# Advanced Probability

-Martingales
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(15th October 2018, Monday)

## Chapter 2. Martingales in Discrete Time

### 2.1. Definitions.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

- A **Filtration** for $(\Omega, \mathcal{F}, \mathbb{P})$ is a sequence $(\mathcal{F}_n)_{n\geq 0}$ of $\sigma$-algebras s.t. for all $n \geq 0$, we have

$$\mathcal{F}_n \subset \mathcal{F}_{n+1} \subset \mathcal{F}$$

  Set $F_\infty = \sigma(\mathcal{F}_n : n \geq 0)$ then $\mathcal{F}_\infty \subset \mathcal{F}$. We allow $\mathcal{F}_\infty \neq \mathcal{F}$. We interpret $n$ as times and $\mathcal{F}_n$ as the extent of knowledge at time $n$.

- A **Random process(in discrete time)** is a sequence of random variables $(X_n)_{n\geq 0}$. It has a natural filtration $(F_n^X)_{n\geq 0}$ given by

$$\mathcal{F}_n^X = \sigma(X_0, \cdots, X_n)$$

  That is, the knowledge obtained from $X_n$ by time $n$. We say $(X_n)_{n\geq 0}$ is **adapted to** $(\mathcal{F}_n)_{n\geq 0}$ if $X_n$ is $\mathcal{F}_n$-measurable for all $n \geq 0$. This is equivalent to having $\mathcal{F}_n^X \subset \mathcal{F}_n$, for all $n \geq 0$. (Here, $X_n$ are real-valued)

- We would say $(X_n)_{n\geq 0}$ is **integrable** if $X_n$ is integrable for all $n \geq 0$.

- A **martingale** is an *adapted, integrable random process* $(X_n)_{n\geq 0}$ s.t. for all $n \geq 0$,

$$\mathbb{E}[X_{n+1}|\mathcal{F}_n] = X_n \quad \text{a.s.}$$

  In the case $\mathbb{E}[X_{n+1}|\mathcal{F}_n] \leq X_n$ a.s., $(X_n)_n$ is called a **super-martingale** and in the case $\mathbb{E}[X_{n+1}|\mathcal{F}_n] \geq X_n$ a.s., $(X_n)_n$ is called a **sub-martingale**.

### Optional Stopping

- A random variable $T : \Omega \to \{0, 1, 2, \cdots\} \cup \{\infty\}$ is a **stopping time** if $\{T \leq n\} \in \mathcal{F}_n$ for all $n \geq 0$.

- For a stopping time $T$, we set $\mathcal{F}_T = \{A \in \mathcal{F}_\infty : A \cap \{T \leq n\} \in \mathcal{F}_n \text{ for all } n \geq 0\}$. It is easy to check $\mathcal{F}_T$ is indeed a $\sigma$-algebra and that if $T(\omega) = n$ for all $\omega \in \Omega$, then $T$ is a stopping time and $\mathcal{F}_T = \mathcal{F}_n$.

- Given $X$, define $X_T(\omega) = X_{T(\omega)}(\omega)$ whenever $T(\omega) < \infty$ and define the **stopped process** $X^T$ by

$$X_n^T(\omega) = X_{T(\omega)\wedge n}(\omega) \quad \text{for } n \geq 0$$

**Proposition 2.2.1.)** Let $X$ be an adapted process. Let $S$, $T$ be stopping times for $X$. Then

(a) $S \wedge T$ is a stopping time for $X$.

(b) $\mathcal{F}_T$ is a $\sigma$-algebra.

(c) If $S \leq T$ then $\mathcal{F}_S \subset \mathcal{F}_T$.

(d) $X_T 1_{T<\infty}$ is an $\mathcal{F}_T$-measurable random variable.

(e) $X^T$ is adapted.

(f) If $X$ is integrable, then $X^T$ is also integrable.

**proof)**

(a) $\{S \wedge T \leq n\} = \{S \leq n\} \cup \{T \leq n\} \in \mathcal{F}_n$ for all $n \geq 0$, so $S \wedge T$ is a stopping times

(b) Directly from the definition, we see that $\phi\mathcal{F}_T$. Also, given $A \in \mathcal{F}_T$ and a sequence $(A_m)_m \subset \mathcal{F}_T$, we have

$$\begin{aligned} A^c \cap \{T \leq n\} &= \{T \leq n\} - A \cap \{T \leq n\} \in \mathcal{F}_n \quad &\Rightarrow A^c \in \mathcal{F}_T \\ (\cup_m A_m) \cap \{T \leq n\} &= \cup_m (A_m \cap \{T \leq n\}) \in \mathcal{F}_n \quad &\Rightarrow \cup_m A_m \in \mathcal{F}_T \end{aligned}$$

hence $\mathcal{F}_T$ is a $\sigma$-algebra.

(c) Let $A \in \mathcal{F}_S$. Then $A \cap \{T \leq n\} = A \cap \{S \leq n\} \cap \{T \leq n\} \in \mathcal{F}_n$, hence $A \in \mathcal{F}_T$.

(d) For each $t \in \mathbb{R}$, we have $\{X_T 1_T > t\} = \cup_m \{X_m > t, T = n\}$ so for any $n \geq 0$,

$$\{X_T 1_T > t\} \cap \{T \leq n\} = \cup_{m=1}^n \{X_m > t, T = n\} \in \mathcal{F}_n$$

and so $X_T 1_T$ is $\mathcal{F}_T$-measurable.

(e) By definition of being a stopping time, for any $t \in \mathbb{R}$,

$$\{(X^T)_n > t\} = \{T > n, X_n > t\} \cup \Big( \cup_{m=0}^n \{T = m, X_m > t\} \Big) \in \mathcal{F}_n$$

so $X^T$ is adapted.

(f) First consider the case where $X$ is non-negative integrable. Then

$$\mathbb{E}(X_n^T) = \mathbb{E}(\mathbb{E}(X_n^T | T)) = \sum_{m \geq n} \mathbb{P}(T = m)\mathbb{E}(X_m) + \mathbb{P}(T > n)\mathbb{E}(X_n) < \infty$$

for any $n$, so we have the result for non-negative $X$.

For the general case, divide $X$ into a non-negative and a negative part.

*(End of proof)* □

**Theorem 2.2.2)** *(Optional stopping theorem)* Let $X$ be a super-martingale and let $S, T$ be *bounded* stopping times with $S \leq T$ a.s. Then

$$\mathbb{E}[X_T] \leq \mathbb{E}[X_S]$$

**proof)** Fix $n \geq 0$ such that $T \leq n$ a.s. Then

$$\begin{aligned} X_T &= X_S + \sum_{S \leq k < T} X_{k+1} - X_k \\ &= X_S + \sum_{k=0}^n (X_{k+1} - X_k) 1_{S \leq k < T} \end{aligned}$$

Now $\{S \leq k\}$ is in $\mathcal{F}_k$ and $\{T > k\}$ is in $\mathcal{F}_k$, so

$$\begin{aligned} \mathbb{E}[(X_{k+1} - X_k) 1_{S \leq k < T}] &= \mathbb{E}[\mathbb{E}[(X_{k+1} - X_k) 1_{S \leq k < T} | \mathcal{F}_k]] \\ &= \mathbb{E}[\mathbb{E}[X_{k+1} - X_k | \mathcal{F}_k] 1_{S \leq k < T}] \end{aligned}$$

but since $(X_n)$ was a super-martingale, $\mathbb{E}[X_{k+1} - X_k | \mathcal{F}_k] \leq 0$ a.s. and therefore $\mathbb{E}[(X_{k+1} - X_k) 1_{S \leq k < T}] \leq 0$ a.s. Hence $\mathbb{E}(X_T) \leq \mathbb{E}(X_S)$.

*(End of proof)* □

•Note that $X$ is a sub-martingale *if and only if* $(-X)$ is a super-martingale, and that $X$ is a martingale *if and only if* $X$ and $(-X)$ are super-martingales. Hence, we obtain sub-martingale and martingale versions of the theorem :

$$\text{If } (X_n) \text{ is a sub-martingale, } \mathbb{E}[X_T] \geq \mathbb{E}[X_S]$$
$$\text{If } (X_n) \text{ is a martingale, } \mathbb{E}[X_T] = \mathbb{E}[X_S]$$

**Theorem 2.2.3.)** Let $X$ be an adapted integrable process. Then the followings are equivalent.

(a) $X$ is a super-martingale.

(b) for all bounded stopping times $T$ and stopping time $S$,

$$\mathbb{E}(X_T|\mathcal{F}_S) \leq X_{S\wedge T} \quad \text{a.s.,}$$

(c) for all stopping times $T$, $X_T$ is a super-martingale,

(d) for all bounded stopping times $T$ and all stopping times $S$ with $S \leq T$ a.s,

$$\mathbb{E}(X_T) \leq \mathbb{E}(X_S)$$

⋆ The theorem gives an inverse statement of the optional stopping theorem.

**proof)**

(a) ⇒ (b) Suppose $X$ is a super-martingale and $S, T$ are stopping times. Let $T \leq n$, for some $n < \infty$. Then

$$X_T = X_{S\wedge T} + \sum_{k=0}^{T} (X_{k+1} - X_k) 1_{S\leq k<T} \cdots\cdots (*)$$

Let $A \in \mathcal{F}_S$. Then $A \cap \{S \leq k\} \in \mathcal{F}_k$ and $\{T > k\} \in \mathcal{F}_k$ so

$$\mathbb{E}[(X_{k+1} - X_k)1_{S\leq k<T}1_A] = \mathbb{E}[\mathbb{E}[(X_{k+1} - X_k)1_{S\leq k<T}1_A|\mathcal{F}_k]] \leq 0$$

and

$$\begin{aligned}\mathbb{E}[(X_T - X_{S\wedge T})1_A] =& \mathbb{E}[\sum_{n=0}^{T}(X_{k+1} - X_k)1_{S\leq k<T}1_A] \leq 0 \\ \Rightarrow& \quad \mathbb{E}[X_T 1_A] \leq \mathbb{E}[X_{S\wedge T}1_A]\end{aligned}$$

But since this inequality is true for any $A \in \mathcal{F}_S$ and noting that $X_{S\wedge T} \in \mathcal{F}_S$), we see

$$\mathbb{E}[X_T|\mathcal{F}_S] \leq X_{S\wedge T} \quad \text{a.s.}$$

The inclusions (b)⇒(c) and (c)⇒(d) are obvious.

(d) ⇒ (a) Let $m \leq n$ and $A \in \mathcal{F}_n$. Set $T = m1_A + n1_{A^c}$. Then $T$ is a stopping with $T \leq n$. Then

$$\mathbb{E}(X_n 1_A - X_m 1_A) = \mathbb{E}(X_n) - \mathbb{E}(X_T) \leq 0$$

(note, if $\omega \in A$ then $(X_n 1_A - X_m 1_A)(\omega) = X_n(\omega) - X_m(\omega)$ and 0 otherwise) so

$$\mathbb{E}[X_n|\mathcal{F}_m] \leq X_m$$

*(End of proof)* □