Def: A finite extension L/F is solvable by radicals if $L \subset$ iterated radical extension.

$F = F_0 \subset F_1 \subset \ldots \subset F_m$

$F_{i+1} = F_i(\sqrt[n_i]{a_i})$

$E = F(\sqrt[n]{(a)}, \zeta_n)$ splitting of $x^n - a$

$F(\sqrt[n]{a}, \zeta_n)$ $F(\sqrt[n]{a})$ $F(\zeta_n)$ F

Know $Gal(F(\zeta_n)/F) \hookrightarrow (\mathbb{Z}/nZ)^\times$

$Gal(\sqrt[n]{a}, F(\zeta_n)/F) \hookrightarrow (\mathbb{Z}/nZ)$ because $\sigma(\sqrt[n]{a}) = \zeta_n^i \sqrt[n]{a}$

Some $1 \leq i \leq n$.

$\sigma \mapsto i_\sigma$ $\tau \mapsto i_\tau$

$\sigma\tau \mapsto i_{\sigma\tau}$

$(\sigma\tau)(\alpha) = \sigma(\tau(\alpha)) = \sigma(\zeta_n^{i\tau} \cdot \alpha) = \sigma(\zeta^{i\tau}) \times \sigma(\alpha) = \zeta^{i_\tau} \cdot \zeta^{i_\sigma} \alpha = i_\tau + i_\sigma$

$\mathbb{Z}/n\mathbb{Z} \supset Gal(F(\alpha, \zeta)/F(\mathbb{Z})) \hookrightarrow Gal(F(\alpha, \zeta)/F) \twoheadrightarrow Gal(F(\zeta_n)/F) \subset (\mathbb{Z}/n\mathbb{Z})^\times$

Definition 0.1 *A finite group G is solvable if $\exists$ subgroups of G_i such that $1 = G_m \subseteq \ldots \subseteq G_1 \subseteq G_0 = G$ s.t. $G_{i+1} \lhd G_i$ and G_i/G_{i+i} and G_i/G_{i+1}.*

Example 0.2 *Abelian Groups*

D_{2n} *dihedral groups.*

$\mathbb{Z}/n\mathbb{Z} \subset D_{2n}$, $D_2n/\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}$

$e \subset \mathbb{Z}/n\mathbb{Z} \subset D_{2n}$, $D_{2n}/\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}$

$D_{10} =< \sigma, \tau >$

$< \sigma > \cong \mathbb{Z}/5\mathbb{Z}$

$< \tau > \cong \mathbb{Z}/2\mathbb{Z}$

$GL_n(F) =$ *invertible matrices.*

$G = \{$*upper triangular matrices*$\} < GL_n(F)$

Fact: S_n is solvable $\iff n < 5$. (Need to prove)

Theorem 0.3 *Let $N \lhd G$. Then G is solvable $\iff G/N$ are solvable.*

Proof: ($\implies$)

$G = G_0 \supseteq G_1 \supseteq G_1 \supseteq \ldots \supseteq G_m = \{1\}$, G_i/G_{i+1} normal.

Need to find $G/N = H_0 \supseteq H_1 \supseteq \ldots \supseteq H_? = 1$

Let H_i image of G_i in G/N.

$G \to G/N$

$\vee \qquad \vee$

$G_i \to G_iN/N = H_i$

$H_{i+1} \triangleleft H_i$

$G_{i+1}N/N \triangleleft G_iN/N$

$g_i nN$

$G_i \twoheadrightarrow \qquad H$

$\downarrow \qquad \downarrow$

$G_i/G_{i+1} \to [F]\ H_i/H$

$\implies H_i/H_{i+1}$ Abelian (quotient of Abelian)

■