0.1 Metric Spaces

Definition 0.1 *A Metric space* (X, d) *is a set* X *together with a function* $d : X \times X \to \mathbb{R}$ *s.t.*

- *Positivity:* $d(x, y) \geq 0, = 0 \iff x = y$
- *Symmetry:* $d(y, x) = d(x, y)$
- *Triangle inequality:* $d(x, y) \leq d(x, z) + d(z, y)$

Example 0.2

- $\mathbb{R}, d(x, y) = |x - y|$
- $R^n, d(x, y) = (\sum_{i+1}^{n} |x_i - y_i|^2)^{\frac{1}{2}}$
  $d_p(x, y) = (\sum_{i+1}^{n} |x_i - y_i|^p)^{\frac{1}{p}}$
  $p = 1$ *is the Manhattan Metric*
  $d_\infty(x, y) = max_{1 \leq i \leq n} |x_i - y_i|$
- *Any normed space* $(V, \|\cdot\|)$
- X *any set* $(\neq \emptyset)$
  $d(x, y) = 1$ *if* $x \neq y, 0$ *if* $x = y$
- *TODO: Insert diagram Globe*
- *TODO: Insert diagram Toruses*
- *Paris metro or post office metric*
  $d(x, y) = \|x - y\|$ *if* $y = tx$ *for some* $t \in \mathbb{R}$
  $|x| + |y|$ *otherwise*
- *P-adic Metric:* $\mathbb{Z}$*, let* p *be a prime.*
  If $x, y \in \mathbb{Z}$*,* $x - y = p^k n$*,* p *does not divide* n
  $d(x, y) = 1/(k + 1)$ *if* $x \neq y$
  0 *if* $x = y$

Definition 0.3 *Let* (X, d) *be the a metric space.*

The open ball of radius r*, centre* a *is the set* $B_r(a) = \{x \in X | d(x, a) < r\}$

Example 0.4

$\mathbb{R}$ *usual metric,* $B_r = (a - r, a + r)$

$\mathbb{R}^n B_1(0) \in \mathbb{R}^n, d$

TODO: Insert Diagrams for different metrics

$B\dot{r}(x) = \{x\}$ *if* $r \leq 1$

$B\dot{r}(x) = X$ *if* $r \geq 1$

Definition 0.5 *$Y \subset X$ is a neighbourhood of a point $x \in X$ if $\exists r > 0$ s.t $B_r(x) \subset Y$*

Definition 0.6 *A set $X \subset X$ is bounded if $S \subset B_r(x)$ for some $x \in X, r > 0$*

Proposition 0.7 *If $S \subset X$ is bounded, then for any $y \in X \exists r > 0$ s.t. $S \subset B_r(y)$*

Proof: Ex: Use triangle ineq ■

Definition 0.8 *Suppose $A \subset X$. A point $x \in X$ is an interior point of A if $\exists r > 0$ s.t. $B_r(x) \subset A$*

$x \in X$ is an exterior point if A if $\exists r > 0$ s.t. $B_r(x) \subset A^c = X - A$

A point $x \in X$ is a boundary point of A if $\forall r > 0$, if $B_r(x)$ contains points of A and points of A^c

Set of interior points = int A exterior = ext A boundary = δA

Proposition 0.9 *$X = intA \cup extA \cup \delta A$. These sets are pairwise disjoint,*

$extA = int(A^c)$

$intA = ext(A^c)$

$intA \subset A$, $extA \subset A^c$