathbf{u}_1, \mathbf{u}_2 : I \to \Omega$ be two differentiable functions which are solutions to the Cauchy problem:

$$\begin{cases} \mathbf{u}'(t) = \mathbf{F}(t, \mathbf{u}(t)), \\ \mathbf{u}(t_0) = \mathbf{u}_0. \end{cases}$$

(a) Show that $\mathbf{u}_1 = \mathbf{u}_2$.
(b) Give an example of an increasing function $\omega \in C(\mathbb{R}_+, \mathbb{R}_+)$ which satisfies (6).

8. (Cauchy-Peano theorem) Consider $\mathbf{F} : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ merely continuous, and the ODE:

$$\mathbf{u}'(t) = \mathbf{F}(t, \mathbf{u}(t)), \quad \mathbf{u}(0) = \mathbf{u}_0 \in \mathbb{R}^d.$$

Prove the existence of a *maximal* C^1 solution on $(-T_c^-, T_c^+)$ with $T_c^-, T_c^+ > 0$.
Hint. From the fundamental theorem of calculus the ODE can be reframed as $\mathbf{u}(t) = \mathbf{u}_0 + \int_0^t \mathbf{F}(s, \mathbf{u}(s))\,\mathrm{d}s$ and we define the *Picard iteration*:

$$\mathbf{u}_{n+1}(t) = \mathbf{u}_0 + \int_0^t \mathbf{F}(s, \mathbf{u}_n(s))\,\mathrm{d}s, \quad \mathbf{u}_{n+1}(0) = \mathbf{u}_0 \in \mathbb{R}^d, \quad n \geq 0,$$

initialised with $\mathbf{u}_0(t) = \mathbf{u}_0$ for all t. Prove that locally around $t = 0$ the sequence is uniformly bounded and uniformly equicontinuous. Recall and use the Arzéla-Ascoli theorem to prove the compactness of the sequence for the uniform convergence. Show that the cluster point is a solution to the ODE.