Statistics 330 - Final Exam

Total marks: 75.

1. Let X_1 and X_2 be independent poisson random variables with $E(X_1) = \lambda$ and $E(X_2) = 5\lambda$.

   (a) Find $P(X_1 + X_2 = m)$ for any positive integer m. [4]

   (b) Assuming that for $Y = [X_1 | X_1 + X_2 = m]$, $P(Y = k) = P(X_1 = k | X_1 + X_2 = m)$. Identify the distribution of Y. [4]

2. Let X be a continuous random variable with the cdf given by

$$F_X(x) = \begin{cases} 1 & 1 \leq x \\ x^2 & 0 \leq x < 1 \\ 0 & x < 0, \end{cases}$$

   (a) find the pdf of X and show that it is a pdf. [3]

   (b) Identify the pdf of X obtained in (a). [2]

3. Let X be a continuous random variable with the pdf given by

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Find the cdf of X. [2]

   (b) Find the pdf of $Y = F_X(u)$, where $F_X(\cdot)$ is the cdf of X obtained in (a). [3]

4. Let X be a discrete random variable with the pmf $P(X = x) > 0$ on the set of non-negative integers $\{0, 1, 2, ...\}$.

   (a) Show that the cdf $F_X(x)$ is a non-decreasing function. [5]

   (b) Prove the following identity. [4]

$$E(X) = \sum_{n=0}^{\infty} P(X > n).$$

5. Let X_1, X_2, X_3 be independent and identically distributed $N(0,1)$ random variables. Then define Y_1, Y_2 and Y_3 as

$$\begin{aligned} X_1 &= Y_1 \cos(Y_2)\sin(Y_3) \\ X_2 &= Y_1 \sin(Y_2)\sin(Y_3) \\ X_3 &= Y_1 \cos(Y_3) \end{aligned}$$

where $0 \le Y_1 < \infty, 0 \le Y_2 < 2\pi$ and $0 \le Y_3 \le \pi$. Then, show that Y_1, Y_2 and Y_3 are mutually independent. [10]

6. Let $X_1, ..., X_{50}$ be a random sample of size 50 from $N(0,1)$. Assume that the moment generating function for a random variable W with distribution $N(\mu, \sigma^2)$ is given by

$$M_W(t) = e^{\mu t + \sigma^2 \frac{t^2}{2}}. \tag{1}$$

Then, (a) without finding the pdf, find the mgf of $Z = 5X_1 + 3X_2 - \frac{1}{9}X_{10}$ and (b) compare it with the mgf given in equation (1) to identify the distribution of Z. [5+3]

7. Let $X_1, ..., X_n$ be a random sample from truncated exponential distribution with the pdf

$$f_X(x) = \begin{cases} e^{-(x-\theta)} & x > \theta \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the maximum likelihood estimator of θ (say $T_1(\underline{X})$). [5]

(b) Find a method of moment estimator of θ (say $T_2(\underline{X})$). [3]

(c) Evaluate $B_{T_1}(\theta)$, $B_{T_2}(\theta)$ and $var(T_1(\underline{X}))$ where, $B_T(\theta) = \mathrm{E}(T) - \theta$. [6]

8. Let $X_1, ..., X_n$ be a random sample from $\Gamma(\alpha, \lambda)$, where α is known.

(a) Find the maximum likelihood estimator of $\theta = \frac{1}{\lambda}$ (say $T(\underline{X})$). [3]

(b) Find the exact distribution of $T(\underline{X})$. [3]

(c) Assuming n is large, find the approximate distribution of $T(\underline{X})$. [10]

Formula Sheet

1. Let $X \sim N(0,1)$, then

$$f_X(x) = \begin{cases} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} & -\infty < x < \infty \\ 0 & otherwise. \end{cases}$$

2. Let $X \sim Beta(\alpha, \beta)$, then

$$f_X(x) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} & 0 < x < 1 \\ 0 & otherwise. \end{cases}$$

3. Let $X \sim \Gamma(\alpha, \lambda)$, then

$$f_X(x) = \begin{cases} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} & 0 < x < \infty \\ 0 & otherwise. \end{cases}$$

4. Let $X \sim Poi(\lambda)$, then

$$P(X = x) = \frac{\lambda^x}{x!} e^{-\lambda} \quad \text{for } x = 0, 1, 2, ... \quad \text{and zero elsewhere.}$$

5. Let $X \sim Bin(n, p)$, then

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k} \quad \text{for } k = 0, 1, 2, ..., n \quad \text{and zero elsewhere.}$$

6. Some properties of $\Gamma(\cdot)$:

(a) $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$, for any positive real number α.

(b) $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

(c) $\Gamma(0) = \Gamma(1) = 1$.

(d) $\Gamma(k + 1) = k!$, for any positive integer k.

7. Determinant of a 3×3 matrix :

$$\text{Let } J = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \text{ then, } det(J) = |J| = a \cdot \left| \begin{pmatrix} e & f \\ h & i \end{pmatrix} \right| - b \cdot \left| \begin{pmatrix} d & f \\ g & i \end{pmatrix} \right| + c \cdot \left| \begin{pmatrix} d & e \\ g & h \end{pmatrix} \right|$$

8. A function $g(\cdot)$ is said to be **non-decreasing** if $g(b) \geq g(a)$ whenever $b > a$.

9. Let $X_1, ..., X_n$ be a random sample such that the pdf of $X_i \sim f(x)$ for $i = 1, ..., n$, then the k-th order statistics is denoted by $X_{(k)}$. The pdf of $Y = X_{(k)}$ is given by

$$f_{X_{(k)}}(x) = \frac{n!}{(k-1)!\,(n-k)!}(F(x))^{k-1}f(x)(1-F(x))^{n-k}$$

for $1 \leq k \leq n$. The joint pdf of $Y = (X_{(i)}, X_{(j)})$ for $i < j$, is given by

$$f_Y(u,v) = \frac{n!}{(i-1)!\,(j-1-i)!\,(n-j)!}(F(u))^{i-1}f(u)(F(v)-F(u))^{j-1-i}f(v)(1-F(v))^{n-j}.$$

10. **Central Limit Theorem** : Let $Z_1, ..., Z_n$ be identically distributed random variables with $E(Z_1) = \mu$ and $var(Z_1) = \sigma^2$, then for large values of n, the distribution of $\frac{\sqrt{n}(\bar{Z}-\mu)}{\sigma}$ can be approximated by $N(0,1)$.

10. **Delta Method** : If Y is a random variable such that $\sqrt{n}(Y-\mu) \sim N(0,\sigma^2)$, then for any one-to-one function $g(\cdot)$, $\sqrt{n}(g(Y)-g(\mu)) \sim N(0,(g'(\mu))^2\sigma^2)$.

11. Let $X_1, ..., X_m$ be a random sample from
(a) $Poi(\lambda)$, then $\sum_{i=1}^{m} X_i \sim Poi(m\lambda)$.
(b) $Bin(k,p)$, then $\sum_{i=1}^{m} X_i \sim Bin(mk,p)$.
(c) $\Gamma(\alpha,\lambda)$, then $\sum_{i=1}^{m} X_i \sim \Gamma(m\alpha,\lambda)$.

12. Trigonometric relations:
(a) $\frac{d}{dx}sin(x) = cos(x)$
(b) $\frac{d}{dx}cos(x) = -sin(x)$
(c) $sin^2(x) + cos^2(x) = 1$.