0.0.1 recap

Defined $\mathbb{N}$

$0 = \emptyset$

$1 = \{\emptyset\}$

$2 = \{\emptyset, \{\emptyset\}\}$

0.0.2 Today

Arithmetic on $\mathbb{N}$

How can we make sense of "1+2 = 3?"

Strategy: Define the function "addition of m" for every $m \in N$

Theorem 0.1 *For every $m \in \mathbb{N}$, there exists a function $S_m \subset \mathbb{N} \times \mathbb{N}$ such that*

- $S_m(0) = m$
- $\forall n \in \mathbb{N}, S_m(n^+) = [S_m(n)]^+$

Proof: Let's construct a relation $S'_m = \mathbb{N} \times \mathbb{N}$ with the "right" properties, then we'll show S'_m is a function.

$\mathcal{S}_m = \{R \subset \mathbb{N} \times \mathbb{N} | (0, m) \in R, \text{ and if } (n, x) \in R \text{ then } (n^+, x^+) \in R\}$

$\mathcal{S}_m \ni \mathbb{N} \times \mathbb{N}$, so $\mathcal{S} \neq \emptyset$

Define $s'_m = \cap_{R \in \mathcal{S}_\Updownarrow} R$

$Notice:$

1. $S + m' \subseteq \mathbb{N} \times \mathbb{N}$
2. $(0, m) \in s'_m$
3. If $(n, x) \in s'_m, then (n^+, x^+) \in s'_m$

Strategy: Show that S'_m is a function.

Show

1. $Domain(s'_m) = \mathbb{N}$
2. If (n, x) and (n, y) are both in s'_m then then $x = y$

Proof: of 1

Let $p(n)$ be true if there is some $x \in \mathbb{N}$ such that $(n, x) \in s'_m$

$p(0)$ is true because $(0, m) \in s'_m$

Suppose $p(n)$ is true, so there is some $x \in \mathbb{N}$ such that $(n, x) \in s\prime m \implies (n^+, x^+)$, so p_n is true.

So domain $s'_m = \mathbb{N}$

■

Proof: of 2 Let $p(n)$ be true if for $\forall r \in n$, there is a unique $x \in \mathbb{N}$ such that $(r, x) \in s'_m$.

$p(0)$ is vacuously true.

Suppose $p(n)$ is true.

We want to show $p(n^+)$, i.e. $\forall r \in \mathbb{N}$, $r \in n+, \exists!\, x \in \mathbb{N}$ such that $(r, x) \in s'_m$

If $r \in n^+$, then either $r \in n$ or $r = n$. Consider any $r \in n^+$.

If $R \in n$, then we know $\exists!\, x \in \mathbb{N}$, s.t. $(r, x) \in s'_m$

Now lets prove the statement for $r = n$.

Suppose that $\exists a, b \in \mathbb{N}$ such that (n, a) and (n, b) are both in s'_m.

Claim: If $a \neq b$, not both of these can have "predecessors".

Suppose $a \neq b$

Suppose that there are $p_1, x_1, p_2, x_2 \in \mathbb{N}$ such that $(p_1, x_1) \in s'_m, (p_2, x_2) \in s'_m and (p_1, x_1) = (n, a), (p_2, x_2) = (n, b)$

$\implies p_1^+ = p_2+, \implies p_1 = p_2 = p$

$(p, x_1) \in s'_m, (p, x_2) \in s'_m$ and $p^+ = n$ so $p \in n$.

Induction hypothesis $x_1 = x_2 \implies a = b$. Contradiciton.

Suppose $a \neq b$. Suppose WLOG that (n, a) doesn't have a predecessor. Define $s''_m := s'_m(n, a)$.

s''_m satisifies the properties of being in $\mathcal{S}_m$ - Exercise.

Problem, because s'_m was the intersection of all $R \in \mathcal{S}_m$.

$\implies s'_m \subseteq s''_m$. Contradiciton.

Write $s'_m = s_m$ Addition by m function. ■

Properties

1. $\forall n \in \mathbb{N}, s_1(n) = n^+$
2. $\forall n \in \mathbb{N}, s_0(n) = n$
3. $\forall m, n, k \in \mathbb{N}$ we have $S_{s_m(n)}(k) = s_m(s_n(k))$ (Assoc)
4. $\forall m, n \in \mathbb{N}$, we have $s_m(n) = s_n(m)$.

- Let $p(n)$ be true if $s_1(n) = n^+$, $p(0)$. Suppose $p(n)$ is true. $s_1(n^+) = [s_1(n)]^+ = (n^+)^+$ $p(n^+)$ is true.

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