Definition 0.1 *A metric space (X, d) is connected if there do not exist non-empty open sets U, V such that $U \cap V = \emptyset$ and $X = U \cup V$.*

X is disconnected if it is not connected.

If $A \subset X$ is open, then $A \neq X$, then $A \cup A^c = X$

A^c is not open.

Proposition 0.2 *A metric space X is connected, if and only if there do not exist non-empty disjoint closed sets E, F s.t. $X = E \cup F$.*

Proof: If $X = U \cup V$, $U, V \neq \emptyset, U \cap V = \emptyset$

$V = U^c$ closed.

$U = V^c$ closed. ■

Proposition 0.3 *A metric space X is connected, if and only if the only subsets of X that are both open and closed are $\emptyset, X$.*

Example 0.4 $\mathbb{Q} = \mathbb{Q} \cap (-\infty, \pi) \cup \mathbb{Q} \cap (\pi, \infty)$

$[0, 1] \cup sin(\frac{1}{x}) \subset \mathbb{R}^2$

Proof: *Suppose $\exists$ open U, V, $U \cap V = \emptyset$, $X = (X \cap U) \cup (X \cap V)$*

Two cases:

If $X_1 \subset U$, then $X_2 \cap U \neq \emptyset$. $X_2 \cap V \neq \emptyset \implies$ red set is disconnected.

If $X_1 \not\subset U, X_1 \cap U \neq \emptyset$, then U, V disconnect X_1. ■

Definition 0.5 *A metric space (X, d) is path connected if $\forall a, b \in X$ there exists a continuous function $f : [0, 1] \to X$ s.t. $f(0) = a, f(1) = b$*

Connected $\not\Longrightarrow$ path connected.

Connectedness in $\mathbb{R}$.

Definition 0.6 *A set $S \subset \mathbb{R}$ is on interval if $a, b \in S, a < c < b \implies c \in S$*

Theorem 0.7 *$S \subset \mathbb{R}$ is connected if and only if it is an interval.*

Proof: Suppose S is not an interval. Then $\exists a, b$ Then $\exists a, b \in S$, say $a < b$ and x s.t. $a < x < b$, $x \notin S$.

Then $S = (S \cap (-\infty, x)) \cup (S \cap (x, \infty))$.

$\therefore S$ is not connected.

Suppose S is an interval. If S is not connected, then $\exists$ open $U, V \subset S, U \cap V = \emptyset$.

$U, V \neq \emptyset$.

Let $a \in U$, $b \in B, a < b$.

Let $c = \sup[a, b] \cap U \subset S$.

Then $c \in U$ or $c \in V$.

If $c \in U$, then $c + \varepsilon \in U$ for small enough $\varepsilon > 0$, because U is open. Contradiction because $c = \sup[a, b] \cap U$.

So $c \in V$. V is open, so $C - \varepsilon \in V$ for small enough $\varepsilon > 0$. Contradiction of c being supremum.

$\therefore$ no such U, V exist.

$\therefore$ S is connected. ■

Theorem 0.8 *In a metric space, path connected $\implies$ connected.*

Proof: Suppose X is path connected but not connected.

Let $a, b \in X$, $f : [0, 1] \to X$ continuous.

$f(0) = a$

$f(1) = b$.

If $X = U \cup V$ open.

$U \cap V = \emptyset$

$U, V \neq \emptyset$.

Then $f^{-1}(U), f^{-1}(V)$ give a disconnection of $[0, 1]$. Contradiction.

$\therefore X$ is connected. ■