Lectures on Schramm–Loewner Evolution

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These notes are based on a course given to Masters-level students in Cambridge. Their scope is the basic theory of Schramm–Loewner evolution, together with some underlying and related theory for conformal maps and complex Brownian motion. The structure of the notes is influenced by our attempt to make the material accessible to students having a working knowledge of basic martingale theory and Itô calculus, trying to keep the complex analysis tools at a minimum.

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We review the notion of conformal isomorphism of complex domains and the question of existence and uniqueness of conformal isomorphisms between proper simply connected complex domains. Then we illustrate, by a simple special case, Loewner's idea of encoding the evolution of complex domains using a differential equation.

1.1 Conformal isomorphisms

A (*complex*) *domain* is a non-empty connected open subset of the complex plane $\mathbb{C}$. A domain D is simply connected if $\mathbb{C} \setminus D$ is connected in $\mathbb{C} \cup \{\infty\}$. This is not equivalent to the notion of being connected in $\mathbb{C}$: for instance, the infinite strip $\{z \in \mathbb{C} : \mathrm{Im}(z) \in (0,1)\}$ is simply connected. A domain is *proper* if it is not the whole of $\mathbb{C}$.

A holomorphic function f on a domain D is a *conformal map on D* if its derivative f' vanishes nowhere. If a conformal map f on D is injective, then it can be shown that its image $D' = f(D)$ is also a domain and its inverse $f^{-1} : D' \to D$ is also a conformal map. We call a bijective conformal map $f : D \to D'$ a *conformal isomorphism*.

We note the following fundamental result. Write $\mathbb{D}$ for the open disc having centre 0 and radius 1.

Theorem 1.1 (Riemann mapping theorem)**.** *Let D be a proper simply connected domain. Then there exists a conformal isomorphism $\Phi : D \to \mathbb{D}$.*

In general it is not easy to find an explicit Riemann map. Even for such a simple domain as a square, the map may not be expressed in terms of usual simple functions. One case where the map is explicit (and this will be useful in the following) is the upper-half plane: let $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$. The map

$$z \mapsto \Psi(z) = i\frac{1+z}{1-z} : \mathbb{D} \to \mathbb{H}$$

is a conformal isomorphism from $\mathbb{D}$ to $\mathbb{H}$. The inverse map is $z \mapsto (z-i)/(z+i)$.

In general, there is no uniqueness for Riemann maps. For instance there are many conformal maps between $\mathbb{H}$ and itself: for $\sigma \in (0,\infty)$ and $b \in \mathbb{R}$, the scaling and translation maps $z \mapsto \sigma z$ and $z \mapsto z + b$ are conformal automorphisms of the upper half-plane $\mathbb{H}$ fixing (the boundary point) ∞. As we will see just below, we get uniqueness of the map by adding certain specific constraints, such as fixing the image of three given points on the boundary (in the case of Jordan domains). To do this, we need to discuss what are the conformal autmorphisms of the disk or the upper half-plane.

1.2 Möbius transformations

We define the *Möbius group* of conformal automorphisms of $\mathbb{D}$ to be the set of conformal maps where each element has the form

$$\Phi_{\theta,w}(z) = e^{i\theta}\frac{z-w}{1-\bar{w}z}, \quad z \in \mathbb{D}$$

for some $\theta \in [0, 2\pi)$ and $w \in \mathbb{D}$. Note that every Möbius transformation extends to a homeomorphism of the closed unit disc $\bar{\mathbb{D}}$, mapping the boundary $\partial\mathbb{D}$ to itself.

As we are about to see, Möbius transformations are the only automorphisms of the unit disk. We first need the following lemma, which is a basic result of complex analysis. We shall give a proof using Brownian motion in the next section.

Lemma 1.2 (Schwarz lemma). *Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic function with $f(0) = 0$. Then $|f(z)| \leqslant |z|$ for all z. Moreover, if $|f(z)| = |z|$ for some $z \neq 0$, then $f(w) = e^{i\theta}w$ for all w, for some $\theta \in \mathbb{R}$.*

Corollary 1.3. *The Möbius transformations are the only conformal automorphisms of $\mathbb{D}$.*

Proof. Let Φ be a conformal automorphism of $\mathbb{D}$. Set $w = \Phi^{-1}(0)$. Then $f = \Phi \circ \Phi_{0,w}^{-1}$ is a conformal automorphism of $\mathbb{D}$ and $f(0) = 0$. Pick $u \in \mathbb{D} \setminus \{0\}$ and set $v = f(u)$. Note that $v \neq 0$. Now, either $|f(u)| = |v| \geqslant |u|$ or $|f^{-1}(v)| = |u| \geqslant |v|$. In any case, by the Schwarz lemma, there exists $\theta \in \mathbb{R}$ such that $f(z) = e^{i\theta}z$ for all z, and so $\Phi = f \circ \Phi_{0,w} = \Phi_{\theta,w}$. □

The Möbius group has three real parameters. Thus, one expect to be able to specify uniquely a conformal automorphism of the disk by imposing three real constraints. Indeed, here are several ways of doing this:

1. For each $w \in \mathbb{D}$, there is a unique automorphism of $\mathbb{D}$ such that $\Phi(w) = 0$ and $\Phi'(w) > 0$, which is $\Phi = \Phi_{0,w}$.
2. Given $w \in \mathbb{D}$ and a boundary point b of $\mathbb{D}$, there is a unique Φ such that $\Phi(w) = 0$ and $\Phi(b) = 1$. This is achieved by choosing θ so that $e^{i\theta}\Phi_{0,w}(b) = 1$ and taking $\Phi = \Phi_{\theta,w}$.
3. Given any three distinct boundary points b_1, b_2, b_3 of $\mathbb{D}$, ordered anticlockwise, there is a unique Φ such that $\Phi(b_1) = 1$, $\Phi(b_2) = i$, and $\Phi(b_3) = -1$. To do this, we rotate to put b_1 at 1, then apply Ψ to map to $\mathbb{H}$, then translate to send $\Psi(b_3)$ to $0 = \Psi(-1)$ and scale to send $\Psi(b_2)$ to $-1 = \Psi(i)$, while fixing $\infty = \Psi(1)$. We finally map back to $\mathbb{D}$ by Ψ^{-1}.

As a consequence, for a given simply connected domain D and $w \in D$, there exists a unique conformal isomorphism $\Phi : D \to \mathbb{D}$ such that $\Phi(w) = 0$ and $\Phi'(w) > 0$. To discuss the equivalent of 2. and 3. above we need to discuss briefly how does the boundary gets transformed under a conformal map.

1.3 Conformal boundary

Let D be a proper simply connected domain. We shall be interested in the 'boundary of D seen from inside D'. This is not simply the set of limit points of D in $\mathbb{C} \setminus D$, and indeed sometimes may not even be identified with a subset of $\mathbb{C}$. Choose a conformal isomorphism $\Phi : D \to \mathbb{D}$. We say that a sequence $(x_n : n \in \mathbb{N})$ in D is *D-Cauchy* if $(\Phi(x_n) : n \in \mathbb{N})$ is

Cauchy in $\mathbb{D}$. Since the image of a Cauchy sequence in $\mathbb{D}$ under any Möbius tranformation is another Cauchy sequence in $\mathbb{D}$, this notion does not depend on the choice of Φ. Call two D-Cauchy sequences $x = (x_n : n \in \mathbb{N})$ and $y = (y_n : n \in \mathbb{N})$ *equivalent* if $(x_1, y_1, x_2, y_2, \dots)$ is also a D-Cauchy sequence. Let $\hat{D}$ denote the set of equivalence classes of D-Cauchy sequences. We can define an injection $\iota : D \to \hat{D}$ by $\iota(z) = [(z, z, z, \dots)]$ and a bijection $\hat{\Phi} : \hat{D} \to \bar{\mathbb{D}}$ by $\hat{\Phi}(x_n : n \in \mathbb{N}) = \lim_n \Phi(x_n)$. Define the *boundary* $\partial D = \hat{D} \setminus \iota(D)$. Note that $\hat{\Phi} \circ \iota = \Phi$ so $\hat{\Phi}$ maps ∂D onto the unit circle $C = \{z \in \mathbb{C} : |z| = 1\}$.

We give $\hat{D}$ the topology of $\bar{\mathbb{D}}$: thus, for $b \in \partial D$, we say that a simply connected subdomain $N \subseteq D$ is a *neighbourhood of* b *in* D if $\{z \in \mathbb{D} : |z - \hat{\Phi}(b)| < \varepsilon\} \subseteq \Phi(N)$ for some $\varepsilon > 0$.

A *Jordan curve* is a continuous one-to-one map $\gamma : C \to \mathbb{C}$. Say D is a *Jordan domain* if $\bar{D} \setminus D$ is the image of a Jordan curve. It can be shown in this case that Φ extends to a homeomorphism of $\bar{D}$ to $\bar{\mathbb{D}}$, so ι extends naturally to a homeomorphism of $\bar{D}$ to $\hat{D}$ and we can identify ∂D with $\bar{D} \setminus D$.

On the other hand, an $\mathbb{H}$-Cauchy sequence is a sequence $(z_n : n \in \mathbb{N})$ in $\mathbb{H}$ which either converges in $\mathbb{C}$ or is such that $|z_n| \to \infty$ as $n \to \infty$. Hence we write $\partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$. In the domain $D = \mathbb{H} \setminus (0, i]$, for $z \in [0, i)$, the D-Cauchy sequences $(z + (1 + i)/n : n \in \mathbb{N})$ and $(z + (-1 + i)/n : n \in \mathbb{N})$ are not equivalent, so their equivalence classes $z+$ and $z-$ are distinct boundary points.

By Corollary 1.3, the conformal isomorphism $\Phi : D \to \mathbb{D}$ is unique up to a Möbius transformation of $\mathbb{D}$. Let $w \in D$ and let $b_1, b_2, b_3 \in \partial D$ be distinct and ordered anti-clockwise. Then Φ may be specified uniquely by imposing any one of the three following additional conditions:

$$\Phi(w) = 0 \text{ and } \Phi'(w) > 0; \quad \Phi(w) = 0 \text{ and } \Phi(b_1) = 1; \quad \Phi(b_1) = 1, \Phi(b_2) = i, \Phi(b_3) = -1.$$

Note also that ∞ can now be seen as a legitimate boundary point of a domain such as $\mathbb{H}$.

1.4 $SLE(0)$

Consider the (deterministic) process $\gamma = (\gamma_t : t \geqslant 0)$ in the closed upper half-plane $\bar{\mathbb{H}}$ given by

$$\gamma_t = 2i\sqrt{t}.$$

This process belongs to the family of processes $(SLE(\kappa) : \kappa \in [0, \infty))$ to which these notes are devoted, corresponding to the parameter value $\kappa = 0$. Think of γ as progressively eating away the upper half-plane so that what remains at time t is the subdomain $H_t = \mathbb{H} \setminus K_t$, where $K_t = \gamma(0, t] = \{\gamma_s : s \in (0, t]\}$. There is a conformal isomorphism $g_t : H_t \to \mathbb{H}$ given by

$$g_t(z) = \sqrt{z^2 + 4t}$$

which has the following asymptotic behaviour as $|z| \to \infty$

$$g_t(z) = z + \frac{2t}{z} + O(|z|^{-2}).$$

As we shall explain in Proposition 4.2, there is only one conformal isomorphism $H_t \to \mathbb{H}$ such that $g_t(z) - z \to 0$ as $|z| \to \infty$. Thus we can think of the family of maps $(g_t : t \geqslant 0)$ as a canonical encoding of the path γ.

Consider the vector field b on $\bar{\mathbb{H}} \setminus \{0\}$ defined by

$$b(z) = \frac{2}{z} = \frac{2(x - iy)}{x^2 + y^2}.$$

Fix $z \in \bar{\mathbb{H}} \setminus \{0\}$ and define

$$\zeta(z) = \inf\{t \geqslant 0 : \gamma_t = z\} = \begin{cases} y^2/4, & \text{if } z = iy \\ \infty, & \text{otherwise.} \end{cases}$$

Then $\zeta(z) > 0$ and $z \in \bar{K}_t$ if and only if $\zeta(z) \leqslant t$. Set $z_t = g_t(z)$. Then $(z_t : t < \zeta(z))$ satisfies the differential equation

$$\dot{z}_t = b(z_t) \tag{1}$$

and, if $\zeta(z) < \infty$, then $z_t \to 0$ as $t \uparrow \zeta(z)$. Thus $(g_t(z) : z \in \bar{\mathbb{H}} \setminus \{0\}, t < \zeta(z))$ is the (unique) maximal flow of the vector field b in $\bar{\mathbb{H}} \setminus \{0\}$. By *maximal* we mean that $(z_t : t < \zeta(z))$ cannot be extended to a solution of the differential equation on a longer time interval.

1.5 Loewner evolutions

Think of $SLE(0)$ as obtained via the associated flow $(g_t : t \geqslant 0)$ by iterating continuously a map $g_{\delta t}$, which nibbles an infinitesimal piece $(0, 2i\sqrt{\delta t}]$ of $\mathbb{H}$ near 0. Note that for small δt,

$$g_{\delta t}(z) = \sqrt{z^2 + 4t} = z + \frac{2\delta t}{z} + o(\delta t).$$

so that if $z_t = g_t(z)$ then

$$\left.\frac{dz_t}{dt}\right|_{t=0} = \frac{2}{z}.$$

Let us derive the differential equation (1) from this. For $t \geqslant 0$ and $\delta t \to 0$, note that $g_t(\gamma(t + \delta t)) = 2i\sqrt{\delta t}$. Thus $g_{t+\delta t}(z) = g_{\delta t} \circ g_t(z) = g_t(z) + 2/g_t(z) + o(\delta t)$. Thus $\dot{g}_t(z) = 2/g_t(z)$.

Charles Loewner, in the 1920's, showed that one could evolve families of complex domains by more general continuous iterations, where the nibbling point γ_t moves over time according to more complicated curves than the straight line above. In that case, he showed (and we will see) that the corresponding differential equation becomes

$$b(t, z) = \frac{2}{z - \xi_t}, \quad t \geqslant 0, z \in \mathbb{H}$$

where $\xi_t \in \mathbb{R}$. ξ_t is the so-called *driving function* or Loewner transform of the curve γ. It describes implicitly a family of domains $(H_t : t \geqslant 0)$, and thus (possibly, although not always) a path γ. Intuitively, the interpretation for ξ_t is that when "$d\xi_t > 0$" the curve "turns to the right" while when "$d\xi_t < 0$" it "turns to the left". Thus the more wide the fluctuations of ξ, the more winding the resulting path γ will be.

Oded Schramm, in 1999, realized that for some conjectural conformally invariant scaling limits γ of planar random process, with a certain spatial Markov property, the process $\xi = (\xi_t : t \geqslant 0)$ would have to be a Brownian motion, of some diffusivity κ. The associated processes γ were at that time totally new and have since revolutionized our understanding of conformally invariant planar random processes.

2

We review briefly some essentials of stochastic calculus and Brownian motion, including optional stopping, Itô's calculus and the strong Markov property of Brownian motion. We give precise definitions and statements, adapted to our later needs, but do not always state the definitive form of a result, nor do we give proofs.

2.1 Martingales and stopping times

For the purposes of our general discussion, we suppose given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $(\mathcal{F}_t)_{t \geqslant 0}$. An adapted integrable process $M = (M_t)_{t \geqslant 0}$ is a *martingale* if $\mathbb{E}(M_t | \mathcal{F}_s) = M_s$ almost surely for all $s, t \geqslant 0$ with $s \leqslant t$. We consider here only continuous martingales. A random variable T in $[0, \infty]$ is a *stopping time* if $\{T \leqslant t\} \in \mathcal{F}_t$ for all $t \geqslant 0$. We define

$$\mathcal{F}_T = \{A \in \mathcal{F} : A \cap \{T \leqslant t\} \in \mathcal{F}_t \text{ for all } t \geqslant 0\}.$$

For a continuous martingale M and a stopping time T, the optional stopping theorem guarantees that $\mathbb{E}(M_T) = \mathbb{E}(M_0)$, provided that the collection of random variables $(M_t^T = M_{t \wedge T}, t \geqslant 0)$ is uniformly integrable, which is automatic if T is e.g. bounded. Even if T can take the value ∞, if M^T is uniformly integrable, then the limit $M_\infty = \lim_{t \to \infty} M_t$ exists almost surely on $\{T = \infty\}$ and the identity $\mathbb{E}(M_T) = \mathbb{E}(M_0)$ remains true. Moreover, if S is another stopping time, with $S \leqslant T$, then $\mathbb{E}(M_T | \mathcal{F}_S) = M_S$ almost surely.

By the optional stopping theorem, if M is a continuous martingale and T is a stopping time, then the stopped process M^T is also a continuous martingale, where $M_t^T = M_{T \wedge t}$. An adapted process M is a *local martingale* if there is a sequence of stopping times $(T_n : n \in \mathbb{N})$ with $T_n \uparrow \infty$ almost surely such that M^{T_n} is a martingale for all n. Then $(T_n : n \in \mathbb{N})$ is called a *reducing sequence* for M. If M is a continuous local martingale starting from 0, then we may obtain a reducing sequence by setting $T_n = \inf\{t \geqslant 0 : |M_t| \geqslant n\}$, which has the useful extra property that the martingales M^{T_n} are uniformly bounded.

To every continuous local martingale M there corresponds a unique continuous non-decreasing adapted process $[M]$ starting from 0, called the *quadratic variation* of M, which is characterized by the property that $(M_t^2 - [M]_t : t \geqslant 0)$ is a local martingale, and is given by

$$[M]_t = \lim_{n \to \infty} \sum_{k=0}^{\lfloor 2^n t \rfloor - 1} (M_{(k+1)2^{-n}} - M_{k2^{-n}})^2.$$

where the limit holds in probability, and uniformly on compact time intervals. From the preceding statement follows a polarized extension: to every pair of continuous local martingales M and N, there corresponds a unique continuous finite-variation adapted process $[M, N]$ starting from 0, called the *covariation* of M and N, which is characterized by the property that $(M_tN_t - [M,N]_t : t \geqslant 0)$ is a local martingale, and is given by

$$[M,N]_t = \lim_{n\to\infty} \sum_{k=0}^{\lfloor 2^n t\rfloor - 1} (M_{(k+1)2^{-n}} - M_{k2^{-n}})(N_{(k+1)2^{-n}} - N_{k2^{-n}})$$

the limit holding in the same sense.

2.2 Brownian motion

A continuous adapted $\mathbb{R}^d$-valued process $B = (B_t)_{t\geqslant 0}$ is an $(\mathcal{F}_t)_{t\geqslant 0}$-*Brownian motion* if, for all $s,t \geqslant 0$ with $s \leqslant t$, the increment $B_t - B_s$ is independent of $\mathcal{F}_s$ and is normally distributed with mean 0 and covariance matrix $(t-s)I$. This property suffices to determine the law of B conditional on $\mathcal{F}_0$: for any non-negative measurable function F on the set of continuous paths $W = C([0,\infty),\mathbb{R}^n)$, we have

$$\mathbb{E}(F(B)|\mathcal{F}_0) = f(B_0) \quad \text{almost surely} \tag{2}$$

where

$$f(x) = \int_W F(w)\mu_x(dw), \quad x \in \mathbb{R}^d$$

and where μ_x is *Wiener measure with starting point* x.

Given a Brownian motion B and a stopping time T, we can define a new filtration $(\tilde{\mathcal{F}}_t)_{t\geqslant 0}$ by $\tilde{\mathcal{F}}_t = \mathcal{F}_{T+t}$ and, on the event $\{T < \infty\}$, we can define a new process $\tilde{B}$ by setting $\tilde{B}_t = B_{T+t}$. Then, conditional on $\{T < \infty\}$, $\tilde{B}$ is a $(\tilde{\mathcal{F}}_t)_{t\geqslant 0}$-Brownian motion. This is called the *strong Markov property* of Brownian motion. It is a powerful way to do computations for Brownian motion, taken in conjunction with equation (2) and using the properties of conditional expectation. We usually omit reference to the filtration unless we wish to make a statement involving more than one filtration, such as the strong Markov property.

A complex-valued process $Z = X + iY$ is a *complex Brownian motion* if (X,Y) is a Brownian motion in $\mathbb{R}^2$.

Lévy's characterization is a useful way to identify Brownian motions: for an $\mathbb{R}^d$-valued process $B = (B^1,\dots,B^d)$, if B^i is a continuous local martingale for all i and if $[B^i,B^j]_t = t\delta_{ij}$ for all i,j and all $t \geqslant 0$, then B is a Brownian motion.

2.3 Itô's integral

Given a continuous finite-variation adapted process A and a continuous adapted process H, we can form the Lebesgue–Stieltjes integral $H \cdot A$. This is a continuous finite-variation adapted process starting from 0 and is given by

$$(H\cdot A)_t = \lim_{n\to\infty} \sum_{k=0}^{\lfloor 2^n t\rfloor - 1} H_{k2^{-n}}(A_{(k+1)2^{-n}} - A_{k2^{-n}})$$

where the limit holds in probability, and uniformly on compact time intervals. Given a continuous local martingale M and a continuous adapted process H, we can form the *Itô integral* $H \cdot M$. This is a continuous local martingale starting from 0 and is given by

$$(H \cdot M)_t = \lim_{n\to\infty} \sum_{k=0}^{\lfloor 2^n t\rfloor - 1} H_{k2^{-n}}(M_{(k+1)2^{-n}} - M_{k2^{-n}})$$

the limit holding in the same sense. It is characterized among continuous local martingales starting from 0 by the property that $[H \cdot M] = H^2 \cdot [M]$. We usually write

$$(H \cdot M)_t = \int_0^t H_s dM_s, \quad (H \cdot A)_t = \int_0^t H_s dA_s.$$

A *continuous semimartingale* X is any process having a decomposition $X_t = X_0 + M_t + A_t$ for all $t \geqslant 0$, where X_0 is an $\mathcal{F}_0$-measurable random variable, M is a continuous local martingale starting from 0, and A is a continuous adapted process of finite variation, also starting from 0 The decomposition $X = X_0 + M + A$ is then unique, and we extend the Itô integral to continuous semimartingales by setting

$$\int_0^t H_s dX_s = \int_0^t H_s dA_s + \int_0^t H_s dM_s.$$

An $\mathbb{R}^d$-valued process is called a *continuous semimartingale* if each of its components is a continuous semimartingale (in the same filtration).

We shall make extensive use of Itô's formula: if D is an open set in $\mathbb{R}^d$ and $f : D \to \mathbb{R}$ is a C^2 function, and if $X = (X^1, \dots, X^d)$ is a continuous semimartingale with values in D, then

$$f(X_t) = f(X_0) + \sum_{i=1}^d \int_0^t \partial_i f(X_s) dX_s^i + \frac{1}{2} \sum_{i,j=1}^d \int_0^t \partial_i \partial_j f(X_s) d[X^i, X^j]_s, \quad t \geqslant 0. \tag{3}$$

This is conveniently written in *differential form*

$$df(X_t) = \partial_i f(X_t) dX_t^i + \frac{1}{2} \partial_i \partial_j f(X_t) dX_t^i dX_t^j$$

where we sum over the repeated indices.

In fact we have the following *local version* which is in practice very useful. Let D be a proper domain. Let $f : D \to \mathbb{R}$ be a $\mathcal{C}^2$ function on D. Then if X is a semimartingale such that $X_0 \in D$ almost surely, and if $T = \inf\{t \geqslant 0 : X_t \notin D\}$ then Itô's formula holds almost surely for all $t < T$. That is, both left and right-hand sides are unambiguously defined for $t < T$ and they coincide almost surely.

3

We discuss the relation of Brownian motion to harmonic functions and then illustrate how this can be used by giving a proof of the Schwarz lemma. Then we show that the image of complex Brownian motion under a holomorphic function is a local martingale, indeed is itself a complex Brownian motion, up to a random change of time-scale. This leads to some useful formulae for harmonic measure.

3.1 Probabilistic solution of the Dirichlet problem

Let u be a C^2 bounded function on $\mathbb{C} = \mathbb{R}^2$ which is harmonic in a domain D. Fix $z \in D$ and let B be a complex Brownian motion starting from z. Consider the stopping time $T = T_D = \inf\{t \geqslant 0 : B_t \notin D\}$, then $T < \infty$ and $B_T \in \bar{D} \setminus D$, almost surely. Set

$$M_t = u(z) + \int_0^t \nabla u(B_s) dB_s$$

then M is a continuous local martingale. By Itô's formula, $u(B_t) = M_t$ for all $t \leqslant T$, so M^T is uniformly bounded and, by optional stopping,

$$u(z) = M_0 = \mathbb{E}(M_T) = \mathbb{E}(u(B_T)).$$

Hence u can be recovered from its restriction to $\bar{D} \setminus D$. (This argument is not special to two dimensions.)

Suppose now that $f = u + iv$ is a holomorphic function defined on a domain D_0 and that D is a bounded domain with $\bar{D} \subseteq D_0$. By the Cauchy–Riemann equations, the real and imaginary parts of f are harmonic in D_0 so, by a simple patching argument, there exist C^2 functions u and v on $\mathbb{R}^2$ which are harmonic on D and such that $f = u + iv$ on D. Thus we obtain the useful formula

$$f(z) = \mathbb{E}_z(f(B_{T_D})).$$

3.2 Mean-value property

Let D be a domain and let $u : D \to \mathbb{R}$ be a function. We say that u satisfies the *mean-value property* if for all $z \in D$, there exists a sequence $r_n > 0$ such that $r_n \to 0$, and such that for all $n \geqslant 1$

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(z + r_n e^{i\theta}) d\theta.$$

That is, $u(z)$ is equal to its average on a circle of radius r_n around z. Note that, since the hitting distribution of Brownian motion on a sphere of radius r around the starting point z is uniform, it follows from the above representation that if u is harmonic on D and $\bar{B}(z, r) \subset D$ then u satisfies the mean-value property for *any* sequence $r_n \leqslant r$. Remarkably, the converse is true under an additional assumption of continuity.

Proposition 3.1. *If u is continuous and has the mean-value property on D then it is harmonic.*

Proof. The main idea is to use the so-called maximum principle, which in essence tells us that a continuous function satisfying the mean-value property in a domain reaches its maximum on the boundary of this domain. Fix $w \in D$ and $R > 0$ such that $\bar{B}(w, R) \subset D$. Let $h(z) = \mathbb{E}_w(u(B_T))$ be the harmonic function defined in $\bar{B}(w, R)$ which coincides with u on $\partial B(w, R)$. Then h is harmonic and thus satisfies the mean-value property. Consider the function $v = u - h$ which is continuous and obviously satisfies the mean-value property on $B(w, R)$. Let $M = \sup\{v(z), z \in \bar{B}(w, R)\}$. Then $M < \infty$ as v is continuous. Assume $M > 0$, and let E be the set of $z \in \bar{B}(w, R)$ for which $v(z) = M$. It is clear that E is closed and is contained in $B(w, R)$ as $v = 0$ on the boundary. Also, E does not contain w as (by the mean-value property) $v(w) = 0$. Thus E is compact, and does not contain w. Let z_0 minimize in E the distance to w, i.e., let $z_0 \in E$ such that

$$|z_0 - w| \leqslant |z - w|$$

for all $z \in E$. When r is small enough, we know that $B(z_0, r) \subset B(w, R)$ and at least half of the circle of radius r about z_0 does not lie in E, as otherwise we found find a point of E closer to w than z_0. For such $r > 0$, we have necessarily that $\frac{1}{2\pi} \int_0^{2\pi} v(z_0 + re^{i\theta}) d\theta < M$. Since v satisfies the mean-value property on al of $B(w, R)$, this is a contradiction. Thus $M = 0$. Likewise, $\min v = 0$ and $v(z) = 0$ for all $z \in B(w, R)$. Thus u is harmonic. □

Remark 3.2. *This proof works in all dimensions. Note the essential assumption that u is continuous; it is easy to construct counterexamples without it. However, it is also possible to show that if u is a function (not necessarily continuous) satisfying the following assumption: for all $z \in D$ and for* all *$r > 0$ such that $B(z, r) \subset D$ then $u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{i\theta}) d\theta$, then u is harmonic.*

3.3 Proof of the Schwarz lemma

A number of results of complex analysis can be understood well using Brownian motion. Here we give a proof of Lemma 1.2. Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic function with $f(0) = 0$. Fix $z \in \mathbb{D}$ and consider a complex Brownian motion B starting from z. Fix $r \in (|z|, 1)$ and $\varepsilon \in (0, 1 - |z|)$ and consider the (almost surely finite) stopping times

$$S = \inf\{t \geqslant 0 : |B_t| = r\}, \quad T = \inf\{t \geqslant 0 : |B_t - z| = \varepsilon\}.$$

Consider the function $g(w) = f(w)/w$. By Taylor's theorem, g is analytic and hence holomorphic in $\mathbb{D}$ so

$$g(z) = \mathbb{E}(g(B_S)) = \mathbb{E}(g(B_T)).$$

Now $|g(B_S)| \leqslant 1/r$, so, letting $r \uparrow 1$, we deduce that $|g(z)| \leqslant 1$ and hence $|f(z)| \leqslant |z|$.

Suppose now that $e^{i\theta} g(z) = 1$. We know that B_T is uniformly distributed on the set $C = \{w : |w - z| = \varepsilon\}$ and $|e^{i\theta} g(w)| \leqslant 1$ for all $w \in \mathbb{D}$. Hence we must have $e^{i\theta} g(w) = 1$ for all $w \in C$. We deduce that the set $A = \{w \in \mathbb{D} : e^{i\theta} g(w) = 1\}$ contains every open ball in $\mathbb{D}$ centred at z. By a finite number of iterations of this argument, it follows that $A = \mathbb{D}$.

3.4 Conformal invariance of Brownian motion

Theorem 3.3. *Let D and D' be complex domains and let $z \in D$ and $z' \in D'$. Let B and B' be complex Brownian motions starting from z and z' respectively. Set*

$$T = \inf\{t \geqslant 0 : B_t \notin D\}, \quad T' = \inf\{t \geqslant 0 : B'_t \notin D'\}.$$

Suppose that there exists a conformal isomorphism $\Phi : D \to D'$ such that $\Phi(z) = z'$. Set $\tilde{T} = \int_0^T |\Phi'(B_t)|^2 dt$ and define for $t < \tilde{T}$

$$\tau(t) = \inf\left\{s \geqslant 0 : \int_0^s |\Phi'(B_r)|^2 dr = t\right\}, \quad \tilde{B}_t = \Phi(B_{\tau(t)}).$$

Then $(\tilde{T}, (\tilde{B}_t)_{t<\tilde{T}})$ and $(T', (B'_t)_{t<T'})$ have the same distribution.

Proof. Assume for now that Φ extends to a conformal map on a neighbourhood of $\bar{D}$ in $\mathbb{C}$. Then we may define a continuous semimartingale Z and a continuous and strictly increasing adapted process A by setting

$$Z_t = \Phi(B_{T\wedge t}) + (B_t - B_{T\wedge t}), \quad A_t = \int_0^{T\wedge t} |\Phi'(B_s)|^2 ds + (t - T \wedge t).$$

and we may extend τ to a continuous function on $[0, \infty)$ by setting

$$\tau(t) = \inf\{s \geqslant 0 : A_s = t\}.$$

Write $\Phi = u + iv$, $B_t = X_t + iY_t$ and $Z_t = M_t + iN_t$. By Itô's formula, for $t \leqslant T$,

$$dM_t = \frac{\partial u}{\partial x}(B_t)dX_t + \frac{\partial u}{\partial y}(B_t)dY_t, \quad dN_t = \frac{\partial v}{\partial x}(B_t)dX_t + \frac{\partial v}{\partial y}(B_t)dY_t$$

and so, using the Cauchy–Riemann equations,

$$dM_t dM_t = |\Phi'(B_t)|^2 dt = dA_t = dN_t dN_t, \quad dM_t dN_t = 0.$$

On the other hand, for $t > T$,

$$dM_t = dX_t, \quad dN_t = dY_t, \quad dM_t dM_t = dt = dA_t = dN_t dN_t, \quad dM_t dN_t = 0.$$

Hence M, N, $M^2 - A$, $N^2 - A$ and MN are all continuous local martingales. Set $\tilde{M}_s = M_{\tau(s)}$ and $\tilde{N}_s = N_{\tau(s)}$. Then, by optional stopping, $\tilde{M}$, $\tilde{N}$, $\tilde{M}^2 - s$, $\tilde{N}^2 - s$ and $\tilde{M}\tilde{N}$ are continuous local martingales for the filtration $(\tilde{\mathcal{F}}_s)_{s\geqslant 0}$, where $\tilde{\mathcal{F}}_s = \mathcal{F}_{\tau(s)}$. Hence, by Lévy's characterization of Brownian motion, $\tilde{Z} = \tilde{M} + i\tilde{N}$ is a complex $(\tilde{\mathcal{F}}_s)_{s\geqslant 0}$-Brownian motion starting from $z' = \Phi(z)$. Since $\tilde{T} = \inf\{t \geqslant 0 : \tilde{Z}_t \notin D'\}$ and $\tilde{B}_t = \tilde{Z}_t$ for $t \leqslant \tilde{T}$, this proves the claimed identity of distributions.

In the case where Φ fails to be C^2 in a neighbourhood of $\bar{D}$, choose a sequence of open sets $D_n \uparrow D$ with $\bar{D}_n \subseteq D$ for all n. Set $D'_n = \Phi(D_n)$ and set

$$T_n = \inf\{t \geqslant 0 : B_t \notin D_n\}, \quad T'_n = \inf\{t \geqslant 0 : B'_t \notin D'_n\}.$$

Set $\tilde{T}_n = \int_0^{T_n} |\Phi'(B_t)|^2 dt$. Then $\tilde{T}_n \uparrow \tilde{T}$ and $T'_n \uparrow T'$ almost surely as $n \to \infty$. Since Φ is C^2 in a neighbourhood of $\bar{D}_n$, we know that $(\tilde{T}_n, (\tilde{B}_t)_{t<\tilde{T}_n})$ and $(T'_n, (B'_t)_{t<T'_n})$ have the same distribution for all n, which implies the desired result on letting $n \to \infty$. □

3.5 First exit distributions and harmonic measure

The conformal invariance property provides an effective means to calculate the distribution of Brownian motion on its first exit from a simply connected proper domain D. Let B be a complex Brownian motion starting from $z \in D$ and set $T = T_D = \inf\{t \geqslant 0 : B_t \notin D\}$, as above. Then $T < \infty$ almost surely. In the case $D = \mathbb{D}$ and $z = 0$, we know that B_t converges in $\bar{\mathbb{D}}$ as $t \uparrow T$, with limit B_T uniformly distributed on the unit circle. We can choose a conformal isomorphism $\Phi : D \to \mathbb{D}$ taking z to 0. By conformal invariance, as $t \uparrow T$, B_t converges in $\hat{D}$ to a limit $B_T^* \in \partial D$. Denote by $h_D(z, .)$ the distribution of B_T^*. This *first exit distribution* is also called the *harmonic measure for D starting from z*. In the case where D is a Jordan domain, we have $B_T^* = B_T$ and, as we showed above, for every harmonic function u in D which extends continuously to $\bar{D}$, we can recover u from its boundary values by

$$u(z) = \mathbb{E}(u(B_T)) = \int_{\partial D} u(s) h_D(z, ds).$$

We can compute the harmonic measure as follows. By conformal invariance, for $s_1, s_2 \in \partial D$ and $\theta_1, \theta_2 \in [0, 2\pi)$ such that $\Phi(s_j) = e^{i\theta_j}$ for $j = 1, 2$, we have

$$\mathbb{P}(B_T^* \in [s_1, s_2]) = \mathbb{P}_0(B_{T_{\mathbb{D}}} \in [e^{i\theta_1}, e^{i\theta_2}]) = \frac{\theta_2 - \theta_1}{2\pi}.$$

We often fix a parametrization of ∂D by some interval $I \subseteq \mathbb{R}$ and then regard $h_D(z, .)$ as a measure on I. For good parametrizations the harmonic measure then has a density given by

$$h_D(z, t) = \frac{1}{2\pi} \frac{d\theta}{dt}.$$

For example, take $D = \mathbb{D}$ and parametrize the boundary as $(e^{it} : t \in [0, 2\pi))$. For $w = x + iy \in \mathbb{D}$, the Möbius transformation $\Phi_{0,w}$ takes w to 0. The boundary parametrizations are then related by $e^{i\theta} = (e^{it} - w)/(1 - \bar{w}e^{it})$, so differentiating this identity we get

$$h_{\mathbb{D}}(w, t) = \frac{1}{2\pi} \frac{1 - |w|^2}{|e^{it} - w|^2} = \frac{1}{2\pi} \frac{1 - x^2 - y^2}{(\cos t - x)^2 + (\sin t - y)^2}, \quad 0 \leqslant t < 2\pi. \tag{4}$$

Or take $D = \mathbb{H}$ with the obvious parametrization of the boundary by $\mathbb{R}$. Fix $w = x + iy \in \mathbb{H}$ and define $\Phi : \mathbb{H} \to \mathbb{D}$ by $\Phi(z) = (z - w)/(z - \bar{w})$ so that $\Phi(w) = 0$. The boundary parametrizations are related by $e^{i\theta} = (t - w)/(t - \bar{w})$, so

$$h_{\mathbb{H}}(w, t) = \frac{1}{\pi} \operatorname{Im}\left(\frac{1}{t - w}\right) = \frac{y}{\pi((x - t)^2 + y^2)}, \quad t \in \mathbb{R}.$$

It is easy to recover the above formula using properties of standard Brownian motion.

3.6 An estimate for harmonic functions

The following lemma allows us to bound the partial derivatives of a harmonic function in terms of its supremum norm. This will later be useful to derive estimate on a conformal map f using only bounds on its real part.

Lemma 3.4. *Let u be a harmonic function in D and let $z \in D$. Then*

$$\left|\frac{\partial u}{\partial x}(z)\right| \leqslant \frac{4\|u\|_\infty}{\pi \operatorname{dist}(z, \partial D)}.$$

Proof. It will suffice to prove, for all $\varepsilon > 0$, that the estimate holds with an extra factor of $1+\varepsilon$ on the right. Fix $\varepsilon > 0$. By affine transformation, we reduce to the case where $z = 0$ and $\operatorname{dist}(0, \partial D) = 1 + \varepsilon$. Then u is continuous on $\bar{\mathbb{D}}$ so, for $z \in \mathbb{D}$,

$$u(z) = \int_0^{2\pi} u(e^{i\theta}) h_{\mathbb{D}}(z, \theta) d\theta.$$

The formula (4) shows that $\nabla h_{\mathbb{D}}(\cdot, \theta)$ is bounded on a neighbourhood of 0, uniformly in θ, with

$$\nabla h_{\mathbb{D}}(0, \theta) = \frac{1}{\pi}(\cos\theta, \sin\theta).$$

Hence we may differentiate under the integral sign to obtain

$$\nabla u(0) = \frac{1}{\pi} \int_0^{2\pi} u(e^{i\theta})(\cos\theta, \sin\theta) d\theta$$

so that

$$\left|\frac{\partial u}{\partial x}(0)\right| \leqslant \frac{\|u\|_\infty}{\pi} \int_0^{2\pi} |\cos\theta| d\theta = \frac{4\|u\|_\infty}{\pi} = \frac{4(1+\varepsilon)\|u\|_\infty}{\pi \operatorname{dist}(0, \partial D)}.$$

□

4

We say that $K \subset \mathbb{H}$ is a *compact $\mathbb{H}$-hull* if K is bounded and $H = \mathbb{H} \setminus K$ is a simply connected domain. Thus $K = \bar{K} \cap \mathbb{H}$ and H is a neighbourhood of ∞ in $\mathbb{H}$, as in Subsection 1.3. For each compact $\mathbb{H}$-hull K, we shall identify a canonical choice of conformal isomorphism $g_K : H \to \mathbb{H}$ and derive some useful properties.

4.1 Existence and uniqueness of g_K

Proposition 4.1. *Let $x \in \mathbb{R}$ and let D be a neighbourhood of x in $\mathbb{H}$ (hence D is simply connected). Let $\Phi : D \to \mathbb{H}$ be a conformal isomorphism which is bounded near x. Then Φ extends analytically to a neighbourhood of x in $\mathbb{C}$ with $\Phi'(x) > 0$. Moreover, all the coefficients in the Taylor expansion of Φ near x are real.*

Proof. It will suffice to consider the case $x = 0$. Fix $\varepsilon > 0$ and consider the set

$$\tilde{D} = \{z \in \mathbb{C} : z \in D \text{ or } \overline{z} \in D \text{ or } z \in (-\varepsilon, \varepsilon)\}.$$

We can and do choose ε sufficiently small so that $\tilde{D}$ is a (proper) simply connected domain. By the Riemann mapping theorem, there exists a unique conformal isomorphism $\Psi : \tilde{D} \to \mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty))$ with $\Psi(0) = 0$ and $\Psi'(0) > 0$. By reflection symmetry and uniqueness, $\overline{\Psi(z)} = \Psi(\bar{z})$. Hence Ψ must map $(-\varepsilon, \varepsilon)$ to a subset of $(-1, 1)$. By considering Ψ^{-1}, the image of $(-\varepsilon, \varepsilon)$ is in fact all of $(-1, 1)$. Since $\Psi'(0) > 0$, Ψ must restrict to a conformal isomorphism from D to $\mathbb{H}$. Write $\Psi(z) = \sum_{i=0}^{\infty} \psi_i z^i$. Then since $\Psi(\bar{z}) = \overline{\Psi(z)}$ we have $\psi_i = \bar{\psi}_i$ so the coefficients in the Taylor expansion of Ψ near 0 are all real.

Consider now any conformal isomorphism $\Phi : D \to \mathbb{H}$ which is bounded near 0. Then $f = \Phi \circ \Psi^{-1}$ is a Mobius transformation and f is bounded near 0, so f is holomorphic in a neighbourhood of 0 in $\mathbb{C}$ with $f'(0) > 0$ as f maps $\mathbb{H}$ to $\mathbb{H}$. Likewise, since f maps a real interval near 0 to the real line, it follows that all the coefficients in the Taylor expansion of f near 0 are real. But $\Phi = f \circ \Psi$ and $\Psi(0) = 0$ so, by the chain rule, Φ extends analytically to a neighbourhood of 0 in $\mathbb{C}$ with $\Phi'(0) > 0$, and all the derivatives of Φ are real. □

Proposition 4.2. *Let K be a compact $\mathbb{H}$-hull. There exists a unique conformal isomorphism $g_K : H \to \mathbb{H}$ such that $g_K(z) - z \to 0$ as $z \to \infty$. Moreover, for some $a_K \in \mathbb{R}$, we have*

$$g_K(z) = z + \frac{a_K}{z} + O(|z|^{-2}), \quad z \to \infty.$$

Proof. Set $D = \{z \in \mathbb{C} : -1/z \in H\}$. Then D is a neighbourhood of 0 in $\mathbb{H}$. By the Riemann mapping theorem, there exists a conformal isomorphism $\Phi : D \to \mathbb{H}$ which is bounded near 0 (the map Ψ constructed in Proposition 4.1 above does this, and there are of course many other ones). Then, by Proposition 4.1, Φ extends analytically to a neighbourhood of 0 in $\mathbb{C}$ with $\Phi'(0) > 0$. By translation and rescaling if necessary, we may choose Φ so that $\Phi(0) = 0$ and $\Phi'(0) = 1$. Then, by Taylor's theorem, for some $b, c \in \mathbb{C}$,

$$\Phi(z) = z + bz^2 + cz^3 + O(|z|^4), \quad z \to 0.$$

Moreover, the coefficients in the Taylor expansion of Φ must be real as Φ sends an interval on the real line to the real line. Thus $b, c \in \mathbb{R}$. Define g_K on H by $g_K(z) = -1/\Phi(-1/z) - b$. It is a straightforward exercise to check that g_K is a conformal isomorphism to $\mathbb{H}$ having the claimed expansion at ∞, with $a_K = b^2 - c \in \mathbb{R}$.

Finally, if $g : H \to \mathbb{H}$ is any conformal isomorphism such that $g(z) - z \to 0$ as $z \to \infty$, then $f = g_K \circ g^{-1}$ is a conformal automorphism of $\mathbb{H}$ with $f(z) - z \to 0$ as $z \to \infty$. But then f is a Möbius tranformation by the Schwarz lemma, so we must have $f(z) = z$ for all z, showing that $g = g_K$. $\square$

Definition 4.3. *Let K be a compact $\mathbb{H}$-hull and let g_K be as in Proposition 4.2. We call the real number a_K, the* half-plane capacity *of K, denoted by*

$$a_K = \text{hcap}(K).$$

Remark 4.4. *We will see below that* $\text{hcap}(K) \geqslant 0$ *and that, in some sense that will be made precise in* (6)*, the half-plane capacity measures the size of a hull "seen from infinity".*

We start by establishing two useful properties (scaling and additive rule) of the half-plane capacity which support the idea that it measures "size" in a sense. Let K and g_K be as in the last proposition, and fix $r \in (0, \infty)$ and $b \in \mathbb{R}$. Set $\tilde{K} = rK + b = \{z \in \mathbb{H} : (z-b)/r \in K\}$, which is also a compact $\mathbb{H}$-hull. Define $g : \tilde{K} \to \mathbb{H}$ by $g(z) = rg_K((z-b)/r) + b$. Then g is a conformal isomorphism and

$$g(z) = z + \frac{r^2 a_K}{z} + O(|z|^{-2}), \quad z \to \infty.$$

Hence, by uniqueness, we have $g = g_{rK+b}$ and

$$\text{hcap}(rK + b) = r^2 \, \text{hcap}(K).$$

Consider now two compact $\mathbb{H}$-hulls K and K_0, with $K_0 \subseteq K$. Set $\tilde{K} = g_{K_0}(K \setminus K_0)$. It is also a compact $\mathbb{H}$-hull, as $\tilde{H} = \mathbb{H} \setminus \tilde{K} = g_{K_0}(H)$ is simply connected. (Note however that $K \setminus K_0$ is not, in general, a compact $\mathbb{H}$-hull). Consider the function $g = g_{\tilde{K}} \circ g_{K_0}$. Now g_{K_0} restricts to a conformal isomorphism $H \to \tilde{H}$ so g is a conformal isomorphism $H \to \mathbb{H}$. Also, as $|z| \to \infty$ we have

$$g(z) = g_{K_0}(z) + \frac{a_{\tilde{K}}}{g_{K_0}(z)} + O(|g_{K_0}(z)|^{-2}) = z + \frac{a_{K_0} + a_{\tilde{K}}}{z} + O(|z|^{-2}).$$

So, by uniqueness, we obtain $g_K = g_{\tilde{K}} \circ g_{K_0}$ and

$$\text{hcap}(K) = \text{hcap}(\tilde{K}) + \text{hcap}(K_0).$$

Remark 4.5. *Assume that $K \subset \mathbb{D}$. Then observe that for all $|z| \geqslant 2$,*

$$|g_K(z) - z - a_K/z| \leqslant C/|z|^2 \tag{5}$$

for some constant $C > 0$ which might depend on K. We will see below that we can choose C depending only on the radius of K – see Proposition 4.8 for a precise statement.

4.2 Estimates on g_K

Proposition 4.6. *Let K be a compact $\mathbb{H}$-hull. For a complex Brownian motion B started at $z = iy$ with $y > 0$, define $T = T_H = \inf\{t \geqslant 0 : B_t \in K \cup \mathbb{R}\}$. Then*

$$\mathrm{hcap}(K) = \lim_{y\to\infty} y\mathbb{E}^{iy}(\mathrm{Im}(B_T)). \tag{6}$$

In particular $\mathrm{hcap}(K) \geqslant 0$.

Remark 4.7. *Thus, half-plane capacity may be considered to measure the average height of the boundary of a hull seen by a Brownian motion started at ∞.*

Proof. Set $\phi(z) = \mathrm{Im}(z - g_K(z))$. Then ϕ is harmonic on H, and note that it is also bounded, as $\Phi(z)$ is clearly bounded on any neighbourhood of infinity, and g_K is also bounded on the complement of a neighbourhood of infinity (as g_K is a conformal isomorphism and has ∞ as a fixed point).[1]

Note that, by conformal invariance, $\mathrm{Im}(g_K(B_t)) \to 0$ almost surely as $t \uparrow T$, so, by a limiting argument and optional stopping,

$$\phi(z) = \mathbb{E}_z(\mathrm{Im}(B_T)) \tag{7}$$

and hence

$$\mathrm{Im}(z) = \mathrm{Im}\, g_K(z) + \mathbb{E}^z(\mathrm{Im}(B_T)). \tag{8}$$

Now, by definition of g_K,

$$g_K(iy) = iy + \frac{a_K}{iy} + o(1/y).$$

Taking the imaginary part and combining with (8) we get

$$\frac{a_K}{y} + o(1/y) = y - \mathrm{Im}\, g_K(iy) = \mathbb{E}^{iy}(\mathrm{Im}(B_T))$$

from which the proposition follows immediately. □

Proposition 4.8. *There is a constant $C < \infty$ with the following properties. For all $r \in (0, \infty)$ and all $\xi \in \mathbb{R}$, for any compact $\mathbb{H}$-hull $K \subseteq r\mathbb{D} + \xi$,*

$$\left| g_K(z) - z - \frac{a_K}{z - \xi} \right| \leqslant \frac{Cr|a_K|}{|z - \xi|^2}, \quad |z - \xi| \geqslant 2r. \tag{9}$$

Remark 4.9. *Note that C is a universal constant, which in particular does not depend on K. If we were not requiring this property, the result would follow from* (5).

[1] See example sheet. Alternatively, here is an elaboration of this argument. Let $r > 0$ such that $K \subset \mathbb{D}(0, r)$. Consider the arc $C = re^{i\theta}, 0 < \theta < \pi$. Then C separates K from ∞, and g maps C to simple curve such that $\mathbb{H} \setminus g(C)$ has exactly two connected components by Jordan's theorem. It follows that if $H_r = \mathbb{D}(0, r) \cap H$, $g(H_r)$ is contained in one of those two components. It may not be the unbounded one, as $g(\infty) = \infty$ and ∞ is separated from H_r by K. Thus $g(H_r)$ is bounded, from which it follows that $g(z) - z$ is bounded on H.

Proof. We shall prove the result in the case $r = 1$ and $\xi = 0$. The general case then follows by scaling and translation. Define for $\theta \in [0, \pi]$

$$a_K(\theta) = \mathbb{E}_{e^{i\theta}}(\mathrm{Im}(B_T)),$$

where B is a complex Brownian motion starting from $e^{i\theta}$ and $T = T_H = \inf\{t \geqslant 0 : B_t \notin H\}$.

Set $D = \mathbb{H} \setminus \bar{\mathbb{D}}$. Consider the conformal isomorphism $w = \Phi(z) = z + z^{-1} : D \to \mathbb{H}$ and note that $\Phi(e^{i\theta}) = 2\cos\theta$. Then, for $z \in D$,

$$h_D(z,\theta) = h_{\mathbb{H}}(w, 2\cos\theta)\frac{d}{d\theta}\Phi(e^{i\theta}) = \mathrm{Im}\left(\frac{1}{2\cos\theta - w}\right)\frac{2\sin\theta}{\pi}. \tag{10}$$

There is a constant $C < \infty$ such that, for all $|z| \geqslant 3/2$ and $\theta \in (0, \pi)$,

$$\left|\frac{1}{w - 2\cos\theta} - \frac{1}{z}\right| = \frac{|2\cos\theta - z^{-1}|}{|z||z + z^{-1} - 2\cos\theta|} \leqslant \frac{C}{|z|^2}.$$

From this and the strong Markov property, it follows that

$$\begin{aligned}\mathbb{E}^{iy}(\mathrm{Im}(B_T)) &= \int_0^\pi h_D(iy,\theta)a_K(\theta)d\theta \\ &\sim \frac{1}{y}\int_0^\pi \frac{2\sin\theta}{\pi}a_K(\theta)d\theta\end{aligned}$$

and thus by Proposition 4.6,

$$\mathrm{hcap}(K) = \int_0^\pi \frac{2\sin\theta}{\pi}a_K(\theta)d\theta. \tag{11}$$

Define

$$f(z) = g_K(z) - z - a_K/z, \quad z \in H$$

where $a_K = \mathrm{hcap}(K)$. Then f is holomorphic in D and $f(z) \to 0$ as $z \to \infty$. Also, writing $f = u + iv$, and letting $\phi(z) = \mathrm{Im}(z - g_K(z)) = \mathbb{E}^z(\mathrm{Im}(B_T))$ as already observed in (7), so

$$v(z) = -\phi(z) - \mathrm{Im}\left(-\frac{a_K}{z}\right) = \int_0^\pi \mathrm{Im}\left(\frac{1}{w - 2\cos\theta} - \frac{1}{z}\right)a_K(\theta)\frac{2\sin\theta}{\pi}d\theta$$

by (10) and the strong Markov property at time T_D. Hence $|v(z)| \leqslant C/|z|^2$ for all $|z| \geqslant 3/2$.

Note that by Proposition 4.1, for every $x \in \mathbb{R}$ with say $|x| \geqslant 3/2$, the function f extends analytically near x and $f(x) \in \mathbb{R}$. It follows that if $z_n \in \mathbb{H}$ with $z_n \to x$ then $v(z_n) \to 0$. If we now define v by reflection in all of $\{|z| > 3/2\}$ (by setting $v(\bar{z}) = -v(z)$), then note that v is continuous and has the mean value property (in the sense of Proposition 3.1) at every point in $\{|z| > 3/2\}$. Thus v is harmonic in this domain. Define $D_z = \{w \in \mathbb{C} :$

$|w| > (3/4)|z|\}$, and note that $\mathrm{dist}(z, \partial D_z) = (1/4)|z|$. Applying Lemma 3.4 in D_z at the point z, we find that for all $z \in H$ with $|z| \geqslant 2$,

$$\left|\frac{\partial v}{\partial x}(z)\right|, \left|\frac{\partial v}{\partial y}(z)\right| \leqslant \frac{Ca_K}{|z|^3}.$$

By the Cauchy-Riemann equations, $(\partial u)/(\partial x) = (\partial v/\partial y)$ and $(\partial u)/(\partial y) = -(\partial v)/(\partial x)$. Thus identical bounds for v replaced by u and hence also for $|f'(z)|$.

So, for all $\rho \geqslant 2$ and $\theta \in (0, \pi)$,

$$|f(\rho e^{i\theta})| = \left|\int_\rho^\infty f'(ye^{i\theta})dy\right| \leqslant \sqrt{2}Ca\int_\rho^\infty s^{-3}ds = \frac{Ca}{\sqrt{2}\rho^2}.$$

This gives the estimate (9). □

As a measure of progress towards the Loewner differential equation, we note the following consequence of Proposition 4.8:

Corollary 4.10. *Assume that $(K_t)_{t\geqslant 0}$ is a collection of compact $\mathbb{H}$-hulls with $K_t \subset \rho_t\mathbb{D}$ and $\rho_t \to 0$ as $t \to 0^+$ and $\mathrm{hcap}(K_t) = 2t$. Then for all $z \in \mathbb{H}$,*

$$\lim_{t\to 0^+} \frac{g_{K_t}(z) - z}{t} = \frac{2}{z}.$$

4.3 Boundary deformation under g_K

The following result illustrates another way that Brownian motion provides a tool to understand conformal maps.

Proposition 4.11. *Let K be a compact $\mathbb{H}$-hull and let $\xi \in \mathbb{R}$. Suppose that $H = \mathbb{H} \setminus K$ is a neighbourhood of ξ in $\mathbb{H}$. Then g_K extends analytically to a neighbourhood of ξ in $\mathbb{C}$, with $g_K'(\xi) \in (0, 1]$. Also, if H is a neighbourhood of every $b \in [\xi, \infty)$, then $g_K(\xi) \geqslant \xi$.*

Proof. Since g_K is bounded near ξ, the possibility of extending g_K analytically near ξ, with $g_K'(\xi) > 0$, is established in Proposition 4.1. Let B be a complex Brownian motion starting from $z = x + iy$ and consider

$$T_H = \inf\{t \geqslant 0 : B_t \notin H\}, \quad T_{\mathbb{H}} = \inf\{t \geqslant 0 : B_t \notin \mathbb{H}\}.$$

Note that, for $a, b \in \mathbb{R}$ with $a < b$, in the limit $y \to \infty$ with $x/y \to 0$, we have

$$\pi y\, \mathbb{P}_z(B_{T_{\mathbb{H}}} \in [a, b]) = \int_a^b \frac{y^2}{y^2 + (t - x)^2}dt \to b - a.$$

Suppose that H is a neighbourhood of $[a, b]$ in $\mathbb{H}$ and that $z \in H$, then by conformal invariance of Brownian motion

$$\mathbb{P}_{g_K(z)}(B_{T_{\mathbb{H}}} \in [g_K(a), g_K(b)]) = \mathbb{P}_z(B_{T_H} \in [a, b]) \leqslant \mathbb{P}_z(B_{T_{\mathbb{H}}} \in [a, b]).$$

Set $z = iy$ and write $g_K(z) = u + iv$. Note that, as $y \to \infty$, we have $v/y \to 1$, $v \to \infty$ and $u/v \to 0$. Hence on multiplying the preceding inequality by πy and letting $y \to \infty$ we obtain

$$g_K(b) - g_K(a) \leqslant b - a.$$

The bound $g'_K(\xi) \leqslant 1$ then follows by the mean value theorem. As $g_K(b) - b \to 0$ as $b \to \infty$, taking $a = \xi$ and letting $b \to \infty$ shows that $g_K(\xi) \geqslant \xi$ if H is a neighbourhood of every point in $[\xi, \infty)$. □

Corollary 4.12. *Let K be a compact $\mathbb{H}$-hull. If $K \subset \mathbb{D}$, then for all $x > 1$,*

$$x \leqslant g_K(x) \leqslant x + 1/x.$$

Assume that $K \subset \xi + r\mathbb{D}$ for some $\xi \in \mathbb{R}$ and $r > 0$. Then

$$|g_K(z) - z| \leqslant 3r \tag{12}$$

for all $z \in H$.

Proof. That $g_K(x) \geqslant x$ comes directly from Proposition 4.11. Now, let $D = \mathbb{D} \cap H$ and consider the map g_D. Let $\tilde{D} = g_K(D \subset K)$ which is a hull. Then, as shown before, $g_D(z) = g_{\tilde{D}} \circ g_K(z)$. It is easy to check that if $x > 1$, $\tilde{H} = \mathbb{H} \setminus \tilde{D}$ is a neighbourhood in $\mathbb{H}$ of $g_K(x)$. Hence $g_D(x) = g_{\tilde{D}}(g_K(x)) \geqslant g_K(x)$. On the other hand, we know the map g_D explicitly: $g_D(z) = z + z^{-1}$ for all $z \in \mathbb{H} \setminus D$. Therefore $g_K(x) \leqslant x + 1/x$.

For similar reasons, still assuming $\xi = 0$ and $r = 1$, we see that $g_K(\bar{K} \cap \mathbb{H}) \subset g_D(\bar{D} \cap H) = [-2, 2]$. Hence we deduce that if $z \in \bar{H}$, $|g_K(z) - z| \leqslant 3$. Since $\phi(z) = g_K(z) - z$ is bounded and harmonic in H, it follows that ϕ is the average of its boundary values, and hence $|\phi(z)| \leqslant 3$ for all $z \in H$. The inequality (12) follows by translation and scaling. □

5

We now discuss two results which are fundamental to the theory of Schramm–Loewner evolutions. The first, due to Loewner, establishes a one-to-one correspondence between continuous real-valued paths $(\xi_t)_{t\geqslant 0}$ and increasing families $(K_t)_{t\geqslant 0}$ of compact $\mathbb{H}$-hulls having a certain local growth property. The null path $\xi_t \equiv 0$ corresponds to $K_t = (0, 2i\sqrt{t}]$. For smooth paths $(\xi_t)_{t\geqslant 0}$ starting from 0, we have $K_t = \{\gamma_s : 0 < s \leqslant t\}$ for some continuous path $(\gamma_t)_{t\geqslant 0}$ in $\bar{\mathbb{H}}$ starting from 0 and such that $\gamma_t \in \mathbb{H}$ for all $t > 0$. More generally, it may be the case that $(K_t)_{t\geqslant 0}$ is *generated by* a continuous path $(\gamma_t)_{t\geqslant 0}$ in $\bar{\mathbb{H}}$, meaning that H_t is the unbounded component of $\mathbb{H} \setminus \{\gamma_s : 0 < s \leqslant t\}$. The second key result, due (for $\kappa \neq 8$) to Rohde and Schramm, tells us that, when $(\xi_t)_{t\geqslant 0}$ is a Brownian motion, of any diffusivity $\kappa \in (0, \infty)$, the corresponding compact $\mathbb{H}$-hulls $(K_t)_{t\geqslant 0}$ are almost surely generated by such a path γ.

5.1 Local growth property and Loewner transform

Let $(K_t)_{t\geqslant 0}$ be a family of compact $\mathbb{H}$-hulls. Say that $(K_t)_{t\geqslant 0}$ is *increasing* if, for all $s, t \geqslant 0$ with $s < t$, we have $K_s \subseteq K_t$ and $K_t \setminus K_s \neq \emptyset$. Assume that $(K_t)_{t\geqslant 0}$ is increasing and set $K_{s,t} = g_{K_s}(K_t \setminus K_s)$. Say that $(K_t)_{t\geqslant 0}$ has the *local growth property* if, for all $T \geqslant 0$,

$$\sup_{s,t\in[0,T],\, 0<t-s\leqslant h} \operatorname{rad}(K_{s,t}) \to 0 \quad \text{as} \quad h \downarrow 0, \tag{13}$$

where $\operatorname{rad} K = \inf\{r > 0 : \exists\, \xi \in \mathbb{R} : K \subset \mathbb{D}(\xi, r)\}$. For such a family, by compactness, there is, for each $t \geqslant 0$, a unique $\xi_t \in \mathbb{C}$ such that $\xi_t \in \overline{K_{t,t+h}}$ for all $h > 0$. Moreover, $\xi_t \in \mathbb{R}$ and, using the estimate (12), we can show that ξ_t depends continuously on t. The process $(\xi_t)_{t\geqslant 0}$ is called the *Loewner transform* of $(K_t)_{t\geqslant 0}$. Note that we have, for each $t \geqslant 0$, as $h \downarrow 0$,

$$\operatorname{hcap}(K_{t+h}) - \operatorname{hcap}(K_t) = \operatorname{hcap}(K_{t,t+h}) \leqslant \operatorname{rad}(K_{t,t+h})^2 \to 0.$$

If we assume that $K_0 = \emptyset$ and $\operatorname{hcap}(K_t) \to \infty$ as $t \to \infty$, then the map $t \mapsto \operatorname{hcap}(K_t)$ will be a homeomorphism of $[0, \infty)$. Then, by a time-reparametrization, we may if we wish assume that

$$\operatorname{hcap}(K_t) = 2t \quad \text{for all } t \geqslant 0. \tag{14}$$

5.2 Loewner's differential equation

Let $(\xi_t)_{t\geqslant 0}$ now be any continuous real-valued function and consider the open set $U = \{(t, z) \in [0, \infty) \times \mathbb{C} : z \neq \xi_t\}$. Define a time-dependent, holomorphic vector field $b : U \to \mathbb{C}$ by

$$b(t, z) = \frac{2}{z - \xi_t} = \frac{2(x - \xi_t - iy)}{|z - \xi_t|^2}.$$

Define $U_n = \{(t, z) \in U : |z - \xi_t| > 1/n\}$ and note that, for all $(t, z), (t, z') \in U_n$,

$$|b(t, z) - b(t, z')| \leqslant 2n^2 |z - z'|.$$

By standard results in the theory of ordinary differential equations, for each $z \in \mathbb{C} \setminus \{\xi_0\}$, there is a unique $\zeta(z) \in (0, \infty]$ and a unique continuous map $t \mapsto z_t : [0, \zeta(z)) \to \mathbb{C}$ such that, for all $t \in [0, \zeta(z))$, we have $(t, z_t) \in U$ with

$$z_t = z + \int_0^t \frac{2}{z_s - \xi_s} ds \tag{15}$$

and, if $\zeta(z) < \infty$, then $|z_t - \xi_t| \to 0$ as $t \uparrow \zeta(z)$. Then $(z_t)_{t<\zeta(z)}$ is the *maximal solution* to *Loewner's differential equation* $\dot{z}_t = 2/(z_t - \xi_t)$ starting from z. Restricting to the upper half-plane, set $H_t = \{z \in \mathbb{H} : \zeta(z) > t\}$ and define $g_t : H_t \to \mathbb{C}$ by $g_t(z) = z_t$. We call the map $\zeta : \mathbb{C} \setminus \{\xi_0\} \to (0, \infty]$ the *lifetime* and we call the family of maps $(g_t)_{t \geqslant 0}$ the *Loewner flow* for driving function $(\xi_t)_{t \geqslant 0}$.

5.3 Loewner's theorem

Theorem 5.1. *Let $(\xi_t)_{t \geqslant 0}$ be a continuous real-valued function. Let ζ be the lifetime and $(g_t)_{t \geqslant 0}$ the Loewner flow for driving function $(\xi_t)_{t \geqslant 0}$. Set $K_t = \{z \in \mathbb{H} : \zeta(z) \leqslant t\}$ and $H_t = \mathbb{H} \setminus K_t$. Then $(K_t)_{t \geqslant 0}$ is an increasing family of compact $\mathbb{H}$-hulls having the local growth property, is parametrized so that* $\operatorname{hcap}(K_t) = 2t$ *for all t, and has Loewner transform $(\xi_t)_{t \geqslant 0}$. Moreover, for all $t \geqslant 0$, g_t is the unique conformal isomorphism $H_t \to \mathbb{H}$ such that $g_t(z) - z \to 0$ as $|z| \to \infty$. Furthermore, we obtain all such families of compact $\mathbb{H}$-hulls in this way. Finally, for all $x \in \mathbb{R} \setminus \{\xi_0\}$, H_t is a neighbourhood of x in $\mathbb{H}$ if and only if $\zeta(x) > t$.*

Proof. Define $H_t^\dagger = \{z \in \mathbb{C} \setminus \{0\} : \zeta(z) > t\}$ and define $g_t^\dagger : H_t^\dagger \to \mathbb{C}$ by $g_t^\dagger(z) = z_t$, where $(z_t)_{t<\zeta(z)}$ is the maximal solution to Loewner's equation starting from z. By standard results for differential equations, the set $H_t^\dagger$ is open and the map $g_t^\dagger$ is holomorphic for all $t \geqslant 0$. By taking the complex conjugate in Loewner's equation, we see that $(\bar{z}_t)_{t<\zeta(z)}$ is the maximal solution starting from $\bar{z}$. Thus, $H_t^\dagger$ is closed under conjugation and $g_t^\dagger(\bar{z}) = \overline{g_t^\dagger(z)}$.

Let $t \geqslant 0$ and $z_0 \in H_t^\dagger$. Then a solution to the differential equation

$$\begin{cases} \dot{w}_s & = -2/(w_s - \xi_{t-s}), 0 \leqslant s \leqslant t, \\ w_0 & = z_t, \end{cases}$$

is given by $w_s = g_{t-s}^\dagger(z_0)$. Since there is a unique maximal solution to this differential equation, it is necessarily the case that the map $g_t^\dagger$ is injective. We deduce that $g_t^\dagger$ is holomorphic and injective, and that, since $g_t^\dagger$ is injective, we must have $g_t^\dagger(H_t) \subseteq \mathbb{H}$.

Further, note that $\operatorname{Im}(-b(t, z)) > 0$ for all $(t, z) \in U$ with $z \in \mathbb{H}$. The above differential equation may thus always be solved for all $0 \leqslant s \leqslant t$. The value w_t of this solution at time t is a point $z_0 \in \mathbb{H}$ such that $g_t(z_0) = z$. Hence g_t maps H_t onto $\mathbb{H}$ and so g_t is a conformal isomorphism from H_t to $\mathbb{H}$.

This implies in particular that H_t is simply connected. We see also that H_t is a neighbourhood of x in $\mathbb{H}$ whenever $\zeta(x) > t$.

In order to establish the remaining claims about $(g_t)_{t\geqslant 0}$ and $(K_t)_{t\geqslant 0}$, we need some simple estimates for the Loewner flow. Fix $T \geqslant 0$ and set $r = \sup_{t\leqslant T} |\xi_t - \xi_0| \vee \sqrt{T}$. Fix $z \in \mathbb{H}$ with $|z - \xi_0| \geqslant 4r$ and let $R \geqslant \max(4r, |z|/2)$. Write z_t for $g_t(z)$ as usual. Define

$$\tau = \inf\{t \in [0, \zeta(z)) : |z_t - z| = r\} \wedge T.$$

Then $\tau < \zeta(z)$ and $|z_t - \xi_t| = |(z_t - z) + (z - \xi_0) + (\xi_0 - \xi_t)| \geqslant R - 2r$ for all $t \leqslant \tau$. Now

$$z_t - z = \int_0^t \frac{2}{z_s - \xi_s} ds, \quad z(z_t - z) - 2t = 2\int_0^t \frac{z - z_s - \xi_s}{z_s - \xi_s} ds.$$

so, for $t \leqslant \tau$,

$$|z_t - z| \leqslant \frac{2t}{R - 2r} \leqslant \frac{t}{r} \leqslant \frac{T}{r} \leqslant r, \quad |z(z_t - z) - 2t| \leqslant \frac{(4r + 2|\xi_0|)t}{R - 2r}.$$

The first estimate implies that $\tau = T$ (else $|z_\tau - z| \leqslant \tau/r < T/r \leqslant r$, a contradiction) and then $\zeta(z) > T$ so $z \in H_T$. Hence

$$|z - \xi_0| \leqslant 4r \text{ for all } z \in K_T. \tag{16}$$

and so K_T is a compact $\mathbb{H}$-hull. Then from the second estimate and bearing in mind that $R \geqslant |z|/2$, we deduce that, for all $t \geqslant 0$, we have $z(g_t(z) - z) \to 2t$ as $|z| \to \infty$. In particular $g_t(z) - z \to 0$ as $|z| \to \infty$, so $g_t = g_{K_t}$ and then $\operatorname{hcap}(K_t) = 2t$ for all t.

Let us now check the local growth property. Fix $s \geqslant 0$. Define for $t \geqslant 0$

$$\tilde{\xi}_t = \xi_{s+t}, \quad \tilde{H}_t = g_s(H_{s+t}), \quad \tilde{K}_t = \mathbb{H} \setminus \tilde{H}_t, \quad \tilde{g}_t = g_{s+t} \circ g_s^{-1}.$$

Then, taking derivatives, it is straightforward that $(\tilde{g}_t)_{t\geqslant 0}$ is the Loewner flow driven by $(\tilde{\xi}_t)_{t\geqslant 0}$, $\tilde{H}_t$ is the domain of $\tilde{g}_t$, and $\tilde{K}_t = g_s(K_{s+t} \setminus K_s) = K_{s,s+t}$. The estimate (16) applies to give

$$|z - \xi_s| \leqslant 4\left(\sup_{s\leqslant u\leqslant s+t} |\xi_u - \xi_s| \vee \sqrt{t}\right) \text{ for all } z \in K_{s,s+t}.$$

Hence $(K_t)_{t\geqslant 0}$ has the local growth property, and has Loewner transform $(\xi_t)_{t\geqslant 0}$.

Conversely, suppose now that $(K_t)_{t\geqslant 0}$ is any increasing family of compact $\mathbb{H}$-hulls having the local growth property, parametrized so that $\operatorname{hcap}(K_t) = 2t$ for all t. Set $g_t = g_{K_t}$ and take $(\xi_t)_{t\geqslant 0}$ to be the Loewner transform of $(K_t)_{t\geqslant 0}$. It is easy to see that $(\xi_t)_{t\geqslant 0}$ is continuous; now we show that $(g_t)_{t<\zeta}$ is the Loewner flow driven by $(\xi_t)_{t\geqslant 0}$, where $\zeta = \inf\{t \geqslant 0 : z \in K_t\}$.

Fix $s, t \geqslant 0$ with $s \leqslant t$ and $z \in H_t$. Recall that $\operatorname{hcap}(K_{s,t}) = \operatorname{hcap}(K_t) - \operatorname{hcap}(K_s) = 2(t-s)$, and $g_{K_{s,t}}(g_s(z)) = g_t(z)$. Applying the estimates of Proposition 4.8 to the compact $\mathbb{H}$-hull $K_{s,t}$, taking $\xi = \xi_s$, $w = g_s(z)$ and $r = 2\operatorname{rad}(K_{s,t})$, to obtain

$$|g_t(z) - g_s(z)| \leqslant 2C \operatorname{rad}(K_{s,t}) \tag{17}$$

and

$$\left| g_t(z) - g_s(z) - \frac{2(t-s)}{g_s(z)-\xi_s} \right| \leqslant \frac{4C\,\mathrm{rad}(K_{s,t})(t-s)}{|g_s(z)-\xi_s|^2} \tag{18}$$

provided $|g_s(z) - \xi_s| \geqslant 4\,\mathrm{rad}(K_{s,t})$. But if $s < \zeta(z)$ then $g_s(z) \in \mathbb{H} \setminus K_{s,t}$ as soon as $t < \zeta(z)$. By the local growth property we can find t sufficiently close to s such that $|g_s(z) - \xi_s| \geqslant 4\,\mathrm{rad}(K_{s,t})$, and hence (18) holds for t sufficiently close to s. Dividing by $t-s$ and letting $t \to s^+$, we find that $t \mapsto g_t(z)$ is differentiable on the right of s with

$$\frac{\partial}{\partial t} g_t(z)|_{t=s^+} = \frac{2}{g_s(z)-\xi_s}.$$

On the other hand, (17) shows that $g_s(z)$ is continuous and the right-derivative is continuous on $[0,t]$ as well. Thus $g_s(z)$ is differentiable and the derivative is as above.

To show that g_t is the Loewner flow, it remains to show that $(g_t(z))_{t<\zeta(z)}$ is the maximal solution. Fix $z \in \mathbb{H}$ and recall that $\zeta(z) = \inf\{t \geqslant 0 : z \in K_t\}$. If $\zeta(z) < \infty$ then for $s < \zeta(z) < t$ we have $z \in K_t \setminus K_s$ so $g_s(z) \in K_{s,t}$ and so $|g_s(z) - \xi_s| \leqslant 2\,\mathrm{rad}(K_{s,t})$; hence by the local growth property $|g_s(z) - \xi_s| \to 0$ as $s \uparrow \zeta(z)$. Thus $z_t = g_t(z)$ cannot be extended beyond $\zeta(z)$ as a solution to the differential equation, hence $(g_t(z))_{t<\zeta}$ is the maximal solution. Hence all such families of compact $\mathbb{H}$-hulls can be obtained by a Loewner flow.

Finally, fix $x \in \mathbb{R}$ and suppose that H_t is a neighbourhood of x in $\mathbb{H}$. Since $K_t = \cap_{s>t} K_s$, there exists $s > t$ such that H_s is a neighbourhood of x in $\mathbb{H}$. Write g_t^* for the extension of g_t as a conformal map on the reflected domain H_t^*. Then $x \in H_s^*$ so $g_t^*(x) \in g_t^*(H_s^*)$. On the other hand $\xi_t \in g_t^*(H_t^* \setminus H_s^*)$, so $g_t(x) \neq \xi_t$. On letting $z \to x$ in (17), we obtain $|g_t^*(x) - g_s^*(x)| \leqslant 2C\,\mathrm{rad}(K_{s,t})$, so $s \mapsto |g_s^*(x) - \xi_s|$ is continuous and hence uniformly positive on $[0,t]$. Now we can pass to the limit $z \to x$ in (the integrated form of) Loewner's equation to see that $(g_s^*(x) : s \leqslant t)$ is a solution, and hence we must have $\zeta(x) > t$. □

5.4 Rohde–Schramm theorem

Theorem 5.2. *Let $\kappa \in [0,\infty)$ and let $(\xi_t)_{t\geqslant 0}$ be a real Brownian motion of diffusivity κ. Let $(K_t)_{t\geqslant 0}$ be the family of compact $\mathbb{H}$-hulls with Loewner transform $(\xi_t)_{t\geqslant 0}$, given by Theorem 5.1. Then there exists a unique continuous random process $(\gamma_t)_{t\geqslant 0}$ in $\overline{\mathbb{H}}$ which generates $(K_t)_{t\geqslant 0}$.*

The importance of the family of such processes $\gamma = (\gamma_t)_{t\geqslant 0}$ was first recognised by Schramm. We call γ a (chordal) *Schramm–Loewner evolution of parameter κ* or $SLE(\kappa)$ for short.

5.5 Scaling and Markov properties of SLE

Proposition 5.3. *Let γ be an $SLE(\kappa)$ for some $\kappa \in [0,\infty)$. Fix $r \in (0,\infty)$ and define a rescaled process $\tilde\gamma = (\tilde\gamma_t)_{t\geqslant 0}$ by*

$$\tilde\gamma_t = r^{-1}\gamma_{r^2 t}.$$

Then $\tilde\gamma$ is also an $SLE(\kappa)$.

Proof. Define for $t \geqslant 0$ and $z \in \mathbb{H}$

$$\tilde{\xi}_t = r^{-1}\xi_{r^2 t}, \quad \tilde{\zeta}(z) = r^{-2}\zeta(rz)$$

and for $t < \tilde{\zeta}(z)$ define

$$\tilde{g}_t(z) = r^{-1} g_{r^2 t}(rz).$$

Then $(\tilde{\xi}_t)_{t\geqslant 0}$ is a Brownian motion of diffusivity κ. Also, $\tilde{g}_0(z) = z$ and from Loewner's equation for $(g_t)_{t\geqslant 0}$ we obtain

$$\dot{\tilde{g}}_t(z) = \frac{2}{\tilde{g}_t(z) - \tilde{\xi}_t}, \quad t < \tilde{\zeta}(z)$$

with $\tilde{g}_t(z) - \tilde{\xi}_t \to 0$ as $t \uparrow \tilde{\zeta}(z)$ for all $z \in \mathbb{H}$. So $(\tilde{g}_t)_{t\geqslant 0}$ is the Loewner flow driven by $(\tilde{\xi}_t)_{t\geqslant 0}$. Define $\tilde{K}_t = r^{-1}K_{r^2 t}$. Then $\tilde{K}_t = \{z \in \mathbb{H} : \tilde{\zeta}(z) \leqslant t\}$ so $(\tilde{K}_t)_{t\geqslant 0}$ is the family of compact $\mathbb{H}$-hulls with Loewner transform $(\tilde{\xi}_t)_{t\geqslant 0}$. Now $\tilde{\gamma}$ generates $(\tilde{K}_t)_{t\geqslant 0}$ so $\tilde{\gamma}$ is an $SLE(\kappa)$. □

It is simplest to frame the (chordal) Markov property in terms of the compact $\mathbb{H}$-hulls $(K_t)_{t\geqslant 0}$ rather than the path γ.

Proposition 5.4. *Let $(K_t)_{t\geqslant 0}$ be (the compact $\mathbb{H}$-hulls of) an $SLE(\kappa)$ for some $\kappa \in [0,\infty)$. Write $(g_t)_{t\geqslant 0}$ and $(\xi_t)_{t\geqslant 0}$ for the associated Loewner flow and transform. Fix $s \geqslant 0$ and for $t \geqslant 0$ set $\tilde{K}_t = K_{s,s+t} = g_s(K_{s+t} \setminus K_s)$. Then the family $(\tilde{K}_t - \xi_s)_{t\geqslant 0}$ is also an $SLE(\kappa)$ and is moreover independent of $(K_u)_{0\leqslant u\leqslant s}$.*

Proof. Let $\tilde{g}_t = g_{\tilde{K}_t} = g_{s+t} \circ g_s^{-1}$. Note that if $\tilde{z}_t = \tilde{g}_t(z)$ then by taking derivative with respect to time, for all $z \in \tilde{H}_t = \mathbb{H} \setminus \tilde{K}_t$,

$$\frac{d}{dt}\tilde{z}_t = \frac{2}{\tilde{z}_t - \tilde{\xi}_t}$$

where $\tilde{\xi}_t = \xi_{s+t}$. Thus $(\tilde{K}_t)_{t\geqslant 0}$ has Loewner transform $(\tilde{\xi}_t)_{t\geqslant 0}$. It follows that $(\tilde{K}_t - \xi_s)_{t\geqslant 0}$ has Loewner transform $(\xi_{s+t} - \xi_s)_{t\geqslant 0}$. This is again a Brownian motion of diffusivity κ, which is independent of $(\xi_u)_{0\leqslant u\leqslant s}$. The result follows. □

The only continuous real-valued processes with independent and stationary increments are of the form $(B_t + \mu t)_{t\geqslant 0}$ for some $\mu \in \mathbb{R}$. Thus the same proof can be used to show Schramm's theorem:

Theorem 5.5. *Let $(K_t, t \geqslant 0)$ be a family of compact $\mathbb{H}$-hulls with local growth and parametrized by half-plane capacity,with Loewner transform ξ. Assume that the scaling property: for all $r > 0$, $(K_{r^2 t})_{t\geqslant 0}$ has the same law as $(rK_t)_{t\geqslant 0}$. Assume also that the chordal Markov property holds: for all $s \geqslant 0$, letting $\tilde{K}_t = K_{s,s+t}$, the family $(\tilde{K}_t - \xi_s)_{t\geqslant 0}$ has the same law as $(K_t)_{t\geqslant 0}$ and is moreover independent of $(K_u)_{0\leqslant u\leqslant s}$. Then $(K_t)_{t\geqslant 0}$ is an $SLE(\kappa)$ for some $\kappa \geqslant 0$.*

Although simple to prove, this result is conceptually important as it identifies $SLE(\kappa)$ as the only possible continuous curves satisfying the conformal Markov property.

6

The restriction to the real line of the Loewner flow associated to $SLE(\kappa)$ is a simple tranformation of a flow of Bessel processes. For these processes, which have scaling properties like Brownian motion and SLE, it is possible to do some explicit calculations. Once translated back in terms of the Loewner flow, we can deduce probabilities for certain events relating to the SLE path γ.

6.1 Hitting probabilities at 0 for the Bessel flow

Let B be a real Brownian motion starting from 0 and fix $a \in (0,\infty)$. Consider for each $x \in \mathbb{R} \setminus \{0\}$ the integral equation

$$X_t(x) = x + B_t + \int_0^t \frac{a}{X_s(x)} ds. \tag{19}$$

By standard results for ordinary differential equations, there exists $\zeta(x) \in (0,\infty)$ and a continuous path $(X_t(x))_{t<\zeta(x)}$ such that (19) holds for all $t \in [0,\zeta(x))$ and, if $\zeta(x) < \infty$, then $X_t(x) \to 0$ as $t \uparrow \zeta(x)$. Moreover, $\zeta(x)$ and $(X_t(x))_{t<\zeta(x)}$ are uniquely determined by these properties. Furthermore, for $x, y \in (0,\infty)$ with $x < y$, we have $\zeta(x) \leqslant \zeta(y)$ and $X_t(x) < X_t(y)$ for all $t < \zeta(x)$. (This can be shown by reversing time and using uniqueness of solutions for ordinary differential equations.)

Each of the processes $(X_t(x))_{t<\zeta(x)}$ is a Bessel process. Note that they are all constructed from a single Brownian motion. We call the whole family of processes the *Bessel flow*. In cases where the lifetime $\zeta(x)$ is finite, the solution $X_t(x)$ hits the point 0 where the drift is singular at time $\zeta(x)$, otherwise $X_t(x)$ has infinite lifetime and never hits 0. The following *scaling property* may be established by the same argument used for Proposition 5.3: fix $r \in (0,\infty)$ and set $\tilde{\zeta}(x) = r^{-2}\zeta(rx)$ and $\tilde{X}_t(x) = r^{-1}X_{r^2t}(rx)$, then the family of processes $(\tilde{X}_t(x))_{t<\tilde{\zeta}(x)}$ for $x \in \mathbb{R} \setminus \{0\}$ is also a Bessel flow.

Proposition 6.1. *Let $x, y \in (0,\infty)$ with $x < y$. Then*

(a) for $a \in (0, 1/4]$, we have

$$\mathbb{P}(\zeta(x) < \zeta(y) < \infty) = 1;$$

(b) for $a \in (1/4, 1/2)$, we have

$$\mathbb{P}(\zeta(x) < \infty) = 1, \quad \mathbb{P}(\zeta(x) < \zeta(y)) = \phi\left(\frac{y-x}{y}\right)$$

where ϕ is given by

$$\phi(\theta) = c\int_0^\theta \frac{du}{u^{2-4a}(1-u)^{2a}}, \quad \phi(1) = 1;$$

(c) for $a \in [1/2, \infty)$, we have

$$\mathbb{P}(\zeta(x) < \infty) = 0$$

and indeed, for $a \in (1/2, \infty)$,

$$\mathbb{P}\left(\inf_{t \geqslant 0} X_t(x) > 0\right) = 1.$$

Proof. Fix $x > 0$ and write $X_t = X_t(x)$ and $\zeta = \zeta(x)$. For $r \in (0, \infty)$ define a stopping time

$$T(r) = \inf\{t \in [0, \zeta) : X_t = r\}.$$

Fix $r, R \in (0, \infty)$ and assume that $0 < r < x < R$. Write $S = T(r) \wedge T(R)$. Note that $T(r) < \zeta$ on $\{\zeta < \infty\}$. Also, $X_t \geqslant B_t + x$ for all $t < \zeta$, so $T(R) < \infty$ almost surely on $\{\zeta = \infty\}$. In particular, $S < \infty$ almost surely. Assume for now that $a \neq 1/2$. Set $M_t = X_t^{1-2a}$ for $t < \zeta$. Note that M^S is uniformly bounded. By Itô's formula

$$dM_t = (1-2a)X_t^{-2a} dX_t - a(1-2a)X_t^{-2a-1} dt = (1-2a)X_t^{-2a} dB_t.$$

Hence M^S is a bounded martingale and by optional stopping

$$x^{1-2a} = M_0 = \mathbb{E}(M_S) = r^{1-2a}\mathbb{P}(X_S = r) + R^{1-2a}\mathbb{P}(X_S = R). \tag{20}$$

Note that as $r \downarrow 0$ we have $\{X_S = R\} \uparrow \{T(R) < \zeta\}$ and so $\mathbb{P}(X_S = R) \to \mathbb{P}(T(R) < \zeta)$. Similarly, $\mathbb{P}(X_S = r) \to \mathbb{P}(T(r) < \infty)$ as $R \to \infty$. For $a \in (0, 1/2)$, we can let $r \to 0$ in (20) to obtain

$$\mathbb{P}(T(R) < \zeta) = (x/R)^{1-2a}.$$

Then, letting $R \to \infty$, we deduce that $\mathbb{P}(\zeta = \infty) = 0$. For $a \in (1/2, \infty)$, we can let $R \to \infty$ in (20) to obtain

$$\mathbb{P}(T(r) < \infty) = (r/x)^{2a-1}$$

which implies $\mathbb{P}\left(\inf_{t \geqslant 0} X_t(x) > 0\right) = 1$ and hence $\mathbb{P}(\zeta = \infty) = 1$. In the case $a = 1/2$, we instead set $M_t = \log X_t$ and argue as above to obtain

$$\log x = \mathbb{P}(X_S = r) \log r + \mathbb{P}(X_S = R) \log R.$$

This forces $\mathbb{P}(X_S = r) \to 0$ as $r \to 0$ and so

$$\mathbb{P}(T(R) < \zeta) = 1.$$

Since $T(R) \to \infty$ as $R \to \infty$, it follows that $\mathbb{P}(\zeta = \infty) = 1$.

Assume from now on that $a \in (0, 1/2)$. It remains to show for $0 < x < y$ that

$$\mathbb{P}(\zeta < \zeta(y)) = \begin{cases} 1, & \text{if } a \leqslant 1/4 \\ \phi(\frac{y-x}{y}), & \text{if } a > 1/4. \end{cases}$$

Define for $\theta \in [0,1]$

$$\chi(\theta) = \int_\theta^1 \frac{du}{u^{2-4a}(1-u)^{2a}}.$$

Note that χ is continuous on $[0,1]$ as a map into $[0,\infty]$, with $\chi(0) < \infty$ for $a \in (1/4, 1/2)$ and $\chi(0) = \infty$ for $a \in (0, 1/4]$. Note also that χ is C^2 on $(0,1)$, with

$$\chi''(\theta) + 2\left(\frac{1-2a}{\theta} - \frac{a}{1-\theta}\right)\chi'(\theta) = 0.$$

Fix $y > x$ and write $Y_t = X_t(y)$. For $t < \zeta$, define $R_t = Y_t - X_t$, $\theta_t = R_t/Y_t$ and $N_t = \chi(\theta_t)$. By Itô's formula

$$dR_t = -\frac{aR_t dt}{X_t Y_t}, \quad d\theta_t = \left(\frac{\theta_t}{Y_t}\right)^2 \left(\frac{1-2a}{\theta_t} - \frac{a}{1-\theta_t}\right) dt - \frac{\theta_t}{Y_t} dB_t$$

so

$$dN_t = \chi'(\theta_t) d\theta_t + \frac{1}{2}\chi''(\theta_t) d\theta_t d\theta_t = -\frac{\chi'(\theta_t)\theta_t dB_t}{Y_t}.$$

Hence $(N_t : t < \zeta)$ is a local martingale.

Consider the random variables

$$A(x) = \int_0^\zeta \frac{1}{X_t^2} dt, \quad A_n(x) = \int_{T(2^{-n+1}x)}^{T(2^{-n}x)} \frac{1}{X_t^2} dt, \quad n \geqslant 1.$$

By the strong Markov property (of the driving Brownian motion), the random variables $(A_n(x) : n \in \mathbb{N})$ are independent. By the scaling property, they all have the same distribution. Hence, since $A_1(x) > 0$ almost surely, we must have $A(x) = \infty$ almost surely.

Every continuous local martingale is a time-change of Brownian motion. Hence, since N is non-negative, both N_t and the quadratic variation

$$[N]_t = \int_0^t \frac{\chi'(\theta_s)^2 \theta_s^2}{Y_s^2} ds$$

converge to a finite limit almost surely as $t \uparrow \zeta$. Hence θ_t must also converge as $t \uparrow \zeta$. If $\zeta < \zeta(y)$, then $\theta_\zeta = 1$ so $N_\zeta = 0$. If $\zeta = \zeta(y)$, then the conjunction of $A(y) = \infty$ and $[N]_\zeta < \infty$ forces $\theta_t \to 0$ as $t \uparrow \zeta$. In the case $a \in (0, 1/4]$, this would imply that $N_t = \chi(\theta_t) \to \infty$ as $t \uparrow \zeta$, a contradiction, so $\mathbb{P}(\zeta < \zeta(y)) = 1$. On the other hand, for $a \in (1/4, 1/2)$, the process N^ζ is a bounded martingale so by optional stopping

$$\chi\left(\frac{y-x}{y}\right) = N_0 = \mathbb{E}(N_\zeta) = \chi(0)\mathbb{P}(\zeta = \zeta(y)).$$

$\square$

A variation of the calculation for $\mathbb{P}(\zeta(x) < \zeta(y))$ allows us to compute $\mathbb{P}(\zeta(x) < \zeta(-y))$.

Proposition 6.2. *Let $x, y \in (0, \infty)$. Then for $a \in (0, 1/2)$ we have*

$$\mathbb{P}(\zeta(x) < \zeta(-y)) = \psi\left(\frac{y}{x+y}\right)$$

where ψ is given by

$$\psi(\theta) = c\int_0^\theta \frac{du}{u^{2a}(1-u)^{2a}}, \quad \psi(1) = 1.$$

Proof. Note that ψ is continuous and increasing on $[0, 1]$ with $\psi(0) = 0$ and $\psi(1) = 1$. Also ψ is C^2 on $(0, 1)$ with

$$\psi''(\theta) + 2a\left(\frac{1}{\theta} - \frac{1}{1-\theta}\right)\psi'(\theta) = 0.$$

Write $X_t = X_t(x)$ and $Y_t = X_t(-y)$ and set $T = \zeta(x) \wedge \zeta(-y)$. For $t \leqslant T$ set $R_t = X_t + Y_t$ and $\theta_t = Y_t/R_t$. Define a process $Q = (Q_t)_{t\geqslant 0}$ by setting $Q_t = \psi(\theta_{T\wedge t})$. Then Q is continuous and uniformly bounded. Note that $Q_T = \theta_T$ and that $\theta_T = 1$ if $\zeta(x) < \zeta(-y)$ and $\theta_T = 0$ if $\zeta(-y) < \zeta(x)$. By Itô's formula, for $t \leqslant T$,

$$dR_t = \frac{aR_t}{X_tY_t}dt, \quad d\theta_t = \frac{a}{R_t^2}\left(\frac{1}{\theta_t} - \frac{1}{1-\theta_t}\right)dt - \frac{dB_t}{R_t}$$

so

$$dQ_t = \psi'(\theta_t)d\theta_t + \frac{1}{2}\psi''(\theta_t)d\theta_t d\theta_t = -\frac{\psi'(\theta_t)dB_t}{R_t}.$$

Hence Q is a bounded martingale. By optional stopping

$$\mathbb{P}(\zeta(x) < \zeta(-y)) = \mathbb{P}(\theta_T = 1) = \mathbb{E}(Q_T) = Q_0 = \psi(\theta_0) = \psi\left(\frac{y}{x+y}\right).$$

□

6.2 Hitting probabilities for $SLE(\kappa)$ on the real line

Fix $\kappa \in [0, \infty)$ and a real Brownian motion B starting from 0. Set $\xi_t = -\sqrt{\kappa}B_t$. Then $\xi = (\xi_t)_{t\geqslant 0}$ is a Brownian motion of diffusivity κ. Recall that ξ determines by Loewner's theorem a flow of conformal isomorphisms $g_t : H_t \to \mathbb{H}$ and that by the Rohde–Schramm theorem there is a continuous random process γ in $\mathbb{H}$ starting from 0 such that H_t is the unbounded connected component of $\mathbb{H} \setminus \gamma(0, t]$ for all t. Then γ is a realization of $SLE(\kappa)$. Recall also that each map g_t extends analytically to the reflected domain H_t^* and that for all $z \in \mathbb{C} \setminus \{0\}$ we have

$$\dot{g}_t(z) = \frac{2}{g_t(z) - \xi_t}, \quad t < \zeta(z)$$

where $\zeta(z) = \inf\{t \geqslant 0 : z \notin H_t^*\}$. Moreover if $\zeta(z) < \infty$ then $g_t(z) - \xi_t \to 0$ as $t \to \zeta(z)$.

For $x \in \mathbb{R} \setminus \{0\}$, set $\zeta^*(x) = \zeta(x\sqrt{\kappa})$ and define for $t < \zeta^*(x)$

$$X_t(x) = \frac{g_t(x\sqrt{\kappa}) - \xi_t}{\sqrt{\kappa}}.$$

Set $a = 2/\kappa$. Then

$$X_t(x) = x + B_t + \int_0^t \frac{a}{X_s(x)} ds$$

and $X_t(x) \to 0$ as $t \to \zeta^*(x)$ on $\{\zeta^*(x) < \infty\}$. So the family of processes $(X_t(x))_{t<\zeta^*(x)}$ is the Bessel flow of parameter a driven by B. The following result is a direct translation of Propositions 6.1 and 6.2. (Note that, since $\zeta(x) = \zeta^*(x/\sqrt{\kappa})$, the appearance of the ratio $y/(x+y)$ below is the result of a cancellation by $1/\sqrt{\kappa}$.)

Proposition 6.3. *Let $x, y \in (0, \infty)$. Then*

(a) for $\kappa \in [0, 4]$, we have

$$\mathbb{P}(\zeta(x) < \infty) = 0$$

and indeed, for $\kappa < 4$,

$$\mathbb{P}\left(\inf_{t \geqslant 0}(g_t(x) - \xi_t) > 0\right) = 1;$$

(b) for $\kappa \in (4, 8)$, we have

$$\mathbb{P}(\zeta(x) < \infty) = 1$$

and

$$\mathbb{P}(\zeta(x) < \zeta(x+y)) = \phi\left(\frac{y}{x+y}\right), \quad \mathbb{P}(\zeta(x) < \zeta(-y)) = \psi\left(\frac{y}{x+y}\right)$$

where ϕ and ψ are defined in Propositions 6.1 and 6.2;

(c) for $\kappa \in [8, \infty)$, we have

$$\mathbb{P}(\zeta(x) < \zeta(x+y) < \infty) = 1.$$

For $x \in (0, \infty)$ and $t \geqslant 0$, we have $t < \zeta(x)$ if and only if H_t is a neighbourhood of x in $\mathbb{H}$. This is true in turn if and only if $\gamma[0, t] \cap [x, \infty) = \emptyset$. Hence

$$\zeta(x) = \inf\{t \geqslant 0 : \gamma_t \in [x, \infty)\}$$

and, for $y \in (0, \infty)$, γ hits $[x, x+y)$ if and only if $\zeta(x) < \zeta(x+y)$. We make a second translation of Propositions 6.1 and 6.2 in terms of the $SLE(\kappa)$ path γ itself.

7

Recall that one can scale a standard Brownian motion, either in time or space, to obtain a Brownian motion of any diffusivity. Thus "all Brownian motions look the same". In contrast, as the parameter κ is varied, $SLE(\kappa)$ runs through three phases where it exhibits markedly different behaviour.

Theorem 7.1. *Let γ be an SLE(κ), with $\kappa \in [0,4]$. Then $\gamma(0,\infty) \subseteq \mathbb{H}$ and γ is a simple curve, almost surely.*

Proof. For $\kappa = 0$, we have $\gamma_t = \sqrt{2t}i$ for all t, so the claim holds. Fix $x \in (0,\infty)$ and set $W_t = -\xi_t/\sqrt{\kappa}$, $a = 2/\kappa$ and $X_t = W_t + g_t(x)/\sqrt{\kappa} = (g_t(x) - \xi_t)/\sqrt{\kappa}$. Then W is a standard Brownian motion, X satisfies the Bessel stochastic differential equation

$$dX_t = dW_t + \frac{a}{X_t}dt, \quad X_0 = \frac{x}{\sqrt{\kappa}},$$

and $\zeta(x) = \inf\{t \geqslant 0 : X_t = 0\}$. Hence, by Lemma 6.1, $\zeta(x) = \infty$ almost surely[2]. Since $\zeta(x) \leqslant \zeta(y)$ whenever $0 < x \leqslant y$, we deduce that almost surely, for all $x > 0$, and then for all $x < 0$ by symmetry, $\zeta(x) = \infty$. We have shown that $\gamma(0,\infty) \subseteq \mathbb{H}$ almost surely.

Assume that $\gamma_t = \gamma_{t'}$ for some $t < t'$. Then for any $s \in (t,t') \cap \mathbb{Q}$, then as $u \uparrow t'$, we have $\mathrm{Im}(\tilde{g}_u)) \to 0$ where $\tilde{g}_u = g_s(\gamma_u) - \xi_s$. (Indeed, limit points exist and must all be in $\mathbb{R}$). It follows in particular that $\tilde{\gamma}(s,\infty) \cap \mathbb{R}$ is non-empty for some rational $s \geqslant 0$, an event of probability zero since $(\tilde{\gamma}_u)_{u \geqslant s}$ is an $SLE(\kappa)$ by the Markov property. □

Proposition 7.2. *Let γ be an SLE(κ), with $\kappa \in [0,4]$. Then $\gamma_t \to \infty$ as $t \to \infty$, almost surely.*

Proof. [Proof for $\kappa \in [0,4)$] Fix $r \in (0,1)$ and set $\tau = \tau_r = \inf\{t \geqslant 0 : |\gamma_t - 1| = r\}$. Let us show that $|g_\tau(1) - \xi_\tau| \leqslant 2r$ whenever $\tau < \infty$. Condition on γ and suppose that $\tau < \infty$. Let $(W_t)_{t \geqslant 0}$ be a complex Brownian motion starting from $z = iy$. Set $W'_t = g_\tau(W_t)$ and $z' = g_\tau(z)$. Write I for the complex line segment of length r from γ_τ to 1, and write I' for the image of this interval by the map g_τ, which is a curve in $\mathbb{H}$ with both ends at ξ_τ and $g_\tau(1)$. Set

$$T = \inf\{t \geqslant 0 : W_t \in \gamma[0,\tau] \cup I \cup \mathbb{R}\},$$
$$T' = \inf\{t \geqslant 0 : W'_t \in \mathbb{R} \cup I'\}, \qquad T'' = \inf\{t \geqslant 0 : W'_t \in \mathbb{R}\}.$$

Note that $z'/z \to 1$ as $y \to \infty$ and, since K_τ is connected. By a topological argument,

$$\{W_{T''} \in [\xi_\tau, g_\tau(1)]\} \subset \{W_{T'} \in I\} = \{W_T \in I\}.$$

Thus by conformal invariance,

$$|g_\tau(1) - \xi_\tau| = \lim_{y\to\infty} \frac{y}{\pi} \mathbb{P}(W'_{T''} \in [\xi_t, g_\tau(1)]) \leqslant \lim_{y\to\infty} \frac{y}{\pi} \mathbb{P}(W_T \in I) \leqslant 2r.$$

[2]The relevant quantity in the notation of Lemma 6.1 is $\zeta(X_0) = \zeta(x/\sqrt{\kappa})$.

This proves that if $\tau < \infty$ then $|g_\tau(1) - \xi_\tau| \leqslant 2r$. On the other hand, we already know $\inf_{t\geqslant 0}(g_t(1)-\xi_t) > 0$ almost surely. So we must have, almost surely, $\tau_r = \infty$ for some $r > 0$ sufficiently.

It can be shown that since γ is simple, 0 has only two images under g_1, which are $g_1(0^-)$ and $g_1(0^+)$. As a consequence, a similar argument shows that, almost surely,

$$\operatorname{dist}(g_1(\gamma[1,\infty)), \{g_1(0-), g_1(0+)\}) > 0$$

so $\operatorname{dist}(\gamma[1,\infty), 0) > 0$, and so $\liminf_{t\to\infty} |\gamma(t)|$ is positive, and hence is infinite by scaling.

The case $\kappa = 4$ follows a similar but more elaborate argument to say that $\tau_r = \infty$ for some $r > 0$. □

Theorem 7.3. *Let γ be an SLE(κ), with $\kappa = 2/a \in (4, 8)$. Almost surely, γ is not a simple curve, nor a space-filling curve.*

Proof. By Lemma 6.1,

$$\mathbb{P}(\gamma \text{ hits } [x,\infty)) = \mathbb{P}(X \text{ hits } 0) = 1,$$

and

$$\mathbb{P}(\gamma \text{ hits } [x,y]) = \mathbb{P}(\zeta(x) < \zeta(y)) = \phi\left(\frac{y-x}{y}\right) \in (0,1).$$

Hence $\gamma_{\zeta(x)} \in (x,\infty)$ almost surely. Moreover, for $y > x$, we have $\{\gamma_{\zeta(x)} < y\} = \{\zeta(y) > \zeta(x)\}$ and $\{\gamma_{\zeta(x)} \geqslant y\} = \{\zeta(y) = \zeta(x)\}$ and both events have positive probability. In particular, we see that γ hits any given interval in $\mathbb{R}$ of positive length with positive probability. Now if S_1 is the set of all limit points of $g_1(\partial K_1 \cap \mathbb{H})$, then S_1 is an interval of positive length Thus we can find a subinterval $I \subset S_1$ such that $d(\xi_1, I) > 0$. Then by the above observation $g_1(\gamma_t)(1,\infty) \cap I$ is nonempty with positive probability. On the other hand, some topological considerations show that $\partial K_1 \cap \mathbb{H} \subseteq \gamma[0,t]$, so γ has double points with positive probability and hence almost surely by a zero-one argument (see below).

On the other hand, on $\{\gamma_{\zeta(x)} > y\}$, there is a neighbourhood of $[x,y]$ in $\mathbb{H}$ which does not meet γ and $\operatorname{dist}([x,y], H_{\zeta(x)}) > 0$. In particular, γ is not space-filling, with positive probability, and then almost surely.

Here is an elaboration of the zero-one argument for double points. Define, for $t > 0$, $A_t = \{\gamma_s = \gamma_{s'}$ for some distinct $s, s' \in [0,t]\}$. Then the sets A_t are non-decreasing in t and all have the same probability, p say, by scaling. But then $p = \mathbb{P}(\cap_t A_t)$ and $\cap_t A_t \in \mathcal{F}_{0+}$, where $\mathcal{F}_{0+} = \cap_{t>0}\,\sigma(\xi_s : s \leqslant t)$. But, by Blumenthal's zero-one law, $\mathcal{F}_{0+}$ contains only null sets and their complements. Hence $p \in \{0,1\}$. □

Proposition 7.4. *For $\kappa \in (4,8)$, $d(0, H_t) \to \infty$ almost surely.*

Proof. The set S of limit points of $g_{\zeta(1)}(z)$ as $z \to 0$, $z \in \mathbb{H}$ is a compact (possibly empty) subset of $(-\infty, \xi_{\zeta(1)})$. Pick $y < \inf S$. With positive probability, $\operatorname{dist}(S, g_{\zeta(1)}(H_{\zeta(y)})) > 0$, so $\operatorname{dist}(0, H_{\zeta(y)}) > 0$, so $\mathbb{P}(\operatorname{dist}(0, H_t) > 0) = \delta$ for some $t > 0$ and $\delta > 0$. This extends to all t by scaling, with the same δ. So $\mathbb{P}(\operatorname{dist}(0, H_t) > 0 \text{ for all } t > 0) = \delta$ and then $\delta = 1$ by a zero-one argument. Finally $\operatorname{dist}(0, H_t)$ is non-decreasing and, for all $r < \infty$, as $t \to \infty$,

$$\mathbb{P}(\operatorname{dist}(0, H_t) \leqslant r) = \mathbb{P}(\operatorname{dist}(0, H_1) \leqslant r/\sqrt{t}) \to 0.$$

□

Theorem 7.5. *Let γ be an SLE(κ), with $\kappa \in [8, \infty)$. Then, almost surely, $\gamma[0, \infty) \cap \mathbb{H} = \mathbb{H}$. In particular, γ is space-filling and $\gamma_t \to \infty$ as $t \to \infty$.*

Proof. We can deduce from Lemma 6.1 that, for $\kappa \in [8, \infty)$, we have $\gamma_{\zeta(x)} = x$ for all $x \in \mathbb{R}$. This implies $\gamma[0, \infty) \cap \mathbb{R} = \mathbb{R}$. For the full result, more arguments are needed that go beyond the scope of the course. □

8

8.1 SLE and the domain Markov property

Let D be a proper simply connected domain and $\Phi : D \to \mathbb{D}$ a conformal isomorphism. Recall that $\hat{D}$ the metric space obtained by completing D with its conformal boundary ∂D, that is, the completion of D with respect to the metric inherited from $\mathbb{D}$ via Φ. If $z_0, z_1 \in \partial D = \hat{D} \setminus D$, a curve from z_0 to z_1 is a path $\gamma : [a, b] \to \hat{D}$ continuous for the metric on $\hat{D}$ such that $\gamma(a) = z_0$ and $\gamma(b) = z_1$. We call the triplet $M = (D, z_0, z_1)$ a domain with marked boundary points. The set of chords C_M in M is the set of curves from z_0 to z_1 modulo time-reparametrization. We endow the set C_M with the σ-algebra $\mathcal{C}_M$ generated by events of the form $A(B_1, \dots, B_n) = \{\gamma \in C_M : \gamma(t_1) \in B_1, \dots, \gamma(t_n) \in B_n \text{ for some } t_1, \dots, t_n\}$, where $B_1, \dots, B_n$ are Borel subsets of $\mathbb{C}$.

We note $\mathcal{M}$ the set of domains with two marked boundary points as above. Let $M = (D, z_0, z_1) \in \mathcal{M}$. Say a curve $\gamma : [a, b] \to \hat{D}$ from z_0 to z_1 is *regular* if $\gamma(t) \neq z_1$ for all $t < b$, and if the following holds: for $t < b$, let D_t be the unique component of $D \setminus \gamma(0, t]$ which is a neighbourhood of z_1, and let $K_t = D \setminus D_t$. Then $K_t^* = \Phi^{-1}(K_t \cap D), t < b$ is a family of compact hulls in $\mathbb{H}$ with local growth. Note that these properties are invariant under time-reparamterization.

If γ is a chord, we say that the function $\tau = \tau(\gamma)$ is parametrization-invariant if for all reparamterizations of time ϕ, $\tau(\gamma \circ \phi) = \phi^{-1}(\tau(\gamma))$. Say further that τ is paramaterization-invariant stopping time if, for a given parametrization ϕ, $\tau(\gamma)$ is a stopping-time for the filtration generated by $\gamma \circ \phi$. Say that (μ_M) has the *domain Markov property* if for all parametrization-invariant stopping time τ, the chord $\theta_\tau \gamma$ has the law μ_{M_t}.

Theorem 8.1. *For $M \in \mathcal{M}$, let μ_M be a probability measure supported on regular chords in $(C_M, \mathcal{C}_M)$. The following statements are equivalent:*

(a) *the family $(\mu_M : M \in \mathcal{M})$ is conformally invariant and has the domain Markov property;*

(b) *there exists $\kappa \in [0, \infty)$ such that, for all $D \in \mathcal{D}$ and any conformal isomorphism $\Phi_D : (\mathbb{H}, 0, \infty) \to D$, μ_D is the law of $[\Phi_D(\gamma)]$, where γ is an SLE(κ).*

Proof. Suppose that (a) holds and that $X \sim \mu_{(\mathbb{H},0,\infty)}$. By the regularity condition we can choose a representative $(\gamma_t)_{t \geqslant 0}$ of X such that $\text{hcap}(\gamma[0, t]) = 2t$ for all t. By scale invariance and the domain Markov property, $(\gamma_t)_{t \geqslant 0}$ satisfies condition (b) of Proposition 5.4 and hence is an SLE(κ) for some κ. Hence we obtain (b) by conformal invariance.

Suppose on the other hand that (b) holds. By scaling of SLE, the law of $[\Phi_D(\gamma)]$ does not depend on the choice of Φ_D and the family of measures $(\mu_D : D \in \mathcal{D})$ is conformally invariant. Finally, the strong Markov property of Brownian motion implies that $\theta_\tau \gamma \sim \gamma$ for any stopping time τ, so we obtain the domain Markov property. □

When (b) holds, we shall refer to μ_D and to any random chord $X \sim \mu_D$ as SLE(κ) in D. The regularity condition used in this result can certainly be replaced by something weaker at the cost of a more involved argument.

9

We discuss the relation of the Loewner transform to conformal isomorphisms $\Phi : N \to N^*$ of neighbourhoods of 0 in $\mathbb{H}$. It is straightforward to see that these preserve the local growth property while K_t remains in N. We obtain some useful formulae for the half-plane capacity and Loewner transform of $\Phi(K_t)$.

9.1 Loewner evolution of conformal isomorphisms

Let $(K_t)_{t \geqslant 0}$ be a family of compact $\mathbb{H}$-hulls with local growth property, paramatrised by half-plane capacity, let $(\xi_t)_{t \geqslant 0}$ be the Loewner transform and $(g_t)_{t \geqslant 0}$ be the Loewner flow. Write as usual $H_t = \mathbb{H} \setminus K_t$.

Let I be an open interval in $\mathbb{R}$ containing ξ_0 and let N be a neighbourhood of I in $\mathbb{H}$. Suppose we are given a conformal isomorphism $\Phi : N \to N^*$, where $N^* \subseteq \mathbb{H}$ and such that $I^* = \Phi(I) \subseteq \mathbb{R}$. Then I^* is an open interval containing $\xi_0^* = \Phi(\xi_0)$ and N^* is a neighbourhood of I^* in $\mathbb{H}$.

Set $T = \inf\{t \geqslant 0 : \bar{K}_t \not\subseteq N \cup I\}$ and define, for $t < T$, $K_t^* = \Phi(K_t)$ and $H_t^* = \mathbb{H} \setminus K_t^*$. Note that K_t^* is a compact $\mathbb{H}$-hull, with $\bar{K}_t^* \subseteq N^* \cup I^*$. Set $g_t^* = g_{K_t^*}$; set also $N_t = g_t(N \setminus K_t)$ and $N_t^* = g_t^*(N^* \setminus K_t^*)$. Set

$$\Phi_t = g_t^* \circ \Phi \circ g_t^{-1}.$$

Then Φ_t is a conformal isomorphism $N_t \to N_t^*$, and note that we can extend Φ_t as a conformal isomorphism of reflected domains $\tilde{N}_t \to \tilde{N}_t^*$, in a manner analogous to that used for Φ above. Using Lemma 3.4 to bound Φ_t', it is not hard to see that $(K_t^*)_{t<T}$ inherits the local growth property from $(K_t)_{t<T}$ and has Loewner transform given by $\xi_t^* = \Phi_t(\xi_t)$.

9.2 Half-plane capacity under conformal isomorphism

Proposition 9.1. *The map $t \mapsto a_t^*$ is differentiable on $[0, T)$ with $\dot{a}_t^* = 2\Phi_t'(\xi_t)^2$.*

Proof. We show first that $t \mapsto a_t^*$ is differentiable on the right at $t = 0$ with right derivative $2\Phi'(\xi_0)^2$. By scaling and translation properties of hcap, it suffices to consider the case where $\xi_0 = \xi_0^* = 0$ and where $\Phi'(0) = 1$. We obtain from (11) by scaling, for $r > 0$ and $K \subseteq r\mathbb{D}$,

$$\operatorname{hcap}(K) = \int_0^\pi \mathbb{E}_{re^{i\theta}}(\operatorname{Im}(B_{T(K)})) \left(\frac{2\sin\theta}{\pi}\right) r d\theta,$$

where $T(K) = \inf\{t \geqslant 0 : B_t \in \mathbb{R} \cup K\}$. Multiply this formula by r and integrate over $r \in [r_1, r_2]$ to obtain

$$\frac{1}{2}(r_2^2 - r_1^2) \operatorname{hcap}(K) = \int_{A(r_1,r_2)} \mathbb{E}_z(\operatorname{Im}(B_{T(K)})) \left(\frac{2\operatorname{Im} z}{\pi}\right) A(dz),$$

where $A(dz)$ denotes area measure and

$$A(r_1, r_2) = \{z \in \mathbb{H} : r_1 \leqslant |z| \leqslant r_2\}.$$

Set $w = \Phi(z)$. By conformal invariance of Brownian motion, for $t < T$,

$$\mathbb{E}_w[\mathrm{Im}(B_{T(K_t^*)})] = \mathbb{E}_z[\mathrm{Im}(\Phi(B_{T(K_t)}))].$$

Since $\Phi'(0) = 1$, given $\varepsilon > 0$, there exists $r > 0$ such that, for $r_1 \leqslant |w| \leqslant r_2 \leqslant r$,

$$A((1+\varepsilon)r_1, (1-\varepsilon)r_2) \subseteq \Phi^{-1}(A(r_1, r_2)) \subseteq A((1-\varepsilon)r_1, (1+\varepsilon)r_2),$$
$$1 - \varepsilon \leqslant \left|\frac{A(dw)}{A(dz)}\right| \leqslant 1 + \varepsilon,$$
$$(1-\varepsilon)\,\mathrm{Im}(z) \leqslant \mathrm{Im}\, w \leqslant (1+\varepsilon)\,\mathrm{Im}(z).$$

So, provided $K_t^* \subseteq r_1\mathbb{D}$, we have

$$\frac{1}{2}(r_2^2 - r_1^2)\,\mathrm{hcap}(K_t^*) = \int_{A(r_1,r_2)} \mathbb{E}_{\Phi^{-1}(w)}(\mathrm{Im}(\Phi(B_{T(K)}))) \left(\frac{2\,\mathrm{Im}\, w}{\pi}\right) A(dw),$$

and hence, by the above estimates,

$$(1-\varepsilon)^5\,\mathrm{hcap}(K_t) \leqslant \mathrm{hcap}(K_t^*) \leqslant (1+\varepsilon)^5\,\mathrm{hcap}(K_t).$$

Since $\mathrm{hcap}(K_t) = 2t$, and ε may be chosen arbitrarily small, this implies that $a_t^*/t \to 2$ as $t \downarrow 0$.

The same argument, applied to $K_{t,t+h}$, shows, for all $t < T$, that a^* is differentiable on the right at t with right derivative $2\Phi_t'(\xi_t)^2$. Since the right derivative is continuous in t, it follows that a^* is in fact differentiable with the claimed derivative. □

Proposition 9.2. *The map $t \mapsto \Phi_t$ on $[0, T)$ is a C^1 map of holomorphic functions, with*

$$\dot{\Phi}_t(\xi_t) = -3\Phi_t''(\xi_t), \quad \dot{\Phi}_t'(\xi_t) = \frac{1}{2}\frac{\Phi_t''(\xi_t)^2}{\Phi_t'(\xi_t)} - \frac{4}{3}\Phi_t'''(\xi_t).$$

Proof. By Propositions 5.1 and 9.1, $(g_t^*)_{t \leqslant t_0}$ satisfies

$$\dot{g}_t^*(z) = 2\Phi_t'(\xi_t)^2/(g_t^*(z) - \xi_t^*), \quad z \in H_t^*.$$

Set $f_t = g_t^{-1}$ and differentiate the equation $f_t(g_t(z)) = z$ in t to obtain

$$\dot{f}_t(z) = -2f_t'(z)/(z - \xi_t), \quad z \in \mathbb{H}.$$

Now $\Phi_t = g_t^* \circ \Phi \circ f_t$. We cannot take the derivative in t directly at the singularity ξ_t, but, by analyticity, it will suffice to differentiate, using the chain rule, at each $z \in N_t$, and then pass to the limit in the derivatives as $z \to \xi_t$. We have

$$\dot{\Phi}_t(z) = \dot{g}_t^*(\Phi(f_t(z)) + (g_t^*)'(\Phi(f_t(z)))\Phi'(f_t(z))\dot{f}_t(z) = \frac{2\Phi_t'(\xi_t)^2}{\Phi_t(z) - \Phi_t(\xi_t)} - \Phi_t'(z)\frac{2}{z - \xi_t}$$

and we can differentiate in z to obtain

$$\dot{\Phi}_t'(z) = 2\left(-\frac{\Phi_t'(\xi_t)^2\Phi_t'(z)}{(\Phi_t(z) - \Phi_t(\xi_t))^2} + \frac{\Phi_t'(z)}{(z-\xi_t)^2} - \frac{\Phi_t''(z)}{z - \xi_t}\right).$$

The desired identities may now be obtained by taking the limits in these equations as $z \to \xi_t$, using l'Hôpital's rule. □

10

10.1 SLE(6) and the locality property

Consider a family $(\mu_D : D \in \mathcal{D})$, where μ_D is a probability measure on $(C_D, \mathcal{C}_D)$ for each $D \in \mathcal{D}$. Recall that $(\mu_D : D \in \mathcal{D})$ has the *locality property* if, for all $D, D' \in \mathcal{D}$, for all initial domains N common to D and D', for $X \sim \mu_D$ and $X' \sim \mu_{D'}$, we have $X^N \sim X'^N$, where X^N is X stopped on hitting the cut. When $(\mu_D : D \in \mathcal{D})$ is conformally invariant, this property is equivalent to the following property of $\mu = \mu_{(\mathbb{H},0,\infty)}$: let N and N^* be initial domains in $(\mathbb{H}, 0, \infty)$ and suppose we are given a conformal isomorphism $\Phi : N \to N^*$, taking 0 to 0, and $\partial N \cap \mathbb{R}$ onto $\partial N^* \cap \mathbb{R}$; if $X \sim \mu$, then $X^{N^*} \sim \Phi(X^N)$.

Theorem 10.1. *SLE*(6) *has the locality property.*

Proof. Let γ be an SLE(6) and let $\Phi : N \to N^*$ be an isomorphism of initial domains in $(\mathbb{H}, 0, \infty)$, as above.

Set $T = \inf\{t \geqslant 0 : \gamma_t \notin N \cup (\partial N \cap \mathbb{R})\}$. For $t \leqslant T$, let $K_t^* = \Phi(K_t)$, $a_t^* = \mathrm{hcap}(K_t^*)$, $g_t^* = g_{K_t^*}$ and $\Phi_t = g_t^* \circ \Phi \circ g_t^{-1}$. Then $(K_t^*)_{t \leqslant T}$ has the local growth property, with Loewner transform given by $\xi_t^* = \Phi_t(\xi_t)$. Moreover, by Proposition 9.1, the map $t \mapsto a_t^*$ is differentiable, with $\dot{a}_t^* = 2\Phi_t'(\xi_t)^2$. Define τ_s and $\tilde{T}$ by

$$s = \int_0^{\tau_s} \Phi_t'(\xi_t)^2 dt, \qquad \text{for } s \leqslant T = \tau_{\tilde{T}}$$

and set $\tilde{K}_s = K_{\tau_s}^*$, $\tilde{g}_s = g_{\tau_s}^*$ and $\tilde{\xi}_s = \xi_{\tau_s}^*$. Then $\mathrm{hcap}(\tilde{K}_s) = 2s$ and $(\tilde{\xi}_s)_{s \leqslant \tilde{T}}$ is the Loewner transform of $(\tilde{K}_s)_{s \leqslant \tilde{T}}$. So, by Proposition 5.1, we have, for all $z \in \tilde{H}_s$,

$$\dot{\tilde{g}}_s(z) = \frac{2}{\tilde{g}_s(z) - \tilde{\xi}_s}.$$

Now, by Itô's formula and Proposition 9.2,

$$d\xi_t^* = \dot{\Phi}_t(\xi_t)dt + \Phi_t'(\xi_t)d\xi_t + \frac{1}{2}\Phi_t''(\xi_t)\kappa dt = \left(\frac{\kappa}{2} - 3\right)\Phi_t''(\xi_t)dt + \Phi_t'(\xi_t)d\xi_t.$$

Hence, for $\kappa = 6$, $(\tilde{\xi}_s)_{s \leqslant \tilde{T}}$ is a Brownian motion of diffusivity 6. Thus $(\Phi(\gamma_{\tau_s}))_{s \leqslant \tilde{T}}$ is an SLE(6), as required. □

Corollary 10.2. *Let U be a simply connected proper domain and let z_0, z_1, z_1' be distinct points in $\hat{U}$. Set $D = (U, z_0, z_1)$ and $D' = (U, z_0, z_1')$. Let X be SLE*(6) *in D and let X' be an SLE*(6) *in D'. Then X^T and $(X')^{T'}$ have the same distribution, where T and T' are, respectively, the first times that X and X' hit the boundary segment in $\hat{U}$ from z_1 to z_1'.*

10.2 $SLE(6)$ in a triangle

While physicists investigated critical percolation using nonrigorous methods, Cardy established a formula for the limiting crossing probabilities of a rectangle. Carleson observed that this formula became considerably simpler on a triangle. The corresponding formula can be stated as a theorem directly for $SLE(6)$. In turn, since Smirnov proved that Cardy's formula holds in the limit for critical percolation, this provides another of identifying $SLE(6)$ as the unique possible limit for the scaling limit of cluster interface exploration process in critical percolation.

Let Δ be the equilateral triangle with vertices $a = 0$, $b = 1$, $c = e^{i\pi/3}$.

Theorem 10.3. *Let γ be $SLE(6)$ in $(\Delta, 0, 1)$, where Δ denotes the triangle with vertices $0, 1, e^{\pi i/3}$. Then the point X at which γ hits the edge $[1, e^{\pi i/3}]$ is uniformly distributed.*

Proof. The Schwarz–Christoffel transformation $(\mathbb{H}, 0, 1, \infty) \to (\Delta, 0, 1, e^{\pi i/3})$ is given by

$$f(z) = c \int_0^z \frac{dw}{w^{2/3}(1-w)^{2/3}}, \quad c = \frac{\Gamma(2/3)}{\Gamma(1/3)^2}.$$

Consider the map $z \mapsto \varphi(z) = 1/(1-z)$. This is a conformal automorphism which cyclically permutes $0, 1, \infty$. The map $z \mapsto g(z) = 1 + e^{2i\pi/3} z$ is a conformal automorphism of Δ which cyclically permutes a, b, c. Thus

$$f(\varphi(z)) = g(f(z)),$$

by uniqueness of the Riemann map. Thus, composing by $\varphi^{-1}(z) = (z-1)/z$, we deduce

$$f(z) = 1 + e^{2i\pi/3} f((z-1)/z)$$

for all $z \in \mathbb{H}$. This identity extends by continuity when $z \to x \in \mathbb{H}$.

Let $x \in [0, 1]$ and choose y so that $f(y/(1+y)) = x$. Then, by conformal invariance and Proposition 6.3,

$$\mathbb{P}(X \in [1, 1 + xe^{2i\pi/3}]) = \mathbb{P}(SLE(6) \text{ in } (\mathbb{H}, 0, \infty) \text{ hits } [1, 1+y]) = \phi\left(\frac{y}{1+y}\right) = x.$$

Thus X is uniform on $[1, e^{i\pi/3}]$. $\square$

11

We discuss a natural property of conformally invariant families of probability measures on fillings called the restriction property. This will be established in two cases, first for the Brownian excursion and then for $SLE(8/3)$. We also discuss at an informal level an analogous property for probability measures on lattice paths, which holds for the self-avoiding random walk.

11.1 Brownian excursion

There is a useful identity for g_K in terms of the *Brownian excursion* $E = (E_t : t \geqslant 0)$ *from* b *to* ∞ *in* $\mathbb{H}$. This process, which we shall study further later, may be realised as follows: take $E_t = X_t + iR_t$ where X is a real Brownian motion starting from b, and $R_t = |W_t|$ with W a Brownian motion in $\mathbb{R}^3$ starting from 0.

Proposition 11.1. *Let K be a compact $\mathbb{H}$-hull and suppose that $b \in \mathbb{R} \setminus \bar{K}$. Then*

$$g'_K(b) = \mathbb{P}_b(E \textit{ does not hit } K).$$

Proof. We reduce by translation to the case where $b = 0$. Let $Z = X + iY$ be a complex Brownian motion starting from $z = x + iy \in \mathbb{H}$. Define for $r \geqslant 0$

$$T_r = \inf\{t \geqslant 0 : Y_t = r\}, \quad T_K = \inf\{t \geqslant 0 : Z_t \in K\}.$$

Fix $r > y$ and set $M_t = y^{-1} Y_{T_0 \wedge T_r \wedge t}$. Then M is a bounded non-negative martingale with $M_0 = 1$ and with final value $Y_{T_0 \wedge T_r} = (r/y) 1_{\{T_0 > T_r\}}$. Define a new probability measure $\tilde{\mathbb{P}}$ by

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = Y_{T_0 \wedge T_r}.$$

Note that, under $\tilde{\mathbb{P}}$, we have $T_0 > T_r$ almost surely. Moreover, under $\tilde{\mathbb{P}}$, by Girsanov's theorem and Lévy's characterization of Brownian motion, X and Y remain independent, with X a Brownian motion and Y satisfying the stochastic differential equation

$$dY_t = dB_t + \frac{dt}{Y_t}, \quad t \leqslant T_r,$$

for some Brownian motion B. By Itô's formula, R also satisfies this equation, so, by uniqueness in law, if we start E from z, then the distributions of E and of Z under $\tilde{\mathbb{P}}$ agree up to T_r. Hence

$$\begin{aligned} p_r(z) &= \mathbb{P}_z(E \text{ does not hit } K \text{ before } T_r) \\ &= \mathbb{E}_z(y^{-1} Y_{T_0 \wedge T_r} 1_{\{T_K > T_r\}}) = (r/y)\mathbb{P}_z(T_r < T_0 \wedge T_K). \end{aligned}$$

Now $g_K(z) - z \to 0$ as $z \to \infty$ so, for r sufficiently large,

$$|\operatorname{Im} g_K(z) - r| \leqslant 1 \text{ whenever } \operatorname{Im}(z) = r,$$

and hence, by conformal invariance of Brownian motion,

$$\frac{\operatorname{Im} g_K(z)}{r+1} = \mathbb{P}_{g_K(z)}(T_{r+1} < T_0) \leqslant \mathbb{P}_z(T_r < T_0 \wedge T_K) \leqslant \mathbb{P}_{g_K(z)}(T_{r-1} < T_0) = \frac{\operatorname{Im} g_K(z)}{r-1}.$$

So

$$\mathbb{P}_z(E \text{ does not hit } K) = \lim_{r\to\infty} p_z(r) = \operatorname{Im} g_K(z)/y.$$

Now, $\operatorname{Im} g_K(z)/y \to g'_K(0) > 0$ as $z \to 0$ in $\mathbb{H}$. Take $\varepsilon > 0$ and set

$$S = \inf\{t \geqslant 0 : |E_t| = \varepsilon\},$$

then $|E_S| = \varepsilon$ and $\operatorname{Im} E_S > 0$ almost surely. Hence, choosing ε so that $K \cap \varepsilon\mathbb{D} = \emptyset$, by the strong Markov property of E and bounded convergence, as $\varepsilon \to 0$,

$$\mathbb{P}_0(E \text{ does not hit } K) = \mathbb{E}(\operatorname{Im} g_K(E_S)/\operatorname{Im}(E_S)) \to g'_K(0).$$

□

11.2 The restriction property of $SLE(8/3)$

We only describe what the restriction property means for $SLE(\kappa)$ on $(\mathbb{H}, 0, \infty)$ with $\kappa < 4$. There is a more general version of this property which may be stated for arbitrary proper domains with two marked boundary points.

Let A be a compact $\mathbb{H}$-hull and assume that $0 \notin \bar{A} \cap \mathbb{R}$. Let γ be an $SLE(\kappa)$ curve in $(\mathbb{H}, 0, \infty)$ Let V_A be the event that $\gamma(0,\infty) \cap A = \emptyset$. Since $\kappa < 4$, $\mathbb{P}(V_A) > 0$. Then we say $SLE(\kappa)$ has the restriction property if, conditionally on the event V_A, the law of γ is chordal $SLE(\kappa)$ in the domain $H = \mathbb{H} \setminus A$. Put it another way, conditionally on V_A, the law of $\Phi_A(\gamma)$ is $SLE(\kappa)$ in $(\mathbb{H}, 0, \infty)$, where Φ is any conformal isomorphism from H to $\mathbb{H}$ such that $\Phi(0) = 0$ and $\Phi(\infty) = \infty$. We will call Φ_A the only such conformal isomorphism with the additional requirement that $\Phi'_A(\infty) = 1$, i.e., $\Phi_A(z) - z$ is bounded as $z \to \infty$ (in which case the limit exists and is real: Φ_A is simply a translate of the map g_A.)

The next theorem is the crucial step in establishing the restriction property.

Theorem 11.2. *Let γ be an $SLE(8/3)$. Then for all compact $\mathbb{H}$-hull A such that $0 \notin \bar{A} \cap \mathbb{R}$, we have*

$$\mathbb{P}(\gamma(0,\infty) \cap A = \emptyset) = \Phi'_A(0)^{5/8},$$

where Φ_A is the unique conformal isomorphism $\mathbb{H} \setminus A \to \mathbb{H}$ with $\Phi_A(0) = 0$ and $\Phi_A(z) - z$ bounded.

Proof. Write $\kappa = 8/3$ and $\alpha = 5/8$. Set $D = \mathbb{H} \setminus A$. Let $T = \inf\{t \geqslant 0 : \gamma_t \in A\}$. For $t < T$, define $\Phi_t = g^*_t \circ \Phi_D \circ g_t^{-1}$, where g_t is the Loewner flow and $g^*_t = g_{K^*_t}$ and $K^*_t = \Phi_A(K_t)$. Set $\Sigma_t = \Phi'_t(\xi_t)$ and $M_t = \Sigma_t^\alpha$ for $t < T$. By Itô's formula and Proposition 9.2,

$$d\Sigma_t = \dot{\Phi}'_t(\xi_t)dt + \Phi''_t(\xi_t)d\xi_t + \frac{1}{2}\Phi'''_t(\xi_t)\kappa dt = \Phi''_t(\xi_t)d\xi_t + \frac{1}{2}\frac{\Phi''_t(\xi_t)^2}{\Phi'_t(\xi_t)}dt + \left(\frac{\kappa}{2} - \frac{4}{3}\right)\Phi'''_t(\xi_t)dt.$$

Note that, since $\kappa = 8/3$, the final term vanishes. Also, by Itô's formula,

$$dM_t = \alpha \Sigma_t^{\alpha-1} d\Sigma_t + \frac{1}{2}\alpha(\alpha-1)\Sigma_t^{\alpha-2} d\Sigma_t d\Sigma_t = \alpha M_t dY_t,$$

where

$$dY_t = \frac{d\Sigma_t}{\Sigma_t} + \frac{1}{2}(\alpha-1)\frac{\Phi_t''(\xi_t)^2}{\Sigma_t^2}\kappa dt = \frac{\Phi_t''(\xi_t)}{\Sigma_t} d\xi_t + \frac{1}{2}(1+(\alpha-1)\kappa)\frac{\Phi_t''(\xi_t)^2}{\Sigma_t^2} dt.$$

Since $\kappa = 8/3$ and $\alpha = 5/8$, the final term vanishes, so $(Y_t)t < T$ and hence also $(M_t)_{t<T}$ is a local martingale.

By Proposition 11.1, conditional on γ, we have

$$\Phi_t'(\xi_t) = \mathbb{P}_{\xi_t}(E \text{ does not leave } g_t(A)).$$

Thus for all $t < T$, $M_t \in [0,1]$ and M is hence a true martingale, which thus necessarily converges as $t \to T$. We claim that the limit, M_∞ say, is necessarily

$$M_\infty = \mathbf{1}_{\{T=\infty\}}, \ a.s.$$

Indeed, assume first that $\{T = \infty\}$ holds. Then since $\lim_{t\to\infty} \mathrm{Im}(\gamma_t) = \infty$ (with probability one), then $d(0, g_t(A)) \to \infty$ by a scaling argument and Kolmogorov's zero-one law. Hence $\Phi_t'(\xi_t) \to 1$ almost surely on the event $\{T = \infty\}$.

On the other hand, if $\{T < \infty\}$, then γ_T lies on the boundary of A. When A is sufficiently regular, it is not hard to see that $g_T(A)$ then contains a piece of an open cone Γ whose apex is $\xi_T \in \bar{\Gamma}$. It is easy to see that if $\Gamma \subset \mathbb{H}$ is an open cone with apex $\xi \in \mathbb{R}$, then ξ is *regular* for Γ and for the Brownian excursion: that is, started from ξ, a Brownian excursion enters Γ immediately almost surely. (This follows from a scaling argument and a zero-one law, just as for standard Brownian motion). Thus, when A is sufficiently regular, $\mathbb{P}_{\xi_t}(E$ does not leave $g_T(A)) = 0$ and hence $\Phi_T'(\xi_T) = 0$. Thus $M_\infty = 0$ on this event, provided that A was sufficiently regular. The general case follows by considering $A_\epsilon = \overline{\cup_{x\in A}\{x + B(x,\epsilon)\}} \cap \mathbb{H}$ and letting $\epsilon \to 0$.

Hence, by optional stopping,

$$\Phi_A'(0)^{5/8} = M_0 = \mathbb{E}(M_T) = \mathbb{P}(T = \infty).$$

This completes the proof of the result. □

Theorem 11.3. *SLE*(8/3) *has the restriction property.*

Proof. Let γ be an SLE(8/3) and let A be a compact $\mathbb{H}$-hull such that $0 \notin \bar{A} \cap \mathbb{R}$ We wish to prove that conditionally on V_A, $\Phi_A(\gamma)$ is an $SLE(8/3)$ chord. It can be seen that the law of a random chord c is entirely characterized by the probability of the form $c(0,\infty) \cap B = \emptyset$, where B is a compact $\mathbb{H}$-hull such that $0 \notin \bar{B} \cap \mathbb{R}$.

Thus pick an arbitrary compact $\mathbb{H}$-hull B with $0 \notin \bar{B} \cap \mathbb{R}$, and observe that

$$\begin{aligned}\mathbb{P}(\Phi_A(\gamma)(0,\infty) \cap B = \emptyset; V_A) &= \mathbb{P}(\gamma(0,\infty) \cap (A \cup \Phi_A^{-1}(B) = \emptyset) \\ &= \Phi'_{A \cup \Phi_A^{-1}(B)}(0)^{5/8}.\end{aligned}$$

But we know that the map $\Phi_{A \cup \Phi_A^{-1}(B)}$ is simply equal to the composition $\Phi_B \circ \Phi_A$. Applying the chain rule, we deduce immediately that

$$\begin{aligned}\mathbb{P}(\Phi_A(\gamma)(0,\infty) \cap B = \emptyset; V_A) &= \Phi'_A(0)^{5/8} \Phi'_B(0)^{5/8} \\ &= \mathbb{P}(V_A)\mathbb{P}(V_B)\end{aligned}$$

so that $\Phi_A(\gamma)$ is an $SLE(8/3)$ chord. This completes the proof of the result. □

Suppose α is a nonnegative integer and A is a compact $\mathbb{H}$-hull such that $0 \notin \bar{A} \cap \mathbb{R}$. Then $\Phi'_A(0)^\alpha$ is the probability that α independent Brownian excursions avoid A, by Proposition 11.1. Hence this is the probability that the hull generated by α independent Brownian excursions does not intersect A. Thus one way to informally interpret the result of Theorem 11.2 is to say that the $SLE(8/3)$ chord can be thought of as 5/8 of a Brownian excursion. More precisely, we have the following result as an immediate corollary to Proposition 11.1 and Theorem 11.2:

Theorem 11.4. *The compact hull generated by* 8 *independent* $SLE(8/3)$ *chords and the compact hull generated by 5 independent Brownian excursions have the same distribution.*

Naturally, one of the particularly striking aspects of this result is that the curves themselves ($SLE(8/3)$ and Brownian excursions) are very different from one another.

This concludes the notes for this course. We hope you've enjoyed it!