0.0.1 Note

Exercise: Subspace topology

$A \subseteq B \subseteq X$

The subspace topology on A coming from X = the subspace topology on A coming from the subspace topology on B coming from X.

0.0.2 Product Topology

Let $(X, \mathcal{O}_x)$ and $(Y, \mathcal{O}_y)$ be topological spaces. Then we give the product topology on the set $X \times Y$ as follows:

The basis $B_{x \times y} = \mathcal{O}_x \times to_y$ and $\mathcal{O}_x \times to_y$ = all possible unions of sets in $B_{x \times y}$

Example 0.1 *$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ w/ product top.*

$\triangle \subseteq \mathbb{R}^2; \triangle = \{(x, x) | x \in \mathbb{R}\}$

Prop: $(\mathbb{R}^2 \backslash \triangle)$ is open in $\mathbb{R}^2$, but cannot be written as $U \times V$ for U, V open in $\mathbb{R}$

E.g. Cylinder $\subset \mathbb{R}^3$ is homeomorphic to Annulus $\subset \mathbb{R}^2$ is homeomorphic to $S^1 \times [0, 1]$.

Definition 0.2 *Connectedness: Defn: A topological space X is called connected if it cannot be written as a union of two non-empty and disjoint open sets.*

Example 0.3 *$X = [0, 1] \cup (3, 4) \subset \mathbb{R}$ is disconnected because $[0, 1], (3, 4)$ both open and disjoint.*

0.0.3 Equivalent Properties

1. X is connected iff it cannot be written as the union of two non-empty disjoint closed subsets.
   If X disconnected, $X = A \cup B$, $A.b \neq \emptyset$, $A \cap B = \emptyset$, $A, B \implies A, B$ are both closed. A, B are clopen.
2. X is connected if and only if the only clopen subsets of X are $\emptyset$ and X.
   If $\emptyset \subsetneq A \subsetneq X$ is clopen, then $(X \backslash A)$ is clopen also. $X = A \cup (X \backslash A)$

   Example 0.4 $\mathbb{Q} = (\mathbb{Q} \cap (-\infty, \sqrt{2})) \cup (\mathbb{Q} \cap (\sqrt{2}, \infty))$

Any interval in $\mathbb{R}_{\text{metric}}$ is connected.

Proof: (Proving closed intervals for now)

Let $X = [a, b]$. If $a = b$, $x = \{a\}$ is connected.

Now let $a \neq b$; so $a < b$.

Suppose X is disconnected. $X = A \cup B$, where A and B are clopen, and A is non-empty.

Suppose WLOG that $a \in A$.

A is open in X, so $A = [a, b] \cap U$, where U open in $\mathbb{R}$.

$a \subset U$, so $\exists \varepsilon > 0$, such that $(a-\varepsilon, a+\varepsilon) \subseteq U$.

$(a-\varepsilon, a+\varepsilon) \cap [a,b] \subset A = [a, a+\varepsilon) \subseteq A$.

So $[a, \frac{\varepsilon}{2}] \subset A$.

Let $C := \{c \in [a,b] | [a,c] \subseteq A\}$

C has an upper bound, namely b.

C has a least-upper bound, $L \leq B$

L is a limit point of A.

$L \in X$ because $a < L \leq B$.

A is clopen in X, so A is closed in X.

$\implies L \in A$

$\implies [a, L] \subset A$

We'd like to show that $L = b$.

$L \in A$

$\implies$ A being open, $\exists \varepsilon > 0$, s.t. $[L, L+\varepsilon) \subseteq A$.

$\implies [L, L+\frac{\varepsilon}{2}] \subseteq A$.

Bad, because it implies that $[a, L+\frac{\varepsilon}{2}] \subseteq A$, which is a contradiction. $\implies L = B$.

Proof $\implies A = X \implies B$ is empty $\implies$ any closed interval is connected in $\mathbb{R}_{\text{metric}}$. ■

Definition 0.5 *Connected Component: Let X be a space.*

$x \in X$. The connected component of $x \in X$ is the union of all connected $Y \subseteq X$ of X, s.t. $x \in Y$.

Remark: The connected component of any $x \in X$ is a connected space. (needs proof)

Definition 0.6 *Path Connectedness*

A space X is path connected if $\forall a, b \in X$, there is a continuous function $f : [0,1] \in R_m \to X$ with $f(0) = a$, $f(1) = b$.

$[0,1]$ is homeomorphic to $[p,q]$ if $p \neq q$

Proposition 0.7 *If X is path connected, then it is connected.*

Proof: If not then $X = A \cup B$, disjoint union on non-empty clopen sets.

Take $a \in A$, $b \in B$.

There exists some $f : [0,1] \to X$, $f(0) = a$, $f(1) = b$. $[0,1] = f^{-1}(A) \cup f^{-1}(B)$ disjoint union, contradiction $[0,1]$ is connected. ■