

# Advanced Probability

-Martingales

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(15th October 2018, Monday)

## Chapter 2. Martingales in Discrete Time

### 2.1. Definitions.

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space.

- A **Filtration** for  $(\Omega, \mathcal{F}, \mathbb{P})$  is a sequence  $(\mathcal{F}_n)_{n \geq 0}$  of  $\sigma$ -algebras s.t. for all  $n \geq 0$ , we have

$$\mathcal{F}_n \subset \mathcal{F}_{n+1} \subset \mathcal{F}$$

Set  $\mathcal{F}_\infty = \sigma(\mathcal{F}_n : n \geq 0)$  then  $\mathcal{F}_\infty \subset \mathcal{F}$ . We allow  $\mathcal{F}_\infty \neq \mathcal{F}$ . We interpret  $n$  as times and  $\mathcal{F}_n$  as the extent of knowledge at time  $n$ .

- A **Random process(in discrete time)** is a sequence of random variables  $(X_n)_{n \geq 0}$ . It has a natural filtration  $(\mathcal{F}_n^X)_{n \geq 0}$  given by

$$\mathcal{F}_n^X = \sigma(X_0, \dots, X_n)$$

That is, the knowledge obtained from  $X_n$  by time  $n$ . We say  $(X_n)_{n \geq 0}$  is **adapted to**  $(\mathcal{F}_n)_{n \geq 0}$  if  $X_n$  is  $\mathcal{F}_n$ -measurable for all  $n \geq 0$ . This is equivalent to having  $\mathcal{F}_n^X \subset \mathcal{F}_n$ , for all  $n \geq 0$ . (Here,  $X_n$  are real-valued)

- We would say  $(X_n)_{n \geq 0}$  is **integrable** if  $X_n$  is integrable for all  $n \geq 0$ .
- A **martingale** is an *adapted, integrable random process*  $(X_n)_{n \geq 0}$  s.t. for all  $n \geq 0$ ,

$$\mathbb{E}[X_{n+1} | \mathcal{F}_n] = X_n \quad \text{a.s.}$$

In the case  $\mathbb{E}[X_{n+1} | \mathcal{F}_n] \leq X_n$  a.s.,  $(X_n)_n$  is called a **super-martingale** and in the case  $\mathbb{E}[X_{n+1} | \mathcal{F}_n] \geq X_n$  a.s.,  $(X_n)_n$  is called a **sub-martingale**.

### Optional Stopping

- A random variable  $T : \Omega \rightarrow \{0, 1, 2, \dots\} \cup \{\infty\}$  is a **stopping time** if  $\{T \leq n\} \in \mathcal{F}_n$  for all  $n \geq 0$ .
- For a stopping time  $T$ , we set  $\mathcal{F}_T = \{A \in \mathcal{F}_\infty : A \cap \{T \leq n\} \in \mathcal{F}_n \text{ for all } n \geq 0\}$ . It is easy to check  $\mathcal{F}_T$  is indeed a  $\sigma$ -algebra and that if  $T(\omega) = n$  for all  $\omega \in \Omega$ , then  $T$  is a stopping time and  $\mathcal{F}_T = \mathcal{F}_n$ .
- Given  $X$ , define  $X_T(\omega) = X_{T(\omega)}(\omega)$  whenever  $T(\omega) < \infty$  and define the **stopped process**  $X^T$  by

$$X_n^T(\omega) = X_{T(\omega) \wedge n}(\omega) \quad \text{for } n \geq 0$$

**Proposition 2.2.1.)** Let  $X$  be an adapted process. Let  $S, T$  be stopping times for  $X$ . Then

- (a)  $S \wedge T$  is a stopping time for  $X$ .
- (b)  $\mathcal{F}_T$  is a  $\sigma$ -algebra.

- (c) If  $S \leq T$  then  $\mathcal{F}_S \subset \mathcal{F}_T$ .
- (d)  $X_T 1_{T < \infty}$  is an  $\mathcal{F}_T$ -measurable random variable.
- (e)  $X^T$  is adapted.
- (f) If  $X$  is integrable, then  $X^T$  is also integrable.

**proof)**

- (a)  $\{S \wedge T \leq n\} = \{S \leq n\} \cup \{T \leq n\} \in \mathcal{F}_n$  for all  $n \geq 0$ , so  $S \wedge T$  is a stopping times
- (b) Directly from the definition, we see that  $\phi\mathcal{F}_T$ . Also, given  $A \in \mathcal{F}_T$  and a sequence  $(A_m)_m \subset \mathcal{F}_T$ , we have

$$\begin{aligned} A^c \cap \{T \leq n\} &= \{T \leq n\} - A \cap \{T \leq n\} \in \mathcal{F}_n \Rightarrow A^c \in \mathcal{F}_T \\ (\cup_m A_m) \cap \{T \leq n\} &= \cup_m (A_m \cap \{T \leq n\}) \in \mathcal{F}_n \Rightarrow \cup_m A_m \in \mathcal{F}_T \end{aligned}$$

hence  $\mathcal{F}_T$  is a  $\sigma$ -algebra.

- (c) Let  $A \in \mathcal{F}_S$ . Then  $A \cap \{T \leq n\} = A \cap \{S \leq n\} \cap \{T \leq n\} \in \mathcal{F}_n$ , hence  $A \in \mathcal{F}_T$ .
- (d) For each  $t \in \mathbb{R}$ , we have  $\{X_T 1_T > t\} = \cup_m \{X_m > t, T = n\}$  so for any  $n \geq 0$ ,

$$\{X_T 1_T > t\} \cap \{T \leq n\} = \cup_{m=1}^n \{X_m > t, T = n\} \in \mathcal{F}_n$$

and so  $X_T 1_T$  is  $\mathcal{F}_T$ -measurable.

- (e) By definition of being a stopping time, for any  $t \in \mathbb{R}$ ,

$$\{(X^T)_n > t\} = \{T > n, X_n > t\} \cup \left( \cup_{m=0}^n \{T = m, X_m > t\} \right) \in \mathcal{F}_n$$

so  $X^T$  is adapted.

- (f) First consider the case where  $X$  is non-negative integrable. Then

$$\mathbb{E}(X_n^T) = \mathbb{E}(\mathbb{E}(X_n^T | T)) = \sum_{m \geq n} \mathbb{P}(T = m) \mathbb{E}(X_m) + \mathbb{P}(T > n) \mathbb{E}(X_n) < \infty$$

for any  $n$ , so we have the result for non-negative  $X$ .

For the general case, divide  $X$  into a non-negative and a negative part.

(End of proof)  $\square$

**Theorem 2.2.2) (Optional stopping theorem)** Let  $X$  be a super-martingale and let  $S, T$  be bounded stopping times with  $S \leq T$  a.s. Then

$$\mathbb{E}[X_T] \leq \mathbb{E}[X_S]$$

**proof)** Fix  $n \geq 0$  such that  $T \leq n$  a.s. Then

$$\begin{aligned} X_T &= X_S + \sum_{S \leq k < T} X_{k+1} - X_k \\ &= X_S + \sum_{k=0}^n (X_{k+1} - X_k) 1_{S \leq k < T} \end{aligned}$$

Now  $\{S \leq k\}$  is in  $\mathcal{F}_k$  and  $\{T > k\}$  is in  $\mathcal{F}_k$ , so

$$\begin{aligned} \mathbb{E}[(X_{k+1} - X_k) 1_{S \leq k < T}] &= \mathbb{E}[\mathbb{E}[(X_{k+1} - X_k) 1_{S \leq k < T} | \mathcal{F}_k]] \\ &= \mathbb{E}[\mathbb{E}[X_{k+1} - X_k | \mathcal{F}_k] 1_{S \leq k < T}] \end{aligned}$$

but since  $(X_n)$  was a super-martingale,  $\mathbb{E}[X_{k+1} - X_k | \mathcal{F}_k] \leq 0$  a.s. and therefore  $\mathbb{E}[(X_{k+1} - X_k) 1_{S \leq k < T}] \leq 0$  a.s. Hence  $\mathbb{E}(X_T) \leq \mathbb{E}(X_S)$ .

(End of proof)  $\square$

•Note that  $X$  is a sub-martingale *if and only if*  $(-X)$  is a super-martingale, and that  $X$  is a martingale *if and only if*  $X$  and  $(-X)$  are super-martingales. Hence, we obtain sub-martingale and martingale versions of the theorem :

$$\begin{aligned} \text{If } (X_n) \text{ is a sub-martingale, } \mathbb{E}[X_T] &\geq \mathbb{E}[X_S] \\ \text{If } (X_n) \text{ is a martingale, } \mathbb{E}[X_T] &= \mathbb{E}[X_S] \end{aligned}$$

**Theorem 2.2.3.)** Let  $X$  be an adapted integrable process. Then the followings are equivalent.

- (a)  $X$  is a super-martingale.
- (b) for all bounded stopping times  $T$  and stopping time  $S$ ,

$$\mathbb{E}(X_T|\mathcal{F}_S) \leq X_{S \wedge T} \quad \text{a.s.},$$

- (c) for all stopping times  $T$ ,  $X_T$  is a super-martingale,
- (d) for all bounded stopping times  $T$  and all stopping times  $S$  with  $S \leq T$  a.s,

$$\mathbb{E}(X_T) \leq \mathbb{E}(X_S)$$

★ The theorem gives an inverse statement of the optional stopping theorem.

**proof)**

(a)  $\Rightarrow$  (b) Suppose  $X$  is a super-martingale and  $S, T$  are stopping times. Let  $T \leq n$ , for some  $n < \infty$ . Then

$$X_T = X_{S \wedge T} + \sum_{k=0}^T (X_{k+1} - X_k) 1_{S \leq k < T} \dots \dots (*)$$

Let  $A \in \mathcal{F}_S$ . Then  $A \cap \{S \leq k\} \in \mathcal{F}_k$  and  $\{T > k\} \in \mathcal{F}_k$  so

$$\mathbb{E}[(X_{k+1} - X_k) 1_{S \leq k < T} 1_A] = \mathbb{E}[\mathbb{E}[(X_{k+1} - X_k) 1_{S \leq k < T} 1_A | \mathcal{F}_k]] \leq 0$$

and

$$\begin{aligned} \mathbb{E}[(X_T - X_{S \wedge T}) 1_A] &= \mathbb{E}\left[\sum_{n=0}^T (X_{k+1} - X_k) 1_{S \leq k < T} 1_A\right] \leq 0 \\ \Rightarrow \mathbb{E}[X_T 1_A] &\leq \mathbb{E}[X_{S \wedge T} 1_A] \end{aligned}$$

But since this inequality is true for any  $A \in \mathcal{F}_S$  and noting that  $X_{S \wedge T} \in \mathcal{F}_S$ , we see

$$\mathbb{E}[X_T | \mathcal{F}_S] \leq X_{S \wedge T} \quad \text{a.s.}$$

The inclusions (b) $\Rightarrow$ (c) and (c) $\Rightarrow$ (d) are obvious.

(d)  $\Rightarrow$  (a) Let  $m \leq n$  and  $A \in \mathcal{F}_n$ . Set  $T = m 1_A + n 1_{A^c}$ . Then  $T$  is a stopping with  $T \leq n$ . Then

$$\mathbb{E}(X_n 1_A - X_m 1_A) = \mathbb{E}(X_n) - \mathbb{E}(X_T) \leq 0$$

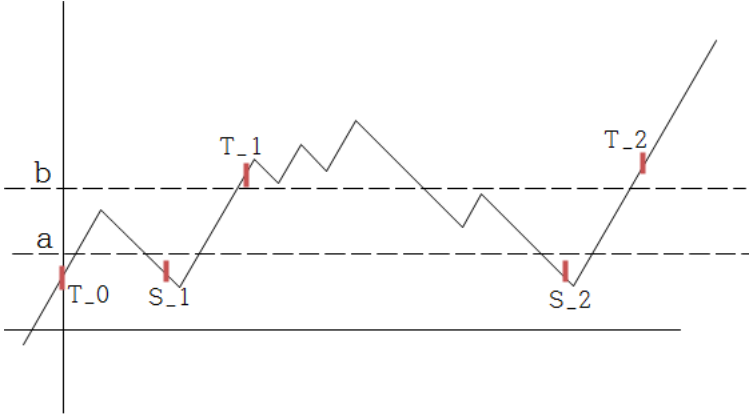
(note, if  $\omega \in A$  then  $(X_n 1_A - X_m 1_A)(\omega) = X_n(\omega) - X_m(\omega)$  and 0 otherwise) so

$$\mathbb{E}[X_n | \mathcal{F}_m] \leq X_m$$

(End of proof)  $\square$

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(17th October, Wednesday)



### 2.3. Doob's upcrossing inequality

- Let  $X$  be a random process and let  $a, b \in \mathbb{R}$  s.t.  $a < b$ . Fix  $\omega \in \Omega$ . By an **upcrossing** of  $[a, b]$  by  $X(\omega)$ , we mean an interval of times  $\{j, j+1, \dots, k\}$  s.t.  $X_j(\omega) < a$ ,  $X_k(\omega) > b$ .
- Write  $U_n[a, b](\omega)$  for the number of disjoint upcrossings contained in  $\{0, 1, \dots, n\}$ , and  $U_n[a, b] \nearrow U[a, b]$  as  $n \rightarrow \infty$ .

**Theorem 2.3.1.)** (Doob's upcrossing inequality) Let  $X$  be a *super-martingale*. Then

$$(b - a)\mathbb{E}[U[a, b]] \leq \sup_{n \geq 0} \mathbb{E}[(X_n - a)^-]$$

(Recall,  $x^- = (-x) \vee 0$ )

**proof)** Set  $T_0 = 0$  and define recursively for  $k \geq 0$ ,

$$S_{k+1} = \inf\{m \geq T_k : X_m < a\}, \quad T_{k+1} = \sup\{m \geq S_{k+1} : X_m > b\}$$

Note that if  $T_k < \infty$ , then  $\{S_k, S_k + 1, \dots, T_k\}$  is an upcrossing of  $[a, b]$  by  $X$ , and  $T_k$  is the time of completion of the  $k$ -th upcrossing. Also note that  $U_n[a, b] \leq n$ . For  $m \leq n$ , we have

$$\{U_n[a, b] = m\} = \{T_m \leq n < T_{m+1}\}$$

On this event,

$$X_{T_k \wedge n} - X_{S_k \wedge n} = \begin{cases} X_{T_k} - X_{S_k} \geq b - a & \text{if } k \leq m \\ X_n - X_{S_k} \geq X_n - a & \text{if } k = m+1, S_{m+1} \leq n \\ 0 & \text{otherwise} \end{cases}$$

Hence

$$\begin{aligned} \sum_{k=1}^n (X_{T_k \wedge n} - X_{S_k \wedge n}) &\geq (b - a)U_n[a, b] + X_n - a \\ &\geq (b - a)U_n[a, b] - (X_n - a)^- \end{aligned}$$

Since  $X$  is a super-martingale and  $T_k \wedge n$  and  $S_k \wedge n$  are *bounded stopping times* with  $S_k \leq T_k$ , by optional stopping theorem, we have

$$\mathbb{E}(X_{T_k \wedge n}) \leq \mathbb{E}(X_{S_k \wedge n})$$

By  $\mathbb{E}(\sum_{k=1}^n (X_{T_k \wedge n} - X_{S_k \wedge n}))$  we get

$$(b - a)\mathbb{E}(U_n[a, b]) \leq \sum_{n \geq 0} \mathbb{E}[(X_n - a)^-]$$

Apply monotone convergence, with  $n \rightarrow \infty$ , then we are done.

(End of proof)  $\square$

This theorem does not seem to have any significance at the moment, but it will turn out to be important later on.

## 2.4. Doob's maximal inequalities.

Define  $X_n^* = \sum_{k \geq n} |X_k|$

In the next two theorems, we see that the martingale (or sub-martingale) property allows us to obtain estimates on this  $X_n^*$  in terms of expectations for  $X_n$ .

**Theorem 2.4.1)** (Doob's maximal inequality) Let  $X$  be a *martingale* or a *non-negative sub-martingale*. Then for all  $\lambda \geq 0$ ,

$$\lambda \mathbb{P}(X_n^* \geq \lambda) \leq \mathbb{E}(|X_n| 1_{\{X_n^* \geq \lambda\}}) \leq \mathbb{E}(|X_n|)$$

**proof)** If  $X$  is a martingale, then  $|X|$  is a non-negative sub-martingale. It suffices to consider the case where  $X$  is a non-negative sub-martingale.

Set  $T = \inf\{k \geq 0 : X_k \geq \lambda\} \wedge n$ . Then  $T$  is a stopping time and  $T \leq n$ , so by optional stopping, has

$$\begin{aligned} \mathbb{E}(X_n) &\geq \mathbb{E}(X_T) = \mathbb{E}(X_T 1_{X_n^* \geq \lambda}) + \mathbb{E}(X_T 1_{X_n^* < \lambda}) \\ &= \mathbb{E}(\lambda 1_{X_n^* \geq \lambda}) + \mathbb{E}(X_n 1_{X_n^* < \lambda}) \end{aligned}$$

and

$$\mathbb{E}(X_n 1_{X_n^* \geq \lambda}) \geq \lambda \mathbb{P}(X_n^* \geq \lambda)$$

(End of proof)  $\square$

**Theorem 2.4.2)** (Doob's  $L^p$ -inequality) Let  $X$  be a *martingale* or a *non-negative sub-martingale*. Then, for all  $p > 1$  and  $q = p/(p-1)$ , we have

$$\|X_n^*\|_p \leq q \|X_n\|_q$$

**proof)** Again, it suffices to consider when  $X$  is a non-negative sub-martingale. Fix  $k < \infty$ . Then

$$\begin{aligned} \mathbb{E}[(X_n^* \wedge k)^p] &= \mathbb{E} \int_0^k p \lambda^{p-1} 1_{\{X_n^* \geq \lambda\}} d\lambda \quad (\text{integration by parts}) \\ &= \int_0^k p \lambda^{p-1} \mathbb{P}(X_n^* \geq \lambda) d\lambda \quad (\text{Fubini}) \\ &\leq \int_0^k p \lambda^{p-2} \mathbb{E}(X_n 1_{X_n^* \geq \lambda}) d\lambda \quad (\text{Doob's maximal inequality}) \\ &= \frac{p}{p-1} \mathbb{E}(X_n (X_n^* \wedge k)^{p-1}) \\ &\leq q \|X_n\|_p \|X_n^* \wedge k\|_p^{p-1} \quad (\text{H\"older's inequality}) \end{aligned}$$

Hence,  $\|X_n^* \wedge k\|_p \leq q \|X_n\|_p$ . Apply monotone convergence theorem with  $k \rightarrow \infty$ , then we have the desired result.

(End of proof)  $\square$

Doob's maximal and  $L^p$  inequalities have different versions which apply under the same hypothesis to

$$X^* = \sum_{n \geq 0} |X_n|$$

since  $X_n^* \nearrow X^*$ . Letting  $n \rightarrow \infty$  in Doob's maximal inequality gives

$$\lambda \mathbb{P}(X^* \geq \lambda) \leq \lim_{n \rightarrow \infty} \lambda \mathbb{P}(X_n^* \geq \lambda) \leq \sup_{n \geq 0} \mathbb{E}(|X_n|)$$

We can then replace  $\lambda \mathbb{P}(X^* > \lambda)$  by  $\lambda \mathbb{P}(X^* \geq \lambda)$  by taking limits from the right in  $\lambda$ .

Similarly, for  $p \in (1, \infty)$  by monotone convergence,

$$\|X^*\|_p \leq q \sup_{n \geq 0} \|X_n\|_p$$

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(19th October, Friday)

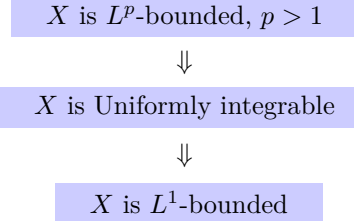
## 2.5. Doob's martingale convergence theorems

We are going to study three different martingale convergence theorems. They are all important.

- We say that a random process  $X$  is  **$L^p$ -bounded** if  $\sum_{n \geq 0} \|X_n\|_p < \infty$ .
- We say that  $X$  is **uniformly integrable** if

$$\sup_{n \geq 0} \mathbb{E}(|X_n| 1_{|X_n| > \lambda}) \rightarrow 0 \quad \text{as } \lambda \rightarrow \infty$$

- If  $X$  is  $L^p$  bounded for some  $p > 1$ , then this implies that  $X$  is uniformly integrable. This again implies that  $X$  is  $L^1$  bounded. The first implication follows from Hölder inequality. The second implication is true because  $\mathbb{E}(|X_n|) = \mathbb{E}(|X_n| 1_{|X_n| \leq \lambda}) + \mathbb{E}(|X_n| 1_{|X_n| > \lambda}) \leq \lambda + \mathbb{E}(|X_n| 1_{|X_n| > \lambda})$ .



**Theorem 2.5.1** (*Almost sure martingale convergence theorem*) Let  $X$  be an  $L^1$ -bounded super-martingale. Then there exists an integrable and  $\mathcal{F}_\infty$ -measurable random variable  $X_\infty$  such that

$$X_n \rightarrow X \quad \text{a.s. as } n \rightarrow \infty$$

**proof**) For a sequence of real numbers  $(x_n)_{n \geq 0}$ , as  $n \rightarrow \infty$ ,  $(x_n)_n$  either converges or  $|x_n| \rightarrow \infty$ , or  $\liminf_n x_n < \limsup_n x_n$ . In the last case, since the rationals are dense in  $\mathbb{R}$ , there exist  $a, b \in \mathbb{Q}$  such that  $\liminf_n x_n < a < b < \limsup_n x_n$ .

Set  $\Omega_0 = \Omega_\infty \cap (\bigcap_{a, b \in \mathbb{Q}, a < b} \Omega_{a, b})$  where  $\Omega_\infty = \{\liminf |X_n| < \infty\}$ ,  $\Omega_{a, b} = \{U[a, b] < \infty\}$  (Recall that  $U[a, b]$  is the number of upcrossings). Then  $X_n(\omega)$  converges for all  $\omega \in \Omega_0$ . By Fatous' lemma,

$$\mathbb{E}(\liminf |X_n|) \leq \liminf \mathbb{E}|X_n| < \infty$$

so this implies  $\mathbb{P}(\Omega_\infty) = 1$ . By Doob's inequality, for  $a < b$ , has

$$(b - a)\mathbb{E}(U[a, b]) \leq |a| + \sup_{n \geq 0} \mathbb{E}|X_n| < \infty$$

and therefore  $\mathbb{P}(\Omega_{a, b}) = 1$ . Putting this together, we deduce that  $\mathbb{P}(\Omega_0) = 1$ , and we can find a random variable  $X_\infty$  defined by

$$X_\infty = \lim_{n \rightarrow \infty} X_n 1_{\Omega_0}$$

Then  $X_n \rightarrow X_\infty$  a.s. Also  $X_\infty$  is  $\mathcal{F}_\infty$ -measurable and  $|X_\infty| \leq \liminf |X_n|$  so  $\mathbb{E}(|X_\infty|) < \infty$ . Hence  $X_\infty$  is integrable.

(End of proof)  $\square$

**Remark :** Every non-negative integrable super-martingale is  $L^1$ -bounded, hence it converges a.s.

**Theorem 2.5.2** ( $L^1$  martingale convergence theorem) Let  $(X_n)_{n \geq 0}$  be a uniformly integrable martingale. Then there exists a random variable  $X_\infty \in L^1(\mathcal{F}_\infty)$  such that

$$X_n \xrightarrow{n \rightarrow \infty} X_\infty \quad \text{a.s. and in } L^1$$

Moreover,  $X_n = \mathbb{E}(X_\infty | \mathcal{F}_n)$  a.s. for all  $n \geq 0$ .

Conversely, for all  $Y \in L^1(\mathcal{F}_\infty)$ , on choosing version  $X_n$  of  $\mathbb{E}(Y | \mathcal{F}_n)$  for all  $n$ , we obtain a uniformly integrable martingale  $(X_n)_{n \geq 0}$  such that

$$X_n \xrightarrow{n \rightarrow \infty} Y \quad \text{a.s. and in } L^1$$

We can think of this theorem as establishing the bijection

$$\text{unif. integrable martingale/a.s.} \leftrightarrow L^1(\mathcal{F}_\infty)$$

**proof)** Let  $(X_n)_{n \geq 0}$  be a uniformly integrable martingale. By the almost sure martingale convergence theorem, there exists  $X_\infty \in L^1(\mathcal{F}_\infty)$  s.t.  $X_n \rightarrow X_\infty$  a.s. Since  $X$  is uniformly integrable, it also follows that  $X_n \rightarrow X_\infty$  in  $L^1$ . (see PM, Thm 2.5.1. and 6.2.3.)

Next, for  $m \geq n$ ,

$$\begin{aligned} \|X_n - \mathbb{E}(X_\infty | \mathcal{F}_n)\|_1 &= \|\mathbb{E}(X_m - X_\infty | \mathcal{F}_n)\|_1 \\ &= \|X_m - X_\infty\|_1 \rightarrow 0 \quad \text{as } m \rightarrow \infty \end{aligned}$$

Hence  $X_n = \mathbb{E}(X_\infty | \mathcal{F}_n)$  a.s.

For the converse statement, suppose  $Y \in L^1(\mathcal{F}_\infty)$  and let  $X_n$  be a version of  $\mathbb{E}(Y | \mathcal{F}_n)$  for all  $n$ . Then  $(X_n)_{n \geq 0}$  is a martingale by the tower property, and is uniformly integrable by **Lemma 1.5.1**. Hence there exists  $X_\infty \in L^1(\mathcal{F}_\infty)$  such that  $X_n \rightarrow X_\infty$  a.s. and in  $L^1$ . For all  $n \geq 0$  and all  $A \in \mathcal{F}_n$ , we have

$$\mathbb{E}(X_\infty 1_A) = \lim_{m \rightarrow \infty} \mathbb{E}(X_m 1_A) = \lim_{n \leq m \rightarrow \infty} \mathbb{E}(\mathbb{E}(Y 1_A | \mathcal{F}_m)) = \mathbb{E}(Y 1_A)$$

where the second equality follows because  $\mathbb{E}(X_m | \mathcal{F}_n) = \mathbb{E}(Y | \mathcal{F}_n)$ . Now  $X_\infty, Y \in L^1(\mathcal{F}_\infty)$  and  $\cup_n \mathcal{F}_n$  is a  $\pi$ -system generating  $\mathcal{F}_\infty$ . Hence, by Dynkin's lemma,

$$X_\infty = Y \quad \text{a.s.}$$

(End of proof)  $\square$

**Theorem 2.5.3)** ( *$L^p$ -martingale convergence theorem*) Let  $p \in (1, \infty)$ . Let  $(X_n)_{n \geq 0}$  be an  $L^p$ -bounded martingale. Then there exists a random variable  $X_\infty \in L^p(\mathcal{F}_\infty)$  s.t.

$$X_n \rightarrow X_\infty \quad \text{a.s. and in } L^p$$

Moreover,  $X_n = \mathbb{E}(X_\infty | \mathcal{F}_n)$  a.s. for all  $n \geq 0$ .

Conversely, for all  $Y \in L^p(\mathcal{F}_\infty)$ , on choosing a version  $X_n$  of  $\mathbb{E}(Y | \mathcal{F}_n)$  for all  $n$ , we obtain an  $L^p$ -bounded martingale such that  $X_n \rightarrow Y$  a.s. and in  $L^p$ .

This is very similar to the statement of  $L^1$ -martingale convergence theorem. Indeed, the proof is also very similar.

**proof)** Let  $(X_n)$  be an  $L^p$ -bounded martingale. By *a.s. martingale convergence theorem*, there exists  $X_\infty \in L^1(\mathcal{F}_\infty)$ ,  $X_n \rightarrow X_\infty$  a.s.

By *Doob's  $L^p$ -inequality*,  $\|X^*\|_p \leq q \sup_{n \geq 0} \|X_n\|_p < \infty$ , where  $X^* = \sup_{n \geq 0} |X_n|$ . Also, since  $|X_n - X_\infty|^p \leq (2X^*)^p$  for all  $n$ , we may apply dominated convergence theorem to deduce that  $X_n \rightarrow X_\infty$  in  $L^p$ . Then  $X_n = \mathbb{E}(X_\infty | \mathcal{F}_n)$  a.s. for all  $n$ , as in the  $L^1$ -convergence.

For the converse statement, suppose  $Y \in L^p(\mathcal{F}_\infty)$  and let  $X_n$  be a version of  $\mathbb{E}(Y | \mathcal{F}_n)$ . Then  $(X_n)_{n \geq 0}$  is a martingale by the tower property and by Jensen inequality,

$$\|X_n\|_p = \|\mathbb{E}(Y | \mathcal{F}_n)\|_p \leq \|Y\|_p$$

Let  $X_n \rightarrow X_\infty$  a.s. and in  $L^p$  for  $X_\infty \in L^p(\mathcal{F}_\infty)$ , using the previous part. Then proceed as in the proof of  $L^1$ -convergence to prove that in fact  $Y = X_\infty$  a.s.

(End of proof)  $\square$