

## 0.1 Axioms of ZFC (Continued)

### 7. Axiom of Specification:

The collection of all elements of a known set, that satisfy a certain predicate, is a set.

E.g.  $\{x \in \mathbb{N} \mid x \text{ is prime}\}$

### 8. Axiom of power set

If  $S$  is a set, then the collection of all subsets of  $S$  is a set, called the power set.  $\mathcal{P}(S)$

$(a, b) = \{\{a\}, \{a, b\}\}$

$A \times B :=$  set of ordered pairs  $(a, b) = \{\{a\}, \{a, b\}\}$

Let  $A, B$  be sets. A function  $f : A \rightarrow B$  is an element of the  $\mathcal{P}(A \times B)$ ,

- For every  $a \in A$ , there is some  $b \in B$  such that  $(a, b) \in f$
- if  $b_1$  and  $b_2 \in B$  such that  $(a, b_1) \in f$  and  $(a, b_2) \in f$  then  $b_1 = b_2$

### 9. Let $X$ be a set. Define $\succ (x) = x^+ := x \cup \{x\}$ . There exists a set $S$ such that

(a)  $\emptyset \in S$ ,

(b)  $x \in S$  then  $x^+ \in S$ .

E.g.  $\text{succ}(\emptyset) = \emptyset \cup \emptyset = \emptyset$   $\{\emptyset\}^+ = \{\emptyset\} \cup \{\{\emptyset\}\} = \{\emptyset, \{\emptyset\}\}$

### 10. Axiom of choice.

A choice function  $f$  defined on a set  $X$  of non-empty sets is a function with the property that if  $a \in X$  then  $f(a) \in a$ .

Choice functions always exist for any  $X$ .

## 0.2 Construction of $\mathbb{N}$

$\mathbb{N}$  includes 0.

Define  $0 := \emptyset$

$1 := \emptyset^+$

$2 := 1^+ = \{\emptyset, \{\emptyset\}\}$

$3 := 2^+ = \{\emptyset, \{\emptyset\}\} \cup \{\{\emptyset, \{\emptyset\}\}\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$

Q: What is  $\mathbb{N}$ .

A9 tells us that there is a set  $S$  such that  $\emptyset \in S$ , if  $x \in S, x^+ \in S$

Consider  $I_S := \{T \in \mathcal{P}(S) \mid \emptyset \in T \text{ and } \forall x \in T, x^+ \in T\}$

Say that a set  $A$  is inductive. If  $\emptyset \in A$  and if  $x \in A, x^+ \in A$ .

$I_s =$  set of all inductive subsets of  $S$ .

$I_s \neq \emptyset$  because  $S \in I_s$

Define  $\mathbb{N} = \cap_{x \in I_s} x = \{x \in S \mid \forall T \in I_s, x \in T\}$

**Theorem 0.1** *Principle of Mathematical Induction*

Let  $p$  be a predicate defined on  $\mathbb{N}$ . Assume that  $p(0)$  holds, and for every  $n \in \mathbb{N}$ ,  $p(n)$  implies  $p(n^+)$ . Then  $p(n)$  holds for every  $n \in \mathbb{N}$ .

**Proof:** Fix  $p$ , with the above properties.

$S = \{x \in \mathbb{N} | p(x) \text{ holds}\}$ . We want to show that  $S = \mathbb{N}$ , i.e. the elements of  $S$  are exactly the elements of  $\mathbb{N}$ .

Observation  $S$  is inductive

1.  $0 \in S$ , because  $p(0) = p(0)$  holds.
2. If  $x \in S$  it means  $p(x)$  holds, therefore  $p(x^+)$  also holds.  
So,  $x^+ \in S$

1 and 2 tell us that  $S$  is inductive.

$S \subseteq \mathbb{N}$  by construction.

$\mathbb{N} \subseteq S$  by construction.

$$\implies S = \mathbb{N}$$

■

**Theorem 0.2** *If  $m, n \in \mathbb{N}$ , such that  $m^+ = n^+$ , then  $m = n$ .*

**Proof:**

**Lemma 0.3** *Let  $x, n \in \mathbb{N}$ . If  $x \in n$  then  $x \subseteq n$*

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