Combinatorics

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Chapter 3. Projections

"If a set has small projections, must it be small?"

- Let $A \subset \mathcal{P}(X)$. For $Y \subset X$, the **projection** or **trace** of A on Y is
$$A|Y = \{x \cap Y : x \in A\}$$
"project A on the coordinates corresponding to Y"
e.g. if $A = \{14, 25, 26, 127, 128\}$, then $A|\{1, 2\} = \{1, 2, 12\}$ so $A|Y \subset \mathcal{P}(Y)$.
- Say A **covers** or **shatters** Y if $A|Y = \mathcal{P}(Y)$.
- The **trace number** or **VC-dimension** of A is (V : Vapnik, C : Cernovnkis)
$$\mathrm{tr}(A) = \max\{|Y| : A \text{ shatters } Y\}$$

Given $|A|$, how small can $\mathrm{tr}(A)$ be?
Equivalentely, if $\mathrm{tr}(A)<k$(i.e. A does not shatter any k-set), how large can A be?

- Trivially, must have $|A| \leq (1 - 2^{-k})2^n$ (else A shatter every k-set)
- Could take $A = X^{(<k)}$-no k-set Y is shattered, as $Y \notin A|Y$.

Aim : $A = X^{(<k)}$ is the best

Remark : very striking as from each k-projection having size $\leq (1 - 2^{-k})$, total we are getting a very small (polynomial in n) bound on $|A|$.

Idea : trivial that $|A| \leq |X^{(<k)}|$ if A is a **down set**(i.e. if $x \in A$ and $y \subset x$ then $y \in x$). Indeed, must have $A \subset X^{(<k)}$, since if A contains a set x with $|x| \geq k$ then $A|x = \mathcal{P}(x)$. So "try to make A into a down-set".

For $A \subset \mathcal{P}(X)$ and $1 \leq i \leq n$ the ***i*-down-compression** of A is defined as follows :

for $x \in \mathcal{P}(x)$, set
$$D_i(x) = \begin{cases} x & \text{if } i \notin x \\ x - \{i\} & \text{if } i \in x \end{cases}$$
and set $D_i(A) = \{D_i(x) : x \in A\} \cup \{x \in A : D_i(x) \in A\}$. "remove element i whenever possible".

Theorem 1) *(Sauer-Shelah lemma)* Let $A \subset \mathcal{P}(X)$ with $\text{tr}(A) < k$. Then $|A| \leq |X^{(<k)}|$.

proof) Given $1 \leq i \leq n$,

Claim : $\text{tr}(D_i(A)) \leq \text{tr}(A)$.

proof) Write $B = D_i(A)$ - we'll show that if B shatters Y (some Y) then A shatters Y.

If $i \notin Y<$ then $B|Y = A|Y$, so may assume $i \in Y$. Given $z \subset Y$ with $i \notin z$, we'll show z, $z \cup \{i\} \in A|Y$. Since $z \cup \{i\} \in B|Y$, have $z \cup \{i\} \cup x \in B$, some $x \subset X \backslash Y$. Hence $z \cup x$ and $z \cup \{i\} \cup x \in A$ (by definition of D_i) whence z, $z \cup \{i\} \in A|Y$.

Now let $D = D_n(D_{n-1}(\cdots D_1(A) \cdots))$. Then $|D| = |A|$, D is a down-set and $\text{tr}(D) \leq \text{tr}(A) < k$. Thus $|D| \leq |X^{(<k)}|$.

(End of proof) $\square$

Remark : we used 1-dimensional compression.

Now, we have : if all k-dimensional projections have size $\leq 2^k - 1$, then A is small ($|A| \leq \sum_{i=0}^{k-1} \binom{n}{k}$)

What about other bounds? For example, what if each k-dimensional projection is $\leq \frac{1}{2}$-sized ($\big|A|Y\big| \leq 2^{k-1}$?)

- A **box** or **brick** in $\mathbb{R}^n$ is a set of the form $[a_1, b_1] \times \cdots \times [a_n, b_n]$, where $a_i \leq b_i$ for all i.
- A **body** $S \subset \mathbb{R}^n$ is a *finite union* of bricks. For S a body, write $|S|$ or $m(S)$ for the volume of S.

Remarks :

1. Everything unchanged if we only assume S compact (or just bounded and measurable)
2. For $A \subset \mathcal{P}(X) \leftrightarrow \{0,1\}^n$, have corresponding body $\widehat{A} \subset \mathbb{R}^n$ with $m(\widehat{A}) = |A|$, namely :

$$\widehat{A} = \cup_{x \in A}[x_1, x_1 + 1] \times \cdots \times [x_n, x_n + 1]$$

For body $S \subset \mathbb{R}^n$ and $Y \subset \{1, 2 \cdots, n\}$, write S_Y for the projection of S onto the subspace spanned by the e_i, $i \in Y$.
e.g. for $S \subset \mathbb{R}^3$, S_1 is the projection of S onto the x-axis. i.e. $S_1 = \{x_1 : (x_1, x_2, x_3) \in S$, some $x_2, x_3\}$. S_{12} is the projection of S onto the xy-plane, i.e. $S_{12} = \{(x_1, x_2) : (x_1, x_2, x_3) \in S$ some $x_3\}$.

Question : When do an upper bound of $|S|$ exist given the values of S_Y?

e.g.

- for $S \subset \mathbb{R}^3$, $|S| \leq |S_1||S_2||S_3|$, as $S \subset S_1 \times S_2 \times S_3$. Also, $|S| \leq |S_{12}||S_3|$, as $S \subset S_{12} \times S_3$.
- But $\{|S_{12}|, S_{13}\}$ does not bound $|S|$, e.g. $S = [0, \frac{1}{N}] \times [0, N] \times [0, N]$.
- How about $\{|S_{12}|, |S_{23}|, |S_{13}|\}$?

Proposition 2) Let S be a body in $\mathbb{R}^3$. Then $|S|^2 \le |S_{12}||S_{23}||S_{13}|$.

Notes

1. Can have equality, e.g. when S is a brick
2. For $S \subset \mathbb{R}^n$, the **sections** of S are the sets $S(x) \subset \mathbb{R}^{n-1} (x \in \mathbb{R})$ given by

$$S(x) = \{(x_1, \cdots, x_{n-1}) \in \mathbb{R}^{n-1} : (x_1, \cdots, x_{n-1}, x) \in S\}$$

proof) Suppose first that every section of S is a square, $S(x) = [0, f(x)] \times [0, f(x)]$ for all x. Then $|S_{12}| = M^2$, where $M = \max_x f(x)$. Also $|S_{13}| = |S_{23}| = \int f(x)dx$. So want : $(\int f^2)^2 \le M^2(\int f)^2$, i.e. $\int f^2 \le M \int f$, which is true because $f(x)^2 \le Mf(x)$ for all x.

For general S, *define* body $T \subset \mathbb{R}^3$ by giving its sections,

$$T(x) = [0, \sqrt{|S(x)|}] \times [0, \sqrt{|S(x)|}]$$

so $|T| = |S|$. Certainly have $|T_{12}| \le |S_{12}|$, since $|T_{12}| \le \max_x |T(x)|$.

Write $g(x) = |S(x)_1|$, $h(x) = |S(x)_2|$, so $|S(x)| \le g(x)h(x)$. Have $|S_{13}| = \int g(x)dx$ and $|S_{23}| = \int h(x)dx$. Also,

$$\begin{aligned} T_{13}| = |T_{23}| = \int \sqrt{|S(x)|}dx &\le \int \sqrt{g(x)h(x)}dx \\ &\le (\int g)^{1/2}(\int h)^{1/2} \quad \text{(Cauchy-Schwarz)} \end{aligned}$$

(End of proof) □

- Sets $Y_1, \cdots, Y_r \subset [n]$ **cover** $[n]$ if $Y_1 \cup \cdots \cup Y_r = [n]$. They are a **k-uniform cover** if each $i \in [n]$ belong to exactly k of $Y_1, \cdots, Y_r$.

  e.g. $\{1\}, \{2\}, \{3\}$ is a 1-uniform cover of [3]. $\{1\}, \{2, 3\}$ is a 1-uniform cover of [3]. $\{1, 2\}, \{1, 3\}$ is not a uniform cover of [3]. $\{1, 2\}, \{2, 3\}, \{1, 3\}$ is a 2-uniform cover of [3].

Aim : $|S|^k \le |S_{Y_1}| \cdots |S_{Y_r}|$ where $Y_1, \cdots, Y_r$ is a k-uniform cover of $[n]$.

(29th November, Thursday)

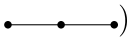
(3rd example class is at 5pm Thursday, 24th - hand in Q1,5,6, directly to the pigeon hole)

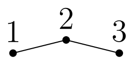
Intersecting Families of Graphs

So far, for n families, our object lived in $[n]$(no structure). What if the ground set has some structure?

For example, ground set= $[n]^{(2)}$ =edges of a graph on $[n]$=subgraphs of H_n, there are $2^{n(n-1)/2}$ possible graphs.

Let $A \subset \mathcal{P}([n]^{(2)})$ be a family of graphs on n vertices. For any fixed graph H, we say A is **H-intersecting** if $\forall G, G' \in A$, $G \cap G'$ contains a copy of H ("$G \cap G' \supset H$")
e.g. $H = P_1$=single edge. Then A is H-intersecting implies that $|A| \leq \frac{1}{2} 2^{n(n-1)/2}$ (as cannot have both $G, G^c \in A$) and can achieve this, e.g. $A = \{G : 12 \in G\}$. (indeed, for any H (non-empty), A being H-intersecting implies $|A| \leq \frac{1}{2} 2^{n(n-1)/2}$).

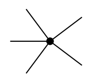
What about $H = P_2$? ($P_2 =$ )

Obvious guess is $A = \{G : G$ contains $H_0\}$ where $H_0 = \{$some fixed copy of $P_2\}$, e.g. $H_0 =$ 

. This has size $A = \frac{1}{4} 2^{n(n-1)/2}$. But can do better : e.g. $A = \{G : d_G(1) \geq \frac{n}{2} + 1\}$ (where $d_G(1) = \#$ edges out of 1). This has

$$|A| = 2^{n(n-1)/2}\left(\frac{1}{2} - \frac{c}{\sqrt{n}}\right) = \left(\frac{1}{2} + o(1)\right)2^{n(n-1)/2}$$

i.e. tends to $\frac{1}{2} 2^{n(n-1)/2}$

Similarly, if H is any star (), we have H-intersecting families of size $(\frac{1}{2} - o(1))2^{n(n-1)/2}$.
$\triangle$-intersecting ($\triangle$=triangle) ?

Obvious guess is $|A| = \frac{1}{8} 2^{n(n-1)/2}$ ($A = \{G : G \supset$ fixed triangles $\}$)

Simonovits-Sos conjecture : If A is $\triangle$-intersecting, then $|A| \leq \frac{1}{8} 2^{n(n-1)/2}$.

Theorem 8) Let $A \subset \mathcal{P}([n]^{(2)})$ be $\triangle$-intersecting. Then $|A| \leq \frac{1}{4} 2^{n(n-1)/2}$.

proof) Say n is even.

Consider the projection of A onto the edge-set $Y = B^{(2)} \cup (B^c)^{(2)}$, any $B \subset [n]$, $B = n/2$. then $G, G' \in A$ implies $G \cap G'$ must meet Y. (Because every triangle meets Y). Thus $A|Y$ is an intersecting family of sets. So

$$\left|A|Y\right| \leq \frac{1}{2} 2^{2\binom{n/2}{2}} = 2^{2\binom{n/2}{2}\left(1 - \frac{1}{2\binom{n/2}{2}}\right)}$$

But the Y form a uniform cover of $[n]^{(2)}$ (as B varies) so by **Corollary 5**, have

$$|A| \leq 2^{2\binom{n/2}{2}} = 2^{\binom{n/2}{2}\left(1 - \frac{1}{2\binom{n/2}{2}}\right)}$$

so done if

$$\binom{n}{2}(1-\frac{1}{2\binom{n/2}{2}})\geq 2$$

But (**LHS**)$=\frac{n(n-1)}{2\cdot\frac{n}{2}\cdot(\frac{n}{2}-1)}=\frac{n-1}{\frac{n}{2}-1}>2$, so done.

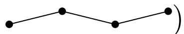
For n odd, the proof is same with $|B|=\frac{n-1}{2}$

(End of proof) $\square$

Simonovits-Sos conjecture was proved in 2010 (Ellis, Filmu, Friedent)

Say H **common** if $\max\{|A| : A\subset \mathcal{P}([n]^{(2)})$ is H-intersecting$\}=(\frac{1}{2}-o(1))2^{n(n-1)/2}$.
e.g. every star is commnon, $\triangle$ is not common. Any disjoint union of stars is also common, *e.g.* take n very large, k large and

$$A=\{G : \text{at least } \frac{k}{2}+3 \text{ of vertices } 1,\cdots k \text{ have degree} \geq \frac{n}{2}+5\}$$

Key question : is P_3(= ) common?
This is an open question!

Easy fact : every G, not a union of stars, contains $\triangle$ or P_3.

So if we know P_3 not common, we would also know -

Alon's common graphs conjecture : H is common $\Leftrightarrow$ H is a union of stars.

But : Christofides (2008) gave a P_3-intersecting family with density $\frac{17}{128}>\frac{1}{8}$.