0.1 Sequences and Convergence

(X, d) a metric space.

A sequences X is a function $f : \mathbb{N} \to X$.

Usually we write a sequence as $(X_n)_{n=1}^{\infty}$, where $x_n = f(n)$

A subsequence of (X_n) is a sequence $(X_{n_i})_{i=1}^{\infty}$, where $(n_i)_{i=1}^{\infty}$

Definition 0.1 *We say (X_n) converges to x if for any $\varepsilon > 0 \exists N$, s.t. $n > N \implies d(x, x_n) < \varepsilon$.*

Write $X_n \to x$

Proposition 0.2 *A sequence in a metric space can have at most one limit.*

Proof: Suppose $X_n \to x$, $X_n \to y$.

Let $\varepsilon = \frac{1}{4} d(x, y) > 0$

Then there exits n such taht $n > N \implies d(x, x_n) < \epsilon$

$\exists \tilde{N}$ s.t. $N \tilde{N} \implies d(y, x_n) < \epsilon$

$4\varepsilon \leq d(x, y) \leq (x, x_n) + d(x_n, y)$

$< \varepsilon + \varepsilon$ if $n > max N, \tilde{N}$

Contradiction.

$\therefore x = y$ ■

Definition 0.3 *A sequence $(x_n)_{n=1}^{\infty}$ is bounded if the set $\{x_n\}_{n_1}^{\infty}$ is bounded in X.*

E.g. $x_n = 1, \forall N$

$(x_n) = 1, 1, 1, 1, 1$

$\{x_n\} = \{1\}$

Proposition 0.4 *A convergent sequence in a metric space (X, d) is bounded.*

Proof: Suppose $x_n \to x$. $\exists N$ s.t. $n > N \implies d(x, x_n) < 1$.

Then at most $x_1, .., x_n$ can lie outside $B_1(x)$

We can take a large enough radius, and it is bounded. ■

Definition 0.5 *A sequence (x_n) in a metric space (X, d) converges to x if $\forall \varepsilon > 0$, $\exists N s.t. n > N \implies d(x, x_n) < \varepsilon$*

A sequence in (X, d) is Cauchy if $\forall \varepsilon > 0, \exists N$ s.t. $m, n > N \implies d(x_m, x_n) < \varepsilon$

Same proof shows that every Cauchy sequence is bounded.

Theorem 0.6 *If $x_n \to x$, $y_n \to y$, then $d(x_n, y_n) \to d(x, y)$*

Theorem 0.7 $d(x,y) \leq d(x,x_n) + d(x_n,y_n) + d(y_n,y)$ *by $\triangle$ ineq applied twice.*

$\implies d(x,y) - d(x_n,y_n) \leq d(x,x_n) + d(y,y_n) \to 0 as n \to \infty$

$d(x_n,y_n) \leq d(x_n,x) + d(x,y) + d(y,y_n)$

$\implies d(x_n,y_n) - d(x,y) \leq d(x_n,x) + d(y_n,y) \to 0$ *as* $n \to 0$

0.1.1 Sequences in $\mathbb{R}$

A sequence $(x_n) \subset \mathbb{R}$ is

increasing (non-decreasing) if $x_n \leq x_{n+1} \forall n$

decreasing (non-increasing) $x_n \geq x_{n+1} \forall n$

strictly increasing if $x_n < x_{n+1} \forall n$

strictly decreasing if $x_n > x_{n+1} \forall n$

A sequence in R is monotone if it is either increasing or decreasing.

Theorem 0.8 *Every sequence in $\mathbb{R}$ contains a monotone subsequence.*

Proof: Let $S = \{k : x_k \geq x_n \forall n \geq k\}$

If S is infinite, the $S = (n_i)_{i=1}^{\infty}$ with

$n_{i+1} > n_i \forall i$

and $x_{n_{i+1}} \geq x_{n_i} \forall I$

$\therefore (x_{n_i})_{i=1}^{\infty}$ is an increasing sequence

$\therefore$ monotone.

Suppose S is finite, say largest element of S is N.

Choose $m_1 > N \implies x_{m_1}$

Choose $m_2 > m_1$s.t.$x_{m_2} \geq x_{m_1}$ (Such m_2 exitsts, if not then $m_1 \in S_1$ not possible)

Choose $m_3 > m_2$s.t.$x_{m_3} \geq x_{m_2}$

$\vdots$

Then (x_{m_i}) is an increasing sequence and therefore monotone. ■

Theorem 0.9 *Every bounded monotone sequence in $\mathbb{R}$ has a limit.*

Proof: See first year notes. ■

Theorem 0.10 (Bolzano-Weierstrauss theorem) *Every bounded sequence of real numbers has a convergent subsequence.*

Proof: Any such sequnce has a monotone subsequence. Such a subsequence is also bounded, therefore converegent. ■