0.0.1 recap

Construction of $\mathbb{R}$. $R =$ set of all cauchy sequences of rational numbers.

We say that $(a_n)\tilde{}(b_n)$ if their difference $(a_n - b_n)$ converges to 0. $\mathbb{R} := R/\tilde{} =$ set of equivalence classes.

Lemma 0.1 *Every cauchy sequence is bdd, i.e. there is some M s.t. $|a_n| \leq M$*

Lemma 0.2 *Every Cauchy sequene $(\dot{a}n)$ that doesn't converge to 0, is bounded away from 0, i.e. $\exists \varepsilon > 0 and N \in \mathbb{N}$ s.t. $\forall n > N, |a_n| > \varepsilon$.*

Proposition 0.3 (˜is an equivalence relation) *1. Reflexivity : (a_n) (a_n)*

2. *Symmetry : (a_n) (b_n) then $a_n - b_n \to 0$*

   $b_n - a_n \to 0$

   so $(b_n)\tilde{}(a_n)$

3. *Transitivity: Suppose (a_n) (b_n) and (b_n) (c_n).*

   So $(a_n - b_n) \to 0$ and $(b_n - c_n) \to 0$

   $\forall \varepsilon > 0 \exists N_1 \in \mathbb{N}$ s.t. if $n > N_1$, then $|a_n - b_n| < \frac{\varepsilon}{2}$ $\forall \varepsilon > 0 \exists N_2 \in \mathbb{N}$ s.t. if $n > N_2$, then $|b_n - c_n| < \frac{\varepsilon}{2}$

   $N = \max N_1, N_2 and n > N$.

   $|a_n - c_n| \leq |a_n - b_n| + |b_n - c_n| < \varepsilon$.

Define $\mathbb{R} =$ set of real numbers $:= R/$. Elements of $\mathbb{R}$ are $[(a_n)]$

0.0.2 p-adics

$p \in \mathbb{N}$ prime.

We can define a new metric on $\mathbb{Q}$. Let $x \in \mathbb{Q}$. Write x uniquely as $x = p^a \frac{r}{s}$, where $r, s, a \in \mathbb{Z}$. and $p \not| r$ and $p \not| s$.

Example 0.4 *$p = 3$, $x = \frac{7}{6}, x = (3^{-1})(\frac{7}{2})$. $|x|_3 = 3$*

$|x|_p := p^{-a}$.

If $x, y \in \mathbb{Q}$, $|x - y|_p$ is the p-adic distance.

0.0.3 $\mathbb{R}$ is well defined.

$\mathbb{Q} \hookrightarrow \mathbb{R}$ $q \mapsto [(q, q, q, q, q)]$

If $p \neq q$ then $i_p \neq i_q$ bc $[p - q, p - q, p - q, ...] \neq [(0, 0, 0, 0, 0)]$

in $\mathbb{R}$ we have a $0 := [(0, 0, 0, 0, 0)]$ and $1 := [(1, 1, 1, 1, .)]$

Properties.

1. Addition: $[(a_n)] + [(b_n)] = [(a_n +_{\mathbb{Q}} b_n)]$ a) Well defined.
   b) $(a_n) + (b_n)$ is a cauchy seq.
   c) commutativity, associativity, $0 + [(a_n)] = [(a_n)]$
2. Multiplication $[a_n] \cdot_{\mathbb{R}} [(b_n)] := [(a_n \cdot_{\mathbb{Q}} b_n)]$ a) This is well defined.
   Proof: LEt $[(a_n)] = [(c_n)]$,$[(b_n)] = [(d_n)]$ Want $[(a_n b_n)] = [(c_n d_n)]$.
   WTS $(a_n b_n - c_n d_n) \to 0$ We know $a_n - c_n \to 0$
   We know $b_n - d_n \to 0$ $a_n b_n - c_n d_n = a_n b_n - b_n c_n + b_n c_n - c_n d_n$
   $= b_n(a_n - c_n) + c_n(b_n - d_n)$.
   $|a_n b_n - c_n d_n| \leq |b_n||(a_n - c_n)| + |c_n||b_n - d_n|$.
   Let M be an upper bound for $|b_n|, |c_n|$
   $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall n > N$, we have $|a_n - c_n < \frac{\varepsilon}{2M}|$
   $|b_n - d_n < \frac{\varepsilon}{2M}|$.
   $|a_n b_n - c_n d_n| < \varepsilon$. ■
3. Division:
   Let $[(a_n)] \in \mathbb{R}$ and $[(a_n) \neq 0]$. There is some $[(b_n)] \in \mathbb{R}$, s.t. $[(a_n)] \cdot [(b_n)] = [(1, 1, 1, 1, 1...)] = 1$
   By Lemma 2, If $[(a_n)] \neq 0$ then some $\varepsilon 0$, $\exists N \in \mathbb{N}$, s.t. $\forall n > N$, $|a_n| > \varepsilon$
   $\implies$ eventually $a_n \neq 0$.
   Define b_n as follows: if $n \leq N$, set $b_n = 1$. If $n > N$, set $b_n = \frac{1}{a_n}$
   $a_n b_n =$ if $n \leq N$, a_n
   if $n > N$, 1
   $\implies [(a_n b_n)] = [(1, 1, 1, 1..)]$
   Prop: b_n is cauchy.
   $|b_m - b_n| = |\frac{1}{a_m} - \frac{1}{a_n}| = |\frac{a_n - a_m}{a_m a_n}|$

   Theorem 0.5 $\mathbb{R}$ *is a field.*
   ($\mathbb{Q}_p$ is a field.)
4. Order Relation
   We say that $[(a_n)] \geq [(b_n)]$ if either $[(a_n)] = [(b_n)]$ or $\exists N \in \mathbb{N}$ s.t. $\forall n > \mathbb{N}$, $a_n - b_n \geq 0$.
   Thm :$\leq$ is a total oreder on $\mathbb{R}$.

0.0.4 Completeness of $\mathbb{R}$

Since $\mathbb{R}$ is ordered, $Completeness \iff lubproperty$