0.1 Connected Spaces

X is connected if there do not exist disjoint open sets $U, V \neq \emptyset$ s.t. $X = U \cup V$.

X is path connected if for all $x, y \in X$ there exists a continuous function $f : [0,1] \to X$ s.t. $f(0) = x$ and $f(1) = y$.

Theorem 0.1 *Path connected $\implies$ connected, but not vice-versa.*

Theorem 0.2 *Suppose U is an open set in $\mathbb{R}^n$. Then U is connected, if and only if it is path connected.*

Proof: Suppose U is connected.

WTS U is path connected.

Let $U \neq \emptyset$. Let $a \in U$.

$E = \{x \in U | \exists \text{a path in } U \text{ from } a \text{ to } x\}$.

Claim: E is both open and closed (in U).

Let $x \in E$. Then $\exists$ a path from a to x.

$B_r(x) \subset U$.

$g(t) = (1-t)x + tz, t \in [0,1]$.

Then $\exists$ a path from a to any $z \in B_r$, by following the path from $atox$ and the radial path from x to z.

$\therefore E$ is open.

E is closed.

Let $(x_n) \subset E$, $x_n \to x \in U$. WTS $x \in E$.

$x \in U \implies \exists r > 0$ s.t. $B_r(x) \subset U$.

We can connect every point, so once we are in $B_r(x)$, take radial path.

E is both open and closed in U.

$\therefore E = U$ because U is connected and $E \neq \emptyset$ ∎

Theorem 0.3 *The continuous image of a connected set is connected.*

Proof: Let X be connected, $f : X \to Y$ continuous.

Suppose $f(X)$ is not connected.

Then $\exists$ open sets $U, V \subset Y$ s.t. $U \cap V = \emptyset$ and $U \cap f(X)$, $V \cap f(X)$.

$U \cup V = Y$.

$f^{-1}(U)$ is non-empty, $f^{-1}(V)$ is non-empty.

$f^{-1}(U) \cap f^{-1}(V)$.

$f^{-1}(U), f^{-1}(V)$ open.

$f^{-1}(U) \cup f^{-1}(V) = X \implies X$ is not connected. ■

Corollary 0.4 *If X is connected, $f : X \to \mathbb{R}$ is continuous.*

Let $x, y \in X,\ f(x) = a$

$f(y) = b$.

Suppose $a < b$.

Then for any $c \in (a, b) \exists z \in X$ s.t. $f(z) = c$.

Proof: $f(X)$ is connect, $\therefore$ it is an interval. ■

0.2 Inverse Function Theorem

Suppose $f : I \to \mathbb{R}$, I an open interval.

f is differentiable at $t_0 \in I$ if $\underset{t \to t_0}{lim} \frac{f(t)-f(t_0)}{t-t_0}$.

Equivalently $lim_{n \to \infty} \frac{f(t_0+h)-f(t_0)}{h}$ exists.

Write $f'(t_0)$ for this limit if it exists.

$f'(t_0) = \lim_{h \to 0} \frac{f(t_0+h)-f(t_0)}{h}$

$f'(t_0) = \frac{f(t_0+h)-f(t_0)}{h} + E(h)$

Then $\frac{f(t_0+h)-[f(t_0)+f'(t_0)h]}{h} + E(h) = 0$

Then $\lim_{h \to \infty} \frac{f(t_0+h)-[f(t_0)+f'(t_0)h]}{h} = 0$

If $f(t_0 + h) - [f(t_0) + f'(t_0)h] = \tilde{E}(h)$, where $\tilde{E}(h) \to 0$ faster than h.

$\frac{\tilde{E}(h)}{h} \to 0$ as $h \to \infty$.

Let $f : U \to \mathbb{R}$. U open in $\mathbb{R}^n$.

Let $x_0 \in U$. f is differentiable at x_0 if $\exists$ a linear fcn $L : \mathbb{R}^n \to \mathbb{R}$ s.t. $f(x_0 + h) - [f(x_0) + Lh] = \tilde{E}(h)$ where $\tilde{E}(h) \to 0$ faster than $|h|$, $|h| \in \mathbb{R}^n$.

$e_i = (0, \dots, 1, \dots, \dots, 0)$

$\lim_{t \to 0} \frac{f(x_0+t_0 i)-f(x_0))}{t} = \frac{\partial f}{\partial x}(x_0)$.

$D_n f(x_0) = \lim_{t \to 0} \frac{f(x_0+t_v)-f(x_0))}{t}$.

$L = [\frac{\partial f}{\partial x_1}(x_0), \dots, \frac{\partial f}{\partial x_n}(x_n)]$