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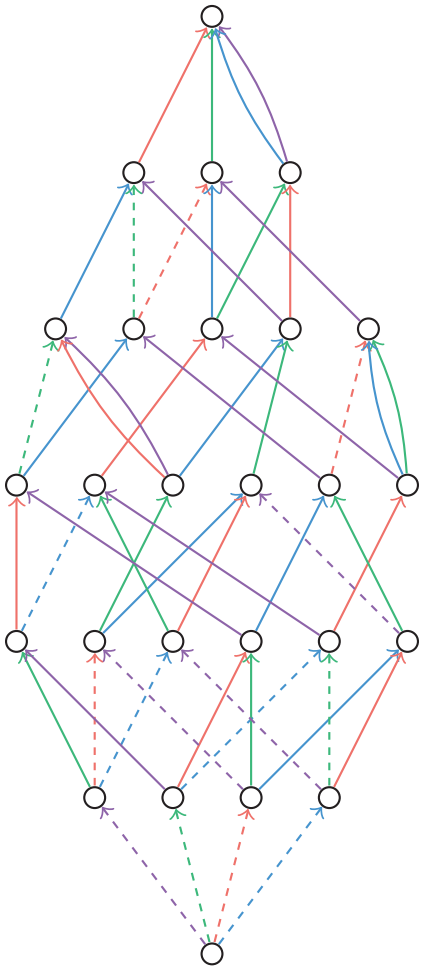

Posets of twisted involutions in Coxeter groups

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Contents

1 Coxeter groups

A Coxeter group, named after Harold Scott MacDonald Coxeter, is an abstract group generated by involutions with specific relations between these generators. A simple class of Coxeter groups are the symmetry groups of regular polyhedras in the Euclidean space.

The symmetry group of the square for example can be generated by two reflections s, t, whose stabilized hyperplanes enclose an angle of $\pi/4$. In this case the map st is a rotation in the plane by $\pi/2$. So we have $s^2 = t^2 = (st)^4 = \mathrm{id}$. In fact, this reflection group is determined up to isomorphy by s, t and these three relations [4, Theorem 1.9]. Furthermore it turns out, that the finite reflection groups in the Euclidean space are precisely the finite Coxeter groups [4, Theorem 6.4].

In this chapter we compile some basic facts on Coxeter groups, based on [4].

1.1 Introduction to Coxeter groups

Definition 1.1. Let $S = \{s_1, \dots, s_n\}$ be a finite set of symbols and

$$R = \{m_{ij} \in \mathbb{N} \cup \infty : 1 \leq i, j \leq n\}$$

a set numbers (or ∞) with $m_{ii} = 1$, $m_{ij} > 1$ for $i \neq j$ and $m_{ij} = m_{ji}$. Then the free represented group

$$W = \langle S \mid (s_i s_j)^{m_{ij}} \rangle$$

is called a ***Coxeter group*** and (W, S) the corrosponding ***Coxeter system***. The cardinality of S is called the ***rank*** of the Coxeter system (and the Coxeter group).

From the definiton we see, that Coxeter groups only depend on the cardinality of S and the relations between the generators in S. A common way to visualize this information are Coxeter graphs.

Definition 1.2. Let (W, S) be a Coxeter system. Create a graph by adding a vertex for each generator in S. Let $(s_i s_j)^m = 1$. In case $m = 2$ the two corrosponding vertices have no connecting edge. In case $m = 3$ they are connected by an unlabed edge. For $m > 3$ they have an connecting edge with label m. We call this graph the ***Coxeter graph*** of our Coxeter system (W, S).

Definition 1.3. Let (W, S) be a Coxeter system. For an arbitrary element $w \in W$ we call a product $s_{i_1} \cdots s_{i_n} = w$ of generators $s_{i_1} \dots s_{i_n} \in S$ an ***expression*** of w. Any expression that can be obtained from $s_{i_1} \cdots s_{i_n}$ by omitting some (or all) factors, is called a ***subexpression*** of w.

The present relations between the generators of a Coxeter group allow us to rewrite expressions. Hence an element $w \in W$ can have more than one expression. Obviously any element $w \in W$ has infinitly many expressions, since any expression $s_{i_1} \cdots s_{i_n} = w$ can be extended by applying $s_1^2 = 1$ from the right. But there must be a smallest number of generators needed to receive w. For example the neutral element e can be expressed by the empty expression. Or each generator $s_i \in S$ can be expressed by itself, but any expression with less factors (i.e. the empty expression) is unequal to s_i.

Definition 1.4. Let (W, S) be a Coxeter system and $w \in W$ an element. Then there are some (not necessarily distinct) generators $s_i \in S$ with $s_1 \cdots s_r = w$. We call r the ***expression length***. The smallest number $r \in \mathbb{N}_0$ for that w has an expression of length r is called the ***length*** of w and each expression of w, that is of minimal length, is called ***reduced expression***. The map

$$l : W \to \mathbb{N}_0$$

that maps each element in W to its length is called ***length function***.

Definition 1.5. Let (W, S) be a Coxeter system. We define

$$D_R(w) := \{s \in S : l(ws) < l(w)\}$$

as the ***right descending set*** of w. The analogue left version

$$D_L(w) := \{s \in S : l(sw) < l(w)\}$$

is called ***left descending set*** of w. Since the left descending set is not need in this paper, we will often call the right descending just ***descending set*** of w.

The next lemma yields some useful identities and relations for the length function.

Lemma 1.6. *Let (W, S) be a Coxeter system, $s \in S$, $u, w \in W$ and $l : W \to \mathbb{N}$ the length function. Then*

1. $l(w) = l(w^{-1})$,
2. $l(w) = 0$ *iff* $w = e$,
3. $l(w) = 1$ *iff* $w \in S$,
4. $l(uw) \leq l(u) + l(w)$,
5. $l(uw) \geq l(u) - l(w)$ *and*
6. $l(ws) = l(w) \pm 1$.

Proof. See [4, Section 5.2]. □

1.2 Exchange and Deletion Condition

We now obtain a way to get a reduced expression of an arbitrary element $s_1 \cdots s_r = w \in W$.

Definition 1.7. Let (W, S) be a Coxeter system. Any element $w \in W$ that is conjugated to an generator $s \in S$ is called ***reflection***. Hence the set of all reflections in W is

$$T = \bigcup_{w \in W} wSw^{-1}.$$

Theorem 1.8 (Strong Exchange Condition)**.** *Let (W, S) be a Coxeter system, $w \in W$ an arbitrary element and $s_1 \cdots s_r = w$ with $s_i \in S$ a not necessarily reduced expression for w. For each reflection $t \in T$ with $l(wt) < l(w)$ there exists an index i for which $wt = s_1 \cdots \hat{s}_i \cdots s_r$, where $\hat{s}_i$ means omission. In case we start from a reduced expression, then i is unique.*

Proof. See [4, Theorem 5.8]. □

The Strong Exchange Condition can be weaken when insisting on $t \in S$ to receive the following corollary.

Corollary 1.9 (Exchange Condition)**.** *Let (W, S) be a Coxeter system, $w \in W$ an arbitrary element and $s_1 \cdots s_r = w$ with $s_i \in S$ a not necessarily reduced expression for w. For each generator $s \in S$ with $l(ws) < l(w)$ there exists an index i for which $ws = s_1 \cdots \hat{s}_i \cdots s_r$, where $\hat{s}_i$ means omission.*

Proof. Directly from Strong Exchange Condition. □

The Exchange Condition immediately yields another corollary for Coxeter groups:

Corollary 1.10 (Deletion Condition)**.** *Let (W, S) be a Coxeter system, $w \in W$ and $w = s_1 \cdots s_r$ with $s_i \in S$ an unreduced expression of w. Then there exist two indices $i, j \in \{1, \cdots, r\}$ with $i < j$, such that $w = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_r$, where $\hat{s}_i$ and $\hat{s}_j$ mean omission.*

Proof. Since the expression is unreduced there must be an index j for that the twisted length shrinks. That means for $w' = s_1 \cdots s_{j-1}$ is $l(w's_j) < l(w')$. Using the Exchange Condition we get $w's_j = s_1 \cdots \hat{s}_i \cdots s_{j-1}$ yielding $w = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_r$. □

This corollary is called ***Deletion Condition*** and allows us to reduce expressions, i.e. to find a subexpression that is reduced. Due to the Deletion Condition any unreduced expression can be reduced by omitting an even number of generators (we just have to apply the Deletion Condition inductively).

The Strong Exchange Condition, the Exchange Condition and the Deletion Condition, are some of the most powerful tools when investigating properties of Coxeter groups. We can use the second to prove a very handy property of Coxeter groups. The intersection of two parabolic subgroups is again a parabolic subgroup.

Definition 1.11. Let (W, S) be a Coxeter system. For a subset of generators $I \subset S$ we call the subgroup $W_I \leq W$, that is generated by the elements in I with the corrosponding relations, a ***parabolic subgroup*** of W.

Lemma 1.12. *Let (W, S) be a Coxeter system and $I, J \subset S$ two subsets of generators. Then $W_I \cap W_J = W_{I \cap J}$.*

Proof. Let $w \in W_{I\cap J}$. Then $w \in W_I$ and $w \in W_J$. To show the other inclusion we induce over the length r. For $r = 0$ we have $w = e$ and so $w \in W_{S'}$ for any $S' \subset S$. Suppose we have proven the assumption for all lengths up to $r - 1$. Let $w \in W_I \cap W_J$ with $l(w) = r$. Then we have two reduced expressions $w = s_1 \cdots s_r = t_1 \cdots t_r$ with $s_i \in I$ and $t_i \in J$. By applying s_r from the right we get $ws_r = s_1 \cdots s_{r-1} = t_1 \cdots t_r s_r$. The expression $t_1 \cdots t_r s_r$ is of length $r - 1$, so Exchange Condition yields $ws_r = s_1 \cdots s_{r-1} = t_1 \cdots \hat{t}_i \cdots t_r$, hence $ws_r \in W_I \cap W_J$. Due to induction we know that $ws_r \in W_{I\cap J}$. **TODO** □

1.3 Finite Coxeter groups

Coxeter groups can be finite and infinite. A simple example for the former category is the following. Let $S = \{s\}$. Due to definition it must be $s^2 = e$. So W is isomorph to $\mathbb{Z}_2$ and finite. An example for an infinite Coxeter group can be obtained from $S = \{s, t\}$ with $s^2 = t^2 = e$ and $(st)^\infty = e$ (so we have no relation between s and t). Obviously the element st has infinite order forcing W to be infinite. But there are infinite Coxeter groups without an ∞-relation between two generators, as well. An example for this is W obtained from $S = \{s_1, s_2, s_3\}$ with $s_1^2 = s_2^2 = s_3^2 = (s_1 s_2)^3 = (s_2 s_3)^3 = (s_3 s_1)^3 = e$. But how can one decide weather W is finite or not?

To provide a general answer to this question we fallback to a certain class of Coxeter groups, the irreducible ones.

Definition 1.13. A Coxeter system is called ***irreducible***, if the corresponding Coxeter graph is connected. Else, it is called ***reducible***.

If a Coxeter system is reducible, then its graph has more than one connection component and each connection component corrosponds to a parabolic subgroup of W.

Proposition 1.14. *Let (W, S) be a reducible Coxeter system. Then there exists a partition of S into I, J with $(s_i s_j)^2 = e$ whenever $s_i \in I, s_j \in J$ and W is isomorph to the direct product of the two parabolic subgroups W_I and W_J.*

Proof. See [4, Proposition 6.1]. □

This proposition tells us, that an arbitray Coxeter system is finite iff its irreducible parabolic subgroups are finite. Therefore we can indeed fallback to irreducible Coxeter systems without loss of generality. If we could categorize all irreducible finite Coxeter systems, we could categorize all finite Coxeter systems. This is done by the following theorem:

Theorem 1.15. *The irreducible finite Coxeter systems are exactly the ones in Figure 1.1.*

Proof. [4, Theorem 6.4] □

Finally we can decide with ease, if a given Coxeter system is finite. Take its irreducible parabolic subgroups and check, if each is one of A_n, B_n, D_n, E_6, E_7, E_8, F_4, H_3, H_4 or $I_2(m)$.

1.4 Bruhat ordering

We now investigate ways to partially order the elements of a Coxeter group. Futhermore, this ordering should be compatible with the length function. The most useful way to achieve this is the Bruhat ordering [4, Section 5.9].

Definition 1.16. Let M be a set. A binary relation, in this case often denoted as "$\leq$", is called a ***partial order*** over M, if fullfills the following conditions for all $a, b, c \in M$:

1. $a \leq a$, called ***reflexivity***

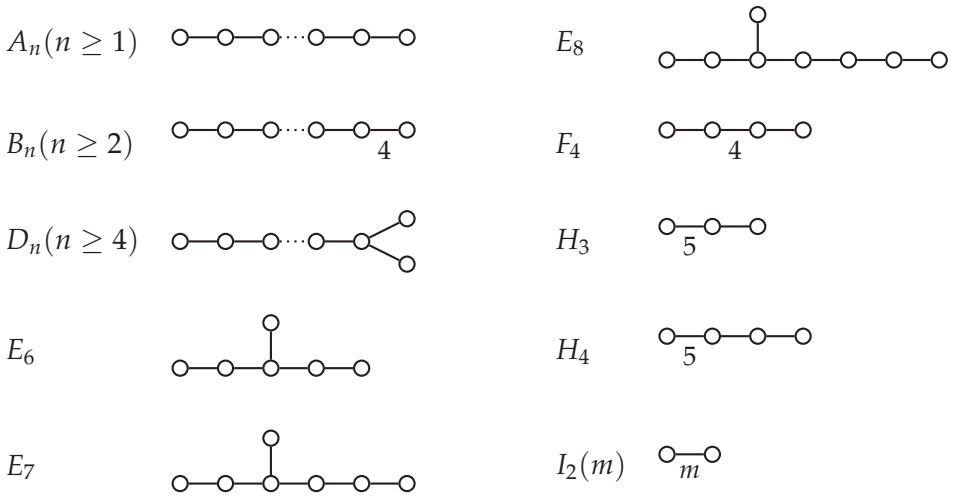

Figure 1.1: All types of irreducible finite Coxeter systems

2. if $a \leq b$ and $b \leq a$ then $a = b$, called ***antisymmetry***
3. if $a \leq b$ and $b \leq c$ then $a \leq c$, called ***transitivity***

In this case $(M, \leq)$ is called a ***poset***. If two elements $a \leq b \in M$ are immediate neighbours, i.e. there is no third element $c \in M$ with $a \leq c \leq b$ we say that b ***covers*** a. A poset is called ***graded poset*** if there is a map $\rho : M \to \mathbb{N}$ so that $\rho(b) - 1 = \rho(a)$ whenever b covers a. In this case ρ is called the ***rank function*** of the graded poset.

Definition 1.17. Let $(M, \leq)$ be a poset. The ***Hasse diagram*** of the poset is the graph obtained in the following way: Add a vertex for each element in M. Then add a directed edge from vertex a to b whenever b covers a.

Example 1.18. Suppose we have an arbitrary set M. Then the powerset $\mathcal{P}(M)$ can be partially ordered by the subset relation, so $(\mathcal{P}(M), \subseteq)$ is a poset. Indeed this poset is always graded with the cardinality function as rank function. In Figure 1.2 we see the Hasse diagram of this poset with $M = \{x, y, z\}$.

Definition 1.19. Let (W, S) be a Coxeter system and $T = \cup_{w \in W} wSw^{-1}$ the set of all reflections in W. We write $w' \to w$ if there is a $t \in T$ with $w't = w$ and $l(w') < l(w)$. If there is a sequence $w' = w_0 \to w_1 \to \ldots \to w_m = w$ we say $w' < w$. The resulting relation $w' \leq w$ is called ***Bruhat ordering***, denoted as $\mathrm{Br}(W)$.

Lemma 1.20. *Let (W, S) be a Coxeter system. Then $\mathrm{Br}(W)$ is a poset.*

Proof. The Bruhat ordering is reflexive by definition. Since the elements in sequences $e \to w_1 \to w_2 \to \ldots$ are strictly ascending in length, it must be antisymmetric. By concatenation of sequences we get the transitivity. □

What we really want is the Bruhat ordering to be graded with the length function as rank function. By definition we already have $v < w$ iff $l(v) < l(w)$, but its not that obvious

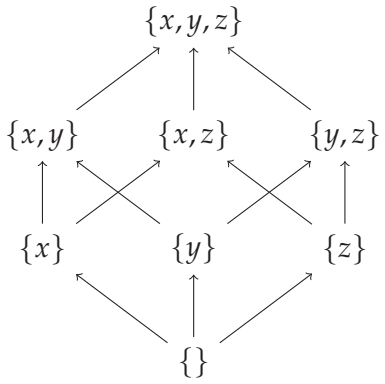

Figure 1.2: Hasse diagram of the set of all subsets of $\{x, y, z\}$ order by the subset relation

that two immediately adjacent elements differ in length by exactly 1. Beforehand let us just mention two other partial orderings, where this property is obvious by definition:

Definition 1.21. Let (W, S) be a Coxeter system. The ordering $\leq_R$ defined by $u \leq_R w$ iff $uv = w$ for some $u \in W$ with $l(u) + l(v) = l(w)$ is called the ***right weak ordering***. The left sided version $u \leq_L w$ iff $vu = w$ is called the ***left weak ordering***.

To ensure the Bruhat ordering is graded as well, we need another characterization of the Bruhat ordering with subexpressions. As we show it is $u \leq w$ iff there is a reduced expression of u that is a subexpression of a reduced expression of w.

Proposition 1.22. *Let (W, S) be a Coxeter system, $u, w \in W$ with $u \leq w$ and $s \in S$. Then $us \leq w$ or $us \leq ws$ or both.*

Proof. We can reduce the proof (**TODO**why?) to the case $u \to w$, i.e. $ut = w$ for a $t \in T$ with $l(v) < l(u)$. Let $s = t$. Then $us \leq w$ and we are done. In case $s \neq t$ there are two alternatives for the lengths. We can have $l(us) = l(u) - 1$ which would mean $us \to u \to w$, so $us \leq w$.

Assume $l(us) = l(u) + 1$. For the reflection $t' = sts$ we get $(us)t' = ussts = uts = ws$. So it is $us \leq ws$ iff $l(us) < l(ws)$. Suppose this is not the case. Since we have assumed $l(us) = l(u) + 1$ any reduced expression $u = s_1 \cdots s_r$ for u yields a reduced expression $us = s_1 \cdots s_r s$ for us. With the Strong Exchange Condition we can obtain $ws = ust'$ from us by omitting one factor. This omitted factor cannot be s since $s \neq t$. This means $ws = s_1 \cdots \hat{s}_i \cdots s_r s$ and so $ws = s_1 \cdots \hat{s}_i \cdots s_r$, contradicting to our assumption $l(u) < l(w)$ □

Theorem 1.23. *Let (W, S) be a Coxeter system and $w \in W$ with any reduced expression $w = s_1 \cdots s_r$ and $s_i \in S$. Then $u \leq w$ (in the Bruhat ordering) iff u can be obtained as a subexpression of this reduced expression.*

Proof. **TODO** □

This characterization of the Bruhat ordering is very handy. With it and the following short lemma we will be in the position to show, that $\mathrm{Br}(W)$ is graded with rank function l.

Lemma 1.24. *Let (W,S) be a Coxeter system, $u,w \in W$ with $u < w$ and $l(w) = l(u)+1$. In case there is a generator $s \in S$ with $u < us$ but $us \neq w$, then both $w < ws$ and $us < ws$.*

Proof. Due to Proposition 1.22 we have $us \leq w$ or $us \leq ws$. Since $l(us) = l(w)$ and $us \neq w$ the first case is impossible. So $us \leq ws$ and because of $u \neq w$ already $us < ws$. In turn, $l(w) = l(us) < l(ws)$, forcing $w < ws$. □

Proposition 1.25. *Let (W,S) be a Coxeter system and $u < w$. Then there are elements $w_0, \ldots, w_m \in W$ such that $u = w_0 < w_1 < \ldots < w_m = w$ with $l(w_i) = l(w_{i-1}) + 1$ for $1 \leq i \leq m$.*

Proof. We induce on $r = l(u) + l(w)$. In case $r = 1$ we have $u = e$ and $w = s$ for an $s \in S$ and are done. Conversely suppose $r > 1$. Then there is a reduced expression $w = s_1 \cdots s_r$ for w. Lets fix this expression. Then $l(ws_r) < l(w)$. Thanks to Theorem 1.23 there must be a subexpression of w with $u = s_{i_1} \cdots s_{i_q}$ for some $i_1 < \ldots < i_q$. We distinguish between two cases:

$u < us$ If $i_q = r$, then $us = s_{i_1} \cdots s_{i_q} s = s_{i_1} \cdots s_{i_{q-1}}$ which is also a subexpression of ws. This yields $u < us \leq ws < w$. Since $l(ws) < r$ there is, by induction, a sequence of the desired form. The last step from ws to w also differs in length by exactly 1, so we are done. If $i_q < r$ then u is itself already a subexpression of ws and we can again find a sequence from u to ws strictly ascending length by 1 in each step and have one last step from ws to w also increasing length by 1.

$us < u$ Then by induction we can find a sequence from us to w, say $us = w_0 < \ldots < w_m = w$, where the lengths of neighboured elements differ by exactly 1. Since $w_0 s = u > us = w_0$ and $w_m s = ws < w = w_m$ there must be a smallest index $i \geq 1$, such that $w_i s < w_i$, which we choose. Suppose $w_i \neq w_{i-1}s$. It is $w_{i-1} < w_{i-1}s \neq w_i$ and due to Lemma 1.24 we get $w_i < w_i s$. This contradicts to the minimality of i. So $w_i = w_{i-1}s$. For all $1 \leq j < i$ we have $w_j \neq w_{j-1}s$, because of $w_j < w_j s$. Again we apply Lemma 1.24 to receive $w_{j-1}s < w_j s$. Alltogether we can construct a sequence

$$u = w_0 s < w_1 s < \ldots < w_{i-1}s = w_i < w_{i+1} < \ldots w_m = w,$$

which matches our assumption. □

Corollary 1.26. *Let (W,S) be a Coxeter system and $\mathrm{Br}(W)$ the Bruhat ordering poset of W. Then $\mathrm{Br}(W)$ is graded with $l : W \to \mathbb{N}$ as rank function.*

Proof. Let $u,w \in W$ with w covering u. Then Proposition 1.25 says there is a sequence $u = w_0 < \ldots < w_m = w$ with $l(w_i) = l(w_{i-1}) + 1$ for $1 \leq i \leq m$. Since w covers u it must be $m = 1$ and so $u < w$ with $l(w) = l(u) + 1$. □

Theorem 1.27 (Lifting Property). *Let (W, S) be a Coxeter system and $v, w \in W$ with $v \leq w$. Suppose $s \in S$ with $s \in D_R(w)$. Then*

1. $vs \leq w$,
2. $s \in D_R(v) \Rightarrow vs \leq ws$.

Proof. We use the alternative subexpression characterization of the Bruhat ordering from Theorem 1.23.

1. Since $s \in D_R(w)$ there exists a reduced expression $w = s_1 \cdots s_r$ with $s_r = s$. Due to $v \leq w$ we can obtain v as a subexpression $v = s_{i_1} \cdots s_{i_q}$ from w. If $i_q = r$ then $vs = s_{i_1} \cdots s_{i_q} s = s_{i_1} \cdots s_{i_{q-1}}$ is also a subexpression of w. Else, if $i_q \neq r$ then v is a subexpression of $ws = s_1 \cdots s_{r-1}$ and so vs is again a subexpression of $w = s_1 \cdots s_{r-1} s$. In both cases we get $vs \leq w$.
2. If we additionally assume $s \in D_R(v)$ then we can always find a reduced expression $w = s_1 \cdots s_r$ with $s_r = s$ having $u = s_{i_1} \cdots s_{i_q}$ as subexpression with $s_{i_q} = s$. This yields $vs = s_{i_1} \cdots s_{i_{q-1}} \leq s_1 \cdots s_{r-1} = ws$. □

The Lifting Property seems quite innocent, but when trying to investigate facts around the Bruhat ordering it proofs to be one of the key tools in many cases.

1.5 Compact hyperbolic Coxeter groups

TODO

2 Twisted involutions in Coxeter groups

In this section we focus on a certain subset of elements in Coxeter groups, the so called twisted involutions. From now on (and in the next sections) we fix some symbols to have always the same meaning (some definitions follow later):

S	A set of generators.
s	A generator in S.
W	A Coxeter group with generators S.
u, v, w	A element in the Coxeter group W.
m_{ij}	The order of the element $(s_i s_j)$ with s_i the i-th generator of W.
(W, S)	The Coxeter system obtained from W and S.
θ	A Coxeter system automorphism of (W, S) with $\theta^2 = \mathrm{id}$.
$\mathcal{I}_\theta$	The set of twisted involutions of W regarding θ.
$\underline{S}$	A set of symbols, $\underline{S} = \{\underline{s} : s \in S\}$.

2.1 Introduction to twisted involutions

Definition 2.1. An automorphism $\theta : W \to W$ with $\theta(S) = S$ is called a ***Coxeter system automorphism*** of (W, S). We always assume $\theta^2 = \mathrm{id}$.

Definition 2.2. Each $w \in W$ with $\theta(w) = w^{-1}$ is called a ***θ-twisted involution*** or just ***twisted involution***, if θ is clear from the context. The set of all twisted involutions in W regarding θ is denoted with $\mathcal{I}_\theta(W)$. Often we just omit the Coxeter group and write $\mathcal{I}_\theta$, when it is clear from the context which W is meant.

Example 2.3. Let $\theta = \mathrm{id}_W$. Then θ is an Coxeter system automorphism and

$$\mathcal{I}_\theta = \{w \in W : w = w^{-1}\}.$$

The next example is more helpfull, since it reveals a way to think of $\mathcal{I}_\theta$ as a generalization of ordinary Coxeter groups.

Example 2.4. Let θ be a automorphism of $W \times W$ with

$$\theta : W \times W \to W \times W : (u, w) \mapsto (w, u).$$

Note that θ is no Coxeter system automorphism, but we can think of it as one if we identify $S \subset W$ with $S \times S \subset W \times W$. Then the set of twisted involutions is

$$\mathcal{I}_\theta = \{(w, w^{-1}) \in W \times W : w \in W\}.$$

This yields a canonical bijection between $\mathcal{I}_\theta$ and W.

The map we define right now is of superior importance to this whole paper, since it is needed to define the poset, the main thesis is about.

Definition 2.5. Let $\underline{S} := \{\underline{s} : s \in S\}$ be a set of symbols. Each element in $\underline{S}$ acts from the right on W by the following definition:

$$w\underline{s} = \begin{cases} ws & \text{if } \theta(s)ws = w, \\ \theta(s)ws & \text{else.} \end{cases}$$

This action can be extended on the whole free monoid over $\underline{S}$ by

$$w\underline{s}_1\underline{s}_2 \dots \underline{s}_k = (\dots((w\underline{s}_1)\underline{s}_2)\dots)\underline{s}_k.$$

If $w\underline{s} = \theta(s)ws$, then we say s ***acts bothsided*** on w. Else we say s ***acts onesided*** on w.

Definition 2.6. Let $k \in \mathbb{N}$ and $s_{i_j} \in S$ for all $1 \leq j \leq k$. Then an expression $w\underline{s}_{i_1} \dots \underline{s}_{i_k}$ is called ***twisted*** w***-expression***. In case $w = e$ we omit w, just write $\underline{s}_{i_1} \dots \underline{s}_{i_k}$ and call it ***twisted expression***.

There is another characterization of this action, distinguishing between one- and both-sided actions by length.

Lemma 2.7. *Let $w \in \mathcal{I}_\theta$ and $s \in S$. Then*

$$w\underline{s} = \begin{cases} ws & \text{if } l(\theta(s)ws) = l(w), \\ \theta(s)ws & \text{else.} \end{cases}$$

Proof. Suppose s acts onesided on w. Then $\theta(s)ws = w$ and so $l(\theta(s)ws) = l(w)$. So let the other way around $l(\theta(s)ws) = l(w)$. **TODO** □

Lemma 2.8. *It is $l(ws) < l(w)$ iff $l(w\underline{s}) < l(w)$.*

Proof. Suppose s acts onesided on w. Then $w\underline{s} = ws$ and there is nothing to prove. So suppose s acts bothsided on w. If $l(ws) < l(w)$, then Lemma 1.6 yields $l(ws) + 1 = l(w)$. Assuming $l(w\underline{s}) = l(\theta(s)ws) = l(w)$ would imply, that s acts onesided on w due to Lemma 2.7, which is a contradiction. So $l(w\underline{s}) = l(\theta(s)ws) < l(w)$. The other way around suppose $l(w\underline{s}) < l(w)$. Then Lemma 1.6 says $l(w\underline{s}) = l(\theta(s)ws) = l(w) - 2$ and so $l(ws) = l(w) - 1$. □

Lemma 2.9. *For all $w \in W$ and $s \in S$ it is $w\underline{s}\underline{s} = w$.*

Proof. For $w\underline{s}$ there are two cases. Suppose s acts onesided on w, i.e. $\theta(s)ws = w$. For $ws\underline{s}$ there are again two possible options:

$$ws\underline{s} = \begin{cases} wss = w & \text{if } \theta(s)wss = ws, \\ \theta(s)wss = ws & \text{else.} \end{cases}$$

The second option contradicts itself.

Now suppose s acts bothsided on w. This means $\theta(s)ws \neq w$ and for $(\theta(s)ws)\underline{s}$ there are again two possible options:

$$(\theta(s)ws)\underline{s} = \begin{cases} \theta(s)wss = \theta(s)w & \text{if } \theta(s)\theta(s)wss = \theta(s)ws, \\ \theta(s)\theta(s)wss = w & \text{else.} \end{cases}$$

The first option is impossible since $\theta(s)\theta(s)wss = w$ and we have assumed $\theta(s)ws \neq w$. Hence the only possible cases yield $w\underline{ss} = w$. □

Remark 2.10. This lemma allows us to to rewrite equations of twisted expressions. For example

$$u = w\underline{s} \iff u\underline{s} = w\underline{ss} = w.$$

This can be iterated to get

$$u = w\underline{s}_1 \dots \underline{s}_k \iff u\underline{s}_k \dots \underline{s}_1 = w.$$

Lemma 2.11. *For all θ, $w \in W$ and $s \in S$ it is $w \in \mathcal{I}_\theta$ iff $w\underline{s} \in \mathcal{I}_\theta$.*

Proof. Let $w \in \mathcal{I}_\theta$. For $w\underline{s}$ there are two cases. Suppose s acts onesided on w. Then we get

$$\theta(ws) = \theta(\theta(s)wss) = \theta^2(s)\theta(w) = sw^{-1} = (ws^{-1})^{-1} = (ws)^{-1}.$$

Suppose s acts bothsided on w. Then we get

$$\theta(\theta(s)ws) = \theta^2(s)\theta(w)\theta(s) = sw^{-1}\theta(s) = (\theta^{-1}(s)ws^{-1})^{-1} = (\theta(s)ws)^{-1}.$$

In both cases $w\underline{s} \in \mathcal{I}_\theta$.

Now let $w\underline{s} \in \mathcal{I}_\theta$. Suppose s acts onesided on w. Then

$$\theta(w) = \theta(\theta(s)ws) = \theta^2(s)\theta(ws) = s(ws)^{-1} = ss^{-1}w^{-1} = w^{-1}.$$

Suppose s acts twosided on w. Then

$$\begin{aligned} \theta(w) &= \theta(\theta(s)\theta(s)wss) = \theta^2(s)\theta(\theta(s)ws)\theta(s) \\ &= s(\theta(s)ws)^{-1}\theta(s) = s(s^{-1}w^{-1}\theta(s^{-1})\theta(s) = w^{-1}. \end{aligned}$$

In both cases $w \in \mathcal{I}_\theta$. □

A remarkable property of the action from Definition 2.5 is its e-orbit. As the following lemma shows, it coincides with $\mathcal{I}_\theta$.

Lemma 2.12. *Fix θ. Then the set of twisted involutions regarding θ coincides with the set of all twisted expressions regarding θ.*

Proof. As already seen in Lemma 2.11, each twisted expression is in $\mathcal{I}_\theta$, since $e \in \mathcal{I}_\theta$. So let $w \in \mathcal{I}_\theta$. If $l(w) = 0$, then $w = e \in \mathcal{I}_\theta$. Induce on the length of w and let $l(w) = r > 0$. Suppose w has a twisted expression ending with $\underline{s}$. Then w also has a reduced expression (in S) ending with s and so $l(ws) < l(w)$. With Lemma 2.8 we get $l(w\underline{s}) < l(w)$. By induction $w\underline{s}$ has a twisted expression and hence $w = (w\underline{s})\underline{s}$ has one, too. □

In the same way, we can use regular expressions to define the length of an element $w \in W$, we can use the twisted expressions to define the twisted length of an element $w \in \mathcal{I}_\theta$.

Definition 2.13. Let $\mathcal{I}_\theta$ be the set of twisted involutions. Then we define $\rho(w)$ as the smallest $k \in \mathbb{N}$ for that a twisted expression $w = \underline{s}_1 \dots \underline{s}_k$ exists. This is called the ***twisted length*** of w.

Lemma 2.14. *The Bruhat ordering, restricted to the set of twisted involutions $\mathcal{I}_\theta$, is a graded poset with ρ as rank function. We denote this poset with* $\mathrm{Br}(\mathcal{I}_\theta)$.

Proof. See [2, Theorem 4.8]. □

We now establish many properties from Section 1 for twisted expressions and $\mathrm{Br}(\mathcal{I}_\theta)$. As seen in Example 2.4 it is $\mathrm{Br}(W) \cong \mathrm{Br}(\mathcal{I}_\theta)$. So the hope, that many properties can be transfered, is eligible.

Lemma 2.15. *Let $w \in \mathcal{I}_\theta$ and $s \in S$. Then $\rho(w\underline{s}) = \rho(w) \pm 1$. In fact it is $\rho(w\underline{s}) = \rho(w) - 1$ iff $s \in D_R(w)$.*

Proof. Since $\mathrm{Br}(\mathcal{I}_\theta)$ is graded with rank function ρ and either $w\underline{s}$ covers w or w covers $w\underline{s}$ it is $\rho(w\underline{s}) = \rho(w) \pm 1$. Now suppose $w\underline{s} < w$. Then $l(w\underline{s}) < l(w)$ and with Lemma 2.8 we have $l(ws) < l(w)$ yielding $s \in D_R(w)$. The other way around suppose $w\underline{s} > w$. Then $l(w\underline{s}) > l(w)$ and again with Lemma 2.8 we have $l(ws) > l(w)$ yielding $s \notin D_R(w)$. □

Lemma 2.16 (Lifting property for $\underline{S}$)**.** *Let $v, w \in W$ with $v \leq w$. Suppose $s \in S$ with $s \in D_R(w)$. Then*

1. $v\underline{s} \leq w$,
2. $s \in D_R(v) \Rightarrow v\underline{s} \leq w\underline{s}$.

Proof. We distinguish between the four cases of one- and bothsided action of s on u and w. Whenever a relation comes from the ordinary Lifting Property, we denote it with $<_{LP}$ in this proof.

$v\underline{s} = vs \wedge w\underline{s} = ws$ Same situation as in Lifting Property.

$v\underline{s} = vs \wedge w\underline{s} = \theta(s)ws$ The first part $v\underline{s} = vs \leq_{LP} w$ is immediate. Suppose $s \in D_R(v)$. Then $vs \leq_{LP} ws \Rightarrow v = \theta(s)vs \leq ws \Rightarrow v\underline{s} = vs \leq \theta(s)ws = w\underline{s}$.

$v\underline{s} = \theta(s)vs \wedge w\underline{s} = ws$ **TODO**

$v\underline{s} = \theta(s)vs \wedge w\underline{s} = \theta(s)ws$ **TODO** □

Proposition 2.17 (Exchange property for twisted expressions)**.** ***TODO***

Proposition 2.18 (Deletion property for twisted expressions)**.** ***TODO***

TODO

2.2 Twisted weak ordering

In this section we introduce the twisted weak ordering $Wk(\theta)$ on the set $\mathcal{I}_\theta$ of θ-twisted involutions.

Definition 2.19. Let (W, S) be a Coxeter system and let $\mathcal{I}_\theta$ be the set of θ-twisted involutions in W. For $v, w \in \mathcal{I}_\theta$ we define $v \preceq w$ iff there are $\underline{s}_1, \ldots, \underline{s}_k \in \underline{S}$ with $w = v\underline{s}_1 \ldots \underline{s}_k$ and $\rho(v) = \rho(w) - k$. We call the poset $(\mathcal{I}_\theta, \preceq)$ ***twisted weak ordering***, denoted with $Wk(W, \theta)$. When the Coxeter group W is clear from the context, we just write $Wk(\theta)$.

Lemma 2.20. *The poset $Wk(\theta)$ is a graded poset with rank function ρ.*

Proof. Follows immediately from the definition of $\preceq$. □

Example 2.21. In Figure 2.1 we see Hasse diagram of $Wk(A_4, \mathrm{id})$. Solid edges represent bothsided actions and dashed edges represent onesided actions.

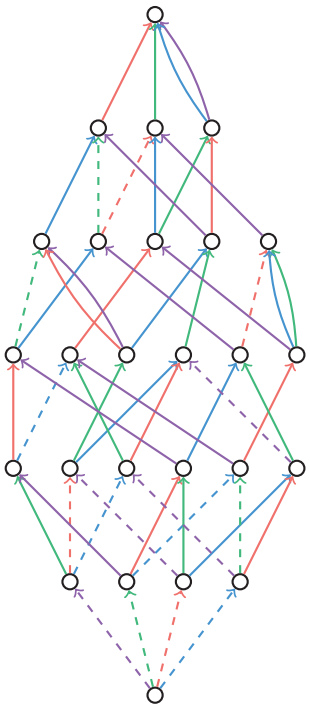

Figure 2.1: Hasse diagram of $Wk(A_4, \mathrm{id})$

Definition 2.22. Let $v, w \in W$ with $\rho(w) - \rho(v) = n$. A sequence $v = w_0 \prec w_1 \prec \ldots \prec w_n = w$ is called a ***geodesic*** from v to w.

TODO

2.3 Residuums

Definition 2.23. Let $w \in W$ and $I \subseteq S$ be a subset of generators. Then we define

$$wC_I := \{w\underline{s}_1 \dots \underline{s}_k : k \in \mathbb{N}_0, s_i \in S\}$$

as the I-***residuum*** of w or just ***residuum***. To emphasize the size of I, say $|I| = n$, we also speak of a ***rank-n-residuum***.

Example 2.24. Let $w \in W$. Then $wC_\emptyset = \{w\}$ and $wC_S = \mathcal{I}_\theta$.

Lemma 2.25. *Let $w \in W$ and $I \subset S$. If $v \in wC_I$, then $vC_I = wC_I$.*

Proof. Suppose $v \in wC_I$. Then $v = w\underline{s}_1 \dots \underline{s}_n$ for some $s_i \in I$. Suppose $u = w\underline{t}_1 \dots \underline{t}_m \in wC_I$ is any other element in wC_I with $t_i \in I$. Then

$$u = w\underline{t}_1 \dots \underline{t}_m = (v\underline{s}_n \dots \underline{s}_1)\underline{t}_1 \dots \underline{t}_m$$

and so $u \in vC_I$. This yields $wC_I \subset vC_I$. Since $w \in vC_I$ we can swap v and w to get the other inclusion. □

Corollary 2.26. *Let $v, w \in W$ and $I \subset S$. Then either $vC_I \cap wC_I = \emptyset$ or $vC_I = wC_I$.*

Proof. Immediately follows from Lemma 2.25. □

Proposition 2.27. *Let $w \in \mathcal{I}_\theta$, $I \subseteq S$ be a set of generators. Then there exists a unique element $w_0 \in wC_I$ with $w_0 \preceq w_0\underline{s}$ for all $s \in I$.*

Proof. Suppose there is no such element. Then for each $w \in wC_I$ we can find a $s \in I$ with $w' = w\underline{s} \preceq w$ and $e' \in wC_I$. By repetition of Deletion property for twisted expressions we get, that $e \in wC_I$, but e has the property, which we assumed, that no element in wC_I has. Hence there must be at least one such element. Now suppose there are two distinct elements u, v with the desired property. Note that this means, that u and w have no reduced twisted expression ending with some $\underline{s} \in I$. Let v have a reduced twisted expression $v = \underline{s}_1 \dots \underline{s}_k$. Since u and v are both in wC_I there must be a twisted v-expression for u

$$u = v\underline{s}_{k+1} \dots \underline{s}_{k+l} = \underline{s}_1 \dots \underline{s}_{k+l}$$

with $s_n \in I$ for $k+1 \leq n \leq k+l$. This twisted expression cannot be reduced, since it ends with $\underline{s}_{k+l} \in I$. Then Deletion property for twisted expressions yields that this twisted expression contains a reduced twisted subexpression for u. It cannot end with $\underline{s}_n$ for $k+1 \leq n \leq k+l$. Hence, it is a twisted subexpression of $\underline{s}_1 \dots \underline{s}_k = v$, too. So $u \leq v$ by Theorem 1.23. Because of symmetry it is also $v \leq u$ and so $u = v$, contradicting to our assumption $u \neq v$. □

Corollary 2.28. *Let $w \in \mathcal{I}_\theta$, $I \subseteq S$ be a set of generators and let $\rho_{min} := \{\rho(v) : v \in wC_I\}$ be the minimal twisted length within the residuum wC_I. Then there is a unique element $w_{min} \in wC_I$ with $\rho(w_{min}) = \rho_{min}$. We denote this element with* $\min(w, I)$.

Proof. The minimal rank ρ_{min} exists, since $\rho : \mathcal{I}_\theta \to \mathbb{N}_0$, $\mathbb{N}_0$ is well-ordered and $wC_I \neq \emptyset$. Suppose we have an element w_{min} with $\rho(w_{min}) = \rho_{min}$. This means, that in particular all $w_{min}\underline{s}$ with $s \in I$ must be of larger twisted length, i.e. $w_{min} \prec w_{min}\underline{s}$ for all $s \in I$. With Proposition 2.27 this element must be unique. □

We proceed with some properties of rank-2-residuums. Our interest in these residuums steems from the fact, that their properties are needed later in Section 2.4 to construct an effective algorithm for calculating the twisted weak ordering, i.e. calculating the Hasse diagram of $Wk(\theta)$ for arbitrary Coxeter systems (W, S) and Coxeter system automorphisms θ.

Definition 2.29. Let $s, t \in S$ be two distinct generators. We define:

$$[\underline{st}]^n := \begin{cases} (\underline{st})^{\frac{n}{2}} & n \text{ even}, \\ (\underline{st})^{\frac{n-1}{2}}\underline{s} & n \text{ odd}. \end{cases}$$

This definition allows us rewrite rank-2-residuums. Suppose we have an element $w \in \mathcal{I}_\theta$ and two distinct generators $s, t \in S$. Thanks to Lemma 2.25 and Corollary 2.28 we can assume, that $w = min(w, \{s, t\})$. Then

$$wC_{\{s,t\}} = \{w\} \cup \{w[\underline{st}]^n : n \in \mathbb{N}\} \cup \{w[\underline{ts}]^n : n \in \mathbb{N}\}.$$

This encourages the following definition.

Definition 2.30. Let $w \in \mathcal{I}_\theta$ and let $s, t \in S$ be two distinct generators. Suppose $w = min(w, \{s, t\})$. Then we call $\{w[\underline{st}]^n : n \in \mathbb{N}\}$ the s-***branch*** and $\{w[\underline{ts}]^n : n \in \mathbb{N}\}$ the t-***branch*** of $wC_{\{s,t\}}$.

One question arises immediately: Are the s- and the t-branch disjoint? With the following propositions, corollaries and lemmas we will get a much better idea of the structure of rank-2-residuums and answer this question.

Proposition 2.31. *Let $w \in W$ and let $s, t \in S$ be two distinct generators. Without loss of generality suppose $w = \min(w, \{s, t\})$. If there is a $v \in wC_{\{s,t\}}$ with $v\underline{s} \prec v$ and $v\underline{t} \prec v$, then it is the unique element with this property forcing $wC_{\{s,t\}}$ to consist of two geodesics from w to v intersecting only in these two elements. Else, the s- and t-branch are disjoint, strictly ascending in twisted length and of infinite size.*

Proof. Suppose there is a v in the s-branch with $v\underline{s} \prec v$ and $v\underline{t} \prec v$, say $v = w[\underline{st}]^n$. Then $w[\underline{st}]^{m+1} \prec w[\underline{st}]^m$ for $n \leq m \leq 2m - 1$ and $w[\underline{st}]^{2n} = w$. This is because assuming the opposite would yield an element $u \neq w$ with $u \prec u\underline{s}$ and $u \prec u\underline{t}$ contradicting to the uniqueness from Proposition 2.27. If no such v exists, then the s- and t-branch must be disjoint, strictly ascending in twisted length and so of infinite size. □

Proposition 2.32. *Let $w \in S$ and $s, t \in S$ be two distinct generators. If s operates onesided on w and $w\underline{s} \prec w$, then either $w\underline{st} \prec w\underline{s}$ or $w\underline{t} \succ w$.*

Proof. We have $\theta(s)ws = w$ and $s \in D_R(w)$. If $t \notin D_R(w)$, then we are done. In return suppose $t \in D_R(w)$. This means $w\underline{s} \leq w$ and $w\underline{t} \leq w$ and [3, Lemma 3.9] yields $w\underline{st} < w$ and $w\underline{ts} < w$. If $t \in D_R(w\underline{s})$, then we are done. Conservely suppose $t \notin D_R(w\underline{s})$. Then $t \in D_R(w\underline{st})$. Together with $w\underline{st} \leq w$ [3, Lemma 3.9(2)] says $(w\underline{st})\underline{t} \leq w\underline{t}$. Finally we get

$$ws = w\underline{s} = (w\underline{st})\underline{t} \leq w\underline{t} = wt.$$

Since $w\underline{s}$ and $w\underline{t}$ are of same twisted length they have to be equal and therefore $s = t$ which contradicts to our assumption of two distinct generators s and t. Now we have $w\underline{st} < w\underline{s}$ or $w\underline{t} > w$ (in $\mathrm{Br}(\mathcal{I}_\theta)$). Both cases can be transfered to the twisted weak ordering. If $w\underline{st} < w\underline{s}$, then $\rho(w\underline{st}) = \rho(w\underline{s}) - 1$ and $w\underline{stt} = w\underline{s}$, yielding $w\underline{st} \prec w\underline{s}$. If $w\underline{t} > w$ then $\rho(w\underline{t}) = \rho(w) + 1$, yielding $w\underline{t} \succ w$. □

Corollary 2.33. *Let $w \in S$ and let $s, t \in S$ be two distinct generators. If w is neither the unique element in $wC_{\{s,t\}}$ of smallest twisted length,i.e. $\min(w, \{s, t\})$, nor the unique (but not necessarily existing) element of largest twisted length, then both s and t act twosided on w.*

Proof. Follows immediately from Proposition 2.32. □

Lemma 2.34. *Let $s, t \in S$ be two distinct generators and $w \in S$ with $w = min(w, \{s, t\})$. Suppose $v \in wC_{\{s,t\}}$ with $v\underline{s} \prec v$ and $v\underline{t} \prec v$. Then the one- and bothsided actions are distributed axisymmetrically or pointsymmetrically like in Figure 2.2.*

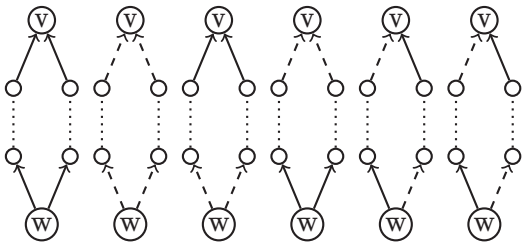

Figure 2.2: Possible distributions of one- and bothsided actions in a rank-2-residuum

Proof. If u covers w, then there are only two edges and the assumption holds. So suppose $wC_{\{s,t\}}$ contains at least four edges. Due to Corollary 2.33 the onesided actions can only occure next to w and v. Hence there are $2^4 = 16$ configurations possible. There are two geodesics from w to v. Suppose they have a different count of onesided actions, say the first has n and the second m onesided actions, and $\rho(v) - \rho(w) = k$. Then

$$l(v) = l(w) + n + 2(k - n) \neq l(w) + m + 2(k - m) = l(v),$$

which is obviously a contradiction. This wipes out ten out of the 16 configurations. The remaining are those from Figure 2.2. □

Lemma 2.35. *Let $w \in S$, $s, t \in S$ be two distinct generators and $m = \mathrm{ord}(st) < \infty$. Then $|wC_{\{s,t\}}| \leq 2m$.*

Proof. **TODO** □

Example 2.36. In Figure 2.3 we see two Hasse diagrams of $Wk(A_4, \mathrm{id})$. The first only contains edges with label s_1, s_3 and the second only edges with label s_2, s_4.

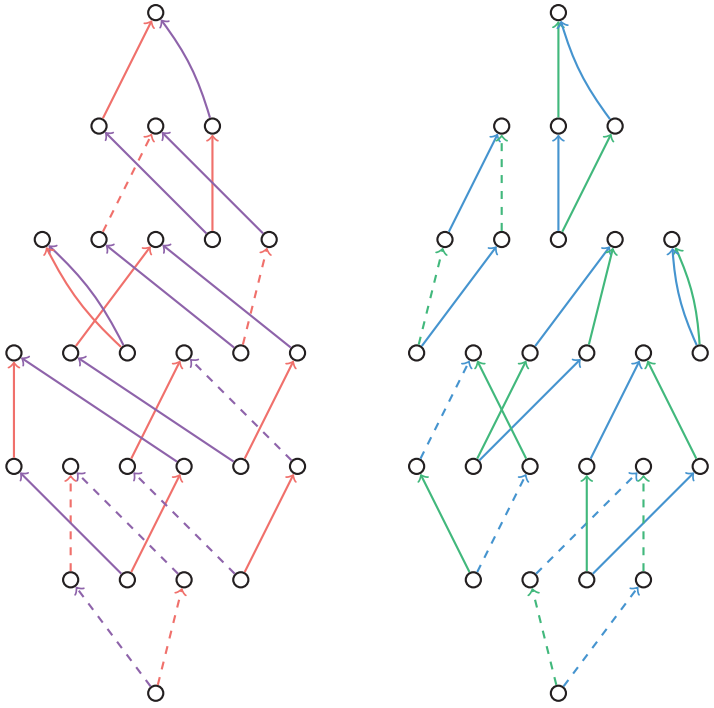

Figure 2.3: Hasse diagrams of $Wk(A_4, \mathrm{id})$ with only s_1, s_3 edges on the left and only s_2, s_4 edges on the right side

2.4 Twisted weak ordering algorithms

Now we address the problem of calculating $Wk(\theta)$ for an arbitrary Coxeter group W, given in form of a set of generating symbols $S = \{s_1, \dots s_n\}$ and the relations in form of $m_{ij} = \mathrm{ord}(s_i s_j)$. From this input we want to calculate the Hasse diagram, i.e. the vertex set $\mathcal{I}_\theta$ and the edges labeled with $\underline{s}$. Thanks to Lemma 2.12 the vertex set can be received by walking the e-orbit of the action from Definition 2.5. The only element of twisted length 0 is e. Suppose we have already calculated the Hasse diagram until the twisted length k, i.e. we know all vertices $w \in \mathcal{I}_\theta$ with $\rho(w) \leq k$ and all edges connecting two vertices u, v with $\rho(u) + 1 = \rho(v) \leq k$. Let $\rho_k := \{w \in \mathcal{I}_\theta : \rho(w) = k\}$. Then all vertices in ρ_{k+1} are of the form $w\underline{s}$ for some $w \in \rho_k, s \in S$. For each $(w, s) \in \rho_k \times S$, we calculate $w\underline{s}$. If $\rho(w\underline{s}) = k + 1$ then $w \prec w\underline{s}$. To avoid having to check the twisted length we use Lemma 2.15. We already know the set $S_w \subseteq S$ of all generators yielding an edge into w. Due to the lemma it is $\rho(w\underline{s}) = k - 1$ for all $s \in S_w$ and $\rho(w\underline{s}) = k + 1$ for all $s \in S \setminus S_w$. Hence we only calculate $w\underline{s}$ for $s \in S \setminus S_w$ and know $w \prec w\underline{s}$ without checking the twisted length explicitly. The last problem to solve is the possibility of two different $(w, s), (v, t) \in \rho_k \times S$ with $w\underline{s} = v\underline{t}$. To deal with this, we have to compare a potential new twisted involution

$w\underline{s}$ with each element of twisted length $k+1$, already calculated. The concrete problem of comparing two elements in a free presented group, called ***wordproblem for groups***, will not be addressed here. We suppose, that whatever computer system is used to implement our algorithm, supplies a suitable way to do that. The only thing to note is, that solving the wordproblem is not a cheap operation. Reducing the count of element comparisions is a major demand to any algorithm, calculating $Wk(\theta)$.

The steps discussed have been compiled in to an algorithm by Haas [1, Algorithm 3.1.1]. We take this as our starting point. Since the runtime is far from being optimal, we use the structural properties of rank-2-residuums from Section 2.3 to improve the algorithm. As we will show, these optimizations yield an algorithm with an asymptotical perfect runtime behavior. TWOA1 shows the algorithm of Haas.

Algorithm 2.37 (TWOA1)**.**

```
1:  procedure TwistedWeakOrderingAlgorithm1((W, S), k_max)
2:      V ← {(e, 0)}
3:      E ← {}
4:      for k ← 0 to k_max do
5:          for all (w, k_w) ∈ V with k_w = k do
6:              for all s ∈ S with ∄(·, w, s) ∈ E do          ▷ Only for s ∉ D_R(w)
7:                  y ← ws
8:                  z ← θ(s)y
9:                  if z = w then                  ▷ Check if s acts one- or bothsided on w
10:                     x ← y
11:                     t ← s
12:                 else
13:                     x ← z
14:                     t ← s̲
15:                 end if
16:                 isNew ← true
17:                 for all (w', k_w') ∈ V with k_w' = k + 1 do      ▷ Check if x already known
18:                     if x = w' then
19:                         isNew ← false
20:                     end if
21:                 end for
22:                 if isNew = true then
23:                     V ← V ∪ {(x, k + 1)}
24:                 end if
25:                 E ← E ∪ {(w, x, t)}
26:             end for
27:         end for
28:         k ← k + 1
29:     end for
```

30: **return** (V, E) ▷ The poset graph
31: **end procedure**

Note, that if W is finite, k_{max} does not have to be evaluated explicitly. When k reaches the maximal twisted length in $Wk(\theta)$, then the only vertex of twisted length k is the unique element $w_0 \in W$ of maximal ordinary length. Since $s \in D_R(w_0)$ for all $s \in S$, there is no $s' \in S$ remaining to calculate $w_0\underline{s}'$ for. This condition can be checked to determine the algorithm without knowing k_{max} before. When W is infinite, there is no maximal element and $\mathcal{I}_\theta$ is infinite, too. In this case k_{max} is used to terminate after having calculated a finite part of $Wk(\theta)$.

Lemma 2.38. *TWOA1 is a deterministic algorithm.*

Proof. The outer loop (line 4) is strictly ascending in $k \in \{0, \dots, k_{max}\}$ and so finite. The innermost loop (line 6) is finite since S is finite. The inner loop (line 5) is finite, since V starts as finite set and in each step there are added at most $|V| \cdot |S|$ many new vertices. Therefore the algorithm terminates. The soundness is due to the arguments at the beginning of Section 2.4. □

Lemma 2.39. *Let $k \in \mathbb{N}$, $n = |\{w \in \mathcal{I}_\theta : \rho(w) \leq k\}|$ and for $0 \leq i \leq k$ let $\rho_i = |\{w \in \mathcal{I}_\theta : \rho(w) = i\}|$. Then $TWOA1 \in \mathcal{O}(n^2/k)$.*

Proof. Our algorithm has to do at least $\rho_i(\rho_i - 1)/2$ many element comparisons (line 17) for each $0 \leq i \leq k$. Set $m = \lfloor \frac{n}{k} \rfloor$. In the most optimistic case it is $\rho_i \geq m$ for all i. In practice the situation will be worse, since some ρ_i will be smaller than m (for example $\rho_0 = 1$) and so some ρ_i will be much larger than m. This optimistic case yields at least $m(m-1)/2 \cdot k$ many element comparisons. Hence regarding the most delimiting operation, the element comparison, our algorithm is in $\Omega(m^2 k) = \Omega(n^2/k)$. The element comparison at line 9 done at most $n \cdot |S|$. Other operations, like for example insertion into or searching in sets can be considered super linear, if for example sets are ordered immediately at insertion and then searching is done with binary search. So the algorithm is in $\mathcal{O}(n^2/k)$. □

Corollary 2.40. *Let $k \in \mathbb{N}$ and $n = |\{w \in \mathcal{I}_\theta : \rho(w) \leq k\}|$. Then any algorithm calculating $Wk(\theta)$ is in $\Omega(n)$.*

Proof. Any algorithm must at least return $\{w \in \mathcal{I}_\theta : \rho(w) \leq k\}$ and this set is of size n. □

Our goal is to improve TWOA1 so that we get an algorithm in $\mathcal{O}(n)$, i.e. an asymptotical perfect algorithm for calculating $Wk(\theta)$. As already seen the element comparison of a potential new element with all already known elements of same twisted length (line 17) is the bottleneck. Here the rank-2-residuums become key. Suppose we have a $w \in \mathcal{I}_\theta$ with $\rho(w) = k$ and $s \in S$. In TWOA1 we would now check, if $w\underline{s}$ is a new vertex, or if we already calculated it by comparing it with all already known vertices of twisted length

$k+1$. Assume it is already known. This means there is another twisted involution v with $\rho(v) = k$ and another generator $t \in S$ with $v\underline{t} = w\underline{s}$. With Proposition 2.31 $w\underline{s}$ is the unique element of maximal twisted length in the rank-2-residuum $wC_{\{s,t\}}$. This yields a necessary condition for $w\underline{s}$ to be equal to a already known vertex, allowing us to replace the ineffective search all method in TWOA1, line 17.

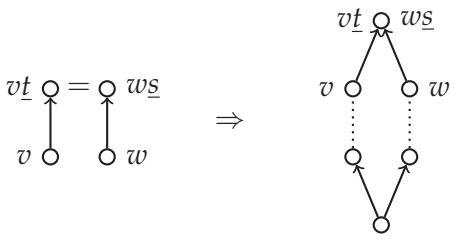

Figure 2.4: Optimization of TWOA1

Lemma 2.41. *Let $k \in \mathbb{N}$ and suppose we are in the sitation described at the beginning of Section 2.4. Let $\rho_i := \{w \in \mathcal{I}_\theta : \rho(w) = i\}$ and ρ'_{k+1} the set of the already calculated vertices with twisted length $k+1$. If $w\underline{s} \in \rho'_{k+1}$ for some $w \in \rho_k, s \in S$, say $w\underline{s} = v\underline{t}$ with $v \in \rho_k$ and $t \in S \setminus \{s\}$, then $w\underline{s} = w[\underline{ts}]^n$ for some $n \in \mathbb{N}$ with $w[\underline{ts}]^j \in \rho_0 \cup \ldots \cup \rho_k \cup \rho'_{k+1}$ for $1 \leq j \leq n$.*

Proof. The equality $w\underline{s} = w[\underline{ts}]^n$ for some $n \in \mathbb{N}$ is due to Proposition 2.31. All vertices in this rank-2-residuum except $v\underline{t}$ have a twisted length of k or lower. For $v\underline{t}$ we supposed it is already known, hence $v\underline{t} \in \rho'_{k+1}$. Therefore all vertices $w[\underline{ts}]^j$, $1 \leq j \leq n$ are in $\rho_0 \cup \ldots \cup \rho_k \cup \rho'_{k+1}$. □

This can be checked effectively. Both, w and s are fixed. Start with $M = \emptyset$. For all already known edges from or to w being labeled with $\underline{t} \in \underline{S} \setminus \{\underline{s}\}$ we do the following: Walk $w[\underline{ts}]^i$ for $i = 0, 1, \ldots$ until $\rho(w[\underline{ts}]^i) = k+1$. Note that walking in this case really means walking the graph. All involved vertices and edges have already been calculated. So there is no need for more calculations in W to find $w[\underline{ts}]^i$. By Proposition 2.31 such a path must exist (in a completely calculated graph). But we could be in the case, where the last step from $w[\underline{ts}]^{i-1}$ to $w[\underline{ts}]^i$ has not been calculated yet. If it is already calculated, then add this element to M by setting $M = M \cup \{w[\underline{ts}]^i\}$. If not, do not add it to M.

Now M contains all already known elements of twisted length $k+1$, satisfying the necessary condition from Lemma 2.41. Furthermore $|M| < |S|$. So for each pair (w, s) we have to do at most $|S| - 1$ many element comparisons the determine, if $w\underline{s}$ is new or already known, no matter how many elements of twisted length $k+1$ are already known. We can get even more information from the rank-2-residuums:

Lemma 2.42. *Let $w \in \mathcal{I}_\theta$ with $\rho(w) = k$, s, t be two distinct generators and $s \notin D_R(w)$. Suppose $n \in \mathbb{N}$ to be the smallest number for that $\rho(w[\underline{ts}]^{2n-1}) = k+1$ holds. Then:*

1. *If $n = \mathrm{ord}(st)$, then $w[\underline{ts}]^{2n-1} = w\underline{s}$.*
2. *If $n \geq 2$ and $l(w[\underline{ts}]^{2n-1}) - l(w[\underline{ts}]^{2n-2}) = 1$, then $w[\underline{ts}]^{2n-1} = w\underline{s}$.*

Proof. 1. Follows immediately from Lemma 2.35.

2. Because of the length difference the step from $w[\underline{ts}]^{2n-2}$ to $w[\underline{ts}]^{2n-1}$ is onesided and because of $n \geq 1$ this step cannot be next to the smallest element in $wC_{\{s,t\}}$. Hence $w[\underline{ts}]^{2n-1} = w\underline{s}$ by Corollary 2.33.

□

Lemma 2.43. *Let $w \in \mathcal{I}_\theta$ with $\rho(w) = k$, s, t be two distinct generators and $s \notin D_R(w)$. Suppose $w[\underline{ts}]^{2n-1} = w\underline{s}$ and suppose n to be the smallest number with this property. Then $w[\underline{ts}]^{n-1}$ is the minimal element $\min(w, \{s,t\})$ and $w[\underline{ts}]^{2n-1}$ is the maximal element. Define*

$$\begin{aligned} a &= l(w\underline{s}) - l(w) - 1, \\ b &= l(w[\underline{ts}]^{n-1}) - l(w[\underline{ts}]^{n-2}) - 1, \\ c &= l(w[\underline{ts}]^{n}) - l(w[\underline{ts}]^{n-1}) - 1 \text{ and} \\ d &= l(w[\underline{ts}]^{2n-1}) - l(w[\underline{ts}]^{2n-2}) - 1. \end{aligned}$$

Note that $a, b, c, d \in \{0, 1\}$ contain the information, if edges next to the minimal and the maximal element of $wC_{\{s,t\}}$ are one- or bothsided. Then each can be deduced from the three remaining ones with the equation $(a + b + c + d) \equiv 0 \mod 2$.

Proof. The minimality of $w[\underline{ts}]^{n-1}$ and the maximality of $w[\underline{ts}]^{2n-1}$ is due to Proposition 2.31. The soundness of the equation follows from the symmetric distribution of one- and bothsided actions from Lemma 2.34. □

Algorithm 2.44 (TWOA2).

```
1:  procedure TwistedWeakOrderingAlgorithm1((W, S), k_max)
2:      V ← {(e, 0)}
3:      E ← {}
4:      for k ← 0 to k_max do
5:          for all (w, k_w) ∈ V with k_w = k do
6:              for all s ∈ S with ∄(·, w, s) ∈ E do      ▷ Only for s ∉ D_R(w)      ▷ TODO
7:              end for
8:          end for
9:          k ← k + 1
10:     end for
11:     return (V, E)                                     ▷ The poset graph
12: end procedure
```

			Timings		Element comparisons	
W	$\lvert Wk(W, \mathrm{id})\rvert$	$\rho(w_0)$	TWOA1	TWOA2	TWOA1	TWOA2
A_9	9496	25	00:02.180	00:01.372	13,531,414	42,156
A_{10}	35696	30	00:31.442	00:06.276	185,791,174	173,356
A_{11}	140152	36	11:04.241	00:29.830	2,778,111,763	737,313
E_6	892	20	00:03.044	00:00.268	85,857	2,347
E_7	10208	35	06:11.728	00:02.840	7,785,186	29,687
E_8	199952	64	–	11:03.278	–	682,227

Table 2.1: Benchmark

3 Main Thesis

Question 3.1. *Let (W, S) be a Coxeter system, $\theta : W \to W$ an automorphism of W with $\theta^2 = \mathrm{id}$ and $\theta(S) = S$, and $K \subset S$ a subset of S generating a finite subgroup of W with $\theta(K) = K$. Futhermore let $T, S_1, S_2, S_3 \subset S$ be four pairwise disjoint sets of generators. For which Coxeter groups W does the implication*

$$w \in w_K C_{T \cup S_i}, i = 1, 2, 3 \Rightarrow w \in w_K C_T \tag{3.1.1}$$

hold for any possible $K, \theta, T, S_1, S_2, S_3$ and w?

Proposition 3.2. *Let (W, S) be a Coxeter system and K, T, S_1, S_2, S_3 be like in Question 3.1. Suppose we have $w \in W$ and $a_1, \dots, a_n \in T \cup S_1$, $b_1, \dots, b_n \in T \cup S_2$, $c_1, \dots, c_n \in T \cup S_3$ with*

$$\begin{aligned} w &= w_K \underline{a_1 \cdots a_n} \\ &= w_K \underline{b_1 \cdots b_n} \\ &= w_K \underline{c_1 \cdots c_n} \end{aligned}$$

and (3.1.1) does not hold for these three expressions, i.e. $w \notin w_K C_T$. Then there exist $t_1, \dots, t_m \in T$ and $a'_1, \dots, a'_{n-m} \in T \cup S_1$, $b'_1, \dots, b'_{n-m} \in T \cup S_2$, $c'_1, \dots, c'_{n-m} \in T \cup S_3$ such that

$$\begin{aligned} w\underline{t_1 \dots t_m} &= w_K \underline{a'_1 \cdots a'_{n-m}} \\ &= w_K \underline{b'_1 \cdots b'_{n-m}} \\ &= w_K \underline{c'_1 \cdots c'_{n-m}} \end{aligned}$$

with $a'_{n-m}, b'_{n-m}, c'_{n-m} \notin T$.

Proof. Suppose at least one element of a_n, b_n, c_n to be in T, for example $a_n \in T$. Then we can apply $\underline{a_n}$ to all three expressions. Since $\rho(w\underline{a_n}) < \rho(w)$ the Exchange property for twisted expressions yields

$$\begin{aligned} w\underline{a_n} &= w_K \underline{a_1 \cdots a_n a_n} = w_K \underline{a_1 \cdots a_{n-1}} \\ &= w_K \underline{b_1 \cdots b_n a_n} = w_K \underline{b_1 \cdots \hat{b}_i \cdots b_n} \\ &= w_K \underline{c_1 \cdots c_n a_n} = w_K \underline{c_1 \cdots \hat{c}_j \cdots c_n} \end{aligned}$$

where $\hat{\ }$ means omission. The omission cannot occur within w_K since all three expressions are still of same twisted length and in the first expression we can see, that $w_K \leq w\underline{a_n}$ still holds. This step can be repeated until $w = w_K$ or $a_n, b_n, c_n \notin T$. □

Lemma 3.3. *A counterexample to Question 3.1 can only exist, if there is an element $u \in wC_T$ and three distinct generators $s_1, s_2, s_3 \in D_R(u)$ such that $u\underline{s_i} \notin wC_T$ for $i = 1, 2, 3$.*

Proof. According to Proposition 3.2. □

A Source codes

```
Read("twistedinvolutionweakordering.gap");

CalculateTwistedWeakOrderings := function()
    local tasks, task, theta, W, matrix;

    tasks := [
        rec(system := CoxeterGroup_An(1), thetas := [ "id" ]),
        rec(system := CoxeterGroup_An(2), thetas := [ "id", "-id" ]),
        rec(system := CoxeterGroup_An(3), thetas := [ "id", "-id" ]),
        rec(system := CoxeterGroup_An(4), thetas := [ "id", "-id" ]),
        rec(system := CoxeterGroup_An(5), thetas := [ "id", "-id" ]),
        rec(system := CoxeterGroup_An(6), thetas := [ "id", "-id" ]),
        rec(system := CoxeterGroup_An(7), thetas := [ "id", "-id" ]),
        rec(system := CoxeterGroup_An(8), thetas := [ "id", "-id" ]),
        rec(system := CoxeterGroup_An(9), thetas := [ "id", "-id" ]),
        rec(system := CoxeterGroup_An(10), thetas := [ "id", "-id" ]),
#        rec(system := CoxeterGroup_An(11), thetas := [ "id", "-id" ]),
#        rec(system := CoxeterGroup_An(10), thetas := [ "id" ]),
#        rec(system := CoxeterGroup_An(12), thetas := [ "id", "-id" ]),
#        rec(system := CoxeterGroup_An(13), thetas := [ "id", "-id" ]),
        rec(system := CoxeterGroup_BCn(2), thetas := [ "id", "-id" ]),
        rec(system := CoxeterGroup_BCn(3), thetas := [ "id" ]),
#        rec(system := CoxeterGroup_BCn(4), thetas := [ "id" ]),
#        rec(system := CoxeterGroup_BCn(5), thetas := [ "id" ]),
#        rec(system := CoxeterGroup_BCn(6), thetas := [ "id" ]),
#        rec(system := CoxeterGroup_BCn(7), thetas := [ "id" ]),
        rec(system := CoxeterGroup_Dn(4), thetas := [ "id" ]),
        rec(system := CoxeterGroup_Dn(5), thetas := [ "id" ]),
        rec(system := CoxeterGroup_Dn(6), thetas := [ "id" ]),
        rec(system := CoxeterGroup_E6(), thetas := [ "id", [6,5,3,4,2,1] ]),
        rec(system := CoxeterGroup_E7(), thetas := [ "id" ]),
#        rec(system := CoxeterGroup_E8(), thetas := [ "id" ]),
        rec(system := CoxeterGroup_F4(), thetas := [ "id" ]),
        rec(system := CoxeterGroup_H3(), thetas := [ "id" ]),
        rec(system := CoxeterGroup_H4(), thetas := [ "id" ]),
#        rec(system := CoxeterGroup_I2m(3), thetas := [ "id", "-id" ]),
#        rec(system := CoxeterGroup_I2m(4), thetas := [ "id", "-id" ]),
#        rec(system := CoxeterGroup_I2m(5), thetas := [ "id", "-id" ]),
#        rec(system := CoxeterGroup_I2m(6), thetas := [ "id", "-id" ]),
    ];

    for task in tasks do
        W := task.system.group;
        matrix := task.system.matrix;

        for theta in List(task.thetas, t -> GroupAutomorphismByPermutation(W, t)) do
            Print(Name(W), " ", Name(theta), "\n");
            TwistedInvolutionWeakOrdering(StringToFilename(Concatenation(Name(W), "-",
                Name(theta))), W, matrix, theta);
        od;
    od;
end;

Benchmark := function ()
    local tasks, task, theta, W, matrix, benchmarks, b, result, startTime, endTime;

    tasks := [
        rec(system := CoxeterGroup_An(9), thetas := [ "id" ]),
```

```
58          #rec(system := CoxeterGroup_An(10), thetas := [ "id" ]),
59          #rec(system := CoxeterGroup_An(11), thetas := [ "id" ]),
60          rec(system := CoxeterGroup_E6(), thetas := [ "id" ]),
61          #rec(system := CoxeterGroup_E7(), thetas := [ "id" ]),
62      ];
63
64      benchmarks := [];
65
66      Print("Benchmark algo 1\n");
67
68      for task in tasks do
69          W := task.system.group;
70          matrix := task.system.matrix;
71
72          for theta in List(task.thetas, t -> GroupAutomorphismByPermutation(W, t)) do
73              Print(Name(W), " ", Name(theta), "\n");
74
75              startTime := Runtime();
76              result := TwistedInvolutionWeakOrdering1(fail, W, matrix, theta);
77              endTime := Runtime();
78
79              Add(benchmarks, rec(name := Name(W), algo := "TWOA1", time := StringTime(
                    endTime - startTime), result := result));
80          od;
81      od;
82
83      Print("Benchmark algo 2\n");
84
85      for task in tasks do
86          W := task.system.group;
87          matrix := task.system.matrix;
88
89          for theta in List(task.thetas, t -> GroupAutomorphismByPermutation(W, t)) do
90              Print(Name(W), " ", Name(theta), "\n");
91
92              startTime := Runtime();
93              result := TwistedInvolutionWeakOrdering(fail, W, matrix, theta);
94              endTime := Runtime();
95
96              Add(benchmarks, rec(name := Name(W), algo := "TWOA2", time := StringTime(
                    endTime - startTime), result := result));
97          od;
98      od;
99
100     for b in benchmarks do
101         Display(b);
102     od;
103 end;
104
105 TestCondition := function ()
106     local tasks, task, S, K, _T, T, K12, K23, K31, wK, parts, part, graph,
107         resS12, resS23, resS31, resT, resDiff, i, j;
108
109     tasks := [
110 #         "H_3-id",
111 #         "H_4-id",
112 #         "F_4-id",
113 #         "D__4_-id",
114 #         "D__5_-id",
115 #         "D__6_-id",
116 #         "E_6-id",
```

```
117 #         "E_7-id",
118           "E_8-id",
119 #         "A__4_-id",
120  #        "A__5_-id",
121  #        "A__6_-id",
122  #        "A__7_-id",
123  #        "BC__4_-id",
124  #        "BC__5_-id",
125  #        "BC__6_-id",
126  #        "A__8_-id",
127  #        "A__9_-id",
128  #        "A__10_-id",
129  #        "BC__7_-id",
130      ];
131
132      for task in tasks do
133          graph := TwistedInvolutionWeakOrderingPersistReadResults(task);
134          S := [1..graph.data.rank];
135          Print(graph.data.name, " ", graph.data.automorphism, "\n");
136          i := 0;
137
138          for K in IteratorOfCombinations(S) do
139              j := 0;
140              i := i + 1;
141
142              if Length(K) <= 2 or Length(K) = Length(S) then continue; fi;
143              parts := PartitionsSet(K, 3);
144              wK := TwistedInvolutionWeakOrderungLongestWord(graph.vertices[1], K);
145
146              for _T in IteratorOfCombinations(K) do
147                  j := j + 1;
148                  Print(i, " ", j, "                          \r");
149
150                  T := Union(_T, Difference(S, K));
151
152                  for part in parts do
153                      K12 := part[1];
154                      K23 := part[2];
155                      K31 := part[3];
156                      #Print("K=", K, " T=", T, " K12=", K12, " K23=", K23, " K31=", K31,
                             "\n");
157
158                      resS12 := TwistedInvolutionWeakOrderungResiduum(wK, Union(K12, T));
159                      resS23 := TwistedInvolutionWeakOrderungResiduum(wK, Union(K23, T));
160                      resS31 := TwistedInvolutionWeakOrderungResiduum(wK, Union(K31, T));
161                      resT := TwistedInvolutionWeakOrderungResiduum(wK, T);
162
163                      resDiff := Difference(Intersection(resS12, resS23, resS31), resT);
164
165                      if Length(resDiff) > 0 then
166                          Print("*** FOUND COUNTEREXAMPLE ***\n",
167                              "W = ", graph.name, "\n",
168                              "theta = ", graph.automorphis, "\n",
169                              "S = ", S, "\n",
170                              "K = ", K, "\n",
171                              "wK = ", wK, "\n",
172                              "T = ", T, "\n",
173                              "K12 = ", K12, "\n",
174                              "K23 = ", K23, "\n",
175                              "K31 = ", K31, "\n",
176                              "w = ", resDiff, "\n\n");
```

```
                    fi;
                od;
            od;
        od;
    od;
end;
```

```
LoadPackage("io");

Read("misc.gap");
Read("coxeter.gap");
Read("twistedinvolutionweakordering-persist.gap");

TwistedInvolutionDeduceNodeAndEdgeFromGraph := function(matrix, startNode, startLabel,
    labels)
    local rank, comb, trace, possibleEqualNodes, e, k, n;

    rank := -1/2 + Sqrt(1/4 + 2*Length(matrix)) + 1;
    possibleEqualNodes := [];

    for comb in List(Filtered(labels, label -> label <> startLabel), label -> rec(
        startNode := startNode, s := [startLabel, label], m := CoxeterMatrixEntry(
        matrix, rank, startLabel, label))) do
        trace := [];
        k := 1;
        n := comb.startNode;

        Add(trace, rec(node := n, edge := rec(label := comb.s[1], type := -1)));

        while k < comb.m do
            e := FindElement(n.inEdges, e -> e.label = comb.s[k mod 2 + 1]);
            if e = fail then break; fi;
            n := e.source;

            Add(trace, rec(node := n, edge := e));
            k := k + 1;
        od;

        while k > 0 do
            e := FindElement(n.outEdges, e -> e.label = comb.s[k mod 2 + 1]);
            if e = fail then break; fi;
            n := e.target;

            Add(trace, rec(node := n, edge := e));
            k := k - 1;
        od;

        if k <> 0 then continue; fi;

        if Length(trace) = 2*comb.m then
            return rec(result := 0, node := trace[Length(trace)].node, type := trace[
                comb.m + 1].edge.type, trace := trace);
        fi;

        if Length(trace) >= 4 then
            if trace[Length(trace) / 2 + 1].edge.type <> trace[Length(trace) / 2].edge.
                type then
                # cannot be equal
            else
                if trace[Length(trace)].edge.type = 0 then
                    return rec(result := 0, node := trace[Length(trace)].node, type :=
```

```
                        0, trace := trace);
50                  else
51                      Add(possibleEqualNodes, trace[Length(trace)].node);
52                  fi;
53              fi;
54          else
55              Add(possibleEqualNodes, trace[Length(trace)].node);
56          fi;
57      od;
58
59      return rec(result := -1, possibleEqualNodes := possibleEqualNodes);
60  end;
61
62  # Calculates the poset Wk(theta).
63  TwistedInvolutionWeakOrdering := function (filename, W, matrix, theta)
64      local persistInfo, maxOrder, nodes, edges, absNodeIndex, absEdgeIndex, prevNode,
            currNode, newEdge,
65          label, type, deduction, startTime, endTime, S, k, i, s, x, y, n;
66
67      persistInfo := TwistedInvolutionWeakOrderingPersistResultsInit(filename);
68
69      S := GeneratorsOfGroup(W);
70      maxOrder := Minimum([Maximum(Concatenation(matrix, [1])), 5]);
71      nodes := [ [], [ rec(element := One(W), twistedLength := 0, inEdges := [], outEdges
             := [], absIndex := 1) ] ];
72      edges := [ [], [] ];
73      absNodeIndex := 2;
74      absEdgeIndex := 1;
75      k := 0;
76
77      while Length(nodes[2]) > 0 do
78          if not IsFinite(W) then
79              if k > 200 or absNodeIndex > 10000 then
80                  break;
81              fi;
82          fi;
83
84          for i in [1..Length(nodes[2])] do
85              Print(k, " ", i, "         \r");
86
87              prevNode := nodes[2][i];
88              for label in Filtered([1..Length(S)], n -> Position(List(prevNode.inEdges,
                    e -> e.label), n) = fail) do
89                  deduction := TwistedInvolutionDeduceNodeAndEdgeFromGraph(matrix,
                        prevNode, label, [1..Length(S)]);
90
91                  if deduction.result = 0 then
92                      type := deduction.type;
93                      currNode := deduction.node;
94                  elif deduction.result = 1 then
95                      type := deduction.type;
96
97                      currNode := rec(element := y, twistedLength := k + 1, inEdges :=
                            [], outEdges := [], absIndex := absNodeIndex);
98                      Add(nodes[1], currNode);
99
100                     absNodeIndex := absNodeIndex + 1;
101                 else
102                     x := prevNode.element;
103                     s := S[label];
104
```

```
105                     type := 1;
106                     y := s^theta*x*s;
107                     if (CoxeterElementsCompare(x, y)) then
108                         y := x * s;
109                         type := 0;
110                     fi;
111
112                     currNode := FindElement(deduction.possibleEqualNodes, n ->
                            CoxeterElementsCompare(n.element, y));
113
114                     if currNode = fail then
115                         currNode := rec(element := y, twistedLength := k + 1, inEdges
                                := [], outEdges := [], absIndex := absNodeIndex);
116                         Add(nodes[1], currNode);
117
118                         absNodeIndex := absNodeIndex + 1;
119                     fi;
120                 fi;
121
122                 newEdge := rec(source := prevNode, target := currNode, label := label,
                        type := type, absIndex := absEdgeIndex);
123
124                 Add(edges[1], newEdge);
125                 Add(currNode.inEdges, newEdge);
126                 Add(prevNode.outEdges, newEdge);
127
128                 absEdgeIndex := absEdgeIndex + 1;
129             od;
130         od;
131
132         TwistedInvolutionWeakOrderingPersistResults(persistInfo, nodes[2], edges[2]);
133
134         Add(nodes, [], 1);
135         Add(edges, [], 1);
136         if (Length(nodes) > maxOrder + 1) then
137             for n in nodes[maxOrder + 2] do
138                 n.inEdges := [];
139                 n.outEdges := [];
140             od;
141             Remove(nodes, maxOrder + 2);
142             Remove(edges, maxOrder + 2);
143         fi;
144         k := k + 1;
145     od;
146
147     TwistedInvolutionWeakOrderingPersistResultsInfo(persistInfo, W, matrix, theta,
            absNodeIndex - 1, k - 1);
148     TwistedInvolutionWeakOrderingPersistResultsClose(persistInfo);
149
150     return rec(numNodes := absNodeIndex - 1, numEdges := absEdgeIndex - 1,
            maxTwistedLength := k - 1);
151 end;
152
153 # Calculates the poset Wk(theta).
154 TwistedInvolutionWeakOrdering1 := function (filename, W, matrix, theta)
155     local persistInfo, maxOrder, nodes, edges, absNodeIndex, absEdgeIndex, prevNode,
            currNode, newEdge,
156         label, type, deduction, startTime, endTime, S, k, i, s, x, y, n;
157
158     persistInfo := TwistedInvolutionWeakOrderingPersistResultsInit(filename);
159
```

```
160     S := GeneratorsOfGroup(W);
161     maxOrder := Minimum([Maximum(Concatenation(matrix, [1])), 5]);
162     nodes := [ [], [ rec(element := One(W), twistedLength := 0, inEdges := [], outEdges
            := [], absIndex := 1) ] ];
163     edges := [ [], [] ];
164     absNodeIndex := 2;
165     absEdgeIndex := 1;
166     k := 0;
167
168     while Length(nodes[2]) > 0 do
169         if not IsFinite(W) then
170             if k > 200 or absNodeIndex > 10000 then
171                 break;
172             fi;
173         fi;
174
175         for i in [1..Length(nodes[2])] do
176             Print(k, " ", i, "         \r");
177
178             prevNode := nodes[2][i];
179             for label in Filtered([1..Length(S)], n -> Position(List(prevNode.inEdges,
                    e -> e.label), n) = fail) do
180                 x := prevNode.element;
181                 s := S[label];
182
183                 type := 1;
184                 y := s^theta*x*s;
185                 if (CoxeterElementsCompare(x, y)) then
186                     y := x * s;
187                     type := 0;
188                 fi;
189
190                 currNode := FindElement(nodes[1], n -> CoxeterElementsCompare(n.element
                        , y));
191
192                 if currNode = fail then
193                     currNode := rec(element := y, twistedLength := k + 1, inEdges :=
                            [], outEdges := [], absIndex := absNodeIndex);
194                     Add(nodes[1], currNode);
195
196                     absNodeIndex := absNodeIndex + 1;
197                 fi;
198
199                 newEdge := rec(source := prevNode, target := currNode, label := label,
                        type := type, absIndex := absEdgeIndex);
200
201                 Add(edges[1], newEdge);
202                 Add(currNode.inEdges, newEdge);
203                 Add(prevNode.outEdges, newEdge);
204
205                 absEdgeIndex := absEdgeIndex + 1;
206             od;
207         od;
208
209         TwistedInvolutionWeakOrderingPersistResults(persistInfo, nodes[2], edges[2]);
210
211         Add(nodes, [], 1);
212         Add(edges, [], 1);
213         if (Length(nodes) > maxOrder + 1) then
214             for n in nodes[maxOrder + 2] do
215                 n.inEdges := [];
```

```
216                 n.outEdges := [];
217             od;
218             Remove(nodes, maxOrder + 2);
219             Remove(edges, maxOrder + 2);
220         fi;
221         k := k + 1;
222     od;
223
224     TwistedInvolutionWeakOrderingPersistResultsInfo(persistInfo, W, matrix, theta,
            absNodeIndex - 1, k - 1);
225     TwistedInvolutionWeakOrderingPersistResultsClose(persistInfo);
226
227     return rec(numNodes := absNodeIndex - 1, numEdges := absEdgeIndex - 1,
            maxTwistedLength := k - 1);
228 end;
229
230 TwistedInvolutionWeakOrderungResiduum := function (vertex, labels)
231     local visited, queue, residuum, current, edge;
232
233     visited := [ vertex ];
234     queue := [ vertex ];
235     residuum := [];
236
237     while Length(queue) > 0 do
238         current := queue[1];
239         Remove(queue, 1);
240         Add(residuum, current);
241
242         for edge in current.outEdges do
243             if edge.label in labels and not edge.target in visited then
244                 Add(visited, edge.target);
245                 Add(queue, edge.target);
246             fi;
247         od;
248     od;
249
250     return residuum;
251 end;
252
253 TwistedInvolutionWeakOrderungLongestWord := function (vertex, labels)
254     local current;
255
256     current := vertex;
257
258     while Length(Filtered(current.outEdges, e -> e.label in labels)) > 0 do
259         current := Filtered(current.outEdges, e -> e.label in labels)[1].target;
260     od;
261
262     return current;
263 end;
```

```
1  GroupAutomorphismByPermutation := function (G, generatorPermutation)
2      local automorphism, generators;
3
4      generators := GeneratorsOfGroup(G);
5
6      if generatorPermutation = "id" or generatorPermutation = [1..Length(generators)]
           then
7          automorphism := IdentityMapping(G);
8          SetName(automorphism, "id");
9
```

```
10          return automorphism;
11      elif generatorPermutation = "-id" then
12          generatorPermutation := Reversed([1..Length(GeneratorsOfGroup(G))]);
13      fi;
14
15      automorphism := GroupHomomorphismByImages(G, G, generators, generators{
            generatorPermutation});
16      SetName(automorphism, Concatenation("(", JoinStringsWithSeparator(
            generatorPermutation, ","), ")"));
17
18      return automorphism;
19  end;
20
21  GroupAutomorphismIdNeg := function (G)
22      return GroupAutomorphismByPermutation(G, Reversed([1..Length(GeneratorsOfGroup(G))
            ]));
23  end;
24
25  GroupAutomorphismId := function (G)
26      return GroupAutomorphismByPermutation(G, [1..Length(GeneratorsOfGroup(G))]);
27  end;
28
29  FindElement := function (list, selector)
30      local i;
31
32      for i in [1..Length(list)] do
33          if (selector(list[i])) then
34              return list[i];
35          fi;
36      od;
37
38      return fail;
39  end;
40
41  StringToFilename := function(str)
42      local result, c;
43
44      result := "";
45
46      for c in str do
47          if IsDigitChar(c) or IsAlphaChar(c) or c = '-' or c = '_' then
48              Add(result, c);
49          else
50              Add(result, '_');
51          fi;
52      od;
53
54      return result;
55  end;
56
57  IO_ReadLinesIterator := function (file)
58      local IsDone, Next, ShallowCopy;
59
60      IsDone := function (iter)
61          return iter!.nextLine = "" or iter!.nextLine = fail;
62      end;
63
64      Next := function (iter)
65          local line;
66
67          line := iter!.nextLine;
```

```
68
69          if line = fail then
70              Error(LastSystemError());
71              return fail;
72          fi;
73
74          iter!.nextLine := IO_ReadLine(iter!.file);
75
76          return Chomp(line);
77      end;
78
79      ShallowCopy := function (iter)
80          return fail;
81      end;
82
83      return IteratorByFunctions(rec(IsDoneIterator := IsDone, NextIterator := Next,
84          ShallowCopy := ShallowCopy, file := file, nextLine := IO_ReadLine(file)));
85  end;
86
87  IO_ReadLinesIteratorCSV := function (file, seperator)
88      local IsDone, Next, ShallowCopy;
89
90      IsDone := function (iter)
91          return iter!.nextLine = "" or iter!.nextLine = fail;
92      end;
93
94      Next := function (iter)
95          local line, lineSplitted, result, i;
96
97          line := iter!.nextLine;
98          if line = fail then
99              Error(LastSystemError());
100             return fail;
101         fi;
102         iter!.nextLine := IO_ReadLine(iter!.file);
103
104         lineSplitted := SplitString(Chomp(line), iter!.seperator);
105         result := rec();
106
107         for i in [1..Minimum(Length(iter!.headers), Length(lineSplitted))] do
108             result.(iter!.headers[i]) := EvalString(lineSplitted[i]);
109         od;
110
111         return result;
112     end;
113
114     ShallowCopy := function (iter)
115         return fail;
116     end;
117
118     return IteratorByFunctions(rec(IsDoneIterator := IsDone, NextIterator := Next,
119         ShallowCopy := ShallowCopy, file := file, seperator := seperator,
120         headers := SplitString(Chomp(IO_ReadLine(file)), seperator),
121         nextLine := IO_ReadLine(file)));
122 end;
```

```
1  Read("coxeter-generators.gap");
2
3  CoxeterElementsCompare := function (w1, w2)
4      return w1 = w2;
5  end;
```

```

CoxeterMatrixEntry := function(matrix, rank, i, j)
    local temp;

    if (i = j) then
        return 1;
    fi;

    if (i > j) then
        temp := i;
        i := j;
        j := temp;
    fi;

    return matrix[(rank-1)*(rank)/2 - (rank-i)*(rank-i+1)/2 + (j-i-1) + 1];
end;
```

```
# Generates a coxeter group with given rank and relations. The relations have to
# be given in a linear list of the upper right entries (above diagonal) of the
# coxeter matrix.
#
# Example:
# To generate the coxeter group A_4 with the following coxeter matrix:
#
# | 1 3 2 2 |
# | 3 1 3 2 |
# | 2 3 1 3 |
# | 2 2 3 1 |
#
# A4 := CoxeterGroup(4, [3,2,2, 3,2, 3]);
CoxeterGroup := function (rank, upperTriangleOfCoxeterMatrix)
    local generatorNames, relations, F, S, W, i, j, k;

    generatorNames := List([1..rank], n -> Concatenation("s", String(n)));

    F := FreeGroup(generatorNames);
    S := GeneratorsOfGroup(F);

    relations := [];

    Append(relations, List([1..rank], n -> S[n]^2));

    k := 1;
    for i in [1..rank] do
        for j in [i+1..rank] do
            Add(relations, (S[i]*S[j])^(upperTriangleOfCoxeterMatrix[k]));
            k := k + 1;
        od;
    od;

    W := F / relations;

    return W;
end;

CoxeterGroup_An := function (n)
    local upperTriangleOfCoxeterMatrix, W;

    upperTriangleOfCoxeterMatrix := Flat(List(Reversed([1..n-1]), m -> Concatenation
        ([3], List([1..m-1], o -> 2))));

```

```
44      #W := CoxeterGroup(n, upperTriangleOfCoxeterMatrix);
45      W := GroupWithGenerators(List([1..n], s -> (s,s+1)));
46
47      SetName(W, Concatenation("A_{", String(n), "}"));
48      SetSize(W, Factorial(n + 1));
49
50      return rec(group := W, rank := n, matrix := upperTriangleOfCoxeterMatrix);
51  end;
52
53  CoxeterGroup_BCn := function (n)
54      local upperTriangleOfCoxeterMatrix, W;
55
56      upperTriangleOfCoxeterMatrix := Flat(List(Reversed([1..n-1]), m -> Concatenation
            ([3], List([1..m-1], o -> 2))));
57      upperTriangleOfCoxeterMatrix[Length(upperTriangleOfCoxeterMatrix)] := 4;
58
59      W := CoxeterGroup(n, upperTriangleOfCoxeterMatrix);
60
61      SetName(W, Concatenation("BC_{", String(n), "}"));
62      SetSize(W, 2^n * Factorial(n));
63
64      return rec(group := W, rank := n, matrix := upperTriangleOfCoxeterMatrix);
65  end;
66
67  CoxeterGroup_Dn := function (n)
68      local upperTriangleOfCoxeterMatrix, W;
69
70      upperTriangleOfCoxeterMatrix := Flat(List(Reversed([1..n-1]), m -> Concatenation
            ([3], List([1..m-1], o -> 2))));
71      upperTriangleOfCoxeterMatrix[Length(upperTriangleOfCoxeterMatrix)] := 2;
72      upperTriangleOfCoxeterMatrix[Length(upperTriangleOfCoxeterMatrix) - 1] := 3;
73      upperTriangleOfCoxeterMatrix[Length(upperTriangleOfCoxeterMatrix) - 2] := 3;
74
75      W := CoxeterGroup(n, upperTriangleOfCoxeterMatrix);
76
77      SetName(W, Concatenation("D_{", String(n), "}"));
78      SetSize(W, 2^(n-1) * Factorial(n));
79
80      return rec(group := W, rank := n, matrix := upperTriangleOfCoxeterMatrix);
81  end;
82
83  CoxeterGroup_E6 := function ()
84      local upperTriangleOfCoxeterMatrix, W;
85
86      upperTriangleOfCoxeterMatrix := [3, 2, 2, 2, 2,  3, 2, 2, 2,  3, 3, 2,  2, 2,  3];
87
88      W := CoxeterGroup(6, upperTriangleOfCoxeterMatrix);
89
90      SetName(W, "E_6");
91      SetSize(W, 2^7 * 3^4 * 5);
92
93      return rec(group := W, rank := 6, matrix := upperTriangleOfCoxeterMatrix);
94  end;
95
96  CoxeterGroup_E7 := function ()
97      local upperTriangleOfCoxeterMatrix, W;
98
99      upperTriangleOfCoxeterMatrix := [3, 2, 2, 2, 2, 2,  3, 2, 2, 2, 2,  3, 3, 2, 2,  2,
            2, 2,  3, 2,  3];
100
101     W := CoxeterGroup(7, upperTriangleOfCoxeterMatrix);
```

```
102
103     SetName(W, "E_7");
104     SetSize(W, 2^10 * 3^4 * 5 * 7);
105
106     return rec(group := W, rank := 7, matrix := upperTriangleOfCoxeterMatrix);
107 end;
108
109 CoxeterGroup_E8 := function ()
110     local upperTriangleOfCoxeterMatrix, W;
111
112     upperTriangleOfCoxeterMatrix := [3, 2, 2, 2, 2, 2, 2,  3, 2, 2, 2, 2, 2,  3, 3, 2,
            2, 2,  2, 2, 2, 2,  3, 2, 2,  3, 2,  3];
113
114     W := CoxeterGroup(8, upperTriangleOfCoxeterMatrix);
115
116     SetName(W, "E_8");
117     SetSize(W, 2^14 * 3^5 * 5^2 * 7);
118
119     return rec(group := W, rank := 8, matrix := upperTriangleOfCoxeterMatrix);
120 end;
121
122 CoxeterGroup_F4 := function ()
123     local upperTriangleOfCoxeterMatrix, W;
124
125     upperTriangleOfCoxeterMatrix := [3, 2, 2, 4, 2, 3];
126
127     W := CoxeterGroup(4, upperTriangleOfCoxeterMatrix);
128
129     SetName(W, "F_4");
130     SetSize(W, 2^7 * 3^2);
131
132     return rec(group := W, rank := 4, matrix := upperTriangleOfCoxeterMatrix);
133 end;
134
135 CoxeterGroup_H3 := function ()
136     local upperTriangleOfCoxeterMatrix, W;
137
138     upperTriangleOfCoxeterMatrix := [5, 2, 3];
139
140     W := CoxeterGroup(3, upperTriangleOfCoxeterMatrix);
141
142     SetName(W, "H_3");
143     SetSize(W, 120);
144
145     return rec(group := W, rank := 3, matrix := upperTriangleOfCoxeterMatrix);
146 end;
147
148 CoxeterGroup_H4 := function ()
149     local upperTriangleOfCoxeterMatrix, W;
150
151     upperTriangleOfCoxeterMatrix := [5, 2, 2, 3, 2, 3];
152
153     W := CoxeterGroup(4, upperTriangleOfCoxeterMatrix);
154
155     SetName(W, "H_4");
156     SetSize(W, 14400);
157
158     return rec(group := W, rank := 4, matrix := upperTriangleOfCoxeterMatrix);
159 end;
160
161 CoxeterGroup_I2m := function (m)
```

```
    local upperTriangleOfCoxeterMatrix, W;

    upperTriangleOfCoxeterMatrix := [m];

    W := CoxeterGroup(2, upperTriangleOfCoxeterMatrix);

    SetName(W, Concatenation("I_2(", String(m), ")"));
    SetSize(W, 2*m);

    return rec(group := W, rank := 2, matrix := upperTriangleOfCoxeterMatrix);
end;

CoxeterGroup_TildeAn := function (n)
    local upperTriangleOfCoxeterMatrix, W;

    upperTriangleOfCoxeterMatrix := Flat(List(Reversed([1..n]), m -> Concatenation([3],
         List([1..m-1], o -> 2))));

    if n = 1 then
        upperTriangleOfCoxeterMatrix[1] := 0;
    else
        upperTriangleOfCoxeterMatrix[n] := 3;
    fi;

    W := CoxeterGroup(n + 1, upperTriangleOfCoxeterMatrix);

    SetName(W, Concatenation("\\tilde A_{", String(n), "}"));
    SetSize(W, infinity);

    return rec(group := W, rank := n + 1, matrix := upperTriangleOfCoxeterMatrix);
end;
```

```
TwistedInvolutionWeakOrderingPersistReadResults := function(filename)
    local fileD, fileV, fileE, csvLine, data, vertices, edges, newEdge, source, target,
         i;

    fileD := IO_File(Concatenation("results/", filename, "-data"), "r");
    fileV := IO_File(Concatenation("results/", filename, "-vertices"), "r", 1024*1024);
    fileE := IO_File(Concatenation("results/", filename, "-edges"), "r", 1024*1024);

    data := NextIterator(IO_ReadLinesIteratorCSV(fileD, ";"));
    vertices := [];
    edges := [];

    i := 1;
    for csvLine in IO_ReadLinesIteratorCSV(fileV, ";") do
        Add(vertices, rec(absIndex := i, twistedLength := csvLine.twistedLength, name
            := csvLine.name, inEdges := [], outEdges := []));
        i := i + 1;
    od;

    i := 1;
    for csvLine in IO_ReadLinesIteratorCSV(fileE, ";") do
        source := vertices[csvLine.sourceIndex + 1];
        target := vertices[csvLine.targetIndex + 1];
        newEdge := rec(absIndex := i, source := source, target := target, label :=
            csvLine.label, type := csvLine.type);

        Add(source.outEdges, newEdge);
        Add(target.inEdges, newEdge);
        Add(edges, newEdge);
```

```
27          i := i + 1;
28      od;
29
30      IO_Close(fileD);
31      IO_Close(fileV);
32      IO_Close(fileE);
33
34      return rec(data := data, vertices := vertices, edges := edges);
35  end;
36
37  TwistedInvolutionWeakOrderingPersistResultsInit := function(filename)
38      local fileD, fileV, fileE;
39
40      if (filename = fail) then return fail; fi;
41
42      fileD := IO_File(Concatenation("results/", filename, "-data"), "w");
43      fileV := IO_File(Concatenation("results/", filename, "-vertices"), "w", 1024*1024);
44      fileE := IO_File(Concatenation("results/", filename, "-edges"), "w", 1024*1024);
45      IO_Write(fileD, "name;rank;size;generators;matrix;automorphism;wk_size;
            wk_max_length\n");
46      IO_Write(fileV, "twistedLength;name\n");
47      IO_Write(fileE, "sourceIndex;targetIndex;label;type\n");
48
49      return rec(fileD := fileD, fileV := fileV, fileE := fileE);
50  end;
51
52  TwistedInvolutionWeakOrderingPersistResultsClose := function(persistInfo)
53      if (persistInfo = fail) then return; fi;
54
55      IO_Close(persistInfo.fileD);
56      IO_Close(persistInfo.fileV);
57      IO_Close(persistInfo.fileE);
58  end;
59
60  TwistedInvolutionWeakOrderingPersistResultsInfo := function(persistInfo, W, matrix,
        theta, numNodes, maxTwistedLength)
61      if (persistInfo = fail) then return; fi;
62
63      IO_Write(persistInfo.fileD, "\"", ReplacedString(Name(W), "\\", "\\\\"), "\";");
64      IO_Write(persistInfo.fileD, Length(GeneratorsOfGroup(W)), ";");
65      if (Size(W) = infinity) then
66          IO_Write(persistInfo.fileD, "\"infinity\";");
67      else
68          IO_Write(persistInfo.fileD, Size(W), ";");
69      fi;
70      IO_Write(persistInfo.fileD, "[", JoinStringsWithSeparator(List(GeneratorsOfGroup(W)
            , n -> Concatenation("\"", String(n), "\"")), ","), "];");
71      IO_Write(persistInfo.fileD, "[", JoinStringsWithSeparator(matrix, ","), "];");
72      IO_Write(persistInfo.fileD, "\"", Name(theta), "\";");
73
74      if (Size(W) = infinity) then
75          IO_Write(persistInfo.fileD, "\"infinity\";");
76          IO_Write(persistInfo.fileD, "\"infinity\"");
77      else
78          IO_Write(persistInfo.fileD, numNodes, ";");
79          IO_Write(persistInfo.fileD, maxTwistedLength, "");
80      fi;
81  end;
82
83  TwistedInvolutionWeakOrderingPersistResults := function(persistInfo, nodes, edges)
84      local n, e, i, tmp, bubbles;
```

```
85
86      if (persistInfo = fail) then return; fi;
87
88      # bubble sort the edges, to make sure, that double edges are neighbours in the list
89      bubbles := 1;
90      while bubbles > 0 do
91          bubbles := 0;
92          for i in [1..Length(edges)-1] do
93              if edges[i].source.absIndex = edges[i+1].source.absIndex and edges[i].
                    target.absIndex > edges[i+1].target.absIndex then
94                  tmp := edges[i];
95                  edges[i] := edges[i+1];
96                  edges[i+1] := tmp;
97                  bubbles := bubbles + 1;
98              fi;
99          od;
100     od;
101
102     for n in nodes do
103         if n.absIndex = 1 then
104             IO_Write(persistInfo.fileV, n.twistedLength, ";\"e\"\n");
105         else
106             IO_Write(persistInfo.fileV, n.twistedLength, ";\"", String(n.element), "\"\
                    n");
107         fi;
108     od;
109
110     for e in edges do
111         IO_Write(persistInfo.fileE, e.source.absIndex-1, ";", e.target.absIndex-1, ";",
                 e.label, ";", e.type, "\n");
112     od;
113 end;
```

B References

[1] Ruth Haas and Aloysius G. Helminck. Algorithms for twisted involutions in weyl groups. *Algebra Colloquium 19*, 2012.

[2] Axel Hultman. Fixed points of involutive automorphisms of the bruhat order. *Adv. Math. 195*, pages 283–296, 2005. Also available at `http://www.ams.org/mathscinet-getitem?mr=2145798`.

[3] Axel Hultman. The combinatorics of twisted involutions in coxeter groups. *Transactions of the American Mathematical Society, Volume 359*, pages 2787–2798, 2007. Also available at `http://www.ams.org/mathscinet-getitem?mr=2286056`.

[4] James E. Humphreys. *Reflection groups and Coxeter groups*. Cambridge University Press, 1992.