Theorem 0.1 *$\exists h > 0$ s.t. IVP/IE has a solution for $t \in [t_0 - h, t_0 + h]$.*

Proof:

(JUST LOOK AT THE NOTES.)

f is continuous on $A_{h,k}(t_0, x_0)$.

$\therefore \exists M > 0$ s.t. $|f(t,x)| \leq M \forall (t,x) \in A_{h,k}(t_0, x_0)$.

By making h spammer if necessary, we may assume $h \leq \frac{k}{M}$.

$x_n(t) = x + (t - t_0) f(t_0, x_0)$ for $t \in [t_0, t_0 + \frac{h}{n}]$.

$x_n(t) = x_n(t_0 + \frac{h}{n}) + [t - (t_0 + \frac{h}{n})] f(t_0 + \frac{h}{n}, x^n(t_0 + \frac{h}{n}))$ for $t \in [t_0, \frac{h}{n}, t_0 + \frac{2h}{n}] \vdots$

$x_n(t) = x_n(t_0 + \frac{(i-1)h}{n}) + [t - (t_0 + \frac{(i-1)h}{n})] f(t_0 + \frac{(i-1)h}{n}, x^n(t_0 + \frac{(i-1)h}{n}))$ for $t \in [t_0, \frac{(i-1)h}{n}, t_0 + \frac{(i)h}{n}]$

The graph of x_n stays inside $A_{h,k}(t_0, x_0)$. Because Lipschitz constant of $x_1 \leq M$.

(x_n) is a bounded sequence, Lipschitz constants uniformly bounded by M. Therefore $(x_m) \subset$ a compact subset of $C[t_0 - h, t_0 + h]$ equipped with the usual uniform metric.

$\therefore \exists$ subsequence (x_n') s.t. $x_{n'} \to x \in C[t_0 - h, t_0 + h]$ by Arzela-Ascoli theorem.

WTS X solves IE.

$x(t) \leftarrow x_{n'}(t) = x_0 + \int_{t_0}^t f(Pn'(s)) ds = x_0 + \int_{t_0}^t Pn'(s) ds = x_0 + \int_{t_0}^t f(s, x(s)) ds + \int_{t_0}^t [f(P_n'(s)) - f(s, x_{n'}(s))] + \int_{t_0}^t [f(s, x_{n'}(s)) - f(s, x(s))] ds$.

Let $t \in [t_0 + \frac{(i-1)h}{n}, t_0 + \frac{ih}{n}]$

$|(t, x_n(t)) - P_n(t)| \leq \sqrt{(\frac{h}{n})^2 + M^2(\frac{h^2}{n^2})} \leq \sqrt{1 + M^2} \frac{h}{n}$

$\frac{h}{n} \to 0$ as $n \to \infty$.

$\therefore |(t, x_n(t)) - P_n(t)| \to 0$ uniformly on $[t_0 - h, t_0 + h]$ as $h \to \infty$

Let $\varepsilon > 0$. f is uniformly cts on $A_{h,k}(t_0, x_0)$ $\exists \delta > 0$ s.t. $P, Q \in A_{h,k}(t_0, x_0), |P - Q| \leq \delta$

$\implies f(P) - f(Q) \leq \varepsilon$.

$\exists N$ s.t. $n > N \implies |f(t, x_n(t)) - f(P_n(t))| < \varepsilon \ \forall t \in [t_0 - h, t_0 + h]$

$\exists \overline{N} s.t. n' \geq N' \implies |x_n'(x) - x(s)| < \delta \forall s \in [t_0 - h, t_0 + h]$

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