0.0.1 Splitting Fields

Definition 0.1 *F field, $f(x) \in F[x]$ non-zero, a splitting field of f is a field extension E/F such that $f(x) = \alpha \Pi_i (x - \alpha_i)$, with $\alpha, \alpha_1, \alpha_2, ..., \alpha_n$ and $E = F(\alpha_1, ..., \alpha_n)$*

F field, A, B rings [e.g. $A = F[x]$, $F \to F[X]$]

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, "i_A"] & B \\ F \arrow[ur, "i_B"] & \end{tikzcd}$$

$Hom_F(A, B) = \{\phi : A \to B \; | i_b = \phi i_A\}$

Proposition 0.2 *$F \to F[x], F \to B$ any ring morphism.*

$Hom_F(F[x], B) \to [\cong] B$

$\phi \mapsto \phi(x)$

Proof: *Given $b \in B$, define $\phi_b : F[x] \to B$ by $\phi(\sum_{n=0}^{m} a_n x^n) = \sum_{n=0}^{m} a_n b^n$*

Check ϕ_b is a ring morphism ■

Cor: Fix $f(x) \in F[X]$, then there is a bijection $Hom_F(\frac{F[x]}{f(x)}, B)$

TODO: Turn scratchwork into proof.

Scratchwork below [Also refer to video]

$$\begin{tikzcd} A \arrow[r, "\bar{\phi}_P"] \arrow[d, "P"] & B \\ A/I \arrow[ur, "\bar{\phi}"] & \end{tikzcd}$$

$\phi(a + I) = \phi(a)$

$a + I = a' + I \implies a - a' \in I$

So $\phi(a) = \phi(a')$

$A = F[x]$

$\downarrow$

$\frac{F[X]}{f} \to B$

Corollary 0.3 *TODO: Write up from notes*

Proof: $f(\alpha) = \frac{F[x]}{(f(x))}$, f min. poly of α ■

Cor: ANy two splitting fields $\frac{E_1}{F}$, and E_2/F of a poly $f(x) \in F[x]$ are F-isomorphic.

Proof: ETS there is an $F-$morphism $\phi : E_1 \to E_2$. Since then $[E_1 : F] \leq [E_2 : F]$ By symmetry there would be a map from E_2 to E_1, so $[E_1 : F] \geq [E_2 : F]$

So ϕ will be an isomorphism.

Let $\alpha_1, ..., \alpha_n$ be the roots in E_1 of f, so $F(\alpha_1, ..., \alpha_n)$.

Assume by induction we ahve $\phi_i : F(\alpha_1, .., \alpha_i) \to E_2$

$F(\alpha_1, ..., \alpha_{i+1}) \supseteq F(\alpha_1, ..., \alpha_i) \to E_2$

Let $g(x)$ be the min poly of α_{i+1} over $F(\alpha_1, ..., \alpha_i)$, then $g|F$. So there exists a root of $g \in E_2$, since F splits there.

User cor. to define ϕ_{i+1} ■

0.0.2 Computing the degree of a splitting field