(X, d) a metric space, $A \subset X$

Definition 0.1 *A set A is open if $A = intA$. Equivalently, A is open if $A \subset intA$ (because int $A = A$)*

$intA = \{x : \exists r > 0, B_r(x) \subset A\}$

Proposition 0.2 *int A is open.*

Proof: Let $x \in intA$. Then $\exists r > 0$ s.t. $B_r(x) \subset A$.

ClaimL $B_r(x) \subset int(A)$

$y \in B_r(x)$.

$s = d(x, y)$ $B_{s-r}(y) \subset B_r(x)$ by triangle inequality. $\therefore y$ is an interior point of $A \therefore B_1(x) \subset intA$ ■

Note: ext A also open by similar argument.

Theorem 0.3 *If $A_1, ...A_k$ are open sets, then $\cap_{j=1}^{k} A_j$ is open.*

If $\{A_i\}_{i \in I}$ is a collection of open sets, then $\cup_{i \in I} A_i$ is open.

Proof: Let $x \in \cap_j = 1^k A_j$. Then for each $j \exists r_j > 0$ s.t. $B_{r_j} \subset A_j$

$\therefore B_r(x) \subset B_{r_j}(x) \subset A_j \forall j = 1, ..., k \therefore B_r(x) \subset \cap_{j=1}^{k} A_j$

ii) If $x \in cup_{i \in I} A_i$ then $x \in A_j$ for some $j \in I \therefore \exists r > 0 s.t. B_r(x) \subset A_j \therefore B_r(x) \subset \cup_{i \in I} A_i$

Definition 0.4 *A set A is closed if A^c is open.*

Theorem 0.5 *A set is closed iff $\bar{A} = A$*

Proof: A closed $\iff$ A^c open $\iff$ $A^c = int(A^c)$ $\iff$ $A^c = ext(A)$ $\iff$ $intA \cup \partial A = \bar{A}$.

Recall $X = int(a) \cup \partial A \cup ext(A)$, but $\bar{A} = intA \cup \partial A$ as the set is pairwise disjoint.

A is closed iff it contains all it's limit points. ■

Theorem 0.6 *i) If $B_1, ..., B_k$ are closed sets, the $\cup_j = 1^k isclosed$.*

ii) $\cap_{i \in I} B_i$ is closed.

Proof: A is closed if A^c is open. $(\cup_i A_i)^c = \cap A_i^c$, $(\cap A_i)^c = \cup A_i^c$ ■

Note: If A is open, then $\forall x \in A, \exists r_x > 0 s.t. \{y \in X : d(x, y) < r_x\} = B_(r_x) \subset A$.

$$\therefore \bigcup_{x \in A} B_r(x) = A$$

Any open set is a union of open balls.

(X, d) a metric space.

If $S \subset X$, then (S, d_s) is a metric space if we define $d_s(x, y) = d(x, y)$ if $x, y \in S$

Proposition 0.7 *Ket $x \in S$. $B_r^S(x) = \{y \in S : d_s(y,x) < r\}$*

$= x \in S$. $B_r^S(x) = \{y \in S : d(y,x) < r\}$

$= S \cap \{y \in S : d_s(y,x) < r\} = S \subset B_r(x)$

Consequence A set $A \subset S$ is open in S iff $\exists$ an open set $U \subset X$ s.t. $A = U \cap S$

A open in $S \implies A = \cup_{x \in A} B_r^s(x) = \cup_{x \in A} S \cap B_r^s(x) = S \cap (\cup_{x \in A} B_r^s(x))$

A set $A \subset S$ is closed in S iff $\exists$ a closed set $C \subset X$ s.t. $A = C \cap S$