0.1 Set Theory

Definition 0.1 *(bad)*

A set is an object S such that for every x, we have exactly one of two things.

1. *$x \in S$ (x belongs to S) or*
2. *$x \notin S$ (x does not belong to S)*

Sets are specified as follows

1. $\{1, 2, 3, 4\}$
2. $\{1, 2, 3, 4, ...\}$
3. $\{x \in \mathbb{N} | x \text{ is prime}\}$

0.1.1 Russel's Paradox

Let S be the following:

$S = \{T | T \notin T\}$

Q: Is $S \in S$?

If yes, then by definition $S \notin S$ If no, then by definition $S \in S$

0.1.2 Foundations of Set Theory/Zermelo-Frankel Set Theory

0.1.2.1 Axioms of ZFC

1. Axiom of Extension: Two sets are equal if and only if they have the same elements.
2. Axiom of Existence: The Empty Set Exists
3. Axiom of Pairing: If X and Y are sets, then there is a set $\{X, Y\}$
   Example : $X = \{1, 2\}$ $Y = \{3, 4\}$ $X, Y = \{\{1, 2\}, \{3, 4\}\}$
   $X = \emptyset$, $Y = \emptyset$
   $\{X, Y\} = \{\emptyset\}$
4. Axiom of Union: If S is a (set of sets), then the union over elements of S is a set.
   Example: $S = \{\{1, 2\}, \{2, 3\}\}$, then $\{1, 2, 3\}$ is a set.
5. Axiom of Intersection: Redundant, as above.
6. Axiom of foundation.
   Every $x \neq \emptyset$ contains a member y such that $x \neq y$.
   Consequence of failure:
   Let $X \neq \emptyset$

$\forall y \in X, y \cap X \neq \emptyset$

Choose any $y_1 \in X$, choose $y_2 \in X \cap Y$. $y_2 \in X$, so $y_2 \cap X \neq \emptyset$. Choose $y_3 \in x \cap y_2$...

7. Axiom Schema of Replacement

   If ϕ is a formula of 1st order logic, A is any set, then ϕ applied to A is contained in a set.