Proof: Arzela-Ascoli Theorem

We need to show $\mathcal{F}$ is complete and totally bounded.

$(C(X, \mathbb{R}^n), d_n)$ is complete, $\mathcal{F}$ is closed in $C(X, \mathbb{R}^n) \implies \mathcal{F}$ is complete

Let $\delta > 0$. WTS $\exists$ a finite collection S of functions in $C(X, \mathbb{R}^m)$ s.t. $\forall f \in \mathcal{F}$, $\exists g \in S \text{s.t.} d_u(f, g) < \delta$.

$\mathcal{F}$ is bounded, so there exists K, s.t. $|f(x)| \leq K$, $\forall x \in X$, $\forall f \in \mathcal{F}$.

By uniform continuity, $\exists \delta > 0$ s.t. $d(u, v) < \delta_1 \implies |f(u) - f(v)| < \frac{\varepsilon}{4}$, $\forall x \in X$, $\forall f \in \mathcal{F}$.

By total boundedness of X, $\exists x_1, \ldots, x_p \in X$ s.t. for some $i \in 1, \ldots, p$ then $|f(x) - f(x_i)| < \frac{\delta}{4} \forall f \in \mathcal{F}$

By total boundedness of $B_k(0) \subset \mathbb{R}^n$, $\exists$ finitely many $y_1, y_2, \ldots, y_q \in \mathbb{R}^n$ s.t. $|y - y_j| \leq \frac{\delta}{4}$

Consider all functions $\alpha : \{x_1, \ldots, x_p\} \to \{y_1, \ldots, y_p\}$.

There are finitely many such α, q^p.

For each α, if $\exists f \in \mathcal{F}$ such that $|f(x_i) - \alpha(x_i)| < \frac{\delta}{4}, \forall i = 1, \ldots, p$

Choose one such f, call it g_α.

Let $S = \{g_\alpha\}$. This is a finite collection.

Then $|g_\alpha(x_i) - \alpha(x_i)| < \frac{\delta}{4} \forall i = 1, \ldots, p$

Let $f \in \mathcal{F}$. For each $i = 1, \ldots, p$ choose one y_j s.t. $|f(x_i) - y_j| < \frac{\delta}{4}$.

Let α be a function that assigns to each x_i the corresponding y_j. Then $|f(x_i) - \alpha(x_i)| < \frac{\delta}{4} \forall i = 1, \ldots, p$

Then $\exists g_\alpha$ as above.

Let $x \in X$. Choose x_i such that $d(x, x_i) < \delta_1$.

$|f(x) - g_\alpha(x)| \leq |f(x) - f(x_i)| + |f(x_i) - \alpha(x_i)| + |\alpha(x_i) - g_\alpha(x_i)| + |g_\alpha(x_i) - g_\alpha(x)| < \frac{\delta}{4} + \frac{\delta}{4} + \frac{\delta}{4} + \frac{\delta}{4} = \delta$

■

∘ $x'(t) = f(t, x(t))$, f cts in $U \subset \mathbb{R} \times \mathbb{R}$.

If in addition f is locally Lipschitz cts with respect to x, then $\exists h > 0$ such that $(*)$ has a unique solution defined for $t \in [t_0 - h, t_0 + h]$.

If f is not locally Lipschitz, then uniqueness can fail.

E.g. $x'(t) = \sqrt{|x(t)|}$

$x(0) = 0$

Existence still holds for small enough h.

Theorem 0.1 (Peano's Theorem) *Suppose* $f \in C(U)$, $(t_0, x_0) \in U$

Then $\exists h > 0$ *s.t.* $(*)$ *has a solution on* $[t_0 - h, t_0 + h]$

∘ $\iff$

∘∘ $x(t) = x_0 + \int_{t_0}^{t} f(s, x(s)) ds$

$x(t) = x_0 + f(t_0, x_0)(t - t0)$

$x(t) = x(t_1) + f(t_1, x(t_1))(t - t_1)$

Approximate w piecewise linear fcns, that limits to the function.