Elliptic Partial Differential Equations

Jiwoon Park

Lent 2019

(26th February, Tuesday)

We have seen in the last lecture how we can find solution for $-\triangle u = f(u)$ using $C^{2,\alpha}$ Schauder estimates (potential thoery).

One famous example of equations of such type is prescribed curvuture equation. That is, for a Riemannian surface (M, g), it solves

$$\operatorname{div}\Big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big) = H(u), \quad \det(D^2 y) = F(\kappa, u) = \tilde{F}(x, u, \nabla u)$$

for curvatures κ, H and with coefficients in te linear regime may be measurable (say L^p).

Goal : to develope a regularity theory for *weak solutions.*

Let L be an operator of form

$$L = -\sum_{i=1}^{d} \partial_{x_i}(a^{ij}(x)\partial_{x_i} u) + c(x) \quad (\text{so that } b^i \equiv 0)$$

and consider equation $Lu = f$ in Ω. We impose conditions

$$\begin{cases} a^{ij} \in L^\infty \cap C^0(\Omega), \\ a^{ij} = a^{ji} \\ a^{ij}(\xi)\xi_i\xi_j \geq \lambda|\xi|^2, \ \forall \xi \in \mathbb{R}^d \\ f \in L^{\frac{2d}{d+2}}(\Omega) \quad (\text{exponent chosen for Sobolev embedding}) \end{cases}$$

u is a weak solution of $Lu = f$ if

$$\int_\Omega \Big(\sum_{i,j=1}^{n} a^{ij}(x)\partial_{x_j} u \partial_{x_i}\varphi + cu\varphi\Big)dx = \int_\Omega \varphi f dx, \quad \forall \varphi \in H_0^1(\Omega)$$

We want to characterize Hölder continuity in terms of the growth of local integrals.

Let $\Omega \subset \mathbb{R}^d$ be bounded and connected. Given $u \in L^1_{loc}(\Omega)$, given $x_0 \in \Omega$, $r > 0$ such that $B(x_0, r) \subset \Omega$, we define

$$u_{x_0,r} = \frac{1}{B(x_0,r)} \int_{B(x_0,r)} u(x)dx$$

Theorem) Assume that $u \in L^2(\Omega)$ and there are $M > 0$, $\alpha \in (0,1)$.

$$\int_{B(x_0,r)} |u(x) - u_{x_0,r}|^2 dx \le M^2 r^{d+2\alpha}, \quad \forall B(x_0,r) \subset \Omega$$

Then u has continuous correction in $C^{0,\alpha}(\Omega)$ and $\forall \overline{\Omega'} \subset \Omega$, we have

$$|u|_{0,\alpha,\Omega'} \le C(M + \|u\|_{L^2(\Omega)})$$

for some $C = C(d, \alpha, \Omega, \Omega') > 0$.

proof) Let $R_0 = \mathrm{dist}(\Omega', \partial\Omega) > 0$. Let $0 < r_1 < r_2 \le R_0$. Then

$$\begin{aligned} |u_{x_0,r_1} - u_{x_0,r_2}|^2 =& \left| \frac{1}{|B(x_0,r_1)|} \int_{B(x_0,r_1)} u(y)dy - \frac{1}{|B(x_0,r_2)|} \int_{B(x_0,r_2)} u(y)dy \right|^2 \\ \le& 2|u(x) - u_{x_0,r_1}|^2_+ 2|u(x) - u_{x_0,r_2}|^2 \end{aligned}$$

Integrate on $B(x_0, r_1)$,

$$\begin{aligned} |B(x_0,r_1)||u_{x_0,r_1} - u_{x_0,r_2}|^2 \le& 2\int |u(x) - u_{x_0,r_1}|^2 dx + 2\int_{B(x_0,r_2)} |u(x) - u_{x_0,r_2}|^2 dx \\ \le& 2M^2 r_1^{d+2\alpha} + 2M^2 r_2^{d+2\alpha} \end{aligned}$$

so

$$|u_{x_0,r_1} - u_{x_0,r_2}|^2 \le \frac{M^2 c(d)}{r_1^d} \left(r_1^{d+2\alpha} + r_2^{d+2\alpha} \right)$$

We want $r_1, r_2 \to 0$. Take $R \le R_0$, $r_{1,j} = \frac{R}{2^{j+1}}$, $r_{2,j} = \frac{R}{2^j}$, $j \in \mathbb{N}$. Then

$$|u_{x_0,R2^{-j-1}} - u_{x_0,R2^{-j}}| \le c(d) \frac{MR_0^\alpha}{2^{j\alpha}}$$

So we have proved that $(u_{x_0,2^{-k}R})_{k\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$. So we may set $\hat{u}(x_0) = \lim_{k\to\infty} u_{x_0,2^{-k}R}$ and moreover $u_{x_0,r}$ converges to $u(x_0)$ with a uniform bound (that does not depend on x_0)

$$|u_{x_0,r} - \hat{u}(x_0)| \le c(d,\alpha) M r^\alpha \quad \cdots\cdots\cdots \ (\otimes)$$

Now by *Lebesgue's differentiation theorem*, $\lim_{r\to 0^+} \int_{B(x_0,r)} \frac{u(x)}{|B(x_0,r)|} dx = u(x_0, r)$ for a.e. x_0, whenever $u \in L^1_{loc}(\Omega) \subset L^2(\Omega)$ so $\hat{u} = u$ a.e. in Ω. But $\hat{u}$ is continuous because it is a uniform limit of continuous functions. Hence u is also continous (has continuous correction) at x_0.

Next, we prove that u is bounded in Ω with estimates. Observe that

$$|u_{x,r} - u(y,r)| = \frac{1}{|B(x,r)|} \left| \int_{B(x,r)} u(\xi)d\xi - \int_{B(y,r)} u(\xi)d\xi \right| \to 0$$

as $|x - y| \to 0$. Also by $(\otimes)$¡

$$\begin{aligned} & |u(x_0)| \le CMR^\alpha + |u_{x,R}| \quad \forall x_0 \in \Omega', \forall R \le R_0 \\ \Rightarrow \quad & |u|_{0,\Omega'} \le MR_0^\alpha + \|u\|_{L^2(\Omega)} \quad \cdots\cdots\cdots \ (\oplus) \end{aligned}$$

where we have second line since

$$|u_{x,R}| = \Big|\frac{1}{|B(x,R)|}\int_{B(x,R)} u(\xi)d\xi\Big| \leq \frac{1}{|B(x,R)|}\Big(\int_{B(x,R)} dx\Big)^{1/2}\Big(\int_{B(x_0,R)} |u(\xi)|^2 d\xi\Big)^{1/2}$$

We now prove that $u \in C^{0,\alpha}$ with estimates. First consider the case $x, y \in \Omega'$, $R := |x-y| < R_0/2$. Then

$$\begin{aligned}|u(x) - u(y)| \leq& |u(x) - u_{x_0,2R}| + |u(y) - u_{y,2R}| + |u_{x,2R} - u_{y,2R}| \\ \leq& 2c(d,\alpha)MR^\alpha + |u_{x,2R} - u_{y,2R}|\end{aligned}$$

using the bound $|u_{x_0,r} - u(x_0)| \leq c(d,\alpha)R^\alpha M$. We now need to estimate $|u_{x,2R} - u_{y,2R}|$. First, write

$$|u_{x,2R} - u_{y,2R}| \leq |u_{x,2R} - u(\zeta)| + |u_{y,2R} - u(\zeta)|$$

Integrating over ζ,

$$|u_{x,2R} - u_{y,2R}| \leq \frac{1}{|B(x,2R)|}\Big(\int_{B(x,2R)} |u(\zeta) - u_{x,2R}|^2 d\zeta + \int_{B(y,2R)} |u(\zeta) - u_{y,2R}|^2 d\zeta\Big) \lesssim M^2R^{2\alpha}$$

So we see that, for R chosen sufficiently small,

$$|u(x) - u(y)| \leq 2c(d,\alpha)MR^\alpha \leq C_d M|x-y|^\alpha$$

If $|x-y| > R_0/2$, we have by $(\oplus)$

$$\begin{aligned}|u(x) - u(y)| \leq& 2\sup_{\Omega'}|u| \leq C\Big(M + \frac{\|u\|_{L^2\Omega}}{R_0^\alpha}\Big)R_0^\alpha \\ \leq& 2^\alpha C\Big(M + \frac{\|u\|_{L^2(\Omega)}}{(R_2/2)^\alpha}\Big)|x-y|^\alpha\end{aligned}$$

(End of proof) □

(28th February, Thursday)

Weak solutions $u \in H^1(\Omega)$ of $Lu = f$ satisfy

$$\sum_{i,j=1}^{d}\int_\Omega a^{ij}(x)\partial_{x_i}u\partial_{x_j}\varphi dx + \int_\Omega c(x)u\varphi dx = \int f\varphi dx \quad \forall\varphi \in H_0^1(\Omega)$$

for $f, c \in L^p(\Omega)$ and $a^{ij} \in C^0(\overline{\Omega})$. We aim to prove that

$$u \in H^1(\Omega) \cap C^{0,\alpha}(\Omega)$$

where $H^1(\Omega)$ comes from Lax-Milgram and $C^{0,\alpha}(\Omega)$ comes from elliptic regularity.

We had proved in the last lecture that if $\int_{B(x_0,r)} |u(t) - u_{x_0,r}|^2 dx \leq M^2 r^{d+2\alpha}$ for all $B(x_0,r) \subset \Omega$, then $u \in C^{0,\alpha}(\Omega)$ and we have estimation in L^2-norm of u. We have a simple corollary of this result :

Corollary) Suppose $u \in H^1_{loc}(\Omega)$ satisfies that for some $\alpha \in (0,1)$,

$$\int_{B(x_0,r)} |\nabla u|^2 dx \le M^2 r^{d-2+2\alpha}, \quad \forall B(x_0,r) \subset \Omega$$

Then $u \in C^{0,\alpha}(\Omega)$ and $\forall \Omega'$ with $\overline{\Omega'} \subset \Omega$,

$$|u|_{0,\alpha,\Omega'} \le C(M + \|u\|_{L^2(\Omega)})$$

for some $C = C(d, \alpha, \Omega', \Omega) > 0$.

proof) We use Poincaré's inequality.

$$\begin{aligned}\int_{B(x_0,r)} |u(x) - u_{x_0,r}|^2 dx \le& C(d) r^2 \int_{B(x_0,r)} |\nabla u|^2 dx \\ \le& C(d) r^2 M^2 r^{d-2+2\alpha} = C(d) M^2 r^{d+2\alpha}\end{aligned}$$

We conclude by applying the last proposition of the last lecture.

(End of proof) □

We expect that if $a^{ij} \in C^0(\overline{\Omega})$, $c = c(x) \in L^d(\Omega)$, $f \in L^{\frac{2d}{d+2}}(\Omega)$ then the weak solution satisfies $u \in H^1(\Omega) \cap C^{0,\alpha}(\Omega)$.

A priori, we study the setting of Ω reduced to balls. So we at the moment insist to work on $B(0,1) = B$, $B(0,r) = B_r$. The idea is to first assume that a^{oij} is *close* to some constant coefficient, say $A = (a^{ij}(x_0))_{i,j=1}^d$ freezing a^{ij} to $a^{ij}(x_0)$. Then we will use perturbation argument.

To use perturbation argument, we may write $u = v + w$ where w is the weak solution of $L_0 w = 0$ where $L_0 w := -\sum_{i,j} \partial_{x_j}(a^{ij}(x_0)\partial_{x_i} w)$ and v solves

$$\sum_{i,j=1}^d \int_B a^{ij}(x_0)\partial_{x_i} v \partial_{x_j}\varphi dx = \int_B (f\varphi - cu\varphi)dx + \sum_{i,j=1}^d \int (a^{ij}(x_0) - a^{ij}(x))\partial_{x_i} u \partial_{x_j}\varphi dx, \quad \forall \varphi \in H^1_0(B)$$

The first step would be to study the constant-coefficient case to have control on w.

Proposition) Suppose that $w \in H^1(B_R)$ is a weak solution of $\sum_{i,j=1}^d a^{ij}(x_0)\partial^2_{x_i x_j} u = 0$ in B_R. Then for all $B(x_0,r) \subset B_R$ and $\rho \in (0,r]$

$$\begin{gathered}\int_{B(x_0,\rho)} |\nabla w|^2 dx \le C\Big(\frac{\rho}{r}\Big)^d \int_{B(x_0,r)} |\nabla w|^2 dx, \\ \int_{B(x_0,\rho)} |\nabla w - (\nabla w)_{x_0,\rho}|^2 dx \le C\Big(\frac{\rho}{r}\Big)^{d+2} \int_{B(x_0,r)} \int |\nabla w - (\nabla w)_{x_0,r}|^2 dx\end{gathered}$$

To show this, we need the following inequality.

Theorem) *(Caccioppoli's inequality for harmonic functions)* If $w \in C^1$ solved $L_0 w = 0$ weakly, *i.e.* it satisfies $\int_B a^{ij}(x_0)\partial_{x_i} w \partial_{x_j}\varphi dx = 0$ for all $\varphi \in H^1_0(B)$, then

$$\int_B |\nabla w|^2 \eta^2 dx \le C \int_B |\nabla \eta|^2 |w|^2 dx, \quad \forall \eta \in C^1_0(B)$$

for $C = C(\lambda, \Lambda) > 0$ where $\lambda|\xi|^2 \le \sum_{ij} a^{ij}(x_0)\xi_i\xi_j \le \Lambda|\xi|^2$.

proof) Let $\eta \in C_0^1(B)$ and choose $\varphi := \eta^2 w$ in the weak formulation. Then, noting that $\nabla\varphi = 2\eta(\nabla\eta)w + \eta^2\nabla w$,

$$\begin{aligned}\lambda \int \eta^2|\nabla w|^2 dx \leq& C(\lambda,\Lambda)\int_B \eta|w||\nabla\eta||\nabla w|dx\\ \leq& C(\lambda,\Lambda)\Big(\int_B \eta^2|\nabla w|^2dx\Big)^{1/2}\Big(\int_B |\nabla\eta|^2|u|^2dx\Big)^{1/2} \quad \text{(Cauchy-Schwarz)}\end{aligned}$$

as desired.

(End of proof) □

Corollary) *(Precis version of Cacciofolli's inequality)* With same choice of w as above, for all $0 < r < R \leq 1$,

$$\int_{B(0,r)} |\nabla w|^2 dx \leq \frac{C}{(R-r)^2}\int_{B(0,R)} |w|^2 dx$$

[This can be thought of as a reverse of Poincaré inequality]

proof) Choose $\eta \in C_0^1(B)$ such that $\eta = 1$ on $B(0,r)$, $\eta \equiv 1$ on $B(0,r)$ and $\eta \equiv 0$ outside $B(0,R)$ and such that $|\nabla\eta| \leq \frac{2}{R-r}$.

(End of proof) □

Proposition) Assume that w is a weak solution of $\sum_{i,j=1}^d \int_B a^{ij}\partial_{x_i}w\partial_{x_j}\varphi dx$ for all $\varphi \in H_0^1(B)$. Then for all $0 < \rho \leq r$,

$$\begin{gathered}\int_{B(0,\rho)} |w|^2 dx \leq C\Big(\frac{\rho}{r}\Big)^d \int_{B(0,r)} |w|^2 dx,\\ \int_{B(0,\rho)} |w - w_{0,\rho}|^2 dx \leq C\Big(\frac{\rho}{r}\Big)^{d+2} \int_{B(0,r)} |w - w_{0,r}|^2 dx\end{gathered}$$

where $C = C(\lambda,\Lambda)$.

proof) Using dilation, without loss of generality, set $r = 1$ and $\rho \in (0, 1/2]$.

♣ **Claim :** $|w|^2_{L^\infty(B_{1/2})} + |\nabla w|^2_{L^\infty(B_{1/2})} \leq C(\lambda,\Lambda)\int_{B_1}|w|^2 dx$.

: first observe that if w satisfies $L_0 w = 0$, then w is automatically smooth (as it is only a dialation of a harmonic function) and $\partial^\alpha w$ satisfies the same equation. So by *Cacciofolli*,

$$\int_{B(0,1/2)} |\nabla(\partial^\alpha w)|^2 dx \leq C\int |\partial^\alpha w|^2 dx \leq \cdots \lesssim \int |w|^2$$

with appropriate integration domains for in between terms. So we see $\|u\|_{H^k(B_{1/2})} \leq C(k,\lambda,\Lambda)\|w\|_{L^2(B_1)}$. Also one may make embedding $H^k \hookrightarrow L^\infty$ for $k > d/2$, with $\|w\|_{L^\infty(B_{1/2})} \leq C'\|w\|_{H^k(B_{1/2})}$, so we have the conclusion.

[A short derivation of embedding $i : H^k(\Omega) \hookrightarrow L^\infty(\Omega)$ for $k > d/2$ and Ω bounded : For $f \in L^\infty(\Omega)$,

$$
\begin{aligned}
|f(x)| =& \Big| \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \tilde{u}(\xi) e^{ix\xi} d\xi \Big| \\
=& \Big| \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{(1+|\xi|^2)^{k/2}}{(1+|\xi|^2)^{k/2}} \hat{u}(\xi) e^{ix\xi} d\xi \Big| \\
\leq& \Big(\int \frac{d\xi}{(1+|\xi|^2)^k} \Big)^{1/2} \Big(\int (1+|\xi|^2)^k |\hat{u}(\xi)|^2 d\xi \Big)^{1/2} \\
\leq& C' \big\| u \big\|_{H^k(\Omega)}
\end{aligned}
$$

Note that the integral converges only if $k > d/2$.]

Having the claim,

$$
\int_{B(0,\rho)} |w|^2 dx \lesssim \rho^d |w|^2_{L^\infty(B_{1/2})} \leq C\rho^d \int_{B_1} |w|^2 dx
$$

so we have the first statement. Also,

$$
\begin{aligned}
\int_{B(0,\rho)} |w - w_{0,\rho}|^2 dx =& \int_{B(0,\rho)} \Big| w - \frac{1}{|B(0,\rho)|} \int_{B(0,\rho)} w(y) dy \Big|^2 dx \\
\leq& \frac{1}{|B(0,\rho)|} \iint_{B(0,\rho)\times B(0,\rho)} |w(x) - w(y)|^2 dx dy \\
\leq& \frac{1}{|B(0,\rho)|} \iint_{B(0,\rho)\times B(0,\rho)} |2\rho|^2 |\nabla w|^2_{L^\infty(B_{1/2})} dx \\
\lesssim& \rho^{d+2} |\nabla w|^2_{L^\infty(B_{1/2})} \\
\lesssim& \rho^{d+2} \int_{B_1} |w|^2 dx \qquad \text{(by Claim)}
\end{aligned}
$$

To conclude, we observe that if w satisfies $L_0 w = 0$, then so does $L_0(w - w_{0,1}) = 0$, so applying this result for $\overline{w} = w - w_{0,1}$, we have

$$
\int_{B(0,\rho)} |w - w_{0,\rho}|^2 dx = \int_{B(0,\rho)} |\overline{w} - \overline{w}_{0,\rho}|^2 dx \lesssim \rho^{d+2} \int_{B_1} |\overline{w}|^2 dx = \rho^{d+2} \int_{B_1} |w - w_{0,1}|^2
$$

(End of proof) □

(5th March, Tuesday)

Recall, we had

Proposition) Assume that w is a weak solution of $\sum_{i,j=1}^d \int_B a^{ij} \partial_{x_i} w \partial_{x_j} \varphi dx$ for all $\varphi \in H_0^1(B)$. Then for all $0 < \rho \leq r$,

$$
\int_{B(0,\rho)} |w|^2 dx \leq C \Big(\frac{\rho}{r}\Big)^d \int_{B(0,r)} |w|^2 dx,
$$
$$
\int_{B(0,\rho)} |w - w_{0,\rho}|^2 dx \leq C \Big(\frac{\rho}{r}\Big)^{d+2} \int_{B(0,r)} |w - w_{0,r}|^2 dx
$$

where $C = C(\lambda, \Lambda)$.

We have a simple corollary of this.

Corollary) Under the previous hypothesis, we have that $\forall u \in H^1(B(x_0,r))$ and $\forall 0<\rho\le r$, we have

$$\int_{B(x_0,\rho)} |\nabla u|^2 dx \le C\Big(\Big(\frac{\rho}{r}\Big)^d \int_{B(x_0,r)} |\nabla u|^2 dx + \int_{B(x_0,r)} |\nabla (u-w)|^2 dx\Big)$$

proof) For $v=u-w$ and $0<\rho\le r$, has

$$\begin{aligned}\int_{B_\rho(x_0)} |\nabla u|^2 dx &\le 2\int_{B_\rho(x_0)} |\nabla w|^2 + 2\int_{B_\rho(x_0)} |Dv|^2 \\ &\le C\Big(\frac{\rho}{r}\Big)^d \int_{B(x_0,r)} |\nabla w|^2 + 2\int_{B_r(x_0)} |Dv|^2 dx \\ &\le C\Big(\Big(\frac{\rho}{r}\Big)^d \int_{B(x_0,r)} |\nabla u|^2 dx + \int_{B(x_0,r)} |\nabla v|^2\Big)\end{aligned}$$

(End of proof) □

Theorem) Let $u \in H^1(B)$ be a weak solution of $Lu=f$.

$$\int_B \sum_{i,j=1}^d a^{ij}(x)\partial_{x_i} u \partial_{x_j}\varphi dx + \int_B c(x)u\varphi dx = \int f\varphi dx, \quad \forall \varphi \in H_0^1(B)$$

with $a^{ij}=a^{ji}$, $a^{ij}\in C^0(\overline{B})$, $c\in L^d(B)$, $f\in L^q$, $q\in(\frac{2}{d},d)$ and $d\ge 2$. Then

$$\int_{B(x,r)} |\nabla u|^2 dx \le Cr^{d-2+2\alpha}\Big(\|f\|^2_{L^q(B_1)} + \|u\|^2_{H^1}\Big)$$

with $\alpha = 2-\frac{d}{q}\in(0,1)$ and $C\equiv C(\lambda,\Lambda,\|c\|_{L^d(B)},\tau)>0$ where $\tau:\mathbb{R}_+\to\mathbb{R}_+\cup\{0\}$ sufficiently chosen so that

$$|a^{ij}(x)-a^{ij}(y)| \le \tau(|x-y|), \quad \forall x,y\in B$$

(End of statement) □

Assume that the weak solution u exists. Last lecture, we took $x_0\in B$, $B(x_0,r)\subset B$ and made decomposition $u=v+w$ where w is the weak solution of $L_0u=0$. Then v must satisfy

$$\begin{aligned}\sum_{i,j=1}^d \int_B a^{ij}(x_0)\partial_{x_i} v\partial_{x_j}\varphi dx &= \int_B f\varphi dx - \int_B c(x)u\varphi dx \\ &+ \sum_{i,j=1}^d \int_B (a^{ij}(x_0)-a^{ij}(x))\partial_{x_i} u\cdot\partial_{x_j}\varphi dx \quad \forall\varphi\in H_0^1(B) \quad \cdots\cdots\cdots \ (WF_v)\end{aligned}$$

proof of Theorem) Take $\varphi = v\in H_0^1(B)$ in (WF_v). Then

$$\sum_{i,j=1}^d \int a^{ij}(x_0)\partial_{x_i} v\cdot\partial_{x_j} v dx = \int fv dx + \int cuv dx + \int \sum (a^{ij}(x_0)-a^{ij}(x))\partial_{x_i} u\cdot\partial_{x_j} v dx$$

Using ellipticity,

$$\int_{B(x_0,\rho)} |\nabla v|^2 dx \le C(\lambda,\Lambda,d)\int |fv|dx + \int |cuv| dx + \int \tau(|x-x_0|)|\nabla u||\nabla v| dx$$

A sensible way to bound this is to separate out terms in v and use Sobolev embedding $H^1 \hookrightarrow L^{\frac{2d}{d-2}}$, $\|g\|_{L^{2d/(d-2)}} \leq C\|\nabla g\|_{L^2}$, so we will keep the power of $|v|$ to be $\frac{2d}{d-2}$. To estimate the first term, use *Höler inequality* to see that

$$\int_{B(x_0,\rho)} |fv|dx \leq \Big(\int |f|^{\frac{2d}{d+2}}dx\Big)^{\frac{d+2}{2d}}\Big(\int |v|^{\frac{2d}{d-2}}dx\Big)^{\frac{d-2}{2d}}$$

For the second term,

$$\int |cuv|dx \leq \Big(\int |cu|^{\frac{2d}{d+2}}\Big)^{\frac{d+2}{2d}}\Big(\int |v|^{\frac{2d}{d-2}}\Big)^{\frac{d-2}{2d}}$$
$$\int |cu|^{\frac{2d}{d+2}}dx \leq \Big(\int |c|^d dx\Big)^{\frac{2}{d+2}}\Big(\int |u|^2 dx\Big)^{\frac{d}{d+2}}$$

Hence, using Young's inequality and Sobolev embedding, with $\theta\frac{d-2}{2d} = 1$,

$$\begin{aligned}\int_{B(x_0,\rho)} |\nabla v|^2 dx \leq &\frac{1}{\epsilon}\Big(\int |f|^{\frac{2d}{d+2}}dx\Big)^{\frac{d+2}{d}} + \epsilon\int_{B(x_0,\rho)} |\nabla v|^2 dx\\ &+ C_\epsilon\Big(\int |c|^d dx\Big)^{\frac{d+2}{d}}\int_{B(x_0,\rho)} |u|^2 dx + C_\epsilon\cdot\tau^2(r)\int |\nabla u|^2 dx + \epsilon\int |\nabla v|^2 dx\end{aligned}$$

so

$$\int |\nabla v|^2 dx \lesssim \Big(\int |f|^{\frac{2d}{d+2}}dx\Big)^{\frac{d+2}{d}} + \Big(\int |c|^d dx\Big)^{\frac{d+2}{d}}\int |u|^2 dx + C(\tau)\int_{B(x_0,\rho)} |\nabla u|^2 dx$$

Now by the corollary, has

$$\begin{aligned}\int_{B(x_0,\rho)} |\nabla u|^2 dx &\leq C\Big[\Big(\frac{\rho}{r}\Big)^d\int_{B(x_0,r)} |\nabla u|^2 dx + \int_{B(x_0,r)} |\nabla v|^2 dx\Big]\\ &\leq C\cdot\Big[\Big(\Big(\frac{\rho}{r}\Big)^d + \tau^2\Big)\int_{B(x_0,r)} |\nabla u|^2 dx + \Big(\int |f|^{\frac{2d}{d+2}}dx\Big)^{\frac{d+2}{d}}\\ &\qquad + \Big(\int_{B(x_0,r)} |c|^d dx\Big)^{\frac{d}{2}}\int_{B(x_0,r)} u^2 dx\Big]\end{aligned}$$

Also by *Hölder inequality*,

$$\Big(\int_{B(x_0,r)} |f|^{\frac{2d}{d+2}}dx\Big)^{\frac{d+2}{d}} \leq \Big(\int_{B(x_0,r)} |f|^q dx\Big)^{\frac{2}{q}} r^{d-2+2\alpha}$$

where q was chosen so that $\alpha = 2 - \frac{n}{q} \in (0,1)$. Hence we have

$$\begin{aligned}\int_{B(x_0,\rho)} |Du|^2 \leq C\Bigg(&\Big[\Big(\frac{\rho}{r}\Big)^d + \tau^2(r)\Big]\int_{B(x_0,r)} |Du|^2 + r^{d-2+2\alpha}\Big\|f\Big\|^2_{L^q(B_1)}\\ &+ \Big(\int_{B(x_0,r)} |c|^d dx\Big)^{\frac{2}{d}}\int_{B(x_0,r)} u^2 dx\Bigg)\end{aligned}$$

To proceed, we note the following lemma :

Lemma) $\phi = \phi(t)$ be a non-negative, non-decreasing function on $[0,R]$ such that

$$\phi(\rho) \leq A\Big(\Big(\frac{\rho}{r}\Big)^\alpha + \epsilon\Big)\phi(r) + Br^\beta, \quad A,\epsilon,B > 0,\ \beta > \alpha$$

Then

$$\phi(r) \le C\Big(\frac{\phi(R)}{R^\gamma} r^\gamma + B r^\beta\Big), \quad \text{for some } \gamma \in (\beta, \alpha)$$

[I am actually bit unsure which version of the lemma I should use. See Han & Lin for reference.]

(End of statement) $\square$

- If in the case of $c \equiv 0$, application of the lemma with $\phi(\rho) = \int_{B(x_0,\rho)} |\nabla u|^2 dx$, $\beta = d-2+2\alpha$, $\gamma = d-2+2\alpha$ gives

$$\begin{aligned}\int_{B(x_0,\rho)} |\nabla u|^2 dx &\le C\Big(\frac{\rho}{r}\Big)^{d-2+2\alpha} \int_{B(x_0,R)} |\nabla u|^2 dx + C\|f\|_{L^q}^2 r^{d-2+2\alpha} \\ &\le \tilde{C} r^{d-2+2\alpha}(\|u\|_{H^1}^2 + \|f\|_{L^q}^2)\end{aligned}$$

- If $c \not\equiv 0$, see example sheet #4.

(7th March, Thursday)

[This lecture is essentially a recap of the last lecture.]

Recall,

Corollary) Under the previous hypothesis, we have that $\forall u \in H^1(B(x_0,r))$ and $\forall 0 < \rho \le r$, we have

$$\int_{B(x_0,\rho)} |\nabla u|^2 dx \le C\Big(\Big(\frac{\rho}{r}\Big)^d \int_{B(x_0,r)} |\nabla u|^2 dx + \int_{B(x_0,r)} |\nabla(u-w)|^2 dx\Big)$$

(End of statement) $\square$

We were working with $\Omega = B$. For a general domain, we can use estimate for balls covering the domain B to get an interior estimate.

$$L = \sum a^{ij}(x)\partial_{x_i}\partial_{x_j} + c(x)$$

with $a^{ij} \in C^0(B)$, $c(x) \in L^d(B)$, and $u \in H^1(B)$ is the weak solution to $Lu = f$, $f \in L^q(B)$. We want to prove

$$\int_{B(x_0,r)} |\nabla u|^2 dx \le C r^{d-2+2\alpha}(\|u\|_{H^1(B)}^2 + \|f\|_{L^q(B)}^2)$$

We have frozen the coefficients of a^{ij} at x_0, so $L_0 = w$ with $L_0 = \sum a^{ij}(x_0)\partial_{x_i}\partial_{x_j}$, and $v = u - w$, so that

$$\sum \int a^{iij}(x_0)\partial_{x_i} v \partial_{x_j}\varphi dx = \int_B f\varphi dx - \int cu\varphi dx + \sum (a^{iij}(x_0) - a^{ij}(x))\partial_{x_i} u \partial_{x_j}\varphi$$

For $B(x_0,R) \subset B(x_0,1)$, $0 < \rho < r \le R$, we had, by choosing φ, v

$$\frac{1}{4}\int_{B(x_0,\rho)} |\nabla v|^2 \le C|\tau|^2 \int_{B(x_0,\rho)} |\nabla u|^2 dx + (\int |c|^d x)^{2/d} \int |u|^2 dx + (\int |f|^{\frac{2d}{d+2}} dx)^{\frac{d+2}{d}}$$

Also by Holder inequality,

$$(\int_{B(x_0,r)} |f|^{\frac{2d}{d+2}} dx)^{\frac{d+2}{2}} \le (\int_{B(x_0,r)} |f|^{\frac{2d}{d+2}p} dx)^{\frac{d+2}{dp}} (\int_{B(x_0,r)} dx)^{\frac{d+2}{dq}}$$

and with choice of $\frac{1}{q} = \frac{2d}{4-2\alpha}$ and $\frac{1}{p} = 1 - \frac{1}{q}$, we have

$$\Big(\int_{B(x_0,r)} |f|^{\frac{2d}{d+2}} dx\Big)^{\frac{d+2}{d}} \leq \Big(\int_{B(x_0,r)} |f|^q dx\Big)^{\frac{2}{q}} r^{d-2+2\alpha}$$

We want to control $\in_{B(x_0,\rho)} |\nabla u|^2 dx$. To do this, we use a corollary from last lecture, that for a fixed r and $u \in H^1(B(x_0,r))$,

$$\int_{B(x_0,\rho)} |\nabla u|^2 dx \leq C\Big[\Big(\frac{\rho}{r}\Big)^d \int_{B(x_0,r)} |\nabla u|^2 dx + \int_{B(x_0,r)} |\nabla(u-w)|^2 dx\Big]$$

for all $0 < \rho < r$, hence

$$\int_{B(x_0,\rho)} |\nabla u|^2 dx \leq C\Big(\Big(\frac{\rho}{r}\Big)^2 + \tau^2(r)\Big) \int_{B(x_0,r)} |\nabla u|^2 dx + \|f\|_{L^q}^2 r^{d-2+2\alpha} + \|c\|_{L^d}^2 \int |u|^2 dx$$

To get the conclusion of the theorem, we want to "replace" r by ρ in the RHS, using the following lemma.

Lemma) Let $\phi(t)$ be a non-negative and non-decreasing function on $[0,R]$ and we assume that

$$\phi(\rho) \leq A\Big[\Big(\frac{\rho}{r}\Big)^\alpha + \epsilon\Big]\phi(r) + Br^\beta$$

for some $A, B, \alpha, \beta, \epsilon \geq 0$ with $\beta < \alpha$ and for all $0 < \rho \leq r < R$. Then for any $\gamma \in (\beta, \alpha)$, there exists $\epsilon_0 = \epsilon_0(A, \alpha, \beta, r)$ such that if $\epsilon < \epsilon_0$, we have

$$\phi(\rho) \leq C\Big(\frac{\rho}{r}\Big)^\gamma \phi(r) + B\rho^\beta, \quad 0 < \rho \leq r \leq R$$

[I am actually bit unsure which version of the lemma I should use. See Han & Lin for reference.]
[Note : This lemma is extremely useless. It only occurs in this context.]

(End of statement) $\square$

- If in the case of $c \equiv 0$, application of the lemma with $\phi(\rho) = \int_{B(x_0,\rho)} |\nabla u|^2 dx$, $\beta = d-2+2\alpha$, $\gamma = d-2+2\alpha$ gives

$$\begin{aligned}\int_{B(x_0,\rho)} |\nabla u|^2 dx &\leq C\Big(\frac{\rho}{r}\Big)^{d-2+2\alpha} \int_{B(x_0,R)} |\nabla u|^2 dx + C\|f\|_{L^q}^2 r^{d-2+2\alpha} \\ &\leq \tilde{C} r^{d-2+2\alpha}(\|u\|_{H^1}^2 + \|f\|_{L^q}^2)\end{aligned}$$

- Will see the case $c \not\equiv 0$ in the fourth Example sheet.

(9th March, Saturday)

De Giorgi's Theorem, Part I

Let $B = B(0,1)$. Let $L = \sum a^{ij}(x)\partial_{ij} + c(x)$ (so that $b = 0$) with λ-uniformly elliptic, $a^{ij} \in L^\infty(B)$(not even continuous) and $c \in L^q(B)$ for $q > d/2$.

Definition) *(weak subsolution)* Let $u \in H^1(B)$ is a **weak subsolution** of $Lu = f$, for f given, if

$$\sum_{i,j=1}^d \int_B a^{ij}(x)\partial_{x_i} u \partial_{x_j} \varphi dx + \int_B c(x) u \varphi dx \leq \int_B f \varphi dx$$

for any $\varphi \in H_0^1(B)$ such that $\varphi \geq 0$ in $B = B(0,1)$.

Theorem) *(De Giorgi, part I)* Under the previous hypothesis, assume in addition that $f \in L^q(B)$, $q > d/2$ and $\exists \Lambda > 0$ suhch that

$$\sup_{i,j} |a^{ij}|_{L^\infty(B)} + \|c\|_{L^q} \leq \Lambda$$

Then, if $u \in H^1(B)$ is a *weak subsolution* of $Lu = f$, then

$$u^+ \in L^\infty_{loc}(B) \quad \text{and}$$
$$\sup_{B(0,1/2)} u^+ \leq C(\|u^+\|^2_{L^2(B)} + \|f\|^2_{L^q(B)})$$

[The same bound was proved by Nash, with a method to which applies also to parabolic equations. But De Giorgi's method gives better insight.]

proof) *(De Giorgi, 1957)* **Idea :** Choose a suitable φ. Let

$$u \in L^\infty(B(0,1/2)), \quad (u-k)^+ = v \quad \int_{B(0,1/2)} (u-k)^2 dx = 0$$

with k large enough.

Take for given $k \in \mathbb{R}_{(>0)}$, and let $v := (u-k)^+$. Let $\zeta \in C_0^1(B)$, $0 \leq \zeta \leq 1$ and put $\varphi = v\zeta^2 \geq 0$. Inject $\varphi = v\zeta^2$ in the weak formulation, with "$\int = \int_{u>k}$" (in this set, would have $u = v + k$ and $\nabla u = \nabla v$ a.e., and if $u < k$, any derivative of v vanishes.) Exploiting that $\partial(v\zeta^2) = (\partial v)\zeta^2 + 2v\zeta\partial\zeta$, we have

$$\begin{aligned}\sum_{i,j=1} \int a^{ij}\partial_{x_i} u \partial_{x_j}(v\zeta^2)dx &\geq \sum_{i,j=1}^{d} \int a^{ij}\partial_{x_i} v \partial_{x_j} v dx - 2\Lambda \int |\nabla v||v||\zeta||\nabla\zeta| dx \\ &\geq \lambda \int |\nabla v|^2\zeta^2 dx - 2\Lambda \int |\nabla v||v||\zeta||\nabla\zeta| dx\end{aligned}$$

Injection of this expression in the weak formulation yields

$$\lambda \int |\nabla v|^2 \zeta^2 dx \leq \int |c||u|v\zeta^2 dx + \int |f|v\zeta^2 dx + C_\Lambda \int |v|^2|\nabla\zeta|^2 dx$$

where we have used $\int |\nabla v||v||\zeta||\nabla\zeta|dx \leq \frac{C_{\Lambda,\lambda}}{2}\int |\nabla\zeta|^2|v|^2 + \frac{\lambda}{2}\int |\nabla v|^2\zeta^2$. Therefore,

$$\begin{aligned}\int |(\nabla v)\zeta|^2 &\lesssim \int |c|v^2\zeta^2 dx + k\int |c|\zeta^2 v dx + \int |f|v\zeta^2 + C_\Lambda \int |\nabla\zeta|^2 v^2 dx \\ &\lesssim \int |c|v^2\zeta^2 dx + k^2 \int_{\{v\zeta\neq 0\}} |c|\zeta^2 dx + \int |f|v\zeta^2 dx + C_\Lambda \int |\nabla\zeta|^2 v^2 dx \quad \cdots\cdots\cdots (*)\end{aligned}$$

just using Young's inequality. *[The integration domain $\{v\zeta \neq 0\}$ looks strange, but it would be useful in a while.]* The goal is to refine this bound.

At this point, recall the Sobolev embedding

$$\Big(\int |v\zeta|^{\frac{2d}{d-2}} dx\Big)^{\frac{d-2}{2d}} \leq C_d \Big(\int |\nabla(u\zeta)|^2 dx\Big)^{1/2}$$

As in the usual discussions, using Hölder inequality multiple number of times to bound the inequality above in terms of $\|v\zeta\|_{L^{\frac{2d}{d-2}}}$ along with Sobolev inequality would give the desired estimate. (Will be doing this in a moment.)

Using *Hölder inequality*, get

$$\int |f|v\zeta^2 dx \le \Big(\int |f|^q dx\Big)^{1/q}\Big(\int |v\zeta|^{q'}|\zeta|^{q'}\Big)^{1/q'}$$
$$\le \|f\|_{L^q}\Big(\int |v\zeta|^{q'p}dx\Big)^{\frac{1}{pq'}}\Big(\int |\zeta|^{q'p'}dx\Big)^{1/p'q'}$$

with $\frac{1}{p}+\frac{1}{p'}=\frac{1}{q}+\frac{1}{q'}=1$, and q is as given in the statement of the theorem. We want $q'p=\frac{2d}{d-2}$ so that $\frac{1}{p'q'}=\frac{1}{q'}(1-\frac{1}{p})=\frac{1}{q'}-\frac{2d}{d-2}=1-\frac{1}{q}-\frac{d-2}{2d}=:\frac{1}{\theta}$, so

$$\int |f|v\zeta^2 dx \le \|f\|_{L^q}\Big(\int |v\zeta|^{\frac{2d}{d-2}}\Big)^{\frac{d-2}{2d}}\Big(\int_{\{\zeta v\neq 0\}}|\zeta|^\theta dx\Big)^{1/\theta}$$

Key idea : it seems dealing with $\left\|\zeta\right\|_{L^\theta}$ is difficult. However, noting that $|\zeta|<1$, then $\Big(\int_{\{\zeta v\neq 0\}}|\zeta|^\theta dx\Big)^{1/\theta}\le \mathrm{meas}(\{\zeta v\neq 0\})^{1/\theta}$. Also, by Sobolev embedding, has $\Big(\int |v\zeta|^{\frac{2d}{d-2}}\Big)^{\frac{d-2}{2d}}\le \|\nabla(v\zeta)\|_{L^2}$. So by Young's inequality,

$$\int |f|v\zeta^2 dx \le C_\delta\left\|f\right\|_{L^q}^2\mathrm{meas}(\{\zeta v\neq 0\})^{2/\theta}+\delta\int|\nabla(v\zeta)|^2dx$$
$$=C_\delta\left\|f\right\|_{L^q}^2\mathrm{meas}(\{\zeta v\neq 0\})^{1+\frac{2}{d}-\frac{2}{q}}+\delta\int|\nabla(u\zeta)|^2dx$$

for some C_δ.

Claim : if $\mathrm{meas}(\{\zeta v\neq 0\})$ is small, then the terms in $(*)$ involving c can be absorbed by the others.

: Using *Hölder* again,

$$\int |c|v^2\zeta^2dx\le\Big(|c|^qdx\Big)^{1/q}\Big(\int_{\{v\zeta\neq 0\}}(v\zeta)^{2q'}dx\Big)^{1/q'}$$
$$\le\|c\|_{L^q}\Big(\int|v\zeta|^{\frac{2d}{d-2}}dx\Big)^{\frac{d-2}{d}}\mathrm{meas}(\{v\zeta\neq 0\})^{1-\frac{d-2}{d}-\frac{1}{q}}$$
$$\le\delta\|c\|_{L^q}^2\int|\nabla(\zeta v)|^2dx+C_\delta\cdot\mathrm{meas}(\{v\zeta\neq 0\})^{\frac{2}{d}-\frac{1}{q}}$$

Recalling $\|c\|_{L^q}\le\Lambda$, we can choose $\delta>0$ such that $\delta\cdot\Lambda<1/100$.

The term $k^2\int_{\{v\zeta\neq 0\}}|c|\zeta^2$ is bounded by

$$k^2\int_{\{v\zeta\neq 0\}}|c|\zeta^2dx\le k^2\|c\|_{L^q}\mathrm{meas}(\{v\zeta\neq 0\})^{1-\frac{1}{q}}$$

Also note that $\mathrm{meas}(\{v\zeta\neq 0\})^{\frac{2}{d}-\frac{1}{q}}$ may be absorbed in $\mathrm{meas}(\{v\zeta\neq 0\})^{1-\frac{1}{q}}$ whenever $\mathrm{meas}(\{v\zeta\neq 0\})$ is small.

Using the claim, we would have $(*)$ with c eliminated and in written terms of $\mathrm{meas}(\{v\zeta\neq 0\})$,

$$\int|\nabla(\zeta v)|^2dx\le C\Big(\int v^2|\nabla\zeta|^2dx+\Big(\|f\|_{L^q}^2+k^2\Big)\mathrm{meas}(\{v\zeta\neq 0\})^{1-\frac{1}{q}}\Big)\quad\cdots\cdots\cdots\ (**)$$

Using Hölder inequality and Sobolev embedding, has

$$\int(v\zeta)^2dx\le\left\|v\zeta\right\|_{L^{\frac{2d}{d-2}}}^2\mathrm{meas}(\{v\zeta\neq 0\})^{\frac{2}{d}}\le C_d\int|\nabla(v\zeta)|^2dx\cdot\mathrm{meas}(\{v\zeta\neq 0\})^{\frac{2}{d}}$$

This yields, along with $(**)$,

$$\begin{aligned}\int (v\zeta)^2 dx &\le \int |\nabla(v\zeta)|^2 dx \cdot \mathrm{meas}(\{v\zeta \neq 0\})^{2/d} \\ &\lesssim \int |v|^2 |\nabla\zeta|^2 dx \cdot \mathrm{meas}(\{v\zeta \neq 0\})^{2/d} + \Big(\|f\|_{L^q}^2 + k^2\Big) \cdot \mathrm{meas}(\{v\zeta \neq 0\})^{1-\frac{1}{q}+\frac{2}{d}}\end{aligned}$$

Then we have proven that $\exists \epsilon = \frac{2}{d} - \frac{1}{q} > 0$ and C such that

$$\int (v\zeta)^2 dx \le C\Big(\int v^2 |\nabla\zeta|^2 dx \cdot \mathrm{meas}(\{v\zeta \neq 0\})^{\epsilon} + (k^2 + \|f\|_{L^q}^2)\mathrm{meas}(\{v\zeta \neq 0\})^{1+\epsilon}\Big)$$

Next time : Choose ζ with $|\nabla\zeta| \le (S)$, and $\{\zeta v \neq 0\} = \{u \ge k, |x| < r\}$. Hence

$$\int_{\{u>k,|x|<r\}} (u-k)^2 dx \le C(k,r)$$

Goal would be to find k_∞ large enough so that $\int (u-k_\infty)^2 dx = 0$. Choose (k_n, r_n) as a sequence such that

$$\int_{\{u>k_n,|x|>r_n\}} (u-k_n)^2 dx \le \gamma(k_n, r_n)^k \int (u-k_0)^2 dr$$

(12th March, Tuesday)

We were proving,

Theorem) *(De Giorgi, part I)* Let $L = \sum_{i,j=1}^d a^{ij}(x)\partial_{x_i x_j} + c(x)$, $a^{ij} \in L^\infty(B)$, $c \in L^q(B)$, $q > \frac{d}{2}$ such that $\sup_{ij} |a^{ij}|_{L^\infty(B)} + \|c\|_{L^q} < \Lambda$ and with usual uniform ellpticity condition.
If u is a weak subsolution of $Lu = f$, $f \in L^q(B)$, then we have $u^+ \in L^\infty_{loc}(B)$ and moreover

$$\sup_{B(0,1/2)} u^+ \le C(\|u^+\|_{L^2(B)} + \|f\|_{L^q(B)})$$

where $C = C(d, \lambda, \Lambda, q) > 0$.

proof continued) Last time, we chose $v = (u-k)^+$ and $\varphi = v\zeta^2$ for some $\zeta \in C_0^\infty(B)$, $0 \le \zeta \le 1$. The goal is to find k such that $\int v^2 dx = 0$. This will imply $u^+ \le k$.

The key result from the last lecture is that by choosing $\epsilon = \frac{2}{d} - \frac{1}{q} > 0$, we have

$$\int (v\zeta)^2 dx \le C\Big(\int v^2 |\nabla\zeta|^2 dx \cdot \mathrm{meas}(\{v\zeta \neq 0\})^{\epsilon} + (k + \|f\|_{L^q})^2 \mathrm{meas}(\{v\zeta \neq 0\})^{1+\epsilon}\Big) \quad \cdots\cdots\cdots (\dagger)$$

Now, choose $\zeta \in C_0^\infty(B)$ with

$$\begin{cases} \zeta = 1 & \text{in } B(0,r) \\ \zeta = 0 & \text{in } B(0,1)\backslash B(0,R) \\ |\nabla\zeta| \le \frac{2}{R-r} & \text{in } B(0,1) \end{cases}$$

for some $0 < r < R < 1$. With such choice of ζ, we have

$$\{v\zeta \neq 0\} = A(k,r) := \{x \in B(0,r) : u \ge k\}$$

We may then recast (†) in terms of $A(k,r)$.

$$\int_{A(k,r)} (u-k)^2 dx \lesssim |A(k,r)|^\epsilon \frac{1}{(R-r)^2} \int_{A(k,r)} (u-k)^2 dx + (k + \|f\|_{L^q})^2 |A(k,r)|^{1+\epsilon} \quad \cdots\cdots\cdots \ (\dagger')$$

whenever $|A(k,r)|$ is small enough. We want to make some sort of bound on the RHS and use iterative scheme to make $\int_{A(h,r)} (u-h)^2 \to 0$ for some fixed h. $|A(h,r)|$ can be estimated as

$$\begin{aligned}
|A(h,r)| &= \text{meas}(\{x \in B(0,r) : u \geq h\}) \\
&= \int_{x \in B_r, u \geq h} dx \leq \frac{1}{h} \int_{A(h,r)} u^+ dx \leq \frac{1}{h} \Big(\int_{A(h,r)} (u^+)^2 dx \Big)^{1/2} \Big(\int_{A(h,r)} dx \Big)^{1/2} \\
&= \frac{1}{h} \Big(\int_{A(h,r)} (u^+)^2 dx \Big) |A(h,r)|^{1/2} \\
\Rightarrow \quad |A(h,r)| &= \frac{1}{h^2} \Big(\int_{A(h,r)} (u^+)^2 dx \Big)
\end{aligned}$$

Take $k_0 := C_0 \|u\|_{L^2(B)}$, for C_0 large enough so that

$$|A(k_0, r)| \leq \frac{1}{(k_0)^2} \|u^+\|_{L^2(B)} \leq \frac{1}{C_0} \ll 1$$

For any $h > k$, has $A(k,r) \supset A(h,r)$, so

$$\int_{A(h,r)} (u-h)^2 dx \leq \int_{A(k,r)} (u-h)^2 dx \leq \int_{A(k,r)} (u-k)^2 dx$$

and

$$\begin{aligned}
|A(h,r)| &= \text{meas}(B(0,r) \cap \{u \geq h\}) \\
&= \int_{B(0,r), u-k \geq h-k} dx \leq \int \frac{(u-k)^2}{(h-k)^2} dx \leq \frac{1}{(h-k)^2} \int_{A(k,r)} (u-k)^2 dx
\end{aligned}$$

For any choice of $h > k \geq k_0$ and $\frac{1}{2} \leq r < R \leq 1$, any we apply $(\dagger')$ with the new estimates.

$$\begin{aligned}
&\text{LHS}(h,r) := \int_{A(h,r)} (u-h)^2 dx \\
&\lesssim \frac{|A(h,r)|^\epsilon}{(R-r)^2} \int_{A(k,r)} (u-k)^2 dx + (h + \|f\|_{L^q})^2 |A(h,r)|^{1+\epsilon} \\
&\leq \frac{1}{(R-r)^2} \frac{1}{(h-k)^{2\epsilon}} \Big(\int_{A(k,r)} (u-k)^2 dx \Big)^\epsilon \Big(\int_{A(k,r)} (u-k)^2 dx \Big) \\
&\qquad + (h + \|f\|_{L^q})^2 \frac{1}{(h-k)^{2(1+\epsilon)}} \Big(\int_{A(k,r)} (u-k)^2 dx \Big)^{1+\epsilon} \\
&\leq \frac{1}{(h-k)^{2\epsilon}} \Big(\int_{A(k,r)} (u-k)^2 dx \Big)^{1+\epsilon} \Big(\frac{1}{(R-r)^2} + \frac{(h + \|f\|_{L^q})^2}{(h-k)^2} \Big) =: \text{RHS}(k,r,R) \quad \cdots\cdots\cdots \ (\dagger'')
\end{aligned}$$

Hence we have an interative scheme :

- Let $k_l = k_0 + k^* \Big(1 - \frac{1}{2^l}\Big)$, so $k_l \leq k_0 + k^*$. The constant k^* would be specified later to be sufficiently large.
- Let $r_l = \tau + \frac{1}{2^l}(1-\tau)$ where $\tau = \frac{1}{2}$.

- As $l \to \infty$, $k_l \nearrow k_0 + k^*$ and $r_l \searrow 1/2$. Also, $\frac{1}{2} \le r_l \le R < 1$ for sufficiently large l so we can apply the new estimate $\text{LHS}(h, r_l) \le \text{RHS}(k_l, r_l, R)$.
- Has $k_l - k_{l-1} = k^*(\frac{1}{2^{l-1}} - \frac{1}{2^l}) = \frac{k}{2^l}$ and $r_{l-1} - r_l = \frac{1-\tau}{2^l}$.
- We let $\varphi(k, r) = \|(u-k)^+\|_{L^2(B(0,r))} = \left(\int_{A(k,r)} (u-k)^2 dx\right)^{1/2}$. We apply ($\dagger''$), then

$$\begin{aligned}
\varphi(k_l, r_l) &\lesssim \left(\frac{1}{(r_{l-1} - r_l)} + \frac{k_l + \|f\|_{L^q}}{k_l - k_{l-1}}\right) \frac{1}{(k_l - k_{l-1})^\epsilon} \varphi(k_{l-1}, r_{l-1})^{1+\epsilon} \\
&= \left(\frac{2^l}{1-\tau} + \frac{k_0 + k^*(1 - 1/2^l) + \|f\|_{L^q}}{k^*/2^l}\right) \frac{1}{(k^*/2^l)^\epsilon} \varphi(k_{l-1}, r_{l-1})^{1+\epsilon} \\
&= \left(\frac{2^l}{1-\tau} + \frac{2^l(k_0 + k^* + \|f\|_{L^q})}{k^*}\right) \frac{2^{l\epsilon}}{(k^*)^\epsilon} \varphi(k_{l-1}, r_{l-1})^{1+\epsilon} \\
&= \frac{k_0 + 3k^* + \|f\|_{L^q}}{(k^*)^{1+\epsilon}} 2^{l(1+\epsilon)} \varphi(k_{l-1}, r_{l-1})^{1+\epsilon} \quad \text{as } \tau = \frac{1}{2}
\end{aligned}$$

Choose $k^* = C_\infty(k_0 + \|f\|_{L^q})$, then, as $r^\epsilon > 2^{1+\epsilon} > 1$, (??? check)

$$\varphi(k_l, r_l) \lesssim \frac{1}{r^l} \varphi(k_0, r_0)^{1+\epsilon} \xrightarrow{l \to \infty} 0$$

Hence

$$\varphi(k_0 + k_*, 1/2) = 0$$

This implies

$$\sup_{B(0,1/2)} u^+ \le k_0 + k^* \le C\left(\|u^+\|_{L^2(B)} + \|f\|_{L^q}\right)$$

(End of proof) □

(14th March, Thursday)

De Giorgi's Theorem, Part II

Set $B = B(0,1)$. We now write Lu in the *divergence form*

$$Lu = \sum_{i,j=1}^d \partial_{x_i}(a^{ij}(x)\partial_{x_j} u) + c(x)$$

Here, we assume $c = 0$. Also let $a^{ij} \in L^\infty(B)$, $a^{ij} = a^{ji}$ and $\lambda|\xi|^2 \le \sum a^{ij}\xi_i\xi_j \le \Lambda|\xi|^2$.

Definition) A function $u \in H^1_{loc}(B)$ is a **(weak) subsolution** of $Lu = 0$ if, $\forall \varphi \in H^1_0(B)$, $\varphi \ge 0$, we have

$$\sum_{i,j=1}^d \int_B a^{ij}(x)\partial_{x_i} u \partial_{x_j} \varphi dx \le 0$$

In *De Giorgi (part I)*, we have proved that wehnever u is a weak subsolution of $Lu = f$, $f \in L^q(B)$, then it is in $L^\infty_{loc}(B)$ and $\|u^+\|_{L^\infty(0,\frac{1}{2})} \le C(\|u\|^2_{H^1} + \|f\|^2_{L^q})$.

Theorem) *(De Giorgi, part II)* If u is a weak solution of $Lu = 0$ in $B(0,1)$, then $u \in C^{0,\alpha}(b)$ and

$$\sup_{x \in B(0,1/2)} |u(x)| + \sup_{x,y \in B(0,1/2)} \frac{|u(x) - u(y)|}{|x - y|^{\alpha}} \leq C(d, \Lambda/\lambda) \|u\|_{L^2(B)}$$

for some $\alpha = \alpha(d, \lambda/\Lambda) \in (0,1)$. (check the statement)

We will need three key ingredients to prove the theorem.

- Poincaré-Sobolev ienquality
- Density theorem
- Oscillation theorem

First, we have the following lemma.

Lemma) Let $\Phi \in C^{0,1}_{loc}(\mathbb{R})$ by *convex* and $\Phi' \geq 0$. If u is a subsolution of $Lu = 0$, then we have that $v = \Phi(u)$ is also a subsolution of $Lu = 0$ whenever $v \in H^1_{loc}(B)$.

proof) Exercise.

Remark : if u is a supersolution and Φ is concave, then $\Phi(u)$ is a subsolution.

Example : if u is a subsolution, then $v = (u - k)^+$ is also a subsoltuion, with choice of $\Phi(s) = (s - k)^+$.

Proposition) *(Poincaré-Sobolev inequality)* For any $\epsilon > 0$, there is $C = C(\epsilon, d) > 0$ such that $\forall u \in H^1(B)$ satisfying $\text{meas}\{x \in B; u(x) = 0\} \geq \epsilon \cdot \text{meas}(B)$, we have

$$\int_B |u|^2 dx \leq C(\epsilon, d) \int_B |\nabla u|^2 dx$$

proof) We prove by contradiction. We assume that there is a sequence $(u_m)_m \subset H^1(B)$ satisfyint the assumption and such that

$$\int_B |\nabla u_m|^2 dx \xrightarrow{m \to \infty} 0 \quad \text{while} \quad \int_B |u_m|^2 dx = 1, \ \forall m$$

This implies (u_m) is bounded in H^1, so we have (up to a subsequence) $u_m \to u_\infty \in H^1(B)$ strongly in L^2 and weakly in $H^1(B)$. Then we should have $\int |\nabla u_\infty|^2 = 0$ which implies u_∞ is a constant almost everywhere. But by the assumption $\text{meas}\{x \in B; u(x) = 0\} \geq \epsilon \cdot \text{meas}(B)$, we have

$$\lim_{m \to \infty} \int_B |u_m - u_\infty|^2 dx \geq \lim_{m \to \infty} \int_{u_m = 0} |u_m - u_\infty|^2 dx = \int_{u_m = 0} |u_\infty|^2 dx \geq \epsilon |u_\infty|_{L^\infty}$$

so this implies u_∞ should be identically 0, which gives a contradiction with the fact that $u_n \to u_\infty$ in L^2.

(End of proof) □

[The difference between the original Poincaré's inequality is that we only assume $u \in H^1(B)$ in place of $u \in H^1_0(B)$. There is another version of this family of inequalities : (Poincaré-Wirtinger) if $u \in H^1(\Omega)$, for Ω bounded(at least in one direction) then

$$\int_\Omega \left| u(x) - \int_\Omega u(y) dy \right|^2 dx \leq C \int_\Omega |\nabla u|^2 dx$$

]

Proposition) *(Density theorem)* Suppose u is a positive supersolution of $Lu = 0$ in $B(0,2)$ satisfying $\text{meas}\{x \in B(0,1); u(x) \geq 1\} \geq \epsilon \cdot \text{meas}(B)$. Then there is $C = C(\epsilon, d, \Lambda/\lambda) > 0$ such that

$$\inf_{B(0,1/2)} u \geq C$$

Similarly, if u is a negative subsolution, then $\sup_{B(0,1/2)} u \leq C$.
[We could have just set u to be a supersolution in $B = B(0,1)$ rather than $B(0,2)$]

proof) Assume that $u \geq \delta > 0$. (We will let $\delta \to 0^+$ later). Choosing $\Phi(s) = (\log(s))^- = \max\{-\log(s), 0\}$, we have $v \leq \log \delta$ and $v = (\log u)^-$ is a *subsolution.* As v is a subsolution, the *De Giorgi (Part I)* guarantees that

$$\sup_{B(0,1/2)} v \leq C\Big(\int_{B(0,1)} |v|^2 dx\Big)^{1/2} \quad (\text{has } f \equiv 0).$$

Also,

$$\text{meas}(\{x \in B(0,1); v = 0\}) = \text{meas}(\{x \in B(0,1); u \geq 1\}) \geq \epsilon \text{meas}(B)$$

By *Poincaré-Sobolev* inequality, has

$$\sup_{B(0,1/2)} v \leq C\Big(\int_B |v|^2 dx\Big)^{1/2} \leq \tilde{C}\Big(\int_B |\nabla v|^2 dx\Big)^{1/2}$$

We want to bound the $\int |\nabla v|^2$ part. We use the weak formulation of u being a supersolution : $\sum \int a^{ij} \partial_{x_i} \partial_{x_j} \varphi dx \geq 0$. We want to choose φ so that $\log u$ appear in the formulation - inject $\varphi = \zeta^2/u$, then

$$0 \leq \sum_{ij} \int_{B(0,2)} a^{ij} \partial_{x_i} u \, \partial_{x_j} \Big(\frac{\zeta^2}{u}\Big) dx = -\sum \int a^{ij} \frac{\zeta^2}{u^2} \partial_{x_i} u \, \partial_{x_j} u dx + 2 \sum \int \frac{\zeta a^{ij} \partial_{x_i} u \partial_{x_j} \zeta}{u} dx$$

so using uniform ellipticity of $(a^{ij})_{ij}$ and AM-GM equality, has

$$\int \zeta^2 |\nabla(\log u)|^2 dx \leq C(\Lambda/\lambda)\Big(\int \frac{\zeta^2}{u^2} |\nabla u|^2 + \int |\nabla \zeta|^2\Big)$$

Fix $\zeta \in C_0^1(B(0,2))$ with $\zeta = 1$ in $B(0,1)$, then

$$\int_{B(0,1)} |\nabla(\log u)|^2 dx \leq C$$

and

$$\sup_{B(0,1/2)} v \leq \|\nabla v\|_{L^2} = \|\nabla(\log u)\|_{L^2} \leq C$$

But

$$\sup v = \sup(\log u)^- \leq C$$

so taking exponential, has $u \geq e^{-C}$.

To see the general case without assuming $u \geq \delta$ for some δ, observe that our result did not depend on δ. Hence, if we take $u = \lim_{\delta \to 0} \max\{u, \delta\} =: \lim_{\delta \to 0} u_\delta$ then each $u_\delta = \max\{u, \delta\}$ is a positive supersolution to $Lu = 0$ so $u_\delta \geq e^{-C}$ uniformly over $\delta > 0$. Therefore, we would also have $u \geq e^{-C}$.

(End of proof) □

Definition) The **oscillation** of u is defined by

$$\operatorname{osc}_{\Omega}(u) = \sup_{\Omega} u - \inf_{\Omega} u$$

Proposition) Assume that u is a bounded solution of $Lu = 0$ in $B(0,2)$, then there is $\gamma = \gamma(d, \Lambda/\lambda) \in (0,1)$ such that

$$\operatorname{osc}_{B(0,1/2)}(u) \leq \gamma \operatorname{osc}_{B(0,1)}(u)$$