0.1 Set Theory

Definition 0.1 (bad)

A set is an object S such that for every x, we have exactly one of two things.

- 1. $x \in S$ (x belongs to S) or
- 2. $x \notin S$ (x does not belong to S)

Sets are specified as follows

- 1. $\{1, 2, 3, 4\}$
- $2. \{1, 2, 3, 4, \ldots\}$
- 3. $\{x \in \mathbb{N} | x \text{ is prime}\}$

0.1.1 Russel's Paradox

Let S be the following:

$$S = \{T | T \not\in T\}$$

Q: Is
$$S \in S$$
?

If yes, then by definition $S \notin S$ If no, then by definition $S \in S$

0.1.2 Foundations of Set Theory/Zermelo-Frankel Set Theory

0.1.2.1 Axioms of ZFC

- 1. Axiom of Extension: Two sets are equal if and only if they have the same elements.
- 2. Axiom of Existence: The Empty Set Exists
- 3. Axiom of Pairing: If X and Y are sets, then there is a set $\{X,Y\}$

Example :
$$X = \{1, 2\}$$
 $Y = \{3, 4\}$ $X, Y = \{\{1, 2\}, \{3, 4\}\}$

$$X=\emptyset,\,Y=\emptyset$$

$$\{X,Y\} = \{\emptyset\}$$

4. Axiom of Union: If S is a (set of sets), then the union over elements of S is a set.

Example:
$$S = \{\{1, 2\}, \{2, 3\}\}\$$
, then $\{1, 2, 3\}$ is a set.

- 5. Axiom of Intersection: Redundant, as above.
- 6. Axiom of foundation.

Every $x \neq \emptyset$ contains a member y such that $x \neq y$.

Consequence of failure:

Let
$$X \neq \emptyset$$

$$\forall y \in X, y \cap X \neq \emptyset$$

Choose any $y_1 \in X$, choose $y_2 \in X \cap Y$. $y_2 \in X$, so $y_2 \cap X \neq \emptyset$. Choose $y_3 \in X \cap Y_2$...

7. Axiom Schema of Replacement

If ϕ is a formula of 1st order logic, A is any set, then ϕ applied to A is contained in a set.