(X, d) is a metric space.

Definition 0.1 *X is sequentially compact if every sequence in X has a convergent sequence.*

Definition 0.2 *X is compact if every covering of X has a finite subcover.*

Theorem 0.3 *X is sequentially compact $\iff$ X compact.*

Proof: Compactness $\implies$ Sequential Compactness

Let (x_n) be a sequence in X.

WTS (x_w) has a convergent subsequence.

Let $A = x_n$

$(x_n) = x_1, x_2, ...$

Case 1 A is finite.

Then some value $x_j \in A$ is repeated infinitely often in the sequence (x_n). Choose the subsequence (x_{n_j}) with $x_{n_j} = \overline{x}\ \forall j$.

Then $(x_{n_j}) isa$ constant sequence $\therefore$ convergent.

Case 2: A is infinite.

Claim: A has a limit point.

Suppose not. Then A is closed. because $\overline{A} = A \cup \{\text{limit points}\}$.

If $a \in A$, then a is not a limit point.

$\therefore \exists$ open set $\cup_a$ containing a, $A \cap U_a = \{a\}$.

$X = A^c \underset{a \in A}{\cup} U_a$.

By compactness $\exists$ a finite subcover of X.

$X = A^c \cup U_{a_1} \cup \ldots \cup U_{a_n}$.

This is a contradiction because A is infinite.

Let x be a limit point of A.

$\forall \varepsilon > 0$, $B_\varepsilon(x)$ contains infinitely points points of A other than X.

Choose $x_{n_1} \in A$, s.t. $d(x_{n_1}, x) < 1$.

Choose $x_{n_2} \in A$, $n_2 > n$,s.t. $d(x_{n_2}, x) < 1/2$.

$\vdots$

Choose $x_{n_k} \in A$, s.t. $n_k > n_{k-1}$ $d(x_{n_k}, x) < 1/k$.

$\vdots$

Then $(x_{n_j})_{j=1}^{\infty}$ is a subsequence of (x_n). $(x_{n_j}) \to x$ as $j \to \infty$

$\therefore (x_n)$ has a convergent subsequence.

Sequential compactness $\implies$ Compactness

Suppose X is sequentially compact.

Claim: For each $k \in \mathbb{N}$, $\exists$ finitely many points $\{x_1, \ldots, x_N\}$ s.t. $x \in X \implies d(x_i, x) < \frac{1}{k}$ for some $x_i \in \{x_1, \ldots, x_N\}$

Choose x_1

Choose $x_2 \in X \backslash B_{\frac{1}{k}}(x_1)$

Choose $x_3 \in X \backslash [B_{\frac{1}{k}}(x_2) \cup B_{\frac{1}{k}}(x_1)]$

Keep going until this stop. This must stop.

If this does not stop, then (x_n) is a sequence with no convergent subsequence.

For each $k \in \mathbb{N}$, let A_k be the set chosen by this process.

Let $A = \underset{k \in N}{\cup} A_k$. Then A is countable, dense.

Definition 0.4 *A is dense in X if $\overline{A} = X$.*

Claim: Every open cover of X has a countable subcover.

Let $\mathcal{F} = \{U_i\}_{i \in I}$ be a covering of X by open sets.

If $\exists x \in A$, some $r > 0$ s.t. $B_r(x) \subset U_i$ for some i, choose one such U_i, call it $U_{x,r}$.

Let $\mathcal{F}^* = \{U_{x,r} : x \in A, r > 0 \text{ is retained}\}$

$\mathcal{F}^*$ countable. $\mathcal{F}^*$ covers X.

Let $y \in X$. Then $\exists s > 0$ s.t. $B_s(y) \subset U_i$ for some i.

Choose rational r

$\frac{s}{4} < r < \frac{s}{2}$.

Choose $x \in A$ s.t. $d(x, y) < r$.

Then $y \in B_r(x)$.

Then $y \in B_r(x) \subset B_s(y) \subset U_i$.

$\implies \exists U_{x,r} \in \mathcal{F}^*$ s.t. $y \in U_{x,r}$ by defintiion of $\mathcal{F}^*$.

If $\{U_n\}_{n \in \mathbb{N}}$ is a countable open cover of X, then $\exists$ a finite subcover.

Let $V_n = U_1 \cup U_2 \ldots \cup U_n$

WTS $V_n = X$ for large enough n.

Suppose not. Then $\forall n \exists x_n \in X \backslash V_n$.

X is sequentially compact $\therefore (X_n)$ has a convergent subsequence.

By relabelling we may assume $x_n \to x$. $x \in U_n$ for some N. $\therefore x_n \in U_n$, $\forall n$ suff large. $\implies$ contradiction because $x_n \in X \backslash V_n$.

∎