0.1 Recap

Definition of a topology on a set X.

$(X, \mathcal{O} \subset P(X))$

E.g. $X = \mathbb{R}$, $\mathcal{O} =$ set of "open" sets of $\mathbb{R}$ (as a metric space.) E.g. X is any metric space, $\mathcal{O} =$ any metric open set

(The standard metric topology in $\mathbb{R}$).

Definition 0.1 *Let $(X, \mathcal{O})$ be a topological space. A subset $\mathcal{B} \subset \mathcal{O}$ is called a basis for the topology, if every $U \in \mathcal{O}$ can be expressed as a union of elements of $\mathcal{B}$.*

E.g. In $\mathbb{R}$ metric, $\mathcal{B} = \{$all open balls$\}$.

Side example: $\cap_{n\in\mathbb{N}}(-\frac{1}{n}, \frac{1}{n}) = 0$. Infinite intersection don't have to be open.

Example 0.2 *The Lower Limit Topology*

$X = \mathbb{R}$

$\mathcal{B}_{ll} = \{[a,b) | a, b \in \mathbb{R}\}$

$\mathcal{O}_{ll} =$ *set of all possible unions of elements of* $\mathcal{B}_{ll}$

Proposition 0.3 *Any interval of the form (a,b) is open in $\mathbb{R}_{ll}$.*

Proof: $(a,b) = \underset{n\in N>0}{\cup}[a+1/n, b)$ ■

Definition 0.4 *A set $Y \subset X$ is closed if $(X\backslash Y) \in O$.*

$[a,b)$ are both open and closed.

$\mathcal{O}_{metric} \subset \mathcal{O}_{ll}$, because $B_{metric} \subset B_{ll}$.

$\mathbb{R}_{ll}$ has "more" open sets that $\mathbb{R}$ metric. Say that the lower limit topology is finer metric topology.

Q) Is $\mathcal{O}_{metric} = \mathcal{O}_{ll}$? Is $[a,b)$ open in the metric topology?

Definition 0.5 *Let $A \subseteq X$. Let $x \in X$.*

There is the following trichotomy:

1) There is some $U \in \mathcal{O}$ s.t. $x \in U$ and $U \subset A$. -¿ X is in the interior of A, $x \in intA$.

2) There is some $U \in \mathcal{O}$ s.t. $x \in U$ and $U \subset (X\backslash A)$. $x \in int(X\backslash A)$.

3) For all open sets $U \in O$ s.t. we have $U \cap A \neq \emptyset$ and $U \cap (X\backslash A) \neq \emptyset$ $x \in U$. X is in the boundary of A.

Example 0.6 *In $\mathbb{R}$ metric.*

$A = [a,b)$. $int(A) = (a,b)$

$boundary(A) = \{a, b\}$.

$int(X \backslash A) = (-\infty, a) \cup (b, \infty)$.

Definition 0.7 *If $A \subset X$, then we say that $x \in X$, is a limit point of A, if either $x \in intA$ or $x \in boundaryA$.*

i.e. x is a limit point if $\forall U \subset \mathcal{O}$, s.t. $x \in U$, $U \cap A \neq \emptyset$

Definition 0.8 *The closure of A, denoted $\overline{A}$ is the set of limit points of A.*

Proposition 0.9 *Let $A \subset X$. Then:*

1. *int(A) is open, i.e. int(A) $\in \mathcal{O}$.*
2. *$\overline{A}$ is closed.*
3. *A is open iff $A = int(A)$*
4. *A is closed iff $A = \overline{A}$.*

0.1.1 Finite Complement Topology

$X = \mathbb{R}$

$\mathcal{O} = \{Y \subseteq X | X \backslash Y \text{ is finite}\}$

Definition 0.10 (Subspace Topology) *If $(X, \mathcal{O})$ is a topological space, and $Y \subset X$ is any set, then we can define a standard topology on Y, called the subspace topology.*

Define $\mathcal{O}_y := \{U \cap Y | U \in \mathcal{O}\}$

Definition 0.11 *Let X and Y be topological spaces. A function $f : X \to Y$ is continuous, if for all $\forall V$ open in Y, $f^{-1}(V)$ is open in X.*

Definition 0.12 *$f : X \to Y$ is called a homeomorphism if it is continuous, one to one, onto and f^{-1} is continuous.*