9. Bistability, Metastability, and Escape Problems

This chapter is devoted to the asymptotic study of systems which can exist in at least two stable states, and to some closely related problems. Such systems are of great practical importance, e.g., switching and storage devices in computers are systems which have this property. So do certain molecules which can isomerise, and more recently, a large number of electronic, chemical and physical systems which demonstrate related properties in rich variety have been investigated.

The problems of interest are:

i) How stable are the various states relative to each other?
ii) How long does it take for a system to switch spontaneously from one state to another?
iii) How is the transfer made, i.e., through what path in the relevant state space?
iv) How does a system relax from an unstable state?

These questions can all be answered relatively easily for one-dimensional diffusion processes—but the extension to several, but few dimensions is only recent. The extension to infinitely many variables brings us to the field of the liquid-gas transition and similar phase transitions where the system can be in one of two phases and arbitrarily distributed in space. This is a field which is not ready to be written down systematically from a stochastic point of view, and it is not treated here.

The chapter is divided basically into three parts: single variable bistable diffusion processes, one-step birth-death bistable systems and many-variable diffusion processes. The results are all qualitatively similar, but a great deal of effort must be invested for quantitative precision.

9.1 Diffusion in a Double-Well Potential (One Variable)

We consider once more the model of Sect. 5.2.7 where the probability density $p(x, t)$ of a particle obeys the Fokker-Planck equation

$$\partial_t p(x, t) = \partial_x[U'(x)p(x, t)] + D\partial_x^2 p(x, t) . \tag{9.1.1}$$

The shape of $U(x)$ is as shown in Fig. 9.1. There are two minima at a and c and in between, a local maximum. The stationary distribution is

$$p_s(x) = \mathcal{N} \exp[-U(x)/D] \tag{9.1.2}$$

and it is this that demonstrates the bistability. Corresponding to a, c and b are

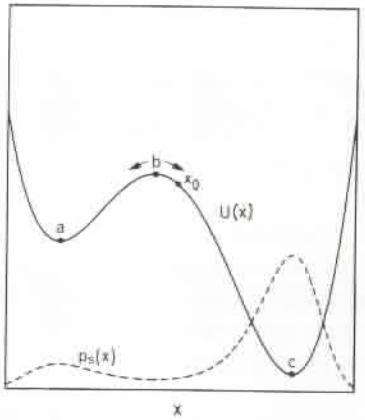

Fig. 9.1. Plot of $p_s(x)$ and $U(x)$ for a double well potential

two maxima, and a central minimum as plotted in Fig. 9.1. The system is thus most likely to be found at a or c.

9.1.1 Behaviour for $D = 0$

In this case, $x(t)$ obeys the differential equation

$$\frac{dx}{dt} = -U'(x), \qquad x(0) = x_0 . \tag{9.1.3}$$

Since

$$\frac{dU(x)}{dt} = U'(x)\frac{dx}{dt} = -[U'(x)]^2 < 0 ,$$

$x(t)$ always moves in such a way as to minimise $U(x)$, and stops only when $U'(x)$ is zero. Thus, depending on whether x_0 is greater than or less than b, the particle ends up at c or a, respectively. The motion follows the arrows on the figure.

Once the particle is at a or c it stays there. If it starts exactly at b, it also stays there, though the slightest perturbation drives it to a or c. Thus, b is an unstable stationary point and a and c are stable. There is no question of *relative* stability of a and c.

9.1.2 Behaviour if D is Very Small

With the addition of noise, the situation changes. The stationary state can be approximated asymptotically as follows. Assuming $U(x)$ is everywhere sufficiently smooth, we can write

$$\begin{aligned} U(x) &\simeq U(a) + \tfrac{1}{2}U''(a)\,(x-a)^2 \qquad |x-a| \text{ small} \\ &\simeq U(c) + \tfrac{1}{2}U''(c)\,(x-c)^2 \qquad |x-c| \text{ small} . \end{aligned} \tag{9.1.4}$$

If D is very small, then we may approximate

$$\begin{aligned} p_s(x) &\simeq \mathcal{N} \exp\left[-U(a)/D - \tfrac{1}{2}U''(a)\,(x-a)^2/D\right] \qquad |x-a| \text{ small} \\ &\simeq \mathcal{N} \exp\left[-U(c)/D - \tfrac{1}{2}U''(c)\,(x-c)^2/D\right] \qquad |x-c| \text{ small} \\ &\simeq 0 \quad \text{(elsewhere)} \end{aligned} \tag{9.1.5}$$

so that

$$\mathcal{N}^{-1} \simeq e^{-U(a)/D}\sqrt{2\pi D/U''(a)} + e^{-U(c)/D}\sqrt{2\pi D/U''(c)}\,. \tag{9.1.6}$$

Suppose, as drawn in the figure,

$$U(a) > U(c)\,. \tag{9.1.7}$$

Then for small enough D, the second term is overwhelmingly larger than the first and $\mathcal{N}^{-1}$ can be approximated by it alone. Substituting into (9.1.5) we find

$$\begin{aligned} p_s(x) &= \sqrt{\frac{U''(c)}{2\pi D}} \exp\left[-\tfrac{1}{2}U''(c)\,(x-c)^2/D\right] \qquad |x-c| \sim \sqrt{D} \\ &= 0 \qquad \text{(otherwise)}\,. \end{aligned} \tag{9.1.8}$$

This means that in the limit of very small D, the deterministic stationary state at which $U(x)$ has an absolute minimum is the more stable state in the sense that in the stochastic stationary state, $p_s(x)$ is very small everywhere except in its immediate vicinity.

Of course this result disagrees with the previous one, which stated that each state was equally stable. The distinction is one of time, and effectively we will show that the deterministic behaviour is reproduced stochastically if we start the system at x_0 and consider the limit $D \to 0$ of $p(x, t)$ for any finite t. The methods of Sect. 6.3 show this as long as the expansion about the deterministic equation is valid. Equation (6.3.10) shows that this will be the case provided $U'(x_0)$ is nonzero, or, in the case of any finite D, $U'(x_0)$ is of order D^0 (here, D replaces the ε^2 in Sect. 6.3.) This is true provided x_0 is not within a neighbourhood of width of order $D^{1/2}$ of a, c, or b. This means that in the case of a and c, fluctuations take over and the motion is given approximately by linearising the SDE around a or c. Around b, the linearised SDE is unstable. The particle, therefore, follows the Ornstein-Uhlenbeck Process until it leaves the immediate neighbourhood of $x = b$, at which stage the asymptotic expansion in $\sqrt{D}$ takes over.

However, for $t \to \infty$, the asymptotic expansion is no longer valid. Or, in other words, the $t \to \infty$ limit of the small noise perturbation theory does not reproduce the $D \to 0$ limit of the stationary state.

The process that can occur is that of escape over the central barrier. The noise $dW(t)$ can cause the particle to climb the barrier at b and reach the other side. This involves times of order $\exp(-\text{const}/D)$, which do not contribute to an asymptotic expansion in powers of D since they go to zero faster than any power of D as $D \to 0$.

9.1.3 Exit Time

This was investigated in Sect. 5.2.7c. The time for the particle, initially near a, to reach the central point b is

$$T(a \to b) = \pi[|U''(b)|\,U''(a)]^{-1/2} \exp\{[U(b) - U(a)]/D\} \tag{9.1.9}$$

(half as long as the time for the particle to reach a point well to the right of b). For $D \to 0$, this becomes exponentially large. The time taken for the system to reach the stationary state will thus also become exponentially large, and on such a time scale development of the solutions of the corresponding SDE in powers of $D^{1/2}$ cannot be expected to be valid.

9.1.4 Splitting Probability

Suppose we put the particle at x_0: what is the probability that it reaches a before c, or c before a? This can be related to the problem of exit through a particular end of an interval, studied in Sect. 5.2.8 We put absorbing barriers at $x = a$ and $x = c$, and using the results of that section, find π_a and π_c, the "*splitting probabilities*" for reaching a or c first. These are (noting that the diffusion coefficient is D, and hence independent of x):

$$\boxed{\begin{aligned} \pi_a(x_0) &= \left[\int_{x_0}^{c} dx\, p_s(x)^{-1}\right] \Big/ \left[\int_a^c dx\, p_s(x)^{-1}\right] \\ \pi_c(x_0) &= \left[\int_a^{x_0} dx\, p_s(x)^{-1}\right] \Big/ \left[\int_a^c dx\, p_s(x^{-1})\right]. \end{aligned}} \tag{9.1.10}$$

The splitting probabilities π_a and π_c can be viewed more generally as simply the probability that the particle, started at x_0, will fall into the left or right-hand well, since the particle, having reached a, will remain on that side of the well for a time of the same order as the mean exit time to b.

We now consider two possible asymptotic forms as $D \to 0$.

a) x_0 a Finite Distance from b

We first evaluate

$$\int_a^c dx\, p_s(x)^{-1} = \int_a^c dx\, \mathcal{N}^{-1} \exp\left[U(x)/D\right]. \tag{9.1.11}$$

This is dominated by the behaviour at $x \sim b$. An asymptotic evaluation is correctly obtained by setting

$$U(x) \simeq U(b) - \tfrac{1}{2}|U''(b)|/(b-x)^2\,. \tag{9.1.12}$$

As $D \to 0$, the limits at $x = a$, c effectively recede to $\pm\infty$ and we find

$$\int_a^c dx\, p_s(x)^{-1} \sim \mathcal{N}^{-1} \sqrt{\frac{2\pi D}{|U''(b)|}} \exp\left[U(b)/D\right]. \tag{9.1.13}$$

Now suppose $x_0 < b$. Then $\int_a^{x_0} dx\, p(x)^{-1}$ can be evaluated by the substitution

$$y = U(x) \tag{9.1.14}$$

with an inverse $x = W(y)$ and is asymptotically

$$\begin{aligned}\mathcal{N}^{-1} \int_{-\infty}^{U(x_0)} e^{y/D} W'(y)dy &\sim \mathcal{N}^{-1} D\, e^{U(x_0)/D} W'[U(x_0)] \\ &= \mathcal{N}^{-1} \frac{D\, e^{U(x_0)/D}}{U'(x_0)}.\end{aligned} \tag{9.1.15}$$

Thus,

$$\pi_c \sim \frac{1}{U'(x_0)} \sqrt{\frac{|U''(b)|D}{2\pi}} \exp\left[\frac{U(x_0) - U(b)}{D}\right] \tag{9.1.16}$$

and

$$\pi_a = 1 - \pi_c. \tag{9.1.17}$$

We see here that the splitting probability depends only on x_0 and b. Thus, the probability of reaching c in this limit is governed entirely by the probability of jumping the barrier at b. The points at a and c are effectively infinitely distant.

b) x_0 Infinitesimally Distant from b

Suppose

$$x_0 = b - y_0 \sqrt{D}. \tag{9.1.18}$$

In this case, we can make the approximation (9.1.12) in both integrals. Defining [9.1]

$$\mathrm{erf}(x) = \sqrt{\frac{\pi}{2}} \int_0^x dt\, e^{-t^2}, \tag{9.1.19}$$

we find

$$\pi_c = 1 - \pi_a \sim \tfrac{1}{2}\{1 - \mathrm{erf}[y_0 \sqrt{|U''(b)|}]\} \tag{9.1.20}$$

$$= \tfrac{1}{2}\left\{1 - \mathrm{erf}\left[(b - x_0)\sqrt{\frac{|U''(b)|}{D}}\right]\right\}. \tag{9.1.21}$$

Equation (9.1.21) is the result that would be obtained if we replaced $U(x)$ by its quadratic approximation (9.1.12) over the whole range.

c) Comparison of Two Regions

The two regions give different results, and we find that a simple linearisation of the SDE [which is what replacing $U(x)$ by a quadratic approximation amounts to] gives the correct result only in the limit of large D and in a region of order of magnitude $\sqrt{D}$ around the maximum b.

9.1.5 Decay from an Unstable State

The mean time for a particle placed at a point on a potential to reach one well or the other is an object capable of being measured experimentally. If we use (9.1.1) for the process, then the mean time to reach a or c from b can be computed exactly using the formulae of Sect. 5.2.8. The mean time to reach a from b is the solution of

$$-U'(x)\partial_x[\pi_a(x)T(a, x)] + D\partial_x^2[\pi_a(x)T(a, x)] = -\pi_a(x) \tag{9.1.22}$$

with the boundary conditions

$$\pi_a(a)T(a, a) = \pi_a(c)T(a, c) = 0 \tag{9.1.23}$$

and $\pi_a(x)$ given by (9.1.10)

The solution to (9.1.22) is quite straightforward to obtain by direct integration, but it is rather cumbersome. The solution technique is exactly the same as that used for (5.2.158) and the result is similar. Even the case covered by (5.2.158) where we do not distinguish between exit at the right or at the left, is very complicated.

For the record, however, we set down that

$$T(a, x) = \frac{\pi_c(x) \int_x^c dx' p_s(x')^{-1} \int_a^{x'} \pi_a(z)p_s(z)dz - \pi_a(x) \int_a^x dx' p_s(x')^{-1} \int_a^{x'} \pi_a(z)p_s(z)dz}{D\pi_a(x)} \tag{9.1.24}$$

where one considers that $\pi_a(x)$ is given by (9.1.10) and $p_s(z)$ by (9.1.2). It can be seen that even for the simplest possible situation, namely,

$$U(x) = -\tfrac{1}{2}kx^2, \tag{9.1.25}$$

the expression is almost impossible to comprehend. An asymptotic treatment is perhaps required. Fortunately, in the cases where $p_s(z)$ is sharply peaked at a and c with a sharp minimum at b, the problem reduces essentially to the problem of relaxation to a or to c with a reflecting barrier at b.

To see this note that

i) the explicit solution for $\pi_a(x)$ (9.1.10) means

$$\begin{aligned}\pi_a(x) &\simeq 1 \quad (x < b) \\ &\simeq \tfrac{1}{2} \quad (x = b) \\ &\simeq 0 \quad (x > b)\end{aligned} \tag{9.1.26}$$

and the transition from 1 to 0 takes place over a distance of order $\sqrt{D}$, the width of the peak in $p_s(x)^{-1}$.

ii) In the integrals with integrand $\pi_a(z)p_s(z)$, we distinguish two cases.

$x' > b$: in this case the estimates allow us to say (9.1.27)

$$\int_a^{x'} \pi_a(z)p_s(z)dz \simeq n_a/2 \,, \tag{9.1.29}$$

where $n_a = \int_{-\infty}^{b} p_s(z)dz$ and represents the probability of being in the left-hand well.

However, when $x' < a$, we may still approximate $\pi_a(z)$ by 1, so we get

$$\int_a^{x'} \pi_a(z)p_s(z)dz \simeq \int_a^{x'} p_s(z)dz = \frac{n_a}{2} - \int_{x'}^{b} p_s(z)dz \,. \tag{9.1.30}$$

Substituting these estimates into (9.1.24) we obtain

$$T(a, b) \simeq D^{-1} \int_a^b dx' p_s(x)^{-1} \int_{x'}^b p_s(z)dz \,. \tag{9.1.31}$$

which is the exact mean exit time from b to a in which there is a reflecting barrier at b. Similarly,

$$T(c, b) \simeq D^{-1} \int_b^c dx' p_s(x)^{-1} \int_b^{x'} p_s(z)dz \,. \tag{9.1.32}$$

9.2 Equilibration of Populations in Each Well

Suppose we start the system with the particle initially in the left-hand well at some position x_i so that

$$p(x, 0) = \delta(x - x_i) \,. \tag{9.2.1}$$

If D is very small, the time for x to reach the centre is very long and for times small compared with the first exit time, there will be no effect arising from the existence of the well at c. We may effectively assume that there is a reflecting barrier at b.

The motion inside the left-hand well will be described simply by a small noise expansion, and thus the typical relaxation time will be the same as that of the deterministic motion. Let us approximate

$$U(x) \simeq U(a) + \tfrac{1}{2}U''(a)x^2 \,;$$

then the system is now an Ornstein-Uhlenbeck process and the typical time scale is of the order of $[U''(a)]^{-1}$.

Thus, we expect a two time scale description. In the short term, the system relaxes to a quasistationary state in the well in which it started. Then on a longer time scale, it can jump over the maximum at b and the long-time bimodal stationary distribution is approached.

9.2.1 Kramers' Method

In 1940, *Kramers* [9.2] considered the escape problem from the point of view of molecular transformation. He introduced what is called the Kramers equation (Sect. 5.3.6a) in which he considered motion under the influence of a potential $V(x)$ which was double welled. In the case of large damping, he showed that a corresponding Smoluchowski equation of the form found in (6.4.18) could be used, and hence, fundamentally, the escape problem is reduced to the one presently under consideration.

Kramers' method has been rediscovered and reformulated many times [9.3]. It will be presented here in a form which makes its precise range of validity reasonably clear.

Using the notation of Fig. 9.1, define

$$M(x, t) = \int_{-\infty}^{x} dx' p(x', t) \tag{9.2.2}$$

$$N_a(t) = 1 - N_c(t) = M(b, t)$$

and (9.2.3)

$$N_0(t) = (c - a)\, p(x_0, t) \,.$$

Further, define the corresponding stationary quantities by

$$\begin{aligned} n_a &= 1 - n_c = \int_{-\infty}^{c} p_s(x')dx' \\ n_0 &= (c-a)\, p_s(x_0) \end{aligned} \tag{9.2.4}$$

From the FPE (9.1.1) and the form of $p_s(x)$ given in (9.1.2) we can write

$$d_t M(x, t) = D\, p_s(x)\partial_x[\, p(x, t)/p_s(x)] \tag{9.2.5}$$

which can be integrated to give

$$d_t \int_a^{x_0} dx\; M(x, t)/p_s(x) = D[\, p(x_0, t)/p_s(x_0) - p(a, t)/p_s(a)] \,. \tag{9.2.6}$$

This equation contains no approximations. We want to introduce some kind of approximation which would be valid at long times.

We are forced to introduce a somewhat less rigorous argument than desirable in order to present the essence of the method. Since we believe relaxation within each well is rather rapid, we would expect the distribution in each well (in a time of order of magnitude which is finite as $D \to 0$) to approach the same shape as the stationary distribution, but the relative weights of the two peaks to be different. This can be formalised by writing

$$\begin{aligned} p(x, t) &= p_s(x)N_a(t)/n_a \qquad x < b \\ &= p_s(x)N_c(t)/n_c \qquad x > b \,. \end{aligned} \tag{9.2.7}$$

This would be accurate to lowest order in D, except in a region of magnitude $\sqrt{D}$ around b.

If we substitute these into (9.2.6), we obtain

$$\begin{aligned} \kappa(x_0)\dot{N}_a(t) &= D[N_0(t)/n_0 - N_a(t)/n_a] \\ \mu(x_0)\dot{N}_c(t) &= D[N_0(t)/n_0 - N_c(t)/n_c] \end{aligned} \tag{9.2.8}$$

with

$$\begin{aligned} \kappa(x_0) &= \int_a^{x_0} p_s(x)^{-1}[1 - \psi(x)]dx \\ \mu(x_0) &= \int_{x_0}^{c} p_s(x)^{-1}[1 - \psi(x)]dx \end{aligned} \tag{9.2.9}$$

and

$$\begin{aligned} \psi(x) &= n_a^{-1} \int_x^b p_s(z)dz \qquad x < b \\ &= n_c^{-1} \int_b^x p_s(z)dz \qquad x > b\,. \end{aligned} \tag{9.2.10}$$

Note that if x is finitely different from a or c, then $\psi(x)$ vanishes exponentially as $D \to 0$, as follows directly from the explicit form of $p_s(x)$. Hence, since x in both integrals (9.2.9) satisfies this condition over the whole range of integration, we can set

$$\psi(x) = 0$$

and use

$$\begin{aligned} \kappa(x_0) &= \int_a^{x_0} p_s(x)^{-1}\,dx \\ \mu(x_0) &= \int_{x_0}^{c} p_s(x)^{-1}\,dx\,. \end{aligned} \tag{9.2.11}$$

a) Three State Interpretation

Equations (9.2.8) correspond to a process able to be written as a chemical reaction of the kind

$$X_a \rightleftharpoons X_0 \rightleftharpoons X_c \tag{9.2.12}$$

except that there is no equation for N_0, the number of X_0. By noting that $N_a + N_c = 1$, we find that

$$N_0(t) = n_0[\mu(x_0)N_a(t) + \kappa(x_0)N_c(t)]/[\kappa(x_0) + \mu(x_0)]\,. \tag{9.2.13}$$

This is the same equation as would be obtained by adiabatically eliminating the variable $N_0(t)$ from (9.2.8) and the further equation for $N_0(t)$

$$\begin{aligned} \dot{N}_0(t) = D\{&N_a(t)/[n_a\kappa(x_0)] + N_c(t)/[n_c\mu(x_0)]\} \\ &- N_0(t)\{[n_0\kappa(x_0)]^{-1} + [n_0\mu(x_0)]^{-1}\}\,. \end{aligned} \tag{9.2.14}$$

Since

$$n_0 = p_s(x_0)(c - a) = \mathcal{N}\exp[-U(x_0)/D](c - a)\,. \tag{9.2.15}$$

we see that the limit $D \to 0$ corresponds to $n_0 \to 0$, and hence the rate constant in (9.2.14) multiplying $N_0(t)$ becomes exponentially large. Hence, adiabatic elimination is valid.

This three-state interpretation is essentially the transition state theory of chemical reactions proposed by *Eyring* [9.4].

b) Elimination of Intermediate States

Eliminating $N_0(t)$ from (9.2.8) by adding the two equations, we get

$$\dot{N}_a(t) = -\dot{N}_c(t) = r_a N_a(t) + r_c N_c(t) \tag{9.2.16}$$

with

$$r_a = D[n_a \int_a^c dx\, p_s(x)^{-1}]^{-1} \qquad r_c = D[n_c \int_a^c dx\, p_s(x)^{-1}]^{-1}\,, \tag{9.2.17}$$

where r_a and r_c are independent of x_0. Thus, the precise choice of x_0 does not affect the interpeak relaxation.

Since $N_a + N_c = 1$, the relaxation time constant, τ_r, is givenby

$$\tau_r^{-1} = r_a + r_c = \frac{D}{n_a n_c \int_a^c dx\, p_s(x)^{-1}}\,. \tag{9.2.18}$$

c) The Escape Probability Per Unit Time for a particle initially near a to reach x_0 is the decay rate of $N_a(t)$ under the condition that an absorbing barrier is at x_0. This means that in (9.2.8) we set $N_0(t) = 0$ [but note that $p_s(x)$ is defined by (9.1.2)]. Similar reasoning gives us

$$\dot{N}_a(t) = -DN_a(t)/n_a\kappa(x_0) \tag{9.2.19}$$

so that the mean first passage time is given by

$$\tau_a = n_a D^{-1} \int_a^{x_0} dx\, p_s(x)^{-1}\,. \tag{9.2.20}$$

This result is essentially that obtained in (5.2.166) by more rigorous reasoning.

d) Dependence of Relaxation Time on Peak Populations

Equation (9.2.18) looks at first glance like a simple formula relating the relaxation time to n_a and $n_c = 1 - n_a$. One might think that all other factors were independent of n_a and $\tau_r \propto n_a(1 - n_a)$. However, a more careful evaluation shows this is not so. If we use the asymptotic evaluation (9.1.13) we find

$$\tau_r = \frac{n_a n_c}{\mathcal{N} D}\sqrt{\frac{2\pi D}{|U''(b)|}}\exp[U(b)/D] \tag{9.2.21}$$

and similarly, $\mathcal{N}$ can be evaluated asymptotically by taking the contribution from each peak. We obtain

$$\mathcal{N}^{-1} = \sqrt{2\pi D}\{[U''(a)]^{-1/2}\exp[-U(a)/D] + [U''(c)]^{-1/2}\exp[-U(c)/D]\} \tag{9.2.22}$$

and similarly, by definition (9.2.4) of n_a and n_c,

$$n_a/n_c = \sqrt{U''(c)/U''(a)}\exp\left[\frac{U(c) - U(a)}{D}\right]. \tag{9.2.23}$$

After a certain amount of algebra, one can rewrite (9.2.21) as

$$\tau_r = 2\pi\, H(b; a, c)[n_a n_c]^{1/2} \tag{9.2.24}$$

with $H(b; a, c)$ a function which depends on the height of $U(b)$ compared to the average of $U(a)$ and $U(c)$: explicitly,

$$H(b; a, c) = [|U''(b)|^{-1/2}U''(a)^{-1/4}U''(c)^{-1/4}]\exp\left[\frac{2U(b) - U(a) - U(c)}{2D}\right]. \tag{9.2.25}$$

9.2.2 Example: Reversible Denaturation of Chymotrypsinogen

Chymotrypsinogen is a protein which can be transformed into a denatured form by applying elevated pressures of up to several thousand atmospheres, as demonstrated by *Hawley* [9.5]. Presumably the molecule collapses suddenly if sufficient pressure is imposed.

A somewhat unrealistic, but simple, model of this process is given by the equation.

$$dx = \frac{-U'(x)}{\gamma}dt + \sqrt{\frac{2kT}{\gamma}}\,dW(t), \tag{9.2.26}$$

where x is the volume of a molecule, $U(x)$ is the Gibbs Free energy per molecule and kT/γ takes the place of D. Here γ is a friction constant, which would arise by an adiabatic elimination procedure like that used to derive the Smoluchowski equation in Sect. 6.4. The stationary distribution is then

$$p_s(x) = \mathcal{N}\exp[-U(x)/kT] \tag{9.2.27}$$

as required by statistical mechanics.

The explicit effect of the variation of pressure is included by writing

$$U(x) = U_0(x) + x\,\delta p \tag{9.2.28}$$

where δp is the pressure difference from the state in which n_a and n_c are equal. The term $x\delta p$ changes the relative stability of the two minima and is equivalent to the work done against the pressure δp. From (9.2.23) this requires $U_0(x)$ to satisfy

$$\sqrt{U_0''(a)}\exp[U_0(a)/kT] = \sqrt{U_0''(c)}\exp[U_0(c)/kT]. \tag{9.2.29}$$

The maxima and minima of $U(x)$ are slightly different from those of $U_0(x)$. If we assume that higher derivatives of $U_0(x)$ are negligible, then the maxima and minima of $U(x)$ are at points where $U'(x) = 0$ and are given by $a + \delta a$, $b + \delta b$, $c + \delta c$, where

$$\left.\begin{aligned}\delta a &= -\delta p/U_0''(a) = \beta_a\delta p\\ \delta b &= \delta p/|U_0''(b)| = \beta_b\delta p\\ \delta c &= -\delta p/U_0''(c) = \beta_c\delta p.\end{aligned}\right\} \tag{9.2.30}$$

We thus identify β_a and β_c as the *compressibilities* $\partial x/\partial p$ of the states a and c, which are negative, as required by stability. The quantity β_b is some kind of *incompressibility* of the transition state. Since this is unstable, β_b is positive.

The values of $U(a)$, $U(b)$, $U(c)$ of these minima are

$$\left.\begin{aligned}U(a + \delta a) &= U_0(a + \delta a) + (a + \delta a)\delta p\\ &= U_0(a) + a\delta p + \tfrac{1}{2}\beta_a(\delta p)^2\\ U(b + \delta b) &= U_0(b) + b\delta p + \tfrac{1}{2}\beta_b(\delta p)^2\\ U(c + \delta c) &= U_0(c) + c\delta p + \tfrac{1}{2}\beta_c(\delta p)^2\end{aligned}\right\} \tag{9.2.31}$$

and from (9.2.23) we obtain

$$\boxed{\frac{n_a}{n_c} = \exp\left[\frac{-(a - c)}{kT}\delta p - \frac{(\beta_a - \beta_c)}{2kT}(\delta p)^2\right].} \tag{9.3.32}$$

This formula is exactly that obtained by thermodynamic reasoning and fits the data well.

The relaxation time τ_r to the stationary distribution has also been measured. We compute it using (9.2.24, 25). We find that

$$\boxed{\begin{aligned}\tau_r(0) &= \gamma^{-1}\pi\beta_b^{1/2}\exp\left[\frac{U(b)}{kT}\right]\\ \tau_r(\delta p) &= (n_a n_c)^{1/2}\tau_r(0)\exp\left[-\frac{a+c-2b}{2kT}\delta p + \frac{\delta p^2}{2kT}(\beta_b - \tfrac{1}{2}\beta_a - \tfrac{1}{2}\beta_c)\right].\end{aligned}} \tag{9.2.33}$$

Notice that, in principle, a and c, the volumes of the two states and β_a and β_c, their compressibilities, are all measurable directly. The transition state data b, $U(b)$ and β_b are left as free parameters to be determined from lifetime measurements. Of course, the quadratic terms will only be valid for sufficiently small δp and for applications, it may be necessary to use a more sophisticated method.

In Fig. 9.2, $\tau_r(\delta p)$ and n_a/n_c are plotted for a set of possible values of parameters, as computed from (9.2.33).

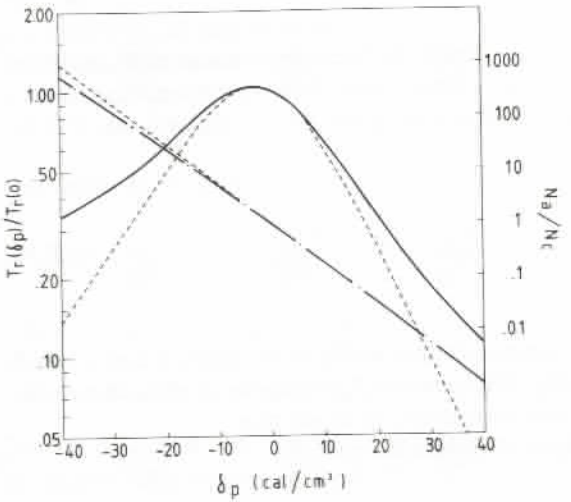

Fig. 9.2. Relaxation rate $\tau_r(\delta p)$ and ratio of concentrations of natural and denatured forms of chymotrypsinogen according to (9.2.33). Dashed line omits compressibility corrections

Notice that the equilibration time reaches a maximum near the point at which natural and denatured forms are in equal concentration. Some skewing, however, can be induced by making the potential asymmetric. Hawley, in fact, notes that measurements in this region are limited by "instrumental stability and the patience of the investigator."

Finally, the curves with zero compressibility are given for comparison. The difference is so large at the wings that it is clear that the quadratic correction method is not valid for the $\tau_r(\delta p)$ curve. However, the corrections almost cancel for the n_a/n_c curve. A realistic treatment in which more variables are included preserves the qualitative nature of this description, but permits as well the possibility $\beta_b < 0$, which is not possible here, as is shown in [9.9].

9.2.3 Bistability with Birth-Death Master Equations (One Variable)

The qualitative behaviour of bistable systems governed by one-step birth-death Master equations is almost the same as that for Fokker-Planck equations.

Consider a one step process with transition probabilities $t^+(x)$, $t^-(x)$ so that the Master equation can be written as

$$\frac{dP(x)}{dt} = J(x+1, t) - J(x, t) \tag{9.2.34}$$

with

$$J(x, t) = t^-(x)P(x, t) - t^+(x-1)P(x-1, t)\,. \tag{9.2.35}$$

Suppose now that the stationary distribution has maxima at a, c and a minimum at b, and in a similar way to that in Sect. 9.2.1, define

$$M(x, t) = \sum_{z=0}^{x-1} P(z, t) \tag{9.2.36}$$

$$N_a(t) = 1 - N_c(t) = M(b, t) \tag{9.2.37}$$

and if x_0 is a point near b,

$$N_0(t) = P(x_0, t)\,. \tag{9.2.38}$$

The corresponding stationary quantities are

$$n_a = 1 - n_c = \sum_{z=0}^{b-1} P_s(z) \tag{9.2.39}$$

$$n_0 = P_s(x_0)\,. \tag{9.2.40}$$

We now sum (9.2.34) from 0 to $x-1$ to obtain

$$\frac{dM(x, t)}{dt} = J(x, t) \tag{9.2.41}$$

[since $J(0, t) = 0$]. We now use the fact that the stationary solution $P_s(x)$ is in a one-step process obtained from the detailed balance equation

$$t^-(x)P_s(x) = t^+(x-1)P_s(x-1) \tag{9.2.42}$$

to introduce an "integrating factor" for (9.2.41). Namely, define

$$\beta(x, t) = P(x, t)/P_s(x) \tag{9.2.43}$$

then (9.2.41) can be written

$$\frac{dM(x, t)}{dt} = P_s(x)t^-(x)[\beta(x, t) - \beta(x-1, t)] \tag{9.2.44}$$

so that

$$\begin{aligned}\frac{d}{dt}\sum_{z=a+1}^{x_0}\left[\frac{M(z, t)}{P_s(z)t^-(z)}\right] &= \beta(x_0, t) - \beta(a, t) \\ &= \frac{P(x_0, t)}{P_s(x_0)} - \frac{P(a, t)}{P_s(a)}\,.\end{aligned} \tag{9.2.45}$$

Equation (9.2.45) is now in almost precisely the form of (9.2.6) for the corresponding Fokker-Planck process. It depends on the solution being obtained via a detailed balance method (9.2.42).

We make the same assumptions as Kramers; namely, we assume that only relaxation between peaks is now relevant and write

$$\begin{aligned} P(x,t) &= P_s(x)N_a(t)/n_a \qquad x < b \\ &= P_s(x)N_c(t)/n_c \qquad x > b \end{aligned} \tag{9.2.46}$$

and obtain relaxation equations exactly like those in (9.2.8)

$$\begin{aligned} \kappa(x_0)\dot{N}_a(t) &= N_0(t)/n_0 - N_a(t)/n_a \\ \mu(x_0)\dot{N}_c(t) &= N_0(t)/n_0 - N_c(t)/n_c \,. \end{aligned} \tag{9.2.47}$$

where

$$\begin{aligned} \kappa(x_0) &= \sum_{z=a+1}^{x_0} [P_s(z)t^-(z)]^{-1}[1-\psi(z)] \\ \mu(x_0) &= \sum_{z=x_0+1}^{c} [P_s(z)t^-(z)]^{-1}[1-\psi(z)] \end{aligned} \tag{9.2.48}$$

with

$$\begin{aligned} \psi(z) &= n_a^{-1} \sum_{y=z}^{b} P_s(y) \qquad z < b \\ &= n_c^{-1} \sum_{y=b+1}^{z+1} P_s(y) \qquad z > b \,. \end{aligned} \tag{9.2.49}$$

The only significant difference is that D appears on the right of (9.2.8) but is here replaced by a factor $t^-(z)^{-1}$ in the definitions of $\kappa(x_0)$ and $\mu(x_0)$.

All the same approximations can be made, the only difficulty being a precise reformulation of the $D \to 0$ limit, which must here correspond to a large number limit, in which all functions change smoothly as x changes by ± 1. This is just the limit of the system size expansion, in which a Fokker-Planck description can be used anyway.

We shall not go into details, but merely mention that exact mean exit times are obtainable by the method of Sect. 7.4. By adapting the methods of Sect. 5.2.8 to this system one finds the splitting probabilities that the system initially at x_0, reaches points a, c are

$$\begin{aligned} \pi_a &= \{\sum_{z=x_0+1}^{c} [P_s(z)t^-(z)]^{-1}\} / \{\sum_{z=a+1}^{c} [P_s(z)t^-(z)]^{-1}\} \\ \pi_c &= \{\sum_{z=a+1}^{x_0} [P_s(z)t^-(z)]^{-1}\} / \{\sum_{z=a+1}^{c} [P_s(z)t^-(z)]^{-1}\} \,. \end{aligned} \tag{9.2.50}$$

Thus, for all practical considerations one might just as well model by means of a Fokker-Planck description. It is rare that one knows exactly what the underlying mechanisms are, so that any equation written down can be no more than an educated guess, for which purpose the simplest is the most appropriate.

9.3 Bistability in Multivariable Systems

There is a wide variety of possibilities when one deals with multivariable systems. If the system is described by a Master equation, the possible variety of kinds of transition and state space is so bewilderingly rich, that one can hardly imagine where to start. However, since we saw in Sect. 9.2.4 that a Master equation description is not very different from a Fokker-Planck description, it seems reasonable to restrict oneself to these, which, it turns out, are already quite sufficiently complicated.

The heuristic treatments of these problems, as developed mainly by the physicists *Langer*, *Landauer* and *Swanson* [9.6] are now in the process of being made rigorous by mathematical treatments by *Schuss* and *Matkowsky* [9.7] and others. The first rigorous treatment was by *Ventsel* and *Freidlin* [9.8] which, however, does not seem to have attracted much attention by applied workers since the rigour is used only to confirm estimates that have long been guessed, rather than to give precise asymptotic expansions, as do the more recent treatments.

We will consider here systems described in a space of l dimensions by a Fokker-Planck equation which is conveniently written in the form

$$\partial_t p = \nabla \cdot [-\boldsymbol{v}(\boldsymbol{x})p + \varepsilon \underline{\boldsymbol{D}}(\boldsymbol{x}) \cdot \nabla p] \tag{9.3.1}$$

whose stationary solution is called $p_s(\boldsymbol{x})$ and which is assumed to be known in much of what follows. It can, of course, be estimated asymptotically in the small ε limit by the method of Sect. 6.3.3.

9.3.1 Distribution of Exit Points

We will treat here only a simplified case of (9.3.1) in which $\underline{\boldsymbol{D}}(\boldsymbol{x})$ is the identity:

$$\underline{\boldsymbol{D}}(\boldsymbol{x}) = \mathbf{1} \,. \tag{9.3.2}$$

This does conceal features which can arise from strongly varying D but mostly, the results are not greatly changed.

We suppose that the system is confined to a region R with boundary $\boldsymbol{S}$, and that the velocity field $\boldsymbol{v}(\boldsymbol{x})$ points inwards to a stationary point $\boldsymbol{a}$. The problem is the asymptotic estimate of the distribution of points $\boldsymbol{b}$ on $\boldsymbol{S}$ at which the point escapes from R. We use (5.4.49) for $\pi(\boldsymbol{b}, \boldsymbol{x})$ (the distribution of escape points on $\boldsymbol{S}$, starting from the point $\boldsymbol{x}$), which in this case takes the form

$$\boldsymbol{v}(\boldsymbol{x}) \cdot \nabla \pi(\boldsymbol{b}, \boldsymbol{x}) + \varepsilon \nabla^2 \pi(\boldsymbol{b}, \boldsymbol{x}) = 0 \tag{9.3.3}$$

with boundary condition

$$\pi(\boldsymbol{b}, \boldsymbol{u}) = \delta_s(\boldsymbol{b} - \boldsymbol{u}) \qquad (\boldsymbol{u} \in \boldsymbol{S}) . \tag{9.3.4}$$

An asymptotic solution, valid for $\varepsilon \to 0$, is constructed, following the method of *Matkowsky and Schuss* [9.7].

a) Solution Near $\boldsymbol{x} = \boldsymbol{u}$ and in the Interior of $\boldsymbol{R}$

Firstly one constructs a solution valid inside R. For $\varepsilon = 0$ we have

$$\boldsymbol{v}(\boldsymbol{x}) \cdot \nabla \pi(\boldsymbol{b}, \boldsymbol{x}) = 0 \tag{9.3.5}$$

which implies that $\pi(\boldsymbol{b}, \boldsymbol{x})$ is constant along the flow lines of $\boldsymbol{v}(\boldsymbol{x})$, since it simply states that the derivative of $\pi(\boldsymbol{b}, \boldsymbol{x})$ along these lines is zero. Since we assume all the flow lines pass through $\boldsymbol{a}$, we have

$$\pi(\boldsymbol{b}, \boldsymbol{x}) = \pi(\boldsymbol{b}, \boldsymbol{a}) \tag{9.3.6}$$

for any $\boldsymbol{x}$ inside. However, the argument is flawed by the fact that $\boldsymbol{v}(\boldsymbol{a}) = 0$ and hence (9.3.5) is no longer an appropriate approximation.

We consider, therefore, the solution of (9.3.3) within a distance of order $\sqrt{\varepsilon}$ of the origin. To assist in this, we introduce new coordinates (z, y_r) which are chosen so that z measures the distance from $\boldsymbol{a}$, while the y_r are a set of $l - 1$ tangential variables measuring the orientation around $\boldsymbol{a}$.

More precisely, choose $z(\boldsymbol{x})$ and $y_r(\boldsymbol{x})$ so that

$$\begin{aligned} &\boldsymbol{v}(\boldsymbol{x}) \cdot \nabla z(\boldsymbol{x}) = -z(\boldsymbol{x}) \\ &\boldsymbol{v}(\boldsymbol{x}) \cdot \nabla y_r(\boldsymbol{x}) = 0 \\ &z(\boldsymbol{a}) = 0 . \end{aligned} \tag{9.3.7}$$

The negative sign in the first of these equations takes account of the fact that $\boldsymbol{a}$ is assumed stable, so that $\boldsymbol{v}(\boldsymbol{x})$ points towards $\boldsymbol{a}$. Thus, $z(\boldsymbol{x})$ *increases* as $\boldsymbol{x}$ travels further from $\boldsymbol{a}$.

Thus, we find, for any function f,

$$\nabla f = \nabla z(\boldsymbol{x}) \frac{\partial f}{\partial z} + \sum_r \nabla y_r(\boldsymbol{x}) \frac{\partial f}{\partial y_r} \tag{9.3.8}$$

and hence,

$$\boldsymbol{v}(\boldsymbol{x}) \cdot \nabla \pi = -z \frac{\partial \pi}{\partial z} \tag{9.3.9}$$

and

$$\begin{aligned} \nabla^2 \pi = {} & \nabla z(\boldsymbol{x}) \cdot \nabla z(\boldsymbol{x}) \frac{\partial^2 \pi}{\partial z^2} + 2 \sum_r \nabla z(\boldsymbol{x}) \cdot \nabla y_r(\boldsymbol{x}) \frac{\partial^2 \pi}{\partial z \, \partial y_r} \\ & + \sum_{r,s} \nabla y_r(\boldsymbol{x}) \cdot \nabla y_s(\boldsymbol{x}) \frac{\partial^2 \pi}{\partial y_r \partial y_s} + \nabla^2 z(\boldsymbol{x}) \frac{\partial \pi}{\partial z} + \sum_r \nabla^2 y_r(\boldsymbol{x}) \frac{\partial \pi}{\partial y_r} . \end{aligned} \tag{9.3.10}$$

We now evaluate π asymptotically by changing to the scaled (or stretched) variable ξ defined by

$$z = \xi \sqrt{\varepsilon} . \tag{9.3.11}$$

Substituting (9.3.8–11) into (9.3.3) we find that, to lowest order in ε,

$$-\xi \frac{\partial \pi}{\partial \xi} + H \frac{\partial^2 \pi}{\partial \xi^2} = 0$$

where $\tag{9.3.12}$

$$H = \nabla z(\boldsymbol{a}) \cdot \nabla z(\boldsymbol{a}) .$$

We can now solve this equation getting

$$\pi(\boldsymbol{b}, \boldsymbol{x}) = C_1 \int_0^{z/\sqrt{\varepsilon}} d\xi \exp(\xi^2/2H) + \pi(\boldsymbol{b}, \boldsymbol{a}) . \tag{9.3.13}$$

Because H is positive, we can only match this solution for π with the constancy of $\pi(\boldsymbol{b}, \boldsymbol{x})$ along flow lines for $\boldsymbol{x} \neq \boldsymbol{a}$ if $C_1 = 0$. Hence, for all $\boldsymbol{x}$ on the interior of R,

$$\pi(\boldsymbol{b}, \boldsymbol{x}) = \pi(\boldsymbol{b}, \boldsymbol{a}) . \tag{9.3.14}$$

[Notice that if $\boldsymbol{v}(\boldsymbol{x})$ points the other way, i.e., *is unstable*, we omit the negative sign in (9.3.9) and find that $\pi(\boldsymbol{b}, \boldsymbol{x})$ is given by (9.3.3) with $H \to -H$, and hence in a distance of order $\sqrt{\varepsilon}$ of $\boldsymbol{a}$, $\pi(\boldsymbol{b}, \boldsymbol{x})$ changes its value].

b) Solution Near the Boundary $\boldsymbol{S}$

We consider, with an eye to later applications, the solution of the slightly more general equation

$$\boldsymbol{v}(\boldsymbol{x}) \cdot \nabla f(\boldsymbol{x}) + \varepsilon \nabla^2 f(\boldsymbol{x}) = 0$$

with $\tag{9.3.15}$

$$f(\boldsymbol{u}) = g(\boldsymbol{u}) \qquad (\boldsymbol{u} \in \boldsymbol{S})$$

of which the boundary value problem (9.3.4) is a particular case. Two situations arise.

i) $\boldsymbol{\nu} \cdot \boldsymbol{v}(\boldsymbol{x}) \neq 0$ *on* $\boldsymbol{S}$ *or anywhere in* R *except* $\boldsymbol{x} = \boldsymbol{a}$ *which is stable:* clearly, in any asymptotic method, the boundary condition (9.3.15) is not compatible with a constant solution. Hence, there must be rapid changes at the boundary.

Near a point $\boldsymbol{u}$ on $\boldsymbol{S}$ we can write

$$\boldsymbol{v}(\boldsymbol{u}) \cdot \nabla f(\boldsymbol{x}) + \varepsilon \nabla^2 f(\boldsymbol{x}) = 0 . \tag{9.3.16}$$

Near $\boldsymbol{S}$ it is most convenient to introduce $\boldsymbol{\nu}(\boldsymbol{u})$, the normal (pointing out) at $\boldsymbol{u}$ to $\boldsymbol{S}$, and to define a variable ρ by

$$\boldsymbol{x} = \boldsymbol{u} - \varepsilon\rho\boldsymbol{v}(\boldsymbol{u}) \tag{9.3.17}$$

and other variables y_r parallel to $\boldsymbol{S}$.

Then to lowest order in ε, (9.3.16) reduces to (at a point near $\boldsymbol{u}$ on $\boldsymbol{S}$)

$$[\boldsymbol{v}\cdot\boldsymbol{v}(\boldsymbol{u})]\frac{\partial f}{\partial \rho} + H(\boldsymbol{u})\frac{\partial^2 f}{\partial \rho^2} = 0 \tag{9.3.18}$$

with $H(\boldsymbol{u}) = \boldsymbol{v}^2 = 1$.

The solution is then

$$f(\boldsymbol{x}) = g(\boldsymbol{u}) + C_1(\boldsymbol{u})\{1 - \exp[-\boldsymbol{v}\cdot\boldsymbol{v}(\boldsymbol{u})\rho]\} . \tag{9.3.19}$$

As $\rho \to \infty$, we approach a finite distance into the interior of R and thus

$$f(\boldsymbol{x}) \to g(\boldsymbol{u}) + C_1(\boldsymbol{u}) = C_0 \tag{9.3.20}$$

from the analysis in (a), so

$$C_1(\boldsymbol{u}) = C_0 - g(\boldsymbol{u}) . \tag{9.3.22}$$

One must now fix C_0, which is the principal quantity actually sought.

This can be done by means of Green's theorem. For let $p_s(\boldsymbol{x})$ be the usual stationary solution of the forward Fokker-Planck equation. We know it can be written

$$p_s(\boldsymbol{x}) = \exp\left|-\frac{1}{\varepsilon}[\phi(\boldsymbol{x}) + O(\varepsilon)\right| \tag{9.3.23}$$

as has been shown in Sect. 6.3.3. We take (9.3.16), multiply by $p_s(\boldsymbol{x})$ and integrate over R. Using the fact that $p_s(\boldsymbol{x})$ satisfies the forward Fokker-Planck equation, this can be reduced to a surface integral

$$0 = \int_R d\boldsymbol{x}\, p_s(\boldsymbol{x})[\boldsymbol{v}(\boldsymbol{x})\cdot\nabla f(\boldsymbol{x}) + \varepsilon\nabla^2 f(\boldsymbol{x})] \tag{9.3.24}$$

$$= \int_S dS\, \{p_s(\boldsymbol{x})\boldsymbol{v}\cdot\boldsymbol{v}(x)f(\boldsymbol{x}) + \varepsilon[p_s(\boldsymbol{x})\boldsymbol{v}\cdot\nabla f(\boldsymbol{x}) - f(\boldsymbol{x})\boldsymbol{v}\cdot\nabla p_s(\boldsymbol{x})]\} . \tag{9.3.25}$$

Noting that, to lowest order in ε.

$$\boldsymbol{v}\cdot\nabla f(\boldsymbol{x}) = -\frac{1}{\varepsilon}\frac{\partial f}{\partial \rho} = -\boldsymbol{v}\cdot\boldsymbol{v}(x)[C_0 - g(\boldsymbol{x})] \tag{9.3.26}$$

and

$$\boldsymbol{v}\cdot\nabla p_s(\boldsymbol{x}) = -\frac{1}{\varepsilon}\boldsymbol{v}\cdot\nabla\phi(\boldsymbol{x})\exp[-\phi(\boldsymbol{x})/\varepsilon] , \tag{9.3.27}$$

we deduce that

$$\boxed{C_0 = \frac{\int_S dS\, e^{-\phi(\boldsymbol{x})/\varepsilon}[2\boldsymbol{v}\cdot\boldsymbol{v}(\boldsymbol{x}) + \boldsymbol{v}\cdot\nabla\phi(\boldsymbol{x})]g(\boldsymbol{x})}{\int_S dS\, e^{-\phi(\boldsymbol{x})/\varepsilon}\boldsymbol{v}\cdot\boldsymbol{v}(\boldsymbol{x})} .} \tag{9.3.28}$$

Recalling that for this problem, $g(\boldsymbol{u}) = \delta_s(\boldsymbol{u} - \boldsymbol{b})$, we find that, if $\boldsymbol{x}$ is well in the interior of R,

$$\boxed{\pi(\boldsymbol{x}, \boldsymbol{b}) = C_0 = \frac{e^{-\phi(\boldsymbol{b})/\varepsilon}[2\boldsymbol{v}\cdot\boldsymbol{v}(\boldsymbol{b}) + \boldsymbol{v}\cdot\nabla\phi(\boldsymbol{b})]}{\int_S dS\, e^{-\phi(\boldsymbol{x})/\varepsilon}\boldsymbol{v}\cdot\boldsymbol{v}(\boldsymbol{x})} .} \tag{9.3.29}$$

We see here that the exit distribution is essentially $\exp[-\phi(\boldsymbol{b})/\varepsilon]$, i.e., approximately the stationary distribution. If the Fokker-Planck equation has a potential solution, then

$$\boldsymbol{v}(\boldsymbol{b}) = -\nabla\phi(\boldsymbol{b}) \tag{9.3.30}$$

and

$$\pi(\boldsymbol{x}, \boldsymbol{b}) = e^{-\phi(\boldsymbol{b})/\varepsilon}\boldsymbol{v}\cdot\boldsymbol{v}(\boldsymbol{b})/[\int_S dS\, e^{-\phi(\boldsymbol{x})/\varepsilon}\boldsymbol{v}\cdot\boldsymbol{v}(\boldsymbol{x})]$$

and we simply have a kind of average flow result.

ii) $\boldsymbol{v}\cdot\boldsymbol{v}(\boldsymbol{x}) = 0$ *on* $\boldsymbol{S}$: this problem is more directly related to bistability, since midway between two stable points $\boldsymbol{a}$ and $\boldsymbol{c}$, a curve $\boldsymbol{v}\cdot\boldsymbol{v}(\boldsymbol{x}) = 0$ which separates the two regions and is known as *a separatrix* is expected.

The method is much the same except that near $\boldsymbol{u}$ on $\boldsymbol{S}$, we expect

$$\boldsymbol{v}\cdot\boldsymbol{v}(\boldsymbol{x}) \sim \boldsymbol{v}\cdot(\boldsymbol{x} - \boldsymbol{u})\kappa(\boldsymbol{u}) . \tag{9.3.31}$$

where $\kappa(\boldsymbol{u})$ is a coefficient which depends on $\boldsymbol{v}(\boldsymbol{x})$ and is assumed to be nonzero.

The situation is now like that at $\boldsymbol{x} = \boldsymbol{a}$ and it is appropriate to substitute

$$\boldsymbol{x} = \boldsymbol{u} - \sqrt{\varepsilon}\,\rho\boldsymbol{v}(\boldsymbol{u}) \tag{9.3.32}$$

and to lowest order in ε (9.3.16) reduces to (at a point near $\boldsymbol{u}$ on $\boldsymbol{S}$)

$$\kappa(\boldsymbol{u})\,\rho\frac{\partial f}{\partial \rho} + \frac{\partial^2 f}{\partial \rho^2} = 0 \tag{9.3.33}$$

so that

$$f(\boldsymbol{x}) = g(\boldsymbol{u}) + C_1\int_0^\rho d\rho\, \exp[-\tfrac{1}{2}\kappa(\boldsymbol{u})\rho^2] \tag{9.3.34}$$

and letting $\rho \to \infty$, we find

$$C_1 = [C_0 - g(\boldsymbol{u})]\sqrt{\frac{2\kappa(\boldsymbol{u})}{\pi}}. \tag{9.3.35}$$

The result for $\pi(\boldsymbol{x}, \boldsymbol{b})$ is now completed as before: one gets

$$\boxed{\pi(\boldsymbol{x}, \boldsymbol{b}) = \frac{e^{-\phi(\boldsymbol{b})/\varepsilon}\left[\left(1 + \sqrt{\frac{2\kappa(\boldsymbol{b})}{\pi}}\right)\boldsymbol{v}\cdot\boldsymbol{v}(\boldsymbol{b}) + \boldsymbol{v}\cdot\nabla\phi(\boldsymbol{b})\right]}{\int dS\, e^{-\phi(\boldsymbol{x})/\varepsilon}\sqrt{\frac{2\kappa(\boldsymbol{x})}{\pi}}\,\boldsymbol{v}\cdot\boldsymbol{v}(\boldsymbol{x})}.} \tag{9.3.36}$$

9.3.2 Asymptotic Analysis of Mean Exit Time

From our experience in one-dimensional systems, we expect the mean exit time from a point within R to be of order $\exp(K/\varepsilon)$; for some $K > 0$, as $\varepsilon \to 0$. We therefore define

$$\tau(\boldsymbol{x}) = \exp(-K/\varepsilon)T(\boldsymbol{x}) \tag{9.3.37}$$

where $T(\boldsymbol{x})$ is the mean escape time from R starting at $\boldsymbol{x}$ and $\tau(\boldsymbol{x})$ satisfies (from Sect. 5.4)

$$\begin{aligned} &\boldsymbol{v}(\boldsymbol{x})\cdot\nabla\tau(\boldsymbol{x}) + \varepsilon\nabla^2\tau(\boldsymbol{x}) = -e^{-K/\varepsilon} \\ &\tau(\boldsymbol{u}) = 0 \qquad \boldsymbol{u} \in \boldsymbol{S}. \end{aligned} \tag{9.3.38}$$

If this scaling is correct, then any expansion of $\tau(\boldsymbol{x})$ in powers of ε will not see the exponential, so the equation to lowest order in ε will be essentially (9.3.16).

As in that case, we show that $\tau(\boldsymbol{x})$ is essentially constant in the interior of R and can be written as [in the case $\boldsymbol{v}\cdot\boldsymbol{v}(\boldsymbol{x}) \neq 0$ on $\boldsymbol{S}$]

$$\tau(\boldsymbol{x}) \sim C_0[1 - \exp[-\boldsymbol{v}\cdot\boldsymbol{v}(\boldsymbol{u})\rho]] \tag{9.3.39}$$

near $\boldsymbol{S}$.

We multiply (9.3.38) by $p_s(\boldsymbol{x}) = \exp[-\phi(\boldsymbol{x})/\varepsilon]$ and use Green's theorem to obtain [in much the same way as (9.3.25) but with $\tau(\boldsymbol{x}) = 0$ on $\boldsymbol{S}$]

$$-e^{-K/\varepsilon}\int_R d\boldsymbol{x}\, e^{-\phi(\boldsymbol{x})/\varepsilon} = -\int_S dS\, e^{-\phi(\boldsymbol{x})/\varepsilon}[C_0\boldsymbol{v}\cdot\boldsymbol{v}(\boldsymbol{x})], \tag{9.3.40}$$

i.e.,

$$C_0 = \int_R d\boldsymbol{x}\, e^{-[K+\phi(\boldsymbol{x})]/\varepsilon}/\int_S dS\, e^{-\phi(\boldsymbol{x})/\varepsilon}\,\boldsymbol{v}\cdot\boldsymbol{v}(\boldsymbol{x}). \tag{9.3.41}$$

By hypothesis, C_0 does not change exponentially like $\exp(A/\varepsilon)$. In the numerator of (9.3.41) the main contribution comes from the minimum of $\phi(x)$ which occurs at the point $\boldsymbol{a}$, whereas in the denominator, it occurs at the point on $\boldsymbol{S}$ where $\phi(\boldsymbol{x})$ is a minimum, which we shall call $\boldsymbol{x}_0$. Thus, the ratio behaves like

$$\exp\{[\phi(\boldsymbol{a}) - \phi(\boldsymbol{x}_0) - K]/\varepsilon\}.$$

and hence for C_0 to be asymptotically constant,

$$K = \phi(\boldsymbol{a}) - \phi(\boldsymbol{x}_0) \tag{9.3.42}$$

and, for $\boldsymbol{x}$ well into the interior of R, we have

$$\boxed{\tau(\boldsymbol{x}) = \int_R d\boldsymbol{x}\, e^{\phi(\boldsymbol{x})/\varepsilon}/\int_S dS\,[e^{-\phi(\boldsymbol{x})/\varepsilon}\,\boldsymbol{v}\cdot\boldsymbol{v}(\boldsymbol{x})].} \tag{9.3.43}$$

In the case where $\boldsymbol{v}\cdot\boldsymbol{v}(\boldsymbol{x}) = 0$ on all of S, we now have

$$\tau(\boldsymbol{x}) \sim C_0\int_0^{\rho} d\rho\, \exp[-\tfrac{1}{2}\kappa(\boldsymbol{u})\rho^2] \tag{9.3.44}$$

and hence in the interior,

$$\tau(\boldsymbol{x}) \sim C_0\sqrt{\frac{\pi}{2\kappa(\boldsymbol{u})}}. \tag{9.3.45}$$

The analysis proceeds similarly and we find, for $\boldsymbol{x}$ well in the interior of R,

$$\boxed{\tau(\boldsymbol{x}) \sim \sqrt{\frac{\pi}{2\kappa(\boldsymbol{u})}}\int_R d\boldsymbol{x}\, e^{\phi(\boldsymbol{x})/\varepsilon}/\int_S dS\, e^{-\phi(\boldsymbol{x})/\varepsilon}.} \tag{9.3.46}$$

9.3.3 Kramers' Method in Several Dimensions

The generalisation of Kramers' method is relatively straightforward. We consider a completely general Fokker-Planck equation in l dimensions [we use $P(\boldsymbol{x})$ for the probability density for notational ease]

$$\partial_t P = \nabla\cdot[-\boldsymbol{v}(\boldsymbol{x})P + \varepsilon\underline{\boldsymbol{D}}(\boldsymbol{x})\cdot\nabla P] \tag{9.3.47}$$

whose stationary solution is to be called $P_s(\boldsymbol{x})$ and can only be exhibited explicitly if (9.3.47) satisfies potential conditions. We assume that $P_s(\boldsymbol{x})$ has two well-defined maxima at $\boldsymbol{a}$ and $\boldsymbol{c}$ and well-defined saddlepoint at $\boldsymbol{b}$ (Fig. 9.3). We assume that the value at the saddlepoint is very much smaller than the values at $\boldsymbol{a}$ and $\boldsymbol{c}$. We introduce a family of $(l-1)$ dimensional planes $\boldsymbol{S}(w)$, where w is a parameter which labels the planes. We choose $\boldsymbol{S}(a)$ to pass through $\boldsymbol{a}$, $\boldsymbol{S}(b)$ through $\boldsymbol{b}$ and $\boldsymbol{S}(c)$ through $\boldsymbol{c}$. The planes $\boldsymbol{S}(w)$ are assumed to be oriented in such a way that $P_s(\boldsymbol{x})$ has a unique maximum when restricted to any one of them. We define, similarly to Sect. 9.2.1

$$M[\boldsymbol{S}(w)] = \int_{L(w)} dx\, P(\boldsymbol{x}), \tag{9.3.48}$$

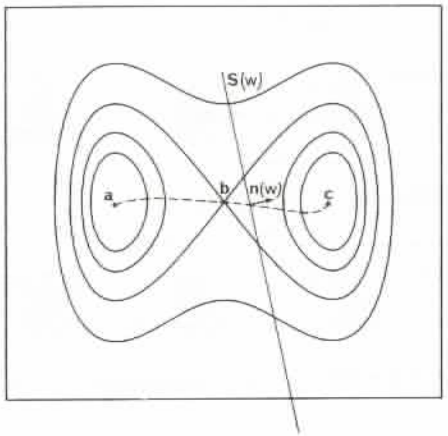

Fig. 9.3. Contours of the stationary distribution function $P_s(x)$. The plane $S(w)$ is oriented so that $P_s(x)$ has a unique maximum there, and the curve $x = u(w)$ (dashed line) is the locus of these maxima

where $L(w)$ is the region of space to the left of the plane $S(w)$; then

$$\dot{M}[S(w)] = \int_{S(w)} dS \cdot [-v(x)P + \varepsilon \underline{D}(x) \cdot \nabla P] . \tag{9.3.49}$$

Current in stationary state is defined by

$$J_s = -v(x)P_s + \varepsilon \underline{D}(x) \cdot \nabla P_s . \tag{9.3.50}$$

Assumption I: we exclude cases in which finite currents J_s occur where P_s is very small. Because of $\nabla \cdot J_s = 0$, we can write

$$J_s = -\varepsilon \nabla \cdot (\underline{A} P_s) \tag{9.3.51}$$

where $\underline{A}$ is an antisymmetric tensor. *We require that $\underline{A}$ be of the some order of magnitude as $\underline{D}(x)$, or smaller.*

Relaxation equations are derived in two stages. Define a quantity $\beta(x)$ by

$$\begin{aligned} \beta(x) = P(x, t)/P_s(x) &\simeq N_a(t)/n_a \qquad (x \text{ near } a) \\ &\simeq N_c(t)/n_c \qquad (x \text{ near } c) . \end{aligned} \tag{9.3.52}$$

This is the assumption that all relaxation within peaks has ceased. Substitute now in (9.3.49), integrate by parts discarding terms at infinity and obtain

$$\dot{M}[S(w)] = \varepsilon \int_{S(w)} dS \cdot [\mathscr{D}(x) \cdot \nabla \beta] \, P_s(x) \tag{9.3.53}$$

with

$$\mathscr{D}(x) = \underline{D}(x) + \underline{A}(x) . \tag{9.3.54}$$

Assumption II: $P_s(x)$ is sharply singly peaked on $S(w)$ so we may make the approximate evaluation

$$\dot{M}[S(w)] = \{\varepsilon[n(w) \cdot \mathscr{D}(x) \cdot \nabla \beta]_{u(w)} + \delta(w)\} \left| \int_{S(w)} dS \, P_s(x) \right| , \tag{9.3.55}$$

where $\delta(w)$ is expected to be very much smaller than the term in square brackets. Here $u(w)$ is the position at which $P_s(x)$ has its maximum value when restricted to $S(w)$, and $n(w)$ is the normal to $S(w)$.

Assumption III: the direction of $n(w)$ can be chosen so that $\mathscr{D}^{\mathrm{T}}(x) \cdot n(w)$ is parallel to the tangent at w to the curve $x = u(w)$ — without violating the other assumptions. Hence,

$$\mathscr{D}^{\mathrm{T}}[u(w)] \cdot n(w) = d(w) \partial_w u(w) . \tag{9.3.56}$$

Defining now

$$p(w) = \left| \int_{S(w)} dS \, P_s(x) \right| , \tag{9.3.57}$$

which is (up to a slowly varying factor) the probability density for the particle to be on the plane $S(w)$ and is expected to have a two-peaked shape with maxima at $w = a$ and $w = c$ and a minimum at $w = b$.

Assumption IV: these are assumed to be sharp maxima and minima. Neglecting $\delta(w)$, making the choice (9.3.56) and noting

$$\partial_w u(w) \cdot \nabla \beta[u(w)] = \partial_w \beta[u(w)] , \tag{9.3.58}$$

we find

$$\frac{1}{\varepsilon} \int_a^{w_0} dw \, \{\dot{M}[S(w)]/[p(w)d(w)]\} = \beta(w_0) - \beta(a) . \tag{9.3.59}$$

Using the sharp peaked nature of $p(w)^{-1}$, (9.3.59) can now be approximated by taking the value at the peak, using (9.3.52) and

$$N(a, t) = M[S(b), t] \tag{9.3.60}$$

as well as defining

$$\kappa(w_0) = \int_a^{w_0} [p(w)]^{-1} dw \tag{9.3.61}$$

$$\mu(w_0) = \int_{w_0}^{b} [p(w)]^{-1} dw , \tag{9.3.62}$$

to obtain the relaxation equations

$$\kappa(w_0) \dot{N}_a(t) = \varepsilon d(w_0)[N_0(t)/n_0 - N_a(t)/n_a] \tag{9.3.63}$$

$$\mu(w_0) \dot{N}_c(t) = \varepsilon d(w_0)[N_0(t)/n_0 - N_c(t)/n_c] . \tag{9.3.64}$$

These are of exactly the same form as those in the one-variable case and all the same interpretations can be made.

9.3.4 Example: Brownian Motion in a Double Potential

We consider Brownian motion in velocity and position as outlined in Sect. 5.3.6. Thus, we consider the Fokker-Planck equation

$$\frac{\partial P(x,p,t)}{\partial t} = -p\frac{\partial P}{\partial x} + U'(x)\frac{\partial P}{\partial p} + \gamma\left[\frac{\partial}{\partial p}pP + \frac{\partial^2 P}{\partial p^2}\right]. \tag{9.3.65}$$

In the notation of the previous section we have

$$\left.\begin{aligned} &\boldsymbol{x} = (x, p)\\ &\boldsymbol{v}(\boldsymbol{x}) = (p, -U'(x) - \gamma p)\\ &\varepsilon = 1\\ &\underline{\boldsymbol{D}}(\boldsymbol{x}) = \begin{bmatrix} 0 & 0\\ 0 & \gamma \end{bmatrix}\\ &P_s(\boldsymbol{x}) = \mathcal{N}_2 \exp\left[-\tfrac{1}{2}p^2 - U(x)\right]\\ &\mathcal{N}_2 = (2\pi)^{-1/2}\mathcal{N}_1\\ &\mathcal{N}_1 = \{\int_{-\infty}^{\infty} dx \exp\left[-U(x)\right]\}^{-1}. \end{aligned}\right\} \tag{9.3.66}$$

Hence, we can write

$$\boldsymbol{v}(\boldsymbol{x}) = \begin{bmatrix} 0 & -1\\ 1 & \gamma \end{bmatrix} \cdot \nabla(\log P_s) \tag{9.3.67}$$

and the current in the stationary state is

$$\boldsymbol{J}_s = -\boldsymbol{v}P_s + \underline{\boldsymbol{D}}\cdot\nabla P_s = -\nabla\cdot\left\{\begin{bmatrix} 0 & -1\\ 1 & 0 \end{bmatrix} P_s\right\} \tag{9.3.68}$$

so that $\underline{\boldsymbol{A}}$ exists, and

$$\underline{\boldsymbol{A}} = \begin{bmatrix} 0 & -1\\ 1 & 0 \end{bmatrix}. \tag{9.3.69}$$

Thus, Assumption I is satisfied.

The plane $\boldsymbol{S}(w)$ can be written in the form

$$\lambda x + p = w. \tag{9.3.70}$$

Assumption II requires us to maximise $P_s(\boldsymbol{x})$ on this plane, i.e., to maximise $-\frac{1}{2}p^2 - U(x)$ on this plane. Using standard methods, we find that maxima must lie along the curve $\boldsymbol{u}(w)$ given by

$$\boldsymbol{u}(w) = \begin{bmatrix} x(w)\\ p(w) \end{bmatrix} = \begin{bmatrix} x(w)\\ w - \lambda x(w) \end{bmatrix} \tag{9.3.71}$$

where $x(w)$ satisfies

$$U'[x(w)] + \lambda^2 x(w) - \lambda w = 0\,; \tag{9.3.72}$$

whether $P_s(X)$ is sharply peaked depends on the nature of $U(x)$.

We now implement Assumption III.

The parameter λ is a function of w on the particular set of planes which satisfy (9.3.56). The tangent to $\boldsymbol{u}(w)$ is parallel to

$$\left[\frac{dx}{dw}, 1 - \lambda\frac{dx}{dw} - x\frac{d\lambda}{dw}\right] \tag{9.3.73}$$

and differentiating (9.3.72) we have

$$\frac{dx}{dw} = (U'' + \lambda^2)^{-1}\left[\lambda - \frac{d\lambda}{dw}(2\lambda x - w)\right]. \tag{9.3.74}$$

The normal to (9.3.70) is parallel to $(\lambda, 1)$. Hence,

$$\mathscr{D}^{\mathrm{T}}\boldsymbol{n} = (1+\lambda^2)^{-1/2}\begin{bmatrix} 0 & 1\\ -1 & \gamma \end{bmatrix}\begin{bmatrix} \lambda\\ 1 \end{bmatrix} = (1+\lambda^2)^{-1/2}\begin{bmatrix} 1\\ \gamma - \lambda \end{bmatrix} \tag{9.3.75}$$

and this is parallel to (9.3.73) if

$$\frac{dx}{dw}/1 = \left[1 - \lambda\frac{dx}{dw} - x\frac{d\lambda}{dw}\right]/(\gamma - \lambda). \tag{9.3.76}$$

We can now solve (9.3.74, 76) simultaneously, to get

$$\frac{dx}{dw} = \frac{1}{\gamma} - \frac{x}{\gamma^2}\left[\frac{U'' - \gamma\lambda + \lambda^2}{x(U'' + \lambda^2) - (2\lambda x - w)}\right] \tag{9.3.77}$$

$$\frac{d\lambda}{dw} = \frac{1}{\gamma}\left[\frac{U'' - \lambda\gamma + \lambda^2}{x(U'' + \lambda^2) - (2\lambda x - w)}\right], \tag{9.3.78}$$

The saddle point is at $(x, p) = (0, 0)$ and thus $w = 0 \Longleftrightarrow x = 0$. Using this in (9.3.77) we see that we must have

$$x \simeq w/\gamma \qquad \text{as } w \simeq 0\,. \tag{9.3.79}$$

Near $x = 0$, we write approximately

$$U[x] \simeq -\tfrac{1}{2}U_2 x^2 \tag{9.3.80}$$

and substituting (9.3.79, 80) in (9.3.72), we see that

$$\lambda^2 - \gamma\lambda + U''(0) = 0 \tag{9.3.81}$$

which determines

$$\lambda(0) = \frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} + U_2}\,. \tag{9.3.82}$$

We now see that (9.3.78) tells us that $d\lambda/dw = 0$ when $w = 0$. Thus, λ will not change significantly from (9.3.82) around the saddle point, and we shall from now on approximate λ by (9.3.82).

Only one of the roots is acceptable and physically, this should be $\lambda \to \infty$ in the high friction limit which would give Kramers' result and requires the positive sign. The other root corresponds to taking a plane such that we get a minimum of $P_s(\boldsymbol{x})$ on it.

We now integrate (9.3.57) and determine $d(w)$. Notice that $d(w)$ must be defined with $\boldsymbol{n}(w)$ a unit vector. Direct substitution in (9.3.75) and using (9.3.79)

$$(1 + \lambda^2)^{-1/2} = \frac{dx}{dw}(w = 0)d(0) = \frac{1}{\gamma}\,d(0) \tag{9.3.83}$$

so that

$$d(0) = \gamma(1 + \lambda^2)^{-1/2}\,. \tag{9.3.84}$$

Further,

$$\begin{aligned} p(w) &= \int_{S(w)} |d\boldsymbol{S}\, P_s(\boldsymbol{x})| = \int_{S(w)} \sqrt{dx^2 + dp^2}\, P_s(x, p) \\ &= \mathcal{N}_2 \frac{[1 + \lambda^2]^{1/2}}{\lambda} \int dp\, \exp\left[-\frac{p^2}{2} - U\left(\frac{w - p}{\lambda}\right)\right]. \end{aligned} \tag{9.3.85}$$

An exact evaluation depends on the choice of $U(x)$. Approximately, we use

$$U(x) \simeq U_0 - \tfrac{1}{2}U_2x^2 \tag{9.3.86}$$

and evaluate the result as a Gaussian: we get

$$p(w) = \frac{(1 + \lambda^2)^{1/2}}{\lambda}\mathcal{N}_2\, e^{-U_0} \exp\left[\frac{U_2w^2}{2(\lambda^2 - U_2)}\right] \tag{9.3.87}$$

and thus

$$\kappa(0) = \int_{-\infty}^{0} p(w)^{-1}dw = \tfrac{1}{2}\mathcal{N}_2^{-1}\frac{\lambda\gamma}{(1 + \lambda^2)^{1/2}}\frac{e^{U_0}}{\sqrt{U_2}} = \mu(0)\,. \tag{9.3.88}$$

Thus, from (9.2.19) adapted to the many dimensional theory, we have for the mean first passage time from one well to the point $x = 0$,

$$\tau_0 = \kappa(0)d(0)^{-1} = \frac{\lambda}{2}\, e^{U_0}\mathcal{N}_1^{-1}\sqrt{\frac{2\pi}{U_2}}, \tag{9.3.89}$$

i.e.,

$$\boxed{\tau_0 = \frac{1}{2}\left(\frac{\gamma}{2} + \sqrt{\frac{\gamma^2}{4} + U_2}\right) e^{U_0}\mathcal{N}_1^{-1}\sqrt{\frac{2\pi}{U_2}}.} \tag{9.3.90}$$

Comparisons with Other Results

a) Exact One-Dimensional Mean First Passage Time (Smoluchowski's Equation)
One reduces Kramers equation in the large friction limit to the Smoluchowski equation for

$$\hat{P}(x, t) = \int dv\, P(x, v, t)\,, \tag{9.3.91}$$

i.e.,

$$\frac{\partial\hat{P}(x, t)}{\partial t} = \frac{1}{\gamma}\frac{\partial}{\partial x}\left[U'(x)\,\hat{P} + \frac{\partial\hat{P}}{\partial x}\right] \tag{9.3.92}$$

and the *exact* result for the mean first passage time from $x = a$ to $x = 0$ for this approximate equation is

$$\tau_1 = \gamma\int_a^0 dx \exp[U(x)] \int_{-\infty}^{x} dz \exp[-U(z)]\,. \tag{9.3.93}$$

This result can be evaluated numerically.

b) Kramers' Result
This is obtained by applying our method to the one-dimensional Smoluchowski equation (9.3.92) and making Gaussian approximations to all integrals. The result is

$$\tau_2 = \tfrac{1}{2}\gamma\, e^{U_0}\mathcal{N}_1^{-1}\sqrt{\frac{2\pi}{U_2}} \tag{9.3.94}$$

which differs from (9.3.90) for τ_0 by the replacement $\lambda \to \gamma$, which is clearly valid in a large γ limit. In this limit,

$$\tau_0 \simeq (1 + U_2\gamma^{-2})\tau_2\,. \tag{9.3.95}$$

c) Corrected Smoluchowski
A more accurate equation than the Smoluchowski equation (9.3.1) is the *corrected Smoluchowski equation* (6.4.108);

$$\frac{\partial\hat{P}}{\partial t} = \frac{1}{\gamma}\frac{\partial}{\partial x}\left\{[1 + \gamma^{-2}U''(x)]\left[U'(x)\,\hat{P} + \frac{\partial\hat{P}}{\partial x}\right]\right\}. \tag{9.3.96}$$

One now calculates the exact mean first passage time for this equation using standard theory; it is

$$\tau_3 = \gamma \int_a^0 dx\,[1 + \gamma^{-2}U''(x)] \exp[U(x)] \int_{-\infty}^{x} dz \exp[-U(z)] . \tag{9.3.97}$$

Note however, that the principal contribution to the x integral comes from near $x = 0$ so that the small correction term $\gamma^{-2}U''(x)$ should be sufficiently accurately evaluated by setting

$$U''(x) \simeq U''(0) = -U_2 \tag{9.3.98}$$

in (9.3.97). We then find the *corrected Smoluchowski result,*

$$\tau_3 = (1 - \gamma^{-2}U_2)^{-1}\tau_1 \simeq (1 + \gamma^{-2}U_2)\tau_1 . \tag{9.3.99}$$

Notice that in this limit,

$$\frac{\tau_3}{\tau_1} = \frac{\tau_0}{\tau_2} \tag{9.3.100}$$

which means that in the limit that all integrals may be evaluated as sharply peaked Gaussians, our result is in agreement with the corrected Smoluchowski.

d) Simulations

By computer simulation of the equivalent stochastic differential equations

$$dx = p\,dt \tag{9.3.101}$$

$$dp = -[\gamma p + U'(x)]dt + \sqrt{2\gamma}\,dW(t) , \tag{9.3.102}$$

we can estimate the mean first passage time *to the plane* S_0, i.e., to the line

$$p = -\lambda x . \tag{9.3.103}$$

The results have to be computed for a given set of potentials. In order to assess the effect of the sharpness of peaking, we consider different temperatures T, i.e., we consider

$$dx = p\,dt \tag{9.3.104}$$

$$dp = -[\gamma p + U'(x)]dt + \sqrt{2\gamma T}\,dW(t) . \tag{9.3.105}$$

By the substitutions

$$\begin{aligned} p &\to pT^{1/2} \\ x &\to x\,T^{1/2} , \end{aligned} \tag{9.3.106}$$

we obtain

$$\left.\begin{aligned} dx &= p\,dt \\ dp &= -[\gamma p + V'(x, T)] + \sqrt{2\gamma}\,dW(t) \end{aligned}\right\} \tag{9.3.107}$$

where

$$V(x, T) = U(xT^{1/2}) . \tag{9.3.108}$$

The simulations were performed with

$$U(x) = \tfrac{1}{4}(x^2 - 1)^2 \tag{9.3.109}$$

and the results are shown in Fig. 9.4. They separate naturally into two sets: curved, or straight lines. The best answer is the corrected Smoluchowski which agrees with the simulations at all temperatures, and at low temperatures, agrees with our method. Thus, we confirm the validity of the method in the region of validity expected, since low temperature corresponds to sharply peaked distributions.

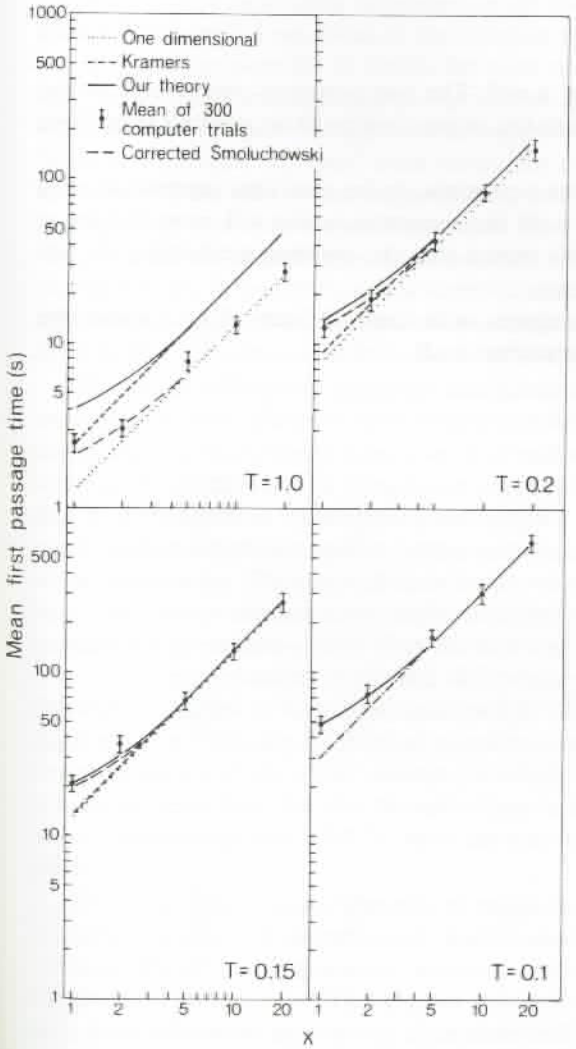

Fig. 9.4. Comparison of various estimates of the mean exit time from the double well potential of Sect. 9.3.4

Notice also that the choice of the plane S_0 as the separatrix is appropriate on another ground. For, near to $x = 0$, $p = 0$, we can write

$$dx = p\,dt \tag{9.3.110}$$

$$dp = (-\gamma p + U_2 x)dt + \sqrt{2\gamma T}\,dW(t)\,. \tag{9.3.111}$$

The condition that the deterministic part of (dx, dp), namely, $(p, -\gamma p + U_2 x)$ is in the direction connecting the point (x, p) to the origin is

$$\frac{p}{-\gamma p + U_2 x} = \frac{x}{p}\,. \tag{9.3.112}$$

Putting $p = -\lambda x$, we find

$$\lambda^2 - \lambda\gamma - U_2 = 0 \tag{9.3.113}$$

which is the same as (9.3.81) near $x = 0$. The two solutions correspond to the deterministic motion pointing towards the origin (+ve root) or pointing away from the origin (−ve root).

Thus, when the particle is on the separatrix, in the next time interval dt, only the random term $dW(t)$ will move it off this separatrix and it will move it right or left with equal probability, i.e., this means that the splitting probability, to left or right, should be 1:1 on this plane.

This separatrix definition also agrees with that of Sects. 9.1, 2 where the $v(x)$ should be perpendicular to the normal to S.

10. Simulation of Stochastic Differential Equations

Simulating stochastic differential equations is something that can now be realistically attempted, in contrast to the situation when this book was first written. While the dramatic increase in the power and availability of computers is the most obvious reason, another is the development of a better understanding of the theoretical basis and efficiency of algorithms for solving stochastic differential equations. Unfortunately there is no easy adaptation of all but the simplest algorithms used for ordinary differential equations to the solution of stochastic differential equations, and it is also not possible to obtain the same order of convergence. However the same basic theoretical approach to algorithms can be used, and the differences can be made clear by quite straightforward arguments. As in the case of ordinary differential equations, the three main issues are *speed*, *accuracy* and *stability*. The first two of these are obvious requirements, and are of course connected with each other—a highly accurate algorithm can be used with a larger time step than a less accurate one, and thus the process of solution should proceed more rapidly. The aspect of stability is less obvious, but soon becomes apparent in practice if an unstable algorithm is used, in which errors in computation can become, often quite suddenly, so large that the solution found bears no relation to the real one.

The major differences in the the simulation of stochastic differential equations arise from the *non differentiability* of the noise term, rather than its stochastic nature, and in this regard one has to keep a sense of realism, since the white noise which we use is an idealisation of a physical noise which may have a finite correlation time and may be differentiable. There is no point in trying to simulate accurately aspects of a model system which are valid on a time scale much shorter than the correlation time of the actual noise. The physical basis for the white noise limit, given in Chap. 6, can be used to assess whether some implementation of a non-white noise source should be used in a given case—very often the best way to implement the non-white noise is to use a supplementary stochastic differential equation. We will find, however, that there is one set of very important cases in which it is possible to use algorithms appropriate to ordinary differential equations, and that is the case when the noise terms are independent of the variable for which one is solving. In fact the class is somewhat wider than that, and the noise form in this class is known as *commutative noise*. As shown in Sect. 10.5.2c, there are non-trivial examples of equations of this kind.

There are no extensive libraries of stochastic differential equation algorithms available for use, and unfortunately uninformed intuition in this field can lead to considerable difficulty, extensive waste of time or even complete failure to solve the problem under study. This chapter is intended to address the main issues and give some advice on appropriate algorithms and strategies. It is not a comprehen-