# Localization and Delocalization Phases for Integer-Valued GFF and relevant models

Jiwoon Park, supervised by Dr. Roland Bauerschmidt

# 1 Different mathematical models of Kosterlitz-Thouless transition

**1.1. The Kosterlitz-Thoulesss transition.** The Kosterlitz-Thouless (or KT) transition is a special family of phase transition in statistical physics context. It was first discussed theoretically by Berezinskii, Kosterlitz and Thouless in a language of physics in a series of researches including [18], [17].

There are number of famous mathematical models for describing the KT transition. The earliest model studied by Kosterlitz and Thouless was the XY model, which places each spin on $\mathbb{Z}^2$ lattice site taking value on $S^1 = \{e^{i\theta} : \theta \in [0, 2\pi)\}$ and they interact as magnetic dipoles. This model has the advantage that the continuum limit can be imagined relatively easily. That is, each point in $Q \subset \mathbb{R}^2$ is endowed with a value on $\mathbb{C}$, say $\theta : Q \to \mathbb{C}$. If we assume that $\theta$ is $C^2$ away from some isolated points, then using a pity knowledge on topology, $\theta$ can be decomposed into two parts, one created by a gradient of a scalar field and one divergence-free field. This divergence-free field accounts for the discontinuity of $\theta$ on the isolated points, and those points are called vortices. However, because it takes high cost energy in creating such discontinuity, such discontinuity can only be present in high-temperature regimes. Appearance of these vortices mixes up the long-range order of the system, so when viewed as a statistical physics context, the transition from the vortices-absent state to the vortices-present state can be thought as a phase transition. This is precisely what we call be the name of KT transition, and is also called the topological phase transition.

Seemingly related models includes the Villain model, or the plane rotator model, where the free energy $\beta \cdot \cos(\theta_i - \theta_j)$ ($i \sim j$ indicates $i$ and $j$ are adjacent) is replaced by

$$-\log\left(\sum_{m\in\mathbb{Z}} e^{-\frac{\beta}{2}(\theta_i-\theta_j-2\pi m)^2}\right)$$

and the **sine-Gordon model** where the state space is replaced to $\mathbb{R}$ and the free energy is

$$\frac{\beta}{2}(\phi(i)-\phi(j))^2 - 2\cos(2\pi\phi(i)) - 2\cos(2\pi\phi(j)).$$

These models are share similar structures, and as for the case of XY model, both the Villain model and the sine-Gordon can be studied roughly using the continuum approximation.

On the other hand, there is also the **integer-valued Gaussian free field (or IV-GFF)**, a fully discrete model, is also commonly used as a physical model for the description of KT transition. Although the discreteness of the IV-GFF makes it available for some useful combinatorial arguments, such as the Peierls' argument, one often finds the analyticity common in the other continuous models lacking in the IV-GFF, and hence making it difficult to apply rigorous analysis. Although it is reasonable to conjecture the phenomenon appearing in the other models in the thermodynamic limit to appear the same in the IV-GFF, the proofs are mostly absent.

For the integer-valued Gaussian free field. If we let $\Lambda \subset \mathbb{Z}^2$ to be a finite set, the interaction

Hamiltonian of the system with state $\phi \in \mathbb{Z}^\Lambda$ is given by

$$H_\Lambda = \frac{1}{2}\sum_{i\sim j}(\phi(i) - \phi(j))^2$$

so if the temperature is sufficiently low, unlike the usual Gaussian free field, the system resides in ensemble where the number of change in $\phi$ is limited. In this case, the states of each lattice sites are highly correlated, and is also called the **localized phase**.

In contrast, when the temperature is sufficiently high, ensembles with higher variety of state choices are preferred, so the system undergoes a phase transition. In this phase, it is expected that the integer-valued GFF will show a behaviour similar to that of a Gaussian free field. In this case, since the Green's function of Laplacian in two dimensions grows logarithmically, the correlation function between states of each lattice sites also fluctuate logarithmically. This state is called the **delocalized phase**.

Just as other phase transition problems, the analysis on the localized and the delocalized phase vary significantly. Indeed, only perturbative arguments, in temperature variable exist for the IV-GFF. But there are also non-perturbative arguments available for the sine-Gordon model, coming by the name of renormalization. In our review, we will see some preceding works on the KT transition of the sine-Gordon model in parallel to the IV-GFF, and see how these methodologies on the sine-Gordon model might be modified to apply on IV-GFF.

**1.2. The IV-GFF.** Consider $\Lambda \equiv \Lambda_N = (a_0, b_0) + [0, 1, \cdots, L^N - 1]^2$ for some $a_0, b_0 \subset \mathbb{Z}$ and $\partial\Lambda = \{(a_0 + x, b_0 + y) \in \Lambda : x \in \{0, L^N - 1\} \text{ or } y \in \{0, L^N - 1\}\}$. $a_0$ and $b_0$ will be chosen depending on the setting. Denote $E(\Lambda)$ the set of undirected edges of $\Lambda$ when $\Lambda$ is considered as a subgraph of $\mathbb{Z}^2$.

**Definition 1.1.** *The Gaussian free field, or GFF with Dirichlet boundary condition (or 0-boundary condition) and inverse-temperature $\beta$ on $\Lambda$ is given by the probability measure*

$$\mathbb{P}^{\mathrm{GFF}}(\phi \in A) = \frac{1}{Z^{\mathrm{GFF}}(\beta, \Lambda)} \int_A \prod_{i\in\Lambda} d\phi(i) \cdot e^{-\frac{\beta}{2}\sum_{i\sim j}(\phi(i)-\phi(j))^2} \cdot \prod_{j\in\partial\Lambda} \delta_0(\phi(j))$$

*for any $A \in \mathbb{R}^\Lambda$ and $Z^{\mathrm{GFF}}(\beta, \Lambda)$ the normalizing constant. The expression $\sum_{i\sim j}$ indicates the summation over pairs $\{i, j\} \subset E(\Lambda)$.*

*Similarly, the GFF with free boundary condition with base point $i_0 \in \Lambda$ is given by*

$$\mathbb{P}^{\mathrm{GFF}}(\phi \in A) = \frac{1}{Z^{\mathrm{GFF}}(\beta, \Lambda)} \int_A \prod_{i\in\Lambda} d\phi(i) \cdot e^{-\frac{\beta}{2}\sum_{i\sim j}(\phi(i)-\phi(j))^2} \cdot \mathbb{I}(\phi(i_0) \in [0, 1)).$$

*The GFF with periodic boundary condition with base point $i_0 \in \Lambda$ is defined similarly.*

*The IV-GFF (integer-valued Gaussian free field) with Dirichlet boundary condition is a random field given by conditioning the Gaussian free field with that $\phi$ is only taking integer values, i.e.*

$$\mathbb{P}^{\mathrm{IV}}(\phi \in A) = \frac{1}{Z^{\mathrm{IV}}(\beta, \Lambda)} \int_A \prod_{i\in\Lambda} \left[d\phi(i)\Big(\sum_{k\in\mathbb{Z}} \delta_k(\phi(i))\Big)\right] \cdot e^{-\frac{\beta}{2}\sum_{i\sim j}(\phi(i)-\phi(j))^2} \cdot \prod_{j\in\partial\Lambda} \delta_0(\phi(j)).$$

*The IV-GFF with free boundary condition and periodic boundary condition are defined accordingly.*

**1.3. The Coulomb Gas.** The lattice Coulomb gas model considers a system of charged particles under Coulomb potential restricted on lattice sites. We assume here that each particle has charge either $+1$ or $-1$. However, the long-ranged behaviour of the Coulomb potential at dimension 2 makes the system ill-defined under taking $|\Lambda| \to \infty$. For this reason, one often replaces the Coulomb potential by a Yukawa potential, namely

$$U(\phi) = (\phi, (-\triangle + m^2)^{-1}\phi)$$

where $\phi(i)$ is the charge at site $i$ and $(-\triangle+m^2)^{-1}$ is the inverse of $(-\triangle+m^2)$ with boundary condition as appropriate. Note that the mass term $m^2$ is introduced to make the limit $|\Lambda| \to \infty$ docile, while actually the massless choice $m^2 = 0$ gives the problem of interest in this review. So we first have to deal with the massive case in the limit and put $m^2 \to 0$.

**Definition 1.2.** *The* **Coulomb gas** *with mass $m^2$ is a random configuration*

$$\omega = (r_j^l, \alpha_j^l)_{j=1}^l \in \Omega = \bigsqcup_{l=0}^{\infty} \Omega_l = \bigsqcup_{l=0}^{\infty} \big(\Lambda \times \{-1, 1\}\big)^l$$

*of charged particles where $l$ is the number of total particles, $j$ the number of positively charged particles, $r_j^l \in \Lambda$ the lattice site and $\alpha_j^l \in \{-1, 1\}$ the selection of the charge, with probability distribution given by*

$$\mathbb{P}(\omega \in A) = \frac{1}{Z(\beta)} \int_A d\omega \cdot z(\omega) e^{-\frac{1}{2}\beta U(\omega)}$$

*where $\int_A d\omega = \sum_{l=0}^{\infty} \frac{1}{l!} \prod_{j=1}^{l} \sum_{\alpha=\pm 1} \sum_{r_j^l \in \Lambda} 1_A(\omega)$ and $z(\omega) = \prod_{j=1}^{N} z(\alpha_j^l)$ is the activity of the particles and $U(\omega)$ is the electric potential formed by the configuration $\omega$.*

The existence of KT transition of Coulomb gas model can be argued in a intuitive level. However, when the problem is set mathematically, it turns out that the actual problem is intricate. In [15] and [21], it was observed that the Coulomb gas model can alternatively expressed using random field via Fourier transform. In order to do this, first observe that, since $U(\phi) = (\phi, (-\triangle + m^2)^{-1}\phi)$ is a quadratic form, there exists a (normalized) Gaussian measure $\mu_0$ on $\mathbb{R}^\Lambda$ such that

$$e^{-\frac{1}{2}U(\phi)} = \int_{\mathbb{R}^\Lambda} d\mu_0(\xi) e^{i\langle \phi, \xi \rangle}$$

where $\langle \phi, \xi \rangle = \sum_{i \in \Lambda} \phi(i)\xi(i)$. In terms of $\omega$,

$$e^{-\frac{1}{2}\beta U(\omega)} = e^{-\frac{1}{2}U(\beta^{1/2}\phi_\omega)} = \int_{\mathbb{R}^\Lambda} d\mu_0(\xi) e^{i\beta^{1/2}\sum_{j=1}^{l} \alpha_j^l \xi(r_j^l)}$$

with $\phi_\omega(i) = \sum_{j=1}^{l} \alpha_j^l \cdot \delta_i(r_j^l)$, and therefore

$$\begin{aligned}
Z(\beta) &= \sum_{l=0}^{\infty} \frac{1}{l!} \int_{\mathbb{R}^\Lambda} d\mu_0(\xi) \cdot \prod_{j=1}^{l} \sum_{r_j^N \in \Lambda} 2z \cos(\beta^{1/2}\xi(r_j^N)) \\
&= \int_{\mathbb{R}^\Lambda} d\mu_0(\xi) \cdot e^{\sum_{r \in \Lambda} 2z \cos(\beta^{1/2}\xi(r))} \\
&= \frac{1}{\Xi} \int_{\mathbb{R}^\Lambda} d\xi \cdot e^{-\frac{1}{2}(\xi, (-\triangle + m^2)\xi)} \cdot e^{\sum_{r \in \Lambda} 2z \cos(\beta^{1/2}\xi(r))}
\end{aligned}$$

if we assume $z(\alpha)$ is constant as $z \in \mathbb{R}$. In the final line, we have used the fact that the Fourier transform of a Gaussian measure has covariance matrix that is an inverse of the original one, so $d\mu_0(\xi)$ has covariance $(-\triangle + m^2)^{-1}$. Scaling $\xi$ by factor of $\beta^{1/2}$, we conclude

$$Z(\beta) = \frac{1}{\Xi} \int_{\mathbb{R}^\Lambda} d\xi \cdot e^{-\frac{1}{2\beta}(\xi,(-\triangle+m^2)\xi)} \cdot e^{\sum_{r\in\Lambda} 2z\cos(\xi(r))}$$

It takes more effort to express certain physical quantities such as correlation function, but in the very elementary level, this indicates that the Coulomb gas model has a dual formulation stated in terms of statistical field theory.

**Definition 1.3.** *A sine-Gordon field is a random field on $\Lambda$ with probability measure*

$$\mathbb{P}(\xi \in A) = \frac{1}{\Xi(\beta)} \int_A d\xi \cdot e^{-\frac{\beta}{2}\langle\xi,(-\triangle+m^2)\xi\rangle} e^{\sum_{i\in\Lambda} z\cos(\xi(i))}$$

*for $A \subset \mathbb{R}^\Lambda$, for inverse temperature $\beta$ and activity $z/2$.*

In the current report, two results on the Coulomb gas would be introduced. The first is the work by Dimock and Hurd, which in fact dealt with a continuum Coulomb gas model, and uses a ultraviolet cutoff. This cutoff is unnecessary in our presentation because the lattice Coulomb gas model does not have ultraviolet catastrophe. The second work by Falco deals with the lattice Coulomb gas model with mass term. But this mass term turns out to be ineffective in the analysis, so the massless limit can be taken.

Put it this way, one readily sees that the final expression can be considered as a partition function of a random field $\xi$ with interaction energy $(\xi, (-\triangle + m^2)\xi)$ and a periodic potential $\sum_{r\in\Lambda} 2z\cos(\beta^{1/2}\xi(r))$. Because of this periodic potential, it is reasonable to expect that partition function would behave similarly as for the case of IV-GFF, and therefore one may also expect that common methodologies could be applied to analyze both IV-GFF and the Coulomb gas. Because of this special behaviour.

# 2 Survey of Different Problems and Approaches

In this section, I describe problems that could be of interest and make earlier attempts to describe the Kosterlitz-Thouless phase of IV-GFF sand the massless Coulomb model or the massless sine-Gordon model. Although these models share the same physics, because of the discrete foundation of IV-GFF, it is often the case that some methods that apply to the sine-Gordon model does not apply on IV-GFF without sufficient amount of modification, hence is left as a conjecture.

## 2.1. The Two Point Function.

*2.1.1. Delocalization regime.* The initial attempts for studying the Kosterlitz-Thouless transition was to prove the bounds for the two point functions in the long range order. Although these computations do not explicitly prove the existence of a phase transitions, these estimates provide useful characterization of the delocalization phase and the localization phase.

The earliest of such series of researches were set by [9] and [10]. Each paper sets the lower and the upper bound of the logarithm of Laplace transformation for IV-GFF that only varies by a factor $(1+\varepsilon)$ with sufficiently low $\beta$.

**Theorem 2.1.** *Fix $\varepsilon > 0$ and let $\beta_0(\varepsilon) > 0$ be sufficiently large, uniform over $|\Lambda|$. Let $\phi$ be the random field on $\Lambda$ with distribution of IV-GFF with inverse temperature $\beta > \beta_0(\varepsilon)$ and free boundary condition or the periodic boundary condition. Then for any $f \in \mathrm{Dom}(-\triangle_\Lambda^{-1})$,*

$$\exp\Big[\frac{1}{2(1+\varepsilon)\beta}\langle f, (-\triangle_\Lambda)^{-1} f\rangle\Big] \leqslant \mathbb{E}\big[e^{\langle f,\phi\rangle}\big] \leqslant \exp\Big[\frac{1}{2\beta}\langle f, (-\triangle_\Lambda)^{-1} f\rangle\Big].$$

Note that under the free boundary condition or the periodic boundary condition, whenever $f \in \mathrm{Dom}(\triangle_\Lambda^{-1})$, for any $c \in \mathbb{R}$,

$$\langle (-\triangle_\Lambda)^{-1} f, g\rangle = -\langle f, \triangle_\Lambda c\rangle = 0$$

so this implies $\sum_{r\in\Lambda} f(r) = 0$. Conversely, whenever $\sum_{r\in\Lambda} f(r) = 0$, the discrete Poisson equation

$$-\triangle_\Lambda g = f$$

can be solved, e.g. for the periodic boundary condition, the discrete Fourier transformation can be used and for the free boundary condition, the variational problem $\min_\phi \frac{1}{2}|\nabla\phi|^2 + \langle f, \phi\rangle$ can be solved to obtain the solution. Therefore, for both boundary conditions, we have

$$\mathrm{Dom}(-\triangle_\Lambda^{-1}) = \{f : \Lambda \to \mathbb{R} : \sum_{r\in\Lambda} f(r) = 0\}.$$

To see how this result implies the bound for two-point function, use the variational argument as follows. Inserting $\delta f$ in place of $f$, we have in the limit $\delta \to 0$

$$\mathbb{E}[e^{\langle \delta f,\phi\rangle}] = \mathbb{E}[1 + \delta\langle f,\phi\rangle + \frac{1}{2}(\langle f,\phi\rangle)^2 + O(\delta^3)]$$

while the bounds are manipulated as

$$1 + \delta^2\frac{1}{2(1+\varepsilon)\beta}\langle f, (-\triangle_\Lambda)^{-1} f\rangle + O(\delta^3) \leqslant \mathbb{E}[e^{\delta\langle f,\phi\rangle}] \leqslant 1 + \delta^2\frac{1}{2\beta}\langle f, (-\triangle_\Lambda)^{-1} f\rangle + O(\delta^3).$$

The inequality in $O(\delta)$ order gives $\mathbb{E}[\langle f,\phi\rangle] = 0$ and $O(\delta^2)$ order gives

$$\frac{1}{(1+\varepsilon)\beta}\langle f, (-\triangle_\Lambda)^{-1} f\rangle \leqslant \mathbb{E}[(\langle f,\phi\rangle)^2] \leqslant \frac{1}{\beta}\langle f, (-\triangle_\Lambda)^{-1} f\rangle.$$

Hence when $f$ is set as $f = \delta_x - \delta_y \in \mathrm{Dom}(-\triangle_\Lambda^{-1})$, we have the bound for $\mathbb{E}[(\phi(x) - \phi(y))^2]$.

The same sort of bound can be obtained for the Coulomb gas model by the inspection of the method that will be studied in **Section 4**. In fact, this method was refined in [19] this bound can be computed up to $\beta > 8\pi$, which is the conjectured critical temperature. In this result, the Fourier transform is discussed instead of the Laplace transform.

**Theorem 2.2.** *Fix $\beta > 8\pi$. Then there exists $z_0(\beta) > 0$ and $\eta > 0$ such that whenever $|z| < z_0(\beta)$, $\omega = (r_i, \alpha_i)_{i=1}^l$ the Coulomb gas configuration with activity $z$ and inverse temperature $\beta$, and $\phi(x) = \sum_{i=1}^l \delta_x(r_i)\alpha_i$, the bound*

$$\mathbb{E}[e^{i(\phi(x)-\phi(0))\cdot\xi}] \leqslant C \cdot \|x\|_2^{-\eta\cdot\mathrm{dist}(\xi,\mathbb{Z})}$$

*hold for a constant $C > 0$ dependent on $\beta$, $z$ and $\xi$.*

*2.1.2. Localization regime.* In the early study [10], it was also observed that the variance bound of the two-point function can also be studied using Peierls' argument. The Peierls' argument was initially developed by [20] to study the Ising model and turns out to be useful in studying many discrete statistical physics problems perturbatively. The result states the following.

**Theorem 2.3.** *There exists $\beta^+ > 0$ and $C > 0$ uniform in $|\Lambda|$ such that the following holds for any $\beta > \beta^+$. Let $\phi$ be the random field on $\Lambda$ with distribution of IV-GFF with inverse temperature $\beta > \beta_0(\varepsilon)$, with any boundary condition. Then*

$$\mathbb{E}[\phi(x)^2] \leqslant \frac{C}{\beta}.$$

A similar sort of results turns out to be true for the Coulomb gas. This was first seen by [4] in 3-dimensional system and was later proved to be true by [23] in 2-dimensions using Peierls' argument and expansion methods, including the Mayer expansion and the Glimm-Jaffe expansion. This result is called by the name Debye screening.

**Theorem 2.4.** *For any $m_0 > 0$, there exists $C > 0$, $\lambda_0 > 0$ and $\beta(\cdot) : (0, \lambda_0) \to \mathbb{R}_+$ such that the follow holds. Let $\omega = (r_i, \alpha_i)_{i=1}^l$ be the random configuration of Coulomb gas with mass $\lambda^2 m_0^2$ For $\lambda > \lambda_0$, inverse temperature $\beta < \beta(\lambda)$ and activity $z > 0$. Let $\phi(x) = \sum_{i=1}^l \delta_x(r_i)\alpha_i$ be the charge at point $x \in \Lambda$. Then the limit of $\mathbb{E}[\phi(x)]$ and $\mathbb{E}[\phi(x)\phi(y)]$ exist under taking $|\Lambda| \to \infty$ and satisfies the bound*

$$\big|\,\mathrm{Cov}[\phi(x), \phi(y)]\,\big| \leqslant C \cdot z \exp\big(-\frac{1}{2} m_0 \|x-y\|_2\big)$$

*holds uniformly in $|\Lambda|$.*

**2.2. The scaling limit.** Recalling that $\mathbb{E}^{\mathrm{GFF}}[e^{\langle f,\phi\rangle}] = \exp\big(\frac{1}{2}\langle f, (-\triangle)^{-1} f\rangle\big)$, the result of **Theorem 2.2** suggests that under some appropriate limit, IV-GFF would be approximately be a GFF. Indeed, it had been conjectured in [16] that with sufficiently small $\beta > 0$, the long-ranged behaviour of the two-dimensional IV-GFF will converge to a Gaussian free field.

However, it is not clear from the start how the limit should be understood. The bound on the Laplace transform or the Fourier transform as was dealt by Fröhlich and Spencer or Marchetti and Klein only conveys limited amount of information about the random field. To correctly take account into all available information in the large length scale, one needs to understand the limit in the context of renormalization.

For the development of the theory, let us think of the Coulomb gas model. Suppose one has a decomposition of the Yukawa potential into scales by

$$(-\triangle + m^2)_\Lambda^{-1} = \sum_{j=1}^{N} \Gamma_j(\,\cdot\,; m).$$

so that $\Gamma_j$ only includes the information about the interaction on scale $j$. Is is not trivial if such choice of $\Gamma_j$ exists, and this would be discussed in more detail later on. So if $\mathbb{P}_j$ is a Gaussian measure on $\mathbb{R}^\Lambda$ with covariance $\Gamma_j$, one has the representation

$$Z(\beta) = \int_{(\mathbb{R}^\Lambda)^N} \prod_{j=1} d\mathbb{P}_j(\phi_j) \exp(-V(\sum_{j=1}^{N} \phi_j))$$

where

$$V(\phi) = -\sum_{r\in\Lambda} 2z\cos(\beta^{1/2}\phi(r))$$

comes from the sine-Gordon transformation. This is equivalent to having $Z(\beta) = Z_N(0;\beta)$ where $Z_j$'s are define inductively by

$$Z_j(\phi_j;\beta) = e^{-V_j(\phi_j)} = \mathbb{E}_j^{\zeta_j}\left[e^{-V_{j-1}(\phi_j+\zeta_j)}\right], \quad V_0 = V$$

where takes average over $\zeta_j$. We can think of each operation $\mathbb{E}_j^{\zeta_j}$, or the renormalization map, to average out the information of the field on the scale $j$, and we are only left with the behaviour of the field at the length scale larger or equal to $j+1$. Hence if we first take $|\Lambda| \to \infty$, and see the limiting behaviour of $V_j$ as $j \to \infty$, one can identify the limit of the long-range fluctuations. However, since the space of possible functions for $V_j$'s can not be parametrized by finite family of variables, setting the notion of convergence would be one of the most important a priori work to obtain convergence of desired physical quantities.

**2.3. Near-Critical Phenomenon.** As was mentioned earlier, most of the results discussed so far had only been perturbative, and hence fails to analyse the behaviour of the systems near the criticality, both for the localization and the delocalization regime. The analysis of the near-critical phenomenon is one of the toughest problems for the study of Kosterlitz-Thouless phase transition. In general, not even the sharpness of the KT transition is proved for the presented models. Here we state some quantitative and qualitative conjectures that characterises the KT transition, although they should be considered as a long-time goal for the time being.

In the following, we conjecture the existence of the sharp transition at $\beta_{\mathrm{IV}}$ for the integer-valued GFF and $\beta_c(0) = 8\pi$ for the Coulomb gas model when activity $z$ is 0. By sine-Gordon transformation, this automatically implies sharp transition of sine-Gordon model at $(\beta_c(0))^{-1}$ when activity is 0.

*2.3.1. Delocalization regime.* The near-critical phenomenon for the delocalization regime had been reasonably well studied for the Coulomb gas by the two papers of Falco, [7], [8].

**Theorem 2.5.** *Given $\eta = 1/2$ (the charge of the augmented particles), there is $L_0 \equiv L_0(z) > 1$, $z_0 \equiv z_0(\eta) > 0$ and a continuous function $\beta_c(z) \geqslant 8\pi$ such that if $L \geqslant L_0$, $|z| < z_0$ and $\beta = \beta_c(z)$, the limit of the two-point energy with two particles with fractional charge $\pm\eta$ augmented exists and the limit is*

$$\rho_{1/2}(x) = \frac{1}{2}\frac{e^{2\pi c_E} + f_a}{|x|}(1 + f \log|x|)^{1/2}(1 + o(1))$$

*where $f = c|z|$, $c > 0$, $f_b = c(\eta)^2 z^2(1 + \tilde{f}_b)$, $c(\eta) > 0$, $f_a, \tilde{f}_b \to 0$ as $z \to 0$.*

We would like to conjecture the same for the IV-GFF. But since we only have a one parameter for the IV-GFF, it doesn't hold any equivalent version of the statement. Instaed, we think of a long-ranged interaction Laplacian defined by

$$\triangle_\rho f(x) = \frac{1}{|B_\rho(0)|}\sum_{|y-x|\leqslant\rho}(f(y) - f(x))$$

and extend the definition of IV-GFF to include parameter $\rho$ whenever the ordinary lattice Laplacian is replaced by $\triangle_\rho$.

**Conjecture 2.6.** *There exists $\beta(\rho)$ defined such that $\beta(1) = \beta_{\mathrm{IV}}$ and the infrared limit of the IV-GFF with $\rho$-ranged Laplacian is a GFF for $\beta \leqslant \beta(\rho)$.*

*2.3.2. Localization regime.* Denote $\xi \equiv \xi(\beta, z)$ for the correlation length of IV-GFF for $\beta > \beta_{\mathrm{IV}}$. The asymptotic form of these correlation function near the critical temperature for the XY model had been argued physically in [17]. If this argument can be also applied for IV-GFF, then this implies

**Conjecture 2.7.** *For each $\beta_0 > \beta_{iv}$, there exists $z_0(\beta_0) > 0$ and $c \equiv c(z)$ such that*

$$\xi \sim \exp\left[c(z) \cdot |\beta - \beta_{\mathrm{IV}}(z)|^{1/2}\right].$$

*whenever $\beta \in (\beta_{\mathrm{IV}}, \beta_0)$ and $|z| < z_0(\beta_0)$.*

This suggests that the order of KT phase transition is infinite in Ehrenfest's phase transition classification. However, there is no work so far that mathematically studies the localization regime near the critical temperature. The greatest obstacle for studying the localization regime non-perturbatively is that the the renormalization flow does not converge to a trivial fixed point in this setting. Furthermore, in contrast to the case of delocalization regime, the limiting distribution would be significantly different for each distribution, which would add the complication when one tries to adapt methodology on one model to the other.

# 3 Peierls' argument

In this section, we see how the Peierl's argument applies in the IV-GFF when the inverse temperature $\beta$ is sufficiently large. The ultimate goal should be proving **Theorem 2.3**, but we look at a seemingly easier problem here for simplicity. Trivial modifications of the statement and the proof would yield **Theorem 2.3**. We fix $\phi$ for denoting a IV-GFF, and assume 0-Dirichlet boundary. We prove a bound on the average proportion of sites with even pairity that is uniform on the lattice size $|\Lambda_N| = L^N$, so make the symbol $N$ explicit. We adopt the formulation of [12]. See [13] for a more detailed account and [11] for an application of Peierls' argument on GFF.

Our aim in this section is the following :

**Theorem 3.1.** *Let $\phi$ be an IV-GFF on $\Lambda$ with inverse-temperature $\beta$ and 0-boundary condition. Consider $V(\phi, \Lambda) = \{x \in \Lambda : e^{i\pi\phi(x)} = 1\}$. Then for each $\varepsilon > 0$, there exists $\beta_0(\varepsilon)$ such that whenever $\beta \geqslant \beta_0(\varepsilon)$, we have*

$$\mathbb{E}\Big[\big|\, V(\phi, \Lambda) \,\big|\Big] \geqslant (1 - \varepsilon)|\Lambda|.$$

The strategy is to prove that whenever there is a point $x \in |\Lambda$ with $e^{i\pi\phi(x)} = -1$, the connected region of $\Lambda$ surrounding $x$ that takes $e^{i\pi\phi(\cdot)} = -1$ can not be too large. To prove this, we need some definitions.

**Definition 3.2.** *Let $V(\phi, \Lambda) = \{x \in \Lambda : e^{i\pi\phi(x)} = 1\}$ and $W(\phi, \Lambda) = \{x \in \Lambda : e^{i\pi\phi(x)} = -1\}$. Denote*

$$\begin{aligned}
&\partial W = \{(i, j) \in E(\Lambda) : i \in W(\phi, \Lambda),\ j \in V(\phi, \Lambda)\},\\
&O^-(i) = \text{connected component of } W(\phi, \Lambda) \text{ contianing } i, \quad i \in \Lambda,\\
&\partial O^-(i) = \text{the boundary of } O^-(i).
\end{aligned}$$

**Lemma 3.3.** *Let $\gamma$ be a closed boundary, i.e. a collection of edges that encloses a region. Then*

$$p(\gamma) := \mathbb{P}(\gamma \in \partial O^-(x) \text{ for some } x \in \Lambda) \leqslant e^{-\frac{1}{2}\beta|\gamma|}$$

*Proof.* The proof just takes account of the energy cost of forming a phase boundary exactly at $\gamma$. Since a phase boundary involves odd number of state change on $\gamma$, the energy cost is at least $\frac{1}{2}\beta|\gamma|$, and hence the result.

□

Having the lemma, the theorem is reduced to a simple counting argument. We consider a particular site $i \in \Lambda$ and let

$$p(i) = \mathbb{P}(\phi(i) \in W(\phi, \Lambda)).$$

Then $p(i)$ is just the probability that there exists a boundary $\gamma$ of some connected component of $W(\phi, \Lambda)$ that encloses site $i$, hence we may write

$$p(i) \leqslant \frac{\sum_\gamma \sum_{\partial W : \gamma \subset \partial W} \sum_{\phi \in \mathbb{Z}^\Lambda : \partial W = \partial W(\phi)} e^{-H(\phi)}}{\sum_{\partial W} \sum_{\phi \in \mathbb{Z}^\Lambda : \partial W = \partial W(\phi)} e^{-H(\phi)}} = \sum_\gamma p(\gamma)$$

where $H(\phi) = \frac{\beta}{2}\langle\phi, (-\triangle)\phi\rangle$, $\sum_\gamma$ runs over the choice of $\gamma$ that encloses the site $i$, $\sum_{\partial W:\gamma\subset\partial W}$ running over all configurations of $\partial W$ such that $\gamma \subset \partial W$ and $\sum_{\phi\in\mathbb{Z}^\Lambda:\partial W=\partial W(\phi)}$ running over all IV-GFF configurations such that the given configuration $\partial W$ matches the configuration $\partial W(\phi, \Lambda)$ given by $\phi$. But by the lemma, the latter quantity is bounded by $e^{-\frac{1}{2}\beta|\gamma|}$ and therefore

$$p(i) \leqslant \sum_\gamma e^{-\frac{1}{2}\beta|\gamma|} = \sum_{r=4}^{\infty} \sum_{\gamma:|\gamma|=r} e^{-\frac{1}{2}\beta r}$$

But the number of configuration of $\gamma$ of length $r$ that encloses the site $i$ is bounded by $r \times r^2$ (the first $r$ is the number of available starting sites and $r^2$ accounts for the number of freedom where $\gamma$ can travel), we conclude

$$p(i) \leqslant \sum_{r=4}^{\infty} e^{-\frac{1}{2}\beta r} \cdot r^3$$

and with sufficiently large $\beta$, this is bounded by $e^{-\beta}$. Therefore

$$\frac{|\Lambda| - \mathbb{E}[|V(\phi, \Lambda)|]}{|\Lambda|} \leqslant e^{-\beta}$$

uniformly in $\Lambda$, proving **Theorem 3.1**.

# 4 Fröhlich-Spencer method

In this section, we see the idea behind the proof of **Theorem 2.2**. The details on the analysis will not be discussed with detail, but we rather focus on the form of the expressions that should are expected.

**4.1. The lower bound.** The Fröhlich-Spencer method uses a special expansion of the Laplace transform of a given function $f : \Lambda \to \mathbb{R}$ with measure $\mathbb{P}^{\mathrm{IV}}$ to estimate the correlation function. In this section, for the sake of simplicity, assume the free boundary condition with base point $x_0 \in \Lambda$. In particular, the constant functions are the only harmonic functions and the orthogonal complement of the harmonic functions is

$$\mathrm{Harm}(\Lambda)^\perp = \{f : \Lambda \to \mathbb{R} \big| \sum_{i\in\Lambda} f(i) = 0\}$$

and therefore a function $f$ has a solution for $(-\triangle)\sigma = f$ with free boundary condition if and only if $f \in \mathrm{Harm}(\Lambda)^\perp$. In the following argument, we will be dealing with $f \in \mathrm{Harm}(\Lambda)^\perp$.

Denote $\sigma = (-\triangle)^{-1} f$ (with $\sigma(x_0) = 0$). First observation is to consider the formal equality

$$\sum_{k\in\mathbb{Z}} \delta_k(\phi(i)) = \frac{1}{2\pi} \sum_{k\in\mathbb{Z}} \cos(2\pi k\phi(i)).$$

So if we let

$$\lambda(\phi) = \prod_{i\in\Lambda} \lambda_i(\phi(i)) = \prod_{i\in\Lambda} \frac{1}{2\pi} \sum_{k\in\mathbb{Z}} \cos(2\pi\phi(i))$$

we may write the Laplace transformation of $f \in \mathrm{Harm}(\Lambda)^\perp$ as

$$\begin{aligned}
\mathbb{E}^{\mathrm{IV}}[e^{\langle f,\phi\rangle}] &= \frac{1}{Z^{\mathrm{IV}}} \int_{\mathbb{R}^\Lambda} \prod_{i\in\Lambda} d\phi(i) \cdot \lambda(\phi) e^{-\frac{\beta}{2}\langle \phi,(-\triangle)\phi\rangle} \cdot 1_{[0,1)}(\phi(j)) \cdot e^{\langle \phi,(-\triangle)\sigma\rangle} \\
&= \frac{e^{\frac{1}{2\beta}\langle \sigma,(-\triangle)\sigma\rangle}}{Z^{IV}} \int_{\mathbb{R}^\Lambda} d\phi \cdot \lambda(\phi+\beta^{-1}\sigma) e^{-\frac{\beta}{2}\langle \phi,(-\triangle)\phi\rangle} 1_{[0,1)}(\phi(j))
\end{aligned}$$

where the second equality uses the change of variable $\phi \mapsto \phi + \beta^{-1}\sigma$. If we let

$$Z^\lambda(\sigma) = \int_{\mathbb{R}^\Lambda} d\phi \cdot \lambda(\phi+\beta^{-1}\sigma) e^{-\frac{\beta}{2}\langle \phi,(-\triangle)\phi\rangle} 1_{[0,1)}(\phi(j)),$$

by taking $f \equiv 0$, one observes that $Z^{\mathrm{IV}}(0) = Z^{\mathrm{IV}}$ and we get the expression

$$\mathbb{E}^{\mathrm{IV}}[e^{\langle f,\phi\rangle}] = \exp\big(\frac{1}{2\beta}\langle f,(-\triangle)^{-1}f\rangle\big)\frac{Z^\lambda(\sigma)}{Z^\lambda(0)}$$

At this point, fix $K \in \mathbb{N}$ and truncate the sum in $\lambda_i(\phi)$ as

$$\lambda_i(\phi(i)) = \frac{1}{2\pi} \sum_{|k|\leqslant K} \cos(2\pi\phi(i))$$

If we can prove an estimate for $\mathbb{E}^{\mathrm{IV}}(e^{\langle f,\phi\rangle})]$ that is uniform in $K$, we can deduce that the same estimate holds when $K$ tends to $\infty$. Now the lower bound is proved modulo the following proposition.

**Proposition 4.1.** *For any $\varepsilon > 0$, there is $\beta_0 \equiv \beta_0(\varepsilon) > 0$, uniform in $|\Lambda|$ such that whenever $\beta < \beta_0$,*

$$\frac{Z^\lambda(\sigma)}{Z^\lambda(0)} \geqslant \exp\big(\frac{\varepsilon}{2\beta(1+\varepsilon)}\langle f,(-\triangle)^{-1}f\rangle\big).$$

For the proof of the proposition, we consider a sort of inverse sine-Gordon transformation so that $Z^\sigma$ can be expressed using a linear combination of function of charge configurations, so that a lower bound can be obtained effectively. For example, suppose we truncate $\lambda_i$ as

$$2\pi\bar{\lambda}_i(\phi(i)) = \sum_{|q|\leqslant 2} \cos(2\pi q\phi(i))$$

Then, for weights $w_1, w_2$ with $w_1 + w_2 = 1$,

$$\begin{aligned}
(2\pi)^{|\Lambda|}\bar{\lambda}(\phi(i)) &= \prod_{j\in\Lambda} \sum_{q=1,2} w_q\big(1 + \frac{2}{w_q}\cos(2\pi q\phi(j))\big) \\
&= \sum_{Q:\Lambda\to\{1,2\}} w_Q \prod_{j\in\Lambda} \big(1 + \frac{2}{w_{Q(j)}}\cos(2\pi Q(j)\phi(j))\big)
\end{aligned}$$

where $w_Q = \prod_{j\in\Lambda} w_{Q(j)}$. Based on the same principle, writing

$$\lambda_j(\phi(j)) = \sum_{q=1}^{K} \frac{e^{-q^2}}{C(K)}\big(1 + 2C(K)e^{q^2}\cos(2\pi q\phi(i))\big)$$

with $C(K) = \sum_{q=1}^{K} e^{-q^2}$, we have

$$\lambda(\phi) = \prod_{j\in\Lambda} \lambda_i(\phi(i)) = \sum_{Q:\Lambda\to\{1,\cdots,K\}} w_Q \prod_{j\in\Lambda}\Big(1 + \frac{2}{w_{Q(j)}}\cos(2\pi Q(j)\phi(j))\Big)$$

with $w_Q, w_{Q)(j)}$ as appropriate. (In fact, it is helpful to choose $K$ depending on the value of $i$ for the sake of analysis. But we do not make this explicit here). Now use the fact that the product $\prod_{i\in\Lambda}(1+\gamma_i\cos(2\pi Q(i)\phi(i)))$ can be written as a linear combination

$$\sum_{\mathcal{Q}\in N} c_{\mathcal{Q}} \prod_{Q\in\mathcal{Q}} (1 + \gamma(Q)\cos(2\pi\langle\phi, Q\rangle)).$$

- this just relies on the fact that, for charge configurations $Q_1$ and $Q_2$,

$$\begin{aligned}
&(1+\gamma_1\cos(2\pi\langle\phi,Q_1\rangle)(1+\gamma_2\cos(2\pi\langle\phi,Q_2\rangle) \\
=&\frac{1}{3}(1+3\gamma_1\cos(2\pi\langle\phi,Q_1\rangle)) + \frac{1}{3}(1+3\gamma_2\cos(2\pi\langle\phi,Q_2\rangle)) \\
&+\frac{1}{6}(1+3\gamma_1\gamma_2\cos(2\pi\langle\phi,Q_1-Q_2\rangle)) + \frac{1}{6}(1+3\gamma_1\gamma_2\cos(2\pi\langle\phi,Q_1+Q_2\rangle))
\end{aligned}$$

and applying this repeatedly. This makes us able to write $Z^\lambda(\sigma)$ as a linear combination of partition functions on charge configurations. However, this form is not most apt for obtaining an estimate for $Z^\lambda(\sigma)$, because the term $\gamma(Q)\cos(2\pi\langle\phi,Q\rangle)$ might exceed 1. Instead, one pursues this expression but make $|\gamma(Q)| \ll 1$ and each $Q$ is neutral. Obtaining the neutrality is not that complicated, because we could have chosen most of the $Q$ to be neutral in the first place. For rescaling the coefficients, the cost to pay is to generate an artificial **spin wave** on the system so that upon adding the spin waves on the original field, the change of Gaussian measure makes each coefficient contract. As a result, one may obtain the expression

$$Z^\lambda(\sigma) = \sum_{N\in\mathscr{N}} c_N Z_N^\lambda(\sigma)$$

for $\sum_{N\in\mathcal{N}} c_N = 1$, $N$ an ensemble of neutral charge configuration, $\mathscr{N}$ a finite collection, and

$$Z_N(\sigma) = \int \prod_{\rho\in N}(1 + z(\rho,N)\cos(\langle\phi,\bar\rho\rangle + \langle\sigma+\bar\sigma(\rho,N),\rho\rangle))d\mathbb{P}^{\mathrm{GFF}}(\phi).$$

Here $\bar\sigma(\rho,N)$ is the spin wave that had been chosen to suppress the coefficient of $\cos(\langle\phi,\rho\rangle)$, and the bound on $|z(\rho,N)|$ that can be suppressed as much as it is desired upon the choice of $\bar\sigma(\rho,N)$.

Finally, we use the inequalities, for $z$ and $\beta$ sufficiently small,

$$\begin{aligned}
&(1)\ :\ 1+z\cos(x+y)\geqslant \exp\Big[-\frac{z\sin x\sin y}{1+z\cos x}-c|z|y^2\Big](1+z\cos x)\\
&(2)\ :\ \sum_{\rho\in N}|z(\rho,N)|\cdot\langle\sigma,\rho\rangle^2\leqslant\frac{\beta}{c_2}\langle\sigma,(-\triangle)\sigma\rangle
\end{aligned}$$

to conclude the proof. That is, by (1), we have

$$\frac{Z_N(\sigma)}{Z_N(0)}\geqslant\exp(-c\sum_{Q\in N}|z(\rho,N)|(\langle\sigma,Q\rangle)^2)\int e^{S(\phi,N)}(1+z(\rho,N)\cos(\langle\phi,Q+\bar\sigma(\rho,N)\rangle))d\mathbb{P}^{\mathrm{GFF}}(\phi)/Z_N(0)$$

where

$$S(\phi,N)=-\sum_{Q\in N}\frac{z(\rho,N)\sin(\langle\phi,Q+\bar\sigma\rangle)\sin(\langle\sigma,Q\rangle)}{1+z(\rho,N)\cos(\langle\phi,Q+\bar\sigma\rangle)}$$

But then we see that $(1+z(\rho,N)\cos(\langle\phi,Q+\bar\sigma(\rho,N)\langle\rangle)d\mathbb{P}^{\mathrm{GFF}}(\phi)/Z_N(0)$ is a probability measure, which we denote by $d\mathbb{P}^{tilt}(\phi)$. If fact, the entire point of using (1) was to obtain this probability measure. Then Jensen's inequality gives

$$\begin{aligned}
\frac{Z_N(\sigma)}{Z_N(0)}&\geqslant\exp(-c\sum_{Q\in N}|z(\rho,N)|(\langle\sigma,Q\rangle)^2)\\
&\geqslant\exp\big(-\frac{\varepsilon\beta}{2(1+\varepsilon)}\langle\sigma,(-\triangle)\sigma\rangle\big)\\
&=\exp\big(-\frac{\varepsilon\beta}{2(1+\varepsilon)}\langle f,(-\triangle)^{-1}f\rangle\big)
\end{aligned}$$

where the first inequality follows because $\int S(\phi,N)d\mathbb{P}^{tilt}(\phi)=0$ using $S(\phi,N)=-S(-\phi,N)$ and $d\mathbb{P}^{tilt}(\phi)=d\mathbb{P}^{tilt}(-\phi)$, and the second inequality follows from (2).

**4.2. The upper bound.** The strategy for obtaining the upper bound is to approximate the IV-GFF with a sequence of sine-Gordon measures with increasing activities. Define $d\mathbb{P}^{SG}(\phi;z,\beta)$ to be the probability distribution of a sine-Gordon field with inverse temperature $\beta$ and activity $z$. Then we get the limit

$$\lim_{z\to\infty}\mathbb{E}^{SG}(\phi;z,\beta/4\pi^2)[e^{\langle\phi,f\rangle/2\pi}]=\mathbb{E}^{\mathrm{IV}}(e^{\langle\phi,f\rangle}).$$

Now observe that

$$\frac{d}{dz}\mathbb{E}^{SG}[e^{\langle\phi,f\rangle};z,\beta]=\sum_{j\in\Lambda}\mathbb{E}[e^{\langle\phi,f\rangle}(\cos\phi(j)-\cos\psi(j))]$$

where $\phi,\psi$ are independent sine-Gordon fields. Under the change of variable $\chi=(\phi+\psi)/\sqrt{2}$ and $\xi=(\phi-\psi)/\sqrt{2}$, $\chi$, $\xi$ are again independent sine-Gordon fields, and the the Taylor expansion of $e^{\langle \phi, f\rangle}\big(\cos(\phi(j)) - \cos(\psi(j))\big)$ about $\cos\chi(j)$ and $\cos\xi(j)$ gives

$$\begin{aligned}\frac{d}{dz}\mathbb{E}^{SG}[e^{\langle \phi, f\rangle}; z, \beta] &= -2\sum_{j\in\Lambda}\sum_{Q:\Lambda\to\mathbb{Z}_+}\mathbb{E}\Big[F(\chi; j, Q)F(\xi; j, Q)\Big] \\ &= -2\sum_{j\in\Lambda}\sum_{Q:\Lambda\to\mathbb{Z}_+}\mathbb{E}\Big[F(\chi; j, Q)\Big]^2 \leqslant 0.\end{aligned}$$

for

$$F(\chi; j, Q) = \frac{1}{\Xi(z, \beta)}e^{\langle \chi, f\rangle}\sin\Big(\frac{\chi(j)}{\sqrt{2}}\Big)\prod_{j\in\Lambda}\frac{\big(\sqrt{2z}\cos(\chi(j)/\sqrt{2})\big)^{Q(j)}}{\sqrt{Q(j)!}}.$$

Therefore $\mathbb{E}^{SG}[e^{\langle \phi, f\rangle}; z, \beta]$ is a decreasing function of $z$, and using $d\mathbb{P}^{SG}(\cdot; 0, \beta) = d\mathbb{P}^{\mathrm{GFF}}(\cdot; \beta)$, we conclude that

$$\mathbb{E}^{\mathrm{IV}}[e^{\langle f, \phi\rangle}] \leqslant \mathbb{E}^{\mathrm{GFF}}[e^{\langle f, \phi\rangle}].$$

# 5 Renormalization methods for analysis of delocalized phase for sine-Gordon model

Since the invention of renormalization group, or the RG method, there had been various attempts to attack fundamental problems of statistical physics and quantum physics using RG. Among these, [3] was the first attempt to formulate the long-range behaviour of the dipole gas in the low temperature regime. The dipole gas has its analogue of sine-Gordon transformation, and the renormalization had been applied on high temperature regime for the sine-Gordon model, and the method can be transferred to the Coulomb gas model. The Brydges-Yau method uses large field and large set regulators so that does not have to separate out the large-field and small-field region. Along with a abstract form of polymer expansion, the method simplifies the RG step significantly, and thus provides a basis for proving accurate long-range behaviour of the sine-Gordon model.

On the other hand, this method also holds clear disadvantages in that it is hard to deduce any other physical results apart from the pressure as a function of the activity $z$. The most successful alternative RG method was that of [7], [8], which provides a critical line for KT transition as a function of $z$ and the aysmptotic form of the correlation function on the critical line.

In this section, I will make a brief overview of what the RG method is, and the following two sections will discuss about two methods mentioned above in more detail. Both the methods work for the Coulomb gas model, and uses two distinct forms of so called the polymer expansion. **Polymers** are defined to be finite collections of boxes in $\mathcal{B}_j$ where we let

$$\begin{aligned}&\Lambda \equiv \Lambda_N = [-(L^N-1)/2, (L^N-1)/2]^2 \cap \mathbb{Z}^2 \\ &B_j(0) = [-(L^j-1)/2, (L^j-1)/2]^2 \cap \mathbb{Z}^2 \\ &\mathcal{B}_j = \{(aL^j, bL^j) + B_j(0) \subset \Lambda_N : a, b \in \mathbb{Z}\}\end{aligned}$$

for $L$ odd, and for each length scale $j \leqslant N$. We denote $\mathcal{P}_j$ for the set of polymers and a **polymer activity** is defined to be the functions of $\mathcal{P}_j$ and the field on $\mathcal{P}_j$. Each of the work requires a great deal of foundational work, which are followed by an algebra and analysis of polymer expansions.

Also note that because many estimates used in renormalzation method are vastly simplified with translational symmetry, we assume the **periodic boundary condition**.

**5.1. Poorly done Renormalization.** In the sine-Gordon model, because of the analyticity of the Hamiltonian in $z$, the renormalization can be done in any number of order in the activity $z$ at least on a formal level. We will use this procedure to explain the intuition behind the renormalization methods and what which results we should be expecting for. This expectation becomes a good guide for how we should formulate the problem algebraically and on how the analysis should be done to show convergence. The analysis done here largely follows a physicist's point of view. To see this in more detail, it is helpful to see [17], [5], [22].

First, consider a field $\phi_0$ with distribution $\mathcal{N}(0, \beta\triangle^{-1})$ on $\Lambda$ as described above. Using Fourier transformation, the covariance $\triangle^{-1}$ can be written as

$$\triangle^{-1}(x-y) \equiv \triangle^{-1}(x,y) = \frac{1}{4\pi^2|\Lambda|} \sum_{p\in\Lambda^*\setminus\{0\}} \frac{e^{2\pi i p(x-y)}}{|p|^2}$$

where $\Lambda^* = [-1/2, 1/2]^2 \cap L^{-N}\mathbb{Z}^2$. The first equivalence follows because we are dealing with the periodic boundary condition. We will divide this into a high frequency mode and a low frequency mode, say

$$\begin{aligned} \triangle^{-1}(x) &= \Gamma_0(x) + \Gamma_{\geqslant 1}(x) \\ &= \frac{1}{4\pi^2|\Lambda|} \sum_{p\in\Lambda^*\setminus\{0\}, |p|\leqslant 1/L^{N-1}} \frac{e^{2\pi i p(x-y)}}{|p|^2} + \frac{1}{4\pi^2|\Lambda|} \sum_{p\in\Lambda^*\setminus\{0\}, |p|> 1/L^{N-1}} \frac{e^{2\pi i p(x-y)}}{|p|^2}. \end{aligned}$$

If $\phi_0 \sim \mathcal{N}(0, \beta\triangle^{-1})$, by the Gaussianity, there is a decomposition $\phi_0 = \phi_1 + \zeta_1$ such that $\zeta_1 \sim \mathcal{N}(0, \beta\Gamma_0)$, $\phi_1 \sim \mathcal{N}(0, \beta\Gamma_{\geqslant 1})$. Then the partition function of the sine Gordon field with inverse temperature $\beta^{-1}$ and activity $z$ can be written as

$$Z(z,\beta) = \mathbb{E}^{\phi_0}\big[e^{2z\sum_{r\in\Lambda}\cos(\beta^{1/2}\phi_0(r))}\big] = \mathbb{E}^{\phi_1}_{\Gamma_{\geqslant 1}}\big[\mathbb{E}^{\zeta_1}_{\Gamma_0}\big[e^{2z\sum_{r\in\Lambda}\cos((\phi_1(r)+\zeta_1(r)))}\big]\big].$$

Note that $Z(z,\beta)$ is actually the original partition function divided by the partition function of the Gaussian random variable. With $\Xi(z,\beta)$ the original partition function, we denote the ratio between $\Xi(z,\beta)$ and $Z(z,\beta)$ as $\Xi(z,\beta)/Z(z,\beta) = Z^{\mathrm{GFF}}(\beta)$. Under the assumption that $|z|$ is sufficiently small, we try to compute first the expectation $\mathbb{E}^{\zeta_1}_{\Gamma_0}$ with the fist order expansion in $z$, called the **fluctuation integral**.

$$\begin{aligned} Z_1(\phi_1) &:= \mathbb{E}^{\zeta_1}_{\Gamma_0}\big[e^{2z\sum_{r\in\Lambda}\cos((\phi_1(r)+\zeta_1(r)))}\big] \\ &= \mathbb{E}^{\zeta_1}_{\Gamma_0}\Big[1 + 2z\sum_{r\in\Lambda}\cos((\phi_1(r)+\zeta_1(r))) + O(z^2)\Big]. \end{aligned} \tag{5.1}$$

We can compute this using $\cos(\theta) = \frac{e^{i\theta}+e^{-i\theta}}{2}$ and

$$\frac{\int d\zeta \cdot e^{-\frac{1}{2\beta}\langle\zeta,\Gamma_0^{-1}\zeta\rangle} e^{i\zeta(r)}}{\int d\zeta \cdot e^{-\frac{1}{2\beta}\langle\zeta,\Gamma_0^{-1}\zeta\rangle}} = e^{-\frac{\beta}{2}\langle\delta_r(x),\Gamma_0\delta_r(x)\rangle} = e^{-\frac{\beta}{2}\Gamma_0(0)}. \tag{5.2}$$

Asymptotically as $L \to \infty$, $\Gamma_0(0)$ is $\frac{1}{2\pi}\log L$, so $e^{-\frac{\beta}{2}\Gamma_0(0)} \approx L^{-\beta/4\pi}$. Using the linear approximation for the periodic potential, we have obtained

$$Z_1(\phi_1) \approx e^{2L^{-\beta/4\pi} z \sum_{r\in\Lambda} \cos(\phi_1(r))}.$$

So the activity $z$ is just replaced by $z_1 = L^{-\beta/4\pi} z$. But just making iterations of any such operations would make $Z_j(\phi_j)$ go to the original partition function, which is not a clever thing to do. To see what we really should have done, for example, consider the thermodynamic limit

$$F_\infty(z,\beta) = \lim_{|\Lambda|\to\infty} \frac{1}{|\Lambda|} \log \Xi(z,\beta;\Lambda)$$

assuming its existence. To obtain an expression for $\mathbb{F}_\infty$, one also needs to progressively expand $\Lambda$, i.e. the real quantity we are interested in is

$$Z(\phi_1;\Lambda_{N+1}) \approx e^{2L^{-\beta/4\pi} z \sum_{r\in\Lambda_{N+1}} \cos(\phi_1(r))}.$$

We now use the procedure called **reblocking** : we view each block made of $L^2$ points, $B \in \mathcal{B}_1(\Lambda_{N+1})$ as a single point. Assuming that there is not too much variation within each block, we may take representative value of $\phi_1$ from each block and label it $\phi_1(B)$, hence we obtain

$$Z(\phi_1;\Lambda_{N+1}) \approx e^{2L^{-\beta/4\pi} \sum_{B\in\mathcal{B}_1(\Lambda_{N+1})} \cos(\phi_1(B))}.$$

We also need a new covariance for the new block-field $\phi_1(B)$. In fact, since the covariance $\Gamma_{\geqslant 1}$ takes all of $\triangle^{-1}(x)$ with scale $\geqslant L$ and $(\phi_1(B))_{B\in\mathcal{B}_1(\Lambda_{N+1})}$ approximately attains all information on the field of scale $\geqslant L$, the covariance remains the same. This can be also be written as

$$(\Gamma_{\geqslant 1})^{-1} F(B) = \sum_{\bar{B}\in\mathcal{B}_1 : \bar{B}\sim B} (F(\bar{B}) - F(B))$$

and $(\Gamma_{\geqslant 1})^{-1}$ can be just viewed as a scaled-up Laplacian. So denote $\Gamma_{\geqslant 1} = (\triangle_{\mathcal{B}_1})^{-1}$. This procedure is also called the **rescaling**. We are now led to write

$$Z(z,\beta;\Lambda_{N+1}) = \mathbb{E}^{\phi_1}_{(\triangle_{\mathcal{B}_1})^{-1}}\left[e^{2L^{2-\beta/4\pi} \sum_{B\in\mathcal{B}_1(\Lambda_{N+1})} \cos(\phi_1(B))}\right]$$

and therefore $\frac{1}{|\Lambda_N|}\log \Xi(z,\beta;\Lambda_N)$ has the limit

$$\lim_{N\to\infty} \frac{1}{|\Lambda_N|} \log Z(z,\beta;\Lambda_N) + \lim_{N\to\infty} \frac{1}{|\Lambda_N|} \log Z^{\mathrm{GFF}}(\beta;\Lambda_N)$$

whenever $\beta > 8\pi$. In this approximation, we have $\lim_{N\to\infty} \frac{1}{|\Lambda_N|} \log Z(z,\beta;\Lambda_N) = 0$, but this is clearly not the case. We shall see how this can be improved very soon.

*5.1.1. Higher degree expansion.* Better results can be seen by increasing the number of degrees in the expansion of the exponential function in (5.1). For example, when the exponential is expanded up to degree 2, we have

$$\begin{aligned}
Z_1(\phi_1) &:= \mathbb{E}_{\Gamma_0}^{\zeta_1}\left[e^{2z\sum_{r\in\Lambda}\cos((\phi_1(r)+\zeta_1(r)))}\right] \\
&= \mathbb{E}_{\Gamma_0}^{\zeta_1}\Big[1+2z\sum_{r\in\Lambda}\cos((\phi_1(r)+\zeta_1(r))) \\
&\qquad\qquad + 2z^2\sum_{r,s\in\Lambda}\cos((\phi_1(r)+\zeta_1(r))\cos((\phi_1(s)+\zeta_1(s))+O(z^3)\Big].
\end{aligned}$$

The expectation of the second degree term can also be computed using $\cos(\alpha)\cos(\beta) = \frac{1}{2}(\cos(\alpha+\beta)+\cos(\alpha-\beta))$ and

$$\begin{aligned}
\frac{\int d\zeta\cdot e^{-\frac{1}{2\beta}\langle\zeta,\Gamma_0^{-1},\zeta\rangle}e^{i(\zeta_1(r)+\zeta_1(s))}}{\int d\zeta\cdot e^{-\frac{1}{2\beta}\langle\zeta,\Gamma_0^{-1},\zeta\rangle}} &= \exp\Big(-\frac{\beta}{2}\langle\delta_r+\delta_s,\Gamma_0(\delta_r+\delta_s)\Big) \\
&= \exp\Big(-\Gamma_0(0)-\Gamma_0(r,s)\Big) \\
\frac{\int d\zeta\cdot e^{-\frac{1}{2\beta}\langle\zeta,\Gamma_0^{-1},\zeta\rangle}e^{i(\zeta_1(r)-\zeta_1(s))}}{\int d\zeta\cdot e^{-\frac{1}{2\beta}\langle\zeta,\Gamma_0^{-1},\zeta\rangle}} &= \exp\Big(-\Gamma_0(0)+\Gamma_0(r,s)\Big).
\end{aligned}$$

Hence we have

$$\begin{aligned}
&\mathbb{E}_{\Gamma_0}^{\zeta_1}\left[\cos((\phi_1(r)+\zeta_1(r))\cos((\phi_1(s)+\zeta_1(s))\right] \\
&= \frac{1}{2}e^{-\beta\Gamma_0(0)}\Big(e^{-\beta\Gamma_0(r,s)}\cos(\phi_1(r)+\phi_1(s))+e^{\beta\Gamma_0(r,s)}\cos(\phi_1(r)-\phi_1(s))\Big).
\end{aligned}$$

Since $\Gamma_0$ is a range-$L$-cutoff of $\triangle^{-1}$, we may compute this approximately by dividing the cases $|r-s|\leqslant L$ and $|r-s|>L$.

- When $|r-s|>L:\ \Gamma_0(r-s)\approx 0$, and we had $\Gamma_0(0)\approx\frac{1}{2\pi}\log L$, so this gives

$$L^{-\beta/2\pi}\cos(\phi_1(r))\cos(\phi_1(s)).$$

- When $|r-s|\leqslant L:$ with $\Gamma_0\neq 0$, we have

$$\begin{aligned}
&L^{-\beta/2\pi}\cos(\phi_1(r))\cos(\phi_1(s)) \\
&\quad+\frac{1}{2}L^{-\beta/2\pi}\Big((e^{-\beta\Gamma_0(r,s)}-1)\cos(\phi_1(r)+\phi_2(s))+(e^{\beta\Gamma_0(r,s)}-1)\cos(\phi_1(r)-\phi_2(s))\Big).
\end{aligned}$$

  Writing $r=s+\delta r$, $\cos(\phi_1(r)+\phi_1(s))\approx\cos(2\phi_1(r))-2\sin(\phi_1(r))\nabla\phi_1(r)\cdot\delta r$ and $\cos(\phi_1(r)-\phi_1(s))\approx\frac{1}{2}|\nabla\phi_1(r)|^2$ and so the second term can be written as

$$\frac{1}{2}L^{-\beta/2\pi}\Big((e^{-\beta\Gamma_0(r,s)}-1)(\cos(2\phi_1(r))-\sin(\phi_1(r))\phi_1(r)\cdot\delta r)+(e^{\beta\Gamma_0(r,s))}-1)\cdot\frac{1}{2}|\nabla\phi_1(r)|^2\Big),$$

Summing over $\{\delta r : |\delta r| \leqslant L\}$, the contribution from $\cos(2\phi_1(r))$ and $|\nabla\phi_1(r)|^2$ survive, but $\sin(\phi_1(r))\phi_1(r)\cdot\delta r$ cancels out because it is odd in $\delta r$. Also, use of erroneous but useful approximation $\Gamma_0(\delta r) \approx \frac{1}{2\pi}\log(L/|\delta|)$ gives $e^{\beta\Gamma_0(\delta r)} \approx \left(\frac{L}{|\delta r|}\right)^{\beta/2\pi}$ and therefore

$$\sum_{|\delta r|\leqslant L}(e^{-\beta\Gamma_0(\delta r)}-1) \approx \int_1^L (2\pi a da)\cdot\Big(\big(L/a\big)^{\beta/2\pi}-1\Big) \approx \frac{\beta/2}{2-\beta/2\pi}L^2$$

$$\sum_{|\delta r|\leqslant L}(e^{\beta\Gamma_0(\delta r)}-1) \approx \int_1^L (2\pi a da)\cdot\Big(\big(a/L\big)^{\beta/2\pi}-1\Big) \approx \frac{-\beta/2}{2+\beta/2\pi}L^2.$$

The exact values are not important, but it is important to keep track of the order of $L$, and the sign depending on the problem of interest.

Putting these together, we obtain

$$\begin{aligned} Z_1(\phi_1,z,\beta;\Lambda_N) &\approx 1 + 2L^{-\beta/4\pi}z\sum_{r\in\Lambda}\cos(\phi_1(r)) + \frac{1}{2}\Big(2L^{-\beta/4\pi}z\sum_{r\in\Lambda}\cos(\phi_1(r))\Big)^2 \\ &\quad + 2\gamma(\beta)z^2L^{2-\beta/2\pi}\sum_{r\in\Lambda}\cos(2\phi_1(r)) - \frac{1}{2}g(\beta)L^{2-\beta/2\pi}\sum_{r\in\Lambda}|\nabla\phi(r)|^2 \\ &\approx e^{-\frac{1}{2}g(\beta)L^{2-\beta/2\pi}|\nabla\phi_1|^2}\exp\Big(2L^{-\beta/4\pi}z\sum_{r\in\Lambda}\cos(\phi_1(r)) + 2\gamma(\beta)z^2L^{2-\beta/2\pi}\sum_{r\in\Lambda}\cos(2\phi_1(r))\Big). \end{aligned}$$

Upon reblocking, we have

$$\begin{aligned} Z_1(\phi_1,z,\beta;\Lambda_{N+1}) &\approx e^{-\frac{1}{2}g(\beta)L^{4-\beta/2\pi}\sum_{B\in\mathcal{B}_1}\phi(B)\cdot(-\triangle_{\mathcal{B}_1}\phi_1(B))} \\ &\quad\times\exp\Big(2L^{2-\beta/4\pi}\sum_{B\in\mathcal{B}_1}\cos(\phi_1(B)) + 2\gamma(\beta)z^2L^{4-\beta/4\pi}\sum_{B\in\mathcal{B}_1}\cos(2\phi_1(B))\Big). \end{aligned} \tag{5.3}$$

There are several important observations to be made here :

(i) We have an additional Gaussian term with covariance $g(\beta)^{-1}L^{\beta/2\pi-4}I$ on each renormalization step. One usually absorb this Gaussian term into the expectation $\mathbb{E}^{\phi_1}_{\Gamma_{\geqslant 1}}$ to define a new covariance

$$\tilde{\Gamma}_{\geqslant 1} = \Gamma_{\geqslant 1} + g(\beta)^{-1}L^{\beta/2\pi-4},$$

a new field $\tilde{\phi}_1 \sim \mathcal{N}(0,\tilde{\Gamma}_0)$ and $\delta E_1 = \frac{1}{|\Lambda|}\log Z^{\text{GFF}}(g(\beta)L^{4-\beta/2\pi})$ so that

$$\begin{aligned} Z(z,\beta,\Lambda_{N+1}) &\approx e^{|\Lambda|\delta E_1}\cdot\mathbb{E}^{\tilde{\phi}_1}_{\tilde{\Gamma}_{\geqslant 1}}\Big[\exp\Big(2L^{2-\beta/4\pi}\sum_{B\in\mathcal{B}_1}\cos(\phi_1(B)) + 2\gamma(\beta)z^2L^{4-\beta/4\pi}\sum_{B\in\mathcal{B}_1}\cos(2\phi_1(B))\Big] \\ &= e^{|\Lambda|\delta E_1}\cdot\tilde{Z}_1(\tilde{\phi}_1,z,\beta,\Lambda_{N+1}). \end{aligned}$$

When $\beta > 8\pi$, the term inside the expectation is expected to tend to 1 upon repeating renormalization steps, and so

$$F_\infty(z,\beta) = \lim_{N\to\infty}\frac{1}{|\Lambda_N|}\log\Xi(z,\beta,\Lambda_N) \approx \sum_{j=1}^{\infty}\delta E_j + \lim_{N\to\infty}\frac{1}{|\Lambda_N|}\log Z^{\text{GFF}}(\beta;\Lambda_N).$$

It seems that $F_\infty(z,\beta)$ has gained additional energy $\sum_{j=1}^{\infty}\delta E_j$ apart from its original Gaussian contribution. For this reason, one usually refers this as a **vacuum energy**.

(ii) We had the additional term $2\gamma z^2 L^{4-\beta/4\pi} \sum_{B\in\mathcal{B}_1} \cos(2\phi_1(B))$ in the effective Hamiltonian in the second order expansion. In fact, higher order expansion gives also the contributions of form $\sum_{B\in\mathcal{B}_1} \cos(q\phi_1(B))$ for $q \in \mathbb{Z}$. Fortunately, these contributions have relatively benign behaviours in the renormalzation analysis when $\beta$ is not too small. For example, if we had $\sum_{r\in\Lambda} \cos(q\phi(r))$ in place of $\sum_{r\in\Lambda} \cos(\phi(r))$ in the definition of sine-Gordon model, then in place of (5.2), we would have

$$\frac{\int d\zeta \cdot e^{-\frac{1}{2\beta}\langle \zeta, \Gamma_0^{-1}\zeta\rangle} e^{iq\zeta(r)}}{\int d\zeta \cdot e^{-\frac{1}{2\beta}\langle \zeta, \Gamma_0^{-1}\zeta\rangle}} = e^{-\frac{\beta}{2}\langle q\cdot\delta_r(x), \Gamma_0 q\cdot\delta_r(x)\rangle} = e^{-\frac{q^2}{2}\beta\Gamma_0(0)} \approx L^{-q^2\beta/4\pi}.$$

Then even after the reblocking, each renormalization step has $L^{2-q^2\beta/4\pi}$ contribution in front of $\sum_{B\in\mathcal{B}_1} \cos(q\phi_1(B))$. So these terms contracts at the expected critical temperature $\beta = 8\pi$ and even lower. Hence in more rigorous analysis, these modes can be grouped into a single factor that is considered to be **irrelevant**.

(iii) As we have seen in the single order expansion, it is not clear whether the model is a Gaussian free field in the long range order at the critical temperature $\beta = 8\pi$. For this reason, the activity $z$ is called a **marginal** parameter under renormalization flow. However, when the expansion is pushed further, it becomes more convincing that the long range order of the sine-Gordon model would be same as that of the Gaussian free field. Therefore, we would necessarily have to take account of $z^2$ at least for RG analysis at the critical temperature.

(iv) This specific choice of renormalization flow has the advantage that it is non-perturbative in $\beta$, i.e. the method covers a wide range of choice for $\beta$ and even suggests a highly plausible critical temperature. However, because we have used a Taylor expansion in $z$, the method is perturabative in the direction of $z$ in nature. That is, we can prove statements only under the assumption that $z$ is small enough. Small $z$ means there is a high cost in creating each particle in the system, and this is the reason why the dual of this model was called the **Coulomb gas** in the first place.

**5.2. Difficulties in Proper Renormalization.** Although this rough sketch of the renormalzation process seems to convey enough details for us to make this into a proper proof, it actually turns out that this is hardly the case. It also seems that increasing the degree of expansion each time increases the mode of difficulty significantly. Here, we introduce some difficulties we should overcome in order to give a proper mathematical way of doing renormalization analysis.

- The reblocking procedure have to be justified in a proper way. In particular, expressions such as "$\phi(B)$" for $B \in \mathcal{B}_j(\Lambda)$ has to be replaced with a legal expression. One usually uses polymer activity in place of functions of $(\phi(B) : B \in \mathcal{B}_j)$. When we have a **polymer activity** of form $A(\phi_j; X)$ for $X \in \mathcal{P}_j$, then this means this is a function of the polymer $X$ and the field restricted on $X$, $\phi_j|_X$.
- To prove that the limit

$$\lim_{N\to} \frac{1}{|\Lambda_N|} \log \Xi(z, \beta; \Lambda_N)$$

exists, we have argued that $Z_j(\tilde{\phi}_j)$ defined inductively by

$$\tilde{Z}_j(\tilde{\phi}_j, z, \beta; \Lambda_N) = e^{-\delta E_j} \mathbb{E}_{\tilde{\Gamma}_j}^{\tilde{\zeta}_{j-1}} \left[ \tilde{Z}_{j-1}(\tilde{\phi}_{j-1}, z, \beta; \Lambda_N) \right] \tag{5.4}$$

converges in an appropriate sense to the unity. This sort of argument should also be valid if we try to compute the limit different extensive quantities, such as two point correlation or two point energy : first we obtain the scaling limit then the limit of a extensive quantity $\mathcal{F}$ is the value of $\mathcal{F}$ of the scaling limit plus some function of the vacuum energy.

But then the question now is how we should prove the implication form convergence of the scaling limit to the convergence an extensive quantity $\mathcal{F}$. As we have learned from analysis classes, this significantly depends on the notion of convergence. The most naive attempt one could make is to endow a norm on the space of functions $\big(F(\mathbb{R}^{\Lambda_N}, z, \beta, \Lambda_N)\big)_{j,N}$. Although this in theory encodes all information about the model of interest, it is also highly inefficient. Fields $\tilde{\phi}_j$ of different scales are concentrated on different regions of $\mathbb{R}^{\Lambda_N}$, and functions of fields restricted in small $X \subset \Lambda$ plays more important role in the analysis. Therefore a practical choice of space would include information about the field, scale and set dependence. In particular, we will consider subspaces of

$$\begin{aligned}
&\{(j, \phi, X, A(\phi; X)) : j \geqslant 0,\ \phi \in \mathbb{R}^{\Lambda},\ X \in \mathcal{P}_j,\ A : \mathbb{R}^{\mathbb{R}^{\Lambda} \times \mathcal{P}_j}\} \\
&\{(j, X, A(\cdot\,; X)) : j \geqslant 0,\ X \in \mathcal{P}_j,\ A : \mathbb{R}^{\mathbb{R}^{\Lambda} \times \mathcal{P}_j}\} \\
&\{(j, A(\cdot\,;\cdot)) : j \geqslant 0,\ A : \mathbb{R}^{\mathbb{R}^{\Lambda} \times \mathcal{P}_j}\}
\end{aligned}$$

endowed with norms as appropriate.

- Although listing of all polymer activities in $j^{th}$ level gives the complete description of the theory in scale larger than $j$, because polymer activities $A(\zeta; \cdot)$ with random field $\zeta$ are correlated in a complicated way, it is often the case that a brute force method would involve exponential amount of combinatorics as the renormalization proceeds. In this consideration, there is a need to invent nice algebraic way to formulate the polymer activities in a compact form. There are several requirements in this algebraic forms :

  (1) It should make clear distinction of the irrelevant terms from the terms that matter. Without this distinction, estimating the contracting of irrelevant terms would fall into a complete pandemonium.

  (2) An invariance of the algebraic expression under renormalization would be helpful for simplifying the combinatorics.

  (3) The expression should have reasonable amount of freedom in choice of parameters, and in how the renormalization acts on the algebraic system.

These requirements are nothing close to being absolute. But they try to follow some effective way of constructing a high-level mathematical theory.

- Last but not least, the specific choice of covariance decomposition is sometimes a serious obstacle in doing the analysis. The most naive choice of the decomposition would be to use

$$\begin{aligned}(-\triangle_\Lambda + m^2)^{-1}(x) &= \frac{1}{4\pi^2|\Lambda|}\sum_{p\in\Lambda^*}\frac{e^{2\pi i x\cdot p}}{|p|^2+m^2}\\ &= \frac{1}{4\pi^2|\Lambda|}\sum_{j=0}^{N}\sum_{p\in\Lambda^*}\mathbb{I}\big(|p|\in[L^{-N+j},L^{-N+j+1})\big)\frac{e^{2\pi i x\cdot p}}{|p|^2+m^2}\end{aligned}$$

  so

$$\begin{aligned}\Gamma'_j(x;m) &= \frac{1}{4\pi^2|\Lambda|}\sum_{p\in\Lambda^*}\mathbb{I}\big(|p|\in[L^{-N+j},L^{-N+j+1})\big)\frac{e^{2\pi i x\cdot p}}{|p|^2+m^2}, \quad j=0,\cdots,N-2,\\ \Gamma'_{\geqslant N-1} &= \frac{1}{4\pi^2|\Lambda|}\sum_{p\in\Lambda^*}\mathbb{I}\big(|p|\geqslant L^{-N+j}\big)\frac{e^{2\pi i x\cdot p}}{|p|^2+m^2}.\end{aligned}$$

  gives $(-\triangle_\Lambda + m^2)^{-1} = \sum_{j=0}^{N-2}\Gamma'_j + \Gamma'_{\geqslant N-1}$. However, this decomposition has an issue any $\phi_j(x),\phi_j(y)$ with $|x-y|\gg L^j$ are correlated under covariance $\Gamma'_j$. Although the covariance might be very small, we still need to keep track of these contributions and add them into account for various estimates. So we rather consider a more intricate but yet more handy decomposition in the form

$$\begin{aligned}\Gamma_j(x;m) &= \frac{1}{4\pi^2|\Lambda|}\sum_{p\in\Lambda^*}(F_j(p;m)-F_{j+1}(p;m))\frac{e^{2\pi i x\cdot p}}{|p|^2+m^2}, \quad j=0,\cdots,N-2,\\ \Gamma_j(x;m) &= \frac{1}{4\pi^2|\Lambda|}\sum_{p\in\Lambda^*}F_{N-1}(p;m)\frac{e^{2\pi i x\cdot p}}{|p|^2+m^2}\end{aligned}$$

  and $F_0(\,\cdot\,;m)\equiv 1$ with choices of $F_j$'s so that $\Gamma_j(x;m)=0$ for $\|x\|_{\geqslant}\frac{1}{2}L^{j+1}$. Because of this property, we call this a **finite range decomposition** of the covariance. Such choice of $F_j$ and its decay properties are discussed in [2] and also the essential estimates are summarized in [7].

# 6 Brydges-Yau method

The most relevant work in our sequel using Brydges-Yau method is the work done by [14], [6]. This method deals with the infrared limit of the a Coulomb gas with ultraviolet cutoff on 2 dimensional lattice when $\beta > \beta_0 > 8\pi$ and $|z|$ is sufficiently small, depending on $\beta_0$. The paper shows the convergence of the intermediate function $Z_j(\phi_j;\Lambda)$ to the identity in some specified norm of polymer activities when $|\Lambda|\to\infty$ is taken a priori. For the unproved results in this section, we make reference to [6]. We will try to focus on how the result is formulated, but no detailed account would be made on the required estimates, because the method is brute-forced in many ways, and involves many combinatorial complications.

**6.1. The Polymer Expansion.** The RG method of Dimock and Hurd is based on the Mayer expansion of the polymer activities. With

$$Z_0(\phi;\Lambda) = \exp(-V_0(\phi;\Lambda)), \quad V_0(\phi;\Lambda) = -2z\sum_{r\in\Lambda}\cos(\phi(r)),$$

one may also write

$$\begin{aligned} V_0(\phi;B) &= -2z\sum_{r\in B}\cos(\phi(r)), \quad B\in\mathcal{B}_j \\ V_0(\phi;X) &= \sum_{B\in X} V_0(\phi;B), \quad X\in\mathcal{P}_j \end{aligned}$$

and so for any scale $j$,

$$\begin{aligned} Z_0(\phi;\Lambda) &= \prod_{B\in\mathcal{B}_j}\exp(-V_0)(\phi;B) \\ &= \prod_{B\in\mathcal{B}_j}\left[1+(\exp(-V_0)(\phi;B)-1)\right] \\ &= \sum_{X\in\mathcal{P}_j}\prod_{B\in X}\left[\exp(-V_0)(\phi;B)-1\right] \\ &= \sum_{X\in\mathcal{P}_j}\prod_{Y\in\mathcal{P}_j^c(X)} A_0(\phi;Y) \end{aligned}$$

In the last equality, $\mathcal{P}_j^c$ is the set of connected components of $X$, where $Y\in\mathcal{P}_j$ is connected if and only if for each $B\in Y$, there is $B'\in Y\backslash\{B\}$ such that the euclidean distance between $B$ and $B'$ is smaller or equal to $\sqrt{2}\cdot 2^j$ and $A_0(Y)$ is the polymer activity given by

$$A_0(\,\cdot\,;Y) = \begin{cases} \prod_{B\in Y}\left[\exp(-V_0^B)-1\right] & \text{if } Y \text{ is connected,} \\ 0 \quad \text{if otherwise} \end{cases}.$$

Using the circle product of polymer activities defined by

$$A_1\circ A_2(\phi;X) = \sum_{Y,Z\subset X}\mathbb{I}(Y,Z \text{ are disconnected})A_1(\phi;Y)A_2(\phi;Z)$$

and the polymer activity exponential

$$\mathcal{E}xp(A) = \mathbb{I}(X=\emptyset) + A + \frac{1}{2!}A^{\circ 2} + \frac{1}{3!}A^{\circ 3} + \cdots$$

this can be simplified in to a form

$$Z_0(\phi;\Lambda) = \sum_{X\in\mathcal{P}_0}\mathcal{E}xp(A_0)(\phi;X)$$

**6.2. The algebra.** The greatest advantage of this abstract expression is that this form is preserved under the renormalization. To see this, we need to make even more definitions.

For a forest structure $\mathcal{F} = \{T_1, \cdots, T_n\}$ endowed on a polymer $X$ and $s = (s_{xy} : x, y \in X)$, define

$$\sigma_{xy}(s, \mathcal{F}) = \sum_{ab \in \gamma(x \to y)} s_{ab}$$

where $\gamma(x \to y)$ is the shortest path connecting $x$ and $y$ if $x$ and $y$ are in the same tree and $\gamma(x \to y) = \emptyset$ is otherwise. Also define, for $x, y \in X$, $\{Y_i\}$ disconnected subsets of $X$ with $\cup_i Y_i = X$,

$$\begin{aligned} C(Y_i, Y_j)(x, y) &= \frac{1}{2}\big[\mathbb{I}(x \in Y_i)\mathbb{I}(y \in Y_j) + \mathbb{I}(x \in Y_j)\mathbb{I}(y \in Y_i)\big] C(x, y) \\ C_X &= \sum_{i,j} C(Y_i, Y_j) \\ C_X(s) &= \sum_{i,j} C(Y_i, Y_j) s_{ij} \end{aligned}$$

and for sufficiently regular $f : \mathbb{R}^X \to \mathbb{R}$,

$$\triangle_C f(\phi) = \frac{1}{2} \sum_{(x,y) \in \mathbb{R}^X \times \mathbb{R}^X} C(x, y) \frac{d^2 f}{d\phi(x) d\phi(y)}(x, y) dx dy.$$

See [1] for more detailed account on cluster expansion. Then we have the following, which is purely combinatorial.

**Lemma 6.1.** *Assume a polymer activity $A$ has sufficient amount of regularity. Let $\mathbb{E}_\Gamma^\zeta$ be an expectation over a random field $\zeta \sin \mathcal{N}(0, \Gamma)$, independent of $\phi$. Then*

$$\mathbb{E}_\Gamma^\zeta \Big[ \sum_{X \in \mathcal{P}_j} \mathcal{E}xp A(\phi + \zeta; X) \Big] = \sum_{X \in \mathcal{P}_j} \mathcal{E}xp(\mathcal{F}A)(\phi; X)$$

*where $\mathcal{F}$ acts on polymer activities by*

$$\mathcal{F}A(\phi; X) = \sum_{\{Y_i\}} \sum_{T \in \mathcal{T}(\{Y_i\})} \mathbb{E}_{\Gamma(X)(s,T)}^\zeta \Big[ \prod_{ij \in T} (-2\triangle_{C(Y_i, Y_j)}) \prod_i A(Y_i) \Big]$$

*where the sum $\sum_{\{Y_i\}}$ runs over collections $\{Y_i\}$ that are disconnected and $\cup_i Y_i = X$. Also, $\mathcal{T}(\{Y_i\})$ denotes the set of tree structures on each $Y_i$.*

**6.3. The analysis.** By the result of **Lemma 6.1**, if we make the decomposition of the covariance matrix by $(-\triangle)^{-1}(x, y) = \sum_{j=0}^{N-2} \Gamma_j(x, y) + \Gamma_{\geqslant N-1}$ and define inductively $Z_{j+1}(\phi_{j+1}) = \mathbb{E}_{\Gamma_j}^{\zeta_{j+1}}[Z_j(\zeta_{j+1} + \phi_{j+1})]$ then we obtain

$$\begin{aligned} A_{j+1} &= \mathcal{F}A_j \\ Z_j(\phi_j) &= \sum_{X \in \mathcal{P}_0} \mathcal{E}xp(A_j)(\phi_j; X). \end{aligned}$$

Hence if we can show that $A_j \to 0$ as $j \to \infty$ in an appropriate sense, then we see that the scaling limit of the field is a Gaussian free field. Hence one needs to define the a norm on the space of polymer activities and develop various estimates on the renormalization steps.

To define a norm on the space of polymer activities, it is worth observing that, because we have defined each $j^{th}$-scale partition function $Z_j$ as a series expansion of sets, controlling the polymer activity at the large set region region would be helpful to prove convergence of the expansion. Also because we have higher probability of having fields with smaller deviation, we would like to penalize those regions with what we call the large set regulator and the large field regulator,

$$\begin{aligned}
&\mathcal{R}(X) = A^{|X|}\Theta(X)\\
&R(\phi; r, X) = \exp\Big(r\sum_{\alpha\leqslant s}\sum_{x\in X}|D^\alpha\phi(x)|^2 + rc\sum_{x\in\partial X}|D\phi(x)|^2\Big)
\end{aligned}$$

for $X \in \mathcal{P}_j$ and $r, c > 0$. So defining

$$\begin{aligned}
\big\|A(X)\big\|_{R.h} &= \sum_{n=0}^{\infty}\frac{h^n}{n!}\big\|D^n A(X)\big\|_\infty\\
\big\|D^n A(X)\big\|_{R,\infty} &= \sup\Big\{\big\|D^n A(X,\phi)\big\|_1 R(X,\phi)^{-1} : \sum_{\alpha\leqslant s}\sum_{X}|D^\alpha\phi|^2 dx < \infty\Big\}\\
\big\|A\big\|_{R,h,\mathcal{R}} &= \sum_{X:X\supset B}\mathcal{R}(X)\big\|K(X)\big\|_h
\end{aligned}$$

where $\|K_n(X,\phi)\|_1$ is defined using dual representation

$$\|D^n K(X,\phi)\|_1 = \sup\Big\{D^n(X,\phi)(f) : \sum_{\alpha\leqslant s}\sum_{X}|D^\alpha f|^2 \leqslant 1\Big\}$$

for some fixed integers $r, s > 0$. Under such choice of norm, we can make an estimate that becomes the milestone for proving contraction property of the renormalization map.

**Lemma 6.2.** *Let $rc^{-1}L^2$ be sufficiently small. Then there is $\gamma > 0$ such that whenever $\delta h \in (\gamma^2 d(C,K), h)$ for a constant $d(C,K)$, one has*

$$\big\|\mathcal{F}A\big\|_{R_l(\phi;r,X),h-\delta h,\mathcal{R}_p} \leqslant 2\big\|A\big\|_{R(r),h,\mathcal{R}_{p+3}}$$

*where*

$$\begin{aligned}
R_l(\phi; r, X) &= R(r, l^{-1}X, \phi(\,\cdot\,/l))\\
\mathcal{R}_p &= (2^p)^{|X|}\Theta(X).
\end{aligned}$$

**6.4. The renormalization.** The RG is consists three steps :

(i) The fluctuation integral : we have already got the expression

$$\sum_{X\in\mathcal{P}_j}\mathcal{E}xp(\mathcal{F}A_j)(\phi, X) = \mathbb{E}^{\zeta_{j+1}}_{Gam\tilde{m}a_j}\Big[Z_j(\phi_{j+1}+\zeta_{j+1})\Big].$$

where $Z_j(\phi) = \sum_{X\in\mathcal{P}_j}\mathcal{E}xp(A_j)(\phi;\Lambda)$. However, the covariance $\tilde{\Gamma}_j$ are determined after each RG step is completed. So we deal with the decomposition with an arbitrary set of covariances a priori and assign the required form them later on.

(ii) Isolation of relevant terms : this step extracts out the terms in $Z_j(\phi)$ that do not contract under the next scaling term, namely the relevant terms from the terms that contract to 0, called the irrelevant terms. By doing so, we learn how to treat the analysis of some terms different from the others.

Let $F$ be some polymer activity with sufficient regularity that satisfies the factorisation

$$F(\phi; X) = \sum_{B \in X} F(\phi, B; X)$$

for some $F(\phi, B; X)$. The $\exp(F)$ will contain the constant and the quadratic part of $\mathcal{E}xp(A_j)$. After the extraction, suppose we are left with

$$\frac{\mathcal{E}xp(A_j)(\phi; \Lambda)}{\exp\left(\sum_{X \in \mathcal{P}_j} F(\phi; X)\right)} = \mathcal{E}xp(G(A_j, F))(\phi; \Lambda)$$

for some new polymer activity $G(A, F)$. Then $\mathcal{E}xp(G(A_j, F))(\phi_1 i; \Lambda_{N+1})$ can be used to play the role of $\tilde{Z}_1(\phi_1; \Lambda_{N+1})$ in (5.4). Now one can check that $G(A_j, F)$ defined by

$$\begin{aligned} \tilde{A}(\phi; X) &= A(\phi; X) - \sum_{\{Z_j\}} \prod_j e^{F(X_j)-1} \\ G(A_j, F)(\phi; Y) &= \sum_{\{X_i\},\{Y_i\}} \prod_i \tilde{A}_j(X_i) \prod_j (e^{-F(Y_j)} - 1) \end{aligned}$$

satisfies the required formula. Here, the sum $\sum_{\{Z_j\}}$ runs over the the collection of distinct polymers with $\cup_j Z_j = X$ and $\cup_j \bar{Z}_j$ is connected, and the sum $\sum_{\{X_i\},\{Y_i\}}$ runs over diconnected collection $\{X_i\}$, distinct collection $\{Y_i\}$ such that $\cup_i \bar{X}_i$ and $\cup_i \bar{Y}_i$ are connected, and $\cup_i X_i = \cup_i Y_i = Y$.

Next, the relevant term of the expansion is given by the constant term and the quadratic function of the gradient which can be obtained from the Taylor expansion of the polymer activity. Call the constant term $\delta E^j(X)$ and the covariance of the quadratic part is $(\delta\sigma^j)$, so that $F$ is given by

$$F_j(\phi_j; X) = \delta E^j(X) - \frac{1}{2\beta} \sum_{x \in X} \delta\sigma^j(X)(D\phi_j(x), D\phi_j(x)i).$$

The constant term can be just separated out, and the quadratic part may be merged in the fluctuation integral. That is, we can let

$$\tilde{\Gamma}_j = (\Gamma_j^{-1} + (\delta\sigma^j)^{-1})^{-1}$$

and write

$$\sum_{X \in \mathcal{P}_j} \mathcal{E}xp(G(A_j, F_j))(\phi, \Lambda) = e^{\sum_{X \in \mathcal{P}_j} \delta E_j(X)} \mathbb{E}_{\tilde{\Gamma}_j}\Big[\sum_{X \in \mathcal{P}_j} \mathcal{E}xp(A_j)\Big](\phi, \Lambda).$$

This completes the extraction step.

(iii) Scaling : this step rescales the field and the polymer activity obtained after previous two steps so that it matches the scales of the original field and the polymer activity. However, because the dimension of the massless Gaussian field at dimension 2 is 0, we just leave the field as it is. For the polymer activity, just repeat the whole process with one higher scale and to obtain instead

$$\sum_{X\in\mathcal{P}_{j+1}} \mathcal{E}xp(\mathcal{F}A_j)(\phi, X) = \mathbb{E}_{\Gamma_j}^{\zeta_{j+1}}\Big[Z_j(\phi_{j+1}+\zeta_{j+1})\Big].$$

Under this scaling and reblocking, we may write

$$Z_j(\phi) = e^{E_j}\sum_{X\in\mathcal{P}_j}\mathcal{E}xp(A_j)(\phi; X)$$

where $E_j = \sum_{i=0}^{j-1}\sum_{X\in\mathcal{P}_i}\delta E_i(X)$.

In this set-up, the main theorem says the following.

**Theorem 6.3.** *For any $\beta > 8\pi$, $L \equiv L(\beta)$ sufficiently large, and $|z|$ is sufficiently small. Let $\delta_0 = C|z|e^{h_0}$ be sufficiently small and $\delta^0 = L^{-j\varepsilon}\delta_0$, then we have*

$$\begin{aligned} &\|A_j\|_{R_j,h_j,\mathcal{R}_j} \leqslant \delta_j, \\ &|\delta E_j| \leqslant \delta_j \end{aligned}$$

*where we have set $R_j, h_j, \mathcal{R}_j$ using the parameters*

$$\begin{aligned} &r = 4, \quad s = 6, \quad c = (8Lc_s)^{-1}, \\ &\kappa_j = \kappa_0\cdot\sum_{k=0}^{j} 2^{-k}, \\ &h_j = \kappa_0^{-1/2}\cdot(1+\sum_{k=j+1}^{\infty} 2^{-k}) \end{aligned}$$

*for $\kappa_0$ sufficiently small and $c_s$ a sufficiently large number.*

# 7 Falco method

Falco uses RG to obtain a phase transition line as a function $\beta_c = \beta_c(z)$ at a small neighbourhood of $z = 0$, and analyses the correlation length of the sine-Gordon model on the region $\beta \geqslant \beta_c(z)$.

As for the case of Brydges-Yau method, the main statement will only be defined after we actually get some algebraic form for the renormalized free energy that is conserved under the flow. The analysis here will be much more heavy because an expansion of another order is required for the analysis on the phase transition line, but this will be compensated by a better polymer expansion, use of renormalization with an implicit parameter and an implicit function theorem that determines the implicit parameters.

The formulations of RG, including polymer expansion follows the layout set by [**BrydgesLecture**].

**7.1. Extraction of Gaussian term, Massless limit.** The spirit of RG is similar for this method, but number of improvements are made by using a more systematics way of polymer expansion that enables a more delicate control of the RG flow by turning the choice of extraction terms implicitly.

The isolation of the relevant terms is done implicitly, so that the integration measure does not have to be modified in each renormalization step. This a priori knowledge on the integration measures (which is in fact just the original measure) able us to treat the measure decomposition with more care. It is useful to recall that a Gaussian probability measure appears from the start of the sine-Gordon transformation,

$$e^{-\frac{\beta}{2}U(\phi)} = \int_{\mathbb{R}_\Lambda} d\mathbb{P}(\xi/\sqrt{\beta}; m) e^{i\langle\phi,\xi\rangle}$$

where $\mathbb{P}(\,\cdot\,; m)$ denotes the probability measure with covariance $(-\triangle + m^2)^{-1}$. Extraction of a quadratic term would result in

$$d\mathbb{P}\Big(\frac{\xi}{\sqrt{\beta}}; m\Big) = e^{\frac{t}{2\beta}\langle\xi,-\triangle\xi\rangle} d\mathbb{P}\Big(\frac{\xi}{\sqrt{\beta/(1+t)}}; \frac{m}{\sqrt{1+t}}\Big) \times \frac{Z(\beta, m)}{Z(\beta/(1+t), m/\sqrt{1+t})}$$

where $Z(\beta, m) = \int_{x\in\mathbb{R}^\Lambda} \exp(-\frac{1}{2\beta}\langle\xi, (-\triangle+m^2)\xi\rangle$ and reparametrizing $1+t = \frac{1}{1-s}$, $\alpha^2 = \beta(1-s)$ for ease of computation, we obtain

$$d\mathbb{P}(\xi/\sqrt{\beta}; m) = \frac{Z(\beta, m)}{Z(\beta(1-s), m\sqrt{1-s})} \exp\Big(\frac{s}{2\alpha^2}\langle\xi, -\triangle, \xi\rangle\Big) d\mathbb{P}(\xi/\alpha; m\sqrt{1-s}).$$

Using this expression, the partition function of the Coulomb gas can be written as

$$\begin{aligned} Z(\beta; m) &= \frac{Z(\beta, m)}{Z(\beta(1-s), m\sqrt{1-s})} \\ &\qquad \times \sum_{l=0}^{\infty} \frac{1}{l!} \int_{\mathbb{R}^\Lambda} d\mathbb{P}(\xi/\alpha; m\sqrt{1-s}) \cdot \prod_{j=1}^{l} \sum_{r_j^N \in\Lambda} 2z\cos(\xi(r_j^N)) \cdot \exp\Big(\frac{s}{2\alpha^2}\langle\xi, -\triangle\xi\rangle\Big) \\ &= \int_{\mathbb{R}^\Lambda} d\mathbb{P}(\xi; m\sqrt{1-s}) e^{\tilde{V}(\xi;m)} \end{aligned}$$

where

$$\tilde{V}(\xi; m) = \frac{s}{2}\langle\xi, -\triangle\xi\rangle + \sum_{r\in\Lambda} 2z\cos(\alpha\xi(r)) + \log\frac{Z(\beta, m)}{Z(\beta(1-s), m\sqrt{1-s})}.$$

Eventually, we would like to compute the limit $m \to 0$, so note that

$$\lim_{m\to 0} \frac{Z(\beta, m)}{Z(\beta(1-s), m\sqrt{1-s})} = \frac{|\Lambda|}{2}\log(1-s)$$

and we can replace $\tilde{V}(\xi; m)$ by

$$\tilde{V}(\xi) = \frac{s}{2}\langle\xi, -\triangle\xi\rangle + \sum_{r\in\Lambda} 2z\cos(\alpha\xi(r)) + \frac{|\Lambda|}{2}\log(1-s)$$

when we compute

$$Z(\beta; m=0) := \lim_{m\to 0} Z(\beta; m) = \lim_{m\to 0} \mathbb{E}^{(\xi;m)}[\exp(\tilde{V}(\xi))].$$

Also if we have the covariance decomposition

$$(-\triangle_\Lambda + m^2)^{-1} = \sum_{j=0}^{N-2} \Gamma_j(m) + \Gamma_{\geqslant N-1}(m),$$

then the divergence in the infrared limit is contained in $(-\triangle_\Lambda + m^2)^{-1}$ is concentrated in $\Gamma_{\geqslant N-1}$, so in the renormalization, one just needs to consider the mass term in $\Gamma_j$ by

$$Z(\beta; m=0) = \lim_{m\to 0} \mathbb{E}_{\Gamma_{\geqslant N-1}(m)} \mathbb{E}_{\Gamma_{N-1}(m=0)} \cdots \mathbb{E}_{\Gamma_1(m=0)}[\exp(\tilde{V}(\xi))].$$

**7.2. Fluctuation integral.** Upon each renormalization step, we define inductively

$$e^{\tilde{V}_{j+1}}(\phi_{j+1}) = \mathbb{E}^{\zeta_{j+1}}_{\Gamma_j(m=0)}\big[e^{\tilde{V}_j}(\phi_{j+1}+\zeta_{j+1})\big], \quad \tilde{V}_0 = \tilde{V}$$

for $j \leqslant N-1$. By writing $\tilde{V}_0 = \tilde{V}$, we also let $s_0 = s_1$, $z_0 = z$, $E_0 = \frac{1}{2}\log(1-s)$. One may expect that $\tilde{V}_1$ inherits the structure of $\tilde{V}_0$ with small deviation, say

$$(E_0 + \delta E_1)|\Lambda| + \frac{s_1}{2}\langle \phi_1, -\triangle \phi_1\rangle + 2z_1 L^{-2} \sum_{r\in\Lambda} \cos(\alpha\phi_1(r)) + \sum_{r\in\Lambda} W(\phi_1, r) + (\text{correction terms}).$$

Recall from (5.3) that usual renormalization procedure actually gives factor $L^{-\alpha^2/4\pi} z_0 +$ (higher order terms) in front of $\sum \cos(\alpha\phi_2(r))$, but we are surveying the near-critical behaviour, so we just fix the factor as $L^{-2}$. We may then write, for $B_1 \in \mathcal{B}_1$, $X \in \mathcal{P}_1$ or $X \in \mathcal{P}_0$,

$$\begin{aligned}
&V_1(\phi_1; B_1) = \frac{s_1}{2} \sum_{x\in B_1} \phi(x)(-\triangle\phi)(x) + 2z_1 L^{-2} \sum_{x\in\Lambda} \cos(\alpha\phi_1(x)) \\
&W_1(\phi_1; B_1) = \sum_{x\in B_1} W_1(\phi_1, r) \\
&U_1(\phi_1; B_1) = V_1(\phi_1; B_1) + W_1(\phi_1; B_1) \\
&U_1(\phi_1; X) = \sum_{B_1\in X} U_1(\phi_1; B_1) = \sum_{B_1\in X} V_1(\phi_1; B_1) + W_1(\phi_1; B_1) = V_1(\phi_1; X) + W_1(\phi_1; X).
\end{aligned}$$

Obtaining an expression for the correction term is not difficult, but obtaining the one that also comes with a good estimate is difficult. To do this, it is helpful to write

$$\begin{aligned}
e^{\tilde{V}_1}(\phi_1) &= \mathbb{E}^{\zeta_1}_{\Gamma_0(m=0)}[e^{\tilde{V}_0}(\phi_1+\zeta_1;\Lambda)] \\
&= \mathbb{E}^{\zeta_1}_{\Gamma_0(m=0)}\Big[\prod_{B\in\mathcal{P}_{j+1}} \Big(\big(e^{\tilde{V}_0}(\phi_1+\zeta_1; B) - e^{(E_0+\delta E_1)|B| + U_1}(\phi_1; B)\big) + e^{(E_0+\delta E_1)|B|+U_1}(\phi_1; B)\Big)\Big]
\end{aligned}$$

and the following lemma is crucial :

**Lemma 7.1.** *Suppose $\Gamma_j$ is chosen such that $\Gamma_j(x,y) = 0$ whenever $\|x-y\|_\infty > L^{j+1}/2$, $\zeta \sim \mathcal{N}(0,\Gamma_j)$, $\phi$ a given field. Also assume that polymer activities $A_1$ and $A_2$ are such that*

$$A_l(\phi; X_{j+1}) = \prod_{B\in X_{j+1}} A_l(\phi; B), \quad \forall l \in \{1,2\},\ X_{j+1} \in \mathcal{P}_{j+1}$$

*Then the equality*

$$\mathbb{E}[\prod_{B\in X}(A_1(\zeta+\phi;B) + A_2(\phi;B))] = \sum_{Y\in\mathcal{P}_{j+1}(X)} A_2(\phi;Y) \prod_{Z\in\mathcal{P}^c_{j+1}(Y)} \mathbb{E}[A_1(\zeta+\phi;Z)]$$

*holds for any $X \in \mathcal{P}_{j+1}$ where $\mathcal{P}_{j+1}(X) = \{Y\cap X : Y \in \mathcal{P}_{j+1}\}$.*

*Proof.* This is just a consequence of writing

$$\begin{aligned}
\prod_{B\in X}\sum_{l=1,2} A_l(B) &= \sum_{Y\in\mathcal{P}_{j+1}(X)} \prod_{B\in X\setminus Y} A_2(B)\cdot\prod_{B\in Y} A_1(B) \\
&= \sum_{Y\in\mathcal{P}_{j+1}X} A_2(X\setminus Y) \prod_{Z\in\mathcal{P}^c_{j+1}(Y)} A_1(Z)
\end{aligned}$$

and for $Z_1$, $Z_2 \in \mathcal{P}_{j+1}(Y)$ disconnected, $A_1(\zeta+\phi;Z_1)$ and $A_1(\zeta;Z_2)$ are independent because of our assumption on $\Gamma_j$, the covariance of $\zeta$.

□

By **Lemma 7.1**, we obtain the formula

$$\begin{aligned}
e^{\tilde{V}_1(\phi_1)} &= e^{E_1|\Lambda|}\sum_{Y\in\mathcal{P}_1} e^{U_1(\phi_1;\Lambda\setminus Y)} \prod_{Z\in\mathcal{P}^c_1} K_1(\phi_1;Z) \\
&= e^{E_1|\Lambda|}(e^{U_1}\circ K_1)(\phi_1;\Lambda)
\end{aligned}$$

where we have defined

$$\begin{aligned}
E_1 &= E_0 + \delta E_1 = \log(1-s_0) + \delta E_1 \\
K_1(\phi_1;Z) &= e^{-\delta E_1|Z|}\cdot\mathbb{E}^{\zeta_1}_{\Gamma_1}\Big[\prod_{B\in Z}\big(e^{\tilde{V}_0}(\phi_1+\zeta_1;B) - e^{\delta E_1|B|+U_1}(\phi_1;B)\big)\Big] \\
K_1(\phi_1;Y) &= \prod_{Z\in\mathcal{P}^c_1(Y)} K_1(\phi_1;Z).
\end{aligned}$$

As it was the case for the case of Brydges-Yau method, this form is conserved under taking fluctuation integral, and we can choose $U_j$ to be

$$U_j(s_j,z_j,\phi;X) = \sum_{r\in X}\frac{s_j}{2}\phi(r)(-\triangle\phi)(r) + 2z_jL^{-2j}\cos(\alpha\phi(r)) + \sum_{B\in X} W_j(s_j,z_j,\phi,B).$$

The most naive way to do this is the following.

**Lemma 7.2.** *Suppose $A_1, A_2$ are $j^{th}$-scale polymer activities such that $A_l(\phi;X_j) = \prod_{B\in X_j} A_l(\phi;B)$ for $l=1,2$, $X_j \in \mathcal{P}_j$, $K_j$ a $j^{th}$-scale polymer activity, $\zeta$ a random field and $\phi$ a given field. Then*

$$\mathbb{E}[(K_j \circ A_1)(\phi+\zeta;\Lambda)] = (K_{j+1}\circ A_2)(\phi;\Lambda)$$

*where*

$$K_{j+1}(\phi;X) = \sum_{Y\in\mathcal{P}_j(X)} \mathbb{I}(\bar{Y}=X)A_2(X\backslash Y)\mathbb{E}[(K_j\circ(A_1(\phi+\zeta)-A_2(\phi)))(\phi+\zeta;Y)] \tag{7.1}$$

*and for $Y\in\mathcal{P}_j$, $\bar{Y} = \{B\in\mathcal{B}_{j+1}; Y\cap B\neq\emptyset\}$, given the functions are integrable.*

*Moreover, if $\zeta\sim\mathcal{N}(0,\Gamma_j)$ where $\Gamma_j(x,y)=0$ whenever $\|x-y\|_\infty > \frac{1}{2}L^{j+1}$ and $K_j$ has factorisation $K_j(\phi+\zeta;X) = \prod_{Y\in\mathcal{P}_j^c(X)} K_j(\phi+\zeta;Y)$ for $X\in\mathcal{P}_j$, then it is also true that*

$$K_{j+1}(\phi;X) = \prod_{Y\in\mathcal{P}_{j+1}^c(X)} K_{j+1}(\phi;Y), \quad \forall X\in\mathcal{P}_{j+1}(X). \tag{7.2}$$

*Proof.* Proof of this lemma also follows from reordering the sums in a correct way. If we denote $\delta A(\zeta+\phi;X) = A_1(\zeta+\phi;X) - A_2(\phi;X)$, then

$$K_j\circ A_1 = K_j\circ(\delta A\circ A_2) = (K_j\circ\delta A)\circ A_2$$

and so

$$\begin{aligned}
\mathbb{E}[(K_j\circ A_1)(\phi+\zeta;X)] &= \sum_{Y\in\mathcal{P}_j(X)} \mathbb{E}[(K\circ\delta I)(\phi+\zeta;Y)]A_2(\phi;X\backslash Y)\\
&= \sum_{Y\in\mathcal{P}_j(X)} \mathbb{E}[(K\circ\delta I)(\phi+\zeta;Y)]A_2(\phi;X\backslash\bar{Y})A_2(\phi;\bar{Y}\backslash Y)\\
&= \sum_{Z\in\mathcal{P}_{j+1}(X)}\sum_{Y\in\mathcal{P}_j(X)} \mathbb{I}(\bar{Y}=Z)\mathbb{E}[(K\circ\delta A)(\phi+\zeta;Y)]A_2(Z\backslash Y)A_2(X\backslash Z)
\end{aligned}$$

The second equality (7.2) also follows from straightforward algebra and the argument of **Lemma 7.1**.

□

With the aid of **Lemma 7.1** and **Lemma 7.2**, we may write the inductive relation

$$e^{\tilde{V}_j(\phi_j;X)} = e^{E_j|X|}\big(e^{U_j}\circ K_j\big)(\phi_j;X)$$

where $U_J$ is as defined above and $K_{j+1}$ can be defined using **Lemma 7.2** with $A_1(\phi_j;Y) = e^{E_j|Y|}e^{U_j(\phi_j;Y)}$, $A_2(\phi_{j+1};X) = e^{E_{j+1}|X|}e^{U_{j+1}(\phi_{j+1};X)}$. Here,$E_{j+1} = E_j + \delta E_{j+1}$ is some chosen constant, and each $K_j$ satisfies the **connected component factorisation** (7.2). Moreover, one may check that, since $\tilde{V}_0(\phi;X)$ is invariant under the translation $\phi\mapsto\phi+2\pi/\alpha$, we have $2\pi/\alpha$-**periodicity**

$$U_j(\phi_j) = U_j(\phi_j+2\pi/\alpha), \quad K_j(\phi_j) = K_j(\phi_j+2\pi/\alpha).$$

This result seems satisfying, but this is not the unique way to define $K_j$, and not the best one either. Consider the following reasoning.

- The proof of contraction of the auxiliary term $K_j$ makes use of the periodicity of $K_j$. Since $K_j(\phi+2\pi/\alpha; X) = K_j(\phi; X)$, one can use Fourier transformation in order to write

$$K_j(\phi + c, X) = \sum_{q\in\mathbb{Z}} e^{i\alpha qc}\hat{K}_j(q, \phi; X).$$

  for a constant field $c$. Recall that upon each renormalization step, the coefficient of $2\pi/\alpha$-periodic functions are scaled by factor of $L^{-\alpha^2/4\pi}$. As such, the term $e^{2\pi i\alpha q}\hat{K}_j(q, \phi; X)$ contracts with scale $L^{-\alpha^2|q|/4\pi}$. This is compensated by the reblocking procedure by a factor of $L^2$, so when at the critical temperature $\alpha^2 = 8\pi$, overall $\hat{K}_j(q, \phi, B_j)$ for $B_j \in \mathcal{B}_j$ contracts for $|q| \geqslant 2$. (And for $\alpha^2 > 8\pi$, the contraction is made for $|q| \geqslant 1$. For this reason, the proof requires less precise control when out of the critical line.) To see the contraction of the remaining $\hat{K}_j(\pm 1, \phi, B_j)$ we will have to see the derivative of $K_{j+1}$ as a function of $K_j$ and choose $E_{j+1}, s_{j+1}, z_{j+1}$ that has best contraction. However, the best bound for $K_{j+1}$ in the form of (7.1) has limitations. So we need a better choice of $K_{j+1}$.

- The part that is making the problem is the contribution of $A_2(X\backslash Y)$ in front of $\mathbb{E}[K_j \circ (\cdots)]$ in (7.2) and the contribution of $K_j(Y)$ in the expansion of $K_j \circ (\cdots)(\phi + \zeta; Y)$. We attempt to exclude each contribution from the definition of $K_{j+1}$. In exchange, $K_{j+1}$ will not be a linear function of $K_j$ anymore.

- The actual version of $K_j$ used is given by the following procedure. First use reblocking procedure that appeared in the proof of **Lemma 7.2** to write, for $X \in \mathcal{P}_{j+1}$,

$$e^{\tilde{V}_j(\phi_j;X)} = \sum_{Y\in\mathcal{P}_{j+1}(X)} e^{U_j(\phi_j;X\backslash Y)} \prod_{Z\in\mathcal{P}_j^c(Y)} K_j^{\uparrow}(\phi; Z)$$

  where

$$K_j^{\uparrow}(\phi; Z) = \sum_{Z'\in\mathcal{P}_j} \mathbb{I}(\bar{Z}' = Z) \cdot e^{U_j(\phi, Z\backslash Z')} \prod_{Y'\in\mathcal{P}_j^c(Z')} K_j(\phi; Z').$$

  From here, we subtract the obstructions, say

$$R_{j+1}(\phi_{j+1} + \zeta_{j+1}; X) = \prod_{Y\in\mathcal{P}_{j+1}^c(X)} \big(K^{\uparrow}(\phi_{j+1} + \zeta_{j+1}; Y) - \sum_{B\in Y} J(\phi_{j+1}; Q, \bar{Q}, B, Y)\big).$$

  where $J$ is the correction, $Q$ indicates the correction for contribution from $\mathbb{E}[K_j]$ and $\bar{Q}$ is the correction for quadratic contribution from $V_j$. They are polymer activities, and will be chosen later. Plugging this back into the equality

$$e^{\tilde{V}_{j+1}}(\phi_{j+1}; X) = \mathbb{E}_{\Gamma_j}^{\zeta_{j+1}}\Big[e^{\tilde{V}_j}(\phi_{j+1} + \zeta_{j+1}; X)\Big]$$

  gives a new definition of $K_{j+1}$.

**7.3. RG flow.** Because we have left a handful of freedom of choice for the definition of $(K_{j+1}(\phi_{j+1}; X))_{X\in\mathcal{P}_{j+1}}$, one should be able to control the size of $K_{j+1}$ by making a suitable choice of $Q, \bar{Q}, J(\phi; Q, \bar{Q}, B, Y), \delta E_{j+1}, s_{j+1}, z_{j+1}$ and $W_{j+1}$ as a function of $K_j$. This process can be divided into several steps.

Step 1. Linearization of $K_j$ : As we have remarked above, we use the linearlized version of $K_j$ to estimate the contraction of the map $(\delta E_j, V_j, K_j) \mapsto K_{j+1}$. But we have to make choice of $J$ first. The following form of $J$ is designed to play exactly the desired role, while being linear in $Q$ and $\bar{Q}$ :

$$J(\phi; Q, \bar{Q}, B, Y) = \frac{Q(\phi; Y)}{|Y|} + \sum_{B'\in\mathcal{B}_j(B)} \sum_{X:B\in X} \mathbb{I}(\bar{X} = Y,\ X \text{ is small}) \frac{\bar{Q}(\phi; X)}{|X|}$$
$$-\mathbb{I}(D = Y) \sum_{Y':D\in Y'} \Big[\mathbb{I}(Y' \text{ is small})$$
$$\times \Big[\frac{Q(\phi; Y')}{|Y'|} + \sum_{B\in\mathcal{B}_j, B\subset D} \sum_{X:B\in X} \mathbb{I}(\bar{X} = Y',\ X \text{ is small}) \frac{\bar{Q}(\phi; X)}{|X|}\Big]\Big]$$

Here, if $X' \in \mathcal{P}_j$, then $|X'|$ is the number of $j^{th}$ scale boxes in $X'$. We say $X' \in \mathcal{P}_j$ is **small** if $|X| \leqslant 4$. Usually, when making a polymer expansion with small polymer activities, the small sets play the role of degree $\leqslant 2$ terms in Taylor expansion. With this choice of $J$, the linear part of $K_j \mapsto K_{j+1}$ becomes, for $Y' \in \mathcal{P}_j$,

$$\mathcal{L}K_j(\phi; Y') = \sum_{X\in\mathcal{P}_j^c(Y')} \mathbb{I}(\bar{X} = Y) \Big[\mathbb{E}^{\zeta}[K_j(\phi+\zeta; X)] - \bar{Q}(\phi; X)\Big]$$
$$+ \Big[\frac{1}{2} \sum_{B_0, B_1\in Y'} \mathbb{I}(\overline{B_0 \cup B_1}) \mathbb{E}^{\zeta}[V_j(\phi+\zeta; B_0); V_j(\phi+\zeta; B_1)] - Q(\phi; Y')\Big]$$
$$- \mathbb{I}(|Y'| = 1) \Big[W_{j+1}(\phi; Y') - \mathbb{E}^{\zeta}[W_j(s_{j+1}, z_{j+1}, \phi+\zeta; Y')]$$
$$- \sum_{Y:Y'\subset Y} \mathbb{I}(Y \text{ is small}) \frac{Q(\phi; Y)}{|Y|}\Big]$$
$$- \sum_{B\in\mathcal{B}_j} \mathbb{I}(\bar{B} = Y') \Big[\delta E_{j+1}|B| + V_{j+1}(\phi, B) - \mathbb{E}^{\zeta}[V_j(\phi+\zeta, B)]$$
$$- \sum_{X:B\in X} \mathbb{I}(X \text{ is small}) \frac{\bar{Q}(\phi; X)}{|X|}\Big].$$

We can choose $\bar{Q}$, $Q$ first and then $W_{j+1}, E_{j+1}, s_{j+1}, z_{j+1}$ from this expression that minimize $\mathcal{L}K_j$.

Step 2. Choice of $Q, \bar{Q}$ : The choice of $Q$ and $\bar{Q}$ are made to make the first and the second term of $\mathcal{L}K_j$ irrelevant. The choice of $Q$ is easy : just make the second term vanish. The choice of $\bar{Q}$ is made to just cancel out the $|q| \leqslant 1$ terms in $K = \sum_{q\in\mathbb{Z}} e^{i\alpha q}\hat{K}(q)$, so choose

$$\bar{Q}(\phi, X) = \frac{1}{|X|}\sum_{z\in X}\mathrm{Tay}_2\mathbb{E}^\zeta[\hat{K}(0, \zeta + \phi - \phi(z); X)] + \frac{1}{|X|}\sum_{p=\pm 1} e^{i\alpha p\phi(z)}\mathrm{Tay}_0\mathbb{E}^\zeta[\hat{K}_j(p, \zeta + \phi - \phi(z); X)].$$

The Taylor expansion is made about the variable $\delta\phi(z) = \phi - \phi(z)$ and $\mathrm{Tay}_j$ indicates Taylor expansion up to degree $j$.

Step 3. Choice of $W_j, E_{j+1}, s_{j+1}, z_{j+1}$ : the three parameters $E_{j+1}, s_{j+1}, z_{j+1}$ and polymer activity $W_{j+1}$ are chosen two make the remaining two lines of $\mathcal{L}K_j$ irrelevant. Denote the fourth line of $\mathcal{L}K_j$ by $D(\delta E_{j+1}, s_{j+1}, z_{j+1}, \phi; Y')$. First choose $\delta E'_{j+1}, s'_{j+1}, z'_{j+1}$ so that $D(\delta E'_{j+1}, s'_{j+1}, z'_{j+1}, \phi; Y')$ is irrelevant. Then choose $W_j$ to be the simplest degree 2 function of $s$ and $z$,

$$W_j(s, z, \phi; B) = s^2 W_a(\phi; B) + z^2 W_b(\phi; B) + zsW_c(\phi; B)$$

and $\delta E_{j+1}, s_{j+1}, z_{j+1}$ as a perturbation of $\delta E'_{j+1}, s'_{j+1}, z'_{j+1}$ that makes

$$- \mathbb{I}(|Y'| = 1)\Big[W_{j+1}(\phi; Y') - \mathbb{E}^\zeta[W_j(s_{j+1}, z_{j+1}, \phi + \zeta; Y')] - \sum_{Y: Y'\subset Y}\mathbb{I}(Y \text{ is small})\frac{Q(\phi; Y)}{|Y|}\Big]$$
$$- D(\delta E_{j+1}, s_{j+1}, z_{j+1}, \phi; Y') + D(\delta E'_{j+1}, s'_{j+1}, z'_{j+1}, \phi; Y')$$

vanish.

As a result, we get a renormalization flow

$$\begin{cases} E_{j+1} = E_j + \delta E_{j+1} = E_j + L^{-2j}(e_{1,j}(K_j) + s_j e_{2,j} + s_j^2 e_{3,j} + z_j^2 e_{4,j}) \\ s_{j+1} = s_j - a_j z_j^2 + \mathcal{F}_j(K_j) \\ z_{j+1} = L^2 e^{-\frac{\alpha^2}{2}\Gamma_j(0,0)}(z_j - b_j s_j z_j + \mathcal{M}_j(K_j)) \\ K_{j+1} = \mathcal{L}K_j + \mathcal{R}_j(z_j, s_j, K_j) \end{cases} \tag{7.3}$$

for some $(e_{k,j})_{k=1}^4$, $a_j$, $b_j$, $\mathcal{L}K_j$, $\mathcal{F}_j$, $\mathcal{M}_j$ and $\mathcal{R}_j$.

**7.4. The analysis.** We now have to show the convergence of the evolution (7.3) given the correct choice of value $s_0$ and $z_0$. This is done by various estimates on the new parameters in (7.3) and some sort of implicit function theorem. By its nature of being a complex dynamics, there is no unique way of doing this. So we only introduce a version of implicit function theorem under appropriate assumptions. The assumptions in the following theorem suggests which estimates we should be seeking for.

**Theorem 7.3** (Stable Manifold Theorem). *Suppose $(x_j)_j, (y_j)_j \in \mathbb{R}$, $(W_j)_j \in N = \prod_j (N_j, \|\cdot\|_j)$ for some product Banach space $N$ with initial data $(x_0, y_0, W_0)$ evolves under*

$$\begin{cases} x_{j+1} - x_j = -y_j^2 + F_j(x_j, y_j, W_j) \\ y_{j+1} - y_j = -x_j y_j + G_j(x_j, y_j, W_j) \\ W_{j+1} = L_j W_j + H_j(x_j, y_j, W_j) \end{cases}$$

*where $F_j(0,0,0) = G_j(0,0,0) = H_j(0,0,0)$ for any $\varepsilon > 0$ there is $\eta_j > 0$ such that whenever $|x_j|, |y_j|, |x_j'|, |y_j'| \leqslant \eta_j$, $\|W_j\|_j, \|W_j'\|_j \leqslant \eta_j^2$, the bounds*

$$
\begin{aligned}
|F_j(x_j, y_j, W_j) - F_j(x_j', y_j', W_j')| &\leqslant C_1(L)L^{-j/4}|y_j^2 - (y_j')^2| \\
&+\varepsilon\Big(\|W_j - W_j'\|_j + \mathbb{I}(j=0)\eta_0(|x_0 - y_0| + |y_0 - y_0'|)\Big) \\
|G_j(x_j, y_j, W_j) - G_j(x_j', y_j', W_j')| &\leqslant C_2 L^{-j/4}|y_j - y_j'| + C_1(L)L^{-j/4}|x_j y_j - x_j' y_j'| \\
&+\varepsilon(\|W_j - W_j'\|_j + \mathbb{I}(j=0)\eta_0(|x_0 - y_0| + |y_0 - y_0'|)) \\
|H_j(x_j, y_j, W_j) - H_j(x_j', y_j', W_j')| &\leqslant C_2(L^{-\theta} + \varepsilon)\mathbb{I}(j=0)\eta_0(|x_0 - y_0| + |y_0 - y_0'|) \\
&+C(\varepsilon, L)(\eta_j^2|x_j - x_j'| + \eta_j^2|y_j - y_j'| + \eta_j\|W_j - W_j'\|_j)
\end{aligned}
$$

*are satisfied for any $L > 0$ large enough and some $C_1(L), C_2, C(\varepsilon, L), \theta > 0$. Then for sufficiently larege $L$, there exists $\varepsilon_1, \varepsilon_2 > 0$ such that whenever $|y_0| \leqslant \varepsilon$, $\|W_0\|_0 \leqslant |y_0|\varepsilon_2$, there is a continuous function $\Sigma : (-\varepsilon, \varepsilon) \to \mathbb{R}$ such that whenever $x_0 = \Sigma(y_0)$, we have*

$$
x_j = |q_j| + O(j^{-3/2}), \quad y_j = q_j + O(j^{-3/2}), \quad \big\|W_j\big\|_j = O(j^{-3})
$$

*with $q_j = y_0/(1 + j|y_0|)$.*

*Proof.* Assume $y_0 > 0$ without loss of generality. First make change of parameter $u_j = x_j - q_j$, $v_j = y_j - q_j$ then $(u_j, v_j, W_j)$ satisfies a new set of equations. Now diagonalize the flow for $(u_j, v_j)$ hence take new variables $w_j^- = u_j - v_j$, $w_j^+ = u_j + 2v_j$, each the unstable and stable direction. They satisfy

$$
\begin{aligned}
w_{j+1}^+ - w_j^+ &= (-2q_{j+1} + q_{j+1}^2)w_j^+ + F_j^+(u_j, v_j, W_j) \\
w_{j+1}^- - w_j^- &= q_j w_j^- + F_j^+(u_j, v_j, W_j)
\end{aligned}
$$

for new $F_j^+, F_j^-$. Using telescoping sum, we then have

$$
\begin{cases}
w_j^+ = \left(\frac{q_j}{q_0}\right)^2 w_0^+ + \sum_{i=0}^{j-1}\left(\frac{q_j}{q_{i+1}}\right)^2 F_i^+(u_j, v_j, W_j) \\
w_j^- = w_\infty^- - \sum_{k \geqslant j} \frac{q_{k+1}}{q_j} F_k^-(u_j, v_j, W_j) \\
W_j = (\prod_{i=0}^{j-1} L_i)W_0 + \sum_{i=0}^{j-1}(\prod_{k=i+1}^{j-1} L_k)F_i^0(u_j, v_j, W_j).
\end{cases}
\tag{7.4}
$$

Then one may write the equation above as a function

$$
T : ((w_j^+)_j, (w_j^-)_j, W_j) \mapsto ((w_j^+)_j, (w_j^-)_j, W_j)
$$

with choice $w_\infty^-$. If the map $T$ has a fixed point such that $w_j^+, w_j^-, W_j$ tends to 0 as $j \to \infty$, then the solution of (7.4) exists with $w_\infty^- = 0$, hence the theorem follows.

To see this, we use the Banach fixed point theorem on space $(\mathbb{W}, \|\cdot\|)$ where

$$
\|(u, v, W)\| = \sup_{j \geqslant 1} \max\{(\tau h_j)^{-1}|u_j|, 2(\tau h_j)^{-1}|v_j|, (\tau h_j)^{-2}\|W_j\|_j\}
$$

for some $\tau > 0$ to be chosen later and $h_j = y_0/(1 + j \cdot y_0)^{3/2}$, and

$$
\mathbb{W} = \{w = (u, v, W) : \|(u, v, W)\| \leqslant 1\}.
$$

Also denote $T(w) = ((T_j^+(w))_j, (T_j^+(w))_j, (T_j^0(w))_j)$. We will show that $\|T(w-w')\| \leqslant \frac{1}{2}\|w-w'\|$ for sufficiently small $\tau > 0$. Since $T(0)=0$, this implies $T(\mathbb{W}) \subset \mathbb{W}$ and hence $T : \mathbb{W} \to \mathbb{W}$ satisfies the condition for Banach fixed point theorem.

First, for $(a_j)_j$, denote $\bar{T}_j^+(a) = \sum_{i=0}^{j-1}(q_j/q_{i+1})^2 a_i$, $\bar{T}_j^-(a) = -\sum_{i\geqslant j}(q_{s+1}/q_j)a_i$ so that

$$\begin{aligned} T_j^+(w) - T_j^+(w') &= \bar{T}_j^+(F^+(w)) - \bar{T}_j^+(F^+(w')), \\ T_j^-(w) - T_j^-(w') &= \bar{T}_j^-(F^-(w)) - \bar{T}_j^-(F^-(w')). \end{aligned}$$

For $\sigma = \pm 1$, our assumptions implies

$$|F_j^\sigma(w) - F_j^\sigma(w')| \leqslant C\|w-w'\|\tau[[h_j^2(\tau+q_1) + C_1(L)\varepsilon h_j^2 + L^{-1/4}p_j q_j]$$

for some new constant $C>0$ and hence

$$|\bar{T}(F_j^\sigma(w)) - \bar{T}(F_j^\sigma(w'))| \leqslant C\|w-w'\|\tau h_j[\tau + q_0 + L^{-1/4} + C_1(L)\varepsilon].$$

So setting $16C\tau \leqslant 1$, $L$ large enough so that $16CL^{-1/4} \leqslant \tau$, $\varepsilon$ small so that $16C(L)\varepsilon C \leqslant \tau$ and $\varepsilon_1 > 0$ small so that $16C(L)\varepsilon_1 C \leqslant \tau$, we have

$$|T_j^\alpha(F^\sigma(w)) - T_j^\alpha(F^\sigma(w'))| \leqslant \|w-w'\|\frac{\tau h_j}{4}.$$

Now setting $\varepsilon_2 > 0$ small enough, we have

$$\|F_j^0(w) - F_j^0(w')\|_j \leqslant C\tau\|w-w'\|(L^{-\theta} + \varepsilon + C(\varepsilon, L)q_1)h_j^2.$$

Setting $L$ large enough and $\varepsilon$ small enough, we have

$$\|T_j^0(F_j^0(w)) - T_j^0(F_j^0(w'))\|_j \leqslant \frac{\tau h_j^2}{2}\|w-w'\|.$$

Putting these together, we conclude that $T$ is a contraction map.

$\square$

Now we are just left to define the norm on the polymer activities $K_j$'s. The strategy is similar as in the Brydges-Yau methods, but we use a bit more complicated regulator.

For polymer activity $\phi$, $\phi$ a field, $X \in \mathcal{P}_j$, we have as before,

$$\|D^n A(\phi, X)\|_{j,\phi,X} = \sup\{|D^n A(\phi,X)(f)| : \sum_{r\in X} L^{2nj}|D^n f(x)|^2 \leqslant 1\}$$

and let

$$\begin{aligned} \|A(\phi,X)\|_{h,j,\phi,X} &= \sum_{n\geqslant 0}\frac{h^n}{n!}\|A(\phi,X)\|_1 \\ \|A(X)\|_{h,j,X} &= \sup\Big\{\frac{\|A(\phi,X)\|_h}{G_j(\phi,X)} : \sum_{n\leqslant 2}\sum_{r\in X} L^{2nj}|D^n\phi(x)|^2 \leqslant 1\Big\} \\ \|A\|_{h,j} &= \sup_{X\in\mathcal{P}_j^c}\gamma^{|X|}\|A(X)\|_{h,j,X}. \end{aligned}$$

The parameter $\gamma > 0$ is set large, and plays the role of $\varepsilon^{-1}$ in **Theorem 7.3**. Only the field regulator differs, with

$$G_j(\phi, X) = \exp(c_1 \sum_{\alpha \leqslant 2} \sum_{r \in X} L^{2j} |D^\alpha \phi(x)|^2 + c_2 \sum_{r \in X} L^{4j} |D^2 \phi(x)|^2 + c_1 \sum_{B \in X} \|\phi\|^2_{L^\infty(B^*)})$$

where $B^*$ is the small-set neighbourhood of $B$, i.e. the union of small sets that intersects $B$. We have denoted $\|K_j\|_j$ for the norm $\|K_j\|_{h,j}$.

**7.5. The result.** Putting these together, we have obtained the following :

**Theorem 7.4.** *Having $\alpha > 0$, $L > 0$, there exists $C > 0$ such that*

$$|\delta E_j| \leqslant C L^{-2j} \varepsilon_1$$

*whenever $|s_j|, |z_j|, \|K_j\|_j \leqslant \varepsilon_1$ for a sufficiently small $\varepsilon_0$. Therefore, there exists $\varepsilon_2 > 0$ small and a continuous function $\Sigma : (-\varepsilon_2, \varepsilon_2) \to \mathbb{R}$ such that $\Sigma(0) = 8\pi$ and whenever $s = s(z) \equiv \Sigma(z)$, we have the limit*

$$\lim_{|\Lambda| \to \infty} \frac{1}{|\Lambda|} \log Z(\beta, z; \Lambda) = \frac{1}{2} \log(1 - s) + \sum_{j=1}^{\infty} \delta E_j(s, z) < \infty$$

*exists.*

The first resulting statement obtained by this renormalization flow does not quite catch the full implication of the method. Indeed, this method can be developed further to estimate the correlation length at a relatively high precision that is hard to be achieved using any other method. In [8], the two point energy fractional charges (non-integer charge) on the critical line is estimated, as was already introduced.

**Theorem 7.5** (Falco, 2013)**.** *Given $\eta = 1/2$, there is $L_0 \equiv L_0(z) > 1$, $z_0 \equiv z_0(\eta) > 0$ and a continuous function $\beta_c(z) \geqslant 8\pi$ such that if $L \geqslant L_0$, $|z| < z_0$ and $\beta = \beta_c(z)$, the limit of the two-point energy of two particles with fractional charges $\pm\eta$,*

$$\lim_{|\Lambda| \to \infty} \frac{Z(x_1, x_2, \eta, \beta; \Lambda)}{Z(\beta; \Lambda)}$$

*exists where $Z(x_1, x_2, \eta, \beta; \Lambda)$ is a partition function of configurations with $+\eta$ charge at $x_1$ and $-\eta$ charge at $x_2$ and the limit is*

$$\rho_{1/2}(x) = \frac{1}{2} \frac{e^{2\pi c_E} + f_a}{|x|} (1 + f \log |x|)^{1/2} (1 + o(1))$$

*where $f = c|z|$, $c > 0$, $f_b = c(\eta)^2 z^2 (1 + \tilde{f}_b)$, $c(\eta) > 0$, $f_a, \tilde{f}_b \to 0$ as $z \to 0$.*

# 8 Discussions

So far, we had seen various methods to prove the KT transition in the sine-Gordon model and the IV-GFF. The Fröhlich-Spencer method and the Peierls argument provides a strong evidence of a phase transition, while both methods were perturbative in the inverse-temperature $\beta$, and nothing could be studied on the intermediate $\beta$ region. On the other hand, for the sine-Gordon model, both the Fröhlich-Spencer method and the renormalization methods could be developed in a non-perturbative way, and hence gives better results for the near-critical delocalization regime. While no information could be retrieved at the near-critical localization regime.

It is plausible that the method of [19] could be applied to prove an improved range of the delocalization regime for IV-GFF, but the exact near-critical analysis relies strongly on the specific structure of the sine-Gordon model, so it is hard to expect such a refined result.

The two renormalization methods could also be a good candidate to study the delocalization regime of the IV-GFF. In addition, the RG methods do not only compute a two-point function, but also proves a scaling limit, enveloping wider range of implications. However, both these methods exploit our a priori knowledge of the renormalized form of the sine-Gordon potential, which is not so clear for the case IV-GFF. This a priori knowledge was a result of a formal renormalization computations, which was only possible because of the analyticity of the sine-Gordon potential. For IV-GFF, the potential is highly singular, so one needs a manipulation step that attains a form close to this. We will see how this could be made into a plausible argument shortly.

**8.1. Scaling Limit of IV-GFF.** The Falco's method suggests a plausible method for proving the scaling limit of IV-GFF for high temperature, or sufficiently small value of $\beta$. As explained previously, the greatest difficulty with dealing with IV-GFF is that the relative density function of IV-GFF with respect to the Gaussian is not a proper function, but is concentrated on particular discrete values.

To go about this technical difficulty, we have seen that [10] used a finite series approximation of the periodic distribution and proved inequalities for the approximations, but such a limiting procedure was only possible because we were trying to prove a mere inequality - this trick does not work in the RG setting. Instead, we try to smooth out the potential using a RG step. Consider the covariance decomposition $(-\triangle + m^2)^{-1} = \sum_{j=0}^{N-2} \Gamma_j(\cdot\,; m) + \Gamma_{\geqslant N-1}(\cdot\,; m)$ and denote $\Gamma_j(\cdot) = \Gamma_j(\cdot\,; 0)$ for $j = 0, \cdots, N-2$. In order to treat the off-criticality, only the first-order analysis is necessary, so we can apply **Lemma 7.1** with

$$\begin{aligned}
A_1(\phi - \zeta; X) &= \sum_{M:\Lambda\to\mathbb{Z}} \sum_{j\in X} \delta_0(\phi(j) - \zeta(j) - M(j)) - e^{\delta E_1 + W(\phi;X)} \\
A_2(\phi; X) &= e^{\delta E_1 |X| + W_1(\phi;X)}
\end{aligned}$$

with $W_1$ some periodic function and $\delta E_1$ left to be chosen. This gives

$$\mathbb{E}^{\zeta_1}_{\Gamma_1}\Big[ \sum_{M:\Lambda\to\mathbb{Z}} e^{-\frac{1}{2}\langle \phi_1 - \zeta_1, \Gamma_0^{-1}(\phi-\zeta)\rangle} \prod_{j\in\Lambda} \delta_0(\phi_1(j) - \zeta_1(j) - M(j))\Big] = e^{\delta E_1|\Lambda|}\big(e^{W_1} \circ K_1\big)(\phi_1; \Lambda) \quad (8.1)$$

where

$$K_1(\phi_1; X) = \sum_{M:\Lambda\to\mathbb{Z}} \exp\big(-\frac{1}{2}\langle(\phi - M)|_X, \Gamma_0^{-1}|_X(\phi - M)|_X\rangle\big) - e^{\delta E_1|X| + W_1(\phi_1;X)}.$$

The most reasonable choice for $\delta E_1$ would now be given by

$$\delta E_1 = \frac{1}{|\Lambda|}\log\Big[\Big(\frac{1}{2\pi}\Big)^{|\Lambda|}\int_{\mathbb{R}^\Lambda} d^\Lambda\theta \sum_{M:\Lambda\to\mathbb{Z}} \exp\Big(-\frac{1}{2}\langle(\phi+\theta-M)|_B, \Gamma_0^{-1}|_B(\phi+\theta-M)|_B\rangle\Big)\Big]$$

for any $B \in \mathcal{B}_0$ so that extraction of $e^{\delta E_1|}$ from $\sum_{M:\Lambda\to\mathbb{Z}}\exp\big(-\frac{1}{2}\langle(\phi - M)|_B, \Gamma_0^{-1}|_B(\phi - M)|_B\rangle\big)$ would only give a exponential of a periodic function. Then an explicit expansion of (8.1) would give

$$e^{\delta E_1|\Lambda|}\Big(e^{W_1}(\phi_1;\Lambda) + \sum_{B\in\mathcal{B}_0} e^{W_1}(\phi_1;\Lambda\backslash B)K_1(\phi;B) + \sum_{Y\in\mathcal{P}_0,\,|Y|\geqslant 2} e^{W_1}(\phi_1;\Lambda\backslash Y)K_1(\phi;Y)\Big).$$

The summation over the non-box set, $\sum_{Y\in\mathcal{P}_0,\,|Y|\geqslant 2}$ could be controlled relatively easily, and $K_1(\phi;B)$ can also be bounded because of the definition of $\delta E_1$.

Once we have obtained the estimates, we could now start again using the formulation of Falco - we first extract

$$\frac{1}{2}s\langle\phi_1|_X, (\Gamma_{\geqslant 1})^{-1}|_X\phi_1|_X\rangle \approx \frac{1}{2}sL^2 \sum_{B\sim B'\in\mathcal{B}_\infty(X)} |\phi_1(B) - \phi_1(B')|^2$$

from $W_1(\phi_1;X)$ for a variable $s$ to be determined implicitly, and proceed the renormalizaiton, using **Lemma 7.2**. We would not need a more complicated renormalization step if we only aim to obtain an off-critical result. But we would need more work to obtain bounds on $W_1$ and $K_1$, because these computations necessarily involve massive combinatorics.

# References


[1] Abdelmalek Abdesselam and Vincent Rivasseau. Trees, forests and jungles: a botanical garden for cluster expansions. *Lecture Notes in Physics*:7–36.

[2] D. Brydges, C. Guadagni, G., and P. K.Mitter. Finite range decomposition of gaussian processes. *Journal of Statistical Physics*, 115(1/2):415–449, April 2004.

[3] D. Brydges and H.T. Yau. Grad $\phi$ perturbations of massless gaussian fields. *Commun. Math. Phys.*, 129:351–392, 1990.

[4] David C. Brydges and Paul Federbush. Debye screening. *Commun. Math. Phys.*, 73:197–246, 1980.

[5] John Cardy. *Scaling and Renormalization in Statistical Physics*. Cambridge University Press, 1996.

[6] J. Dimock and T.R. Hurd. Sine-gordon revisited. *Annales Henri Poincaré*, 1(3):499–541, July 2000.

[7] Pierluigi Falco. Kosterlitz-thouless transition line for the two dimensional coulomb gas. *Commun. Math. Phys.*, 312:559–609, 2012.

[8] Pierluigi Falco. Critical exponents of the two dimensional coulomb gas at the berezinskii-kosterlitz-thouless transition, 2013.

[9] Jürg Fröhlich and Yong Moon Park. Remarks on exponential interactions and the quantum sine-gordon equation in two space-time dimensions. *Helvetica Physica Acta*, 50, 1977.

[10] Jürg Fröhlich and Thomas Spencer. The kosterlitz-thouless transition in two-dimensional abelian spin systems and the coulomb gas. *Commun. Math. Phys.*, 81:527–602, 1981.

[11] Christophe Garban and Avelio Sepúlveda. Statistical reconstruction of the gaussian free field and kt transition. 2020. (Visited on ).

[12] J. Glimm and A. Jaffe. *Quantum Physics : A Functional Integral Point of View*. Springer-Verlag, New York, 1981.

[13] Markus Göpfert and Gerhard Mack. Proof of confinement of static quarks in 3-dimensional u(1) lattice gauge theory for all values of the coupling constant. *Commun. Math. Phys.*, 82:545–606, 1982.

[14] T.R. Hurd J. Dimock. A renormalization group analysis of the kosterlitz-thouless phase. *Commun. Math. Phys.*, 137:263–287, 1991.

[15] M. Kac. On the partition functions of one dimensional gas. *Phys. Fluids*, 2:8–12, 1959.

[16] Vital Kharash and Ron Peled. The fröhlich-spencer proof of the berezinskii-kosterlitz-thouless transition, 2017.

[17] J. M. Kosterlitz. The critical properties of the two-dimensional xy model. *J. Phys. C : Solid State Phys.*, 7, 1974.

[18] J. M. Kosterlitz and D. J. Thouless. Ordering, metastability and phase transitions in two-dimensional systems. *J. Phys. C : Solid State Phys.*, 6, 1973.

[19] Domingos H. U. Marchetti and Abel Klein. Power-law fall off in two-dimensional coulomb gases at inverse temperature $\beta > 8\pi$. *Journal of Statistical Physics*, 64(1/2), 1991.

[20] R. Peierls. On ising's model of ferromagnetism. *Proc. Cambridge Phil. Soc*, 32(477), 1936.

[21] A.J.F. Siegert. Partition functions as averages of functionals of gaussian random functions. *Physica*, 26:S30–S35, December 1960.

[22] D. Tong. Statistical field theory. Available at `http://www.damtp.cam.ac.uk/user/tong/sft.html`.

[23] Wei-Shih Yang. Debye screening for two-dimensional coulomb systems at high temperatures. *Journal of Statistical Physics*, 49(1/2):1–32, 1987.