Analysis of PDEs

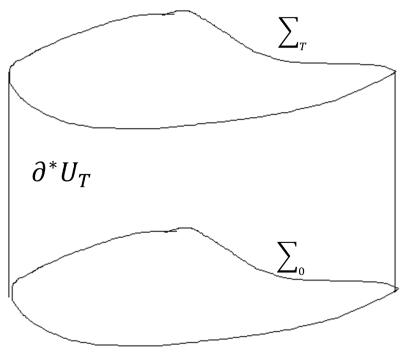
(23rd November, Friday)

Initial-Boundary Value Problems for Wave Equations

Suppose $U \subset \mathbb{R}^n$ is open with C^1-boundary. We define

$$U_T = U \times (0,T), \quad \Sigma_t = U \times \{t\}, \quad \partial^* U_T = \partial U \times [0,T]$$

So $\partial U_T = \Sigma_0 \sqcup \Sigma_T \sqcup \partial^* U_T$. We define



$$Lu = -\sum_{i,j=1}^{n} (a^{ij} u_{x_i})_{x_j} + \sum_{i=1}^{n} b^i u_{x_i} + b u_t + cu$$

where $a^{ij}, b^i, b, c \in C^1(\overline{U}_T)$. Further assume a^{ij} satisfy the uniform ellipticity condition

$$\sum_{i,j=1}^{n} a^{ij}(x,t)\xi_i \xi_j \geq \theta |\xi|^2$$

for some $\theta > 0$, all $(x,t) \in U_T$, $\xi \in \mathbb{R}^n$.

The **initial-boundary value problem(IBVP)** we consider is:

$$\begin{cases} u_{tt} + Lu = f & \text{in } U_T \\ u = \psi, \ u_t = \psi' & \text{on } \Sigma_0 \\ u = 0 & \text{on } \partial^* U_T \end{cases} \tag{1}$$

e.g. The model in our mind is solving wave equation on a string given boundary conditions. If $L = -\Delta$, $f = 0$, this is the wave equation on a bounded domain with specified initial conditions.

As with the elliptic boundary value problem, we first find a weak formulation of the problem. Suppose $u \in C^2(\overline{U}_T)$ is a solution of (1) and multiply the equation by $v \in C^2(\overline{U}_T)$ satisfying $v = 0$ on $\partial^* U_T \cup \Sigma_T$. Then integrate over U_T.

$$\int_0^T dt \int_U dx(u_{tt}v + Luv) = \int_0^T dt \int_U dx f v$$

Integrating by parts, we find

$$\int_{U_T} \Big(- u_t v_t + \sum_{i,j} a^{ij} u_{x_i} v_{x_j} + \sum_i b^i u_{x_i} v + b u_t v + cuv \Big) dxdt - \int_{\Sigma_0} \psi' v dx = \int_{U_T} f v dxdt \qquad (2)$$

Conversely, if (2) holds for all $v \in C^2(\overline{U}_T)$ which vanish on $\Sigma_T \cup \partial^* U_T$ and $u \in C^2(\overline{U}_T)$ satisfies $u = \psi$ on Σ_0, $u = 0$ on $\partial^* U_T$, undoing the integration by parts gives

$$\int_{U_T} (u_{tt}v + Lv - fv)dxdt + \int_{\Sigma_0} (u_t - \psi')vdx = 0$$

Taking $v \in C_c^\infty(U_T)$, the Σ_0 term vanishes and we deduce $u_{tt} + Lu = f$ in U_T. This implies

$$\int_{\Sigma_0} (u_t - \psi')vdx = 0 \quad \forall v \in C_c^\infty(\Sigma_0) \quad \Rightarrow \quad u_t = \psi'$$

The expression (2) makes sense if $u \in H^1(U_T)$, $v \in H^1(U_T)$. This motivates the definition :

Definition) Suppose $f \in L^2(U_T)$, $\psi \in H_0^1(\Sigma_0)$, $\psi \in L^2(\Sigma_0)$ and $a^{ij}, b^i, b, c \in C^1(\overline{U}_T)$ with a^{ij} satisfying uniform ellipticity condition in U_T. We say $u \in H^1(U_T)$ is a weak solution of the IBVP (1) if

$$\begin{cases} u = \psi & \text{on } \Sigma_0 & \text{in the trace sense} \\ u = 0 & \text{on } \partial^* U_T & \text{in the trace sense} \end{cases}$$

and (2) holds for all $v \in H^1(U_T)$ with $v = 0$ on $\Sigma_T \cup \partial^* U_T$ in the trace class.

Note that, we could not say $\partial_t u = \psi'$ on Σ_0 in trace sense, because $\partial_t u$ is just a L^2-function while we do not have trace theorem for L^2 functions.

We cannot use Lax-Milgrim theorem as it is. But we can do something different to show unique existence of the solution in a different way.

Theorem) A weak solution to (1), if it exists, is unique.

Motivation : Suppose we consider the standard wave equation

$$u_{tt} - \Delta u = 0 \quad \text{in } U_T$$

with the initial and boundary conditions as in (1). Assume $u \in C^2(U_T)$. To show the solution is unique, sufficient to consider $\psi = \psi' = 0$. Multiply by u_t and integrate over $x \in U$.

$$\int_U u_{tt}u_t - \Delta u \cdot u_t dx = \int_U u_{tt}u_t + Du \cdot Du_t dx = \frac{d}{dt} \int_U \frac{1}{2} u_t^2 + \frac{1}{2} |Du|^2 dx$$

So if $u = u_t = 0$ initially, then

$$\int_{\Sigma_t} \frac{1}{2} u_t^2 + |Du|^2 dx = 0 \quad \forall t \in (0, T)$$

and therefore $u = 0$ in U_T.

We work in the same spirit for the general case where $u \in H^1(U_T)$, but we have to be more careful when doing this.

proof of theorem) Note that by linearity, sufficient to prove that if $\psi = 0$, $\psi' = 0$, $f = 0$ then $u = 0$. We want to use u_t as a test function but it is not regular enough (does not vanish on Σ_T). Take

$$v(x, y) = \int_t^T e^{-\lambda s} u(x, s) ds$$

for $\lambda \in \mathbb{R}$ we choose later. We find $v \in H^1(U_T)$, $v = 0$ on $\partial^* U_T \cup \Sigma_T$ and $v_t = -e^{-\lambda t} u \in H^1(U_T)$. Putting this into (2) with $\psi = \psi' = f = 0$, we have

$$\int_{U_T} \Big[u_t u e^{-\lambda t} - \sum_{i,j} a^{ij} v_{tx_i} v_{x_j} e^{\lambda t} + \sum_i b^i u_{x_i} v - b v^2 e^{\lambda t} + (c-1) uv - v v_t e^{\lambda t} \Big] dxdt = 0$$

Rewriting,

$$\begin{aligned}
\textbf{(A)} = \int_{U_T} \Big[&\frac{d}{dt} \Big(\frac{1}{2} u^2 e^{-\lambda t} - \frac{1}{2} \sum_{i,j} a^{ij} v_{x_i} v_{x_j} e^{\lambda t} - \frac{1}{2} v^2 e^{\lambda t} \Big) \\
&+ \frac{\lambda}{2} \Big(u^2 e^{-\lambda t} + \frac{1}{2} \sum_{i,j} a^{ij} v_{x_i} v_{x_j} e^{\lambda t} + v^2 e^{\lambda t} \Big) \Big] dxdt \\
= \int_{U_T} \Big[&\frac{1}{2} \sum_{i,j} \dot{a}^{ij} v_{x_i} v_{x_j} e^{\lambda t} - \sum_i b^i u_{x_i} v + b v^2 e^{\lambda t} - (c-1) uv \Big] dxdt = \textbf{(B)}
\end{aligned}$$

and

$$\begin{aligned}
\textbf{(A)} = &\int_{\Sigma_T} \frac{1}{2} u^2 e^{-\lambda T} dx + \int_{\Sigma_0} \Big(\frac{1}{2} \sum_{i,j} v_{x_i} v_{x_j} + \frac{1}{2} v^2 \Big) \\
&+ \frac{\lambda}{2} \int_{U_T} \Big(u^2 e^{-\lambda t} + \frac{1}{2} \sum_{i,j} a^{ij} v_{x_i} v_{x_j} + v^2 e^{\lambda t} \Big) dxdt
\end{aligned}$$

and (using AM-GM inequality and that a, b, c are of C^1)

$$\textbf{(B)} \le C \int_{U_T} u^2 e^{-\lambda t} + (\sum_{i,j} a^{ij} v_{x_i} v_{x_j} + v^2) e^{\lambda t} dxdt$$

for some constant C independent of λ. Putting these together and taking λ large enough, we have

$$(\lambda - 2C) \int_{U_T} u^2 e^{-\lambda t} + (\sum_{i,j} a^{ij} v_{x_i} v_{x_j} + v^2) e^{\lambda t} dxdt \le 0$$

With $\lambda - 2C \ge 0$, we have $u \equiv 0$

(End of proof) $\square$

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(26th November, Monday)

Theorem) Given $\psi \in H_0^1(U)$, $\psi' \in L^2(U)$ and $f \in L^2(U_T)$, there exists a weak solution $u \in H^1(U_T)$ and

$$\|u\|_{H^1(U_T)} \le C(\|\psi\|_{H^1(U)} + \|\psi'\|_{L^2(U)} + \|f\|_{L^2(U_T)}) \tag{3}$$

for some $C = C(U, T, a^{ij}, a^i, b, c)$ not depending on u.

proof) We use *Galerkin's method.* The idea is to project the equation onto a finite dimensional subspace of $L^2(U)$, spanned by the first N eigenfunctions of the Dirichlet Laplacian (or some other convenient basis for $L^2(U)$). We assume that $\psi, \psi' \in C_c^\infty(U)$, $f \in C_c^\infty(U_T)$. Since these spaces are dense in $H_0^1(U), L^2(U), L^2(U_T)$ respectively, we can recover the result for general ψ, ψ', f using a continuity argument once (3) is established.

Let $\{\varphi_k\}_{k=1}^\infty$ be an orthonormal basis for $L^2(U)$ with $\varphi_k \in H_0^1(U)$, e.g. take φ_k to be the k^{th} eigenfunction of

$$\begin{cases} -\Delta\varphi_k = \lambda_k\varphi_k & \text{in } U \\ \varphi_k = 0 & \text{on } \partial U \end{cases}$$

Now, define

$$u^N(x,t) = \sum_{k=1}^N u_k(t)\varphi_k(x)$$

where $u_k(t)$ are determined by solving the ordinary differential equation :

$$\Big(\frac{d^2u^N}{dt^2}, \varphi_k\Big) + \int_{\Sigma_t} \Big[\sum_{i,j} a^{ij}u^N_{x_j}(\varphi_k)_{x_j} + \sum_i b^i u^N_{x_i}\varphi_k + bu^N_t\varphi_k + cu^N\varphi_k\Big]dx = \int_{\Sigma_t} f\varphi_k dx \tag{4}$$

and $u_k^N(0) = (\psi, \varphi_k)_{L^2(U)}$, $\dot{u}_k^N(0) = (\psi', \varphi_k)_{L^2(U)}$ for $k = 1, \cdots, N$. This is the projection of the PDE onto $\langle\varphi_1, \cdots, \varphi_N\rangle$. Note that (4) is a system of N-ODE's for the unknowns $u_k^N(t)$, $k = 1, \cdots, N$ which is linear in u_k^N with coefficients which are C^1 in t. By Picard-Lindelöf, a unique solution exists for $u_k^N : [0,T] \to \mathbb{R}$. We now estimate u^N. Multiply (4) by $e^{-\lambda t}\dot{u}_k^N(t)$ and sum over $k = 1, \cdots, N$. Noting that $\sum_{k=1}^N e^{-\lambda t}\dot{u}_k^N(t)\varphi_k(x) = \dot{u}^N e^{-\lambda t}$, we find after integrating over $t \in [0,\tau]$ for some $\tau \in (0,T]$,

$$\begin{aligned}\int_0^\tau dt \int_U dx \Big[\ddot{u}^N\dot{u}^N e^{-\lambda t} + \sum_{i,j=1} a^{ij}u^N_{x_i}\dot{u}^N_{x_j}e^{-\lambda t} + \sum_i b^i u^N_{x_i}\dot{u}^N_{x_j}e^{-\lambda t} \\ + b(\dot{u}^N)^2e^{-\lambda t} + u^N\dot{u}^Ne^{-\lambda t} + (c-1)u^N\dot{u}^Ne^{-\lambda t}\Big] = \int_0^\tau dt\int_U dx(f\dot{u}^Ne^{-\lambda t})\end{aligned}$$

Rearranging,

$$\begin{aligned}\textbf{(A)} = \int_0^\tau dt\int_U dx\Big[\frac{d}{dt}\Big[\Big(\frac{1}{2}(\dot{u}^N)^2 + \frac{1}{2}\sum_{i,j} a^{ij}u^N_{x_i}u^N_{x_j} + \frac{1}{2}(u^N)^2\Big)e^{-\lambda t}\Big] \\ + \frac{\lambda}{2}\Big((\dot{u}^N)^2 + \frac{1}{2}\sum_{i,j} a^{ij}u^N_{x_i}u^N_{x_j} + (u^N)^2\Big)e^{-\lambda t}\Big] \\ = \int_0^\tau dt\int_U dx\Big[\frac{1}{2}\sum_{ij}\dot{a}^{ij}u^N_{x_i}u^N_{x_j} - \sum_i b^iu^N_{x_i}\dot{u}^N - b(\dot{u}^N)^2 + (1-c)u^N\dot{u}^N + f\dot{u}^N\Big]e^{-\lambda t} = \textbf{(B)}\end{aligned}$$

We may write

$$\begin{aligned}\textbf{(A)} =& \frac{e^{-\lambda\tau}}{2}\int_{\Sigma_\tau}(\dot{u}^N)^2 + \sum_{i,j}a^{ij}u^N_{x_i}u^N_{x_j} + (u^N)^2)dx - \frac{1}{2}\int_{\Sigma_0}(\dot{u}^N)^2 + \sum_{i,j}a^{ij}u^N_{x_i}u^N_{x_j} + (u^N)^2)dx \\ &+ \frac{\lambda}{2}\int_0^\tau dt\int_U dx\Big[(\dot{u}^N)^2 + \frac{1}{2}\sum_{i,j}a^{ij}u^N_{x_i}u^N_{x_j} + (u^N)^2\Big]e^{-\lambda t}\end{aligned}$$

so can bound

$$\begin{aligned}\textbf{(A)} \geq & \frac{e^{-\lambda\tau}}{2}\int_{\Sigma_\tau}((\dot{u}^N)^2+(u^N)^2+\theta|Du^N|^2)dx \\ & +\frac{\lambda}{2}\int_0^\tau dt\int_U dx\Big((\dot{u}^N)^2+(u^N)^2+\theta|Du^N|^2\Big)e^{-\lambda t}-C_1\big(\|\psi'\|^2_{L^2(U)}+\|\psi\|^2_{H^1(U)}\big)\end{aligned}$$

where C_1 is independent of N,λ. On the other hand,

$$\textbf{(B)} \leq C_2\int_0^\tau dt\int_U dx\Big((\dot{u}^N)^2+(u^N)^2+\theta|Du^N|^2\Big)e^{-\lambda t}+C_3\int_0^\tau\int_U dx f^2e^{-\lambda t}$$

with C_2, C_3 again independent of N,λ, where the last term is estimated using Cauchy-Schwarz inequality. Combining these estimates, choosing $\lambda > C_2$, we conclude

$$\begin{aligned}&\sup_{\tau\in[0,T]}\big(\|u^N\|^2_{H^1(\Sigma_\tau)}+\|\dot{u}^N\|^2_{L^2(\Sigma_\tau)}\big)+\|u^N\|^2_{H^1(U_T)} \\ &\leq C_4\big(\|\psi'\|^2_{L(U)}+\|\psi\|^2_{H^1(U)}+\|f\|^2_{L^2(U_T)}\big)\end{aligned}$$

where C_4 is independent of N. Thus we can extract a subsequence $u^{N_m}\xrightarrow{w} u$ in $H^1(U_T)$.

It remains to show u is a weak solution. To see this, consider v of form $v=\sum_{k=1}^M v_k(t)\varphi_k(x)$. For some $v_k\in C^1([0,T])$ with $v_k(T)=0$. Multiply the ODE for u^N by $v_K(t)$, summing over $k=1,\cdots,M$ and integrating over $[0,T]$ in t. We can integrate the $\ddot{u}^N$ term by parts to find

$$\int_{U_T}\Big(-u^N_tv_t+\sum_{i,j}a^{ij}u^N_{x_i}v_{x_j}+\sum_i b^iu^N_{x_i}v+b\dot{u}^Nv+c\cdot uv\Big)dxdt-\int_{\Sigma_0}u^N_tvdx=\int_{U_T}fvdxdt$$

Now if $N>M$, we have

$$\int_{\Sigma_0}u^N_tvdx=\int_{\Sigma_0}\psi'vdx$$

Setting $N=N_m$ and sending $m\to\infty$, we find

$$\int_{U_T}\Big(-u_tv_t+\sum_{i,j}a^{ij}u_{x_i}v_{x_j}+\sum_i b^iu_{x_i}v+b\dot{u}v+c\cdot uv\Big)dxdt-\int_{\Sigma_0}\psi'vdx=\int_{U_T}fvdxdt$$

Note that v's of the form $v=\sum_{k=1}^M v_k(t)\varphi_k(x)$ are dense in $H^1(U_T)$ with $v=0$ on $\Sigma_T\cup\partial^*U_T$ so u satisfies the identity to be a weak solution.

Finally, we check the boundary conditions. We note for $k=1,2,\cdots$, we have

$$w\mapsto\int_{\Sigma_0}w\varphi_ndx$$

is a bounded linear functional on $H^1(U_T)$ so we can conclude

$$\int_{\Sigma_0}u\varphi_kdx=\lim_{M\to\infty}\int_{\Sigma_0}u^{N_M}\varphi_kdx=(\psi,\varphi_k)_{L^2(U)}$$

so $u=\psi$ on Σ_0.

Note we actually have established a stronger estimate :

$$\sup_{\tau\in[0,T]}(\|u\|^2_{H^1(\Sigma_\tau)}+\|\dot{u}\|^2_{L^2(\Sigma_t)})+\|u\|^2_{H^1(U_T)}\leq C_4(\|\psi'\|^2_{L^2(U)}+\|\psi\|^2_{H^1(U)}+\|f\|^2_{L^2(U_T)})$$

(End of proof) $\square$