Analysis of PDEs

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Michealmas Term 2017

These notes are produced entirely from the course I took, and my subsequent thoughts. They are not necessarily an accurate representation of what was presented, and may have in places been substantially edited. Please send any corrections to pdtwm2@cam.ac.uk

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1. Introduction

Let $U \subset \mathbb{R}^n$ be open.

Definition 1.1. *A* ***partial differential equation (PDE) of rank/order k*** *is a relation of the form:*

$$F\left(x, u(x), Du(x), \dots, D^k u(x)\right) = 0 \tag{1.1}$$

where $F : U \times \mathbb{R} \times \mathbb{R}^n \times R^{n^2} \times \cdots \times \mathbb{R}^{n^k} \to \mathbb{R}$ is a given function, and u is the unknown.

Here, $Du := \left(\frac{\partial u}{\partial x_i}\right)_i$, etc.

We say that $u \in C^k(U)$ is a **classical solution** if (1.1) is identically satisfied on U, when $u, Du, \dots, D^k u$ are substituted in.

We can also consider the case when $u(x) \in \mathbb{R}^p$ and F is valued in $\mathbb{R}^q$. In this case, we talk about a **system of PDEs**.

PDEs come in all kinds of different forms, including Poisson and Laplace's equation, Maxwell's equations, the minimal surface equation, Ricci flow, etc. So it is unlikely that we will be able to prove anything about the whole class of PDE. So we need to restrict to smaller classes of PDEs so that we can say more.

1.1. Data and Well-Posedness.

In all the above examples of PDEs, we need some additional information to solve them. For example, we may need:

- Boundary values of u for the Laplace equation
- An initial temperature distribution for the heat equation.

We broadly refer to this information as the **data**.

An important part of studying PDE's is to understand what data is required for a certain problem. A guiding principle to this is the idea of *well-posedness*.

Definition 1.2. *A PDE problem (i.e. equation + data) is* ***well-posed*** *if:*

(i) A solution exists (in some function space, e.g. $C(U)$, $C^1(U)$, etc)
(ii) The solution is unique (in some function space, i.e. uniquely determined by the data)
(iii) The solution depends continuously on the data (in some function space, with the data in some function space).

The last part (iii) is a subtle but important one for physical problems - due to slight inaccuracies in measurements we make, we don't want the solution to change drastically with slight error in the boundary conditions!

There is some freedom to choose which function space we work in. We want to find the largest function space for which a solution exists, but a small enough one so that the solution is unique! For example, requiring the solution to be C^1 might give two solutions, whilst requiring it to be C^2 might give just one. So spaces like C^k for $k \in \mathbb{R}$, might help us get around this to find the biggest space!

The details of whether a problem is well-posed or not can (very easily) depend on the choice of function space.

Notation: We will use **multi-index notation** (introduced by Schwartz). We have $\mathbb{N} = \{0, 1 \dots\}$. Then, $\alpha \in \mathbb{N}^n$ is called a **multi-index**, and if $\alpha = (\alpha_1, \dots, \alpha_n)$, we define:

- $D^\alpha f(x) := \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}(x)$, where $|\alpha| := |\alpha_1| + \cdots |\alpha_n|$ (and we assume all partial derivatives commute). [This $|\alpha|$ definition is **just** for multi-indices.]
- If $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, then $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.
- $\alpha! := \alpha_1! \cdots \alpha_n!$.

1.2. **Classifying PDEs.**

There are subclasses of PDE for which (1.1) has a simpler structure.

Definition 1.3. *We say that (1.1) is a **linear PDE** if F is a linear function of u and its derivatives, and so hence we can re-write (1.1) as:*

$$\sum_{\alpha : |\alpha| \leq k} a_\alpha(x) D^\alpha u(x) = f(x).$$

Definition 1.4. *We say that (1.1) is a **semilinear PDE** if it is linear in the highest order derivatives, i.e. it is of the form:*

$$\sum_{|\alpha| = k} a_\alpha(x) D^\alpha u(x) + \underbrace{a_0[x, u, Du, \dots, D^{k-1}u]}_{\text{can be non-linear in lower order}} = 0.$$

[Note that the coefficients can depend on x.]

Definition 1.5. *We say that (1.1) is a **quasi-linear PDE** if it is of the form:*

$$\sum_{|\alpha| = k} a_\alpha[x, u, Du, \dots, D^{k-1}u] D^\alpha u(x) + a_0[x, u, \dots, D^{k-1}u] = 0.$$

So the highest order derivatives appear linearly, but the coefficients can depend on the lower order derivatives of u.

Definition 1.6. *We say that (1.1) is* ***fully non-linear*** *if it is not of one of the above forms.*

Example 1.1. *The following PDEs demonstrate each of the above forms.*

(i) $\Delta u = 0$ *is linear.*

(ii) $\Delta u = u_x^2$ *is semi-linear.*

(iii) $uu_{xx} + u_{yy} = u_x^2$ *is quasi-linear.*

(iv) $u_{xx}u_{yy} - u_{xy}^2 = 0$ *is fully non-linear.*

Note that here we write: $u_x \equiv \partial_x u := \frac{\partial u}{\partial x}$.

2. The Cauchy-Kovalevskaya Theorem

Here we specify data on some codimension 1 surface, and then try to find some solution on a neighbourhood of the surface by evolving the data via the PDE. Such scenarios tend to arise in evolution problems, where we want to evolve our solution over a small time step.

To motivate this theorem, let us recall some ODE theory.

Fix $U \subset \mathbb{R}^n$ an open subset and assume that $f : U \to \mathbb{R}^n$ is given. Consider then the ODE:

$$\dot{u}(t) = f(u(t)), \quad \text{where} \quad u(0) = u_0 \in U. \tag{2.1}$$

Here, $u : (a,b) \to U$ is the unknown, and $a < 0 < b$. Solving (2.1) is the **Cauchy problem** for this ODE, as we have specified the initial data.

We then have the following:

Theorem 2.1 (Picard-Lindelöf). *Suppose that $\exists r, K > 0$ such that $B_r(u_0) \subset U$, and*

$$\|f(x) - f(y)\| \leq K\|x - y\| \quad \forall x, y \in B_r(u_0)$$

i.e. suppose that f is Lipschitz about u_0 locally.

Then, $\exists \varepsilon = \varepsilon(K, r) > 0$ and $\exists!$ C^1 function $u : (-\varepsilon, \varepsilon) \to U$ solving (2.1).

Proof (Sketch). If u solves (2.1), then by the fundamental theorem of calculus, we have:

$$u(t) = u_0 + \int_0^t f(u(s)) \tag{2.2}$$

[this is the weak formulation of the problem]. Conversely, if u is a C^0 solution of (2.2), then it solves (2.1) - so we do not need so assume as much on u in this formulation (i.e. just C^0 instead of C^1), which is the reason why it is useful].

Thus u, if it exists, is a fixed point of the map:

$$F : C^0(U) \to C^1(U), \quad \text{with} \quad (F(\omega))(t) := u_0 + \int_0^t f(\omega(s))\, \mathrm{d}s.$$

So let $\mathscr{C} = \{\omega : [-\varepsilon, \varepsilon] \to \overline{B_{r/2}(u_0)} : \omega \in C^0\}$ (i.e. continuous functions here).

This set is then a closed ball in a Banach space, when equipped with the supremum norm. Hence it is a complete metric space.

We now wish to choose $\varepsilon > 0$ small enough so that $F : \mathscr{C} \to \mathscr{C}$ is a contraction map (**Note:** we must verify that F does indeed map $\mathscr{C}$ into $\mathscr{C}$). To see that F maps into $\mathscr{C}$ note that we have:

$$\|F(\omega)(t) - u_0\| = \left\| \int_0^t f(\omega(s))\, \mathrm{d}s \right\| \leq \varepsilon \cdot \sup_{s \in [-\varepsilon, \varepsilon]} \|f(\omega(s))\| \leq \varepsilon \left(\frac{Kr}{2} + \|f(u_0)\| \right)$$

where we have used the Lipschitz property. Hence if we choose ε small enough so that the RHS of this inequality is $\leq r/2$, we see that F does map $\mathscr{C} \to \mathscr{C}$.

To see that F is a contraction map, simply note that:

$$\|F(\omega)(t)-F(\nu)(t)\| \leq \int_0^t \|f(\omega(s))-f(\nu(s))\| \,\mathrm{d}s \leq K\int_0^t \|\omega(s)-\nu(s)\| \,\mathrm{d}s \leq K\varepsilon\|\omega-\nu\|.$$

So now taking $\varepsilon > 0$ small enough so that $\varepsilon K < 1$ we get that F is a contraction map.

Then by the contraction mapping theorem ($\equiv$ the Banach fixed point theorem, since $\mathscr{C}$ is complete), we get that $\exists$ a fixed point of this map, which is then, by the above, the solution we were after. So we are done.

$\square$

This theorem tells us that a unique C^1 solution exists locally. If f were better behaved (e.g. C^∞), then we might expect the u to be more regular as well (and indeed this is the case).

Note also that the use of the CMT (contraction mapping theorem) also gives us a way of approximation the solution, by repeated iteration of the map F.

Let us now consider an alternative approach to solving (2.1). Suppose that $f \in C^\infty$. First we note that we know $u(0) = u_0$, and so from the ODE, we see that $\dot{u}(0) = f(u_0)$.

Now differentiating (2.1) (which we shall assume we can do for now), gives:

$$\ddot{u}(t) = \dot{u}(t)\cdot Df(u(t)) =: f_2(u(t), \dot{u}(t)).$$

So hence we can find $\ddot{u}(0)$ since we know everything on the RHS at $t = 0$. So proceeding iteratively, we can determine that $u^{(k)}(t) = f_k(u, \dot{u}, \ldots, u^{(k-1)})$ from (2.1), and so in principle, we can find $u_k = u^{(k)}(0)$ for all k.

So formally at least (i.e. not worrying about convergence, which is a different question), we can write:

$$u(t) = \sum_{k=0}^{\infty} u_k \cdot \frac{t^k}{k!}.$$

In order to obtain a genuine solution, we then would need to show that this series converges (in at least some neighbourhood of $t = 0$), so then we can differentiate this series term by term and check it solves the ODE (2.1).

The Cauchy-Kovalevskaya theorem (applied to this scenario) makes this statement precise. We will then show that this approach can be extended to a larger class of PDE.

Theorem 2.2 (Cauchy-Kovalevskaya for (simple) ODEs). *If f is real analytic on a neighbourhood of u_0, then the series given above converges in a neighbourhood of $t = 0$ to the Picard-Lindelöf solution (found above)*[(i)] *of the Cauchy problem (2.1).*

Proof. None given. $\square$

[(i)] This is because we know that the Picard-Lindelöf solution is the unique solution of this problem.

Definition 2.1. *Let $U \subset \mathbb{R}^n$ be open, and suppose $f : U \to \mathbb{R}$. Then f is called* ***real analytic near $\boldsymbol{x_0}$*** *$\in U$ if $\exists r > 0$ and constants $f_\alpha \in \mathbb{R}$ (α a multi-index) such that:*

$$f(x) = \sum_{\alpha \in \mathbb{N}^n} f_\alpha (x - x_0)^\alpha \quad \text{for} \quad |x - x_0| < r$$

where the sum is over all multi-indices. [Recall that $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.]

i.e. this just says that f is a real-valued analytic function, i.e. on a ball about x_0, we can write f as a convergent power series.

Exercise: Show that if f is real-analytic near x_0, then f is C^∞ near x_0.

[*Hint:* Use the Weierstrass M-test, or something for the convergence. Weierstrass M-test gives that this series converges on $B_{r/2}(x_0)$, and so we can differentiate term by term and the result has the same radius of convergence, etc.]

Furthermore, the constants f_α are given by (as we can differentiate the series term by term):

$$f_\alpha = \frac{D^\alpha f(x_0)}{\alpha!}.$$

In other words, f equals its Taylor series expansion about x_0, i.e.

$$f(x) = \sum_\alpha \frac{D^\alpha f(x_0)}{\alpha!} \cdot (x - x_0)^\alpha.$$

By translation, we can usually assume that $x_0 = 0$, and we do this when we can (i.e. consider $g(x) = f(x + x_0)$ instead).

Example 2.1 (Important Example)**.** *If $r > 0$, set:*

$$f(x) := \frac{r}{r - (x_1 + \cdots + x_n)} \quad \text{for} \quad \|x\| < r/\sqrt{n}.$$

[this is essentially just $\frac{1}{1-x} = \sum_i x^i$ generalised to arbitrary balls and dimension.]

Then, we have $f(x) = \sum_{k=0}^\infty \left(\frac{x_1 + \cdots + x_n}{r}\right)^k$, by summing this geometric series, which is summable by Cauchy-Schwarz (which gives $|x_1 + \cdots + x_n| \leq \|x\| \cdot \|\mathbf{1}\| = \sqrt{n}\|x\| < r$). So hence, expanding each term using the multinomial theorem, we have:

$$f(x) = \sum_{k=0}^\infty \frac{1}{r^k} \left(\sum_{|\alpha| = k} \binom{|\alpha|}{\alpha} x^\alpha \right)$$

where $\binom{|\alpha|}{\alpha} = \frac{|\alpha|!}{\alpha!}$.

So as the sums are absolutely convergent for $\|x\| < r/\sqrt{n}$, we get (as we can rearrange the sums):

$$f(x) = \sum_\alpha \frac{|\alpha|!}{r^{|\alpha|}\alpha!} \cdot x^\alpha$$

i.e. $D^\alpha f|_0 = |\alpha|!/r^{|\alpha|}$. *So indeed, just be working the above steps backward, but now with* $|x_i|$, *we have:*

$$\sum_\alpha \frac{|\alpha|!|x|^\alpha}{r^{|\alpha|}\alpha!} = \sum_{k=0}^{\infty}\left(\frac{|x_1|+\ldots+|x_n|}{r}\right)^k = \frac{r}{r-(|x_1|+\cdots+|x_n|)} < \infty$$

as $\|x\| < r/\sqrt{n}$. *So this series converges absolutely on this neighbourhood of 0.*

$\square$

This above example is important as it allows us to check/confirm the convergence of other power series, by comparison with this one.

Definition 2.2. *Let* $f = \sum_\alpha f_\alpha x^\alpha$ *and* $g = \sum_\alpha g_\alpha x^\alpha$ *be two formal power series. Then we say that* ***g majorizes f****, or g is a* **majorant** *of f, written* $g \gg f$, *if:* $g_\alpha \geq |f_\alpha|$ *for all multi-indices* α.

If f, g *are vector-valued functions, then:*

$$g \gg f \iff g^i \gg f^i \quad \forall i.$$

Note that in particular, if g majorizes f, then we must have $g_\alpha \geq 0$ for all multi-indices α.

Lemma 2.1 (Properties of Majorants).

(i) If $g \gg f$ *and* g *converges for* $\|x\| < r$, *then* f *converges for* $\|x\| < r$ *also.*

(ii) If $f(x) = \sum_\alpha f_\alpha x^\alpha$ *converges for* $\|x\| < r$, *and* $s \in (0, r/\sqrt{n})$, *then* $\exists$ *a majorant of* f *which converges on* $\|x\| < s/\sqrt{n}$.

Proof. (i): Note that:

$$\begin{aligned}\sum_{|\alpha|\leq k} |f_\alpha x^\alpha| &= \sum_{|\alpha|\leq k} |f_\alpha| \cdot |x_1|^{\alpha_1}\cdots|x_n|^{\alpha_n} \\ &\leq \sum_{|\alpha|\leq k} g_\alpha \cdot |x_1|^{\alpha_1}\cdots|x_n|^{\alpha_n} \\ &\leq \sum_\alpha g_\alpha |x_1|^{\alpha_1}\cdots|x_n|^{\alpha_n} = g(\tilde{x})\end{aligned}$$

where $\tilde{x} = (|x_1|, \ldots, |x_n|)$. So as $\|\tilde{x}\| = \|x\|$, and so hence if $\|x\| < r$, g converges at $\tilde{x}$, and so hence we get a uniform bound on these partial sums. So hence we can take $k \to \infty$ to get $\sum_\alpha |f_\alpha x^\alpha| < \infty$, i.e. $|f(x)| < \infty$, and so f converges here. [Alternative we can use the Weierstrass M-test.]

(ii): Let $s \in (0, r/\sqrt{n})$ and set $y = (s, \ldots, s) \in \mathbb{R}^n$. Then, clearly $\|y\| = s\sqrt{n}$. So by assumption, $f(y) = \sum_\alpha f_\alpha y^\alpha$ converges, as $\|y\| = s\sqrt{n} < r$.

Now a convergent power series has uniformly bounded terms, and so $\exists\, C$ such that $|f_\alpha y^\alpha| \le C$ for all α. Hence:

$$|f_\alpha| \le \frac{C}{|y^\alpha|} = \frac{C}{|y_1|^{\alpha_1}\cdots|y_n|^{\alpha_n}} = \frac{C}{|s|^{|\alpha_1|+\cdots+|\alpha_n|}} = \frac{C}{s^{|\alpha|}} \le \frac{C}{s^{|\alpha|}}\cdot\frac{|\alpha|!}{\alpha!}$$

just from what y is, and since $\alpha! \le |\alpha|!$. So then if we set:

$$g(x) := \frac{Cs}{s-(x_1+\cdots+x_n)} = C\sum_\alpha \frac{|\alpha|!}{s^{|\alpha|}\alpha!}\cdot x^\alpha$$

(where we have used the above Important Example), and we know this converges for $\|x\| < s/\sqrt{n}$. Hence this shows that f has a majorant (given by g) which converges for $\|x\| < s/\sqrt{n}$, and so we are done.

$\square$

Note: Real analyticity is a local property, i.e.

$$f \text{ is real analytic about } x_0 \;\Rightarrow\; f \text{ is real analytic on a neighbourhood of } x_0.$$

[See Example Sheet 1 for proof.]

2.1. **Cauchy-Kovalevskaya for 1st Order Systems.**

We start our study of PDE by looking at a class of problems similar to the Cauchy problem for ODEs we saw previously. We consider $\mathbb{R}^n$ with coordinates $(x^1,\dots,x^n) = (x', x_n)$, where $x' = (x^1,\dots,x^n) \in \mathbb{R}^{n-1}$. We will consider a system of equations for an unknown $u(x) \in \mathbb{R}^m$.

We shall seek a solution to the PDE:

$$u_{x_n} = \sum_{j=1}^{n-1} B_j(u,x')u_{x_j} + C(u,x') \tag{2.3}$$

on the subset: $B_r(0) = \{x \in \mathbb{R}^n : \|x\| < r\}$, subject to the condition:

$$u = 0 \quad \text{on} \quad \{x\in\mathbb{R}^n : \|x\|<r,\ x^n = 0\} = B_r(0)\cap\{x^n = 0\}.$$

Here we assume that the functions $B_j : \mathbb{R}^m \times \mathbb{R}^{n-1} \to \mathrm{Mat}_{m\times m}(\mathbb{R})$ and $C : \mathbb{R}^m\times\mathbb{R}^{n-1}\to\mathbb{R}^m$, are analytic and are given.

[Note that this is an evolution equation for u in the x_n-direction, in the interior of a ball or radius r. The derivatives in the other directions u_{x_j} are multiplied by matrices which can depend on everything <u>except</u> the x_n coordinate, and similarly for the vector C.]

Sometimes it is useful to write the sets in question in terms of x' and x^n instead of x, i.e. we work on $\{x\in\mathbb{R}^n : \|x'\|^2 + (x^n)^2 < r^2\}$, etc.

Note: Both B_j, C are independent of x^n. Note that <u>this is not a restriction</u>, as we can always add a new coordinate, i.e. set $u^{m+1} = x^n$ and enlarge the tangent space to $\mathbb{R}^{m+1}$.

[Then any x^n dependence can be replaced by u^{m+1}, which is allowed (as the coefficients can depend on all of u), and the extra $(m+1)$'th equation for u^{m+1} is just: $u^{m+1}_{x_n} = 1$, i.e. no B_j here and $C^{m+1} = 1$.]

This type of PDE might seem quite specific, but a lot of PDE's can actually be written in this form.

Then the Cauchy-Kovalevskaya Theorem in this case becomes:

Theorem 2.3 (Cauchy-Kovalevskaya for 1st Order Systems). *Assume that $\{B_j\}_{j=0}^{n-1}$ and C are real analytic. Then for sufficiently small $r > 0$, $\exists$ a real analytic function $u = \sum_\alpha u_\alpha x^\alpha$ solving (2.3), and it is unique among real analytic functions.*[ii]

Note: An overview of the proof is as follows. First we will write B, C as power series and find all the derivatives of u as a universal polynomial (with non-negative coefficients). Then we will show convergence via majorising. Then we shall use Lemma 2.1 on B, C to get a majorised problem, and show that the solution to the majorised problem is a majorant of the series found for the original problem. To see that it is then a solution to the original problem we will compare power series on $\{x^n = 0\}$.

Proof. This proof will not be very enlightening, but it needs to be done. The idea is simple enough though.

We will write: $B_j = (b_j^{kl})_{1\leq k,l\leq m}$ for $j = 1, \dots, n$, and $C = (C^1, \dots, C^m)$. So in components, (2.3) reads:

$$\text{(2.4)} \qquad \text{For } \|x'\|^2 + x_n^2 < r^2 : \quad u^k_{x_n} = \sum_{j=1}^{n-1}\sum_{l=1}^{m} b_j^{kl}(u, x')u^l_{x_j} + C^k(u, x')$$

$$\text{and for } \|x'\| < r, x_n = 0 : \quad u^k(x) = 0$$

The idea of the proof is to compute $u_\alpha = \frac{D^\alpha u(0)}{\alpha!}$ in terms of B_j and C and show that the resulting series converges. We will do this by induction, computing all derivatives with no x_n-derivatives (i.e. $\alpha_n = 0$ in multi-index notation), and then those with ∂_{x_n} (i.e. $\alpha_n = 1$), etc.

So as B_j and C are real analytic, we can write them as power series:

$$B_j(z, x') = \sum_{\gamma,\delta} B_{j,\gamma,\delta} \cdot z^\gamma x^\delta, \qquad C(z, x') = \sum_{\gamma,\delta} C_{\gamma,\delta} \cdot z^\gamma x^\delta,$$

where these power series converge if $\|x'\| + |z| < s$, for some small $s > 0$ (note that $\gamma \in \mathbb{N}^m$ is a m-multi-index whilst $\delta \in \mathbb{N}^{n-1}$ is a $(n-1)$-multi-index, which we can think of as in $\mathbb{N}^n$ with n-th component 0).

Thus we have:

$$\text{(2.5)} \qquad B_{j,\gamma,\delta} = \frac{D_z^\gamma D_x^\delta B_j(0,0)}{\gamma!\delta!}, \quad \text{and} \quad C_{\gamma,\delta} = \frac{D_z^\gamma D_x^\delta C(0,0)}{\gamma!\delta!}$$

for each $j = 1, \dots, n-1$, and for all δ, γ.

[ii] There could be different solutions which are just, say, smooth, but not real analytic. Thus this function space is small enough to guarantee uniqueness, but large enough to guarantee existence.

Thus since $u \equiv 0$ on the surface $\{x_n = 0\}$, we have $u(x', 0) = 0$. So hence we can differentiate with respect to $x_1, \dots, x_{n-1}$ to see:

$$u_\alpha = \frac{D^\alpha u(0)}{\alpha!} = 0$$

for all multi-indices α with $\alpha_n = 0$.

So by setting $x_n = 0$ and using $u_{x_j} = 0$ for $j = 1, \dots, n-1$, we have by (2.4) that:

$$u_{x_n}(x', 0) = C(0, x').$$

Thus now differentiating this relation in directions tangent to $\{x_n = 0\}$ (i.e. $x_1, \dots, x_{n-1}$), we find that if α is a multi-index of the form: $\alpha = (\alpha_1, \dots, \alpha_{n-1}, 1) = (\alpha', 1)$, then:

$$D^\alpha u(0) = D^{\alpha'} C(0, 0).$$

So hence we have found all u_α for multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$ when $\alpha_n = 0$ or 1.

So now suppose $\alpha_n = 2$, i.e. $\alpha = (\alpha', 2)$. Then:

$$\begin{aligned}
D^\alpha u^k &= D^{\alpha'}(u^k_{x_n})_{x_n} \qquad \text{(as two } x_n \text{ derivatives)}\\
&= D^{\alpha'}\left(\sum_{j=1}^{n-1}\sum_{l=1}^{m} b^{kl}_j u^l_{x_j} + C^k\right)_{x_n} \qquad \text{, by (2.4)}\\
&= D^{\alpha'}\left(\sum_{j=1}^{n-1}\sum_{l=1}^{m}\left(b^{kl}_j u^l_{x_j x_n} + u^l_{x_j} \cdot \sum_{p=1}^{m} (b^{kl}_j)_{z_p} \cdot u^p_{x_n}\right) + \sum_{p=1}^{m} C^k_{z_p} u^p_{x_n}\right)
\end{aligned}$$

where we have used (2.4) in the second line and then used the chain rule when differentiating everything with respect to x_n for the second time.

Thus we have by setting $x = 0$ and using what we already know:

$$D^\alpha u^k(0) = D^{\alpha'}\left(\sum_{j=1}^{n-1}\sum_{l=1}^{m} b^{kl}_j u^l_{x_j x_n} + \sum_{p=1}^{m} C^k_{z_p} u^p_{x_n}\right)\Bigg|_{x=0}.$$

Now crucially, the RHS can be expanded to produce a polynomial with non-negative (in fact integer) coefficients, involving the derivatives of B_j, C, and $D^\beta u$, where $\beta_n \leq 1$ (crucial, as by induction we will know what these $D^\beta u$ derivatives are in terms of B_j and C).

More generally, for each α and each $k \in \{1, \dots, m\}$, we can compute by induction on the number of x_n-derivatives [**Exercise** to check:]

$$D^\alpha u^k(0) = P^k_\alpha(\dots, D^\gamma_z D^\delta_x B_j, \dots, D^\gamma_z D^\delta_x C, \dots, D^\beta u, \dots)\big|_{u=x=0} \tag{2.6}$$

where $\beta_n \leq \alpha_n - 1$ (just the n'th order one), and P^k_α is some polynomial with non-negative coefficients (i.e. a polynomial in the derivatives of B_j, C, and lower derivatives of u (in the x_n-direction)).

Then by rescaling (2.5) and (2.6), we have for each α, k (i.e. divide by $\alpha!$ and rescale coefficients):

$$u^k_\alpha = q^k_\alpha(\dots, B_{j,\gamma,\delta}, \dots, C_{\gamma,\delta}, \dots, u_\beta)$$

where q^k_α is a polynomial with non-negative coefficients and $\beta_n \leq \alpha_n - 1$. So hence by induction, we know the LHS in terms of the B_j, C, and so hence we can express the u_α in terms of B_j and C.

So now we just need to show that the series of u converges somewhere, so that this shows (as we can then differentiate the series term by term) that this is a solution to the original PDE.

So we have showed that <u>if</u> an analytic solution, $u = \sum_\alpha u_\alpha x^\alpha$, exists, then:

$$u_\alpha^k = q_\alpha^k(\dots, B_{j,\gamma,\delta}, \dots, C_{\gamma,\delta}, \dots, u_\beta, \dots)$$

where q_α^k is a universal polynomial (i.e. does not depend on B, C except through its arguments, just because of how q comes from the product rule, etc), and q_α^k has non-negative coefficients, and $\beta_n \leq \alpha_n - 1$, for all multi-indices β, α.

Now, this is a standard procedure for showing existence of solutions to PDEs: we find an a priori estimate, and then show that it is indeed a solution.

So we will now show that the series $u = \sum_\alpha u_\alpha x^\alpha$ above converges near $x = 0$. Currently, the only way we have of doing this is by majorization.

Let us first suppose that $B_j^* \gg B_j$ and $C^* \gg C$, where:

$$B_j^*(z,x) = \sum_{j,\gamma,\delta} B_{j,\gamma,\delta}^* z^\gamma x^\delta, \quad \text{and} \quad C^*(z,x) = \sum_{\gamma,\delta} C_{\gamma,\delta}^* z^\gamma x^\delta,$$

with these series converging for $|z| + \|x'\| < s$, as by Lemma 2.1, we can find some neighbourhood (which might be smaller) where B^*, C^* converge.

[i.e. B, C are analytic at $x = 0$, and so converge on some neighbourhood. So by Lemma 2.1 (ii), we can find majorants B^*, C^* for both, which wlog by shrinking, converge on some neighbourhood.]

So then we have, by definition of majorants:

$$0 \leq |B_{j,\gamma,\delta}^{k,l}| \leq (B^*)_{j,\gamma,\delta}^{k,l}$$

and

$$0 \leq |C_{\gamma,\delta}^k| \leq (C^*)_{\gamma,\delta}^k$$

for each component k, l. We then consider the modified problem of solving:

$$u_{x_n}^* = \sum_{j=1}^{n-1} B_j^*(u^*, x') u_{x_j}^* + C^*(u^*, x') \quad \text{on} \quad \|x'\|^2 + (x^n)^2 < r^2 \tag{2.7}$$

$$\text{and} \quad u^* = 0 \quad \text{on} \quad \|x'\| < r,\ x^n = 0,$$

i.e. the same problem as before but with *'s now (if need be, reduce r so that B_j^*, C^* converge).

Again, we see a real analytic solution $u^* = \sum_\alpha u_\alpha^* x^\alpha$.

> **Claim:** $u^* \gg u$, i.e. $0 \leq |u_\alpha^*| \leq (u^*)_\alpha^k$ for all multi-indices α and all $k = 1, \dots, m$
>
> i.e. the solution to this problem is a majorant to the solution of the original problem.
>
> *Proof of Claim.* We proceed by induction on $\alpha_n \in \mathbb{N}$, where $\alpha = (\alpha_1, \dots, \alpha_n)$.
>
> If $\alpha_n = 0$, as before, from the boundary conditions we have $u_\alpha^k = (u^*)_\alpha^k = 0$, and so the majorant condition is true here.

For the induction step, suppose $0 \leq |u^k_\alpha| \leq (u^*)^k_\alpha$ holds for all $\alpha_n \leq a-1$. Suppose now that $\alpha_n = a$. Then we have:

$$\begin{aligned}|u^k_\alpha| &= \left|q^k_\alpha(,\dots,B^{k,l}_{j,\gamma,\delta},\dots,C^k_{\gamma,\delta},\dots,u^k_\beta,\dots)\right| \\ &\leq q^k_\alpha(,\dots,|B^{k,l}_{j,\gamma,\delta}|,\dots,|C^k_{\gamma,\delta}|,\dots,|u^k_\beta|,\dots) \\ &\leq q^k_\alpha(,\dots,(B^*)^{k,l}_{j,\gamma,\delta},\dots,(C^*)^{k,l}_{\gamma,\delta},\dots,(u^*)^k_\beta,\dots) \\ &= (u^*)^k_\alpha\end{aligned}$$

where the first inequality follows by the triangle inequality, as q^k_α is a polynomial with non-negative coefficients[(iii)] and the second inequality is true because q^k_α has non-negative coefficients and so is monotone increasing in each coordinates, and by the induction assumption we have $|u^k_\alpha| \leq (u^*)^k_\alpha$, and we know B^*, C^* majorize B, C. The last equality follows since the polynomial q^k_α is universal, i.e. the same for both u and u^* since it is derived in the same way for both (the only difference is the presence of *'s in its arguments).

Thus this shows, by induction, that $u^* \gg u$ as formal power series, and so done.

□

So hence by Lemma 2.1 (i), we will be done if we can show that $u^* = \sum_\alpha u^*_\alpha x^\alpha$ converges near 0.

To show this, we will make a particular choice of B^*_j, C^* for which we can solve the resulting PDE (2.7) explicitly.

Recall that in the proof of Lemma 2.1, we showed that:

$$B^*_j(z,x') = \frac{Ar}{r-(x_1+\cdots+x_{n-1})-(z_1+\cdots+z_m)}\begin{bmatrix}1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1\end{bmatrix}$$

$$C^*(z,x') = \frac{Ar}{r-(x_1+\cdots+x_{n-1})-(z_1+\cdots+z_n)}\cdot(1,\dots,1)^T$$

will majorize B_j, C, provided the constant A is large enough and r is small enough (i.e. majorize each component with an expression of this form, and get A and r for each. Then take the maximum A and the minimum r to get that the same A, r can be used for all components).

These will converge for $\|x'\|^2 + \|z\|^2 < s^2$ for s sufficiently small. With these choices, the modified PDE (2.7) becomes:

$$(u^*)^k_{x_n} = \frac{Ar}{r-(x_1+\ldots+x_{n-1})-((u^*)^1+\cdots+(u^*)^m)}\left(\sum_{j=1}^{n-1}\sum_{l=1}^{m}(u^*)^l_{x_j}+1\right)$$

which holds on $\|x'\|^2+(x^n)^2<r^2$, and $(u^*)^k = 0$ on $\|x'\|<r,\ x^n=0$.

(iii) i.e. $|f(x)| = \left|\sum_k a_k x^k\right| \leq \sum_k |a_k|\cdot|x^k| = \sum_k a_k|x|^k = f(|x|)$

It turns out that this problem has an explicit solution, $u^* = v^* \cdot (1, \dots, 1)$ (i.e. same solution in each component), where:

$$v^*(x) = \frac{1}{mn}\left(r - (x_1 + \cdots + x_{n-1}) - \sqrt{(r - (x_1 + \cdots + x_{n-1}))^2 - 2mnArx_n}\right)$$

which can be found by the method of characteristics (or by plugging this in and checking!) [**Exercise** to check].

One thing that is clear is that if $\|x'\| < r$ and $x_n = 0$, then $u^* = 0$, i.e. the boundary condition is satisfied.

Then clearly v^* is real analytic for $\|x\| < \rho$, provided ρ is sufficiently small (as we only need to keep the square root away from 0, which it will be as the square root in v^* tends to $\sqrt{r^2}$ as $\|x\| \to 0$).

Thus, u^* is given by a convergent series, and hence $u = \sum_\alpha u_\alpha x^\alpha$ converges on some neighbourhood of 0 by Lemma 2.1 (as $u^* \gg u$).

So now we have convergence of our a priori estimate of the solution to the original problem. So how do we know it converges to a solution of the original problem? Well, the Taylor series of u_{x_n} and of the function: $\sum_{j=1}^{n-1} B_j(u, x')u_{x_j} + C(u, x')$ both converge near 0, and have equal Taylor series, by construction. So hence they must agree (by the identity principle) on a neighbourhood of 0, i.e. we have $u_{x_n} = \sum_{j=1}^{n-1} B_j(u, x')u_{x_j} + C(u, x')$ locally.

[i.e. analytic functions about a point are determined by their convergent Taylor series. So if the Taylor series agree, then so must the functions.]

Thus this shows that u is a solution locally, and so we have shown that a real analytic solution of (2.3) exists, and is unique since any other solution will have the same Taylor series about 0 (from how we determined the coefficients u_α, i.e. they had to be these). So hence done.

$\square$

So we can solve PDE's of this form! Now we give an example to demonstrate the process of the proof, to hopefully make what is going on more transparent.

Example 2.2. *Consider the PDE system:*

$$\begin{cases} u_y = v_x - f \\ v_y = -u_x \end{cases}$$

with $u = v = 0$ on $\{y = 0\}$ (i.e. the x-axis, which is a codimension 1 surface).

So hence we have $u(x,0) = v(x,0) = 0$, and so differentiating with respect to x, we have:

$$(\partial_x)^n u(x,0) = 0 = (\partial_x)^n v(x,0) \quad \textit{for all } n,$$

i.e. all derivatives of u, v with no y-derivatives are 0 on $y = 0$.

Then from the original equations, we have:

$$u_y(x,0) = -f(x,0), \quad \textit{and} \quad v_y(x,0) = 0.$$

So we can differentiate these expressions as much as we like (assuming that $f \in C^\infty$) in the x-direction to find:

$$(\partial_x)^n \partial_y u(x,0) = -(\partial_x)^n f(x,0), \quad \textit{and} \quad (\partial_x)^n \partial_y v(x,0) = 0$$

for all n. So we have found all derivatives with one ∂_y derivative. So to find all derivatives including ∂_y^2, from our original equation we have:

$$u_{yy} = v_{xy} - f_y \quad \textit{and} \quad v_{yy} = -u_{xy}$$

and so hence using the above, we have:

$$u_{yy}(x,0) = -f_y(x,0) \quad \textit{and} \quad v_{yy}(x,0) = f_x(x,0)$$

i.e.

$$\partial_x^n \partial_y^2 u(x,0) = -\partial_x^n \partial_y f(x,0) \quad \textit{and} \quad \partial_x^n \partial_y^2 v(x,0) = \partial_x^{n+1} f(x,0).$$

So we have the expressions for the ∂_y^2-derivatives, in terms of just f which we know. So in principle by iterating this process, we can see how we could determine all of these derivatives by induction just in terms of f and its derivatives, which we know.

This is the idea the proof of Cauchy-Kovalevskaya uses.

□

Now we give examples of how to reduce some PDE's into ones of the form we had in Cauchy-Kovalevskaya, so that we can apply the Cauchy-Kovalevskaya Theorem to them to show that solutions exist.

2.2. Reduction to a 1st Order System.

Example 2.3. *Consider the problem:*

$$\begin{cases} u_{tt} = uu_{xy} - u_{xx} + u_t, & \textit{with} \\ u|_{t=0} = u_0, \quad u_t|_{t=0} = u_1 \end{cases}$$

where we assume that u_0, u_1 are real analytic at 0 (if not, then we would not expect to find a real analytic solution, and so we would not use Cauchy-Kovalevskaya).

Note that if u_0, u_1 are real analytic at 0, then so is $f = u_0 + tu_1$, and $f|_{t=0} = u_0$ and $f_t = u_1$. So set: $w = u - f$. Then our equation becomes:

$$w_{tt} = ww_{xy} - w_{xx} + w_t + fw_{xy} + f_{xy}w + F$$

where:

$$w|_{t=0} = 0, \quad \textit{and} \quad w_t|_{t=0} = 0$$

(i.e. trivial boundary conditions). Here, $F = ff_{xy} - f_{xx} + f_t$ (i.e. stuff that does not involve w or its derivatives, and so we do not care about it other than the fact it is real analytic).

So let $(x,y,t)=(x^1,x^2,x^3)$*, and let:*

$$u=(w,w_x,w_y,w_t)=:(u^1,u^2,u^3,u^4)$$

(i.e. vector of all derivatives except highest order ones). Then we have:

$$\begin{aligned} u^1_t &= w_t = u^4 \\ u^2_t &= w_{xt} = u^4_{x_1} \\ u^3_t &= w_{yt} = u^4_{x_2} \\ u^4_t &= w_{tt} = u^1 u^2_{x_2} - u^2_{x_1} + u^4 + f u^2_{x_2} + f_{xy} u^1 + F \end{aligned}$$

which is in the Cauchy-Kovalevskaya form: i.e. of a first order system.

Exercise: *Find the* B_j*'s and* C *as in the Cauchy-Kovalevskaya theorem for the above system, and thus show that we can apply the Cauchy-Kovalevskaya theorem to show that a (unique) solution to this does exist.*

□

We now look more at how to reduce PDE's into the system of (2.3), i.e.

$$\begin{cases} u_{x_n} = \sum_{j=1}^{n-1} B_j(u,x')u_{x_j} + C(u,x') & \text{in } \|x'\|^2 + (x^n)^2 < r^2 \\ u = 0 & \text{on } \|x'\| < r,\ x^n = 0. \end{cases} \tag{2.3}$$

Note that the RHS of (2.3) depends on x', and not on x^n. We can reduce this case (i.e. where it does not depend on x^n) by introducing (as before) a new variable, u^{m+1}, satisfying $(u^{m+1})_{x_n} = 1$, and $u^{m+1}|_{x_n=0} = 0$.

Let us consider the scalar, quasilinear problem:

$$\sum_{\alpha:|\alpha|=k} a_\alpha(D^{k-1}u,\dots,Du,u,x)D^\alpha u + a_0(D^{k-1}u,\dots,Du,u,x) = 0, \tag{2.8}$$

where:

$$u:\underbrace{B_r(0)}_{\subset\mathbb{R}^n}\to\mathbb{R}, \quad \text{and} \quad u = \frac{\partial u}{\partial x_n} = \cdots \left(\frac{\partial}{\partial x_n}\right)^{k-1} u = 0. \tag{2.9}$$

We will seek a solution of (2.8) in $B_r(0)$ (the open ball), subject to (2.9) holding on $\{\|x'\| < r,\ x_n = 0\}$. We will turn this into a system of the form of (2.3).

Introduce the vector:

$$u = (u^1,\dots,u^m) := \left(u, \frac{\partial u}{\partial x_1},\dots,\frac{\partial u}{\partial x_n},\frac{\partial^2 u}{\partial x_1^2},\dots,\frac{\partial^2 u}{\partial x_1\partial x_n},\dots,\frac{\partial^2 u}{\partial x_n^2},\dots,\left(\frac{\partial}{\partial x_n}\right)^{k-1}u\right)$$

i.e. $u = \left(\frac{\partial^i u}{\partial x^\alpha}\right)_{0\le i\le k-1,\ \alpha\in\mathbb{N}^i}$, is a vector which contains all partial derivatives up to order $k-1$.

We need to find $\frac{\partial u^j}{\partial x_n}$ for all j. For $j\in\{1,\dots,m-1\}$, we can compute $\frac{\partial u^j}{\partial x_n}$ in terms of either u^l or $\frac{\partial u^l}{\partial x^p}$ for some $l\in\{1,\dots,m\}$ and $p<n$.

[This is because $\frac{\partial u^j}{\partial x_n}$ can have at most k derivatives - say there are N. Then if $N < k$, this will just be some u^l, as u contains all derivatives of order N. If $N = k$, then we can obtai this by taking one of the $(k-1)$'th order derivatives in u and differentiating it with respect to some x_i, $i < n$, to get it.]

So now we just need to compute $\frac{\partial u^m}{\partial x_n} \equiv \frac{\partial^k u}{\partial x_n^k}$ in terms of these: to do this, we use Equation (2.8). Suppose that $a_{(0,\ldots,0,k)}(0,\ldots,0) \neq 0$ (i.e. multi-index subscript, for this coefficient in Equation (2.8)). Then we can re-write (2.8) as:

$$\frac{\partial^k u}{\partial x_n^k} = -\frac{1}{a_{(0,\ldots,0,k)}(D^{k-1}u,\ldots,u,x)}\left[\sum_{\alpha:|\alpha|=k,\ \alpha_n<k} a_\alpha \cdot D^\alpha u + a_0\right],$$

where, at least near $x = 0$, the denominator doesn't vanish (since it is non-zero at 0, and is continuous).

Note that by the above, we know that the RHS of this can be written in terms of u^l and $\frac{\partial u^l}{\partial x^p}$ for $p < n$.

So hence this shows that we have cast (2.8) into the form of a first order system of the Cauchy-Kovalevskaya type. So, (2.9) implies that $u = 0$ at $x_n = 0$ (i.e. this vector of all partial derivatives vanishes at $x = 0$). So provided the a_α's and a_0 are real analytic about the origin, and provided that $\boxed{a_{(0,\ldots,0,k)}(0,\ldots,0) \neq 0}$, we can solve (2.8) subject to (2.9), by the Cauchy-Kovalevskaya theorem.

The boxed condition above is important. It motivates the definition (which is the reduced version of a more general notion we will see briefly):

Definition 2.3. *If $a_{(0,\ldots,0,k)}(0,\ldots,0) \neq 0$, then we say that the plane $\{x_n = 0\}$ (i.e. where the data is specified) is **non-characteristic**.*

*Otherwise, it is **characteristic**. [i.e. not zero is not characteristic, which is good ('two wrongs make a right'). Characteristic surfaces are 'bad'.]*

Now we look into more exotic boundary conditions, and not those specified on a plane, but instead on a surface (e.g. a sphere).

2.3. **Exotic Boundary Conditions.**

It turns out that the surface we want to specify our data on cannot just be any surface/manifold, but that the surface needs to be itself *real analytic*. We define this notion now.

Definition 2.4. *We say that $\Sigma \subset \mathbb{R}^n$ is a **real analytic hypersurface, near $x \in \Sigma$**, if $\exists \varepsilon > 0$ and a real analytic map $\Phi : B_\varepsilon(x) \to U \subset \mathbb{R}^n$ (note that $B_\varepsilon(x)$ won't entirely lie in Σ), where $U = \Phi(B_\varepsilon(x))$, such that:*

(i) Φ is a bijection

(ii) $\Phi^{-1} : U \to B_\varepsilon(x)$ is real analytic

(iii) $\Phi(\Sigma \cap B_\varepsilon(x)) = \{x_n = 0\} \cap U$, *and* $\Phi(x) = 0$ *(wlog),*

i.e. we need a real analytic coordinate chart mapping Σ *locally to a hyperplane.*

So Φ in some sense straightens out Σ about x into a plane.

Example 2.4. *Spheres, planes, tori, etc, are real analytic surfaces.*

We work on such a real analytic surface. Let γ be the unit normal to Σ, and suppose that u solves the scalar, quasilinear problem:

$$\begin{cases} \sum_{\alpha:|\alpha|=k} a_\alpha(D^{k-1}u,\dots,u,x)D^\alpha u + a_0(D^{k-1}u,\dots,u,x) = 0 \\ u = \gamma^i\partial_i u = \cdots = (\gamma^i\partial_i)^{k-1}u = 0 \quad \text{on } \Sigma \end{cases}$$

i.e. all $(k-1)$ normal derivatives are 0 (this is saying that the solution is not evolving out of the surface to these orders. So the k'th order is the first order at which the solution evolves out of the surface).

Note that we have used summation convention, i.e. $\gamma^i\partial_i u = \sum_{i=1}^n \gamma^i\partial_i u = \gamma\cdot\nabla u$: so this is just simply the derivative of u in the γ-direction, i.e. normal to Σ.

Then define:

$$w(y) := u(\Phi^{-1}(y)), \quad \text{i.e.} \quad u(x) = w(\Phi(x))$$

where Φ is the transformation given in Definition 2.4, i.e. w is the problem on the plane (locally). So $w(y) = u(x)$, $\Phi(x) = y$.

Then note that:

$$\frac{\partial u}{\partial x_i} = \frac{\partial w}{\partial y_j}\cdot\frac{\partial \Phi^{(j)}}{\partial x_i} \quad \text{(sum over } j\text{)}$$

by the chain rule, and so plugging this into our equation for u, we find that w satisfies:

$$\begin{cases} \sum_{\alpha:|\alpha|=k} b_\alpha D^\alpha w + b_0 = 0 & \text{for some } b_0, b_\alpha, \text{ with:} \\ w = \frac{\partial w}{\partial y} = \cdots = \left(\frac{\partial}{\partial y}\right)^{k-1} w = 0 & \text{on } \{x_n = 0\}. \end{cases} \tag{2.10}$$

Note that we do this to reduce the problem to the one we considered before, on the plane. So hence we know (as Φ is real analytic and so all of the coefficients b_0, b_α will be) that we just need to check whether $b_{(0,\dots,0,k)}$ is 0 or not to determine whether we can apply Cauchy-Kovalevskaya to find a solution for w, and hence a solution u of the original problem.

Note that if $|\alpha| = k$, then:

$$D^\alpha u = \frac{\partial^k w}{\partial y_n^k}(D\Phi^{(n)})^\alpha + \left(\text{terms not involving } \frac{\partial^k w}{\partial y_n^k}\right)$$

where $\Phi^{(n)}$ is the n'th coordinates of Φ, and so hence we see that the coefficient of $\frac{\partial^k w}{\partial y_n^k}$ in (2.10) is:

$$b_{(0,\dots,0,k)} = \sum_{|\alpha|=k} a_\alpha (D\Phi^{(n)})^\alpha.$$

Hence this leads us to define, from our previous discussion of the planar case:

Definition 2.5. *We say that the real-analytic surface Σ is* ***non-characteristic at $x \in \Sigma$*** *(for <u>this</u> PDE), if:*

$$b_{(0,\dots,0,k)} = \sum_{\alpha:|\alpha|=k} a_\alpha (D\Phi^{(n)})^\alpha \neq 0.$$

Otherwise, it is ***characteristic****.*

Note: The surface being non-characteristic is equivalent to:

$$\sum_{|\alpha|=k} a_\alpha \gamma^\alpha \neq 0,$$

as in the transform, we know that $\Sigma = \{x : \Phi^{(n)}(x) = 0\}$ is just the zero set of the n-th coordinate of Φ (from definition 2.4 (iii)), and so this is just the normal.

[Note that here, $a_\alpha = a_\alpha(0,\dots,0,x)$, as all derivatives are zero here, from (2.9).]

So we have proven the following improved version of Cauchy-Kovalevskaya, where the data can be specified on a real-analytic characteristic surface, instead of a plane.

Theorem 2.4 (Cauchy-Kovalevskaya for Data on Non-Characteristic Surfaces)**.** *Suppose $\Sigma \subset \mathbb{R}^n$ is a real analytic hypersurface, with normal γ. Then consider the Cauchy problem for the sesquilinear system:*

$$\begin{cases} \sum_{\alpha:|\alpha|=k} a_\alpha(D^{k-1}u,\dots,u,x)D^\alpha u + a_0(D^{k-1}u,\dots,u,x) = 0 & \textit{subject to:} \\ u = \frac{\partial u}{\partial \gamma} = \cdots = \left(\frac{\partial}{\partial \gamma}\right)^{k-1} u = 0 & \textit{on } \Sigma \end{cases}$$

where γ is the unit normal to Σ. Suppose that a_α, a_0 are real analytic at $x \in \Sigma$, and that Σ is non-characteristic at x.

Then the Cauchy problem admits a unique real analytic solution in a neighbourhood of x.

Proof. As above. □

Now we look more into characteristic surfaces.

2.4. **Characteristic Surfaces.**

Consider the operator:

$$L = \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j}$$

where wlog $a_{ij} = a_{ji}$ (as antisymmetric parts cancel in the sum), and $a_{ij} \in \mathbb{R}$. Then consider the PDE:

$$\begin{cases} Lu = f \\ \text{subject to:} \quad u = \gamma^i \frac{\partial u}{\partial x^i} = 0 \quad \text{on } \Pi_\gamma := \{x : x \cdot \gamma = 0\} \end{cases}$$

i.e. with boundary condition on the plane with normal γ, where $|\gamma| = 1$ (here we sum over i).

Then as above, we know that Π_γ is non-characteristic if: $\sum_{i,j=1}^{n} a_{ij}\gamma^i\gamma^j \neq 0$. So to try and find the characteristic surfaces, we can try to solve:

$$\sum_{i,j=1}^{n} a_{ij}\gamma^i\gamma^j = 0 \quad \text{subject to} \quad |\gamma| = 1.$$

To see when this holds, we diagonalise the matrix $A = (a_{ij})_{ij}$, so that if the eigenvalues of A are $\{\lambda_i\}_i$, this becomes (in relevant coordinates):

$$\sum_i \lambda_i v_i^2 = 0.$$

So hence if <u>all</u> eigenvalues are > 0, (or all < 0), then we see that $\sum_{i,j} a_{ij}\gamma^i\gamma^j \neq 0 \ \forall \gamma$, and so the problem has no characteristic surface (i.e. all surfaces are non-characteristic, which is good). We call such an operator an **elliptic operator**.

If $(a_{ij})_{ij}$ has <u>one</u> negative eigenvalue and the rest are > 0, then we say that L is **elliptic**.

Example 2.5. *We have:*

(i) $L = \Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ *is elliptic.*

(ii) $L = -\partial_t^2 + \Delta$ *is hyperbolic.*

If we consider the simpler problem of just $Lu = 0$ (forgetting about the Cauchy data for the moment), then we can look for solutions of the form: $e^{ik\cdot x}$, i.e. wave-like solutions. Indeed, we have:

$$L(e^{ik\cdot x}) = -e^{ik\cdot x} \sum_{i,j=1}^{n} a_{ij}k^i k^j$$

i.e. for this to be zero, we need k to be parallel to the normal of a characteristic surface. So in particular, if L is elliptic, then we need $k = 0$ here (as there are no characteristic surfaces). So this shows us that elliptic operators of this form do not admit wave-like solutions.

So if L is hyperbolic, we can have non-trivial wave-like solutions provided that $k \propto \gamma$, for some γ satisfying $|\gamma| = 1$ and $\sum_{i,j} a_{ij}\gamma^i\gamma^j = 0$.

So if we set: $u_\lambda(x) = e^{i(\lambda\gamma)\cdot x}$ (i.e. parallel to this γ, $k = \lambda\gamma$) for such a γ, then by taking λ large, we can arrange for u_λ to have large deviations in the γ (characteristic) direction.

By contrast, we will see later that if L is elliptic and u satisfies $Lu = 0$, then we have $u \in C^\infty$, i.e. we get control on the growth rate of u. Hence we see that despite these operators being very similar in form, their solutions can have very different behaviours.

We finish this section discussing the limitations of Cauchy-Kovalevskaya. Despite its power, it has several limitations, including:

(i) We have no control over "how long" a solution exists for (as it only guarantees existence locally on some small time interval).

(ii) It does not guarantee well-posedness: whilst it does guarantee existence and uniqueness, it does not guarantee continuous dependence on the data.

Example 2.6. *Consider the problem:* $u_{xx} + u_{yy} = 0$ *(i.e.* $\Delta u = 0$ *on* $\mathbb{R}^2$*). Then this admits a solution:*

$$u(x,y) = \cos(kx)\cosh(ky)$$

which we can think of as coming from the Cauchy problem:

$$u(x,0) = \cos(kx), \quad \text{and} \quad u_y(x,0) = 0.$$

Note that this is real analytic, and so must be the solution given by Cauchy-Kovalevskaya (by the uniqueness in Cauchy-Kovalevskaya).

Then note that as $k \to \infty$*, we have* $\sup_{x\in\mathbb{R}} |u(x,0)| < 1$ *(i.e. the data is bounded uniformly), but:*

$$\sup_{x\in\mathbb{R}} |u(x,\varepsilon)| = \infty$$

for any $\varepsilon > 0$ *(as* $u(0,\varepsilon) \to \infty$ $\forall \varepsilon > 0$*). So we cannot control the size of the solution in terms of the size of the data [see Example Sheet 1 for more details] and so this problem is note well-posed. Hence Cauchy-Kovalevskaya cannot guarantee that a problem is well-posed.*

□

(iii) Real analyticity is a very strong assumption on the solution - we may only care about a C^2 solution, for example, so Cauchy-Kovalevskaya may not work whilst something else will.

Plus, in principle, from the equations governing electromagnetic fields, Cauchy-Kovalevskaya says that we could construct an EM field everywhere in the universe (i.e. the solution) just from measuring values in a small ball (i.e. the boundary conditions), which is absurd.

So hence there is a need for more general theory, which we look into now.

3. Function Space Theory

We want to study elliptic boundary value problems.

Suppose $U \subset \mathbb{R}^n$ is a bounded open set with smooth boundary. Suppose (for physics reasons) that ∂U is a perfect conductor, and $\rho : U \to \mathbb{R}$ is the charge density of U. Then the electrostatic field φ is known to satisfy:

$$\begin{cases} \Delta\varphi = \rho & \text{in } U \\ \varphi|_{\partial U} = 0 & \text{(as a perfect conductor).} \end{cases}$$

Then Cauchy-Kovalevskaya is of no use to us here, as we do not have enough boundary data (as we would require knowledge of $\nabla\varphi|_{\partial U}$ as well).

This is an example of an **elliptic boundary value problem** (i.e. an elliptic operator, like the Laplacian, and the initial data specified on a boundary).

Before we attack these problems, we need to introduce some function space theory.

3.1. Hölder Spaces $C^{k,\gamma}(\overline{U})$.

Hölder spaces are spaces which are based on classical continuity and differentiability of functions.

Suppose $U \subset \mathbb{R}^n$ is open.

Definition 3.1. *We write, for each $k \in \mathbb{N}$:*
$\boldsymbol{C^k(U)} := \{u : U \to \mathbb{R} \mid u$ *is k-times differentiable, and* $D^\alpha u : U \to \mathbb{R}^{n|\alpha|}$ *is continuous on U for* $|\alpha| \leq k\}$
i.e. the set of continuous functions of U with continuous partial derivatives up to order k.

However this is not a Banach space[iv]: this condition does not stop u from blowing up on ∂U. So we need to define a stronger version.

So we write:

$$\boldsymbol{C^k(\overline{U})} := \{u \in C^k(U) : D^\alpha u \text{ are bounded and uniformly continuous on } U, \text{ for } |\alpha| \leq k\}$$

i.e. this is just the set of functions in $C^k(U)$ whose derivatives of order $\leq k$ have continuous extensions to all of $\overline{U}$.

Note: $C^k(\overline{U})$ is not just C^k defined as above on the closure of U: the definition of C^k is for open sets U. Then definition of $C^k(\overline{U})$ still only works with functions on U, not ones on $\overline{U}$.

[If U is bounded, then they might turn out to be the same. However if U is unbounded then they won't necessarily be.]

[iv] It is however, a Fréchet space.

Then we define:

$$\|u\|_{C^k(\overline{U})} := \sum_{|\alpha|\leq k} \sup_{x\in U} |D^\alpha u(x)|.$$

Then the space $(C^k(\overline{U}), \|\cdot\|_{C^k(\overline{U})})$ is a Banach space [**Exercise** to check].

Definition 3.2. *We say that $u : Y \to \mathbb{R}$ is* ***Hölder continuous with index*** γ *(or* ***γ-Hölder continuous****) if:*

$$|u(x)-u(y)| \leq C|x-y|^\alpha \quad \forall x, y \in U.$$

Note: If $\gamma \in (0,1]$, then this is a weaker condition than Lipschitz. If $\gamma > 1$, then we get $u' = 0$ and so $u = $ constant.

Definition 3.3. *For $\gamma \in (0,1]$, we define the* ***0-Hölder space*** *by:*

$$C^{0,\gamma}(\overline{U}) := \{u \in C^0(\overline{U}) : u \text{ is } \gamma\text{-Hölder continuous}\}.$$

We define the **γ-Hölder seminorm** by:

$$[u]_{C^{0,\gamma}(\overline{U})} := \sup_{x,y\in U} \frac{|u(x)-u(y)|}{|x-y|^\gamma}$$

i.e. the smallest C such that the Hölder continuity condition holds.

However this is only a seminorm, as constant functions vanish under it. So to make it a norm, we can add the C^0-norm to it. So set:

$$\|u\|_{C^{0,\gamma}(\overline{U})} := \|u\|_{C^0(\overline{U})} + [u]_{C^{0,\gamma}(\overline{U})}.$$

This then makes $(C^{0,\gamma}(\overline{U}), \|\cdot\|_{C^{0,\gamma}(\overline{U})})$ a normed space (which is a Banach space) [**Exercise** to check].

Then in the obvious way, we can extend this to higher order:

Definition 3.4. *For $\gamma \in (0,1]$, the* ***k'th-Hölder space*** *is:*

$$\boldsymbol{C^{k,\gamma}(\overline{U})} := \{u \in C^k(\overline{U}) : D^\alpha u \in C^{0,\gamma}(\overline{U}) \quad \forall |\alpha| = k\}.$$

One should think of $C^{k,\gamma}(\overline{U})$ as filling in the gaps between $C^k(\overline{U})$ and $C^{k+1}(\overline{U})$, and so $u \in C^{k,\gamma}(\overline{U})$ means in some sense that: "u is $(k+\gamma)$-times differentiable".

We can then define a norm on $C^{k,\gamma}(\overline{U})$ by:

$$\|u\|_{C^{k,\gamma}(\overline{U})} = \|u\|_{C^k(\overline{U})} + \sum_{|\alpha|=k} [D^\alpha u]_{C^{0,\gamma}(\overline{U})}.$$

Then, $\left(C^{k,\gamma}(\overline{U}), \|\cdot\|_{C^{k,\gamma}(\overline{U})}\right)$ is a Banach space [**Exercise** to check - use Arzela-Ascoli].

Note: $C^{0,1}(\overline{U})$ is the set of (uniformly bounded) Lipschitz functions on U.

3.2. The Lebesgue Spaces $L^p(U)$, $L^p_{\text{loc}}(U)$.

The *Sobolev spaces* are defined in terms of the L_p spaces, and not the Hölder spaces $C^{k,\gamma}$. So we quickly recall L^p.

Let $U \subset \mathbb{R}^n$ be open, and suppose $p \in [1, \infty]$.

Definition 3.5. *We define the* ***Lebesgue spaces*** $\boldsymbol{L^p(U)}$ *by:*

$$L^p(U) := \{u : U \to \mathbb{R} \mid u \text{ is measurable, and } \|u\|_{L^p(U)} < \infty\}/\sim$$

where we quotient out by the equivalence relation: $u_1 \sim u_2$ *if* $u_1 = u_2$ *a.e.*

Here, we have the $\boldsymbol{L^p}$ **norms** given by:

$$\|u\|_{L_p(U)} := \begin{cases} \left(\int_U |u(x)|^p \, dx\right)^{1/p} & \text{if } p \in [1,\infty) \\ \text{ess.sup}|u(x)| & \text{if } p = \infty. \end{cases}$$

Then we know that $L^p(U)$ is a Banach space when endowed with the L^p norm. [**Exercise** to check - follows by the Dominated Convergence Theorem].

These are also known as the **global** $\boldsymbol{L_p(U)}$ **spaces**. We can also define local versions.

Definition 3.6. *The* ***local*** $\boldsymbol{L^p}$ ***space*** *is:*

$$L^p_{loc}(U) := \{u : u \in L^p(V) \ \ \forall V \subset\subset U\} = \bigcap_{V : V \subset\subset U} L^p(V).$$

Here, $V \subset\subset U$ means that V is **compactly contained in** U, i.e. $\exists K$ compact such that $V \subset K \subset U$.

Note that if $K \subset U$ is compact, with U open, then:

$$d(K, \partial U) := \inf\{|x-y| : x \in K, \ y \in U^c\} > 0.$$

Note also that $L^p_{\text{loc}}(U)$ is <u>not</u> a Banach space (indeed, it is not even a normed space), but it is a Fréchet space.

3.3. Weak Derivatives.

We define a generalised notion of differentiability for L^p functions through their integrals: this is called a *weak derivative*.

Definition 3.7. *Suppose $u, v \in L^1_{loc}(U)$, and α is a multi-index. Then we say that v is the* ***α'th weak derivative of u*** *if:*

$$\forall \varphi \in C^\infty_c(U), \qquad \int_U u D^\alpha \varphi \, \mathrm{d}x = (-1)^{|\alpha|} \int_U v\varphi \, \mathrm{d}x.$$

i.e. u, v obey the correct integration by parts formula when tested against any **test function** $\varphi \in C^\infty_c(U)$ - so the 'boundary terms' vanish due to the compact support in U, and we can differentiate the test function. as much as we like.

Note: This is the same as the distributional derivative of u (i.e. can get Dirac delta), except here we require that $v \in L^1_{\text{loc}}(U)$ as well.

We also note that v (the weak derivative), when it exists, is unique. The following lemma verifies this.

Lemma 3.1. *Suppose $v, \tilde{v} \in L^1_{loc}(U)$ are both α'th weak derivatives of $u \in L^1_{loc}(U)$. Then, $v = \tilde{v}$ a.e..*

Proof. $\forall \varphi \in C^\infty_c(U)$ we have:

$$\int_U v\varphi \, \mathrm{d}x = (-1)^{|\alpha|} \int_U u D^\alpha \varphi \, \mathrm{d}x = \int_U \tilde{v}\varphi \, \mathrm{d}x$$

i.e.

$$\int_U (v - \tilde{v})\varphi \, \mathrm{d}x = 0 \quad \forall \varphi \in C^\infty_c(U)$$

$\Rightarrow v - \tilde{v} = 0$ a.e. [where this last implication is seen by taking φ to be approximations to the unit, so as these integrals become convolutions, and then using the limiting properties of approximations to the unit].

□

So hence if u is smooth, then the weak derivative will just be the usual derivative of u (up to a.e. equality). In general, we write $v = D^\alpha u$ for the α'th weak derivative.

Definition 3.8. *We say that $u \in L^1_{loc}(U)$ belongs to the* ***Sobolev space $W^{k,p}(U)$*** *if $u \in L^p(U)$ and the weak derivatives $D^\alpha u$ exist and belong to $L^p(U)$ for all $|\alpha| \leq k$.*

i.e.

$$W^{k,p}(U) = \{u \in L^1_{loc}(U) : D^\alpha u \text{ exist and are in } L^p(U) \;\; \forall 0 \leq |\alpha| \leq k\}.$$

If $p = 2$, then we write:

$$H^k(U) := W^{k,2}(U)$$

where we use the letter 'H' because this turns out to be a Hilbert space.

Note: We <u>require</u> that $u \in L^1_{\text{loc}}$ for Sobolev functions or else we cannot make sense of the integrals in weak derivatives.

We define the **Sobolev norm** on $W^{k,p}(U)$ by:

$$\|u\|_{W^{k,p}(U)} := \begin{cases} \left(\sum_{|\alpha|\leq k} \int_U |D^\alpha u|^p \, \mathrm{d}x\right)^{1/p} & \text{if } p \in [1,\infty) \\ \sum_{|\alpha|\leq k} \|D^\alpha u\|_{L^\infty(U)} & \text{if } p = \infty \end{cases}$$

i.e. the finite L^p-norm of the vector: $\left(\|u\|_{L^p}, \|D^1 u\|_{L^p}, \ldots, \|D^\alpha u\|_{L^p}\right)$ of all relevant derivatives.

Definition 3.9. *We denote by* $\boldsymbol{W_0^{k,p}(U)}$ *the completion of* $C_c^\infty(U)$ *in the above Sobolev norm, i.e.*

$$W_0^{k,p}(U) := \overline{C_c^\infty(U)}^{\|\cdot\|_{W^{k,p}(U)}}.$$

We call $C_c^\infty(U)$ the set of **test functions** on U. We also define:

$$H_0^k := W_0^{k,2}(U).$$

[The point of these Sobolev spaces is that they are the completion of smooth functions in some sense. So we can extract limits in this larger function space, which we can then work with to try and show that they are, e.g. smooth.]

Example 3.1. *Let $U = B_1(0)$ be the open unit ball in $\mathbb{R}^n$. Then set:*

$$u(x) = \begin{cases} |x|^{-\alpha} & \text{for } x \in U\backslash\{0\} \\ \text{anything} & \text{for } x = 0. \end{cases}$$

[Note that the value of u at $x = 0$ is relevant in L^p as this is a set of measure zero.]

Then for $x \neq 0$, the classical derivative is: $D_i u = -\frac{\alpha x_i}{|x|^{\alpha+2}}$. Then by considering $\varphi \in C_c^\infty(\underbrace{B_1(0)\backslash\{0\}}_{\text{open}})$, it is clear that, if u is weakly differentiable, due to the uniqueness of the weak derivative, then the weak derivative of u must be: $D_i u = -\frac{\alpha x_i}{|x|^{\alpha+2}}$.

So we can check that: $u \in L^1_{loc}(U) \iff \alpha < n$, and $\frac{x_i}{|x|^{\alpha+2}} \in L^1_{loc}(U) \iff \alpha < n-1$.

So hence if we assume $\alpha < n-1$, then for $\varphi \in C_c^\infty(U)$, we have (by integrating by parts):

$$-\int_{U\backslash B_\varepsilon(0)} u \cdot \varphi_{x_i} \, \mathrm{d}x = \int_{U\backslash B_\varepsilon(0)} (D_i u)\varphi \, \mathrm{d}x - \int_{\partial B_\varepsilon} u\varphi \nu^i \, \mathrm{d}S_i,$$

where $\nu = (\nu^1, \dots, \nu^n)$ is the outward normal [since the other boundary term at ∂U vanishes, as $\varphi = 0$ there]. Then we can estimate:

$$\left| \int_{\partial B_\varepsilon(0)} u\varphi \nu^i \, \mathrm{d}S_i \right| \leq \|\varphi\|_{L^\infty} \cdot \overbrace{\varepsilon^{-\alpha}}^{\text{from } u} \cdot \underbrace{C \cdot \varepsilon^{n-1}}_{=vol(B_\varepsilon(0))} \leq \tilde{C}\varepsilon^{n-1-\alpha} \to 0$$

as $\varepsilon \downarrow 0$. So hence if $\alpha < n-1$, we see that u is weakly differentiable (i.e. obeys integration condition and is in the appropriate L^p space), and we have $D_i u = -\frac{\alpha x_i}{|x|^{\alpha+2}}$. □

So this example demonstrates that we can have weakly differentiable functions, even if they are not continuous (as sets of measure 0 do not matter).

Note: In the above example, we have $D_i u \in L^p(U) \iff p(\alpha+1) < n$, i.e. $\alpha < \frac{n-p}{p}$. So if $\alpha < \frac{n-p}{p} = n/p - 1$, then we have $u \in W^{1,p}(U)$, and if $\alpha > n/p$, then $u \notin W^{1,p}(U)$, i.e. we want $\alpha < \frac{n-p}{p}$.

So notice that if $p > n$, then this $\Rightarrow \alpha < 0$, and so u is in fact continuous on $B_1(0)$. We will see more examples of this behaviour (i.e. if the dimension is large or small) later → see "Sobolev Embeddings".

Theorem 3.1. *For each $k \in \{0, 1, \dots\}$ and for each $p \in [1, \infty]$, the Sobolev space $(W^{k,p}(U), \|\cdot\|_{W^{k,p}(U)})$ is a Banach space.*

Proof. We need to show that this is actually a normed space, and then that it is complete

Normed: Homogeneity and positivity of $\|\cdot\|_{W^{k,p}(U)}$ are obvious. The triangle inequality follows from Minkowski's inequality:

$$\left(\sum_{i=1}^m |a_i + b_i|^p\right)^{1/p} \leq \left(\sum_{i=1}^m |a_i|^p\right)^{1/p} + \left(\sum_{i=1}^m |b_i|^p\right)^{1/p}$$

and as: $\|u+v\|_{L^p(U)} \leq \|u\|_{L^p(U)} + \|v\|_{L^p(U)}$.

Completeness: Note that $\|D^\alpha u\|_{L^p(U)} \leq \|u\|_{W^{k,p}(U)}$ for all $|\alpha| \leq k$. So hence if $(u_j)_{j=1}^\infty$ is Cauchy in $W^{k,p}(U)$, then $(D^\alpha u_j)_{j=1}^\infty$ is Cauchy in $L^p(U)$ for all $|\alpha| \leq k$. So hence by completeness of $L^p(U)$, we get that $\exists u^\alpha \in L^p(U)$ such that

$$D^\alpha u_j \to u^\alpha \quad \text{in } L^p(U) \text{ for each } \alpha.$$

Then let $u = u^{(0,\dots,0)}$, i.e. the limit of the u_j in $L^p(U)$. Then we just need to show that the derivatives of the limits are just the limits of the derivatives, i.e. $u^\alpha = D^\alpha u$.

To see this, let $\varphi \in C_c^\infty(U)$. Then:

$$(-1)^{|\alpha|} \int_U u_j D^\alpha \varphi \, \mathrm{d}x = \int_U (D^\alpha u_j)\varphi \, \mathrm{d}x$$

for all j, by definition of the weak derivative. So hence taking $j \to \infty$ in this, using that $D^\alpha u_j \to u^\alpha$ in $L^p(U)$, we have that (by Dominated Convergence)

$$(-1)^{|\alpha|} \int_U u D^\alpha \varphi \, \mathrm{d}x = \int_U u^\alpha \varphi \, \mathrm{d}x.$$

So hence by uniqueness of weak derivatives, this shows that $D^\alpha u = u^\alpha \in L^p(U)$, and so hence $u_j \to u$ in $W^{k,p}(U)$. So done.

□

So we have seen that elements in these Sobolev spaces can be "badly behaved" (i.e. blow up at points). For this reason, it is useful to be able to approximate them by nicer functions, which are easier to work with. We look into this now.

3.4. **Approximation of Functions in Sobolev Spaces.**

There is a lot of information in the handout "Approximation to the Identity" we were given. Be sure to consult it.

3.4.1. *Convolutions and Smoothing.*

A useful trick to improve regualrity of a function is to convolve it with a *smooth mollifier*.

Definition 3.10. *Let:*

$$\eta(x) := \begin{cases} C \cdot e^{-\frac{1}{1-|x|^2}} & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1. \end{cases}$$

where C is chosen so that $\int_{\mathbb{R}^n} \eta(x) \, \mathrm{d}x = 1$. Then for each $\varepsilon > 0$, set:

$$\eta_\varepsilon(x) := \frac{1}{\varepsilon^n} \eta\left(\frac{x}{\varepsilon}\right).$$

*Then we call η_ε the **standard mollifier**.*

Note: η, and so hence η_ε, is smooth on $\mathbb{R}^n$. η_ε has support contained in $B_\varepsilon(0)$, and $\int_{\mathbb{R}^n} \eta_\varepsilon(x) \, \mathrm{d}x = 1$ for all $\varepsilon > 0$.

So the key points are:

- $\eta_\varepsilon \in C_c^\infty(\mathbb{R}^n)$
- $\mathrm{supp}(\eta_\varepsilon) \subset \overline{B_\varepsilon(0)}$
- $\int_{\mathbb{R}^n} \eta_\varepsilon(x) \, \mathrm{d}x = 1$.

So suppose $U \subset \mathbb{R}^n$ is open. Then let

$$U_\varepsilon := \{x \in U : \mathrm{dist}(x, \partial U) > \varepsilon\}$$

i.e. the points in U further than ε from the boundary.

Definition 3.11. *If $f \in L^1_{loc}(U)$, then we define the* ***mollification of f****, denoted $f_\varepsilon : U_\varepsilon \to \mathbb{R}$, by:*

$$f_\varepsilon := \eta_\varepsilon * f$$

where the $*$ denotes the ***convolution****, i.e.*

$$f_\varepsilon(x) = \int_U \eta_\varepsilon(x-y)f(y)\,\mathrm{d}y = \int_{B_\varepsilon(x)} \eta_\varepsilon(x-y)f(y)\,\mathrm{d}y$$

where everything in the integral is defined for $x \in U_\varepsilon$.

So one can think of f_ε as the 'average of f', weighted by η_ε, on an ε-ball about each point.

The following theorem demonstrates why mollifications are so useful:

Theorem 3.2. *Mollifiers have the following properties:*

- *(i)* $f_\varepsilon \in C^\infty(U_\varepsilon)$ *is smooth.*
- *(ii)* $f_\varepsilon \to f$ *a.e. on U as $\varepsilon \downarrow 0$, i.e. we can approximate f by smooth functions.*
- *(iii) If $f \in C(U)$, then $f_\varepsilon \to f$ uniformly on compact subsets of U.*
- *(iv) If $p \in [1, \infty)$, and $f \in L^p_{loc}(U)$, then $f_\varepsilon \to f$ in $L^p_{loc}(U)$, i.e.*

$$\|f_\varepsilon - f\|_{L^p(V)} \to 0 \quad \forall V \subset\subset U.$$

Proof. See handout. These are immensely important properties though. To give the idea for (i), the derivatives of f_ε fall on the η_ε, which is smooth, and so we need no extra assumptions on f.

□

3.4.2. *Approximation of Sobolev Functions.*

So our first approximation result is the following:

Lemma 3.2 (Local Smooth Approximation of Sobolev functions away from the boundary, ∂U)**.** *Let $u \in W^{k,p}(U)$ for some $p \in [1, \infty)$. Then set $u_\varepsilon = \eta_\varepsilon * u$ in U_ε. Then:*

- *(i)* $u_\varepsilon \in C^\infty(U_\varepsilon)$ *is smooth for all $\varepsilon > 0$.*
- *(ii) If $V \subset\subset U$, then $u_\varepsilon \to u$ in $W^{k,p}(V)$ as $\varepsilon \downarrow 0$.*

Proof. (i): See handout/part (i) of Theorem 3.2.

(ii): This follows easily from Theorem 3.2 and the following claim:

Claim: $D^\alpha u_\varepsilon = \eta_\varepsilon * D^\alpha u$ in U_ε, $\forall |\alpha| \leq k$, i.e. we have

$$D^\alpha(\eta_\varepsilon * u) = \eta_\varepsilon * D^\alpha u$$

Proof. Note that:

$$\begin{aligned}
D_x^\alpha u_\varepsilon(x) &= D_x^\alpha\left(\int_U \eta_\varepsilon(x-y)u(y)\,\mathrm{d}y\right)\\
&= \int_U (D_x^\alpha \eta_\varepsilon(x-y))u(y)\,\mathrm{d}y \quad \text{(see handout for justification)}\\
&= (-1)^{|\alpha|}\int_U (D_y^\alpha \eta_\varepsilon(x-y))u(y)\,\mathrm{d}y \quad \text{, as } \eta_\varepsilon \text{ is smooth}\\
&= \int_U \eta_\varepsilon(x-y)D^\alpha u(y)\,\mathrm{d}y \quad \text{, by definition of weak derivative}\\
&= \eta_\varepsilon * D^\alpha u(x).
\end{aligned}$$

So hence $D_\alpha u_\varepsilon = \eta_\varepsilon * D^\alpha u$ in U_ε, for all $|\alpha| \leq k$. □

So now fix $V \subset\subset U$. Then by Theorem 3.2 (iv), we see that $D^\alpha u_\varepsilon = \eta_\varepsilon * D^\alpha u \to D^\alpha u$ in $L^p(V)$, as $\varepsilon \downarrow 0$, for all $|\alpha| \leq k$. Then,

$$\|u_\varepsilon - u\|^p_{W^{k,p}(V)} = \sum_{|\alpha|\leq k} \|D^\alpha u_\varepsilon - D^\alpha u\|^p_{L^p(V)} \to 0$$

as $\varepsilon \downarrow 0$, and so done.

□

So this works in the interior of U. Again we need to be careful about the boundary. The next result starts helping with the global interior.

Theorem 3.3 (First Global Approximation Result: Smooth approximation globally away from ∂U)**.** *Suppose $U \subset \mathbb{R}^n$ is open and bounded. Suppose $u \in W^{k,p}(U)$ for some $k \in \mathbb{N}$ and $p \in [1,\infty)$. Then, $\exists (u_m)_m \in C^\infty(U) \cap W^{k,p}(U)$ such that*

$$u_m \to u \quad \text{in } W^{k,p}(U).$$

Note: It is not too hard to remove the boundedness assumption on U in this result. Note that we do not quite yet have $u_m \in C^\infty(U)$ as arbitrary. We need $u \in W^{k,p}(U)$ as well as this is not guaranteed, as the derivatives could blow up at the boundary (but this cannot happen in the boundary case).

Proof. We have $U = \bigcup_{i=1}^\infty U_i$, where

$$U_i = \{x \in U : \mathrm{dist}(x, \partial U) > 1/i\}.$$

Then let $V_i := U_{i+3}\backslash\overline{U}_{i+1}$, and choose $V_0 \subset\subset U$ such that $U = \bigcup_{i=0}^\infty V_i$ (i.e. V_0 fills in the rest of the interior).

Then let $\{\xi_i\}_i$ be a **partition of unity subordinate** to the open sets V_i, so that:

$$0 \leq \xi_i \leq 1, \quad \xi_i \in C_c^\infty(V_i), \quad \text{and} \quad \sum_{i=1}^\infty \xi_i = 1 \quad \text{on } U$$

(note that this sum is always a finite sum due to the compact support condition).

So suppose $u \in W^{k,p}(U)$. Then, we have $\xi_i u \in W^{k,p}(U)$ [need verification - **Exercise** to check], and we know $\text{supp}(\xi u) \subset V_i$ for all i.

Now fix $\delta > 0$. Then for each i, choose $\varepsilon_i > 0$ sufficiently small so that:

$$u^i := \eta_{\varepsilon_i} * (\xi_i u)$$

satisfies: $\text{supp}(u^i) \subset W_i$, where $W_i := U_{i+4} \backslash \overline{U}_i \supset\supset V_i$, and $\|u^i - \xi_i u\|_{W^{k,p}(U)} = \|u^i - \xi_i u\|_{W^{k,p}(W_i)} \leq \delta/2^{i+1}$ (we can do this because of the convergence $u^i \to \xi_i u$).

Now write:

$$v = \sum_{i=0}^\infty u^i.$$

Then we have $v \in C^\infty(V)$, as for each open set $V \subset\subset U$, the sum is a finite[v] sum of smooth functions.

[i.e. the idea is to take our function, and split it up on different regions, so it decays. Then we take a smooth mollifier on each region and smooth it out, and then sum again. This then gives our approximation.]

Then since $\sum_{i=0}^\infty \xi_i = 1$, we have:

$$u = u \cdot 1 = \sum_{i=0}^\infty \xi_i u$$

for each $V \subset\subset U$, and so we have:

$$\|v - u\|_{W^{k,p}(V)} \leq \sum_{i=0}^\infty \|u^i - \xi_i u\|_{W^{k,p}(U)} \leq \delta \cdot \sum_{i=0}^\infty 2^{-(i+1)} = \delta$$

where in the first inequality we have used the triangle inequality. Note that δ here does not depend on $V \subset\subset U$. Hence taking the supremum over all such V, we can conclude that $\|v - u\|_{W^{k,p}(U)} \leq \delta$, and so as $\delta > 0$ as arbitrary, we are done.

□

However if we want to get a better approximation result, we need more conditions on the boundary ∂U, since we could have $\partial U =$ "the Cantor set" or something equally nasty. So we define:

[v]This is why we need to work on the V_i and not the U_i, since if we worked with the U_i we would end up with an infinite sum.

Definition 3.12. *Suppose $U \subset \mathbb{R}^n$ is open and bounded. Then we say that ∂U is $\boldsymbol{C^{k,\delta}}$ if $\forall p \in \partial U$, $\exists r > 0$ and $\gamma \in C^{k,\delta}(\mathbb{R}^{n-1})$ such that we have (possibly after relabelling axes):*

$$U \cap B_r(p) = \{(x', x_n) \in B_r(p) : x_n > \gamma(x')\}$$

i.e. locally ∂U is the graph of a $C^{k,\delta}$-function (as $\{x_n = \gamma(x')\}$ is $\partial U \cap B_r(p)$).

[Recall that $C^{k,\delta}$ is the set Hölder continuous functions.]

Theorem 3.4 (Smooth Approximation of Sobolev Functions up to ∂U)**.** *Suppose U is open and bounded, and ∂U is Lipschitz (i.e. $C^{0,1}$).*

Then let $u \in W^{k,p}(U)$ for some $p \in [1, \infty)$. Then, $\exists$ smooth functions $u_m \in C^\infty(\overline{U})$ such that $u_m \to u$ in $W^{k,p}(U)$.

Proof. The idea is as follows: get the approximation locally about each point on the boundary, and then use the compactness of ∂U to reduce to a finite number. Then use Theorem 3.3 to get an approximation in the interior (away from the boundary) and then combine all of these with a partition of unity.

Fix $x^0 \in \partial U$. Then since ∂U is $C^{0,1}$, we know $\exists r > 0$ and $\gamma \in C^{0,1}(\mathbb{R}^{n-1})$ such that

$$U \cap B_r(x^0) = \{(x', x_n) \in B_r(x^0) : x_n > \gamma(x')\}.$$

So set $V = U \cap B_{r/2}(x^0)$. Then define the **shifted point**, $x^\varepsilon := x + \lambda\varepsilon e_n$, for $x \in V$ and $\varepsilon > 0$ (the point is to translate into U to get an approximation function).

[i.e. ∂U is $C^{0,1}$ $\Rightarrow$ at any point of ∂U, we can find a cone based at that point which lies locally in U.]

Then for $\lambda > 0$ large enough, we have $B_\varepsilon(x^\varepsilon) \subset U \cap B_r(x^0)$, for all $\varepsilon > 0$ sufficiently small, as $\gamma \in C^{0,1}$.

So define: $u_\varepsilon(x) := u(x^\varepsilon)$, for $x \in V$, i.e. a translation of u. Then set $v^{\varepsilon,\tilde{\varepsilon}} = \eta_{\tilde{\varepsilon}} * u_\varepsilon$, for $0 < \tilde{\varepsilon} < \varepsilon$. Then we have $v^{\varepsilon,\tilde{\varepsilon}} \in C^\infty(\overline{V})$.

Now fix $\delta > 0$. We then note:

$$\|v^{\varepsilon,\tilde{\varepsilon}} - u\|_{W^{k,p}(V)} = \|v^{\varepsilon,\tilde{\varepsilon}} - u_\varepsilon + u_\varepsilon - u\|_{W^{k,p}(V)} \le \|v^{\varepsilon,\tilde{\varepsilon}} - u_\varepsilon\|_{W^{k,p}(V)} + \|u_\varepsilon - u\|_{W^{k,p}(V)}.$$

Then since the translation operator is continuous in the L^p norms (as $p < \infty$ - see Handout for details), we can pick $\varepsilon > 0$ such that $\|u_\varepsilon - u\|_{W^{k,p}(V)} \le \delta/2$. Having fixed this ε, we can then pick $0 < \tilde{\varepsilon} < \varepsilon$ small enough such that $\|v^{\varepsilon,\tilde{\varepsilon}} - u_\varepsilon\|_{W^{k,p}(V)} \le \delta/2$, by previous results (as this is a smooth mollification). So hence we see that, for $\varepsilon, \tilde{\varepsilon}$ sufficiently small, we have $\|v^{\varepsilon,\tilde{\varepsilon}} - u\|_{W^{k,p}(V)} \le \delta$.

Now note that these V sets for each $x^0 \in \partial U$ cover ∂U. Then since ∂U is compact, we can find finitely many $x_i^0 \in \partial U$ and radii $r_i > 0$ and sets $V_i = U \cap B_{r_i/2}(x_i^0)$ (as above) and functions $v_i \in C^\infty(\overline{V}_i)$, for $i = 1, \ldots, N$, such that:

$$\partial U \subset \bigcup_{i=1}^{N} B_{r_i/2}(x_i^0) \quad \text{and} \quad \|v_i - u\|_{W^{k,p}(V_i)} \leq \delta \quad \text{for each } i.$$

Now we can approximate u all along the boundary by the above. But we may be missing some of the interior. But by Theorem 3.3, we know how things work in $\text{Int}(U)$, away from the boundary. So we can take an open set $V_0 \subset\subset U$ such that $U \subset \bigcup_{i=0}^{N} V_i$.

Then by Theorem 3.3, we can choose $v_0 \in C^\infty(\overline{V}_0)$ such that $\|v_0 - u\|_{W^{k,p}(V_0)} \leq \delta$.

So now let $\{\xi_i\}_{i=0}^{N}$ be a partition of unity subordinate to the cover $\{V_0, B_{r_1}(x_1^0), \ldots, B_{r_N}(x_N^0)\}$ (the balls can leak out of U, just the supports can't).

Then define:

$$v = \sum_{i=0}^{N} \xi_i v_i.$$

Then clearly $v \in C^\infty(\overline{U})$. Further, we have:

$$\begin{aligned}
\|D^\alpha v - D^\alpha u\|_{L^p(U)} &= \left\| D^\alpha \left(\sum_{i=0}^{N} \xi_i v_i \right) - D^\alpha \left(\sum_{i=0}^{N} \xi_i u \right) \right\|, \quad \text{as } \sum_i \xi_i = 1 \text{ on } U \\
&\leq C_k \sum_{i=0}^{N} \|v_i - u\|_{W^{k,p}(V_i)} \quad \text{for some constants } C_k \text{ [}\textbf{Exercise}\text{ to check]} \\
&\leq C_k(1+N)\delta \quad \text{by the above construction.}
\end{aligned}$$

So hence taking $\delta \downarrow 0$ and picking a sequence of v_m's (i.e. $\delta = 1/m$ gives v_m), we are done.

[This is a reall nice trick to patch together solutions on different reasons. As the ξ_i die off in the balls, we do not need to worry about patching together solutions on overlaps, as long as we have a vector space structure and so can add/multiply!]

□

3.5. **Extensions and Traces.**

There are two things we look into here: extending Sobolev functions, and traces. We have already looked at approximation them by smooth functions. Extensions involve trying to extend a given function to a larger space on which we do not change its Sobolev norm too much. Traces deal with incorporating boundary conditions (which tend to only be defined on sets of measure zero, which does not change a Sobolev function) into the function space itself.

3.5.1. *Extensions.*

Theorem 3.5 (Extensions of $W^{1,p}$ functions)**.** *Suppose U is open, bounded, and ∂U is C^1. Then choose a bounded V with $U \subset\subset V$. Let $p \in [1, \infty]$. Then, $\exists$ a bounded linear operator, called the* ***extension operator****, $E : W^{1,p}(U) \to W^{1,p}(\mathbb{R}^n)$, such that $\forall u \in W^{1,p}(U)$ we have:*

(i) $E(u)|_U = u$ *a.e.*

(ii) *supp*$(E(u)) \subset V$

(iii) $\|E(u)\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(U)}$, *where* $C = C(U, V, p)$

i.e. this operator is a bounded operator which extends u to a larger region.

Note: We can generalise this result to $W^{k,p}(U)$, but for larger k this gets messy, so we just stick with the simple case.

Remark: To prove this, we first need to look a bit more at ∂U. Since ∂U is assumed to

be $C^1 \equiv C^{1,0}$, for any $q \in \partial U$, $\exists r > 0$ and $\gamma \in C^1(R^{n-1})$ such that $U \cap B_r(q) = \{(x', x_n) \in B_r(q) : x_n > \gamma(x')\}$.

It is often convenient to first "straighten out ∂U". So define a map $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ via: $\Phi(x) = y$, where, in components:

$$y^i := \begin{cases} x^i & \text{if } i = 1, \ldots, n-1 \\ x_n - \gamma(x_1, \ldots, x_{n-1}) & \text{if } i = n. \end{cases}$$

So hence $\partial U \mapsto \{y : y_n = 0\}$. We can then write down the inverse of Φ, which is $\Psi : \mathbb{R}^n \to \mathbb{R}^n$, where $\Psi(y) = x$, where $x^i = y^i$ for $i = 1, \ldots, n-1$, and $x^n = y^n + \gamma(y^1, \ldots, y^{n-1})$.

Clearly then $\Phi \circ \Psi = \Psi \circ \Phi = \text{id}$, and $\Phi(U \cap B_r(q)) \subset \{y : y_n > 0\}$, i.e. the upper-half space.

Now Φ, Ψ are both C^1, and $\det(D\Phi) = \det(D\Psi) = 1$. So Φ is a C^1-diffeomorphism.

We will use these facts in the proof of Theorem 3.5, which we give now.

Proof of Theorem 3.5. Fix $x^0 \in \partial U$. We first prove the case when ∂U is flat near x^0, and when u is C^1. Then we remove the flatness assumption, and then the C^1 assumption.

Claim: The result is true when ∂U is flat near x^0 and $u \in C^1(\overline{U})$.

Proof of Claim. ∂U being flat near x^0 means that it lies in the plane $\{x : x_n = 0\}$ (wlog n). We may assume that $\exists r > 0$ small enough so that:

$$B_+ := B_r(x^0) \cap \{x_n \geq 0\} \subset \overline{U} \quad \text{and} \quad B_- := B_r(x^0) \cap \{x_n \leq 0\} \subset \mathbb{R}^n \backslash U.$$

Then we define:

$$\bar{u}(x) := \begin{cases} u(x) & \text{if } x \in B_+ \\ -3u(x', -x_n) + 4u(x', -x_n/2) & \text{if } x \in B_- \end{cases}$$

where as usual, $x = (x', x_n)$, $x' = (x_1, \ldots, x_{n-1})$. This is an example of a **higher order reflection**.

[The idea here is that if we know u is C^1 on B_+, we want to extend u to a C^1 function on the whole ball $B_r(x^0)$. Making a continuous extension is easy, as we can just reflect the solution along the boundary (i.e. $\bar{u} = u(x', -x_n)$ if below). But what is harder is making the derivatives also continuous - this is what the higher order reflection ensures, and gives the above formula.]

So note that clearly $\bar{u}$ is continuous at $x_n = 0$, as both expressions becomes $u(x', 0)$.

We claim that also $\bar{u} \in C^1(B_r(x^0))$. By the above, we know that $\bar{u} \in C^0(B_r(x^0))$. For the partial derivatives, we have:

For $1 \leq k \leq n-1$:

$$\bar{u}_{x_k} = \begin{cases} u_{x_k} & \text{in } B_+ \\ -3u_{x_k}(x', -x_n) + 4u_{x_k}(x', -x_n/2) & \text{in } B_- \end{cases}$$

and for $k = n$, we have:

$$\bar{u}_{x_n} = \begin{cases} u_{x_n}(x', x_n) & \text{in } B_+ \\ 3u_{x_n}(x', -x_n) - 2u_{x_n}(x', -x_n/2) & \text{in } B_- \end{cases}$$

which are all clearly continuous on $x_n = 0$, and so hence we see $\bar{u} \in C^1(B_r(x^0))$.

Then we can easily check directly from these expressions that:

$$\|\bar{u}\|_{W^{1,p}(B_r(x^0))} \leq C\|u\|_{W^{1,p}(B_+)}$$

for some $C \geq 0$, independent of u. So hence we are done in this case by letting $E(u) := \bar{u}$ (potentially multiplied by a smooth cut-off function which is 1 on B_+).

□

So we need to remove the flatness condition on ∂U, and the C^1 condition on u. We first remove the flatness assumption (this should be easy as ∂U is C^1 and so we can straighten it out).

Claim: The result is true for ∂U not necessarily flat near $x^0 \in \partial U$.

Proof of Claim. Since ∂U is C^1, we can then find an open set $W \ni x^0$ and a map Φ such that $\Phi(W) = B_r(y^0)$ and $\Phi(U \cap W) = B_r(y^0) \cap \{y_n > 0\}$.

[i.e. take a ball around x^0. Then as Φ is an open map (as its inverse Ψ is continuous), $\Phi(V)$ is an open set containing $y^0 = \Phi(x^0)$. So hence we can find a ball $B_r(y^0) \subset \Phi(V)$. Then as $\Psi = \Phi^{-1}$ is an open map, we know $\Psi(B_r(y^0))$ is an open set containing x^0: call this set W, and then this has the required properties.]

We want a ball on the flat space so that we can apply the previous case. So write $y = \Phi(x)$, so that $\Psi(y) = x$. Then set:

$$\tilde{u}(y) := u(\Psi(y))$$

i.e. a change of coordinates for u. Let B_+, B_- be the corresponding sets as in the previous case for y^0. Then, we know $\tilde{u} \in C^1(B_+)$ (as U is mapped into here), and so by the previous step, we can find an extension of $\tilde{u}$, say $\bar{\tilde{u}} \in C^1(B_r(y^0))$, such that $\bar{\tilde{u}}$ extends $\tilde{u}$, and that:

$$\|\bar{\tilde{u}}\|_{W^{1,p}(B_r(y^0))} \leq C\|\tilde{u}\|_{W^{1,p}(B_+)}.$$

So converting this extension back into the x-coordinates via Φ, define:

$$\bar{u}(x) = \bar{\tilde{u}}(\Phi(x)).$$

Then, $\bar{u} \in C^1(W)$, $\bar{u}$ extends u (as $\Phi \circ \Psi = \text{id}$), and:

$$\|\bar{u}\|_{W^{1,p}(W)} \leq C\|u\|_{W^{1,p}(U)}$$

for some C (as Φ is C^1).

[See Example Sheet 2. for details of deriving this last inequality from the above.]

So now we have local extensions at each point on the boundary, and so we can use the compactness of ∂U to get global ones.

Indeed, since ∂U is compact, and these W sets above for each $x \in \partial U$ form an open cover of ∂U, we can take a finite number of points $x_i^0 \in \partial U$ and sets W_i and extensions $\bar{u}_i \in C^1(W_i)$ extending u, such that $\partial U \subset \bigcup_{i=1}^N W_i$.

Now pick $W_0 \subset\subset U$ such that $U \subset \bigcup_{i=0}^N W_i$ (i.e. fill in the interior).

Now let $(\xi_i)_{i=0}^N$ be a partition of unity subordinate to the cover $\{W_i\}_{i=0}^N$. Then write:

$$\bar{u} = \sum_{i=0}^N \xi_i \bar{u}_i, \quad \text{where } \bar{u}_0 = u.$$

Then on U, we have $\bar{u} = u$, since $\bar{u}_i = u$ on U for all i (as they are extensions) and $\sum_{i=0}^N \xi_i = 1$. By construction, we know that $\bar{u} \in C^1(\mathbb{R}^n)$ ($\mathbb{R}^n$, as the ξ_i die outside of some compact set).

[We do not need to check agreement on overlaps, because that is not what we are constructing/doing - a partition of unity enables us to sum over solutions instead.]

We also have: $\|\bar{u}\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(U)}$.

Then by multiplying $\bar{u}$ by a cut-off function if necessary, we may assume that $\text{supp}(\bar{u}) \subset V$ for some $V \supset\supset U$. So hence done in this case.

□

Finally, we need to establish the result holds for $u \in W^{1,p}(U)$, and not necessarily with $u \in C^1(\overline{U})$ [note that we could just say that $C^1(\overline{U}) \subset W^{1,p}(U)$ is dense (since it contains all the test functions which are dense from previous results) and so our functional extends uniquely by continuity, by we shall give more details].

We write for $u \in E(u) = \bar{u}$, given as above. This is then a linear operator [**Exercise** to check this from the construction (easy)].

Now suppose $u \in W^{1,p}(U)$. Then by a previous density result, $\exists (u_m)_m \subset C^\infty(\overline{U})$ such that $u_m \to u$ in $W^{1,p}(U)$.

Now consider the sequence $(E(u_m))_{m=1}^\infty$. Then by the above, case, as u_m is smooth, we know $E(u_m) \in W^{1,p}(\mathbb{R}^n)$, and by linearity:

$$\|E(u_m) - E(u_k)\|_{W^{1,p}(\mathbb{R}^n)} = \|E(u_m - u_k)\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u_m - u_k\|_{W^{1,p}(U)},$$

as by the construction above, E is bounded (i.e. $\|\bar{u}\| \leq C\|u\|$).

But $(u_m)_m$ is convergent in $W^{1,p}(U)$, and thus is Cauchy. Hence $(E(u_m))_m$ is Cauchy in $W^{1,p}(\mathbb{R}^n)$, which is a complete space, and so $\exists$ a limit of $(E(u_m))_m$ in $W^{1,p}(\mathbb{R}^n)$.

We can then check that this limit is independent of the sequence approximating it [**Exercise** to check - see Linear Analysis Part II for the general result]. So hence we can define:

$$E(u) = \lim_{m\to\infty} E(u_m).$$

We can then check that this obeys the conditions of the theorem [**Exercise** to check] and so we are done.

□

Note: In general over $W^{k,p}$ we would consider:

$$\sum_{i=1}^{k} c_i u\left(x', -\frac{x_n}{i}\right) \quad \text{on } x_n < 0.$$

Then we would need (for the continuity of u and the relevant derivatives at the boundary)

$$\sum_{i=1}^{k} c_i \left(-\frac{1}{i}\right)^m = 1 \quad \forall m = 0, 1, \ldots, k-1.$$

3.5.2. *Trace Theorems.*

Since $u \in W^{1,p}(U)$ is only defined up to sets of measure zero, and as ∂U is typically a set of measure zero (as it will be a codimension 1 space), we cannot naively define $u|_{\partial U}$, as this is a function defined on a set of measure 0, and so we can change its values completely without leaving the equivalence class of u in $W^{1,p}(U)$.

But we still want to be able to talk about boundary values for elliptic PDE problems. However this is fixed if we consider $W^{1,p}(\partial U)$. This leads us to:

Theorem 3.6 (The Trace Theorem). *Assume that U is bounded, and ∂U is C^1. Then, $\exists$ a bounded linear operator, called the* ***trace****, $T : W^{1,p}(U) \to L^p(\partial U)$, for $p \in [1, \infty)$, such that:*

(i) $T(u) = u|_{\partial U}$ *if* $u \in W^{1,p}(U) \cap C(\overline{U})$.

(ii) $\|T(u)\|_{L^p(\partial U)} \leq C\|u\|_{W^{1,p}(U)}$ *for each $u \in W^{1,p}(U)$, where $C = C(U,p)$ (i.e. T is bounded).*

Note: We want (i) to be true for $u \in C(\overline{U})$ since then u is defined on ∂U and so we can define the restriction.

Proof Sketch. We give a sketch proof of this result - the details are similar to those in the extension theorems. See Example Sheet 2 for details.

Suppose $U = \{x_n > 0\}$ and $u \in C^\infty(\overline{U})$ with $\text{supp}(u) \subset B_R(0) \cap \overline{U}$ [i.e. the boundary is flat near x and u is smooth here]. As always, write $x = (x', x_n)$, where $x' \in \mathbb{R}^{n-1}$.

Then by the fundamental theorem of calculus, as $u|_{\partial U} = 0$ (as its suppose is contained in U) we have:

$$\begin{aligned}\int_{\mathbb{R}^{n-1}} |u(x',0)|^p \, \mathrm{d}x' &= \int_{\mathbb{R}^{n-1}} \int_0^\infty -\frac{\partial}{\partial x_n}\left(|u(x',x_n)|^p\right) \, \mathrm{d}x_n \mathrm{d}x' \\ &= \int_U p|u|^{p-1} u_{x_n} \cdot \text{sign}(u) \, \mathrm{d}x.\end{aligned}$$

Then by Young's inequality, $|ab| \leq \frac{|a|^p}{p} + \frac{|b|^q}{q}$, if $1/p + 1/q = 1$, we have:

$$\int_{\mathbb{R}^{n-1}} |u(x',0)|^p \, \mathrm{d}x' \leq C_p \int_U |u|^p + |u_{x_n}|^p \, \mathrm{d}x \leq C_p \|u\|^p_{W^{1,p}(U)}.$$

So hence the map: $T(u) := u_{\partial U}$ is bounded, and we have from the above:

$$\|T(u)\|_{L^p(\partial U)} \leq C_p \|u\|_{W^{1,p}(U)}$$

for $u \in C^\infty(\overline{U})$.

Then to complete the proof, we use a partition of unity argument to reduce the general boundary case to the above case (as ∂U is compact), and then we use the approximation theorems to get this for $W^{1,p}(\partial U)$, just like in the proof of Theorem 3.5 (i.e. extends uniquely from $C^1 \cap W^{1,p}$ to $W^{1,p}$).

□

Remarks:

- If $u \in W_0^{1,p}(U)$, then (by definition of $W_0^{1,p}(U)$ by the $W^{1,p}$-closure of test functions in U) $\exists$ $u_m \in C_c^\infty(U)$ with $u_m \to u$ in $W^{1,p}(U)$. So hence:

$$T(u) = T\left(\lim_{m\to\infty} u_m\right) = \lim_{m\to\infty} T(u_m) = 0$$

  by continuity of T, and since the u_m are smooth compactly supported in U, and so by the above, $T(u_m) = u_m|_{\partial U} = 0$.

  In fact the converse is also true: if $T(u) = 0$, then $u \in W_0^{1,p}(U)$.
- If $u \in W^{k,p}(U)$, then we can similarly define trace operators for $Du, \dots, D^{k-1}u$.

3.6. **Sobolev Inequalities.**

The Sobolev inequalities are a collection of inequalities that allow us to 'trade' differentiability for integrability. The basic result is the Gagliardo-Nirenberg-Sobolev inequality (GNS). Before we state and prove it, we need the following lemma:

Lemma 3.3. *Let $n \geq 2$ and $f_1, \ldots, f_n \in L^{n-1}(\mathbb{R}^{n-1})$. Then for $1 \leq i \leq n$, denote $\tilde{x}_i := (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ (i.e. remove the i'th coordinate). Set:*

$$f(x) := \prod_{i=1}^{n} f_i(\tilde{x}_i).$$

Then, $f \in L^1(\mathbb{R}^n)$, and $\|f\|_{L^1(\mathbb{R}^n)} \leq \prod_{i=1}^n \|f_i\|_{L^{n-1}(\mathbb{R}^{n-1})}$.

Proof. We proceed by induction on n.

If $n = 2$, then we have $f(x_1, x_2) = f_1(x_2) f_2(x_1)$, and both parts behave independently. So:

$$\|f\|_{L^1(\mathbb{R}^2)} = \int_{\mathbb{R}^2} |f(x_1, x_2)| \, \mathrm{d}x_1 \mathrm{d}x_2 = \int_{\mathbb{R}} |f_1(x_2)| \, \mathrm{d}x_2 \cdot \int_{\mathbb{R}} |f_2(x_1)| \, \mathrm{d}x_1 = \|f_1\|_{L^1(\mathbb{R})} \cdot \|f_2\|_{L^1(\mathbb{R})}$$

where we have used Fubini's theorem in the second equality. So the result is true here.

So now suppose that it is true for $n \geq 2$, and consider the $n+1$ case. Then we have:

$$f(x) = f_{n+1}(\tilde{x}_{n+1}) F(x)$$

where $F(x) = f_1(\tilde{x}_1) \cdots f_n(\tilde{x}_n)$. Now fix x_{n+1}, and note that, by Hölder 's inequality ($\|fg\|_{L^1} \leq \|f\|_{L^p} \|f\|_{L^q}$ for p, q Hölder conjugate):

$$\int_{x_1, \ldots, x_n} |f(x, x_{n+1})| \, \mathrm{d}x_1 \cdots \mathrm{d}x_n \leq \|f_{n+1}\|_{L^n(\mathbb{R}^n)} \cdot \|F(x, x_{n+1})\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)}$$

(i.e. take $p = n$, so $q = \frac{1}{1 - 1/n}$, in Hölder 's inequality).

Now apply the induction hypothesis to $F(x, x_{n+1})^{n/(n-1)} = f_1(x, x_{n+1})^{n/(n-1)} \cdots f_n(x, x_{n+1})^{n/(n-1)}$ (recalling that these a functions of $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, as x_{n+1} is fixed), to get:

$$\begin{aligned}
\int_{x_1, \ldots, x_n} |f(x, x_{n+1})| \, \mathrm{d}x_1 \cdots \mathrm{d}x_n &\leq \|f_{n+1}\|_{L(\mathbb{R}^n)} \cdot \left(\prod_{i=1}^n \|f_i^{\frac{n}{n-1}}(\cdot, x_{n+1})\|_{L^{n-1}(R^{n-1})} \right)^{\frac{n-1}{n}} \\
&= \|f_{n+1}\|_{L^n(\mathbb{R}^n)} \cdot \prod_{i=1}^n \|f_i(\cdot, x_{n+1})\|_{L^n(\mathbb{R}^{n-1})}
\end{aligned}$$

where the last equality comes from playing around with the exponents. Now integrate this inequality over x_{n+1} to get:

$$\begin{aligned}
\|f\|_{L^1(\mathbb{R}^{n+1})} &\leq \|f_{n+1}\|_{L^n(\mathbb{R}^n)} \cdot \int_{\mathbb{R}} \prod_{i=1}^n \|f_i(\cdot, x_{n+1})\|_{L^n(\mathbb{R}^{n-1})} \, \mathrm{d}x_{n+1} \\
&\leq \|f_{n+1}\|_{L^n(\mathbb{R}^n)} \cdot \prod_{i=1}^n \left(\int_{\mathbb{R}} \|f_i(\cdot, x_{n+1})\|^n_{L^n(\mathbb{R}^{n-1})} \, \mathrm{d}x_{n+1} \right)^{1/n}
\end{aligned}$$

where we have used the generalised Hölder inequality[(vi)]. So hence:

$$= \|f_{n+1}\|_{L^n(\mathbb{R}^n)} \cdot \prod_{i=1}^{n} \|f_i\|_{L^n(\mathbb{R}^n)} = \prod_{i=1}^{n+1} \|f_i\|_{L^n(\mathbb{R}^n)}$$

where we have used that

$$\int_{\mathbb{R}} \|f(\cdot, x_{n+1})\|^n_{L^n(\mathbb{R}^{n-1})}\, \mathrm{d}x_{n+1} = \int_{\mathbb{R}} \left(\int_{\mathbb{R}^{n-1}} |f(x)|^n \, \mathrm{d}x_1 \cdots \widehat{\mathrm{d}x_i} \cdots \mathrm{d}x_n \right) \mathrm{d}x_{n+1} = \|f\|^n_{L^n(\mathbb{R}^n)},$$

where we miss out $\mathrm{d}x_i$ for some i.

Hence this proves the $n+1$ case, and so hence done by induction.

□

Theorem 3.7 (The Gagliardo-Nirenberg-Sobolev (GNS) Inequality)**.** *Assume we are on $\mathbb{R}^n$, and $p < n$. Then we have:*

$$W^{1,p}(\mathbb{R}^n) \subset L^{p^*}(\mathbb{R}^n)$$

*where $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$ is the **Sobolev conjugate to p** for this n. Moreover, $\exists C = C(n,p) > 0$ such that:*

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \le C \|Du\|_{L^p(\mathbb{R}^n)} \quad \forall u \in W^{1,p}(\mathbb{R}^n)$$

i.e. the embedding $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^{p^*}(\mathbb{R}^n)$ is continuous.

Remark: Note that $p^* = \frac{np}{n-p} > p$. So, this is saying that, if a function is weakly differentiable and so in some $W^{1,p}(\mathbb{R}^n)$ for some p, if $p < n$, then it represents also some function in $L^{p^*}(\mathbb{R}^n)$, for some $p^* > p$. So as in L^p, we do not know if things are weakly differentiable, this is saying that we can drop the differentiability, and instead have more integrability, as $p^* > p$. Note that $\|u\|_{L^{p^*}(\mathbb{R}^n)} \le C\|Du\|_{L^p} \le C\|u\|_{W^{1,p}(U)}$.

Note: Adding constants to u does not change $\|Du\|_{L^p(\mathbb{R}^n)}$. However, we need $|u| \to 0$ as $|x| \to \infty$ to remain in L^p.

Note: The GNS inequality can simply be stated via, given a derivative, your function is more integrable. Nothing is said about extra integrability of the derivative, however.

Proof. Assume that $u \in C_c^\infty(\mathbb{R}^n)$ and consider the case $p = 1$. Then by the fundamental theorem of calculus and compact support, we have:

$$u(x) = \int_{-\infty}^{x_i} u_{x_i}(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_n)\, \mathrm{d}y_i \quad \text{for each } i = 1, \dots, n.$$

So,

$$|u(x)| \le \int_{-\infty}^{\infty} |Du(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_n)|\, \mathrm{d}y_i =: f_i(\tilde{x}_i).$$

[(vi)] The generalised Hölder inequality states: $\left\|\prod_{i=1}^n f_i\right\|_{L^1} \le \prod_{i=1}^n \|f_i\|_{L_{p_i}}$, where $\sum_i \frac{1}{p_i} = 1$. So here we have used the case where $p_i = 1/n$ for all i.

Thus, $|u(x)|^{\frac{n}{n-1}} = (\underbrace{|u|\cdots|u|}_{n \text{ times}})^{\frac{1}{n-1}} \leq \prod_{i=1}^n f_i(\tilde{x}_i)^{\frac{1}{n-1}}$.

Then integrating this over x and using Lemma 3.3 above, we have:

$$\|u\|^{\frac{n}{n-1}}_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \leq \prod_{i=1}^n \|f_i^{\frac{1}{n-1}}\|_{L^{n-1}(\mathbb{R}^{n-1})}.$$

Then using the definition of the f_i and the L^p-norms, we get:

$$\prod_{i=1}^n \|f_i^{\frac{1}{n-1}}\|_{L^{n-1}(\mathbb{R}^{n-1})} = \prod_{i=1}^n \|Du\|^{\frac{1}{n-1}}_{L^1(\mathbb{R}^n)} = \|Du\|^{\frac{n}{n-1}}_{L^1(\mathbb{R}^n)}.$$

So hence we get:

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq \|Du\|_{L^1(\mathbb{R}^n)}$$

here, as $p^* = \frac{n}{n-1}$ in this case (as $p = 1$).

So we have proven the rest for $p = 1$ and $u \in C_c^\infty(\mathbb{R}^n)$. But then as $C_c^\infty(\mathbb{R}^n)$ is dense in $W^{1,1}(\mathbb{R}^n)$, the general result for $p = 1$ follows by taking limits/approximation.

So <u>now suppose $p \neq 1$</u>.

Then we can apply the $p = 1$ case to: $v = |u|^\gamma$, for some $\gamma > 1$ we will choose later.

So, $Dv = \gamma \cdot \text{sign}(u) \cdot |u|^{\gamma-1} \cdot Du$. So by the $p = 1$ case, we get:

$$\begin{aligned}
\left(\int_{\mathbb{R}^n} |u|^{\frac{\gamma n}{n-1}} \, dx\right)^{\frac{n-1}{n}} &= \||u|^\gamma\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \leq \|D(|u|^\gamma)\|_{L^1(\mathbb{R}^n)} \\
&= \gamma \int_{\mathbb{R}^n} |u|^{\gamma-1}|Du| \, dx \\
&\leq \gamma \cdot \left(\int_{\mathbb{R}^n} |u|^{(\gamma-1)\cdot\frac{p}{p-1}} \, dx\right)^{1-\frac{1}{p}} \cdot \left(\int_{\mathbb{R}^n} |Du|^p \, dx\right)^{1/p}
\end{aligned}$$

where in the last line we have used Hölder 's inequality (as $q = \frac{p}{p-1}$ here). So hence we should choose γ to match the powers of u in the integrals, i.e.

$$\frac{\gamma n}{n-1} = (\gamma - 1) \cdot \frac{p}{p-1} \;\Rightarrow\; \gamma = \frac{p(n-1)}{n-p} > 1.$$

Then, $\frac{\gamma n}{n-1} \equiv (\gamma - 1) \cdot \frac{p}{p-1} = \frac{np}{n-p} = p^*$. So we get:

$$\begin{aligned}
\left(\int_{\mathbb{R}^n} |u|^{p^*} \, dx\right)^{\frac{n-1}{n}} &\leq \frac{p(n-1)}{n-p}\left(\int_{\mathbb{R}^n} |u|^{p^*} \, dx\right)^{\frac{p-1}{p}} \cdot \|Du\|_{L^p(\mathbb{R}^n)} \\
\Rightarrow \quad \left(\int_{\mathbb{R}^n} |u|^{p^*} \, dx\right)^{1/p^*} &\leq \underbrace{\frac{p(n-1)}{n-p}}_{=C} \|Du\|_{L^p(\mathbb{R}^n)}
\end{aligned}$$

i.e.

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|Du\|_{L^p(\mathbb{R}^n)}.$$

This argument holds $\forall u \in C_c^\infty(\mathbb{R}^n)$, and then by approximating by $C_c^\infty(\mathbb{R}^n)$, we can extend this result to $W^{1,p}(\mathbb{R}^n)$. So we are done.

$\square$

We then have two corollaries of this result:

Corollary 3.1 (GNS Inequality for $U \subset \mathbb{R}^n$.). *Suppose $U \subset \mathbb{R}^n$ is open and bounded with a C^1 boundary. Suppose $p \in [1,n)$. Then if p^* is the Sobolev conjugate of p (i.e. $\frac{1}{p^*} + \frac{1}{n} = \frac{1}{p}$), then we have:*

$$W^{1,p} \subset L^{p^*}(U)$$

and in particular, $\exists C = C(U,p,n)$ such that $\|u\|_{L^{p^*}(U)} \leq C\|u\|_{W^{1,p}(U)}$ for all $u \in W^{1,p}(U)$.

Proof. By the extension theorem, Theorem 3.5, we can find $\bar{u} \in W^{1,p}(\mathbb{R}^n)$ with $u = \bar{u}$ a.e. on U, and $\|\bar{u}\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(U)}$. So then:

$$\|u\|_{L^{p^*}(U)} = \|\bar{u}\|_{L^{p^*}(U)} \leq \|\bar{u}\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|D\bar{u}\|_{L^p(\mathbb{R}^n)} \leq C\|\bar{u}\|_{W^{1,p}(\mathbb{R}^n)} \leq \tilde{C}\|u\|_{W^{1,p}(U)}$$

where the first inequality is true since $U \subset \mathbb{R}^n$, the second is true from the GNS inequality (Theorem 3.7), the third is true since $\|Dv\|_{L^p} \leq \|v\|_{W^{1,p}}$ is true in general (by definition of the Sobolev norm), and the final inequality is from the extension theorem inequality above. So done.

$\square$

Corollary 3.2 (Poincaré Inequality). *Suppose $U \subset \mathbb{R}^n$ is open and bounded. Suppose $u \in W_0^{1,p}(U)$ for some $p \in [1,n)$. Then:*

$$\|u\|_{L^q(U)} \leq C\|Du\|_{L^p(U)}$$

for all $q \in [1,p^*]$. In particular, as $p \leq p^*$, we have $\|u\|_{L^p(U)} \leq C\|Du\|_{L^p(U)}$

i.e. in $W_0^{1,p}$, control of derivatives gives control on u.

Proof. Since $u \in W_0^{1,p}(U) := \overline{C_c^\infty(U)}^{\|\cdot\|_{W^{1,p}(U)}}$, we know that $\exists (u_m)_m \subset C_c^\infty(U)$ converging to u in $W^{1,p}(U)$.

Extending each u_m to vanish on U^c, we have that $u_m \in C_c^\infty(\mathbb{R}^n)$ (can do since each u_m vanishes near ∂U).

Su we can apply the GNS inequality (Theorem 3.7) to each u_m to find:

$$\|u_m\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|Du_m\|_{L^p(\mathbb{R}^n)}.$$

Then since $u_m = 0$ outside U, this gives:

$$\|u_m\|_{L^{p^*}(U)} \leq C\|Du_m\|_{L^p(U)}.$$

So sending $m \to \infty$ gives: $\|u\|_{L^{p^*}(U)} \leq C\|Du\|_{L^p(U)}$. So this proves the result for $q = p^*$.

Then since U is bounded, by Hölder 's inequality, we have

$$\int_U |u|^q \, \mathrm{d}x \leq \left(\int_U 1 \, \mathrm{d}x\right)^{1/r} \cdot \left(\int_U |u|^{qs} \, \mathrm{d}x\right)^{1/s}$$

where $\frac{1}{s} + \frac{1}{r} = 1$. So setting $qs = p^*$, i.e. $s = p^*/q > 1$ (for $q \in [1, p^*]$), we see:

$$\left(\int_U |u|^q \, \mathrm{d}x\right)^{1/q} \leq C \cdot \left(\int_U |u|^{p^*} \, \mathrm{d}x\right)^{1/p^*} = C\|u\|_{L^{p^*}(U)}$$

i.e.

$$\|u\|_{L^q(U)} \leq C\|u\|_{L^{p^*}(U)}$$

i.e. $L^{p^*}(U) \subset L^q(U)$. So hence:

$$\|u\|_{L^q(U)} \leq C\|u\|_{L^{p^*}(U)} \leq \tilde{C}\|Du\|_{L^p(U)}$$

by the GNS inequality (Theorem 3.7), and so done.

□

So we have dealt with the case $p \in [1, n)$. So now we look at $p \in (n, \infty)$: we might hope that if $u \in W^{1,p}(\mathbb{R}^n)$, then u is "better than L^∞". We have:

Theorem 3.8 (Morrey's Inequality). *Suppose $p \in (n, \infty)$. Then, $\exists C = C(n, p)$ such that:*

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)}$$

for all $u \in C_c^\infty(\mathbb{R}^n)$, where $\gamma = 1 - \frac{n}{p} < 1$,

i.e. such functions are Hölder continuous:

$$W^{1,p}(\mathbb{R}^n) \subset C^{0,\gamma}(\mathbb{R}^n).$$

Proof. We first prove the Hölder part of the estimate. Let Q be an open cube of side length r containing the origin, O. Then define:

$$\bar{u} = \frac{1}{|Q|}\int_Q u(x) \, \mathrm{d}x$$

the average of u on Q. Then:

$$|\bar{u} - u(0)| = \left|\frac{1}{|Q|}\int_Q [u(x) - u(0)] \, \mathrm{d}x\right| \leq \frac{1}{|Q|}\int_Q |u(x) - u(0)| \, \mathrm{d}x.$$

Now note, as we are working with $u \in C_c^\infty(\mathbb{R}^n)$ smooth, we can differentiate, and so have (by the fundamental theorem of calculus):

$$u(x) - u(0) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}(u(tx)) \, \mathrm{d}t = \sum_i \int_0^1 x^i \cdot \frac{\partial u}{\partial x^i}(tx) \, \mathrm{d}t$$

where we have used the chain rule. So as $|x^i| < r$ for $x \in Q$, we get:

$$|u(x) - u(0)| \leq r \int_0^1 \sum_i \left| \frac{\partial u}{\partial x^i}(tx) \right| \, \mathrm{d}t.$$

So hence combining, we get:

$$\begin{aligned}
|\bar{u} - u(0)| &\leq \frac{r}{|Q|} \int_Q \int_0^1 \sum_i \left| \frac{\partial u}{\partial x^i}(tx) \right| \, \mathrm{d}t\mathrm{d}x \\
&\leq \frac{r}{|Q|} \int_0^1 \left(\int_{tQ} \sum_i \left| \frac{\partial u}{\partial x_i}(y) \right| \, t^{-n} \mathrm{d}y \right) \mathrm{d}t \quad \text{by changing the order of integration } (y = tx) \\
&\leq \frac{r}{|Q|} \int_0^1 t^{-n} \left(\sum_{i=1}^n \left\| \frac{\partial u}{\partial x_i} \right\|_{L^p(tQ)} \cdot |tQ|^{1/q} \right) \mathrm{d}t
\end{aligned}$$

where in the last line we have applied Hölder 's inequality to $\left|\frac{\partial u}{\partial x_i}\right| \cdot 1$, where q is the Hölder conjugate of p. [The factor of t^{-n} in the second line comes from changing variables, $y = tx$.]

Then since $|tQ| = t^n r^n$, we get:

$$|\bar{u} - u(0)| \leq C \cdot r^{1-n+\frac{n}{q}} \cdot \|Du\|_{L^p(\mathbb{R}^n)} \int_0^1 t^{-n+\frac{n}{q}} \, \mathrm{d}t \leq \frac{C}{1 - \frac{n}{p}} \cdot r^{1-\frac{n}{p}} \cdot \|Du\|_{L^p(\mathbb{R}^n)}$$

i.e.

$$\boxed{|\bar{u} - u(0)| \leq C r^\gamma \cdot \|Du\|_{L^p(\mathbb{R}^n)}.}$$

So now suppose $x, y \in \mathbb{R}^n$, with $|x - y| = \frac{r}{2}$. Then pick a box of side length r containing x, y. Then by applying the above result, shifting so that x, y play the role of 0, we have, by the triangle inequality:

$$|u(x) - u(y)| \leq |\bar{u} - u(x)| + |\bar{u} - u(y)| \leq C \cdot r^\gamma \|Du\|_{L^p(\mathbb{R}^n)}$$

by the above, as $2r = |x - y|$. So hence:

$$\frac{|u(x) - u(y)|}{|x - y|^\gamma} \leq C \cdot 2^{-\gamma} \|Du\|_{L^p(\mathbb{R}^n)}.$$

Note that the left hand side is independent of r. Hence this shows that this is true for any x, y, and so:

$$[u]_{C^{0,\gamma}(\mathbb{R}^n)} \leq C_{n,p} \cdot \|Du\|_{L^p(\mathbb{R}^n)}.$$

Finally, to see that u is bounded (to bound the other part of the Hölder norm), note that any $x \in \mathbb{R}^n$ belongs to some cube Q of side length 1. So,

$$|u(x)| \leq |\bar{u}| + |\bar{u} - u(x)| \leq |\bar{u}| + C \cdot \|Du\|_{L^p(\mathbb{R}^n)},$$

and we know:

$$|\bar{u}| \leq \int_Q |u(x)| \, \mathrm{d}x \leq \|u\|_{L^p(\mathbb{R}^n)} \cdot \|1\|_{L^q(Q)},$$

by Hölder 's inequality (noting that $|Q| = 1$ here). So hence:

$$|u(x)| \leq \tilde{C} \left(\|u\|_{L^p(\mathbb{R}^n)} + \|Du\|_{L^p(\mathbb{R}^n)} \right)$$

for some $\tilde{C}$, independent of x. So hence:

$$\|u\|_{C^0(\mathbb{R}^n)} := \sup_x |u(x)| \leq \tilde{C} \|u\|_{W^{1,p}(\mathbb{R}^n)}.$$

So hence combining the bounds on $\|u\|_{C^0(\mathbb{R}^n)}$ and $[u]_{C^{0,\gamma}(\mathbb{R}^n)}$ gives the result.

□

Note: The Fundamental Theorem of Calculus is key to the Sobolev embeddings: it relates a function to the integral of its derivative, which is just like trading differentiability for integrability.

Corollary 3.3. *Suppose $u \in W^{1,p}(U)$ for some $U \subset \mathbb{R}^n$ open and bounded with C^1-boundary. Let $p \in (n, \infty)$ and $\gamma = 1 - \frac{n}{p} < 1$.*

Then, $\exists u^* \in C^{0,\gamma}(U)$ such that $u = u^*$ a.e., and:

$$\|u^*\|_{C^{0,\gamma}(U)} \leq C\|u\|_{W^{1,p}(U)}$$

for some C independent of u.

i.e. $\exists$ a continuous embedding $W^{1,p}(U) \hookrightarrow C^{0,\gamma}(U)$.

Proof. **Exercise**. Although we give the following hint:

> *Hint:* The idea is to just extend u to $\mathbb{R}^n$, and then approximate it by $C_c^\infty(\mathbb{R}^n)$. Then use Theorem 3.8 (Morrey's inequality) to get the inequality for the sequence. Then we can extract a limit in the Hölder space (as it is complete and this is bounded), and then take a limit to get the result.

□

So in summary, we have seen:

Let U be open, bounded, and with ∂U C^1. Then:

- If $p \in [1, n)$, then $W^{1,p}(U) \hookrightarrow L^{p^*}(U)$ embeds continuously, where $\frac{1}{p^*} + \frac{1}{n} = \frac{1}{p}$.
- If $p \in (n, \infty)$, then $W^{1,p}(U) \hookrightarrow C^{0,\gamma}(U)$ embeds comtinuously, for $\gamma = 1 - \frac{n}{p}$.

These are the **Sobolev embeddings**. By applying them iteratively, we can establish higher order versions, e.g. $W^{k,p}(U) \hookrightarrow L^q(U)$ for some q related to k, p, n.

Remark: The point is that $p^* > p$ for $p < n$. So we move up the L^p spaces with the Sobolev embedding, but lost some differentiability. But then if we still have some differentiability left over, we can repeat this, and embed L^{p^*} in some $L^{p^{**}}$, etc. If the function is "regular enough", then we might eventually have $p^* > n$, and so then we can apply the second Sobolev embedding, and so get that the function is in fact (Hölder) continuous. If we can apply this to the derivatives, we may even get, say, that in fact the function is C^2, and so a classical solution of a PDE. So this is the reason why we work with Sobolev spaces and the Sobolev embeddings: being Banach spaces, we can extract convergent subsequences and limits in Sobolev spaces. Then if we can prove things about their integrability, then we can use the Sobolev embeddings to prove that they are actually regular, and so classical solutions to the PDE.

Now we move onto looking at a certain class of PDE, which encompasses many standard PDE's, such as Laplace's equation, the heat equation and the wave equation.

4. Second Order Elliptic Boundary Value Problems

Let $U \subset \mathbb{R}^n$ be a open, bounded, with C^1 boundary **throughout** §4.

For $u \in C^2(\overline{U})$, we define:

$$Lu := -\sum_{i,j=1}^{n} (a^{ij} u_{x_i})_{x_j} + \sum_{i=1}^{n} b^i u_{x_i} + cu$$

where a^{ij}, b^i, c are given functions on U (wlog we can assume $a^{ij} = a^{ji}$, as we can combine the symmetric partial derivatives). Typically we assume that they are all at least $L^\infty(U)$, but sometimes we will require more.

This form of PDE is called the **divergence form**, due to the $\nabla \cdot (A\nabla u)$ term at the front. This form of PDE is better for *energy methods*.

If $a^{ij} \in C^1(U)$, then we can rewrite this as:

$$Lu = -\sum_{i,j=1}^{n} a^{ij} u_{x_i x_j} + \sum_{i=1}^{n} \tilde{b}^i u_{x_i} + cu$$

which is known as the **non-divergence form** of the PDE. The form is better suited for the *maximal principle* approach.

We shall further assume that L is *elliptic*, i.e.

Definition 4.1. *We say that* L *is* ***elliptic*** *if:*

$$\sum_{i,j} a^{ij}(x)\xi_i \xi_j > 0 \quad \forall x \in U,\ \xi \in \mathbb{R}^n \backslash \{0\}.$$

However this ellipticity condition is typically not a strong enough condition for what we want. We will therefore also assume a stronger condition, *uniform ellipticity*, for our purposes:

Definition 4.2. *We say that* L *is* ***uniformly elliptic*** *if:*

$$\sum_{i,j} a^{ij}(x)\xi_i \xi_j \geq \theta \|\xi\|^2$$

for all $x \in U,\ \xi \in \mathbb{R}^n$*, for some* $\theta > 0$ *(independent of* x, ξ*)*

Note: This condition is sometimes called *strict ellipticity*, and uniformly elliptic sometimes means something slightly different. Note that the above condition implies that the quadratic form $L(\xi) := \xi^T A \xi$ is bounded below, i.e. $L(\xi) \geq \theta \|\xi\|^2$.

4.1. **Weak Formulations and Lax-Milgram.**

We shall consider the following boundary value problem:

$$\begin{cases} Lu = f & \text{in } U \\ u|_{\partial U} = 0. \end{cases} \tag{4.1}$$

This form of the equation is not very amenable to study by functional analytic methods.

So suppose $u \in C^2(\overline{U})$ solves (4.1), and suppose $v \in C^2(\overline{U})$ is <u>any</u> fucntion <u>with</u> $v|_{\partial U} = 0$. Then multiplying "$Lu = f$" by v and integrating by parts gives:

$$\begin{aligned} \int_U vf \, \mathrm{d}x = \int_U v \cdot Lu = \int_U v \left[-\sum_{i,j} (a^{ij} u_{x_i})_{x_j} + \sum_i b_i u_{x_i} + cu \right] \mathrm{d}x \\ = \underbrace{-\int_{\partial U} \sum_{i,j} v a^{ij} u_{x_i} n_j \, \mathrm{d}S}_{=0} + \int_U \left[\sum_{i,j} a^{ij} u_{x_i} v_{x_j} + \sum_i b^i v u_{x_i} + cuv \right] \mathrm{d}x \end{aligned}$$

where the surface term vanishes since $v|_{\partial U} = 0$. Hence we get:

$$\int_U vf \, \mathrm{d}x = B[u,v] \qquad \forall v \in C^2(\overline{U}) \text{ such that } v|_{\partial U} = 0 \tag{4.2}$$

where:

$$B[u,v] := \int_U \left[\sum_{i,j} a^{ij} u_{x_i} v_{x_j} + \sum_i b^i v u_{x_i} + cuv \right] \mathrm{d}x.$$

So we see that if $u \in C^2(\overline{U})$ solves (4.1), then (4.2) holds. Conversely, suppose $u \in C^2(\overline{U})$ with $u|_{\partial U} = 0$ and (4.2) holds. Then, undoing the integration by parts, we have:

$$\int_U (f - Lu)v \, \mathrm{d}x = 0$$

for all $v \in C^2(\overline{U})$ with $v|_{\partial U} = 0$.

In particular, this holds for all $v \in C_c^\infty \overline{U}$, and so hence this implies that $Lu = f$.

So hence if $u \in C^2(\overline{U})$ with $u|_{\partial U} = 0$, then:

$$u \text{ satisfies (4.1) } \iff u \text{ satisfies (4.2).}$$

However note that (4.2) makes sense (by approximation by smooth functions) for any $u, v \in H_0^1(U) := W_0^{1,2}(U)$, and so using (4.2) means we can reduce the number of assumptions we must make on u (i.e. our function space) which allows for stronger techniques to be used. This motivates the definition:

Definition 4.3. *We say that $u \in H_0^1(U)$ is a* ***weak solution of system (4.1)****, for $f \in L^2(U)$, if:*

$$B[u,v] = (f,v)_{L^2} := \int_U f v \, \mathrm{d}x \qquad \forall v \in H_0^1(U)$$

where here $(\cdot,\cdot)_{L^2}$ denotes the L^2 inner product.

So hence by the above we see that a $C^2(\overline{U})$ weak solution is a classical solution, i.e. a solution in the usual sense. The idea is then to prove the existence of a weak solution, and then show that, if a weak solution exists, then it is $C^2(\overline{U})$ (i.e. sufficiently regular), and so is a classical solution, which proves existence of solutions. So we start by looking at weak solutions.

The point of this is that we have moved into $H^1_0(U)$, which is a Hilbert space, to help us find solutions. So hence if we can find a sequence of nicer functions (e.g. smooth test functions) converging to the weak solution, then we can extract a convergent subsequence to find a limit, which we hope to prove is a weak solution.

Our first main result exploiting the Hilbert space structure is the following:

Theorem 4.1 (The Lax-Milgram Theorem). *Let H be a real Hilbert space with inner product $(\cdot,\cdot)$, and suppose $B: H\times H\to\mathbb{R}$ is a bilinear map. Suppose that $\exists$ constants $\alpha,\beta>0$ such that:*

$$|B[u,v]|\le \alpha\|u\|\cdot\|v\| \qquad \forall u,v\in H \qquad \textit{(boundedness)}$$

$$\beta\|u\|^2\le B[u,u] \qquad \forall u\in H \qquad \textit{(coercivity)}$$

Then, if $f: H\to\mathbb{R}$ is a bounded linear map (i.e. $f\in H^*\simeq H$), then $\exists!\ u\in H$ such that:

$$B[u,\cdot]=\langle f,\cdot\rangle$$

where $\langle\cdot,\cdot\rangle$ is the pairing between H^* ($\simeq H$) and H (i.e. $\langle f,v\rangle = f(v)$).

Remark: B is like an inner product (although we do not have symmetry). So, since B is a pairing of H with itself, it gives a map $H\to H^*$ via $v\mapsto B[v,\cdot]$. By the Riesz Representation theorem, we know that $f=(w,\cdot)$ for some $w\in H$ for each $f\in H^*$. So we just need to show that the map is bijective, so we get that $\exists v\in H$ we $B[v,\cdot]=(w,\cdot)=f$. Although if B was an inner product on H, then the result simply follows from the Riesz Representation theorem applied to $(H,B[\cdot,\cdot])$.

Proof. We split this proof up into lots of mini steps, to make it transparent.

Step 1: For each fixed $u\in H$, the map $B[u,\cdot]$ is a bounded linear map on H. So hence by the Riesz representation theorem (for Hilbert spaces, which tells us $H\cong H^*$ via $x\mapsto(\cdot,x)$), $\exists!\ w\in H$ with:

$$B[u,\cdot]=(w,\cdot)$$

i.e. we know by the RRT, every element of the dual is of this form. We hence we get a map $A: H\to H$, sending $u\mapsto w$. We write: $Au=w$.

Step 2: We claim that $A : H \to H$ is a bounded linear operator, i.e. $A \in B(H)$. Indeed, if $\lambda_1, \lambda_2 \in \mathbb{R}$ and $u_1, u_2 \in H$, then for each $v \in H$, we have:

$$\begin{aligned}(A(\lambda_1 u_1 + \lambda_2 u_2), v) &= B[\lambda_1 u_1 + \lambda_2 u_2, v] \\ &= \lambda_1 B[u_1, v] + \lambda_2 + B[u_2, v] \\ &= \lambda_1 (Au_1, v) + \lambda_2 (Au_2, v) \\ &= (\lambda_1 Au_1 + \lambda_2 Au_2, v)\end{aligned}$$

by linearity and B and the inner product. So hence as this is true for all $v \in H$, by the non-degeneracy of the inner product, we have $A(\lambda_1 u_1 + \lambda_2 u_2) = \lambda_1 Au_1 + \lambda Au_2$, and so A is linear.

To see that A is bounded, simply note that:

$$||Au||^2 = (Au, Au) = B[u, Au] \leq \alpha ||u|| \cdot ||Au||$$

by the boundedness of B. So hence $||Au|| \leq \alpha ||u||$.

Step 3: Next we assert that A is injective and that Image(A) is closed. So note:

$$\beta ||u||^2 \leq B[u, u] = (Au, u) \leq ||Au|| \cdot ||u||$$

where the last inequality is by the Cauchy-Schwarz inequality for the inner product. So hence we see:

$$||u|| \leq \frac{1}{\beta} ||Au||$$

which shows that A is injective. This also shows that Image(A) is closed, since if $(A(u_n))_n$ converges, then this tells us that $(u_n)_n$ is a Cauchy sequence in H, which is a Hilbert space, and so hence complete. Hence $(u_n)_n$ converges. Then by continuity of A (as it is bounded and linear), we get that $\lim_{n\to\infty} A(u_n) = A(\underbrace{\lim_{n\to\infty} u_n}_{\in H})$, and so the limit lies in H. Hence Image(A) contains all of its limit points, and so is closed.

Step 4: Now we show that Image(A) $= H$.

Since Image(A) is closed and H is a Hilbert space, we have:

$$H = \text{Image}(A) \oplus \text{Image}(A)^{\perp}$$

where $\perp$ denotes the orthogonal complement. Hence if Image(A) $\neq H$, then $\exists w \in \text{Image}(A)^{\perp}$ with $w \neq 0$. But then:

$$\beta ||w||^2 \leq B[w, w] = (Aw, w) = 0$$

since $Aw \in \text{Image}(A)$ and $w \in \text{Image}(A)^{\perp}$ and so these are orthogonal. Hence we get $||w|| = 0$, i.e. $w = 0$, a contradiction. So hence we must have Image(A) $= H$, i.e. A is surjective.

So hence A is a bijection, and so is invertible. But then if $u = A^{-1}(w)$, we have:

$$||u|| \leq \frac{1}{\beta} ||Au|| \quad \Longrightarrow \quad ||A^{-1}(w)|| \leq \frac{1}{\beta} ||w||$$

which tells us that A^{-1} is bounded.

Step 5: Now consider the problem we want to solve: $B[u, \cdot] = \langle f, \cdot \rangle$.

From the Riesz representation applied to $\langle f, \cdot \rangle \equiv f$, we know $\exists!\ w \in H$ such that:

$$(w, \cdot) = f.$$

So let $u = A^{-1}(w)$, which exists by the above. Then:

$$B[u,v] = (Au,v) = (w,v) = f(v) \quad \forall v \in H$$

i.e. $B[u,\cdot] = f$. So hence we have shown the existence of such a u.

Step 6: For uniqueness, just note that if both u, u' satisfy $B[u,\cdot] = f$, then we see that $B[u-u',v] = 0$ for all $v \in H$. So set $v = u - u'$. Then we get:

$$\|u-u'\|^2 \leq \frac{1}{\beta} B[u-u', u-u'] = 0$$

and so hence $u = u'$. So u is unique, and so done.

□

Note: If B is symmetric, then we get a much simpler proof without as many assumptions on B. Indeed, in this case we can just define:

$$((u,v)) := B[u,v]$$

and then $((\cdot,\cdot))$ is a new inner product on H. So hence we can just directly apply the Riesz representation theorem (for Hilbert spaces) to this new inner product, to get the result.

So the Lax-Milgram theorem is important because it does not require the symmetry of B.

We now prove what we can about our scenario for Lax-Milgram.

Theorem 4.2 (Energy Estimates for Weak Solutions, $B(\cdot,\cdot)$). *Suppose $a^{ij} = a^{ji}, b^i, c \in L^\infty(U)$, and that L is uniformly elliptic. Then, if B is given as in Definition 4.3, then $\exists \alpha, \beta > 0$ and $\gamma \geq 0$ such that:*

(i) $|B[u,v]| \leq \alpha \|u\|_{H^1(U)} \cdot \|v\|_{H^1(U)}$ *for all* $u, v \in H_0^1(U)$.

(ii) $\beta \|u\|^2_{H^1(U)} \leq B[u,u] + \gamma \|u\|^2_{L^2(U)}$. *[Garding's Inequality]*

Proof. (i): We have:

$$\begin{aligned}
|B[u,v]| &\leq \sum_{i,j} \|a^{ij}\|_{L^\infty(U)} \cdot \int_U |Du||Dv| \, \mathrm{d}x + \sum_i \|b^i\|_{L^\infty(U)} \cdot \int_U |Du||v| \, \mathrm{d}x + \|c\|_{L^\infty(U)} \cdot \int_U |u||v| \, \mathrm{d}x \\
&\leq C_1 \|Du\|_{L^2(U)} \|Dv\|_{L^2(U)} + C_2 \|Du\|_{L^2(U)} \|v\|_{L^2(U)} + C_3 \|u\|_{L^2(U)} \|v\|_{L^2(U)} \\
&\leq C(\|u\|_{L^2(U)} + \|Du\|_{L^2(U)}) \cdot (\|v\|_{L^2(U)} + \|Dv\|_{L^2(U)}) \\
&\leq \alpha \|u\|_{H^1(U)} \|v\|_{H^1(U)}
\end{aligned}$$

for some α, where in the second line we have used the Cauchy-Schwarz inequality for integrals (i.e. Hölder 's inequality for $p = 2$), and in the last line we have used the definition of the Sobolev norm since $\|u\|_{L^2(U)}, \|Du\|_{L^2(U)} \leq \|u\|_{H^1(U)}$.

(ii): We need to use the uniform ellipticity condition now. So note that we have:

$$\begin{aligned}\theta \int_U |Du|^2 \, \mathrm{d}x &\leq \int_U \sum_{i,j=1}^n a^{ij}(x) u_{x_j} u_{x_i} \, \mathrm{d}x \\ &= B[u,u] - \int_U \left(\sum_i b^i u_{x_i} u + cu^2 \right) \mathrm{d}x \\ &\leq B[u,u] + \sum_{i=1}^n \|b^i\|_{L^\infty(U)} \int_U |Du||u| \, \mathrm{d}x + \|c\|_{L^\infty} \int_U |u|^2 \, \mathrm{d}x\end{aligned}$$

where we have used uniform ellipticity in the first line (here, θ is the constant in the uniform ellipticity condition). Now by "Young's inequality with ε"[(vii)] we have:

$$\int_U |Du||u| \, \mathrm{d}x \leq \varepsilon \int_U |Du|^2 \, \mathrm{d}x + \frac{1}{4\varepsilon} \int_U |u|^2 \, \mathrm{d}x \quad \forall \varepsilon > 0.$$

So choose $\varepsilon > 0$ small enough so that $\varepsilon \cdot \sum_{i=1}^\infty \|b^i\|_{L^\infty(U)} < \frac{\theta}{2}$. Then:

$$\theta \int_U |Du|^2 \, \mathrm{d}x \leq B[u,u] + \frac{\theta}{2} \int_U |Du|^2 \, \mathrm{d}x + \gamma \int_U |u|^2 \, \mathrm{d}x$$

i.e.

$$\frac{\theta}{2} \int_U |Du|^2 \, \mathrm{d}x \leq B[u,u] + \gamma \|u\|^2_{L^2(U)}$$

i.e.

$$\frac{\theta}{2} \|Du\|^2_{L^2(U)} \leq B[u,u] + \gamma \|u\|^2_{L^2(U)}.$$

Then adding $\frac{\theta}{2}\|u\|^2_{L^2(U)}$ to each side (so that the LHS becomes $\frac{\theta}{2}\|u\|^2_{H^1(U)}$), or alternatively by using the Poincaré inequality, we are done.

□

Remark: If $B[\cdot,\cdot]$ corresponds to an operator where $b^i = 0$ for all i and $c \geq 0$, then by the above we immediately get:

$$\theta \|Du\|^2_{L^2(U)} \leq B[u,u].$$

Then recalling Poincaré's inequality, $\|u\|^2_{L^2(U)} \leq C\|Du\|^2_{L^2(U)}$ for all $u \in H^1_0(U)$, we don't have to add $\|u\|_{L^2(U)}$ at the end, and so we can take $\gamma = 0$ in Garding's inequality (Theorem 4.2 (ii)) above, and hence we get exactly the assumptions of the Lax-Milgram theorem. So in particular, we can use Lax-Milgram to solve Laplace's equation.

So observe that if $\gamma > 0$ in these energy estimates (Theorem 4.2), then $B[\cdot,\cdot]$ does not precisely satisfy the conditions of Lax-Milgram (Theorem 4.1). The following result confronts this possibility.

[(vii)] i.e. $0 \leq \left(a\sqrt{\varepsilon} - \frac{1}{2\sqrt{\varepsilon}} b\right)^2 = \varepsilon a^2 - ab + \frac{1}{4\varepsilon} b^2$

Theorem 4.3 (A First Existence Theorem for Weak Solutions). *Let U, L be as above. Then, $\exists \gamma \geq 0$ such that, for any $\mu \geq \gamma$ and any $f \in L^2(U)$, $\exists!$ weak solution $u \in H_0^1(U)$ of the system:*

$$\begin{cases} Lu + \mu u = f & \text{in } U \\ u = 0 & \text{on } \partial U. \end{cases} \tag{4.3}$$

Moreover, $\|u\|_{H^1(U)} \leq C\|f\|_{L^2(U)}$ for some $C \geq 0$ independent of u ($C = C(L,u)$).

Proof. Apply Theorem 4.2 to L, to get the existence of some γ as in Theorem 4.2. Then let $\mu \geq \gamma$. Then define $L_\mu = L + \mu$ (so this system is just as before, with this new L). Then define the bilinear form:

$$B_\mu[u,v] := B[u,v] + \mu \cdot (u,v)_{L^2(U)}$$

for all $u, v \in H^1(U)$, i.e. this is just the bilinear form corresponding to L_μ.

Then by Theorem 4.2 for $B[\cdot,\cdot]$, we can see that this means that $B_\mu[\cdot,\cdot]$ satisfies the conditions of Lax-Milgram, i.e. the RHS of Theorem 4.2 (ii) is $\leq B_\mu[u,u]$ (as $\mu \geq \gamma$).

So now fix $f \in L^2(U)$ and set: $\langle f, v\rangle := (f,v)_{L^2(U)}$ (i.e. each such f defines an element of $\left(L^2(U)\right)^*$ via the inner product). This is a bounded linear functional on $L^2(U)$ (i.e. it is just $(f,\cdot)_{L^2(U)}$), and thus is a bounded linear functional on $H_0^1(U)$ as well.

So apply the Lax-Milgram theorem to find a unique $u \in H_0^1(U)$ with:

$$B_\mu[u,\cdot] = \langle f, \cdot\rangle := (f,\cdot)_{L^2(U)}$$

on $H_0^1(U)$, i.e. u is the unique weak solution to (4.3).

To establish the last claim, from the Garding inequality (Theorem 4.2 (ii)) we have:

$$\beta\|u\|^2_{H^1(U)} \leq B_\mu[u,u] = (f,u)_{L^2(U)} \leq \|f\|_{L^2(U)}\|u\|_{L^2(U)}$$

where we have used that $B_\mu[u,\cdot] = (f,\cdot)_{L^2(U)}$ and the Cauchy-Schwarz inequality for L^2. Hence this shows

$$\beta\|u\|_{H^1(U)} \leq \|f\|_{L^2(U)}.$$

□

Remark: This gives us a solution the the BVP (Boundary Value Problem), but at the moment, we just have $u \in H_0^1(U)$. Addressing this (i.e. to get better regularity) will take us to *elliptic regularity*.

We also had to introduct μ, which was not good (although if $\gamma = 0$ in Garding's inequality then there is no need to introduce μ). To fix this we will look at compactness (i.e. weak compactness of Hilbert spaces).

[i.e. for Hilbert spaces, $H^* \cong H$ canonically, and so all Hilbert spaces are reflexive. Hence the unit ball in a Hilbert space is weakly compact (this is Banach-Alaoglu). If H is separable, then we get that the weak topology is also metrizable, and so get weak sequential compactness.]

4.2. **Compactness Results in PDE.**

In the study of PDE, compactness plays as important role. The basic result to bear in mind is Balzano-Weierstrass, which tells us that any bounded real sequence has a convergent subsequence. We will see how similar results arise in the study of PDE.

Definition 4.4. *Suppose H is a Hilbert space with inner product $(\cdot,\cdot)$, and $(u_n)_n \subset H$ is a sequence. Then we say that $(u_n)_n$* ***converges weakly to $u \in H$****, written $u_n \rightharpoonup u$, if:*

$$(u_n, w) \to (u, w) \quad w \in H.$$

[Note that this is what weak convergence becomes in Hilbert spaces, since we can identify H^* with $\{(\cdot, w) : w \in H\}$, i.e. all linear functionals are just of this form.]

Remark: Contrast weak convergence with strong convergence, which says $u_n \to u$ if $\|u_n - u\| \to 0$.

Note: Weak limits are unique when they exist: indeed, suppose

$$u_n \rightharpoonup u \quad \text{and} \quad u_n \rightharpoonup u'.$$

Then for any $w \in H$, we have:

$$(w, u') - (w, u) = \lim_{m\to\infty}(w, u_m) - \lim_{m\to\infty}(w, u_m) = 0$$

by definition of weak convergence and by the uniqueness of limits in $\mathbb{R}$. So hence we see $(w, u' - u) = 0$ for all $w \in H$, and so hence $u' = u$ by non-degeneracy of the inner product (or by, e.g. Hahn-Banach). $\square$

Note: Strong convergence $\Rightarrow$ Weak convergence (by continuity/dominated convergence).

Theorem 4.4 (Separable Banach-Alaoglu for Hilbert Spaces)**.** *Let H be a* ***separable*** *Hilbert space, and suppose $(u_n)_n \subset H$ is a bounded sequence, i.e. $\|u_n\| \le K$ for all n.*

Then, $(u_n)_n$ has a weakly convergent subsequence, i.e. $\exists (u_{m_j})_j$ such that $u_{m_j} \rightharpoonup u$ for some $u \in H$ with $\|u\| \le K$.

Proof. This result is immediate from Banach-Alaoglu and related results, which tells us that the unit ball in H is weakly compact (by reflexivity) and it is weakly metrizable (by separability), and so hence is weakly sequentially compact.

Alternatively, we can go into more detail and use a diagonal argument, which we give now.

Let $(e_i)_{i=1}^{\infty}$ be an orthonormal basis for H (a countable such one exists by separability - throw away all linearly dependent stuff from a countable dense set and then use Gram-Schmidt).

Then consider the real sequence $((e_1, u_n))_n \subset \mathbb{R}$. Then by Cauchy-Schwarz, we know:

$$|(e_1, u_n)| \le \|e_1\|\|u_n\| \le K$$

i.e. this is a bounded sequence in $\mathbb{R}$. So hence by Balzano-Weierstrass, we can find a convergent subsequence $(e_1, u_{m_n}) \to c_1$, for some $|c_1| \leq K$. Set $u_{1,n} = u_{m_n}$.

Then repeat the same argument with $(e_2, u_{1,n})$ to get a further subsequence u_{1,m_k} such that $(e_2, u_{1,m_k}) \to c_2$, and $|c_2| \leq K$. Note that as this is a subsequence, we still have $(e_1, u_{1,m_k}) \to c_1$.

Now set $u_{2,n} := u_{1,m_k}$ and proceed inductively, constructing for each $l \in \mathbb{N}$, a sequence $(u_{l,k})_k$, where $(u_{l+1,k})_{k=1}^{\infty}$ is a subsequence of $(u_{l,k})_{k=1}^{\infty}$. So we have:

$$(e_j, u_{l,k}) \to c_j \quad \text{as } k \to \infty \quad \text{for some } |c_j| \leq K, \quad \forall j \leq l.$$

So we have the convergence for any number of finitely many coordinates. Now we just need to extend to the infinite.

So take the diagonal sequence: let $v_l := u_{l,l}$. Then by construction, $(v_l)_l$ is a subsequence of $(u_m)_m$, and we have $(e_j, v_l) \to c_j$ as $l \to \infty$ for all j, as eventually always $(v_j)_j$ is a subsequence of $(u_{l,n})$.

[Now we want to set $u = \sum_i c_i e_i$ to be our candidate weak limit - but first we must check that this is in the space.]

So note:

$$\begin{aligned}
\sum_{j=1}^{p} |c_j|^2 = \sum_{j=1}^{p} \lim_{l\to\infty} |(e_j, v_l)|^2 &= \lim_{l\to\infty} \sum_{j=1}^{p} |(e_j, v_l)|^2 \\
&\leq \sup_l \sum_{j=1}^{p} |(e_j, v_l)|^2 \\
&\leq \sup_l \|v_l\|^2 \quad \text{, by Bessel's inequality} \\
&\leq K^2 \qquad \text{as this is a subsequence of the original sequence}
\end{aligned}$$

where in the first line, we have exchanged the limit with the sum as the sum is finite. Hence we get a uniform bound in these partial sums, and so hence taking $p \to \infty$ we get:

$$\sum_{j=1}^{\infty} |c_j|^2 \leq K^2 < \infty.$$

So hence $u = \sum_{i=1}^{\infty} c_i e_i$ converges in H (as the partial sums are Cauchy by the above and H is complete), and by the above we have $\|u\| \leq K$.

We also know by the above that $(e_j, v_l) \to c_l = (e_j, u)$ for all j.

So we know the weak convergence condition holds on an orthonormal basis. So we just need to check that

$$(w, v_l) \to (w, u) \quad \text{as } l \to \infty \quad \forall w \in H.$$

So fix $w \in H$. Then write $w = \sum_{i=1}^{p} (e_i, w) e_i + w_p$ in H. Then as the $(e_j)_j$ form an orthonormal basis, we know $\sum_{i=1}^{p} (e_i, w) e_i \to w$, i.e. $w_p \to 0$ in H as $p \to \infty$ (as $\|w\| < \infty$). So:

$$|(w, v_l) - (w, u)| \leq |(w - w_p), v_l - u)| + |(w_p, v_l - u)|.$$

Now, we have by Cauchy-Schwarz,

$$|(w_p, v_l - u)| \leq \|w_p\| \cdot \|v_l - u\| \leq (\|u\| + \|v_l\|)\|w_p\| \leq 2K\|w_p\| \to 0$$

as $p \to \infty$, and this is true for all l (i.e. the convergence is independent of l).

So hence fix $\varepsilon > 0$. Then pick p such that $|(w_p, v_l - u)| < \varepsilon/2$ (independent of l by the above). Now, $w - w_p$ is a finite linear combination of the e_j's, and we know $(e_j, v_l - u) \to 0$ as $l \to \infty$. So hence we can pick L such that $\forall l \geq L$, we have $|(w - w_p, v_l - u)| < \varepsilon/2$.

Then, for all $l \geq L$, we have $|(w, v_l) - (w, u)| < \varepsilon$. So hence we are done, as $\varepsilon > 0$ was arbitrary. So $v_l \rightharpoonup u \in H$, and so done.

□

Lemma 4.1 (The Poincaré Inequality Revisited). *Suppose $u \in H^1(\mathbb{R}^n)$. Then let $Q = \prod_{i=l}^n [\xi_i, \xi_i + L]$ be a box of side length L. Then we have:*

$$\|u\|^2_{L^2(Q)} \leq \frac{1}{|Q|}\left(\int_Q u \, \mathrm{d}x\right)^2 + \frac{nL^2}{2}\|Du\|^2_{L^2(Q)}.$$

Note: Since these integrals are only on Q, if consider $u - c\zeta$, where ζ is a smooth function in $H^1(\mathbb{R}^n)$ which vanishes outside of some compact set containing Q and is 1 on Q, then $\int_Q (u - c\zeta) = 0$ if we choose $c = \frac{1}{|Q|}\int_Q u \, \mathrm{d}x$. So hence applying the above Poincaré inequality to this new function $u - c\zeta$, we get:

$$\|u - \lambda\|^2_{L^2(Q)} \leq \frac{nL^2}{2}\|Du\|^2_{L^2(Q)}$$

where $\lambda = \frac{1}{|Q|}\int_Q u \, \mathrm{d}x$ is the average of u on Q. So the Poincaré inequality tells us that we can bound the difference between u and its average by its derivative.

Proof. By approximation we can assume $u \in C^\infty(\overline{Q})$ (i.e. such functions are dense in $H^1(\mathbb{R}^n)$). Then for $x, y \in Q$, write:

$$u(x) - u(y) = \int_{y_1}^{x_1} \frac{\mathrm{d}}{\mathrm{d}t}u(t, x_2, \ldots, x_n) \, \mathrm{d}t + \int_{y_2}^{x_2} \frac{\mathrm{d}}{\mathrm{d}t}u(y_1, t, x_3, \ldots, x_n) \, \mathrm{d}t + \cdots$$

$$\cdots + \int_{y_n}^{x_n} \frac{\mathrm{d}}{\mathrm{d}t}u(y_1, \ldots, y_{n-1}, t) \, \mathrm{d}t$$

(as all cross-terms cancel when integrating). So squaring, using Cauchy-Schwarz in $\mathbb{R}^n$[(viii)] we get:

$$u(x)^2 + u(y)^2 - 2u(x)u(y) \leq n\left[\left(\int_{y_1}^{x_1} \frac{\mathrm{d}}{\mathrm{d}t}u(t, x_2, \ldots, x_n) \, \mathrm{d}t\right)^2 + \cdots + \left(\int_{y_n}^{x_n} \frac{\mathrm{d}}{\mathrm{d}t}u(y_1, \ldots, y_{n-1}, t) \, \mathrm{d}t\right)^2\right]$$

[(viii)] i.e. $a_1 + \cdots + a_n = (1, 1, \ldots, 1) \cdot (a_1, \ldots, a_n) \leq \sqrt{(1^2 + \cdots + 1^2)(a_1^2 + \cdots + a_n^2)} = \sqrt{n}\|a\|$

Now integrating this over Q, i.e. acting on by $\int_Q \mathrm{d}x \int_Q \mathrm{d}y$, the LHS becomes:

$$\int_Q \mathrm{d}x \int_Q \mathrm{d}y[u(x)^2+u(y)^2-2u(x)u(y)] = 2|Q|\cdot\|u\|^2_{L^2(Q)} - 2\left[\int_Q u(x)\ \mathrm{d}x\right]^2.$$

Now note that if $I_1 = \left[\int_{y_1}^{x_1} \frac{\mathrm{d}}{\mathrm{d}t}u(t,x_2,\dots,x_n)\ \mathrm{d}t\right]^2$, then by Cauchy-Schwarz for integrals, we have (in general):

$$\left(\int_y^x g\ \mathrm{d}t\right)^2 \leq \left(\int_y^x 1^2\ \mathrm{d}t\right)\left(\int_y^x g^2\ \mathrm{d}t\right) = |x-y|\int_y^x g^2\ \mathrm{d}t$$

and so as $|x_i - y_i| \leq L$ here, we get:

$$I_1 \leq L\int_{y_1}^{x_1}\left(\frac{\mathrm{d}}{\mathrm{d}t}u(t,x_2,\dots,x_n)\right)^2\ \mathrm{d}t \leq L\int_{\xi_1}^{\xi_1+L}\left(\frac{\mathrm{d}}{\mathrm{d}t}u(t,x_2,\dots,x_n)\right)^2\ \mathrm{d}t.$$

Thus as this bound does not depend on y, only x, and it does not depend on x_1, we have:

$$\int_Q\int_Q \mathrm{d}x\mathrm{d}y\ I_1 \leq L^2|Q|\cdot\|D_1u\|^2_{L^2(Q)}$$

where the factor of $|Q|$ comes from the no y dependence and the extra factor of L comes from the no x_1-dependence. So similarly estimating the other terms on the RHS, we find:

$$2|Q|\cdot\|u\|^2_{L^2(Q)} - 2\left[\int_Q u\ \mathrm{d}x\right]^2 \leq L^2 n|Q|\sum_{i=1}^n \|D_iu\|^2_{L^2(Q)} = nL^2|Q|\cdot\|Du\|^2_{L^2(Q)}$$

which is what we wanted to show.

□

Remark: An alternative proof of the Poincaré inequality above is as follows. Due to convexity of Q and the fundamental theorem of calculus we can write:

$$u(x)-u(y) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}(u(tx+(1-t)y))\ \mathrm{d}t = \int_0^1 (x-y)\cdot Du(tx+(1-t)y)\ \mathrm{d}t.$$

Then squaring and integrating we get:

$$\begin{aligned} 2|Q|\cdot\|u\|^2_{L^2(Q)} - 2\left(\int_Q u\ \mathrm{d}x\right)^2 &\leq \int_Q\int_Q \underbrace{|x-y|^2}_{\leq nL^2}\cdot|Du(tx+(1-t)y)|^2\ \mathrm{d}t\mathrm{d}x\mathrm{d}y \\ &\leq nL^2\int_Q\left(\int_0^1\int_{tQ}|Du(z)|^2\ \mathrm{d}z\mathrm{d}t\right)\mathrm{d}y \\ &\leq nL^2\cdot|Q|\cdot 1\cdot\|Du\|^2_{L^2(Q)} \end{aligned}$$

which upon rearranging gives us the result.

□

With the above Poincaré estimate, we are ready to prove the most important compactness result for us:

Theorem 4.5 (Rellich Compactness/Rellich-Kondrachov). *Suppose $U \subset \mathbb{R}^n$ is open, bounded with C^1 boundary. Then let $(u_m)_m \subset H^1(U)$ be a bounded sequence, i.e. $\|u_m\|_{H^1(U)} \leq K$.*

Then, $\exists u \in H^1(U)$ and a subsequence $(u_{m_j})_j$ such that:

$$u_{m_j} \rightharpoonup u \quad \text{in } H^1(U) \quad \textbf{and} \quad u_{m_j} \to u \quad \text{in } L^2(U).$$

Proof. By the extension theorem (Theorem 3.5) we can assume that $u_m \in H_0^1(Q)$ for some large cube Q, with $U \subset\subset Q$. So we can work in $H_0^1(U)$.

Then by weak compactness of $H_0^1(U)$, we know that $\exists u \in H_0^1(U)$ and a subsequence $(u_{m_j})_j$ such that $u_{m_j} \rightharpoonup u$ (by Theorem 4.4 - we know $H_0^1(U)$ is separable as $W^{k,p}(U)$ is separable for all $p < \infty$, as it is inherited from $L^p(U)$).

So set $w_j = u_{m_j}$. Then we just need to show: $\|w_j - u\|_{L^2(Q)} \to 0$.

[So the weak convergence is easy. THe hard bit is getting the L^2 convergence as well, but this is what the Poincaré Inequality gives.]

Fix $\delta > 0$. Then we can cover Q exactly by $k = k(\delta)$ (some number) cubes of side length $l < \delta$, such that the cubes intersect only a their faces. Call these (finitely many) cubes $\{Q_a\}_{a=1}^k$. Then by applying the Poincaré inequality, Lemma 4.1, we get (as the covering here is disjoint except on sides, which are sets of measure zero):

$$\begin{aligned}
\|w_j - u\|_{L^2(Q)} &= \sum_{a=1}^k \|w_j - a\|_{L^2(Q_a)} \\
&\leq \sum_{a=1}^k \left[\frac{1}{|Q_a|} \left(\int_{Q_a} (w_j - u) \, \mathrm{d}x \right)^2 + \frac{n^2\delta^2}{2} \|Dw_j - Du\|_{L^2(Q_a)}^2 \right] \\
&= \frac{n^2\delta^2}{2} \|Dw_j - Du\|_{L^2(Q)}^2 + \sum_{a=1}^k \frac{1}{|Q_a|} \left(\int_{Q_a} (w_j - u) \, \mathrm{d}x \right)^2.
\end{aligned}$$

Now, $\|Dw_j - Du\|_{L^2(Q)}^2 \leq \tilde{K}$, as all of these are in $H_0^1(U)$ and so have bounded derivatives (also, the w_j converge weakly by assumption, and so have bounded $\|\cdot\|_{H^1(U)}$ norm). So hence for $\delta > 0$ small enough, we have (for $\varepsilon > 0$ given)

$$\frac{n^2\delta^2}{2} \|Dw_j - Du\|_{L^2(Q)}^2 < \frac{\varepsilon}{2}$$

for all j large (i.e. choose this δ which is independent of j).

Now note that the map: $u \longmapsto \int_Q u(x) \, \mathrm{d}x$ is a bounded linear map on $H^1(Q)$, and so by the weak convergence $w_j \rightharpoonup u$, we have $\int_{Q_a} w_j - u \, \mathrm{d}x \to 0$ for all a (as by definition of weak convergence, all functionals $\to 0$). So hence for j large enough, we have (i.e. now for this δ choose j sufficiently large)

$$\sum_{a=1}^{k(\delta)} \frac{1}{|Q_a|}\left(\int_{Q_a}(w_j - a)\,\mathrm{d}x\right)^2 \leq \frac{\varepsilon}{2}.$$

So hence combining, we see that for all j sufficiently large, we have $\|w_j - u\|^2_{L^2(Q)} < \varepsilon$, and so done.

□

Remarks:

- $H^1(U)$ is separable (see Example Sheet 3).
- The same result (Theorem 4.5) holds with $H^1(U)$ replaced by $H^1_0(U)$.

With this, we are ready to start looking at solving elliptic BVP's.

4.3. **Fredholm Alternative and Spectra of Elliptic PDEs.**

Definition 4.5. *Let H be a Hilbert space. Then a bounded operator $K : H \to H$ is* ***compact*** *if for each bounded sequence $(u_m)_m \subset H$ (i.e. $\|u_m\| \leq M$), $\exists$ a subsequence $(u_{m_j})_j$ such that $(K(u_{m_j}))_j$ converges strongly in H.*

Example 4.1 (Key Example)**.** *Suppose $K : L^2(U) \to H^1(U)$ is a bounded linear operator. Then since we know $H^1(U) \hookrightarrow L^2(U)$ (as it is a subset), we can think of K as a map $L^2(U) \to L^2(U)$.*

> ***Claim:*** *$K : L^2(U) \to L^2(U)$ is a compact operator.*
> *Proof.* If $(u_m)_m \subset L^2(U)$ is a bounded sequence, so $\|u_m\| \leq C$, then as K is bounded, we have $\|K(u_m)\|_{H^1(U)} \leq \|K\| \cdot \|u_m\|_{L^2(U)} \leq \tilde{C}$. So hence by Rellich compactness, $\exists$ a subsequence $u_{m_j} \to u$ in $L^2(U)$. So hence K is compact.
>
> □

So hence Rellich compactness "helps to improve regularity", and allows us to use spectral theory of such maps.

This is useful for us, since a differential operator is just a map $D : H^{n-1}(U) \to L^2(U)$, where n is the order of the PDE (i.e. map a function to a given f, i.e. $Lu = f$). So if $n = 2$, finding a solution is just like finding an inverse map $D : L^2(U) \to H^1(U)$, i.e. a map sending f to u, the data to the solution. Hence to find the spectrum of the differential operator D, we can apply the above example to D^{-1} to see that D^{-1} is automatically a compact operator, and so has a nice spectrum. We can then use this to find the spectrum of D.

□

We now give some standard results from compact operators.

Theorem 4.6 (Fredholm Alternative for Compact Operators). *Let $K : H \to H$ be a real compact operator, with H a Hilbert space. Then:*

(i) $\ker(I-K)$ *is finite dimensional.*

(ii) *Image*$(I-K)$ *is closed.*

(iii) *Image*$(I-K) = \ker(I-K^\dagger)^\perp$.

(iv) $\ker(I-K) = \{0\} \iff$ *Image*$(I-K) = H$.

(v) $\dim(\ker(I-K)) = \dim(\ker(I-K^\dagger))$

where $I : H \to H$ is the identity operator, and $K^\dagger$ is the adjoint map of K.

Proof. None given.

[e.g. for (i), suppose not, then $\exists$ an infinite orthonormal set $(u_n)_n$ in $\ker(I-K)$, i.e. $K(u_n) = u_n$ for all n. But then $\|u_n - u_m\|^2 = 2$, and so we have a bounded sequence $(u_n)_n$ with $(K(u_n))_n = (u_n)_n$ having no convergent subsequence (as no subsequence is Cauchy and so no subsequence can converge).]

□

[In some sense, compact operators are"operators for which Physicist arguments work", i.e. "if something holds for matrices ($\equiv$ finite rank operators), then it works for Hilbert spaces". This is because compact operators are the closure of finite rank operators with respect to the operator norm [**Exercise** to check.]]

The Fredholm alternative tells us that the spectrum of compact operators behaves nicely.

Definition 4.6. *Suppose $A : H \to H$ is a bounded linear operator. Then, the* ***resolvent set of A*** *is:*

$$\rho(A) := \{\lambda \in \mathbb{R} : A - \lambda I \text{ is bijective/invertible}\}.$$

Definition 4.7. *The* ***real spectrum*** *of A is:* $\sigma(A) := \mathbb{R}\backslash\rho(A)$.

Definition 4.8. *We say that $\eta \in \sigma(A)$ belongs to the* ***point spectrum of A****, denoted* $\boldsymbol{\sigma_p(A)}$*, if* $\ker(A-\eta I) \neq \{0\}$. *[i.e. the reason it is not invertible is because it is not injective.]*

In this case, $w \in \ker(A-\eta I)$ is called an associated ***eigenvector****.*

Theorem 4.7 (The Spectrum of a Compact Operator)**.** *Assume* $\dim(H) = \infty$ *(or else* H *is a matrix and it is simple), and let* $K : H \to H$ *be compact. Then:*

(i) $0 \in \sigma(K)$

(ii) $\sigma(K)\backslash\{0\} = \sigma_p(K)\backslash\{0\}$

(iii) $\sigma(K)\backslash\{0\}$ *is either finite, or a sequence tending to 0 (in particular,* $\sigma(K)$ *is countable).*

Moreover, if K *is self-adjoint, i.e.* $K = K^\dagger$*, then* $\exists$ *a countable orthonormal basis of* H *consisting of eigenvectors of* K*.*

Proof. See Linear Analysis Part II. $\square$

Recall that $Lu = -\sum_{i,j=1}^n (a^{ij}(x)u_{x_j})_{x_i} + \sum_{i=1}^n b^i(x)u_{x_i} + c(x)u$ is assumed to be uniformly elliptic on U, which is open, bounded with C^1-boundary. The associated bilinear form is:

$$B[u,v] = \int_U \left(\sum_{i,j=1}^n a^{ij}(x)u_{x_i}v_{x_j} + \sum_{i=1}^n b^i(x)u_{x_i}v + c(x)uv \right) \mathrm{d}x.$$

We are interested in the BVP:

$$\begin{cases} Lu = f & \text{in } U \\ u = 0 & \text{on } \partial U. \end{cases}$$

We define the **formal adjoint of L** by:

$$L^\dagger v := -\sum_{i,j=1}^n \left(a^{ij}v_{x_i}\right)_{x_j} - \sum_{i=1}^n b^i(x)v_{x_i} + \left(c - \sum_{i=1}^n b^i_{x_i} \right) v$$

which satisfies:

$$(L\varphi, \psi)_{L^2(U)} = (\varphi, L^\dagger\psi)_{L^2(U)} \quad \forall \varphi, \psi \in C^\infty_{\mathrm{c}}(U)$$

(indeed this is how the adjoint is defined - integrate by parts to verify that $L^\dagger$ does take this form).

The **adjoint bilinear form** is defined to be:

$$B^\dagger[v,u] := B[u,v]$$

(note that $B^\dagger$ is only the same as the bilinear defined by $L^\dagger$ (as in the L case) if $b^i \in C^1(\overline{U})$[(ix)], but this $B^\dagger$ makes sense for any $b^i \in L^\infty(U)$).

Definition 4.9. *We say that* $v \in H^1_0(U)$ *is a* ***weak solution to the adjoint problem****, which is:*

$$\begin{cases} L^\dagger v = f & \textit{in } U \\ v = 0 & \textit{on } \partial U \end{cases}$$

if it satisfies:

$$B^\dagger[\cdot, v] = (f, \cdot)_{L^2(U)} \quad \textit{as maps on } H^1_0(U).$$

[(ix)] Since then $B[u,v] = (Lu,v) = (v, L^\dagger v) = (L^\dagger v, u) = B^\dagger[v,u]$.

Now we can state the Fredholm atlernative here:

Theorem 4.8 (Fredholm Alternative for Elliptic PDEs)**.** *Consider the 'usual' BVP:*

$$\begin{cases} Lu = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$$

for L a uniformly elliptic operator on an open, bounded set U with C^1-boundary. Then either:

(a) For each $f \in L^2(U)$, $\exists!$ weak solution $u \in H_0^1(U)$ to this usual BVP, ***or***

(b) $\exists$ a weak solution $u \in H_0^1(U)$ to the homogeneous problem (i.e. when $f \equiv 0$), where $u \not\equiv 0$.

Moreover, if (b) holds, then the dimension of the space $N \subset H_0^1(U)$, where $N = \{$Solutions to the homogeneous BVP$\}$ equals the dimension of the space $N^\dagger \subset H_0^1(U)$, of solutions to the homogeneous adjoint problem, i.e.

$$\begin{cases} L^\dagger u = 0 & \text{in } U \\ u = 0 & \text{on } U \end{cases}$$

and moreover both $\dim(N) = \dim(N^\dagger) < \infty$ are finite.

Finally, the usual BVP has a weak solution $\iff (f,v)_{L^2(U)} = 0 \ \forall v \in N^\dagger$.

[**Note:** Compare this with the matrix equation $Ax = b$: either $\exists!$ solution, or $\ker(A) \neq 0$. We want to solve $B[u,\cdot] = (f,\cdot)_{L^2(U)}$, which is similar.]

Proof. By Theorem 4.3, we know that $\exists \gamma > 0$ such that for any $f \in L^2(U)$, $\exists$ a unique weak solution to:

$$\begin{cases} L_\gamma u = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$$

where $L_\gamma u := Lu + \gamma u$. Hence we know $\exists! u \in H_0^1(U)$ such that:

$$B[u,v] + \gamma(u,v)_{L^2(U)} = (f,v)_{L^2(U)} \quad \forall v \in H_0^1(U),$$

and moreover $||u||_{H^1(U)} \leq C||f||_{L^2(U)}$.

So write: $L_\gamma^{-1}(f) := u$ for this unique solution. In this case, we have that

$$L_\gamma^{-1} : L^2(U) \to H_0^1(U)$$

is a linear and bounded operator, and so hence from before, it is a compact operator $L^2(U) \to L^2(U)$.

Now suppose that $u \in H_0^1(U)$ is a weak solution to the 'usual' BVP, i.e. $B[u,v] = (f,v)_{L^2(U)}$ for all $v \in H_0^1(U)$. Then we see:

$$B_\gamma[u,v] := B[u,v] + \gamma(u,v)_{L^2(U)} = (f,v)_{L^2(U)} + \gamma(u,v)_{L^2(U)} = (f + \gamma u, v)_{L^2(U)}$$

for all $v \in H_0^1(U)$. So hence:

$$\begin{aligned} u \text{ is a weak solution of the 'usual' BVP } &\iff u = L_\gamma^{-1}(f + \gamma u) = L_\gamma^{-1}(f) + \gamma L_\gamma^{-1}(u) \\ &\iff u - K(u) = h \end{aligned}$$

where $K = \gamma L_\gamma^{-1}$ (the **Green's function**) and $h = L_\gamma^{-1}(f)$, where in the first line we used the linearity of L_γ^{-1}.

Then we see that $K : L^2(U) \to H_0^1(U)$ is linear and bounded (as L_γ^{-1} is) and thus is a compact map $L^2(U) \to L^2(U)$. So hence by the Fredholm alternative for compact operators (Theorem 4.6), **either** (as we are trying to solve $(I-K)(u) = h$ to find a weak solution):

(a) $u - K(u) = h$ admits a unique solution $u \in L^2(U)$ for all $h \in L^2(U)$, thus giving a unique weak solution, <u>or</u>

(b) $\exists u \in L^2(U)$ such that $u - K(u) = 0$, $u \neq 0$.

[Note that the u we find is always in $L^2(U)$, as we are viewing K as a compact operator $L^2(U) \to L^2(U)$. We will show u lies in the appropriate Sobolev space by using the fact that L_γ^{-1} maps $L^2(U)$ into $H_0^1(U)$.]

So suppose (a) holds. Then setting $h = L_\gamma^{-1}(f)$[(x)], we have $\exists\, u \in L^2(U)$ such that $u - K(u) = h$, i.e. $u = \gamma L_\gamma^{-1}(u) + L_\gamma^{-1}(f)$. Then since $L_\gamma^{-1} : L^2(U) \to H_0^1(U)$ maps into $H_0^1(U)$, we get that (since we know $u \in L^2(U)$ now) $L_\gamma^{-1}(u) \in H_0^1(U)$, and so hence u is the sum of two things in $H_0^1(U)$, and so $u \in H_0^1(U)$. So hence we have by the above that it is the unique weak solution to the 'usual' BVP, and it is in $H_0^1(U)$, so done.

If (b) holds, then $\exists u \in L^2(U)$ such that $u = K(u) = \gamma L_\gamma^{-1}(u)$, and so again as L_γ^{-1} maps into $H_0^1(U)$ and $u \in L^2(U)$, we get that in fact $u \in H_0^1(U)$, and by definition of L_γ^{-1},

$$B[u,v] + \gamma(u,v)_{L^2(U)} = (\gamma u, v)_{L^2(U)} \quad \forall v \in H_0^1(U)$$

(since $f = \gamma u$ here), i.e. $B[u,v] = 0$ for all $v \in H_0^1(U)$, i.e. u is a weak solution to the homogeneous version of the usual BVP.

Further, the dimension of the space of such solutions is finite, and equals the dimension of the space of solutions to $v - K^\dagger(v) = 0$ (by the Fredholm alternative (iii), i.e. Image$(I-K) = \ker(I-K^\dagger)^\perp$.)

So the proof of the first 'moreover' part of the theorem will be complete once we have the following claim:

Claim: If $v \in L^2(U)$, then:

$$v - K^\dagger(v) = 0 \iff v \text{ is a weak solution to the homogeneous adjoint problem.}$$

Proof. Note that:

$$\begin{aligned} v - K^\dagger(v) = 0 &\iff v = K^\dagger(v) \\ &\iff (v,w)_{L^2(U)} = (K^\dagger(v), w) = (v, K(w))_{L^2(U)} \quad \forall w \in L^2(U) \\ &\iff (v,w)_{L^2(U)} = (v, \gamma L_\gamma^{-1}(w))_{L^2(U)} \quad \forall w \in L^2(U) \end{aligned}$$

(x) i.e. we have (a) for all $h \in L^2(U)$, so in particular as L_γ^{-1} maps into L^2, we have this for everything in the image of L_γ^{-1}, i.e. for $L_\gamma^{-1}(f)$ for all $f \in L^2(U)$.

where we have used the defining property of the adjoint map. But then from the definition of a weak solution to:

$$\begin{cases} L_\gamma \tilde{w} = \tilde{f} & \text{on } U \\ \tilde{w} = 0 & \text{on } \partial U \end{cases}$$

such a weak solution obeys:

$$B[\tilde{w}, v] + \gamma(\tilde{w}, v)_{L^2(U)} = (\tilde{f}, v)_{L^2(U)}$$

and so taking $\tilde{f} = w$, then we have $\tilde{w} = L_\gamma^{-1}(w)$, and so this becomes:

$$B[L_\gamma^{-1}(w), v] + \gamma(L_\gamma^{-1}(w), v)_{L^2(U)} = (w, v)_{L^2(U)}.$$

[Note that the above expression is always true, by definition of L_γ^{-1} giving the weak solution for this problem.]

So hence combining, we have:

$$\begin{aligned} v - K^\dagger(v) = 0 &\Longleftrightarrow (v, w)_{L^2(U)} = (v, \gamma L_\gamma^{-1}(w))_{L^2(U)} \quad \forall w \in L^2(U) \\ &\Longleftrightarrow B[L_\gamma^{-1}(w), v] + \gamma(L_\gamma^{-1}(w), v)_{L^2(U)} = (v, \gamma L_\gamma^{-1}(w))_{L^2(U)} \quad \forall w \in L^2(U) \\ &\Longleftrightarrow B[L_\gamma^{-1}(w), v] = 0 \quad \forall w \in L^2(U) \\ &\Longleftrightarrow B^\dagger[v, L_\gamma^{-1}(w)] = 0 \quad \forall w \in L^2(U). \end{aligned}$$

where in the second line we have used the fact that we have two expressions for $(v, w)_{L^2(U)} = (w, v)_{L^2(U)}$ and equated them.

But then v is a weak solution to the homogeneous adjoint problem $\Longleftrightarrow B^\dagger[v, u] = 0$ for all $u \in H_0^1(U)$, whereas the above only gets this for $L_\gamma^{-1}(w)$, i.e. the image of L_γ^{-1}.

But then $\{L_\gamma^{-1}(w) : w \in L^2(U)\} = \text{Image}(L_\gamma^{-1})$ is dense in $H_0^1(U)$ [**Exercise** to check - see Example Sheets], and so by continuity of L_γ^{-1} (as it is linear and bounded), the above gives:

$$\begin{aligned} v - K^\dagger(v) = 0 &\Longleftrightarrow B^\dagger[v, L_\gamma^{-1}(w)] = 0 \quad \forall w \in L^2(U) \\ &\Longleftrightarrow B^\dagger[v, u] = 0 \quad \forall u \in H_0^1(U) \\ & v \text{ is a weak solution to the homogeneous adjoint problem.} \end{aligned}$$

So hence done.

$\square$

So all that remains to prove is that:

The 'usual' BVP has a weak solution $\Longleftrightarrow (f, v)_{L^2(U)} = 0 \;\; \forall v$ weak solutions to the homogeneous adjoint BVP.

So recall that from the Fredholm alternative for compact operators, $\text{Image}(I - K) = \ker(I - K^\dagger)^\perp$. So:

$$\text{The usual BVP has a solution } \Longleftrightarrow (I - K)(u) = L_\gamma^{-1}(f) \Longleftrightarrow L_\gamma^{-1}(f) \in \text{Image}(I - K)$$

$$\Longleftrightarrow L_\gamma^{-1}(f) \in \ker(I - K^\dagger)^\perp \Longleftrightarrow (v, L_\gamma^{-1}(f))_{L^2(U)} = 0 \;\; \forall v \in \ker(I - K^\perp).$$

But we have:

$$(v, L_\gamma^{-1}(f))_{L^2(U)} = (v, \gamma^{-1}K(f))_{L^2(U)} = \frac{1}{\gamma}(K^\dagger(v), f)_{L^2(U)} = \frac{1}{\gamma}(v, f)_{L^2(U)}$$

for all $v \in \ker(I - K^\dagger)$, where we have used that $K := \gamma L_\gamma^{-1}$, the defining property of the adjoint, and that if $v \in \ker(I - K^\dagger)$, then $v = K^\dagger(v)$.

So hence we see $(v, L_\gamma^{-1}(f))_{L^2(U)} = 0 \;\; \forall v \in \ker(I - K^\perp) \iff (v, f)_{L^2(U)} \;\; \forall v \in \ker(I - K^\perp)$.

Then simply noting by all that we did at the start of the proof, we know that $v \in \ker(I - K^\dagger) \iff v$ is a solution to the homogeneous adjoint problem, and so hence this shows that:

The usual BVP has a solution $\iff (f, v)_{L^2(U)} = 0$ for all v solving the hom. adjoint problem,

and so done.

□

We will now show a couple more closely related results before moving on.

Theorem 4.9 (Spectrum of L). *For everything as above, we have:*

(i) $\exists$ *an* <u>*at most countable*</u> *set* $\Sigma \subset \mathbb{R}$ *such that: the BVP:*

$$\begin{cases} Lu = \lambda u + f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases} \tag{4.4}$$

has a weak solution $\forall f \in L^2(U) \iff \lambda \notin \Sigma$.

(i.e. the λ for which we <u>*can't*</u> *find a weak solution to (4.4) for all f is at most countable. So we can essentially always solve this problem.)*

(ii) If Σ is infinite, then $\Sigma = \{\lambda_k\}_{k=1}^\infty$ (as countable by (i)) and this is (after re-ordering) a non-decreasing sequence with $\lambda_k \to \infty$ as $k \to \infty$.

(this is only increasing to ∞ as opposed to decreasing to 0 since they come from the <u>*inverse*</u> *of a compact operator.)*

(iii) To each $\lambda \in \Sigma$, there is an associated <u>*finite dimensional*</u> *space:*

$$\mathscr{E}(\lambda) = \{u \in H_0^1(U) : u \text{ is a weak solution to: } Lu = \lambda u \text{ in } U,\ u = 0 \text{ on } \partial U\}.$$

Note: We say that $\lambda \in \Sigma$ is an **eigenvalue of L**, and $u \in \mathscr{E}(\lambda)$ is an **associated eigenfunction**.

Remark: We will see that this is essentially just from the Fredholm alternative, since we just need to consider the case $f \equiv 0$ and so we have $L_\gamma u = (\lambda + \gamma)u$, i.e. u is an eigenfunction of the compact operator L_γ^{-1}, and so we can apply general spectral theory.

Proof. Pick $\gamma > 0$ as we did in the proof of Theorem 4.8 (i.e. pick γ as in Theorem 4.3). So, $L_\gamma^{-1} : L^2(U) \to H_0^1(U)$ is well-defined (as before).

Then for $\lambda \leq -\gamma$ (i.e. $\mu := -\lambda \geq \gamma$), the problem:

$$\begin{cases} Lu - \lambda u = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$$

admits a unique weak solution $\forall f \in L^2(U)$ (by Theorem 4.3), and so we get $\Sigma \subset \{\lambda > -\gamma\}$.

Now consider $\lambda > -\gamma$. Then by the Fredholm alternative (Theorem 4.8), Equation (4.4) has a solution for all $f \in L^2(U) \iff u \equiv 0$ is the only solution to:

$$\begin{cases} Lu = \lambda u & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$$

i.e. when $f = 0$, the homogeneous problem. But then this holds

$$\iff u = 0 \text{ is the only solution to } \begin{cases} (L+\gamma)u = (\lambda+\gamma)u & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$$

$$\iff u = L_\gamma^{-1}((\lambda+\gamma)u) = \frac{\gamma+\lambda}{\gamma} K(u)$$

where $K = \gamma L_\gamma^{-1}$ as before (and we have used the linearity pf L_γ^{-1}).

Thus we see that (4.4) has a solution for each $f \in L^2(U) \iff K(u) = \left(\frac{\gamma}{\gamma+\lambda}\right)u$ only has $u \equiv 0$ as a solution, i.e. if and only if $\frac{\gamma}{\gamma+\lambda}$ is not an eigenvalue of K.

So in particular, we get $\lambda \in \Sigma \iff \frac{\gamma}{\gamma+\lambda}$ is an eigenvalue of K.

Then by the general theory of compact operators (as K is a compact operator), $\mu := \frac{\gamma}{\gamma+\lambda}$ is an eigenvalue of K for at most countably many μ_k, $k = 1, 2, \ldots$, and if $(\mu_k)_k$ is infinite, the $\mu_k \to 0$. This then implies $\lambda_k = \frac{\gamma}{\mu_k} - \gamma \to \infty$.

The fact that $\mathscr{E}(\lambda)$ is finite dimensional is an immediate corollary of the Fredholm alternative (Theorem 4.8).

$\square$

Remark: By complexifying our Hilbert space to consider $u : U \to \mathbb{C}$, with:

$$(u,v)_{H_0^1(U)} := \int_U \left[\left(\sum_{i=1}^n \overline{D_i u} D_i v \right) + \overline{u} v \right] \mathrm{d}x$$

as our inner product, we can consider $L - z$, for $z \in \mathbb{C}$ (i.e. $L - zI$ with I the identity). In this case:

$$\Sigma = \left\{ z \in \mathbb{C} : \text{ The system } \begin{cases} (L-z)u = 0 & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases} \text{ has a non-zero solution } u \right\}$$

is the set of eigenvalues, and $\exists w \in \mathbb{R}$ such that $\Sigma \subset \{\Re(z) > w\}$. Σ is again a discrete set, accumulating only at ∞.

4.4. Formally Self-Adjoint, Positive Operators.

Definition 4.10. *Our operator L (from before) is called* ***formally self-adjoint*** *if: $L = L^\dagger$, where $L^\dagger$ is the formal adjoint of L (as defined before).*

This turns out to be equivalent to: $b^i \equiv 0$ for all i.

Definition 4.11. *L is* ***positive*** *if:*

$$\|u\|^2_{H^1_0(U)} \leq C \cdot B[u,u] \quad \forall u \in H^1_0(U)$$

for some C (i.e. coercive).

Then the main result about such operators is:

Theorem 4.10 (Eigenvalues of Positive, Formally Self-Adjoint, Operators)**.** *Suppose L is a formally self-adjoint, positive, uniformly elliptic operator on U, where U is an open, bounded set with C^1-boundary. Then, we can represent the eigenvalues of L as:*

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots$$

where each eigenvalue appears according to its multiplicity, $\dim(\mathscr{E}(\lambda))$, and $\exists$ an orthonormal basis of $L^2(U)$ of eigenfunctions, $(w_n)_n$, with $w_n \in H^1_0(U)$ an eigenfunction of L with eigenvalue λ_n.

Proof. We have $L^{-1} : L^2(U) \to H^1_0(U)$, sending the data f to a solution u, as a bounded operator. Thus we have $S = L^{-1} : L^2(U) \to L^2(U)$ (as $H^1_0(U) \hookrightarrow L^2(U)$) is a compact operator.

Claim: S is symmetric.

Proof. Pick $f, g \in L^2(U)$. Then, $S(f) = u$ means that u is a weak solution to:

$$\begin{cases} Lu = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$$

and similarly for $S(g) = v$. So, hence by the definition of a weak solution, we have:

$$(S(f), g)_{L^2(U)} = (u, g)_{L^2(U)} = B[v, u],$$

and also

$$(f, S(g))_{L^2(U)} = (f, v)_{L^2(U)} = B[u, v].$$

But as L is self-adjoint, we have $B[u,v] = B[v,u]$ and so this shows

$$(S(f), g)_{L^2(U)} = (f, S(g))_{L^2(U)}.$$

So hence as f, g were arbitrary, this shows that S is symmetric.

□

Thus we can apply the result for compact symmetric operators (see Linear Analysis Part II) to get $\exists\ (\mu_n)_n \subset \mathbb{R}$ such that $\mu_n \to 0$, and $\exists w_n \in L^2(U)$ such that $(w_n)_n$ is an orthonormal basis of $L^2(U)$ with $Sw_n = \mu_n w_n$, i.e. $L^{-1} w_n = \mu_n w_n$.

So w_n is an eigenfunction of L with eigenvalue μ^{-1}. The positivity of the eigenvalues comes from positivity of L and thus S.

$\square$

4.5. **Elliptic Regularity.**

We want to improve the regularity of our solutions u from $H_0^1(U)$ to something like $C^2(\overline{U})$, possibly requiring that L, f are more regular.

Motivation: [Poisson's Equation]

Suppose $u \in C_c^\infty(\mathbb{R}^n)$, and $-\Delta u = f$, for some $f \in C_c^\infty(\mathbb{R}^n)$. Then we can compute:

$$\begin{aligned}\int_{\mathbb{R}^n} f^2 \, \mathrm{d}x = \int_{\mathbb{R}^n} (\Delta u)^2 &= \sum_{i,j} \int_{\mathbb{R}^n} (D_i D_i u)(D_j D_j u) \\ &= -\sum_{i,j} \int_{\mathbb{R}^n} (D_j D_i D_i u) D_j u \\ &= \sum_{i,j} \int_{\mathbb{R}^n} (D_i D_j u) \cdot (D_i D_j u) \\ &= \int_{\mathbb{R}^n} \sum_{i,j} |D_{ij} u|^2 = \|D^2 u\|^2_{L^2(\mathbb{R}^n)}\end{aligned}$$

where we have integrated by parts twice.

So hence since $f^2 = (\Delta u)^2$, this shows that:

$$\|D^2 u\|_{L^2(\mathbb{R}^n)} \leq \|\Delta u\|_{L^2(\mathbb{R}^n)}$$

(in general we get an inequality, so we include it here).

Now this is a bit surprising: $\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2}$ only contains some of the 2nd order derivatives, but the above says that just by controlling these derivatives in L^2, we can control <u>all</u> 2nd order derivatives in L^2.

This remarkable fact/observation is the basis of elliptic regularity.

To make this argument for weak solutions (and not just $C_c^\infty(\mathbb{R}^n)$ as above) we have to contend with the fact that for $u \in H^1(U)$, $D^2 u$ does not necessarily exist in a weak sense (as recall H^1 only guarantees the existence of 1st order weak derivatives). To get around this, we use <u>difference quotients</u>.

So suppose $U \subset \mathbb{R}^n$ is open, and $\underline{V \subset\subset U}$. Then:

Definition 4.12. *For $0 < |h| < dist(V, \partial U)$, we define the **difference quotient** by:*

$$\Delta_i^h u(x) := \frac{u(x + he_i) - u(x)}{h} \quad \textit{for } i = 1, \ldots, n$$

and we write

$$\Delta^h u := (\Delta_1^h u, \dots, \Delta_n^h u).$$

[Note this restriction on h is just so $x \pm he_i$ does not leave U.]

Then if $u \in L^2(U)$, clearly we have $\Delta^h \in L^2(V)$ (as it is the sum of two things which are in some L^2 space).

Also, if $u \in H^1(U)$, then we have $\Delta^h u \in H^1(V)$, and clearly we have

$$D(\Delta^h u) = \Delta^h(Du)$$

which is starting to look like $D^2 u$ if h is small.

Lemma 4.2 (Important)**.** *Suppose $u \in L^2(U)$. Then:*

$$u \in H^1(V) \iff \|\Delta^h u\|_{L^2(V)} \leq C \quad \forall h \text{ s.t. } 0 < \frac{1}{2}h < dist(V, \partial U), \text{ for some } C > 0$$

and moreover, $\exists \tilde{C}$ such that:

$$\frac{1}{\tilde{C}}\|Du\|_{L^2(V)} \leq \|\Delta^h u\|_{L^2(V)} \leq \tilde{C}\|Du\|_{L^2(V)}$$

(i.e. the difference quotient is "equivalent" to the (weak) derivative, in some sense).

Proof. None given - See Example Sheet 3.

□

Now the big important result.

Theorem 4.11 (Interior Regularity)**.** *Suppose L is uniformly elliptic on $U \subset \mathbb{R}^n$ an open set, and assume $a^{ij} \in C^1(U)$ and $b^i, c \in L^\infty(U)$, and $f \in L^2(U)$.*

Suppose further that $u \in H^1(U)$ is a weak solution, i.e. it satisifes $B[u,v] = (f,v)$ $\forall v \in H_0^1(U)$.

Then, $u \in H^2_{loc}(U)$, and for each $V \subset\subset U$, we have:

$$\|u\|_{H^2(V)} \leq C\left(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}\right)$$

with $C = C(a,b,c,V,U,n)$ (but no dependence on f,u).

Note: This result says that we get 2 weak derivatives in this case, which is very good. It is useful to write the inequality as:

$$\|u\|_{H^2(V)} \leq C\left(\|Lu\|_{L^2(U)} + \|u\|_{L^2(U)}\right)$$

(which we can do since $u \in H^2_{\text{loc}}$), which shows that this point holds true for all $u \in H^2(U)$ in general for such an L. Note that we can actually replace the norms on the RHS with $\|\cdot\|_{L^2(W)}$ for any $V \subset\subset W \subset\subset U$, as a consequence of the proof.

Proof. This proof involves a lot of calculation and subscripts, but is fundamentally quite easy. We break it down into steps to make is easier to follow.

Step 1: Fix $V \subset\subset U$ and choose W such that $V \subset\subset W \subset\subset U$. Then take $\xi \in C^\infty_c(W)$ such that $\xi \equiv 1$ on V (and so $\xi \equiv 0$ on ∂W, as ξ is in $C^\infty_c(W)$).

Rewriting the weak solution equation, $B[u,v] = (f,v)$, we have:

$$\sum_{i,j=1}^n \int_U a^{ij} u_{x_i} v_{x_j} \, \mathrm{d}x = \int_U \tilde{f} v \, \mathrm{d}x$$

for all $v \in H^1_0(U)$, where

$$\tilde{f} = f - \sum_{i=1}^n b^i u_{x_i} - cu.$$

We will take $v = -\Delta_k^{-h}\left(\zeta^2 \Delta_k^h u\right)$ in this identity, for some $k \in \{1,\dots,n\}$ fixed (note how this is a very close approximation to the second derivative of u). Then write:

$$A = \sum_{i,j=1}^n \int_U a^{ij} u_{x_i} v_{x_j} \, \mathrm{d}x, \quad \text{and} \quad B = \int_U \tilde{f} v \, \mathrm{d}x.$$

Note: In general, if v, w are supported in W, then:

$$\int_U w(\Delta_k^{-h} v) \, \mathrm{d}x = -\int_U (\Delta_k^h w) v \, \mathrm{d}x$$

which we will refer to as *integration by parts for difference quotients*, and we have:

$$\begin{aligned}\Delta_k^h(wv) &:= \frac{w(x+e_k h)v(x+e_k h) - w(x)v(x)}{h} \\ &= (\tau_k^h w)\cdot \Delta_k^h v + (\Delta_k^h w)\cdot v\end{aligned}$$

where $\tau_k^h w(x) := w(x + he_k)$ is the **translation operator** [**Exercise** to check these]. We will refer to this second equation as the *product rule for difference quotients*.

We will look at both A and B separately, and bound them both in such as way that it implies the result.

Step 2: Bounding A.

So looking at A for this choice of v, we have:

$$\begin{aligned}
A &= -\sum_{i,j}\int_U a^{ij}u_{x_i}\cdot\Delta_k^{-h}\left(\xi^2\Delta_k^h u\right)_{x_j}\ \mathrm{d}x\\
&= \sum_{i,j}\int_U \Delta_k^h(a^{ij}u_{x_i})\cdot(\xi^2\Delta_k^h u)_{x_j}\ \mathrm{d}x\\
&= \sum_{i,j}\int_U \left((\tau_k^h a^{ij})\Delta_k^h u_{x_i} + (\Delta_k^h a^{ij})u_{x_i}\right)\cdot\left(\xi^2\Delta_k^h u_{x_j} + 2\xi\xi_{x_j}\Delta_k^h u\right)\ \mathrm{d}x
\end{aligned}$$

where in the second line, we have used integration by parts for difference quotients, and from the second to third line, for the first term we have used the product rule for difference quotients, and for the second term we have used the product rule for weak derivatives, as ξ is smooth (see Example Sheet 2).

Then by expanding the brackets/product, we have $A = A_1 + A_2$, where:

$$A_1 = \sum_{i,j}\int_U \xi^2\left(\tau_k^h a^{ij}\right)\Delta_k^h u_{x_i}\cdot\Delta_k^h u_{x_j}\ \mathrm{d}x$$

and

$$A_2 = \sum_{i,j}\int_U\left[(\Delta_k^h a^{ij})u_{x_i}\xi^2\Delta_k^h u_{x_j} + (\Delta_k^h a^{ij})u_{x_i}\cdot 2\xi\xi_{x_i}\Delta_k^h u + (\tau_k^h a^{ij})\Delta_k^h u_{x_i}\cdot 2\xi\xi_{x_j}\Delta_k^h u\right]\ \mathrm{d}x.$$

Then by uniform ellipticity, we have $(\tau_k^h a^{ij})\zeta_i\zeta_j \geq \theta|\zeta|^2\ \forall\xi\in\mathbb{R}^n$, and so we have (applying this with $\zeta_i = \Delta_k^h u_{x_i} = (\Delta_k^h u)_{x_i}$):

$$A_1 \geq \theta\int_U \xi^2|\Delta_k^h(Du)|^2\ \mathrm{d}x.$$

Then since $a^{ij}\in C^1(U)$ by assumption, and ξ is supported in W (and W is compact, and so continuous functions on W are bounded), we have:

$$|A_2| \leq C\int_W\left[\xi|\Delta_k^h(Du)|\cdot|Du| + \xi|Du|\cdot|\Delta_k^h u| + \xi|\Delta_k^h(Du)|\cdot|\Delta_k^h u|\right]\ \mathrm{d}x$$

where we have bounded $\xi, D\xi$ and the a^{ij} on W (so as to have exactly one factor of ξ on each term), and then we have changed the integral to one over W as $\xi\equiv 0$ outside W (and each term has a factor of ξ).

So then by Young's inequality (with ε) we have further:

$$|A_2| \leq \varepsilon\int_W \xi^2|\Delta_k^h(Du)|^2\ \mathrm{d}x + \frac{C}{\varepsilon}\int_W |Du|^2 + |\Delta_k^h u|^2\ \mathrm{d}x$$

for any $\varepsilon > 0$. Then note that, by definition of the difference quotient and by Lemma 4.2, we have

$$\int_W |\Delta_k^h u|^2\ \mathrm{d}x \leq C\int_W |Du|^2 \tag{4.5}$$

and so setting $\varepsilon = \frac{\theta}{2}$, we conclude that:

$$A = A_1 + A_2 \geq \frac{\theta}{2}\int_U \xi^2|\Delta_k^h(Du)|^2 - C\int_W |Du|^2\ \mathrm{d}x$$

for some C (i.e. by lower bounding A_1 and lower bounding A_2 by $-|A_2|$, using the above upper bound).

Step 3: Now look at B.

We have, from the definition of $\tilde{f}$ and from our choice of v:

$$|B| \leq C \int_U (|f| + |Du| + |u|) \cdot \Delta_k^{-h}(\xi^2 \Delta_k^h u)\, \mathrm{d}x$$

for some constant C depending on b, c.

Then by Lemma 4.2, we have:

$$\begin{aligned}\int_U \left[\Delta_k^{-h}(\xi^2 \Delta_k^h u)\right]^2 \, \mathrm{d}x &\leq C \int_U |D(\xi^2 \Delta_k^h u)|^2 \, \mathrm{d}x \\ &\leq C \int_U 2\xi_{x_i}\xi |\Delta_k^h u|^2 + \xi^2 |\Delta_k^h (Du)|^2 \, \mathrm{d}x \\ &\leq C \int_U |Du|^2 \, \mathrm{d}x + C \int_U \xi^2 |\Delta_k^h (Du)|^2 \, \mathrm{d}x\end{aligned}$$

for some constants C(xi). Note in the second line we have used the product rule for difference quotients (and then bounded $(a+b)^2 \leq 2a^2 + 2b^2$), and then in the last line we have used (4.5) again.

Then using Young's inequality (with ε) again but this time on B (to split the integral of the product up into integrals of the separate parts), we see that:

$$|B| \leq \varepsilon \int_U \xi^2 |\Delta_k^h (Du)|^2 \, \mathrm{d}x + C \int_W \left(f^2 + u^2 + |Du|^2\right) \, \mathrm{d}x$$

for any $\varepsilon > 0$. So setting $\varepsilon = \frac{\theta}{4}$, we can conclude (as $A + B = 0$, so $|A| = |B|$ - carry on in Step 4...):

Step 4: Since $|A| = |B|$, we get from the above with this ε:

$$\int_U \xi^2 |\Delta_k^h (Du)|^2 \, \mathrm{d}x \leq C \int_W f^2 + u^2 + |Du|^2 \, \mathrm{d}x$$

and so hence as $\xi \equiv 1$ on $V \subset\subset U$, this gives:

$$\int_V |\Delta_k^h (Du)|^2 \, \mathrm{d}x \leq C \int_W f^2 + u^2 + |Du|^2 \, \mathrm{d}x$$

with C being independent of h. So hence by Lemma 4.2, this shows that $Du \in H^1(U)$ (as so hence $u \in H^2(V)$), and we have (taking $h \downarrow 0$ in the above):

$$\|D^2 u\|_{L^2(V)} \leq C \int_W f^2 + u^2 + |Du|^2 \, \mathrm{d}x.$$

Step 5: Finish.

So the above shows that we have $u \in H^2_{\mathrm{loc}}(U)$, with:

$$\|u\|_{H^2(V)} \leq C \left(\|f\|_{L^2(W)} + \|u\|_{H^1(W)}\right).$$

(xi)**Note:** C will keep changing throughout, by its dependencies won't. So to stop us having to use different letters, we will just call it C

So to complete the proof, we need to show that the $\|Du\|_{L^2(W)}$ term on the RHS (coming from the $\|\cdot\|_{H^1(W)}$ norm) is unnecessary.

To see this, let $\xi \in C_c^\infty(U)$ with $\xi \equiv 1$ on W. Then set $v = \xi^2 u$ in the equation for a weak solution (i.e. $B[u,v] = (f,v)$) to get:

$$\int_U \left[\sum_{i,j=1}^n a^{ij} u_{x_i} (\xi^2 u)_{x_j} + \sum_i b^i u_{x_i} \cdot \xi^2 u + c\xi^2 u^2 \right] \mathrm{d}x = \int_U f \cdot \xi^2 u \; \mathrm{d}x.$$

Then proceeding as in the proof of Garding's inequality (i.e. Theorem 4.2 (ii)) we deduce that:

$$\|Du\|_{L^2(W)} \leq C\left(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}\right)$$

for some C, where we have used that $W \subset U$ to bound the norms on W by those on U (since we just get $\|Du\|_{L^2}^2 \leq B[u,u] + \gamma\|u\|_{L^2}$ and then $B[u,u] = \int fu \leq \|f\|_{L^2}\|u\|_{L^2}$). So done.

□

Remarks:

- Notice that this is a <u>local result</u>, i.e. in order to have $u \in H^2(V)$, it is enough to just have $f \in L^2(W)$ for some W slightly larger than V.

  So this shows/tells us that <u>singularities do not propagate</u> either in from the boundary, or from regions where f is not well-behaved (i.e. if f is well-behaved about some point, so will the solution, because of this locality).
- This result allows us to understand the equation as holding pointwise a.e.

  <u>i.e.</u> take $v \in C_c^\infty(U)$ as a test function in $B[u,v] = (f,v)_{L^2(U)}$. Then as $u \in H^2_{\mathrm{loc}}(U)$, we can integrate by parts to obtain:

  $$(Lu - f, v)_{L^2(U)} = 0$$

  and this is true for all $v \in C_c^\infty(U)$, which implies $Lu = f$ a.e. in U.
- The proof required uniform ellipticity to hold on compact subsets of U (and so we can allow for degeneracy towards ∂U).

We can inductively improve this inner regularity result:

Theorem 4.12 (Improved Interior Regularity)**.** *If $a^{ij}, b^i, c \in C^{m+1}(U)$ for some $m \in \mathbb{N}$, and $f \in H^m(U)$, then we have $u \in H^{m+2}_{loc}(U)$, and for $V \subset\subset W \subset\subset U$,*

$$\|u\|_{H^{m+2}(V)} \leq C\left(\|f\|_{H^m(W)} + \|u\|_{L^\infty(W)}\right).$$

Proof. By induction on Theorem 4.11 (see Example Sheet 4 for details).

□

Remarks:

- There is a Hölder theory of elliptic regularity, which roughly gives:
$$f \in C^{k,\alpha}(U) \Rightarrow u \in C^{k+2,\alpha}(U) \quad \forall 0 \leq \alpha < 1.$$
- Combining Theorem 4.12 with the Sobolev embeddings, we can conclude that if m is large enough (i.e. $a^{ij}, b^j, c \in C^{m+1}$, $f \in H^m$), then we have $u \in H^{m+2}_{\text{loc}}(U) \hookrightarrow C^2_{\text{loc}}(U)$, at which point we have a classical solution (here, we are after $m > n/p = n/2$).

  This is the reason for all of this elliptic regularity and the Sobolev embeddings. We get that if our data is, say $C^\infty(U)$, then $f \in H^m(U)$ for all m, and so hence $u \in H^{m+2}_{\text{loc}}(U)$ for all m. Hence by the Sobolev embeddings, this tells us that $u \in C^k(U)$ for all k, i.e. u is smooth.

If ∂U is nice enough, then we can extend the regularity result up to the boundary, as the next theorem tells us.

Theorem 4.13 (Boundary H^2-Regularity). *Assume that $a^{ij} \in C^1(\overline{U})$, and $b^i, c \in L^\infty(U)$ and $f \in L^2(U)$. Suppose that $u \in H^1_0(U)$ is a weak solution of:*
$$\begin{cases} Lu = f & \text{in } U \\ u = 0 & \text{on } \partial U. \end{cases}$$
*Assume that ∂U is C^2. Then, we have $u \in H^2(U)$ [**no local**] and we have:*
$$\|u\|_{H^2(U)} \leq C\left(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}\right).$$
Moreover, if u is the unique weak solution, then we can drop the $\|u\|_{L^2(U)}$ from the RHS of this inequality, i.e. we have $\|u\|_{H^2(U)} \leq C\|Lu\|_{L^2(U)}$.

Proof (Sketch). We give a sketch proof of this result - the general result requires a boundary straightening argument (full details can be found in Evans).

Here, we restrict ourselves to the case where $U = B_1(0) \cap \{x_n > 0\}$, i.e. the boundary is planar.

Let $V = B_{1/2}(0) \cap \{x_n > 0\}$, and choose $\xi \in C^\infty_c(B_1(0))$ with $\xi \equiv 1$ on V, and $0 \leq \xi \leq 1$.

Then since u is a weak solution of the above problem, as before in the proof of Theorem 4.11, for any $v \in H^1_0(U)$, we can write:
$$\sum_{i,j=1}^n \int_U a^{ij} u_{x_i} v_{x_j} \, dx = \int_U \tilde{f} v \, dx$$
where $\tilde{f} := f - \sum_{i=1}^n b^i u_{x_i} - cu$.

Now, let $|h| > 0$ be small and fix $k \in \{1, \ldots, n-1\}$, and write (as before):
$$v = -\Delta_k^{-h}(\xi^2 \Delta_k^h u).$$

Then note that, from the definition of the difference quotient,

$$\begin{aligned} v(x) &= -\frac{1}{h}\Delta_k^{-h}\left(\xi^2(x)\cdot[u(x+he_k)-u(x)]\right) \\ &= -\frac{1}{h^2}\left(\xi^2(x-he_k)[u(x)-u(x-he_k)]-\xi^2(x)[u(x+he_k)-u(x)]\right) \end{aligned}$$

for $x \in U$.

Now, since $u = 0$ along $\{x_n = 0\}$ (in the trace sense), and $\xi = 0$ near the curved part of ∂U, we have $v \in H_0^1(U)$. So we choose this v in the above integral equation. We then can proceed to manipulate this identity as in the proof of Theorem 4.11, to deduce:

$$\int_V |\Delta_k^h(Du)|^2 \, \mathrm{d}x \le \int_U f^2 + u^2 + |Du|^2 \, \mathrm{d}x. \tag{4.6}$$

Even though we do not have $V \subset\subset U$, the results concerning difference quotients in directions tangent to the boundary still hold, and thus we control all 2nd order derivatives of the form: $D_k D_i u$, with $k \in \{1,\dots,n-1\}$, $i \in \{1,\dots,n\}$.

So we just need to control $u_{x_n x_n}$. We return to our original equation $Lu = f$, and recall that by a remark above, it holds a.e. in U. So hence:

$$-\sum_{i,j=1}^n (a^{ij}u_{x_i})_{x_j} + \sum_i b^i u_{x_i} + cu = f \quad \text{(a.e.)}$$

$$\Rightarrow a^{nn}u_{x_n x_n} = F \quad \text{(a.e.)}$$

where:

$$F = -\sum_{i,j\,:\,i+j<2n} a^{ij}u_{x_i x_j} + \sum_{i=1}^n \tilde{b}^i u_{x_i} + cu - f.$$

We then already know that $F \in L^2(U)$, since it is the sum of things which are in $L^2(U)$. Then, the uniform ellipticity condition on L implies that $a_{nn} > 0$ a.e. (take $\zeta = (0,\dots,0,1)$ in the uniform ellipticity condition to see this), and so we get that $u_{x_n x_n} \in L^2(U)$.

[This is because $F \in L^2(U)$, and then since $a^{ij} \in C^1(\overline{U})$ by assumption, we have that a^{nn} attains its minimum on $\overline{U}$. But as $a^{nn} > 0$ a.e., we must have that this minimum is > 0, i.e. $a^{nn} \ge C > 0$ for some C on U. Hence since $u_{x_n x_n} = F/a^{nn}$, this shows that $u_{x_n x_n} \in L^2(U)$.]

Then Equation (4.6) $\Rightarrow \|D_k D_i u\|_{L^2(V)} \le C\left(\|f\|_{L^2(U)} + \|u\|_{H^1(U)}\right)$. Therefore, we get

$$\|F\|_{L^2(U)} \le C\left(\|f\|_{L^2(U)} + \|u\|_{H^1(U)}\right),$$

and so hence we can bound $\|u_{x_n x_n}\|_{L^2(U)}$ by an expression of this form. So hence we have bounded all 2nd order derivatives by expressions like this, and so hence we get:

$$\|u\|_{H^2(V)} \le C\left(\|f\|_{L^2(U)} + \|u\|_{H^1(U)}\right).$$

Then again, (the proof of) Garding's inequality (Theorem 4.2 (ii)) $\Rightarrow$ the $\|\cdot\|_{H^1(U)}$ on the RHS can be replaced by $\|\cdot\|_{L^2(U)}$.

This proves the straight boundary case. Then the general/full result follows by straightening out the boundary and using a partition of unity (see Evan's for details).

□

We now finish this section with some remarks.

Remarks:

- Higher regularity versions of these results can be shown, such as:

  "If $a^{ij}, b^i, c \in C^{m+1}(\overline{U})$ and $f \in H^m(U)$, and ∂U is C^{m+2}, then we have $u \in H^{m+2}(U)$ (**again, not just locally**), and:
  $$\|u\|_{H^{m+2}(U)} \leq C\left(\|f\|_{H^m(U)} + \|u\|_{L^2(U)}\right)."$$
- In particular, this shows that if $a^{ij}, b^j, c \in C^\infty(\overline{U})$, $f \in C^\infty(U)$ and ∂U is C^∞, then by the above result and the general Sobolev embeddings, we have that $u \in C^\infty(U)$, i.e. u is smooth.

  In particular, the solutions of $Lu = \lambda u$ will be smooth under these assumptions (as this is just $\tilde{L}u = 0$, where $\tilde{L} = L - \lambda$, and so $f = 0 \in C_c^\infty(U)$ here).

Now we move into a different type of PDE: *hyperbolic PDEs*.

5. Hyperbolic Equations

We will look at 2nd order, linear, **hyperbolic PDEs**. Consider:

$$\sum_{i,j=1}^{n+1}(a^{ij}(y)u_{y_j})_{y_i}+\sum_{i=1}^{n+1}a^i(y)u_{y_i}+a(y)u=f \tag{5.1}$$

where $y\in\mathbb{R}^{n+1}$, $a^{ij}=a^{ji},a^i,a\in C^\infty(\mathbb{R}^{n+1})$.

Definition 5.1. *This equation (5.1) is said to be* ***hyperbolic*** *if the quadratic form:*

$$q(\xi):=\sum_{i,j=1}^{n+1}a^{ij}(y)\xi_i\xi_j$$

has signature $(+,-,\cdots,-)$ $\forall y\in\mathbb{R}^{n+1}$. *This means that, after perhaps a change of basis at each point, we can write:*

$$q(\xi)=\lambda_{n+1}^2\xi_{n+1}^2-\sum_{i=1}^{n}\lambda_i^2\xi_i^2$$

where $\lambda_i>0$ *for all* i.

[i.e. the matrix $(a^{ij}(y))_{ij}$ *is diagonalisable with exactly one positive eigenvalue, for all* y*.]*

Definition 5.2. *This quadratic form* q *is called the* ***principle symbol*** *of the PDE.*

So by a coordinate transform, we can locally put Equation (5.1) in the form:

$$f=u_{tt}-\sum_{i,j=1}^{n}(a^{ij}(x,t)u_{x_i})_{x_j}+\sum_{i=1}^{n+1}b^i(x,t)u_{x_i}+c(x,t)u$$

where $(x_1,\dots,x_n,t)=(y_1,\dots,y_{n+1})$. We can only do this locally, but we can build up global solutions from local solutions for hyperbolic PDEs.

Note: $\{(x,t):t=0\}$ is a non-characteristic surface of this PDE (assuming uniform ellipticity, i.e. $\sum_{i,j=1}^n a^{ij}\xi_i\xi_j\geq\theta|\xi|^2$). So we can hope to solve a Cauchy problem.

So will specify $u|_{t=0}$ and $u_t|_{t=0}$ (i.e. initial data on this non-characteristic surface). We will look at an initial boundary value problem.

5.1. Hyperbolic Initial Boundary Value Problem (IBVP).

Let $U\subset\mathbb{R}^n$ be open, bounded with C^1 boundary (as usual). Then define:

$$U_T:=(0,T)\times U \quad\text{and}\quad \partial^*U_T=[0,T]\times\partial U$$

(note that we need a new notation, not just ∂U_T, as this would mean including the boundaries $\{0\}\times U$, etc). Then for $t\in[0,T]$, let $\Sigma_t=\{t\}\times U$. So:

$$\partial U_T=\Sigma_0\cup\Sigma_T\cup\partial^*U_T$$

with these sets being pairwise disjoint.

Let us look at the following IBVP. Suppose $u \in C^2(U_T)$ satisfies:

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } U_T \\ u = \psi & \text{on } \Sigma_0 \\ u_t = \psi' & \text{on } \Sigma_0 \\ u = 0 & \text{on } \partial^* U_T \end{cases} \tag{5.2}$$

(i.e. the last condition tells us that u cannot escape from the sides of U_T).

We will perform an energy estimate. Multiply the equation $u_{tt} - \Delta u = 0$ by u_t and integrate it over $(0,t) \times U$ to get:

$$\begin{aligned} 0 &= \int_{(0,t)\times U} u_{tt}u_t - u_t \Delta u \; \mathrm{d}x\mathrm{d}t \\ &= \int_{(0,t)\times U} \frac{\partial}{\partial t}\left(\frac{1}{2}u_t^2\right) - \mathrm{div}(u_t \cdot Du) + Du_t \cdot Du \; \mathrm{d}x\mathrm{d}t \\ &= \int_{(0,t)\times U} \frac{\partial}{\partial t}\left(\frac{1}{2}u_t^2 + \frac{1}{2}|Du|^2\right) - \mathrm{div}(u_t \cdot Du) \; \mathrm{d}x\mathrm{d}t \\ &= \frac{1}{2}\int_{\Sigma_t} u_t^2 + |Du|^2 \; \mathrm{d}x - \frac{1}{2}\int_{\Sigma_0} u_t^2 + |Du|^2 \; \mathrm{d}x - \int_{\partial^* U_t} u_t \cdot \frac{\partial u}{\partial \nu} \; \mathrm{d}S \end{aligned}$$

where in the second line we have used $\nabla \cdot (g\nabla h) = \nabla g \cdot \nabla h + g\Delta h$ (since "div" here means the divergence), in the third line we have used $gg' = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(g^2)$, and then on the last time we have just integrated with respect to t for one term and used the divergence theorem for the other (here, ν is the unit normal to $\partial^* U_t$).

But we know $u \equiv 0$ on $\partial^* U_t \Rightarrow u_t \equiv 0$ on $\partial^* U_t$. Thus we get:

$$\int_{\Sigma_t} u_t^2 + |Du|^2 \; \mathrm{d}x = \int_{\Sigma_0} u_t^2 + |Du|^2 \; \mathrm{d}x \quad \left(= \int_{\Sigma_0} (\psi')^2 + |D\psi|^2 \; \mathrm{d}x \right)$$

i.e. the energy of u is conserved with time.

In particular, this shows that our solution is unique: indeed, if we had two solutions, then by considering their difference, v say,, we would find that we have a solution to the problem where $\psi, \psi' \equiv 0$. So hence the above would give that $\int_{\Sigma_t} v_t^2 + |Dv|^2 \; \mathrm{d}x = 0$, i.e. $v_t = 0$ and $Dv = 0$. So hence v is a constant, which must be zero from the boundary conditions, and so hence the two solutions must agree.

This also shows us that is a solution exists, then we can control $\|u_t\|_{L^2(\Sigma_t)}$ and $\|u\|_{H^1(\Sigma_t)}$ in terms of the initial data, $\|\psi\|_{H^1(\Sigma_0)}$ and $\|\psi'\|_{L^2(\Sigma_0)}$.

Estimates such as this where we can control a solution without actually needing to construct it are called **a priori estimates** (i.e. we can write them down just straight from the equation). They are often crucial to establishing the existence of solutions (see Garding's inequality, Theorem 4.2 (ii), for example).

So let us define:

$$Lu := -\sum_{i,j=1}^{n}\left(a^{ij}(x,t)u_{x_j}\right)_{x_i} + \sum_{i=1}^{n} b^i(x,t)u_{x_i} + bu_t + c(x,t)u \tag{5.3}$$

for $a^{ij} = a^{ji}, b^i, c \in C^1(\overline{U}_T)$. Note the u_t term in L. We also assume **uniform ellipticity** of L, i.e. $\exists\theta > 0$ such that:

$$\sum_{i,j=1}^{n} a^{ij}(x,t)\xi_i\xi_j \geq \theta|\xi|^2 \quad \forall (x,t)\in U_T,\ \xi\in\mathbb{R}^n.$$

Then the IBVP we will consider is:

$$\begin{cases} u_{tt} + Lu = f & \text{in } U_T \\ u = \psi & \text{on } \Sigma_0 \\ u_t = \psi' & \text{on } \Sigma_0 \\ u = 0 & \text{on } \partial^* U_T. \end{cases} \tag{5.4}$$

We would like to find a weak formulation for this problem, with $u \in H^1(U_T)$. So suppose that $u \in C^2(\overline{U}_T)$ solves (5.4). Then, multiply the equation "$u_{tt}+Lu = f$" by some $v \in C^2(\overline{U}_T)$ which satisfies: $v = 0$ on $\partial^* U_T \cup \Sigma_T$ (we do not want $v = 0$ on σ_0 since we want to be able to recover the boundary conditions on the derivative at σ_0 from the weak formulation. Thus the boundary conditions become part of the function space). The integrate the resulting expression to get:

$$\begin{aligned}\int_0^T \mathrm{d}t \int_U \mathrm{d}x\ fv &= \int_0^T \mathrm{d}t\int_U \mathrm{d}x\ (u_{tt}v + (Lu)v) \\ &= \int_0^T \mathrm{d}t \int_U \mathrm{d}x\ \left(-u_t v_t + \sum_{i,j=1}^{n} a^{ij}u_{x_i}v_{x_j} + \sum_{i=1}^{n} b^i u_{x_i}v + bu_t v + cuv\right) \\ &\quad + \left[\int_U u_t v\ \mathrm{d}x\right]_{t=0}^{t=T} - \int_0^T \mathrm{d}t\int_{\partial U}\sum_{i,j} a^{ij}u_{x_j}v\ \mathrm{d}S_i\end{aligned}$$

where we have integrated by parts and used the divergence theorem. So hence:

$$\begin{aligned}\int_{U_T} fv\ \mathrm{d}x\mathrm{d}t = \int_{U_T}&\left(-u_t v_t + \sum_{i,j=1}^{n} a^{ij}u_{x_i}v_{x_j} + \sum_{i=1}^{n} b^i u_{x_i} v + bu_t v + cuv\right)\mathrm{d}x\mathrm{d}t \\ &- \int_{\Sigma_0}\psi' v\ \mathrm{d}x\end{aligned} \tag{5.5}$$

and, $u|_{\Sigma_0} = \psi$, $u|_{\partial^* U_T} = 0$.

Conversely, suppose that $u \in C^2(\overline{U}_T)$ satisfies (5.5) for all $v \in C^2(\overline{U}_T)$ with $v = 0$ on $\partial^* U_T \cup \Sigma_T$. First suppose that $v \in C_c^\infty(U_T)$. Then undoing the integration by parts, we get:

$$0 = \int_{U_T}(u_{tt} + Lu - f)v\ \mathrm{d}x\mathrm{d}t$$

and hence as v was arbitrary, we have $u_{tt} + Lu = f$ in U_T (get equality everywhere, not just a.e., since u is $C^2(\overline{U}_T)$).

So now pick $v \in C^\infty(\overline{U}_T)$ vanishing on $\partial^* U_T \cup \Sigma_T$ (i.e. may not vanish on Σ_0, the "bottom" on U_T). Then we get instead that:

$$\int_{U_T} (u_{tt} + Lu - f)v \; \mathrm{d}x\mathrm{d}t = \int_{\Sigma_0} (\psi' - u_t)v \; \mathrm{d}x.$$

But then by the above case, we know that $u_{tt} + Lu = f$, and so the LHS vanishes. So hence we get:

$$\int_{\Sigma_0} (\psi' - u_t)v \; \mathrm{d}x = 0 \quad \forall v \in C^\infty(\overline{U}_T) \text{ such that } v|_{\partial^* U_T \cup \Sigma_T} = 0.$$

But then this space of v includes v such that $v|_{\Sigma_0} = \varphi$ for any $\varphi \in C_c^\infty(\Sigma_0)$ (e.g. by taking $v(x,t) = \varphi(x)\chi(t)$, where $\chi(t) = 0$ near $t = T$ and $\chi(t) = 1$ near $t = 0$), and so hence we deduce that we have $\psi' = u_t$ on Σ_0 (thus recovering the boundary condition. Again, first we get this is true a.e., and then by continuity it is true everywhere).

So this shows that we can recover the initial problem just by assuming $u|_{\Sigma_0} = \psi$ and $u|_{\partial^* U_T} = 0$, i.e. no assumption on u_t. This therefore motivates our definition of a weak solution:

Definition 5.3. *Suppose that $f \in L^2(U_T)$ and $\psi \in H_0^1(\Sigma_0)$ (and so hence $\psi' \in L^2(\Sigma_0)$). Then we say that $u \in H^1(U_T)$ is a* ***weak solution to (5.4)*** *if:*

- *$u|_{\Sigma_0} = \psi$ in a trace sense*
- *$u|_{\partial^* U_T} = 0$ in a trace sense*
- *Equation (5.5) holds $\forall v \in H^1(U_T)$ with $v \equiv 0$ on $\partial^* U_T \cup \Sigma_T$, in a trace sense.*

Theorem 5.1. *A weak solution as above, if it exists, is unique.*

Proof. We want to take v in (5.5) with $v_t = u$, but we cannot necessarily do this (as we may not have the right assumption on v). So instead we define something close to this: let

$$v(x,t) := \int_t^T e^{-\lambda s} u(x,s) \; \mathrm{d}s.$$

where $\lambda \in \mathbb{R}$ is a constant we will choose later. Then $v \in H^1(U_T)$, and $v \equiv 0$ on $\partial^* U_T \cup \Sigma_T$, and we have $v_t = -e^{-\lambda t} u(x,t)$.

Note that in proving uniqueness, we can assume wlog that ψ, ψ', f all vanish, and then show that we must have $u = 0$ a.e. (since if we had two solutions, their difference will satisfy the problem in this case).

So we can take this v as a test function in (5.5), we get:

$$\int_{U_T} \left[u_t u e^{-\lambda t} - \sum_{i,j} a^{ij} v_{tx_i} v_{x_j} e^{\lambda t} + \sum_i b^i u_{x_i} v + (c-1)uv - v v_t e^{\lambda t} \right] \; \mathrm{d}x\mathrm{d}t = 0$$

where we have exchanged u with $v_t e^{\lambda t}$ in places, so that we will have time derivatives coming in. Getting all the derivatives of u on one side we can rewrite this as:

$$\int_{U_T} \Bigg[\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}u^2e^{-\lambda t} - \frac{1}{2}\sum_{i,j} a^{ij} v_{x_i} v_{x_j} e^{\lambda t} - \frac{1}{2}v^2 e^{\lambda t}\right) + \frac{\lambda}{2}\left(u^2e^{-\lambda t} + \sum_{i,j} a^{ij} v_{x_i} v_{x_j} e^{\lambda t} + v^2 e^{\lambda t}\right)}_{=\tilde{A}}$$

$$\underbrace{+\frac{1}{2}e^{\lambda t}\sum_{i,j} a^{ij}_t v_{x_i} v_{x_j} - \sum_i b^i_{x_i} uv - \sum_i b^i v_{x_i} u + (c-1)uv}_{=\tilde{B}} \Bigg] \,\mathrm{d}x\mathrm{d}t = 0.$$

where we have also used: $\frac{\mathrm{d}}{\mathrm{d}x_i}(b^i uv) = b^i_{x_i} uv + b^i u_{x_i} v + b^i u v_{x_i}$ to remove the derivative on the u (the LHS of this also integrates to 0 by assumption). The t-derivatives are also included in the latter two sums on the second line. Note that we also can't use uniform ellipticity on the term involving a^{ij}_t. Then write this as: $A = B$, where:

$$A = \int_{U_T} \tilde{A}\,\mathrm{d}x\mathrm{d}t \quad \text{and} \quad B = -\int_{U_T} \tilde{B}\,\mathrm{d}x\mathrm{d}t.$$

Integrating the time-derivative in A, using that $v = 0$ on Σ_T (i.e. when $t = T$) and $u = 0$ on Σ_0 (i.e. when $t = 0$) (and so only one of the two terms survives in each case), we get:

$$A = \underbrace{e^{\lambda T}\int_{\Sigma_T} \frac{1}{2}u^2\,\mathrm{d}x}_{\geq 0} + \underbrace{\frac{1}{2}\int_{\Sigma_0}\left(\sum_{i,j} a^{ij} v_{x_i} v_{x_j} + v^2\right)\mathrm{d}x}_{\geq 0 \text{ by uniform ellipticity}} + \frac{\lambda}{2}\int_{U_T}\left(u^2 e^{-\lambda t} + \underbrace{\sum_{i,j} a^{ij} v_{x_i} v_{x_j}}_{\geq \theta |Dv|^2 \text{ by uniform ellipticity}} e^{\lambda t} + v^2 e^{\lambda t}\right)\mathrm{d}x\mathrm{d}t$$

i.e.

$$A \geq \frac{\lambda}{2}\int_{U_T}\left[u^2 e^{-\lambda t} + \theta |Dv|^2 e^{\lambda t} + v^2 e^{\lambda t}\right]\mathrm{d}x\mathrm{d}t.$$

Then we also have (bounding the $b^i_t uv$ as usual and bounding $b^i u v_t = -b^i u^2 e^{-\lambda t} \leq c u^2 e^{-\lambda t}$):

$$B \leq \frac{C}{2}\int_{U_T}\left[u^2 e^{-\lambda t} + \theta |Dv|^2 e^{\lambda t} + v^2 e^{\lambda t}\right]\mathrm{d}x\mathrm{d}t$$

with C independent of λ (to get this, we have used 'Young's inequality with ε' on all terms and then choose the constant large enough so we can get $C\theta$ on the $|Dv|^2$ term. We have also used $uv = (ue^{\lambda t/2})(ve^{-\lambda t/2})$, etc). Hence taking this altogether, we have (as $A = B$):

$$\frac{1}{2}(\lambda - C)\int_{U_T}\underbrace{\left[u^2 e^{-\lambda t} + \theta |Dv|^2 e^{\lambda t} + v^2 e^{\lambda t}\right]}_{\geq 0}\mathrm{d}x\mathrm{d}t \leq 0.$$

So hence taking $\lambda > C$, we then need the integral to be equal to 0, and so in particular we need

$$\int_{U_T} u^2 e^{-\lambda t}\,\mathrm{d}x\mathrm{d}t = 0$$

which implies $u = 0$ a.e.. So done.

□

So we have uniqueness of weak solutions: now we work towards existence.

[The above proof if using the vector field method (seen in General Relativity), as we use $X = e^{-\lambda t}\partial_t$ as a multiplier, and ${}^X\Pi = -\lambda \mathrm{d}t \otimes \mathrm{d}t + O_X(1)$.]

5.2. **Existence of Solutions to Hyperbolic Equations.**

We now prove a result about the existence of solutions to hyperbolic equations.

Theorem 5.2 (Existence of Solutions to Hyperbolic Equations: Galerkin's Method of Finite Dimensional Projections). *Given $\psi \in H_0^1(U)$ with $\psi' \in L^2(U)$ and $f \in L^2(U_T)$, then $\exists!$ weak solution $u \in H^1(U_T)$ of (5.4), with:*

$$\|u\|_{H^1(U_T)} \leq C\left(\|\psi\|_{H^1(U)} + \|\psi'\|_{L^2(U)} + \|f\|_{L^2(U_T)}\right). \tag{5.6}$$

Proof. We will use Galerkin's Method: the idea is to project our equation onto a finite dimensional subspace of $H_0^1(U) \times L^2(U)$ and then take a limit.

Note that if we can show the result holds for $\psi \in C_c^\infty(U)$ (so also $\psi' \in C_c^\infty(U)$), and $f \in C_c^\infty(U_T)$, then the general result will follow by density arguments, provided we can first establish (5.6).

So let $\{\varphi_k\}_{k=1}^\infty$ be an orthonormal basis for $L^2(U)$, with $\varphi_k \in H_0^1(U)$ (e.g. take $\{\varphi_k\}_k$ to be the eigenfunctions of $-\Delta$ with Dirichlet boundary conditions - then this family works by Theorem 4.10).

Now define:

$$u^N(x,t) := \sum_{k=1}^N u_k(t)\varphi_k(x)$$

(i.e. separation of variables - x components form eigenfunction basis). Then from our equation ($Lu + u_{tt} = f$), the $u_k(t)$ can be determined from (subbing u^N in):

$$\left(\frac{\partial^2 u^N}{\partial t^2}, \varphi_k\right)_{L^2(U)} + \int_{\Sigma_t}\left[\sum_{i,j} a^{ij}u^N_{x_i}(\varphi_k)_{x_j} + \sum_i b^i u^N_{x_i}\varphi_k + cu^N\varphi_k\right]\mathrm{d}x = (f,\varphi_k)_{L^2(U)} \tag{5.7}$$

for each $k = 1,\ldots,N$. Here, $u_k(0) = (\psi,\varphi_k)_{L^2(U)}$ and $u_k'(0) = (\psi',\varphi_k)_{L^2(U)}$ (clearly, from $u_N(0)$, $(u^N)'(0)$).

This is a system of ODE's for the functions $u_k(t)$. This equation is uniform in c and t, and linear in the u_k's. So by Picard-Lindelöf, a solution exists for $t \in [0,T]$.

So for each N, we have an approximate solution u^N, which solves the equations when it is projected onto $\langle\varphi_1,\ldots,\varphi_N\rangle$ (i.e. project the boundary conditions onto here, and project the domain here as well).

So to construct the complete solution, we require an estimate. Multiply (5.7) by $e^{-\lambda t}\dot{u}_k(t)$ (here, $\dot{u} = \frac{\partial}{\partial t}u$), and sum it for $k = 1,\ldots,N$ and integrate from 0 to τ, for some $\tau \in (0,T)$, to get:

$$\int_0^\tau \mathrm{d}t \int_U \mathrm{d}x \left[\ddot{u}^N \dot{u}^N + \sum_{i,j} a^{ij} u^N_{x_i} \dot{u}^N_{x_j} + \sum_i b^i u^N_{x_i} \dot{u}^N + c u^N \dot{u}^N\right] e^{-\lambda t} = \int_0^\tau \mathrm{d}t \int_U \mathrm{d}x\ [f \dot{u}_N e^{-\lambda t}].$$

Rearranging, we get:

$$\begin{aligned}\int_0^\tau \mathrm{d}t \int_U \mathrm{d}x &\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\left(\frac{1}{2}(\dot{u}_N)^2 + \frac{1}{2}\sum_{i,j} a^{ij} u^N_{x_i} u^N_{x_j} + \frac{1}{2}(u^N)^2\right) e^{-\lambda t}\right)\right. \\ &\qquad\qquad \left. + \frac{\lambda}{2}\left((\dot{u}^N)^2 + \sum_{i,j} a^{ij} u^N_{x_i} u^N_{x_j} + (u^N)^2\right) e^{-\lambda t}\right] \\ = \int_0^\tau \int_U \mathrm{d}x &\left[\frac{1}{2}\sum_{i,j} \dot{a}^{ij} u^N_{x_i} u^N_{x_j} - \sum_i b^i u^N_{x_i} \dot{u}^N + (1-c) u^N \dot{u}^N + f \dot{u}^N\right] e^{-\lambda t}.\end{aligned}$$

Thus integrating the t-derivatives and estimating as before, for λ sufficiently large, we get:

$$\begin{aligned}\frac{1}{2}\int_{\Sigma_\tau} (\dot{u}^N)^2 + |Du^N|^2 + (u^N)^2\ \mathrm{d}x + \int_0^\tau \int_U (\dot{u}^N)^2 + |Du^N|^2 + (u^N)^2\ \mathrm{d}x\mathrm{d}t \\ \leq C\left(\|u^N\|^2_{H^1(U)} + \|\dot{u}^N\|_{L^2(U)} + \|f\|_{L^2(U_T)}\right)\end{aligned}$$

as here the term at $t = 0$ does not vanish. Thus as this is true for all $\tau \in [0, T]$, we get that:

$$\sup_{\tau\in[0,T]}\left(\|u^N\|_{H^1(\Sigma_\tau)} + \|\dot{u}^N\|_{L^2(\Sigma_\tau)}\right) + \|u^N\|_{H^1(U_T)} \leq C\left(\|u^N\|_{H^1(\Sigma_0)} + \|\dot{u}^N\|_{L^2(\Sigma_0)} + \|f\|_{L^2(U_T)}\right). \tag{5.8}$$

So since $u^N(0) = \sum_{k=1}^N (\psi, \varphi_k)\varphi_k \to \psi$ in $H^1(U)$ (and $\psi \neq 0$, so $\|\psi\|_{H^1(U)} > 0$), for N large enough (as the norms are close) we must have $\|u^N\|_{H^1(\Sigma_0)} \leq 2\|\psi\|_{H^1(U)}$.

(i.e. as eventually always we must have $\|u^N - \psi\|_{H^1(U)} \leq \varepsilon := \frac{1}{2}\|\psi\|_{H^1(U)}$, etc.)

Similarly, for N large enough we must have $\|\dot{u}^N\|_{L^2(U)} \leq 2\|\psi'\|_{L^2(U)}$.

So in (5.8), we have a uniform bound on $\|u^N\|_{H^1(U_T)}$ (as for N large enough we can bound uniformly, and so take the maximum of this with all $\|u^n\|_{H^1(U_T)}$ for $n \leq N$). So hence as $H^1(U_T)$ is weakly compact, we can extract a weakly convergent subsequence:

$$u^{N_m} \rightharpoonup u \quad \text{as } m \to \infty, \quad \text{in } H^1(U),$$

for some $u \in H^1_0(U)$. Note that u is such that (taking a limit in (5.8)):

$$\|u\|_{H^1(U)} \leq C\left(\|\psi\|_{H^1(U)} + \|\psi'\|_{L^2(U)} + \|f\|_{L^2(U_T)}\right).$$

So we have the inequality. Now we claim that u is the the weak solution to our problem.

Wlog we can relabel the sequence so that we have: $u^N \rightharpoonup u$ in $H^1(U)$.

So suppose that $v = \sum_{k=1}^m v_k(t)\varphi_k$, for some $v_k \in H^1((0,T))$ with $v(T) = 0$. Then multiply (5.7) by v_k, and sum from $k = 1, \ldots, m$ (taking $N > m$) to get:

$$\left(\frac{\mathrm{d}^2 u^N}{\mathrm{d}t^2}, v\right)_{L^2(U)} + \int_{\Sigma_t} \sum_{i,j} a^{ij} u^N_{x_i} v_{x_j} + \sum_i b^i u^N_{x_i} v + c u^N v \ \mathrm{d}x = (f, v)_{L^2(U)}.$$

Then integrating over $t \in [0,T]$, using that $v(T) = 0$, we get:

$$-\int_{\Sigma_0} \dot{u}^N v \ \mathrm{d}x + \int_{U_T} -u^N_t v_t + \sum_{i,j} a^{ij} u^N_{x_i} v_{x_j} + \sum_i b^i u^N_{x_i} v + c u^N v \ \mathrm{d}x\mathrm{d}t = \int_{U_T} f v \ \mathrm{d}x\mathrm{d}t.$$

So if $N > m$,

$$\int_{\Sigma_0} \dot{u}^N v \ \mathrm{d}x = \int_{\Sigma_0} \psi' v \ \mathrm{d}x$$

and then now, passing to the weak limit gives (i.e. taking $N \to \infty$ in the above):

(5.9) $$-\int_{\Sigma_0} \psi' v \ \mathrm{d}x + \int_{U_T} -u_t v_t + \sum_{i,j} a^{ij} u_{x_i} v_{x_j} + \sum_i b^i u_{x_i} v + c u v \ \mathrm{d}x\mathrm{d}t = \int_{U_T} f v \ \mathrm{d}x\mathrm{d}t.$$

i.e. what we want for a weak solution, for these specific v's.

Now note that for $k = 1, \ldots, m$, the map $H^1(U_T) \to \mathbb{R}$, sending $w \mapsto \int_{\Sigma_0} w\varphi_k \ \mathrm{d}x$, is a bounded linear map, and so we can conclude (by weak convergence):

$$\int_{\Sigma_0} u\varphi_k \ \mathrm{d}x = \lim_{N\to\infty} \int_{\Sigma_0} u^N \varphi_k \ \mathrm{d}x = (\psi, \varphi_k)_{L^2(\Sigma_0)}.$$

So hence this shows: $(u - \psi, \varphi_k)_{L^2(\Sigma_0)} = 0$ for all $k \Rightarrow u = \psi$ on Σ_0, i.e. $u|_{\Sigma_0} = \psi$, giving a boundary condition.

Also, v of the form considered above are dense in $\{v \in H^1(U_T) : v = 0 \text{ on } \partial^* U_T \cup \Sigma_T\}$, and so hence (5.9) holds for all v in this set, and thus we see u obeys all the conditions to be a weak solution, and so we are done.

□

Remark: In fact we have:

$$\text{ess sup}_{t\in(0,T)} \left(\|\dot{u}\|_{L^2(\Sigma_t)} + \|u\|_{H^1(\Sigma_t)}\right) \le C\left(\|\psi\|_{H^1(U)} + \|\psi'\|_{L^2(U)} + \|f\|_{L^2(U_T)}\right)$$

and so in fact we get: $u \in L^\infty\left((0,T); H^1(U)\right)$ and $\dot{u} \in L^\infty\left((0,T); L^2(U)\right)$ ($u \in H^1(U)$ implies $\dot{u} \in L^2(U)$), where:

Notation: We write for X a Banach space,

$$L^p((0,T),X) := \{u : (0,T) \to X : \|u\|_{L^p((0,T);X)} < \infty\}$$

where

$$\|u\|_{L^p((0,T);X)} := \left(\int_0^T \|u(t)\|_X^p \ \mathrm{d}t\right)^{1/p}$$

[see Example Sheet 4 for more details].

Note: Although the energy on each time slice is bounded, we cannot conclude that the energy is continuous.

5.3. **Hyperbolic Regularity.**

We now want to improve the regularity of our solution from $H^1(U_T)$. So suppose $u \in C^\infty(\overline{U}_T)$ satisfies:

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } U_T \\ u = \psi & \text{on } \Sigma_0 \\ u_t = \psi' & \text{on } \Sigma_0 \\ u = 0 & \text{on } \partial^* U_T. \end{cases}$$

We want a qualitative estimate for $u \in H^2(\Sigma_t)$. We differentiate the equation with respect to t, and let $w = u_t$ (we can do this as we assume $u \in C^\infty$), then we have:

$$\begin{cases} w_{tt} - \Delta w = 0 & \text{in } U_T \\ w = \psi' & \text{on } \Sigma_0 \\ w_t = \Delta\psi & \text{on } \Sigma_0 \\ w = 0 & \text{on } \partial^* U_T \end{cases}$$

where $\Delta\psi := \psi_{tt}$ (as 1 variable). Then immediately, from Theorem 5.2 applied to this system, we have:

$$\|w_t\|_{L^2(\Sigma_t)} + \|w\|_{H^1(\Sigma_t)} \leq C\left(\|\psi'\|_{H^1(U)} + \|\Delta\psi\|_{L^2(U)}\right).$$

So we have control of u_{tt}, u_{tx_i}, in $L^2(\Sigma_t)$ in terms of the initial data.

Then by an elliptic estimate, we know:

$$\|u\|_{H^2(\Sigma_t)} \underbrace{\leq}_{\text{always true}} C\|\Delta u\|_{L^2(\Sigma_t)} \equiv C\|u_{tt}\|_{L^2(\Sigma_t)}$$

(by Theorem 4.11, as we can write the equation as $\Delta u = u_{tt}$, and so we have $f \mapsto u_{tt}$), and so we recover that we can control all 2nd order derivatives of u in terms of the data.

[i.e. we have changed from a qualitative statement of "derivatives exist" to a quantitative one of "we can control the derivatives by things we know".]

So this is what we want, and works in the case where $u \in C^\infty$. The following makes this precise.

Theorem 5.3 (Hyperbolic Regularity). *If $a^{ij}, b^i, c \in C^2(U_T)$, and if ∂U is C^2, then for $\psi \in H^2(U)$, $\psi' \in H^1(U)$, $f, f_t \in L^2(U_T)$ (as before), we have:*

$$u \in H^2(U_T) \cap L^\infty((0,T); H^2(U)), \quad u_t \in L^\infty((0,T); H^1_0(U)), \quad u_{tt} \in L^\infty((0,T); L^2(U))$$

(i.e. as a map $t \mapsto u_t$, etc).

Proof. We return to the Galerkin approximation, Equation (5.7). Now by assumption, this is a linear system with C^2 coefficients, and so we know (from before/previous results) that $u^N \in C^3((0,T))$.

So differentiating (5.7) with respect to t (assuming as we can via approximation, $f, f_t \in C^0(\overline{U}_T)$) gives:

$$\left(\frac{\mathrm{d}^3 u^N}{\mathrm{d}t^3}, \varphi_k\right)_{L^2(U)} + \int_{\Sigma_t} \sum_{i,j} a^{ij} \dot{u}^N_{x_i} (\varphi_k)_{x_j} + \sum_i b^i \dot{u}^N_{x_i} \varphi_k + c\dot{u}^N \varphi_k \, \mathrm{d}x$$
$$= (\dot{f}, \varphi_k)_{L^2(U)} - \int_{\Sigma_t} \left[\sum_{i,j} \dot{a}^{ij} u^N_{x_i} (\varphi_k)_{x_j} + \sum_i \dot{b}^i u^N_{x_i} \varphi_k + \dot{c} u \varphi_k\right] \mathrm{d}t$$

Multiplying by $\frac{\mathrm{d}^2 u_k}{\mathrm{d}t^2} e^{-\lambda t}$ and summing from $k = 1, \dots, N$ and integrating $\int_0^\tau \mathrm{d}t$, and recalling that we already control $u \in H^1(U_T)$ (from Theorem 5.2), we have:

$$\sup_{t \in (0,T)} \left(\|u^N_t\|_{H^1(\Sigma_t)} + \|u^N_{tt}\|_{L^2(\Sigma_t)}\right) + \|u^N_t\|_{H^1(U_T)}$$
$$\leq C\left(\|u^N_t\|_{H^1(\Sigma_0)} + \|u^N_{tt}\|_{L^2(\Sigma_0)} + \|\psi\|_{H^1(\Sigma_0)} + \|\psi'\|_{L^2(\Sigma_0)} + \|f\|_{L^2(U_T)} + \|f_t\|_{L^2(U_T)}\right).$$

But we also know: $u^N_t|_{t=0} = \sum_{k=1}^N (\psi', \varphi_k)_{L^2(\Sigma_0)} \varphi_k$, and so since the φ_k are a basis of H^1, we have from this:

$$\|u^N_t\|_{H^1(\Sigma_0)} \leq \|\psi'\|_{H^1(\Sigma_0)}$$

via the proof of Bessel's inequality (i.e. uniform control on u^N_t).

To control u^N_{tt} uniformly, let us assume that in fact the φ_k are the eigenfunctions of $-\Delta$ (i.e. this is our basis). Then from (5.7), integrating by parts in the first term, multiplying by $\ddot{u}^N \equiv u^N_{tt}$ and summing over k, we get:

$$\|u^N_{tt}\|_{L^2(\Sigma_0)} \leq C\left(\|u^N\|_{H^2(\Sigma_0)} + \|f\|_{L^2(U_T)} + \|f_t\|_{L^2(U_T)}\right).$$

So hence to get uniform control on u^N_{tt}, it suffices to control $\|u^N\|_{H^2(\Sigma_0)}$ by $\|\psi\|_{H^2(\Sigma_0)}$. So note, since Δ is self-adjoint:

$$(\Delta u^N, \Delta u^N) = (u^N, \Delta^2 u^N) = (\psi, \Delta^2 u^N) = (\Delta\psi, \Delta u^N)$$

where these inner products are those on $L^2(\Sigma_0)$, and in the second equality we have used the fact that $\Delta\varphi_k|_{\partial U} = 0$ (as these are eigenfunctions of the BVP for Laplace's equation) and u^N is a finite sum of the φ_k's.

So hence: $\|u^N\|_{H^2(\Sigma_0)} \leq C\|\Delta u^N\|_{L^2(\Sigma_0)} \leq C\|\psi\|_{H^2(U)}$, which gives the control we wanted.

Thus when we pass to the (weak) limit, $N \to \infty$, we see that $u_t \in H^1(U_T)$, and $u_t \in L^\infty((0,T); H^1_0(U))$, and $u_{tt} \in L^\infty((0,T); L^2(U))$.

Then since $u_{tt} + Lu = f$, by an elliptic estimate on almost every $t =$ constant slice, we obtain that $u \in L^\infty((0,T); H^2(U))$. So hence we are done.

□

So we have now shown:

"If $a^{ij}, b^i, c \in C^2(\overline{U}_T)$, ∂U is C^2, and $u \in H^1(U_T)$ is a weak solution of:

$$\begin{cases} u_{tt} + Lu = f & \text{in } U_T \\ u = \psi & \text{on } \Sigma_0 \\ u_t = \psi' & \text{on } \Sigma_0 \\ u = 0 & \text{on } \partial^* U_T \end{cases}$$

then if $\psi \in H^2(U) \cap H_0^1(U)$, $\psi' \in H_0^1(U)$ and $f, f_t \in L^2(U_T)$, then:

$$\begin{aligned} &u \in L^\infty((0,T); H^2(U) \cap H_0^1(U)) \cap H^2(U_T) \\ &u_t \in L^\infty((0,T); H_0^1(U)) \\ &u_{tt} \in L^\infty((0,T); L^2(U)).\text{"} \end{aligned}$$

Remarks:

(i) We can now understand the equation as holding pointwise a.e. by undoing the integrations by parts that gave us the definition of the weak solution (as we now know by the above that u has enough weak derivatives).

The initial conditions can be understood in a trace sense.

(ii) Returning to the case where $\psi \in H_0^1(U)$, $\psi' \in L^2(U)$, by approximation them in $H^2(U)$ and $H_0^1(U)$ respectively, we can show that a weak solution can be constructed as a strong limit in $H^1(U_T)$. This implies the energy identity from before, and so in fact the weak solutions satisfy:

$$\begin{aligned} &u \in C^0((0,T); H_0^1(U)) \\ &u_t \in C^0((0,T); L^2(U)). \end{aligned}$$

This requires slightly stronger regularity assumptions on a^{ij}, b^i, c. Such solutions are said to be in the **energy class**.

(iii) We can iterate the argument above for Theorem 5.3 to get higher regularity results, such as the following theorem:

Theorem:[Higher Hyperbolic Regularity] *For $k \geq 1$, if $a^{ij}, b^i, c \in C^{k+1}(\overline{U}_T)$ with ∂U being C^{k+1} and:*

$$\left.\frac{\partial^i u}{\partial_t^i}\right|_{\Sigma_0} \in H_0^1(U) \quad \text{for } i = 0, \dots, k$$

$$\left.\frac{\partial^{k+1} u}{\partial t^{k+1}}\right|_{\Sigma_0} \in L^2(U)$$

$$\frac{\partial^i f}{\partial t^i} \in L^2((0,T); H^{k-i}(U)) \quad \text{for } i = 0, \dots, k.$$

Then, $u \in H^{k+1}(U_T)$ and $\frac{\partial^i u}{\partial t^i} \in L^\infty\left((0,T); H^{k+1-i}(U)\right)$ for $i = 0, \dots, k+1$.

Proof. None given.

□

Note that since $u_{tt}|_{\Sigma_0} = (f - Lu)|_{\Sigma_0}$, we see that: $u_{ttt}|_{\Sigma_0} = (f_t - (Lu)_t)|_{\Sigma_0}$, etc, we can therefore re-express the initial conditions in terms of ψ, ψ'. The conditions then imply that $\psi \in H^{k+1}(U)$, $\psi' \in H^k(U)$.

But this also encodes some compatibility conditions. Indeed, consider:

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } U_T \\ u = \psi & \text{on } \Sigma_0 \\ u_t = \psi' & \text{on } \Sigma_0 \\ u = 0 & \text{on } \partial^* U_T. \end{cases}$$

If we have a smooth solution, then we know by the above that $u_{tt} = 0$ on $\partial^* U_T$, and so (from $u_{tt} = \Delta u$) we must have $\Delta u = 0$ on $\partial^* U_T$, and thus we must have $\Delta \psi = 0$ on $\partial^* U_T$ (as $u = \psi$ on Σ_0) for compatibility of the solution on the different parts of ∂U_T (i.e. need these parts to agree on $\partial^* U_T \cap \Sigma_0$).

Also, it is obvious that if everything is smooth (∂U, a^{ij}, etc), and all compatibility conditions hold, then we get a smooth solution (by the above Theorem and the Sobolev embeddings).

5.4. Finite Speed of Propagation.

A crucial feature of hyperbolic equations is "**finite speed of propagation**": this is the principle that information can only travel at a certain speed.

[This leads to the fact that "nothing can travel faster than the speed of light", by applying this principle to a certain hyperbolic PDE.]

We can make this precise.

Let $S_0 \subset U$ be an open set with smooth boundary, and let:

$$D = \{(t,x) \in U_T : x \in S_0,\ 0 < t < \tau(x)\}$$

where $\tau : S_0 \to (0,T)$ is a smooth function, vanishing on ∂S_0 (this is something like a surface with boundary ∂S_0 above S_0). Let $S' := \text{Graph}(\tau) := \{(\tau(x), x) : x \in S_0\}$ (note the t-coordinate is first).

Definition 5.4. *We say that S' is **spacelike** if:*

$$\sum_{i,j=1}^{n} a^{ij}(x)\tau_{x_i}\tau_{x_j} < 1 \quad \forall x \in S_0.$$

Theorem 5.4. *If S' is spacelike and if u is a weak solution of*

$$\begin{cases} u_{tt} + Lu = f & \textit{in } U_T \\ u = \psi & \textit{on } \Sigma_0 \\ u = \psi' & \textit{on } \Sigma_0 \\ u = 0 & \textit{on } \partial^* U_T \end{cases}$$

then, $u|_D$ only depends on the values of $\psi|_{S_0}$, $\psi'|_{S_0}$ and $f|_D$.

Proof. This proof is identical to the uniqueness of weak solutions proof (Theorem 5.1), although first we modify v to be 0 outside D. The expression for v is the same, just made to vanish on S', etc.

Returning to the definition of a weak solution, we have:

$$\int_{U_T} -u_t v_t + \sum_{i,j} a^{ij} u_{x_i} v_{x_j} + \sum_i b^i u_{x_i} v + cuv \, \mathrm{d}x\mathrm{d}t - \int_{\Sigma_0} \psi' v \, \mathrm{d}x = \int_{U_T} f v \, \mathrm{d}x\mathrm{d}t \tag{5.10}$$

for all $v \in H^1(U_T)$ such that $v = 0$ on $\Sigma_T \cup \partial^* U_T$.

By linearity, it suffices to prove that $u|_D = 0$ if $\psi|_{S_0} = 0$, $\psi'|_{S_0} = 0$, and $f|_D = 0$.

So in this case, take:

$$v(t,x) := \begin{cases} \int_t^{\tau(x)} e^{-\lambda s} u(s,x) & \text{if } (t,x) \in D \\ 0 & \text{if } (t,x) \notin D \end{cases}$$

(note that this is in $H^1(U_T)$ and has $v = 0$ on $\Sigma_T \cap \partial^* U_T$. v is also continuous on S_0 as $u = \psi = 0$ there, and on S', i.e. at $t = \tau(x)$).

We then have:

$$v_{x_i} = \tau_{x_i} e^{-\lambda\tau(x)} u(x, \tau(x)) + \int_t^{\tau(x)} e^{-\lambda s} u_{x_i}(x,s) \, \mathrm{d}s$$

and

$$v_t = -e^{-\lambda t} u(x,t).$$

Then inserting this into (5.10), arguing as in the proof of Theorem 5.1, we have:

$$\int_D \left[\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{1}{2} u^2 e^{-\lambda t} - \frac{1}{2} \sum_{i,j} a^{ij} v_{x_i} v_{x_j} e^{\lambda t} - \frac{1}{2} v^2 e^{\lambda t} \right) + \frac{\lambda}{2} \left(u^2 e^{-\lambda t} + \sum_{i,j} a^{ij} v_{x_i} v_{x_j} e^{\lambda t} + v^2 e^{\lambda t} \right) \right] \mathrm{d}x\mathrm{d}t$$

$$= \int_D \left[\frac{1}{2} \sum_{i,j} \dot{a}^{ij} v_{x_i} v_{x_j} e^{\lambda t} - \sum_i b^i u_{x_i} v - (c-1)uv \right] \mathrm{d}x\mathrm{d}t.$$

Then noting that be Fubini's theorem, $\int_D \mathrm{d}x\mathrm{d}t = \int_{S_0} \mathrm{d}x \left(\int_0^{\tau(x)\mathrm{d}t} \right)$, we can perform the integral of the $\frac{\mathrm{d}}{\mathrm{d}t}$ term. The only difference from before when performing this integral (where we set $A = \text{LHS}$, $B = \text{RHS}$) is that here we get a contribution at S', which is:

$$I_{S'} = \int_{S_0} \mathrm{d}x \, \left(\frac{1}{2} u^2(\tau(x), x) e^{-\lambda\tau(x)} - \frac{1}{2} \sum_{i,j} a^{ij} \tau_{x_i} \tau_{x_j} u^2 e^{-\lambda\tau} \right)$$

where we have used the fact that $v = 0$ on S', and $v_{x_i} = \tau_{x_i} u e^{-\lambda\tau}$ on S'. So we just need to show (as then the proof goes as before in Theorem 5.1) that $I_{S'} \geq 0$. But note that the integral is of:

$$\underbrace{\frac{1}{2}u^2 e^{-\lambda\tau}}_{\geq 0} \underbrace{\left(1 - \sum_{i,j} a^{ij}\tau_{x_i}\tau_{x_j}\right)}_{\geq 0 \text{ as } S' \text{ is spacelike}} \geq 0$$

and so hence $I_{S'} \geq 0$. So hence the rest of the argument goes through as before (i.e. LHS ≥ 0, RHS ≤ 0), and so we can conlude that $u \equiv 0$ on D. So done.

$\square$

Remark: This result implies that no signal can travel faster than a certain speed, due to the t-dependence of D (i.e. choose a point x_0, then this will only depend on the data once t is large enough so that $D \ni x_0$). So in particular, if $\sum_{i,j} a^{ij}\xi_i\xi_j \leq \mu|\xi|^2$, then no signal can travel faster than $\sqrt{\mu}$.

This observation allows us to reduce the problem of solving a PDE on an unbounded domain to that of solving one on a sufficiently large bounded domain (so long as we restrict our time as well so that the solution does not evolve out of this domain), i.e. 'everything is local' in hyperbolic equations.

Remark: We only developed elliptic PDE theory for boundary value problems. However we have only developed hyperbolic PDE theory for initial data problems.

End of Lecture Course