Stochastic Calculus and Applications

Lectured by Dr. Roland Bauerschmidt, Typed by Jiwoon Park

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1 Introduction

1.1 Motivation

1.2 The Wiener Integral

$(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space
Definition) Gaussian space $S \subset L^2(\Omega, \mathcal{F}, \mathbb{P})$

Example : Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which a sequence of independent random variables $X_i \sim N(0,1)$ is defined. Then the X_i are an orthonormal system in $L^2(\Omega, \mathcal{F}, \mathbb{P})$:

$$\mathbb{E}(X_i X_j) = 0 \quad \text{for } i \neq j \text{ and } \quad \mathbb{E}(X_i^2) = 1$$

and $S = \overline{\text{span}\{X_i\}}$ is a Gaussian space. (*Exercise* : the limit in L^2 of Gaussian random variables is Gaussian.)

Proposition) Let H be a separable Hilbert space and $(\Omega, \mathcal{F}, \mathbb{P})$ as in the example. Then there is an isomtery $I : H \to S$. In particular, for every $f \in H$, there is a random variable $I(f) \in S$ such that

$$I(f) \sim N(0, (f,f)_H) \quad \text{and} \quad \mathbb{E}(I(f)I(g)) = (f,g)_H$$

Moreover, $I(\alpha f + \beta g) = \alpha I(f) + \beta I(g)$ a.s.

Definition) A Gaussian white noise on $\mathbb{R}_+$

Proposition)

(1) For $A \subset \mathbb{R}_+$, Borel, $|A| < \infty$, $WN(A) \sim N(0, |A|)$.

(2) For $A, B \subset \mathbb{R}_+$ Borel, $A \cap B = \phi$ then $WN(A)$ and $WN(B)$ are independent.

(3) If $A = \cup_{i=1}^{\infty} A_i$ for disjoint sets A_i with $|A_i| < \infty$, $|A| < \infty$, then

$$WN(A) = \sum_{i=1}^{\infty} WN(A_i) \quad \text{in } L^2 \text{ and a.s.} \quad \cdots\cdots\cdots (\star)$$

For $t \geq 0$, define the Brownian motion as $B_t = WN([0,t])$, just like the integration of white noise from 0 to t.
- Justify that this indeed (up to a modification) a BM

1.3 The Lebesgue-Stieltjes Integral

Definition) signed measure (on $[0,T] \subset \mathbb{R}_{\geq 0}$), Hahn-Jordan decomposition, total variation

Proposition) *(Hahn-Jordan)* For any positive measures μ_1, μ_2 on $[0,T]$(we do not require them to have disjoint support), there is a signed measure μ s.t. $\mu = \mu_1 - \mu_2$.

Definition) càdlàg function, total variation, of bounded variation on $[0,T]$

Proposition)

(i) Let μ be a signed measure on $[0,T]$. Then $a(t) = \mu([0,t])$ is càdlàg and $|\mu|((0,t]) = v_a(0,t)$ (i.e. $|\mu|([0,T])) = |a(0)| + v_a(0,t)$)
In particular, $a \in BV([0,T])$.

(ii) Let $a : [0,T] \to \mathbb{R}$ be càdlàg of bounded variation. The there is a signed measure μ such that $a(t) = \mu([0,t])$.

Definition) Let $a : [0,T] \to \mathbb{R}$ be càdlàg of bounded variation, Lebesgue-Stieltjes integral respect to a.

Fact : Let $a : [0,T] \to \mathbb{R}$ be càdlàg and BV (bounded variation), $h \in L^2([0,T], |da|)$. Then

$$\Big| \int_0^t h(s)da(s) \Big| \leq \int_0^t |h(s)||da(s)|$$

and the function $h \cdot a : [0,T] \to \mathbb{R}$ is càdlàg and BV with signed measures $h(s)da(s)$, $|h(s)da(s)| = |h(s)||da(s)|$.

Proposition) Let $a : [0,T] \to \mathbb{R}$ be càdlàg and BV. Let $h : [0,T] \to \mathbb{R}$ be *left-continuous* and bounded. Then

$$\int_0^t h(s)da(s) = \lim_{m\to\infty} \sum_{i=1}^{n_m} h(t_{i-1}^{(m)})(a(t_i^{(m)}) - a(t_{i-1}^{(m)})), \quad t \leq T$$
$$\int_0^t h(s)|da(s)| = \lim_{m\to\infty} \sum_{i=1}^{n_m} h(t_{i-1}^{(m)})\Big|a(t_i^{(m)}) - a(t_{i-1}^{(m)})\Big|$$

for a sequence of subdivisions $0 = t_0^{(m)} < \cdots < t_{n_m}^{(m)} = t$ with $\max_i |t_i^{(m)} - t_{i-1}^{(m)}| \to 0$ as $m \to \infty$.

Definition) finite variation (FV) function

2 Semimartingales

From now on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{\geq 0}, \mathbb{P})$ is a filtered probability space.

Definition) A **càdlàg adapted process** X
Notation : write $X \in \mathcal{F}$ to denote that a random variable X is measurable with respect to the sigma algebra $\mathcal{F}$.

2.1 Finite variation process

Definition) A finite variation process, total variation process

Fact : The total variation process V of càdlàg adpated process A is also càdlàg adapted and it is also increasing.

Definition) $H \cdot A$ for A a finite variation process and H with an integrability condition (to be stated)

Definition) previsible(predictable) σ-algebra, predictable process.

Definition) simple process

Fact : *Simple processes and their pointwise limits are predictable.*

Fact : *Adapted left-continuous processes are predictable*

Fact : *Let H be predictable. Then $H_t \in \mathcal{F}_{t^-}$ where $\mathcal{F}_{t^-} = \sigma(\mathcal{F}_s : s < t)$. (See Example Sheet #1)*

Fact : *Let X be adapted càdlàg. Then $X_{t^-} = \lim_{s\to t^-} X_s$ is left-continuous, predictable.*

Examples Brownian motion is predictable. / A Poisson process (N_t) is *not* predictable.

Proposition) Let A be a finite variation process, and let H be a predictable process such that $\int_0^t |H_s||dA_s| < \infty$ for all t and ω. Then $H \cdot A$ is also a finite variation process.

2.2 Local Martingale

From now on, we assume that $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfies the *usual conditions* (state)

Theorem) *(Optional Stopping Theorem, OST)* Let X be a càdlàg adapted integrable process. Then the following are equivalent : state / proof in Advanced probability

Definition) local martingale

Example :

(*i*) Every martingale is a local martingale (Take $T_n = n$ and use OST).

(*ii*) Let (B_t) be a standard Brownian motion on $\mathbb{R}^3$. Then $(X_t)_{t\geq 1} = (1/|B_t|)_{t\geq 1}$ is a local martingale, but not a martingale. (prove)

Proposition) Let X be a *local martingale* and $X_t \geq 0$ for all $t \geq 0$. Then X is a supermartingale.

Proposition) Let X be a local martingale and suppose that there is $Z \in L^1$ such that $|X_t| \leq Z$ for all $t \geq 0$. Then X is a martingale. In particular, bounded local martingales are martingales.

Fact : Let X be a *continuous adapted process* with $X_0 = 0$. Then

$$S_n = \inf\{t \geq 0 : |X_t| = n\}$$

are stopping times and $S_n \nearrow \infty$ as $n \to \infty$.

Proposition) Let X be a continuous local martingale with $X_0 = 0$. Then the sequence (S_n) defined above reduces X.

Theorem) Let X be a *continuous local martingale* with $X_0 = 0$. If X is also a *finite variation process*, then $X_t = 0$ for all $t \geq 0$ a.s.

2.3 L^2 bounded martingales

Definition) M^2, M_c^2, $\|X\|_{M^2}$ - why is this a norm on M^2?
In fact, $(X,Y)_{M^2} = \mathbb{E}[X_\infty Y_\infty]$ is an inner product on M^2 that induces the inner product - prove this.

Proposition) M^2 is a *Hilbert space* and M_c^2 is a closed subspace.

2.4 Quadratic Variation

Definition) convergent uniformly on compact intervals in probability

Theorem) Let M be a *continuous* local martingale. Then there exists a unique (up to indistinguishability) *continuous adapted increasing process* $\langle M\rangle = \big(\langle M\rangle_t\big)_t$ such that (is uniquely characterized by) $\langle M\rangle_0 = 0$ and $M^2 - \langle M\rangle$ is a continuous local martingale.
Moreover, with $0 = t_0^m < t_1^m < \cdots$ given by $t_i^m = 2^{-m}i$,

$$\langle M\rangle_t^{(m)} \xrightarrow{\text{ucp}} \langle M\rangle_t \quad \text{where } \langle M\rangle_t^{(m)} = \sum_{i=1}^{\lfloor 2^m t\rfloor} (M_{t_i} - M_{t_{i-1}})^2$$

[In fact, the convergence is true for all locally finite subdivision of $[0,\infty)$ with $\max_i |t_i^m - t_{i-1}^m| \to 0$ as $m \to \infty$.]

Definition) quadratic variation of M, $\langle M\rangle$.

Example : $\langle B\rangle_t = t$ for B a standard Brownian motion. - prove

Lemma) *(bounded case)* The theorem is true under the additional assumption $|M_t| \leq C$ for all (ω, t), $M_t = M_{t\wedge T}$ for C, T deterministic constants.

Lemma) Suppose M is a continuous local martingale for which $\langle M\rangle$ exists. Let T be a stopping time. Then $\langle M^T\rangle$ exists and is given by $\langle M^T\rangle_t = \langle M\rangle_{t\wedge T}$ (up to indistinguishability).

Fact : Let M be a continuous local martingale with $M_0 = 0$. Then $M \equiv 0$ *iff* $\langle M\rangle = 0$.

Proposition) Let $M \in M_c^2$ with $M_0 = 0$. Then $M^2 - \langle M\rangle$ is a *uniformly integrable* martingale and

$$\big\|M\big\|_{M^2} = \big(\mathbb{E}\big[\langle M\rangle_\infty\big]\big)^{1/2}$$

In particular, the norm only depends on the quadratic variation.

2.5 Covariation

Definition) For M and N continuous local martingales, define covariation $\langle M, N\rangle$.

Proposition)

(i) $\langle M, N\rangle$ is the unique (up to indistinguishability) finite variation process such that $MN - \langle M, N\rangle$ is a continuous local martingale.

(ii) We have $\langle M, N\rangle_t^{(m)} \xrightarrow{ucp} \langle M, N\rangle_t$ where

$$\langle M, N\rangle_t^{(m)} = \sum_{i=1}^{\lfloor 2^m t\rfloor} (M_{i2^{-m}} - M_{(i-1)2^{-m}})(N_{i2^{-m}} - N_{(i-1)2^{-m}})$$

(iii) The mapping $M, N \mapsto \langle M, N\rangle$ is bilinear and symmetric.

(iv) For every stopping time T, $\langle M^T, N^T\rangle_t = \langle M^T, N\rangle_t = \langle M, N\rangle_{T\wedge t}$.

(v) If $M, N \in M_c^2$ with $M_0 = N_0 = 0$, then $M_T N_t - \langle M, N\rangle$ is a uniformly integrable martingale and

$$(M, N)_{M^2} = \mathbb{E}\langle M, N\rangle_\infty$$

Proposition) *(Kunita-Watanabe inequality)* Let M and N be continuous local martingale,s and let H and K be measurable processes. Then a.s.

$$\int_0^\infty |H_s||K_s||d\langle M, N\rangle_s| \le \Big(\int_0^\infty |H_s|^2 d\langle M\rangle_s\Big)^{1/2}\Big(\int_0^\infty |K_s|^2 d\langle N\rangle_s\Big)^{1/2} \quad \cdots\cdots\cdots \text{(KW)}$$

2.6 Semimartingales

Definition) (continuous) semimartingale, its quadratic variation.

Exercise : We again have limit expression

$$\langle X, Y\rangle_t^{(m)} = \sum_{i=1}^{\lfloor 2^m t\rfloor} (X_{i2^{-m}} - X_{(i-1)2^{-m}})(Y_{i2^{-m}} - Y_{(i-1)2^{-m}}) \xrightarrow{ucp} \langle X, Y\rangle_t$$

3 The Itô integral

3.1 Simple processes

Definition) simple process, Ito integral for simple processes respect to a M_c^2-martingale.

Proposition) Let $M \in M_c^2$ and $H \in \mathcal{E}$. Then $H \cdot M \in M_c^2$ and

$$\|H \cdot M\|_{M^2}^2 = \mathbb{E}\Big(\int_0^\infty H_s^2 d\langle M\rangle_s\Big) \quad \text{(Itô isometry for simple process)}$$

What is the critical point about this proposition?

Proposition) Let $M \in M_c^2$ and let $H \in \mathcal{E}$. Then

$$\langle H \cdot M, N\rangle = H \cdot \langle M, N\rangle, \quad \forall N \in M_c^2$$

i.e. $\langle \int_0^\cdot H_s dM_s, N\rangle = \int_0^\cdot H_s d\langle M, N\rangle_s$.

3.2 Itô isometry

Definition) $L^2(M)$ for $M \in M_c^2$, norm and inner product in $L^2(M)$ - why is $(H, K)_{L^2(M)}$ finite for $H, K \in L^2(M)$?

Fact : $L^2(M) = L^2(\Omega \times [0, \infty), \mathcal{P}, d\mathbb{P}d\langle M\rangle)$ is a Hilbert space. (Recall $\mathcal{P}$ is the previsible σ-algebra)

Proposition) Let $M \in M_c^2$. Then $\mathcal{E}$, the space of simple processes, is dense in $L^2(M)$.

Theorem/Definition) Let $M \in M_c^2$. Then

(i) The map $H \in \mathcal{E} \mapsto H \cdot M \in M_c^2$ extends uniquely to an isometry $L^2(M) \to M_c^2$, the *Itô isomtery.*

(ii) $H \cdot M$ is the unique martingale in M_c^2 such that

$$\langle H \cdot M, N \rangle = H \cdot \langle M, N \rangle, \quad \forall N \in M_c^2$$

$(H \cdot M)_t = \int_0^t H_s dM_s$ is then called the **Itô integral** of H with respect to M.

Corollary) If T is a stopping time, then

$$(1_{[0,T]} H) \cdot M = (H \cdot M)^T = H \cdot M^T$$

Corollary) $\langle H \cdot M, K \cdot N \rangle = (HK) \cdot \langle M, N \rangle$, *i.e.*

$$\langle \int_0^\cdot H_s dM_s, \int_0^\cdot K_s dN_s \rangle_t = \int_0^t H_s K_s d\langle M, N \rangle_s$$

Corollary) One has, if $t > u$,

$$\begin{aligned} &\mathbb{E}\Big(\int_0^t H_s dM_s\Big) = 0 \\ &\mathbb{E}\Big(\int_0^t H_s dM_s | \mathcal{F}_u\Big) = \int_0^u H_s dM_s \\ &\mathbb{E}\Big(\int_0^t H_s dM_s \int_0^t K_s dN_s\Big) = \mathbb{E}\Big(\int_0^t H_s K_s d\langle M, N \rangle_s\Big) \end{aligned}$$

Corollary) *(Associativity of Itô integral)* Let $H \in L^2(M)$. Then $KH \in L^2(M)$ *iff* $K \in L^2(H \cdot M)$ and then

$$(KH) \cdot M = K \cdot (H \cdot M)$$

3.3 Extension to local martingales

Definition) Let M be a *continuous local martingale*, define $L^2_{loc}(M)$.

Theorem) Let M be a continuous local martingale.

(i) For every $H \in L^2_{loc}(M)$, there is a unique (up to indistiguishability) contiuous local martingale $H \cdot M$ with $(H \cdot M)_0 = 0$ such that

$$\langle H \cdot M, N \rangle = H \cdot \langle M, N \rangle \quad \forall N \text{ continuous local martingale}$$

(ii) If $H \in L^2_{loc}(M)$ and K is predictable then $K \in L^2_{loc}(H \cdot M)$ *iff* $HK \in L^2_{loc}(M)$ and then

$$H \cdot (K \cdot M) = (HK) \cdot M$$

(iii) If T is a stopping time,

$$(1_{[0,T]} H) \cdot M = (H \cdot M)^T = H \cdot M^T$$

Finally, if $M \in M_c^2$ and $H \in L^2(M)$ then this definition is consistent with the previous one.

3.4 Extension to Semimartingales

Definition) locally bounded process

Fact : If H is locally bounded and predictable and if A is a finite variation process,

$$\forall t > 0, \quad \int_0^t H_s |dA_s| < \infty \quad \text{a.s.}$$

In particular, for such H, and M a continuous local martingale, it follows that $H \in L^2_{loc}(M)$.

Definition) Let $X = X_0 + M + A$ be a continuous semimartingale, and let H be a predictable locally bounded process. Define $H \cdot X$.

Proposition) *(Stochastic Dominated Convergence Theorem, Stochastic DCT)* Let X be a continuous semimartingale, and let H be locally bounded predictable process and let K be a predictable non-negative process. Let $t > 0$ and assume that

(i) $H_s^n \xrightarrow{n\to\infty} H_s$ for all $s \in [0,t]$.

(ii) $|H_s^n| \le K_s$ for all $s \in [0,t]$ and $n \in \mathbb{N}$.

(iii) $\int_0^t K_s^2 d\langle M\rangle + \int_0^t K_s |dA_s| < \infty$ (where $X = X_0 + M + A$). *[This condition is always true if K is locally bounded]*

Then $\int_0^t H_s^n dX_s \xrightarrow{ucp} \int_0^t H_s dX_s$ as $n \to \infty$.

Corollary) Let X be a continuous semimartingale, and let H be a locally bounded adapted left-continuous processs. Then for any subdivision $0 = t_0^{(m)} < \cdots < t_{n_m}^{(m)} = t$ of $[0,t]$ with $\max_i |t_i^{(m)} - t_{i-1}^{(m)}| \to 0$ as $m \to \infty$, has :

$$\lim_{m\to\infty} \sum_{i=1}^{n_m} H_{t_{i-1}^{(m)}} \left(X_{t_i^{(m)}} - X_{t_{i-1}^{(m)}} \right) = \int_0^t H_s dX_s$$

where the convergence is made ucp. *[Taking the left end-point $t_{i-1}^{(m)}$ is important, and is consistent with the choice of Itô integral. Different choice corresponds to what the integral means.]*

Remark : Suppose H is continuous. Unlike the case that X is of finite variation, it is essential here that H is evaluated at the left end point. - state why

3.5 Itô formula

Theorem) *(Integration by parts)* Let X and Y be continuous semimartingales. Then a.s,

$$X_t Y_t - X_0 Y_0 = \int_0^t X_s dY_s + \int_0^t Y_s dX_s + \langle X, Y\rangle_t$$

The last term $\langle X, Y\rangle$ is called the **Itô correction.**

Theorem) *(Itô formula)* Let $X^1, \cdots, X^p$ be continuous semimartingales, and let $f \in C^2(\mathbb{R}^p; \mathbb{R})$. Then, writing $X = (X^1, \cdots, X^p)$, a.s,

$$f(X_t) = f(X_0) + \sum_{i=1}^p \int_0^t \frac{\partial f}{\partial x^i}(X_s) dX_s^i + \frac{1}{2} \sum_{i,j=1}^p \int_0^t \frac{\partial^2 f}{\partial x^i \partial x^j}(X_s) d\langle X^i, X^j\rangle_s \quad \cdots\cdots\cdots \ (\star)$$

Informally, we may write

$$df(X_t) = \sum_{i=1}^p \frac{\partial f}{\partial x^i}(X_t) dX_t^i + \frac{1}{2} \sum_{i,j=1}^p \frac{\partial^2 f}{\partial x^i \partial x^j}(X_t) d\langle X^i, X^j\rangle_t$$

Summary of calculation rules for the Itô integral :

Let us adopt the notations

$$Z_t - Z_0 = \int_0^t H_s dX_{S_t} \quad \Leftrightarrow \quad dZ_t = H_t dX_t$$
$$Z_t - Z_0 = \langle X, Y\rangle_t = \int_0^t d\langle X, Y\rangle_s \quad \Leftrightarrow \quad dZ_t = dX_t dY_t$$

Then, :

"Associativity"	$H_t(K_t dX_t) = (H_t K_t) dX_t$, (*i.e.* $H \cdot (K \cdot X) = (HK) \cdot X$)
"Kunita-Watanabe equality"	$H_t dX_t dY_t = (H_t dX_t) dY_t$, (*i.e.* $H \cdot \langle X, Y\rangle = \langle H \cdot X, Y\rangle$)
"Itô formula"	$df(X_t) = \sum_i \frac{\partial f}{\partial x^i} dX_t^i + \frac{1}{2} \sum_{i,j} \frac{\partial^2 f}{\partial x^i \partial x^j}(X_t) dX_t^i dX_t^j$

4 Applications to Brownian Motion and Martingales

4.1 Lévy's characterisation of Brownian motion

Theorem) Let $X = (X^1, \cdots, X^p)$ be continuous local martingales. Suppose $X_0 = 0$ and that $\langle X^i, X^j \rangle_t = \delta_{ij} t$ for all $t \geq 0$. Then X is a standard p-diemnsional Brownaim motion. That is, *the covariation singles out the Brownian motion.*

4.2 Dubins-Schwarz Theorem

Theorem) Let M be a continuous local martingale with $M_0 = 0$ and $\langle M \rangle_\infty = \infty$ a.s. Let $T_s = \inf\{t \geq 0 : \langle M \rangle_t > s\}$ be the right-continuous inverse of $\langle M \rangle$.

$$B_s = M_{T_s}, \quad \mathcal{G}_s = \mathcal{F}_{T_s}$$

Then T_s is an $(\mathcal{F}_t)$ stopping times, $\langle M \rangle_{T_s}$ for all $s \geq 0$, B is a $(\mathcal{G}_s)_{s \geq 0}$-Brownian motion and

$$M_t = B_{\langle M \rangle_t},$$

(needs the following lemma)

Lemma) Let M be a continuous local martingale. Almost surely for all $u < v$, M is constant on $[u, v]$ *iff* $\langle M \rangle$ is constant on $[u, v]$.

4.3 Girsanov's Theorem

Definition) stochastic exponential of a continuous local martingale L

Fact : $Z = \mathcal{E}(M)$ is a continuous local martingale and it satisfies

$$dZ_t = Z_t dL_t$$

-prove

Theorem) *(Girsanov)* Let L be a continuous local martingale with $L_0 = 0$. Suppose that $\mathcal{E}(L)$ is a *UI(uniformly integrable)* martingale. Define a probability measure $\mathbb{Q}$ by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathcal{E}(L)_\infty$$

If M is a continuous local martingale with respect to $\mathbb{P}$, then $\tilde{M} = M - \langle M, L \rangle$ is a continuous local martingale with respect to $\mathbb{Q}$.

Remark : The quadratic variation does not change, $\langle M \rangle = \langle \tilde{M} \rangle$. (also prove this)

Proposition) Suppose that $\langle L \rangle$ is bounded, say $\langle L \rangle_\infty \leq C$. Then $\mathcal{E}(L)$ is a UI martingale.

The proof needs the following result.

Proposition) Let M be a continuous local martingale with $M_0 = 0$. Then $M \in M_c^2$ *iff* $\mathbb{E}\langle M \rangle_\infty < \infty$ and then $M^2 - \langle M \rangle$ is a UI martingale and $\|M\|_{M^2} = (\mathbb{E}\langle M \rangle_\infty)^{1/2}$. (Proof in ES)

Theorem) *(Novikov)* Let M be a continuous local martingale with $M_0 = 0$. Then $\mathbb{E}(e^{\frac{1}{2}\langle M \rangle_\infty}) < \infty$ implies that $\mathcal{E}(M)$ is a UI martingale.
(not proving)

Corollary) *(corollary of Girsanov's Theorem)* Let B be a standard Brownian motion (under $\mathbb{P}$) and let L be a continuous local martingale with $L_0 = 0$ such that $\mathcal{E}(L)$ is a UI martingale. Then $\tilde{B} = B - \langle B, L \rangle$ is a standard Brownian motion under the measure $\mathbb{Q}$ where

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathcal{E}(L)_\infty$$

Example : too long...

4.4 The Cameron-Martin formuala

Definition) canonical Wiener space, canonical version of Brownian motion, Cameron-Martin space

Exercise : $\mathcal{H}$ is a Hilbert space with inner product

$$(h, f)_{\mathcal{H}} = \int_0^\infty \dot{h}(s)\dot{f}(s)ds$$

The dual space of $\mathcal{H}$ can be identified with

$$\mathcal{H}^* = \{\mu \in \mathcal{M}(\mathbb{R}_+) : \int_0^\infty (s \wedge t)\mu(ds)\mu(dt) = (\mu, \mu)_{\mathcal{H}^*} < \infty, \mu(\{0\}) = 0\}$$

in the sense that for any $l : \mathcal{H} \to \mathbb{R}$ bounded and linear, there is $\mu \in \mathcal{H}$ such that $l(h) = \int_0^\infty h(t)\mu(dt)$ and vice-versa.

Remark : We would like to think of a Brownian motion as the standard Gaussian measure on $\mathcal{H}$. This measure does not exist. But the next theorem shows it almost does.

Theorem) *(Cameron-Martin)* Let $h \in \mathcal{H}$ and define P^h by (P^h is going to be a canonical measure on the Wiener space)

$$P^h(A) = P\big(\{w \in W : w + h \in A\}\big)$$

for $A \in \mathcal{W}$. Then the measure P^h is absolutely continuous with respect to the Wiener measure P and

$$\frac{dP^h}{dP} = \exp\Big(\int_0^\infty \dot{h}(s)dX_s - \frac{1}{2}\int_0^\infty \dot{h}(s)^2 ds\Big)$$

5 Stochastic Differential Equations

5.1 Notions of Solutions

Definition) stochastic diffrential equation (SDE) $E(\sigma, b)$, weak solution, strong solution, weak uniqueness(uniqueness in law), pathwise uniqueness,
Example : *(Tanaka)* The SDE

$$dX_t = \mathrm{sign}(X_t)dB_t, \quad X_0 = x \quad \cdots\cdots\cdots \text{ (TK)}$$

where $\mathrm{sign}(x) = 1$ if $x > 0$, $\mathrm{sign}(x) = -1$ if $x \le 0$, has a weak solution that is unique in law, but it is pathwise uniqueness does not hold.

Theorem) *(Pathwise uniqueness for SDEs with Lipschitz coefficients)* Suppose that b and σ are *locally Lipschitz* (in space variable), *i.e.*, for each $n > 0$, there exists $K_n > 0$ such that forall $|x|, |y| \le n$, $t \ge 0$, has

$$\begin{aligned} &|b(t,x) - b(t,y)| \le K_n|x - y| \quad \text{and} \\ &|\sigma(t,x) - \sigma(t,y)| \le K_n|x - y|. \end{aligned}$$

Then pathwise uniqueness holds for $E(\sigma, b)$.

Gronwall's Lemma) (on *Example Sheet #3*) Let $T > 0$ and let $f : [0, T] \to \mathbb{R}$ be non-negative *bounded* Borel function. Assume $f(t) \le a + b\int_0^t f(s)ds$ for all $t \le T$. Then

$$f(t) \le ae^{bt} \quad \text{for all } t \le T$$

(see ES 3)

5.2 Strong existence for Lipschitz coefficients

Recall, we denote $E(\sigma, b)$ for $dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t$.

Theorem) Assume b and σ are globally Lipschitz, *i.e.* there is $K > 0$ such that for all $x, y \in \mathbb{R}^d$, $t \ge 0$,

$$|b(t,x) - b(t,y)| \le K|x - y|, \quad |\sigma(t,x) - \sigma(t,y)| \le K|x - y|$$

For any $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ (obeying usual condition), any $(\mathcal{F}_t)$-Brownian motion B, any $x \in \mathbb{R}$, there is a unique strong solution to $E_x(\sigma, b)$.

Proposition) Under the assumptions of the theorem, let X^x be the solution with initial condition $X_0^x = x$. Then for any $p \geq 2$,

$$\mathbb{E}\big(\sup_{s\leq t} |X_t^x - X_s^y|^p\big) \leq C_p|x-y|^p e^{C_p(t\vee 1)^p t}$$

We need :

Lemma) *(Burkholder-Davis-Gundy (BDG) inequality)* For every real $p > 0$, there exists $C_p > 0$ depending only on p such that, for every continuous local martingale M with $M_0 = 0$ and every stopping time T,

$$\mathbb{E}[\sup_{0\leq s\leq T} (M^s)^p] \leq C_p \mathbb{E}[\langle M\rangle_T^{p/2}]$$

Strong solution can be considered functions of Brownian motion in the following sense. Recall the *(d-dimensional) Wiener space* $(W^d, \mathcal{W}^d, P^d)$ where

$$W^d = C(\mathbb{R}_+, \mathbb{R}^d), \quad \mathcal{W} = \sigma(X_t^i :\in \mathbb{R}_+, i = 1, \cdots, d), \quad \text{where } X_t(w) = w(t) \text{ for } w \in W^d$$

and P^d is the probability measure on $(W^d, \mathcal{W}^d)$ such that $(X_t)_{t\geq 0}$ is a standard Brownian motion with $X_0 = 0$.

The space $C(\mathbb{R}_+, \mathbb{R}^d)$ can be given the topology of uniform convergence on compact intervals. This topology is induced by the metric

$$d(w, \tilde{w}) = \sum_{k=1}^{\infty} \alpha_k(\|w - \tilde{w}\|_{L^\infty([0,t];\mathbb{R}^d)} \wedge 1)$$

for any seqeunce $(\alpha_k) \subset \mathbb{R}_+$ with $\sum_{k=1}^{\infty} \alpha_k = 1$.

Remark : This metric makes $C(\mathbb{R}_+\mathbb{R}^d)$ a complete separable metric space (a so called *Polish space*).

Theorem) Under the assumptions of the last theorem (strong solution for Lipschitz coefficients), for $x \in \mathbb{R}^d$, there exists maps

$$F_x : W^m = C(\mathbb{R}_+, \mathbb{R}^m) \to W^d = C(\mathbb{R}_+, \mathbb{R}^d)$$

measurable with respect to the completion of $\mathcal{W}^m$ on W^m and w.r.t. $\mathcal{W}^d$ on W^d such that

(i) $\forall t \geq 0$, $F_x(w)_t$ is a measurable function of $\sigma(w(s) : s \leq t)$ for P^d-a.s. $w \in W^m$.

(ii) $\forall w \in C(\mathbb{R}_+, \mathbb{R}^m) : x \in \mathbb{R}^d \mapsto F_x(w) \in C(\mathbb{R}_+, \mathbb{R}^d)$ is continuous.

(iii) $\forall x \in \mathbb{R}^d$, $\forall(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions, every $(\mathcal{F}_t)$-Brownian motion $\hat{B}$ with $\hat{B}_0 = 0$, the unique solution to $E_x(\sigma, b)$ is $\hat{X}_t = F_x(\hat{B})_t$.

(iv) In the set-up of (iii), if U is $\mathcal{F}_0$-measurable, then $F_U(\hat{B})_t$ is the unique solution to $E(\sigma, b)$ with $X_0 = U$.

(Such F is called the **Itô map**.)

Corollary) The solutions to $E_x(\sigma, b)$ can be constructed for all $x \in \mathbb{R}^d$ simultaneously such that a.s. X^x is continuous in the initial condition.

proof) Direct from the theorem.

5.3 Some examples of SDEs

Describe the following.

Geometric Brownian motion

The Ornstein-Uhlenbeck process

Let X_t be an Ornstien-Uhlenbeck process

Fact : If $X_0 = x$, then $\mathbb{E}(X_t) = e^{-\lambda t}x$, $\mathrm{Cov}(X_t, X_s) = \frac{1}{2\lambda}(e^{-\lambda|t-s|} - e^{-\lambda|t+s|})$.

proof) Clearly, $\mathbb{E}X_t = e^{-\lambda t}\mathbb{E}X_0 + \mathbb{E}\int_0^t e^{-\lambda(t-s)}dB_s = e^{-\lambda t}\mathbb{E}X_0$. Also, by *Itô isomtery*,

$$\begin{aligned}
\mathrm{Cov}(X_t, X_s) &= \mathbb{E}\big((X_t - \mathbb{E}X_t)(X_s - \mathbb{E}X_s)\big) \\
&= \mathbb{E}\Big(\int_0^t e^{-\lambda(t-u)}dB_u \int_0^s e^{-\lambda(s-u)}dB_u\Big) \\
&= \int_0^\infty \mathbf{1}_{u<t}e^{-\lambda(t-u)}\mathbf{1}_{u<s}e^{-\lambda(s-u)}du \\
&= e^{-\lambda(t+s)}\int_0^{t+s} e^{2\lambda u}du = \frac{1}{2\lambda}e^{-\lambda(t+s)}(e^{2\lambda(s\wedge t)} - 1)
\end{aligned}$$

(End of proof) □

Corollary) $X_t \sim N(e^{-\lambda t}x, \frac{1-e^{-2\lambda t}}{2\lambda})$

Fact : If $X_0 \sim N(0, \frac{1}{\lambda})$, then $X_t \sim N(0, \frac{1}{2\lambda})$ for all $t > 0$, and X_t is a *stationary* Gaussian process with $\mathrm{Cov}(X_s, X_t) = \frac{1}{2\lambda}e^{-\lambda|t-s|}$

5.4 Local Solutions

Proposition) *(Local Itô formula)* Let $X = (X^1, \cdots, X^d)$ be semimartingales. Let $U \subset \mathbb{R}^d$ be open, and let $f : U \to \mathbb{R}^d$ be C^2. Set $T = \inf\{t \geq 0 : X_t \notin U\}$. Then for all $t < T$,

$$f(X_t) = f(X_0) + \sum_{i=1}^d \frac{\partial f}{\partial x^i}(X_s)dX_s^i + \frac{1}{2}\frac{\partial^2 f}{\partial x^i \partial x^j}(X_s)d\langle X^i, X^j\rangle_s$$

Example : Let B be a standard Brownian motion with $B_0 = 1$ (in dimension 1), then

$$\sqrt{B_t} = 1 + \frac{1}{2}\int_0^t B_s^{-1/2}dB_s - \frac{1}{8}\int_0^t B_s^{-3/2}ds$$

for $t < T = \inf\{t \geq 0 : B_t = 0\}$.

Theorem) Let $U \subset \mathbb{R}^d$ be open and $b : \mathbb{R}_+ \times U \to \mathbb{R}^d$ and $\sigma : \mathbb{R}_+ \times U \to \mathbb{R}^{d\times m}$ be *locally Lipschitz continuous*. Then for every $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$, a Brownian motion B adapted to this filtration, and every $x \in U$, there exists a stopping time T such that, for $t < T$,

$$X_t = x + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s$$

where T is such that for all $K \subset U$ compact, we have $\sup\{t < T : X_t \in K\} < T$.
Such T is called the **explosion time**.

Example : Consider the SDEs

$$dX_t^i = -\nabla_i H(X_t)dt + dB_t^i, \quad X_0 = x$$

Assume that there are $a \geq 0$, $b \geq 0$ such that

$$x \cdot \nabla H(x) \geq -a|x|^2 - b$$

Then, the SDE has a global solution, *i.e.* $T = \infty$ a.s.

proof) Let $T_n = \inf\{t \geq 0 : |X_t|^2 > n\}$. Then by *Itô's formula* to X^{T_n},

$$\begin{aligned}\mathbb{E}|X_{t\wedge T_n}|^2 &= \mathbb{E}|X_0|^2 - \mathbb{E}\Big(2\int_0^{t\wedge T_n} X_s \cdot \nabla H(X_s)ds - t \wedge T_n\Big) \\ &\leq \mathbb{E}|X_0|^2 + 2a\mathbb{E}\Big(\int_0^{t\wedge T_n} |X_s|^2 ds\Big) + (1+2b)\mathbb{E}(t\wedge T_n) \\ &\leq \mathbb{E}|X_0|^2 + (1+2b)t + 2a\int_0^t \mathbb{E}|X_{s\wedge T_n}|^2 ds\end{aligned}$$

By *Gronwall's lemma*,

$$\mathbb{E}|X_{t\wedge T_n}|^2 \leq (\mathbb{E}|X_0^2| + (1+2b)t)e^{2at}$$

If $\mathbb{P}(T<\infty) > 0$, then for sufficiently large t, $|X_{t\wedge T_n}|^2 \to \infty$ as $n \to \infty$ with positive probability, so it follows that $\mathbb{P}(T<\infty) = 0$.

(End of proof) $\square$

6 Applications to PDEs and Markov Processes

6.1 Prababilistic representations of solutions to PDEs

Exercise : Let $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times m}$ be (locally) bounded Borel functions and let $x \in \mathbb{R}^d$. Assume that X is a solution to $E_x(\sigma, b)$. Then for every $f \in C^1(\mathbb{R}_+) \otimes C^2(\mathbb{R}^d)$,

$$M_t^f = f(t, X_t) - f(0, X_0) - \int_0^t \Big(\frac{\partial}{\partial s} + L\Big) f(s, X_s)ds$$

is a continuous local martingale where

$$Lf(y) = \frac{1}{2}\sum_{i,j=1}^d a_{ij}(y)\frac{\partial^2 f}{\partial y^i \partial y^j} + \sum_{i=1}^d b_i(y)\frac{\partial f}{\partial y^i}$$

where $a(y) = \sigma(y)\sigma(y)^T \in \mathbb{R}^{d\times d}$.

Definition) The L is called the **(infinitesimal) generator** of X.

Example :

- $dX = dB$, a Brownian motion has $L = \frac{1}{2}\triangle$.
- $dX = -Xdt + dB$, an Orstein-Uhlenbeck process has $L = \frac{1}{2}\triangle - x \cdot \nabla$.

Drichlet-Poisson problem

Let $U \subset \mathbb{R}^d$, $U \neq \phi$ be open and bounded. Given $f \in C(\overline{U})$ and $g \in C(\partial U)$, a (DP) asks to find $u \in C^2(\overline{U}) = C^2(U) \cap C(\overline{U})$ such that

$$\begin{cases} -Lu(x) = f(x) & \text{for } x \in U \\ u(x) = g(x) & \text{for } x \in \partial U \end{cases} \quad \cdots\cdots\cdots \text{ (DP)}$$

This is called a **Poisson problem** if $f = 0$ and called a **Dirichlet Problem** if $g = 0$.

Definition) uniform ellipticity

Theorem) Assume that U has a smooth boundary, that a and b are Hölder continuous functions, and that a is uniformly elliptic. Then for every Hölder continuous $f : \overline{U} \to \mathbb{R}$ and every continuous $g : \partial U \to \mathbb{R}$, (DP) has a solution.
[See PDE textbooks. Can also use probabilistic method to prove this.]

Theorem) Let $U \subset \mathbb{R}^d$ be open, bounded and non-empty. Let b and σ be bounded measurable, assume $a = \sigma\sigma^T$ is uniformly elliptic and let u be the solution of (DP) with coefficients σ and b. Let $x \in U$, let X be a solution to $E_x(\sigma, b)$. Let $T_U = \inf\{t \geq 0 : X_t \notin U\}$. Then $\mathbb{E}[T_U] < \infty$ and

$$u(x) = \mathbb{E}_x\Big(u(X_{T_U}) - \int_0^{T_U} Lu(X_s)ds\Big) = \mathbb{E}_x\Big(g(X_{T_U}) + \int_0^{T_U} f(X_s)ds\Big)$$

Cauchy Problem

Given $f \in C_b^2(\mathbb{R}^d)$, find $u \in C(\mathbb{R}_+) \otimes C^2(\mathbb{R}^d)$ such that

$$\begin{cases} \frac{\partial u}{\partial t} = Lu & \text{on } (0,\infty) \times \mathbb{R}^d \\ u(0,\cdot) = f & \text{on } \mathbb{R}^d \end{cases} \quad \cdots\cdots\cdots \text{(CP)}$$

where L is given as above.

Theorem) For $f \in C_b^2(\mathbb{R}^d)$, there exists a solution to (CP).
[Again, refer to a standard PDE texts, such as Evans.]

Theorem) Let u be a (bounded) solution to (CP). Let $x \in \mathbb{R}^d$, let X be any solution to $E_x(\sigma, b)$, $0 \leq s \leq t$, then

$$\mathbb{E}(f(X_t)|\mathcal{F}_s) = u(t-s, X_s)$$

In particular,

$$\mathbb{E}_x f(X_t) = u(t,x)$$

Theorem) *(Feynman-Kac formula)* Let L, b, σ as before. Let $f \in C_b^2(\mathbb{R}^d)$, $V \in C_b(\mathbb{R}^d)$ and suppose that $u : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ satisfies

$$\begin{cases} \frac{\partial u}{\partial t} = Lu + Vu & \text{on } \mathbb{R}_+ \times \mathbb{R}^d \\ u(0,\cdot) = f & \text{on } \mathbb{R}^d \end{cases}$$

(Vu here is just a pointwise multiplication). Let X be a solution to $E_x(\sigma, b)$ for some $x \in \mathbb{R}^d$. Then for all $t \geq 0$,

$$u(t,x) = \mathbb{E}_x\Big[f(X_t)\exp\Big(\int_0^t V(X_s)ds\Big)\Big]$$

6.2 Markov property

Let $B(\mathbb{R}^d)$ be the Banach space of **bounded Borel functions** on $\mathbb{R}^d$, with $\|f\| = \sup_{x\in\mathbb{R}^d} |f(x)|$ for $f \in B(\mathbb{R}^d)$.

Definition)

(i) A collection of bounded linear operators Q_t on $B(\mathbb{R}^d)$ is a **transition semigroup** if $Q_t f \geq 0$ if $f \geq 0$ (pointwise), $Q_t\mathbf{1} = \mathbf{1}$, $\|Q_t\| \leq 1$, and

$$Q_{t+s} = Q_t Q_s \quad \forall t, s \geq 0$$

(ii) An $(\mathcal{F}_t)$-adapted process X is a **Markov process** with transition semigroup $(Q_t)_t$ if

$$\mathbb{E}(f(X_{s+t})|\mathcal{F}_s) = Q_t f(X_s) \quad \forall s, t \geq 0,\ f \in B(\mathbb{R}^d)$$

Theorem) Let $b : \mathbb{R}^d \to \mathbb{R}$, $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times m}$ be *Lipschitz* (this can be weakened). Assume X is a solution to $E(\sigma, b)$ on some $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ and B. Then $X = (X_t)_{t\geq 0}$ is a Markov process with semigroup

$$Q_t f(x) = \mathbb{E}(f(X_t^x)) = \int f(F_x(w)_t)P^m(dw)$$

where X_t^x is an arbitrary solution to $E_x(\sigma, b)$, and F_x is the *Itô solution map*, P^m is the Wiener measure.

Definition) Let Q_t be the transition semigroup, invariant probability measure, reversible probability measure.

Fact : Reversibility of μ implies it is invariant. (Take $g = 1$ and use $Q_t 1 = 1$.)

Example : Consider the transition semigroup associated to the SDE, with suitable on H,

$$dX_t = -\frac{1}{2}\nabla H(X_t)dt + dB_t$$

(Note thet, if taking $H(x) = \lambda|x|^2$, this gives an Orstein-Uhlenbeck process.) Then the measure $\mu(dx) = \frac{1}{Z}e^{-H(x)}dx$, where $Z = \int e^{-H(x)}dx$ is reversible for $(*)$.

Lemma) Assume that the explosion time for $(*)$ is infinite. Then for $f : C([0,T], \mathbb{R}^) d \to \mathbb{R}$,

$$\mathbb{E}\Big(f(X|_{[0,T]})\Big) = \mathbb{E}^{\mathrm{BM}}\Big[f(X|_{[0,T]})\exp\Big(\frac{1}{2}H(X_0) - \frac{1}{2}H(X_T) - \int_0^T (\frac{1}{8}|\nabla H|^2 - \frac{1}{4}\triangle H)(X_s)ds\Big)\Big]$$

(where $\mathbb{E}^{\mathrm{BM}}$ takes average over law under which X is a Brownian motion with same initial condition.)
(not proved in the lecture)