0.0.1 recap

We defined "+" on $\mathbb{N}$.

0.0.2 Properties of $\mathbb{N}$

[Exercises]

1. If $m, n \in \mathbb{N}$, then $m \in n$ iff $m \subsetneq n$
2. If $m, n \in \mathbb{N}$, we say $m \leq n$ if either $m \in n$ or $m = n$ For any two $m, n \in \mathbb{N}$, either $m < n$ or $n < m$ or $n = m$ (Exactly one is true - Trichotomy)
3. For every $m, n \in \mathbb{N}$, we have $m \leq n \iff$ there is a unique $k \in \mathbb{N}$ such that $m + k = n$ - Induction proof
4. If $m, n, k \in \mathbb{N}$ such that $n + m = n + k$, then $m = k$. - Induction proof

Definition 0.1 (Subtraction) *If $m, n \in \mathbb{N}$, and $m \leq n$, set $n - m :=$ the unique k, such that $m + k = n$*

Definition 0.2 (Multiplication) *Let $m \in \mathbb{N}$. Let p_m be a relation on $\mathbb{N} \times \mathbb{N}$ with two properties:*

1. $p_m(0) = 0$
2. $p_m(n+) = p_m(n) + m$

Proceeding exactly with S_m, we can show that there is a function that satisfies the above properties. Properties:

1. Commutativity
2. Associativity etc.

0.1 Construction of the Integers $\mathbb{Z}$

[Equivalence classes of pairs of naturals]

Let $Z := \mathbb{N} \times \mathbb{N}$.

Define the following relation R on $Z \times Z$ as follows: $R \subset Z \times Z$

$R := \{((a, b), (c, d)) \in Z \times Z | a + d = b + c\}$

$((1, 5), (5, 9))$ etc.

Claim 0.3 *R is an equivalence relation.*

Proof:

- Reflexivity: $\forall(a,b) \in z$, notice: $a+b=b+a \implies ((a,b),(a,b)) \in R$ (comm)
- Symmetry: $\forall((a,b),(c,d)) \in R, ((c,d),(a,b)) \in R$ (comm)
- Transitivity: Suppose $((a,b),(c,d)) \subset R$ and $((c,d),(p,q)) \in R$ $a+d=b+candc+q=d+q$ (assoc and comm) $(a+d)+(c+q)=(b+c)+(d+p)$ $(a+q)+(c+d)=(b+p)+(c+d)$ (property 4) $a+q=b+p=((a,b),(p,q)) \in R$

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R is an equivalence relation on $Z \times Z$.

If $x \in Z$, then [x] = equivalence class of x = $\{y \in Z|(x,y) \in R\}$ $[x] \subseteq Z$ and $[x] \neq \emptyset$

Define $\mathbb{Z} := Z/R =$ set of equivalence classes of R on Z

$= [x]|x \in Z$

E.g. $[(1,3)] = [(2,4)] \in \mathbb{Z}$

$i : \mathbb{N} \to \mathbb{Z}$, given by $n \to [(n,0)]$ $[(a,b)]+[(c,d)] := [(a+c,b+d)]$

2) Negative:

If $[(a,b)]$, define $-[(a,b)] := [(b,a)]$

3) Subtraction.

If $[(a,b)],[(c,d)] \in \mathbb{Z}$ then

$[(a,b)]-[(c,d)] := [(a,b)](-[(c,d)]) = [(a,b)]+[(d,c)] = [(a+d,b+c)]$

4) Order relation:

$[(a,b)] < [(c,a)]$ if $a+d < b+c$. Well defined, and strict/total order. That is, for all $[(a,b)],[(c,d)]$ well defined, or $(c,d) < (a,b)$ or $[(a,b)=(c,d)]$. Exactly one satisfied].

5 Multiplication $(a,b) : [(c,d)] = [ac+bd, ad+bc]$

6) Absolute value $|[(a,b)| = a-bif a \geq q], beqotherwise$

From now on, we'll write integers the usual way, $[(a,b)]$ = (a-b)

$-(b-a)ifb > a$

0.2 Constructions of the rationls $\mathbb{Q}$

$\mathbb{Q} = \{(a,b) \in \mathbb{Z} \times \mathbb{Z}|b=0\}$

Define relation R on $Q \times Q[R \subset Q \times Q]$.

$((a,b),(c,d)) \in R$ iff $ad=bc$