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Technische Universität Braunschweig
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Posets of twisted involutions in Coxeter groups

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1 Coxeter groups

1.1 Introduction to Coxeter groups

A Coxeter group, named after Harold Scott MacDonald Coxeter, is an abstract group generated by involutions with specific relations between these generators. A simple class of a Coxeter groups are the symmetry groups of regular polyhedras in the Euclidean space.
The symmetry group of the square for example can be generated by two reflections s, t, whose stabilized hyperplanes enclose an angle of $\pi/4$. In this case the map st is a rotation in the plane by $\pi/2$. So we have $s^2 = t^2 = (st)^4 = \mathrm{id}$. In fact this reflection group is determined up to isomorphy by s, t and these three relations [Humphreys, 1992, Theorem 1.9]. Furthermore it turns out, that the finite reflection groups in the Euclidean space are precisely the finite Coxeter groups [Humphreys, 1992, Theorem 6.4].
In this chapter we will compile some basic facts on Coxeter groups. First of all the definition:

Definition 1.1. Let $S = \{s_1, \ldots, s_n\}$ be a finite set of symbols and

$$R = \{m_{ij} \in \mathbb{N} \cup \infty : 1 \leq i, j \leq n\}$$

a set numbers (or ∞) with $m_{ii} = 1$, $m_{ij} > 1$ for $i \neq j$ and $m_{ij} = m_{ji}$. Then the free represented group

$$W = \langle S \mid (s_i s_j)^{m_{ij}} \rangle$$

is called a **Coxeter group** and (W, S) the corrosponding **Coxeter system**. The cardinality of S is called the **rank** of the Coxeter system (and the Coxeter group).

From the definiton we see, that Coxeter groups only depend on the cardinality of S and the relations between the generators in S. A common way to visualize this information are **Coxeter graphs**. A Coxeter graph is a graph containing a vertex for each generator in S. Let $(s_i s_j)^m = 1$. In case $m = 2$ the two corrosponding vertices have no connecting edge. In case $m = 3$ they are connected by an unlabed edge. For $m > 3$ they have an connecting edge with label m.
For a arbitrary element $w \in W$, (W, S) a Coxeter system, we call a product $s_{i_1} \cdots s_{i_n} = w$ of generators $s_{i_1} \ldots s_{i_n} \in S$ an **expression** of w. The present relations between the generators of a Coxeter group allow us to rewrite expressions. Hence an element $w \in W$ can have more than one expression. Obviously any element $w \in W$ has infinitly many expressions, since any expression $s_{i_1} \cdots s_{i_n} = w$ can be extended by applying $s_1^2 = 1$ from the right. But there must be a smallest number of generators needed to receive w. For example the neutral element e can be expressed by the empty expression. Or each generator $s_i \in S$ can be expressed by itself, but any expression with less factors (i.e. the empty expression) is unequal to s_i.

Definition 1.2. Let (W, S) be a Coxeter system and $w \in W$ an element. Then there are some (not neseccarily distince) generators $s_i \in S$ with $s_1 \cdots s_r = w$. We call r the **expression length**. The smallest number $r \in \mathbb{N}_0$ for that w has an expression of length r

is called the **length** of w and each expression of w, that is ob minimal length, is called **reduced expression**. The map

$$l : W \to \mathbb{N}_0$$

that maps each element in W to its length is called **length function**.

1.2 Exchange and Deletion Condition

We now obtain a way to get a reduced expression of an arbitrary element $s_1 \cdots s_r = w \in W$. But first we define what a reflection is. Any element $w \in W$ that is conjugated to an generator $s \in S$ is called **reflection**. Hence the set of all reflections in W is

$$T = \bigcup_{w \in W} wSw^{-1}.$$

Theorem 1.3. *Let (W, S) be a Coxeter system, $w \in W$ an arbitrary element and $s_1 \cdots s_r = w$ with $s_i \in S$ a not neseccarily reduced expression for w. For each reflection $t \in T$ with $l(wt) < l(w)$ there exists an index i for which $wt = s_1 \cdots \hat{s}_i \cdots s_r$, where $\hat{s}_i$ means omission. In case we started from an reduced expression, then i is unique.*

Proof. See [Humphreys, 1992, Theorem 5.8]. □

This theorem is called the **Strong Exchange Condition**. The same theorem can be stated for $t \in S$ (since $S \subset T$). This weaker theorem is called **Exchange Condition**. The Exchange Condition immediatly yields another corollary for Coxeter groups:

Corollary 1.4. *Let (W, S) be a Coxeter system, $w \in W$ and $w = s_1 \cdots s_r$ with $s_i \in S$ a unreduced expression of w. Then there exist two indices $i, j \in \{1, \cdots, r\}$ with $i < j$, such that $w = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_r$, where $\hat{s}_i$ and $\hat{s}_j$ mean omission.*

Proof. Since the expression is unreduced there must be an index j for that the twisted length shrinks. That means for $w' = s_1 \cdots s_{j-1}$ is $l(w's_j) < l(w')$. Using the Exchange Condition we get $w's_j = s_1 \cdots \hat{s}_i \cdots s_{j-1}$ yielding $w = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_r$. □

This corollary is called **Deletion Condition** and allows us to reduce expressions. To explain what is exactly meant by this we need another definition. For an expression $w = s_1 \cdots s_r$ and any subset of indices $S' \subset S$ we call $w' = t_1 \cdots t_r$ with $t_i = \hat{s}_i$ for $i \in S'$ and $t_i = s_i$ for $i \notin S'$ a **subexpression** of w. With this definition we can precisily define the action of reducing expressions. Reducing an unreduced expression means to extract a reduced subexpression. Due to the Deletion Condition any unreduced expression can be reduced by omitting a even number of generators (we just have to apply the Deletion Condition inductively).

Both, the Exchange Condition and the Deletion Condition, are two of the most powerful tools when investigating properties of Coxeter groups.

1.3 Finite Coxeter groups

Coxeter groups can be finite and infinite. A simple example for the former category is the following. Let $S = \{s\}$. Due to definition it must be $s^2 = e$. So W is isomorph to $\mathbb{Z}_2$ and finite. An example for an infinite Coxeter group can be obtained from $S = \{s, t\}$ with $s^2 = t^2 = e$ and $(st)^\infty = e$ (so we have no relation between s and t). Obviously the element st has infinite order forcing W to be infinite. But there are also infinite Coxeter groups without an ∞-relation between two generators. An example for this is W obtained from $S = \{s_1, s_2, s_3\}$ with $s_1^2 = s_2^2 = s_3^2 = (s_1s_2)^3 = (s_2s_3)^3 = (s_3s_1)^3 = e$. But how can it be seen that this W is infinite?
To provide a general answer to this question we fallback to a certain class of Coxeter groups, the irreducible ones.

Definition 1.5. A Coxeter system is called **irreducible**, if the corrosponding Coxeter graph is connected. Else it is called **reducible**.

If a Coxeter system is reducible, then its graph has more than one connection component and each connection component corrosponds to a subgroup of W.

Definition 1.6. Let (W, S) be a Coxeter system. For a subset of generators $I \subset S$ we call the subgroup $W_I \leq W$ that is generated by the elements in S_I with the corrosponding relations a **parabolic subgroup** of W.

Proposition 1.7. *Let (W, S) be a reducible Coxeter system. Then there exists a partition of S into I, J with $(s_is_j)^2 = e$ whenever $s_i \in I, s_j \in J$ and W is isomorph to the direct product of the two parabolic subgroups W_I and W_J.*

Proof. See [Humphreys, 1992, Proposition 6.1]. □

This proposition tells us, that an arbitray Coxeter system is finite iff its irreducible parabolic subgroups are finite. Therefor we can indeed fallback to irreducible Coxeter systems without loss of generality. If we could categorize all irreducible finite Coxeter systems, we could categorize all finite Coxeter systems. This is done by the following theorem:

Theorem 1.8. *The irreducible finite Coxeter systems are excatly the ones in Figure* **??**.

Proof. [Humphreys, 1992, Theorem 6.4] □

Finally we can decide with ease, if a given Coxeter system is finite. Take its irreducible parabolic subgroups and check, if each is one of A_n, B_n, D_n, E_6, E_7, E_8, F_4, H_3, H_4 or $I_2(m)$.

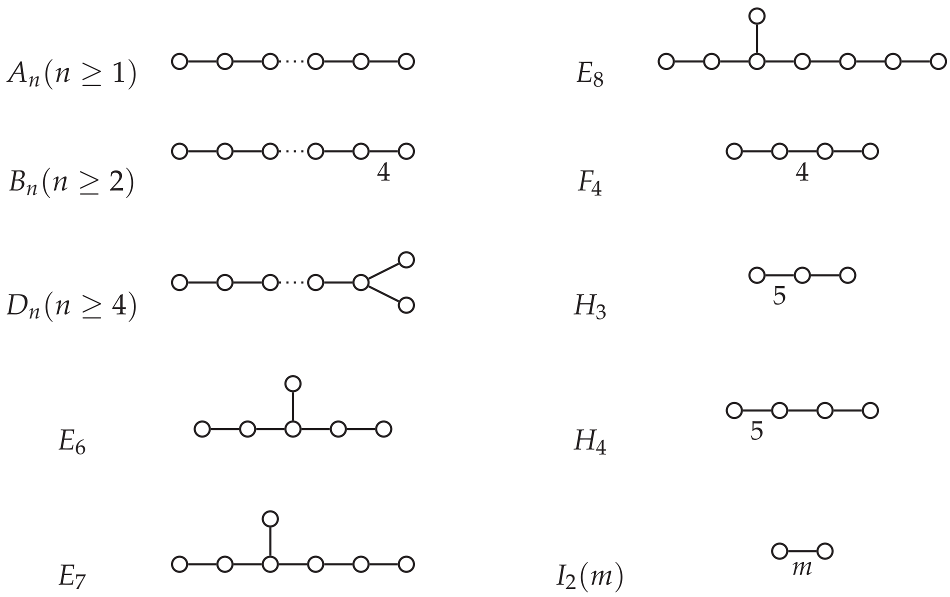

Figure 1: All types of irreducible finite Coxeter systems

2 The twisted weak ordering in Coxeter groups

In this section we will introduce the twisted weak ordering $Wk(\theta)$ on Coxeter groups.

3 Residuums of rank 2

Definition 3.1 (One- and twosided operation)**.** Let (W,S) be a Coxeter system and $w \in W, s \in S$. If $w\underline{s} = \theta(s)ws$, then we say s to operate twosided on w. Else we say s to operate onesided on w.

Definition 3.2 (Ein- und beidseitige endende Gesamtwirkung)**.** Seien (W,S) ein Coxetersystem, $w \in W$ und $s_1,\dots,s_n \in S$. Falls $w\underline{s_1\cdots s_n} = \theta(s_n)(w\underline{s_1\cdots s_{n-1}})s_n$ ist, so sagen wir, dass $s_1\cdots s_n$ eine beidseitig endende Gesamtwirkung auf w hat. Andernfalls sagen wir $s_1\cdots s_n$ hat eine einseitig endende Gesamtwirkung auf w.

Definition 3.3. Let (W,S) be a Coxeter system and $s,t \in S$ two distinct generators. We define:

$$[st]^n := \begin{cases} (st)^{\frac{n}{2}}, & n \text{ even} \\ (st)^{\frac{n-1}{2}}s, & n \text{ odd} \end{cases}$$

Assumption 3.4. Seien (W,S) ein Coxetersystem und $s,t \in S$ zwei verschiedene Erzeuger von W. Dann gilt:

1. Sei $m = \operatorname{ord}(st) < \infty$. Falls $w\underline{[st]^n} \neq w$ ist für alle $n \in \mathbb{N}, n < 2m$, dann gilt $w\underline{(st)^{2m}} = w$.

2. In $wC_{\{s,t\}}$ existieren keine drei Elemente derselben getwisteten Länge.

3. Falls s einseitig auf w wirkt, dann gilt $w\underline{st} < w\underline{s}$ oder $w\underline{t} > w$.

4. Sei $w\underline{[st]^n} = w$ für ein $n \in \mathbb{N}$. Dann ist n gerade und es gilt eine der beiden folgenden Eigenschaften:
   a) Für jedes $m \in \mathbb{N}$ hat das Element $[st]^m$ genau dann eine beidseitig endende Gesamtwirkung auf w, wenn $[st]^{n/2+m}$ eine beidseitig endende Gesamtwirkung auf w hat.
   b) Für jedes $m \in \mathbb{N}$ hat das Element $[st]^m$ genau dann eine beidseitig endende Gesamtwirkung auf w, wenn $[st]^{n-m+1}$ eine beidseitig endende Gesamtwirkung auf w hat.

Lemma 3.5. *Let (W,S) be a Coxeter system, $w \in W$ and $s,t \in S$ two distinct generators. Then $wC_{\{s,t\}}$ does not contain three elements of same twisted length.*

Proof. Let (W,S) be a Coxeter system, $w \in W$ with $\operatorname{rank} w = k$, $s,t \in S$ with $s \neq t$. Without loss of generality we can choose w such that $w < w\underline{s}$ and $w < w\underline{t}$. Assume the existence of an element $u \in wC_{\{s,t\}}$ with $u\underline{s} < u$ and $u\underline{t} < u$. Then [Hultman, 2007, Lemma 3.8] yields $s,t \in D_R(u)$. By using [Hultman, 2007, Lemma 3.9] we conclude that $w\underline{s} \leq u$ and $w\underline{t} \leq u$. Hence there cannot exist more than two Elements of same twisted length.
If no such u exists, then $wC_{\{s,t\}} = w \mathbin{\dot\cup} \{w\underline{[st]^n} : n \in \mathbb{N}\} \mathbin{\dot\cup} \{w\underline{[ts]^n} : n \in \mathbb{N}\}$ and the assumption still holds. □

Lemma 3.6. *Let (W,S) be a Coxeter system, $w \in S$ and $s,t \in S$ two distinct generators. If s operates onesided on w and $w\underline{s} < w$, then either $w\underline{st} < w\underline{s}$ or $w\underline{t} > w$.*

Proof. We have $\theta(s)ws = w$ and $s \in D_R(w)$. If $t \notin D_R(w)$, then we are done. So suppose $t \in D_R(w)$. This means $w\underline{s} \leq w$ and $w\underline{t} \leq w$ and [Hultman, 2007, Lemma 3.9] yields $w\underline{st} < w$ and $w\underline{ts} < w$. If $t \in D_R(w\underline{s})$, then we are done. So suppose $t \notin D_R(w\underline{s})$. Then $t \in D_R(w\underline{st})$. Together with $w\underline{st} \leq w$ [Hultman, 2007, Lemma 3.9(2)] says $(w\underline{st})\underline{t} \leq w\underline{t}$. Finally we get

$$ws = w\underline{s} = (w\underline{st})\underline{t} \leq w\underline{t} = wt.$$

Since $w\underline{s}$ and $w\underline{t}$ are of same twisted length they have to be equal and therefore $s = t$ which contradicts to our assumption of two distinct generators s and t. $\square$

4 Twisted weak ordering

Wir wollen nun einen Algorithmus zur Berechnung der getwisteten schwachen Ordnung $Wk(\theta)$ einer beliebigen Coxetergruppe W erarbeiten. Also Ausgangspunkt werden wir den Algorithmus aus [Haas and Helmnick, 2012, Algorithm 3.1.1] verwenden, der im wesentlichen benutzt, dass für jede getwistete Involution $w \in \mathcal{I}_\theta$ entweder $w\underline{s} < w$ oder aber $w\underline{s} > w$ gilt.

Algorithm 4.1 (Algorithmus 1)**.**

procedure TwistedWeakOrderingAlgorithm1(W) ▷ W sei die Coxetergruppe
  $V \leftarrow \{(e,0)\}$
  $E \leftarrow \{\}$
  for $k \leftarrow 0$ **to** $k_{\max}$ **do**
    for all $(w, k_w) \in V$ **with** $k_w = k$ **do**
      for all $s \in S$ **with** $\nexists(\cdot, w, s) \in E$ **do** ▷ Nur die s, die nicht schon nach w führen
        $y \leftarrow ws$
        $z \leftarrow \theta(s)y$
        if $z = w$ **then**
          $x \leftarrow y$ ▷ s operiert ungetwistet auf w
          $t \leftarrow s$
        else
          $x \leftarrow z$ ▷ s operiert getwistet auf w
          $t \leftarrow \underline{s}$
        end if
        $isNew \leftarrow$ **true**
        for all $(w', k_{w'}) \in V$ **with** $k_{w'} = k + 1$ **do** ▷ Prüfen, ob x nicht schon in V liegt
          if $x = w'$ **then**
            $isNew \leftarrow$ **false**
          end if
        end for
        if $isNew =$ **true then**
          $V \leftarrow V \cup \{(x, k+1)\}$
          $E \leftarrow E \cup \{(w, x, t)\}$
        else
          $E \leftarrow E \cup \{(w, x, t)\}$
        end if
      end for
    end for
    $k \leftarrow k + 1$
  end for

			Timings		Element compares	
W	$\lvert Wk(\mathrm{id}, W)\rvert$	$\rho(w_0)$	TWOA1	TWOA2	TWOA1	TWOA2
A_9	9496	25	00:02.180	00:01.372	13,531,414	42,156
A_{10}	35696	30	00:31.442	00:06.276	185,791,174	173,356
A_{11}	140152	36	11:04.241	00:29.830	2,778,111,763	737,313
E_6	892	20	00:03.044	00:00.268	85,857	2,347
E_7	10208	35	06:11.728	00:02.840	7,785,186	29,687
E_8	199952	64	–	11:03.278	–	682,227

Table 1: Benchmark

```
        return (V, E)                                        ▷ The poset graph
    end procedure
```

Dieser Algorithmus berechnet alle getwisteten Involutionen und deren getwistete Länge (w, k_w) und deren Relationen (w', w, s) bzw. $(w', w, \underline{s})$. Zu bemerken ist, dass zur Berechnung der getwisteten Involutionen der Länge k nur die Knoten aus V benötigt werden, mit der getwisteten Länge $k-1$ und k sowie die Kanten aus E, die Knoten der Länge $k-2$ und $k-1$ bzw. $k-1$ und k verbinden. Alle vorherigen Ergebnisse können schon persistiert werden, so dass nie das komplette Ergebnis im Speicher gehalten werden muss.
Eine Operation, die hier als elementar angenommen wurde ist der Vergleich von Elementen in W. Für bestimmte Gruppen wie z.B. die A_n, welche je isomorph zu $Sym(n+1)$ sind, lässt sich der Vergleich von Element effizient implementieren. Will man jedoch mit Coxetergruppen im Allgemeinen arbeiten, so liegt W als frei präsentierte Gruppe vor und der Vergleich von Element is eine sehr aufwendige Operation. Bei Algorithm 4.1 muss jedes potentiell neue Element x mit allen schon bekannten w' von gleicher getwisteter Länge verglichen werden um zu bestimmen, ob x wirklich ein noch nicht bekanntes Element aus $\mathcal{I}_\theta$ ist.

Algorithm 4.2 (Algorithmus 2)**.**

```
    procedure TwistedWeakOrderingAlgorithm2(W)     ▷ W sei die Coxetergruppe
        V ← {(e, 0)}
        E ← {}
        for k ← 0 to k_max do
            TODO
        end for
        return (V, E)                                        ▷ The poset graph
    end procedure
```

Im Anhang findet sich eine Implementierung der Algorithms 4.1 and 4.2 in GAP 4.5.4. Table **??** zeigt ein Benchmark anhand von fünf ausgewählten Coxetergruppen.
Dabei sind die A_n als symmetrische Gruppen implementiert und die E_n als frei präsentierte Gruppen. Ausgeführt wurden die Messungen auf einem Intel Core i5-3570k mit

vier Kernen zu je 3,40 GHz. Der Algorithmus ist dabei aber nur single threaded und kann so nur auf einem Kern laufen. Um die Messergebnisse nicht durch Limitierungen des Datenspeichers zu beeinflussen, wurden die Daten in diesem Benchmark nicht stückweise persistiert sondern ausschließlich berechnet.
Wie zu erwarten ist der Geschwindigkeitsgewinn bei den Coxetergruppen vom Typ E_n deutlich größer, da in diesem Fall die Elementvergleiche deutlich aufwendiger sind als bei Gruppen vom Typ A_n.

5 Miscellaneous

Definition 5.1 (Geodesic)**.** Let (W, S) be a Coxeter system and $w, u \in W$ with $\rho(u) - \rho(w) = n$. Each sequence $w = w_0 < w_1 < \ldots < w_n = u$ is called a geodesic from w to u.

Question 5.2. *Let (W, S) be a Coxeter system, $\theta : W \to W$ an automorphism of W with $\theta^2 = \mathrm{id}$ and $\theta(S) = S$, and $K \subset S$ a subset of S generating a finite subgroup of W with $\theta(K) = K$. Futhermore let $T, S_1, S_2, S_3 \subset S$ be four pairwise disjoint sets of generators. For which Coxeter groups W does the implication*

$$w \in w_K C_{T \cup S_i}, i = 1, 2, 3 \Rightarrow w \in w_K C_T \tag{5.2.1}$$

hold for any possible $K, \theta, T, S_1, S_2, S_3$ and w?

Proposition 5.3. *Let (W, S) be a Coxeter system and K, T, S_1, S_2, S_3 be like in Question 5.2. Suppose we have $w \in W$ and $a_1, \ldots, a_n \in T \cup S_1$, $b_1, \ldots, b_n \in T \cup S_2$, $c_1, \ldots, c_n \in T \cup S_3$ with*

$$\begin{aligned} w &= w_K \underline{a_1 \cdots a_n} \\ &= w_K \underline{b_1 \cdots b_n} \\ &= w_K \underline{c_1 \cdots c_n} \end{aligned}$$

and (5.2.1) does not hold for these three expressions, i.e. $w \notin w_K C_T$. Then there exist $t_1, \ldots, t_m \in T$ and $a'_1, \ldots, a'_{n-m} \in T \cup S_1$, $b'_1, \ldots, b'_{n-m} \in T \cup S_2$, $c'_1, \ldots, c'_{n-m} \in T \cup S_3$ such that

$$\begin{aligned} w\underline{t_1 \ldots t_m} &= w_K \underline{a'_1 \cdots a'_{n-m}} \\ &= w_K \underline{b'_1 \cdots b'_{n-m}} \\ &= w_K \underline{c'_1 \cdots c'_{n-m}} \end{aligned}$$

with $a'_{n-m}, b'_{n-m}, c'_{n-m} \notin T$.

Proof. Suppose at least one element of a_n, b_n, c_n to be in T, for example $a_n \in T$. Then we can apply $\underline{a_n}$ to all three expressions. Since $\rho(w\underline{a_n}) < \rho(w)$ the exchange condition for $\mathcal{I}_\theta$ [Hultman, 2007, Proposition 3.10]yields

$$\begin{aligned} w\underline{a_n} &= w_K \underline{a_1 \cdots a_n a_n} = w_K \underline{a_1 \cdots a_{n-1}} \\ &= w_K \underline{b_1 \cdots b_n a_n} = w_K \underline{b_1 \cdots \hat{b}_i \cdots b_n} \\ &= w_K \underline{c_1 \cdots c_n a_n} = w_K \underline{c_1 \cdots \hat{c}_j \cdots c_n} \end{aligned}$$

where $\hat{}$ means omission. The omission cannot occur within w_K since all three expressions are still of same twisted length and in the first expression we can see, that $w_K \leq w\underline{a_n}$ still holds. This step can be repeated until $w = w_K$ or $a_n, b_n, c_n \notin T$. □

Lemma 5.4. *A counterexample to Question 5.2 can only exist, if there is an element $u \in wC_T$ and three distinct generators $s_1, s_2, s_3 \in D_r(u)$ such that $u\underline{s_i} \notin wC_T$ for $i = 1, 2, 3$.*

Proof. According to Proposition 5.3. □

Lemma 5.5. *A counterexample to Question 5.2 can only exist, if there are three not neseccarily distinct elements* $a, b, c \in w_K C_{S\setminus T}$*, three distinct generators* $s_1 \in A_r(a)$, $s_2 \in A_r(b)$, $s_3 \in A_r(c)$ *and an element* $u \notin w_K C_{S\setminus T}$ *such that*

$$a\underline{s_1} = b\underline{s_2} = c\underline{s_3} = u.$$

Proof. If there is a counterexample, then the two residuums $w_K C_{S\setminus T}$ and wC_T are disjunct. Since we are only interested in w with $w_K \leq w$ it follows, that any geodesic from w_K to w is contained in the union set of both residuums. Hence having one element in $u \in wC_T$ with three distinct generators s_1, s_2, s_3 with $u\underline{s_i} \notin wC_T$ is equivalent to having three elements $a, b, c \notin wC_T$ and the same three generator s_1, s_2, s_3 with $a\underline{s_1} = b\underline{s_2} = c\underline{s_3} = u \in wC_T$. □

Lemma 5.6. *Let* (W, S) *be a Coxeter system,* $w \in W$ *and* $s \in S$*. Then* $s \in D_R(w)$ *iff* $w\underline{s} < w$*.*

Proof. TODO □

A Source codes

```
LoadPackage("io");

Read("misc.gap");
Read("coxeter.gap");
Read("twistedinvolutionweakordering-persist.gap");

TwistedInvolutionDeduceNodeAndEdgeFromGraph := function(matrix, startNode, startLabel, labels)
    local rank, comb, trace, possibleEqualNodes, e, k, n;

    rank := -1/2 + Sqrt(1/4 + 2*Length(matrix)) + 1;
    possibleEqualNodes := [];

    for comb in List(Filtered(labels, label -> label <> startLabel), label -> rec(startNode := startNode, s := [startLabel, label], m := CoxeterMatrixEntry(matrix, rank, startLabel, label))) do
        trace := [];
        k := 1;
        n := comb.startNode;

        Add(trace, rec(node := n, edge := rec(label := comb.s[1], type := -1)));

        while k < comb.m do
            e := FindElement(n.inEdges, e -> e.label = comb.s[k mod 2 + 1]);
            if e = fail then break; fi;
            n := e.source;

            Add(trace, rec(node := n, edge := e));
            k := k + 1;
        od;

        while k > 0 do
            e := FindElement(n.outEdges, e -> e.label = comb.s[k mod 2 + 1]);
            if e = fail then break; fi;
            n := e.target;

            Add(trace, rec(node := n, edge := e));
            k := k - 1;
        od;

        if k <> 0 then continue; fi;

        if Length(trace) = 2*comb.m then
            return rec(result := 0, node := trace[Length(trace)].node, type := trace[comb.m + 1].edge.type, trace := trace);
        fi;

        if Length(trace) >= 4 then
            if trace[Length(trace) / 2 + 1].edge.type <> trace[Length(trace) / 2].edge.type then
                # cannot be equal
            else
                if trace[Length(trace)].edge.type = 0 then
                    return rec(result := 0, node := trace[Length(trace)].node, type := 0, trace := trace);
                else
                    Add(possibleEqualNodes, trace[Length(trace)].node);
                fi;
```

```
            fi;
        else
            Add(possibleEqualNodes, trace[Length(trace)].node);
        fi;
    od;

    return rec(result := -1, possibleEqualNodes := possibleEqualNodes);
end;

# Calculates the poset Wk(theta).
TwistedInvolutionWeakOrdering := function (filename, W, matrix, theta)
    local persistInfo, maxOrder, nodes, edges, absNodeIndex, absEdgeIndex, prevNode,
        currNode, newEdge,
        label, type, deduction, startTime, endTime, S, k, i, s, x, y, n;

    persistInfo := TwistedInvolutionWeakOrderingPersistResultsInit(filename);

    S := GeneratorsOfGroup(W);
    maxOrder := Minimum([Maximum(Concatenation(matrix, [1])), 5]);
    nodes := [ [], [ rec(element := One(W), twistedLength := 0, inEdges := [], outEdges
         := [], absIndex := 1) ] ];
    edges := [ [], [] ];
    absNodeIndex := 2;
    absEdgeIndex := 1;
    k := 0;

    while Length(nodes[2]) > 0 do
        if not IsFinite(W) then
            if k > 200 or absNodeIndex > 10000 then
                break;
            fi;
        fi;

        for i in [1..Length(nodes[2])] do
            Print(k, " ", i, "          \r");

            prevNode := nodes[2][i];
            for label in Filtered([1..Length(S)], n -> Position(List(prevNode.inEdges,
                e -> e.label), n) = fail) do
                deduction := TwistedInvolutionDeduceNodeAndEdgeFromGraph(matrix,
                    prevNode, label, [1..Length(S)]);

                if deduction.result = 0 then
                    type := deduction.type;
                    currNode := deduction.node;
                elif deduction.result = 1 then
                    type := deduction.type;

                    currNode := rec(element := y, twistedLength := k + 1, inEdges :=
                        [], outEdges := [], absIndex := absNodeIndex);
                    Add(nodes[1], currNode);

                    absNodeIndex := absNodeIndex + 1;
                else
                    x := prevNode.element;
                    s := S[label];

                    type := 1;
                    y := s^theta*x*s;
                    if (CoxeterElementsCompare(x, y)) then
                        y := x * s;
```

```
                        type := 0;
                    fi;

                    currNode := FindElement(deduction.possibleEqualNodes, n ->
                        CoxeterElementsCompare(n.element, y));

                    if currNode = fail then
                        currNode := rec(element := y, twistedLength := k + 1, inEdges
                            := [], outEdges := [], absIndex := absNodeIndex);
                        Add(nodes[1], currNode);

                        absNodeIndex := absNodeIndex + 1;
                    fi;
                fi;

                newEdge := rec(source := prevNode, target := currNode, label := label,
                    type := type, absIndex := absEdgeIndex);

                Add(edges[1], newEdge);
                Add(currNode.inEdges, newEdge);
                Add(prevNode.outEdges, newEdge);

                absEdgeIndex := absEdgeIndex + 1;
            od;
        od;

        TwistedInvolutionWeakOrderingPersistResults(persistInfo, nodes[2], edges[2]);

        Add(nodes, [], 1);
        Add(edges, [], 1);
        if (Length(nodes) > maxOrder + 1) then
            for n in nodes[maxOrder + 2] do
                n.inEdges := [];
                n.outEdges := [];
            od;
            Remove(nodes, maxOrder + 2);
            Remove(edges, maxOrder + 2);
        fi;
        k := k + 1;
    od;

    TwistedInvolutionWeakOrderingPersistResultsInfo(persistInfo, W, matrix, theta,
        absNodeIndex - 1, k - 1);
    TwistedInvolutionWeakOrderingPersistResultsClose(persistInfo);

    return rec(numNodes := absNodeIndex - 1, numEdges := absEdgeIndex - 1,
        maxTwistedLength := k - 1);
end;

# Calculates the poset Wk(theta).
TwistedInvolutionWeakOrdering1 := function (filename, W, matrix, theta)
    local persistInfo, maxOrder, nodes, edges, absNodeIndex, absEdgeIndex, prevNode,
        currNode, newEdge,
        label, type, deduction, startTime, endTime, S, k, i, s, x, y, n;

    persistInfo := TwistedInvolutionWeakOrderingPersistResultsInit(filename);

    S := GeneratorsOfGroup(W);
    maxOrder := Minimum([Maximum(Concatenation(matrix, [1])), 5]);
    nodes := [ [], [ rec(element := One(W), twistedLength := 0, inEdges := [], outEdges
        := [], absIndex := 1) ] ];
```

```
    edges := [ [], [] ];
    absNodeIndex := 2;
    absEdgeIndex := 1;
    k := 0;

    while Length(nodes[2]) > 0 do
        if not IsFinite(W) then
            if k > 200 or absNodeIndex > 10000 then
                break;
            fi;
        fi;

        for i in [1..Length(nodes[2])] do
            Print(k, " ", i, "         \r");

            prevNode := nodes[2][i];
            for label in Filtered([1..Length(S)], n -> Position(List(prevNode.inEdges,
                e -> e.label), n) = fail) do
                x := prevNode.element;
                s := S[label];

                type := 1;
                y := s^theta*x*s;
                if (CoxeterElementsCompare(x, y)) then
                    y := x * s;
                    type := 0;
                fi;

                currNode := FindElement(nodes[1], n -> CoxeterElementsCompare(n.element
                    , y));

                if currNode = fail then
                    currNode := rec(element := y, twistedLength := k + 1, inEdges :=
                        [], outEdges := [], absIndex := absNodeIndex);
                    Add(nodes[1], currNode);

                    absNodeIndex := absNodeIndex + 1;
                fi;

                newEdge := rec(source := prevNode, target := currNode, label := label,
                    type := type, absIndex := absEdgeIndex);

                Add(edges[1], newEdge);
                Add(currNode.inEdges, newEdge);
                Add(prevNode.outEdges, newEdge);

                absEdgeIndex := absEdgeIndex + 1;
            od;
        od;

        TwistedInvolutionWeakOrderingPersistResults(persistInfo, nodes[2], edges[2]);

        Add(nodes, [], 1);
        Add(edges, [], 1);
        if (Length(nodes) > maxOrder + 1) then
            for n in nodes[maxOrder + 2] do
                n.inEdges := [];
                n.outEdges := [];
            od;
            Remove(nodes, maxOrder + 2);
            Remove(edges, maxOrder + 2);
```

```
        fi;
        k := k + 1;
    od;

    TwistedInvolutionWeakOrderingPersistResultsInfo(persistInfo, W, matrix, theta,
        absNodeIndex - 1, k - 1);
    TwistedInvolutionWeakOrderingPersistResultsClose(persistInfo);

    return rec(numNodes := absNodeIndex - 1, numEdges := absEdgeIndex - 1,
        maxTwistedLength := k - 1);
end;

TwistedInvolutionWeakOrderungResiduum := function (vertex, labels)
    local visited, queue, residuum, current, edge;

    visited := [ vertex ];
    queue := [ vertex ];
    residuum := [];

    while Length(queue) > 0 do
        current := queue[1];
        Remove(queue, 1);
        Add(residuum, current);

        for edge in current.outEdges do
            if edge.label in labels and not edge.target in visited then
                Add(visited, edge.target);
                Add(queue, edge.target);
            fi;
        od;
    od;

    return residuum;
end;

TwistedInvolutionWeakOrderungLongestWord := function (vertex, labels)
    local current;

    current := vertex;

    while Length(Filtered(current.outEdges, e -> e.label in labels)) > 0 do
        current := Filtered(current.outEdges, e -> e.label in labels)[1].target;
    od;

    return current;
end;
```

B Bibliography

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