Last Time: Connectedness and path connectedness.

Proposition 0.1 *If X is path-connected, then it is connected.*

Proposition 0.2 *If $f : X \to Y$ is continuous and surjective, then*

If X is connected, so is Y.

If X is path connected, so is Y.

Corollary 0.3 *1) If $f : X \to Y$ is a homeomorphism, then X is connected if and only if Y is connected.*

2) X is path connected if and only if Y is path connected.

Proposition 0.4 *If $n > 1$, then $\mathbb{R}_{metric}$ and R^n_{metric} are homeomorphic.*

Proof: If $f : \mathbb{R} \to \mathbb{R}^n$ were a homeomorphism, then $f : \mathbb{R}\backslash\{0\} \to \mathbb{R}^n\{f(0)\}$ would still be a homeomorphism.

But $\mathbb{R}\backslash\{0\}$ is disconnected as $\mathbb{R}\backslash\{0\} = (-\infty, 0) \cup (0, \infty)$

Since $n > 1$, $\mathbb{R}^n\backslash\{f(0)\}$ is connected, because it is path-connected. ■

0.0.1 Compactness

Look at $\mathbb{R}$, $\mathbb{Z} \subset \mathbb{R}$ is an infinite set of isolated points.

Open in $\mathbb{Z} \leftarrow \{5\} = \mathbb{Z} \cap (4.5, 5.5)$

Definition 0.5 *An open cover of X is a set $\{U_\alpha\}_{\alpha \in I}$ of open subsets of X such that $X = \underset{\alpha \in I_\alpha}{U}$*

Remark: If $Y \subseteq X$, sometimes we specify an open cover of Y by giving a set $U_{\alpha \alpha \in I}$ of open sets of X whose union contains Y.

Definition 0.6 *Compactness: A space X is compact, if every open cover of X has a finite subcover. That is, a finite subset that also an open cover of X.*

Ex. $\mathbb{R}_{\text{metric}}$ is not compact.

Take the open cover $\{(n, n+2) | n \in \mathbb{Z}\}$.

Similarly $(0, 1)$ is not compact in $\mathbb{R}_{\text{metric}}$. Take $\{(\frac{1}{n+2}, \frac{1}{n})$ for $n \geq 1\}$

Theorem 0.7 *Every closed interval $[a, b]$ is compact in the $\mathbb{R}_{metric}$.*

Proof: Let $[a, b] \subset \mathbb{R}_m etric.$

If $a = b$ then $[a, b] = \{a\}$, all finite sets are compact.

Suppose $a < b$.

Let $\{U_\alpha\}_{\alpha\in I}$ be an open cover of $[a,b]$. $A \in U_\alpha$ for $\alpha \in I$

Since U_α is open, $\exists c \in \mathbb{R}, c > a$, s.t. $[a,c] \subset U_\alpha$.

Let $C = \{x \in (a,b] | [a,x]$ is contained in a finite union of the Us$\}$

$C \neq \emptyset$ b/c $c \in C$.

C is bounded above by b.

Let $L = \text{lub}(C)$

We'll show

1) $L \in C$

2) $L = b$

1) Suppose $L \notin C$. But $L \in [a,b]$ so $L \in U_\beta$ for some $\beta \in I$.

U_β is open, so there is some $\varepsilon > 0$, s.t. $[L-\varepsilon, L] \subset U_\beta$

$L - \varepsilon < L$, but $L = \text{lub}(C)$.

$\implies [a, L-\varepsilon]$ is contained in a finite union of the Us.

$\implies [a, L]$ is contained in the above union U_β, still finite.

Contradiction $\implies L \in C$

$\implies [a, L]$ is covered by finitely many Us.

2) WTS $L = b$.

If $L \neq b$, then $\exists\varepsilon > 0$ s.t. $[a, L+\varepsilon]$ is also in a finite union of the Us. Contradiction!

■

Proposition 0.8 *If $Y \subseteq X$ is a closed subspace and X is compact, then Y is also compact.*

Proof: Let $\{U_\alpha\}_{\alpha\in I}$ be an open cover of Y. [Where U_α is open in X].

$\{U_\alpha\} \cup (X \backslash Y)$ is an open cover of X. This has a finite subcover, so Y has a finite subcover. ■

Proposition 0.9 *If $f: X \to Y$ is continuous and surjective and X is compact, then Y is compact.*

$\implies \circ \subset \mathbb{R}^2$ is compact.

Theorem 0.10 *If X and Y are both compact then $X \times Y$ is compact.*

Proof: Let $\{U_\alpha\}_{\alpha\in I}$ be an open cover of $X \times Y$.

If $(x,y) \in X \times Y$ then it's contained in U_α.

Then there is some $A_{xy} \times B_{x,y}$ containing (x,y), lying in U_α.

Fix $x \in X$, and consider $\{B_{xy} | y \in Y\}$. THis is an open cover of Y.

Take a finite subcover. So $B_{xy}, B_{xy_1}, \ldots B_{xy_n}$.

Exists associated $A_{xy}, A_{xy_1}, \ldots A_{xy_n}$. Set $A := \cap A_{xy_o}$

Let $A_{x1}, \ldots, A_{xn}$ be a finite subcover of X.

$A_{x_i} \times B_{x_i y_j}$ forms a finite subcover of $X \times Y$.

■

Theorem 0.11 *The Heine-Borel Theorem: A subspace of* $\mathbb{R}^n$ *is compact if and only if is closed and bounded.*