0.1 Axioms of ZFC (Continued)

7. Axiom of Specification:

   The collection of all elements of a known set, that satisfiy a certain predicate, is a set.

   E.g $\{x \in \mathbb{N} | x \text{ is prime}\}$

8. Axiom of power set

   If S is a set, then the collection of all subsets of S is a set, called the power set. $\mathcal{P}(S)$

   $(a, b) = \{\{a\}, \{a, b\}\}$

   $A \times B :=$ set of ordered pairs$(a, b) = \{\{a\}, \{a, b\}\}$

   Let A, B be sets. A function $f : A \to B$ is an element of the $\mathcal{P}(A \times B)$,

   - For every $a \in A$, there is some $b \in B$ such that $(a, b) \in F$
   - if b_1 and $b_2 \in B$ such that $(a, b_1) \in\in F$ and $(a, b_2) \in f$ then $b_1 = b_2$

9. Let X be a set. Define $\succ (x) = x^+ := x \cup \{x\}$. There exists a set S such that

   (a) $\emptyset \in S$,

   (b) $x \in S$ then $x^+ \in S$.
   E.g. $succ(\emptyset) = \emptyset \cup \emptyset = \emptyset$ $\{\emptyset\}^+ = \{\emptyset\} \cup \{\{\emptyset\}\} = \{\emptyset, \{\emptyset\}\}$

10. Axiom of choice.

    A choice function f defined on a set X of non-empty sets is a function with the property that if $a \in X$ then $f(a) \in a$.

    Choice functions always exist for any X.

0.2 Construction of $\mathbb{N}$

$\mathbb{N}$ includes 0.

Define $0 := \emptyset$

$1 := \emptyset^+$

$2 := 1^+ = \{\emptyset, \{\emptyset\}\}$

$3 := 2^+ = \{\emptyset, \{\emptyset\}\} \cup \{\{\emptyset, \{\emptyset\}\}\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$

Q: What is $\mathbb{N}$.

A9 tells us that there is a set S such that $\phi \in s$, if $x \in S, x^+ \in S$

Consider $I_S := \{T \in \mathcal{P}(S) | \emptyset \in T \text{ and } \forall x \in T, x^+ \in T\}$

Say that a set A is inductive. If $\emptyset \in A$ and if$x \in A, x^+ \in A$.

$I_s =$ set of all inductive subsets of S.

$I_s \neq \emptyset$ because $S \in I_S$

Define $\mathbb{N} = \cap_{x \in I_S} x = \{x \in S | \forall T \in I_s, x \in T\}$

Theorem 0.1 *Principle of Mathematical Induction*

Let p be a predicate defined on $\mathbb{N}$*. Assume that* $p(0)$ *holds, and for every* $n \in \mathbb{N}, p(n)$ *implies* $p(n^+)$*. Then* $p(n)$ *holds for every* $n \in N$*.*

Proof: Fix p, with the above properties.

$S = \{x \in \mathbb{N} | p(x) \text{ holds}\}$. We want to show that $S = N$, i.e. the elements of S are exactly the elements of $\mathbb{N}$.

Observation S is inductive

1. $\emptyset \in S$, because $p(0) = p(\emptyset)$ holds.
2. If $x \in S$ it means $p(x)$ holds, therefore $p(x^+)$ also holds.
   So, $x^+ \in S$

1 and 2 tell us that S is inductive.

$S \subseteq \mathbb{N}$ by construction.

$N \subseteq S$ by construction.

$\implies S = \mathbb{N}$ ■

Theorem 0.2 *If* $m, n \in \mathbb{N}$*, such that* $m^+ = n^+$*, then* $m = n$*.*

Proof:

Lemma 0.3 *Let* $x, n \in \mathbb{N}$*. If* $x \in n$ *then* $x \subseteq n$

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