Definition 0.1 *$f : U \subset \mathbb{R}^n \to \mathbb{R}$, U open is differentiable at $a \in U$ if $\exists$ a linear map $L : \mathbb{R}^n \to \mathbb{R}$ s.t. $f(a+h) - (f(a) + f(h)) + E(h) = 0$, for $h \in \mathbb{R}^n$, (h) small, $\frac{E(h)}{|h|} \to 0$ as $h \to 0$.*

L is unique if it exists.

$L = (\frac{\partial f}{\partial x_1}(a), \ldots, \frac{\partial f}{\partial x_n}(a))$

Equivalently,

$\frac{f(a+h)-(f(a)+Lh)}{|h|}$.

If $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$.

f is differentiable if every component is differentiable,

$f = (f_1, \ldots, f_m)^T$.

Equivalently, f is differentiable if $\exists$ a linear map $L : \mathbb{R}^n \to \mathbb{R}^m$ s.t. $f(a+h) - (f(0) + Lh) + E(h) = 0$, where $\frac{E(h)}{|h|} \to 0$

Theorem 0.2 *(Inverse Function Theorem)*

Suppose $f : U \subset \mathbb{R}^n \to \mathbb{R}^n$.

$(i.e. frac\partial f \partial x_1, \ldots, \frac{\partial f}{\partial x_n})$ are continuous in U

Let $a \in U$, and suppose $df(a) : \mathbb{R}^n \to \mathbb{R}^n$ is invertible.

Then $\exists$ open sets containing a and W containing $f(a)$, s.t. $f|_V : V \to W$ is invertible.

$f^{-1} : W \to V$ is also C^1 and $d(f^{-1})(y) = [df(f^{-1}(y))]^{-1}$.

If $f \in C^k(U)$ for some $k \in \mathbb{N}$ then $f^{-1} \in C^k(W)$.

Proof: By replacing f by $[df(a)]^{-1} \circ f$, we can assume $df(a) = I$.

$d[df(a)^{-1} \circ f](a) = df(a)^{-1} \cdot df(a) = I$.

Let $x_1, x_2 \in B_r(a) \subset U$

$f(x_1) - f(x_2) = df(z) \cdot (x_1 - x_2)$ for some $z \in [x_1, x_2]$.

$(x_1 - x_2) + (d(f(z) - I)(x_1 - x_2)$

$\therefore$ for small enough r, we have $\frac{1}{2}|x_1 - x_2| < |f(x_1) - f(x_2)| \leq 2|x_1 - x_2|$, for all $x_1, x_2 \in B_r(0)$.

f is one to one on $\overline{B_r(a)}$.

$f|_{\overline{B_r(a)}}$ has an inverse f^{-1} defined on $f(\overline{B_r(a)})$.

$|f^{-1}(y_1) - f^{-1}(y_2)| \leq 2|y_1 - y_2|$.

Next step: Show $f(B_r(a))$ contains an open neighbourhood of $f(a)$.

f is one to one on $\overline{B_r(a)}$.

$\therefore f(a) \notin f(\partial B_r(a))$.

$f(\partial B_r(a))$ is compact.

$\therefore \exists \delta > 0$ such that $|f(x) - f(a)| > \delta \forall x \in \delta B_r(a)$

Let $W = B_{S/2}(f(a))$.

If $y \in W, x \in \partial B_r(a)$.

$|f(x) - y| = |f(x) - f(a) + f(a) - y| \geq |f(x) - f(a)| - |f(a) - y| \geq \delta - \frac{\delta}{2} = \frac{\delta}{2} > |f(a) - y|$.

Let $g(x) = |f(x) - y|^2$.

$g(x) > g(a)$ for all x $\in B_r(a)$.

$\therefore$ g must achieve it's minimum at an interior point.

At that point, $\frac{\partial g}{\partial x_i} = 2\sum_{i=1}^n (f^i - y^i)\frac{\partial f^i}{\partial x_j}$.

In matrix form $0 = [\frac{\partial f^i}{\partial x_j}](f'(x) - y', \ldots, f^n(x) - y^n)^T$.

$\implies f(x) = Y$.

This shows for every $y \in W$, $\exists x \in B_r(a)$ s.t. $f(x) = y$.

Define $V = f^{-1(W)} \cap B_r(a)$.

Let $x_1, x_2 \in V$. $y_1 = f(x_1)$, $y_2 = f(x_2) \in W$.

$f(x_1) - f(x_2) \subset df(x_2) \times (x_1 - x_2) + E(x_1 - x_2)$. Multiply both sides by $[df(x_2)]^{-1}$.

$[df(x_2)]^{-1} \to 0 = f^{-1} - f^{-1}(y_2) + [df(x_2)]^{-1}E(x_1 - x_2)$.

$df^{-1}(y_2) = [df(f^{-1}(y_2))]^{-1}$.

$[df(x_2)]^{-1}$ is bounded.

$\frac{E(x_1 - x_2)}{|x_1 - x_2|} \times \frac{|x_1 - x_2|}{|y_1 - y_2|} \to 0$ as $y_1 \to y$.

$\frac{1}{3}|x_1 - x_2| \leq |y_1 - y_2| \leq 2|x_1 - x_2|$

$\implies f^{-1}$ is differentiable at x_2. ∎