2-10

B.Sc. Engg. / HD CSE 4th Semester

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ISLAMIC UNIVERSITY OF TECHNOLOGY (IUT)
ORGANISATION OF ISLAMIC COOPERATION (OIC)
Department of Computer Science and Engineering (CSE)

SEMESTER FINAL EXAMINATION **SUMMER SEMESTER, 2017-2018**

DURATION: 3 Hours **FULL MARKS: 150**

Math 4441: Probability and Statistics

Programmable calculators are not allowed. Do not write anything on the question paper.

There are 8 (eight) questions. Answer any 6 (six) of them.

Figures in the right margin indicate marks.

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1. a) Suppose that four balls are selected one at a time, without replacement, from a box containing r red balls and b blue balls ($r \geq 2, b \geq 2$). Determine the probability of obtaining the sequence of outcomes red, blue, red, blue. 8

   b) Suppose that a box contains 10 balls. At the start, 3 are white and 7 are blue. Whenever a ball is selected from the box, a layer of blue paint is applied to it, so blue balls stay blue, and white balls become blue; afterward, the ball is returned to the box, so that 10 balls always in the box.

      i. Find the probability that a blue ball is chosen at the beginning of round 3. 8

      ii. Suppose at the start of round 3 (i.e., before any painting in round 3 is performed), a blue ball is selected. Find the probability that the blue ball was originally blue (it was blue at the beginning and was not colored at round 1 or 2). 9

2. a) A business trip is equally likely to take 2, 3, or 4 days. After a d-day trip, the change in the traveler's weight, measured as an integer number of pounds, is uniformly distributed between $-d$ and d pounds. For one such trip, denote the number of days D and the change in weight by W. Find the joint PMF $P_{DW}(d, w)$. 12

   b) First a point Y is selected at random from the interval (0,1). Then another point X is selected at random from the interval $(Y, 1)$. Find the probability density function of X. 13

3. a) A prisoner is trapped in a cell containing three doors. The first door leads to a tunnel that returns him to his cell after two days of travel. The second door leads to a tunnel that returns him to his cell after three days of travel. The third door leads immediately to freedom.

      i. Assume that the prisoner will always select doors 1, 2, and 3 with probabilities 0.5, 0.3, and 0.2, respectively. Find the expected number of days until the prisoner reaches freedom. 7

      ii. Assume that the prisoner is always equally likely to choose among those doors that he has not used previously. Find the expected number of days until he reaches freedom. Note: If the prisoner initially tries door 1, then when he returns to the cell, he will now select only from doors 2 and 3. 8

   b) A game is often played in carnivals and gambling houses is called chuck-a-luck, where a player bets on any number 1 through 6. Then three fair dice are tossed. If one, two, or all three land the same number as the player's, then he or she receives one, two, or three times the original stake plus his or her original bet, respectively. Otherwise, the player loses his or her stake. Let X bet the net gain of the player per unit stake. First find the probability mass function of X; then determine the expected amount that the player will lose per unit of stake. 10

4. a) The amount of time that a customer spends waiting at an airport check-in counter is a random variable with mean 8.2 minutes and standard deviation 1.5 minutes. Suppose that a random sample of $n = 49$ customers is observed. Find the approximate probability that the average time waiting in line for these customers is:
   i. Less than 10 minutes. 6
   ii. Between 5 and 10 minutes. 8

   b) Let $X_1, X_2, \dots, X_n$ be a sample from a population with density function given by 11

$$f(x) = \begin{cases} \theta x e^{(-\theta x)}, & x \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

   Determine the maximum likelihood estimator (MLE) of the parameter θ.

5. a) A college admissions officer wanted to know the average Scholastic Aptitude Test (SAT) score of this year's class of entering students. Instead of checking all student folders, she decided to use a randomly chosen sample. It is known that students' scores are normally distributed with a mean μ and standard deviation σ. If the value of σ is 70, how large a random sample is needed if the admissions officer wants to obtain a 95% confidence interval estimate that is of length 4 or less? 10

   b) An important issue for a retailer is to decide when to reorder stock from a supplier. A common policy used to make the decision is of a type called d, D: The retailer orders at the end of a period if the on-hand stock is less than d, and orders enough to bring the stock up to D. The appropriate values of d and D depend on different cost parameters, such as inventory holding costs and the profit per item sold, as well as the distribution of the demand during a period. Consequently, it is important for the retailer to collect data relating to the parameters of the demand distribution. Suppose that the following data give the numbers of a certain type of item sold in each of 30 weeks. 15

   14, 8, 12, 9, 5, 22, 15, 12, 16, 7, 10, 9, 15, 15, 12,
   9, 11, 16, 8, 7, 15, 13, 9, 5, 18, 14, 10, 13, 7, 11

   Assume that the numbers of items order in each week are independent random variables from a common distribution. Use the data to obtain a 95 percent confidence interval for the mean number ordered in a week, if $d = 10$ and $D = 20$.

6. a) A bakery was taken to court for selling loaves of bread that were under-weight. These loaves were advertised as weighing 24 ounces. In its defense, the bakery claimed that the advertised weight was meant to imply not that each loaf weighted exactly 24 ounces, but rather that average value over all loaves was 24 ounces. The prosecution in a rebuttal produced evidence that a randomly chosen sample of 20 loaves had an average weight of 22.8 ounces with a sample standard deviation of 1.4 ounces. In her ruling, the judge stated that advertising a weight of 24 ounces would be acceptable if the mean weight were at least 23 ounces.
   i. State the null and alternative hypothesis to be tested for the claim. 5
   ii. For the 5 percent level of significance, what should be the judge's rule? 8

   b) Suppose that if a signal of intensity μ is emitted from a particular star, then the value received at an observatory on earth is a normal random variable with mean μ and standard deviation 4. In other words, the value of the signal emitted is altered by random noise, which is normally distributed with mean 0 and standard deviation 4. It is suspected that the intensity of the signal is equal to 10. Test whether this hypothesis is plausible if the same signal is independently received 20 times and the average of the 20 values received is 11.6. Assume a 5 percent level of significance. 12
   Find the probability that the null hypothesis (that the signal intensity is equal to 10) will not be rejected when the actual signal level is 9.2.

7. a) When a two-way paging system transmits a message, the probability that the message will be received by the pager it is sent to is p. When the destination pager receives the message, it transmits an acknowledgment signal (ACK) to the source pager. If the source pager does not receive the ACK, it sends the message again.

   i. The paging company wants to limit the number of times it has to send the same message. It has a goal of $P[N \leq 3] \geq 0.95$. What is the minimum value of p necessary to achieve the goal? 8

   ii. As an alternative, assume that the company wants to limit the number of times the paging system has to send the same message to K, by discarding the message after the K-th unsuccessful attempts. Suppose the message takes 1 ms to transmit and the source pager waits an additional 1 ms to receive the ACK. 9
   Find the PMF of D, the delay of a successfully delivered message.

   b) Customer arrives at a bookstore at a Poisson rate of six per hour. Given that the store opens at 9:30 am, find the probability that exactly one customer arrives by 10:00 am and at least one customer by 10:30 am. 8

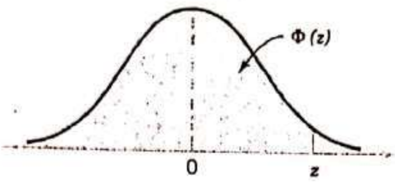
8. a) A professor pays 25 cents for each blackboard error made in the lecture to the student who points out the error. In a career of n years filled with blackboard errors, the total amount in dollars paid by the professor can be approximated by a Gaussian random variable Y_n with expected value $40n$ and variance $100n$. What is the probability that Y_{20}exceeds 1000? 8

   b) A circle of radius 1 is inscribed in a square with sides of length 2. A point is selected at random from the square. What is the probability that it is inside the circle? Note that by a point being selected at random from the square we mean that the point is selected in a way that all the subsets of equal areas of the square are equally likely to contain the point. 8
   Hints: Coordinate of the point selected at random can be represented by two random variables.

   c) Two fair dice are rolled. The maximum and minimum of the outcomes are denoted by X and Y, respectively. Calculate the joint probability mass function of X and Y. Find the marginal probability mass functions of X. 9

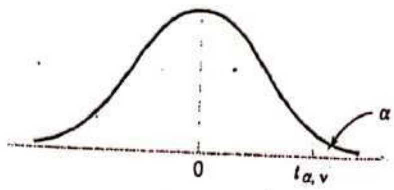
Appendix A: CDF of Standard Normal Distribution

$$\Phi(z) = P(Z \le z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2}\, du$$



z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.500000	0.503989	0.507978	0.511967	0.515953	0.519939	0.532922	0.527903	0.531881	0.535856
0.1	0.539828	0.543795	0.547758	0.551717	0.555760	0.559618	0.563559	0.567495	0.571424	0.575345
0.2	0.579260	0.583166	0.587064	0.590954	0.594835	0.598706	0.602568	0.606420	0.610261	0.614092
0.3	0.617911	0.621719	0.625516	0.629300	0.633072	0.636831	0.640576	0.644309	0.648027	0.651732
0.4	0.655422	0.659097	0.662757	0.666402	0.670031	0.673645	0.677242	0.680822	0.684386	0.687933
0.5	0.691462	0.694974	0.698468	0.701944	0.705401	0.708840	0.712260	0.715661	0.719043	0.722405
0.6	0.725747	0.729069	0.732371	0.735653	0.738914	0.742154	0.745373	0.748571	0.751748	0.754903
0.7	0.758036	0.761148	0.764238	0.767305	0.770350	0.773373	0.776373	0.779350	0.782305	0.785236
0.8	0.788145	0.791030	0.793892	0.796731	0.799546	0.802338	0.805106	0.807850	0.810570	0.813267
0.9	0.815940	0.818589	0.821214	0.823815	0.826391	0.828944	0.831472	0.833977	0.836457	0.838913
1.0	0.841345	0.843752	0.846136	0.848495	0.850830	0.853141	0.855428	0.857690	0.859929	0.862143
1.1	0.864334	0.866500	0.868643	0.870762	0.872857	0.874928	0.876976	0.878999	0.881000	0.882977
1.2	0.884930	0.886860	0.888767	0.890651	0.892512	0.894350	0.896165	0.897958	0.899727	0.901475
1.3	0.903199	0.904902	0.906582	0.908241	0.909877	0.911492	0.913085	0.914657	0.916207	0.917736
1.4	0.919243	0.920730	0.922196	0.923641	0.925066	0.926471	0.927855	0.929219	0.930563	0.931888
1.5	0.933193	0.934478	0.935744	0.936992	0.938220	0.939429	0.940620	0.941792	0.942947	0.944083
1.6	0.945201	0.946301	0.947384	0.948449	0.949497	0.950529	0.951543	0.952540	0.953521	0.954486
1.7	0.955435	0.956367	0.957284	0.958185	0.959071	0.959941	0.960796	0.961636	0.962462	0.963273
1.8	0.964070	0.964852	0.965621	0.966375	0.967116	0.967843	0.968557	0.969258	0.969946	0.970621
1.9	0.971283	0.971933	0.972571	0.973197	0.973810	0.974412	0.975002	0.975581	0.976148	0.976705
2.0	0.977250	0.977784	0.978308	0.978822	0.979325	0.979818	0.980301	0.980774	0.981237	0.981691
2.1	0.982136	0.982571	0.982997	0.983414	0.983823	0.984222	0.984614	0.984997	0.985371	0.985738
2.2	0.986097	0.986447	0.986791	0.987126	0.987455	0.987776	0.988089	0.988396	0.988696	0.988989
2.3	0.989276	0.989556	0.989830	0.990097	0.990358	0.990613	0.990863	0.991106	0.991344	0.991576
2.4	0.991802	0.992024	0.992240	0.992451	0.992656	0.992857	0.993053	0.993244	0.993431	0.993613
2.5	0.993790	0.993963	0.994132	0.994297	0.994457	0.994614	0.994766	0.994915	0.995060	0.995201
2.6	0.995339	0.995473	0.995604	0.995731	0.995855	0.995975	0.996093	0.996207	0.996319	0.996427
2.7	0.996533	0.996636	0.996736	0.996833	0.996928	0.997020	0.997110	0.997197	0.997282	0.997365
2.8	0.997445	0.997523	0.997599	0.997673	0.997744	0.997814	0.997882	0.997948	0.998012	0.998074
2.9	0.998134	0.998193	0.998250	0.998305	0.998359	0.998411	0.998462	0.998511	0.998559	0.998605
3.0	0.998650	0.998694	0.998736	0.998777	0.998817	0.998856	0.998893	0.998930	0.998965	0.998999
3.1	0.999032	0.999065	0.999096	0.999126	0.999155	0.999184	0.999211	0.999238	0.999264	0.999289
3.2	0.999313	0.999336	0.999359	0.999381	0.999402	0.999423	0.999443	0.999462	0.999481	0.999499
3.3	0.999517	0.999533	0.999550	0.999566	0.999581	0.999596	0.999610	0.999624	0.999638	0.999650
3.4	0.999663	0.999675	0.999687	0.999698	0.999709	0.999720	0.999730	0.999740	0.999749	0.999758
3.5	0.999767	0.999776	0.999784	0.999792	0.999800	0.999807	0.999815	0.999821	0.999828	0.999835
3.6	0.999841	0.999847	0.999853	0.999858	0.999864	0.999869	0.999874	0.999879	0.999883	0.999888
3.7	0.999892	0.999896	0.999900	0.999904	0.999908	0.999912	0.999915	0.999918	0.999922	0.999925
3.8	0.999928	0.999931	0.999933	0.999936	0.999938	0.999941	0.999943	0.999946	0.999948	0.999950
3.9	0.999952	0.999954	0.999956	0.999958	0.999959	0.999961	0.999963	0.999964	0.999966	0.999967

Appendix B: Percentage Points of the t-distribution



Percentage Points $t_{\alpha,\nu}$ of the t-Distribution

ν \ α	.40	.25	.10	.05	.025	.01	.005	.0025	.001	.0005
1	.325	1.000	3.078	6.314	12.706	31.821	63.657	127.32	318.31	636.62
2	.289	.816	1.886	2.920	4.303	6.965	9.925	14.089	23.326	31.598
3	.277	.765	1.638	2.353	3.182	4.541	5.841	7.453	10.213	12.924
4	.271	.741	1.533	2.132	2.776	3.747	4.604	5.598	7.173	8.610
5	.267	.727	1.476	2.015	2.571	3.365	4.032	4.773	5.893	6.869
6	.265	.718	1.440	1.943	2.447	3.143	3.707	4.317	5.208	5.959
7	.263	.711	1.415	1.895	2.365	2.998	3.499	4.029	4.785	5.408
8	.262	.706	1.397	1.860	2.306	2.896	3.355	3.833	4.501	5.041
9	.261	.703	1.383	1.833	2.262	2.821	3.250	3.690	4.297	4.781
10	.260	.700	1.372	1.812	2.228	2.764	3.169	3.581	4.144	4.587
11	.260	.697	1.363	1.796	2.201	2.718	3.106	3.497	4.025	4.437
12	.259	.695	1.356	1.782	2.179	2.681	3.055	3.428	3.930	4.318
13	.259	.694	1.350	1.771	2.160	2.650	3.012	3.372	3.852	4.221
14	.258	.692	1.345	1.761	2.145	2.624	2.977	3.326	3.787	4.140
15	.258	.691	1.341	1.753	2.131	2.602	2.947	3.286	3.733	4.073
16	.258	.690	1.337	1.746	2.120	2.583	2.921	3.252	3.686	4.015
17	.257	.689	1.333	1.740	2.110	2.567	2.898	3.222	3.646	3.965
18	.257	.688	1.330	1.734	2.101	2.552	2.878	3.197	3.610	3.922
19	.257	.688	1.328	1.729	2.093	2.539	2.861	3.174	3.579	3.883
20	.257	.687	1.325	1.725	2.086	2.528	2.845	3.153	3.552	3.850
21	.257	.686	1.323	1.721	2.080	2.518	2.831	3.135	3.527	3.819
22	.256	.686	1.321	1.717	2.074	2.508	2.819	3.119	3.505	3.792
23	.256	.685	1.319	1.714	2.069	2.500	2.807	3.104	3.485	3.767
24	.256	.685	1.318	1.711	2.064	2.492	2.797	3.091	3.467	3.745
25	.256	.684	1.316	1.708	2.060	2.485	2.787	3.078	3.450	3.725
26	.256	.684	1.315	1.706	2.056	2.479	2.779	3.067	3.435	3.707
27	.256	.684	1.314	1.703	2.052	2.473	2.771	3.057	3.421	3.690
28	.256	.683	1.313	1.701	2.048	2.467	2.763	3.047	3.408	3.674
29	.256	.683	1.311	1.699	2.045	2.462	2.756	3.038	3.396	3.659
30	.256	.683	1.310	1.697	2.042	2.457	2.750	3.030	3.385	3.646
40	.255	.681	1.303	1.684	2.021	2.423	2.704	2.971	3.307	3.551
60	.254	.679	1.296	1.671	2.000	2.390	2.660	2.915	3.232	3.460
120	.254	.677	1.289	1.658	1.980	2.358	2.617	2.860	3.160	3.373
∞	.253	.674	1.282	1.645	1.960	2.326	2.576	2.807	3.090	3.291

ν = degrees of freedom.

Appendix C: PMF/PDF and the expected values of some Random Variables

Distribution	PMF/PDF	Expected value	Variance
Bernoulli	$P_X(x) = \begin{cases} 1-p & x=0 \\ p & x=1 \\ 0 & otherwise \end{cases}$	$E[X] = p$	$Var[X] = p(1-p)$
Geometric	$P_X(x) = \begin{cases} p(1-p)^{x-1} & x \geq 1 \\ 0 & otherwise \end{cases}$	$E[X] = 1/p$	$Var[X] = (1-p)/p^2$
Binomial	$P_X(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & x = 1, \ldots, n \\ 0 & otherwise \end{cases}$	$E[X] = np$	$Var[X] = np(1-p)$
Pascal	$P_X(x) = \begin{cases} \binom{x-1}{k-1} p^k (1-p)^{x-k} & x = k, k+1, \ldots \\ 0 & otherwise \end{cases}$	$E[X] = k/p$	$Var[X] = k(1-p)/p^2$
Poisson	$P_X(x) = \begin{cases} \frac{(\lambda T)^x e^{-(\lambda T)}}{x!} & x \geq 0 \\ 0 & otherwise \end{cases}$	$E[X] = \alpha$ $\alpha = \lambda T$	$Var[X] = \alpha$
Uniform (discrete)	$P_X(x) = \begin{cases} \frac{1}{b-a+1}, & x = a, a+1, a+2, \ldots, b \\ 0, & \text{otherwise} \end{cases}$	$E[X] = \frac{a+b}{2}$	$Var[X] = \frac{(b-a)(b-a+2)}{12}$
Exponential	$f_X(x) = \begin{cases} ae^{-ax} & x \geq 0 \\ 0 & otherwise \end{cases}$	$E[X] = 1/a$	$Var[X] = 1/a^2$
Gaussian	$f_X(x) = \begin{cases} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}} & \sigma > 0 \\ 0 & otherwise \end{cases}$	$E[X] = \mu$	$Var[X] = \sigma^2$
Uniform (Continuous)	$f_X(x) = \begin{cases} \frac{1}{b-a}, & a \leq x < b \\ 0, & otherwise \end{cases}$	$E[X] = \frac{a+b}{2}$	$Var[X] = \frac{(b-a)^2}{12}$
Sample variance		$s^2 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1}$	
Sample mean		$\bar{x} = \frac{\sum_{i=1}^{n} Xi}{n}$	
Variance		$Var[X] = E[(X-\mu)^2] = E[X^2] - (E[X])^2$	
Standardization of Normal Random Variable		$Z = \frac{X-\mu}{\sigma}$	
Joint distribution of $\bar{X}$ and S^2		$(n-1)\frac{S^2}{\sigma^2} \sim {\chi^2}_{n-1}$	
Distribution of $\bar{X}$ with unknown σ^2		$\sqrt{n}\frac{(\bar{X}-\mu)}{s} \sim t_{n-1}$	
Joint PDF of X and Y		$f_{XY}(x,y) = f_{X\|Y}(x\|y) f_Y(y)$	
Joint PMF of X and Y		$P_{XY}(x,y) = P_{X\|Y}(x\|y) P_Y(y)$	