Continuous Order Polygonal Waveform Synthesis

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ABSTRACT

A method of generating musical waveforms based on polygon traversal is introduced, which relies on sampling a variable polygon in polar space with a rotating phasor. Due to the steady angular velocity of the phasor, the generated waveform automatically exhibits constant pitch and complexly shaped amplitudes. The order and phase of the polygon can be freely adjusted in real-time, allowing for a wide range of harmonically rich timbres with modulation frequencies up to the FM range.

1. INTRODUCTION

Connections between geometric shapes and properties of associated sounds has long been an appealing field of interest for engineers and artists alike, ranging from the strictly physical visualizations of Chladni [1] to the text-based descriptions of Spectromorphology [2]. Highly complex patterns emerge from seemingly simply ideas and formulations, such as Lissajous figures [3] or phase space representations [4]. The relation between visual patterns, motion and sound has been both an inspiration and expression for decades [5, 6].

Vieira-Barbosa produced some excellent animations of polygonal wave generators [7]. While only working with integer order polygons, he also animated the concept of polygon phase modulation and produced interactive sonifications of the resulting waveforms.

Chapman extended this idea to arbitrary orders, but instead of sampling with a phasor into the time domain he uses direct geometric projection, resulting in sharp angular waveforms [8]. He also introduced a more rigid mathematical framework and uses the Schläfli symbol $\{p, q\}$ to denote the geometric properties of regular polygons as a ratio of integer values p and q [9].

Sampath provides a standalone application which allows the user to design a large set of waveforms from geometric generators [10]. Among others, these include Bezier curves, spirals, n-gons, fractals and Lissajous curves.

In the less graphical oriented domain, digital waveshaping synthesis by Le Brun might produce the most similar results to the synthesis method proposed here [11].

2. SYNTHESIS METHOD

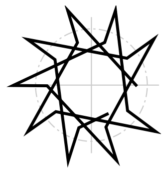


Figure 1: Example of a polygon which requires more than one cycle for a closed shape ($n = 3.33$, $T = 0.2$).

Polygonal waveform synthesis is based on sampling a closed-form polygon P of amplitude p with a rotating phasor $e^{j\phi}$. The fundamental pitch of the generated waveform is based on the angular velocity of the phase $\phi(t) = 2\pi f t = \omega t$ with sampling time t and fundamental frequency f. The polygonal expression $P(\phi, n, T, \Phi)$ simultaneously draws the polygon in the complex plane and generates the waveform when projected into the time-domain, as shown in Fig. 2.

2.1 Polygon

To create the polygon P, a corresponding order-dependent amplitude $p(\phi, n, T)$ is generated:

$$p(\phi, n, T) = \frac{\cos\left(\frac{\pi}{n}\right)}{\cos\left[\frac{2\pi}{n} \cdot \mathrm{mod}\left(\frac{\phi n}{2\pi}, 1\right) - \frac{\pi}{n} + T\right]}, \quad (1)$$

with the angle $\phi(t)$, order of the polygon n and a parameter T for offsetting the vertices, descriptively called *teeth*, adapted from [12].

Non-integer rational values of the order n require multiple cycles c of the phasor to yield a closed shape as depicted in Fig. 1. The number of cycles depends on the smallest common multiple between the decimal digits of the order and 1.

In Schläfli notation $\{a, b\}$, the rotations c corresponds to the second integer a. All polygons of the Schläfli symbol $\{a, b\}$ where $a > 2b$ may be produced. Then, the order is simply

$$n = \frac{a}{b}. \quad (2)$$

Furthermore, non-integer order polygons don't necessarily need to close to avoid discontinuities, only the projection does. Figure 3 shows this for the bottom three waveforms.

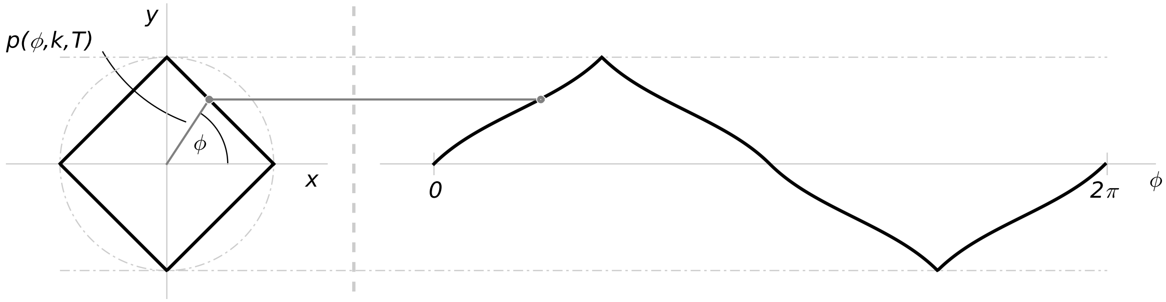


Figure 2: Projection of a square polygon (order $n = 4$) from the two-dimensional x/y plane into the time-domain.

2.2 Projection

The projection from the complex plane onto the time domain is done by simply taking the real (or imaginary) part of the polygon P:

$$\begin{aligned} P(\phi, n, T, \Phi) &= p(\phi, n, T) \cdot e^{j(\phi+\Phi)} && (3) \\ x &= \Re\{\, P(\phi, n, T, \Phi)\,\} && (4) \\ y &= \Im\{\, P(\phi, n, T, \Phi)\,\} && (5) \end{aligned}$$

Although Fig. 2 only depicts the extraction of the y component, it should be noted that the only difference to the x component is a phase shift by 90°. The additional phase offset Φ rotates the polygon in the complex plane and allows for phase modulation of the time domain signal.

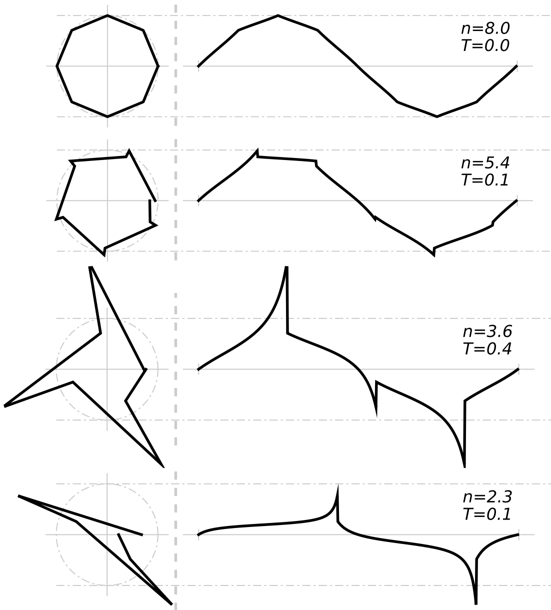


Figure 3: Projections of polygons $P(\phi, n, T, \Phi)$ from the 2D space into the time domain, $\Phi = 0$.

3. EVALUATION

In this section we discuss the three synthesis parameters order n, phase ϕ and teeth T and their influence on the sonic properties of the waveforms.

3.1 Order n

As the order $n \in]2, \infty]$ of the polygon is specifically not bound to be an integer, the shape of the polygon may be changed quasi-continuously in real-time. There is a hard lower limit of 2, corresponding to the polygon collapsing into a line, which, depending on the phase offset and projection, results in zero or infinite amplitude. For $n \to \infty$, the waveform approaches a pure sine wave. Figure 4 shows the spectrogram of a logarithmic sweep over the orders $n \in]2, 11]$ with a constant fundamental pitch of $f_0 = 100\,\mathrm{Hz}$.

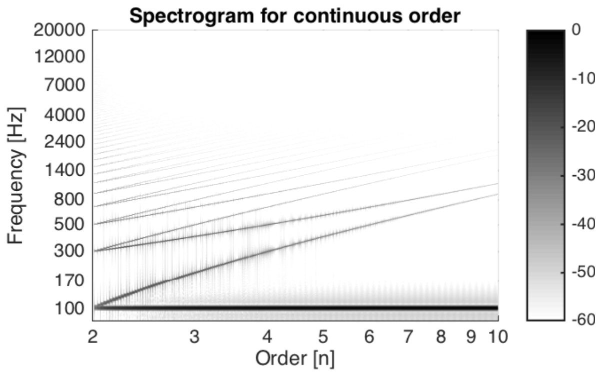


Figure 4: Spectrogram over order $n \in]2, 11]$, $f_0 = 100\,\mathrm{Hz}$.

n	h_1	h_2	h_3	h_4	h_5	h_6
2.001	$1\,f$	$3\,f$	$3\,f$	$5\,f$	$5\,f$	$7\,f$
3	$2\,f$	$4\,f$	$5\,f$	$7\,f$	$8\,f$	$10\,f$
4	$3\,f$	$5\,f$	$7\,f$	$9\,f$	$11\,f$	$13\,f$
5	$4\,f$	$6\,f$	$9\,f$	$11\,f$	$14\,f$	$16\,f$
6	$5\,f$	$7\,f$	$11\,f$	$13\,f$	$17\,f$	$19\,f$

Table 1: Ratios of harmonic overtones to the fundamental f with increasing order. For non-integer orders, overtones are continuously interpolated.

At lower orders, strong harmonics form at specific ratios as noted in Table 1. They split and drift upwards with increasing order, until only the fundamental is left and the waveform is recognized as a pure sine wave.

3.2 Phase offset Φ

Modulating the phase of a polygon P by adjusting Φ is non-trivial for non-integer orders, as discussed in Section 2.1. For closed loop polygons, phase modulation results in interesting spectral behavior as shown in Figure 5, where the fundamental and even overtones are bend up while odd overtones are bend down.

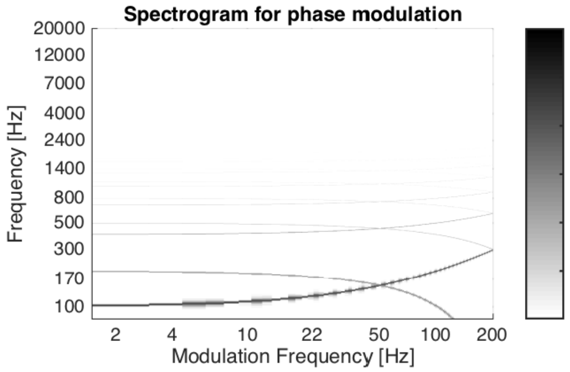


Figure 5: Spectrogram over phase modulation frequency, $f_{mod} \in [2, 200]$ Hz with order $n = 3$, $f_0 = 100$ Hz. Visual artifacts due to non-continuities in sweep.

Depending on the speed of a continuous phase modulation, both slowly evolving shapes or harsh, metallic sounds may be generated.

3.3 Teeth T

The parameter T, named for its visual effect on the polygon, allows the over-extension of the polygon's vertices.

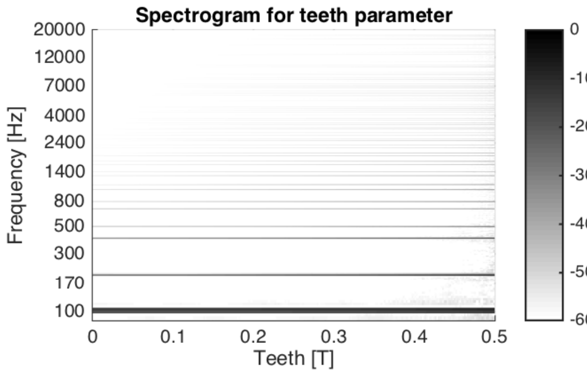


Figure 6: Spectrogram over the parameter Teeth $T \in [0, 0.5]$ with order $n = 3$, $f_0 = 100$ Hz.

Increasing T can make the polygon exceed the unit circle and consequently overdrive the output amplitude. For lower values, this will only amplify present harmonics as shown in Figure 6, which shows the sonic effects of sweeping the parameter $T \in [0, 0.5]$. Depending on the employed limiting technique, higher values will drive the oscillator into saturation, allowing fine-grain control of additional harmonic partials.

4. IMPLEMENTATION

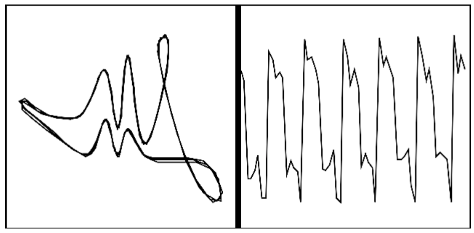
A monophonic version of the proposed synthesis method was implemented in Max/MSP [13] to explore the physical interaction with the available parameters. Figure 7 shows the GUI, with both the polygon and the time-domain signal drawn in real-time. Figure 8 shows two visually and sonically interesting polygons, their time-domain representation and their settings. [1]

[1] Please find a small selection of audio samples at `https://ccrma.stanford.edu/~chohner/polygon_samples.zip`

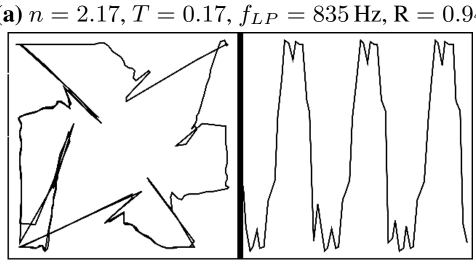
Most of the challenges when porting synth engines into a usable device, both virtual and hardware, are rooted in robustness and edge behavior. We highlight several aspects that need to be taken into account here:

4.1 (Anti)-Aliasing

One general artifact of digital synthesis is aliasing, see [14]. This specifically holds true for most of the waveforms generated here: they often contain discontinuities in the slope, which in turn result in high frequency content that is beyond the typical audio Nyquist limit of 22.05 kHz. To alleviate this, we propose generating the waveforms at four times the final audio sampling rate, lowpass filtering to the Nyquist limit using a 128th-order FIR, then decimating by a factor of 4. Such $4\times$ oversampling dropped the aliasing below the noise level in our tests.



(a) $n = 2.17$, $T = 0.17$, $f_{LP} = 835$ Hz, R $= 0.94$



(b) $n = 5.72$, $T = 0.91$, $f_{LP} = 835$ Hz, R $= 94$

Figure 8: Polygons and corresponding waveforms as visualized by the proof-of-concept implementation.

4.2 Lookup Table vs. Phase Accumulation

Digital waveforms can be either generated as a lookup table or on the fly, or in a mixed approach. Lookup tables might generally be faster and can be pre-antialiased, but in this case require a sophisticated layout or interpolation to accommodate the various lengths of the waveforms due to the required cycles c to close a non-integer shape. In the prototype we chose to evaluate Equation (5) with a continuously varying angle ϕ, accepting an increase of high frequency noise when rotating non-integer orders.

4.3 Amplitude Limiting

As mentioned in Section 3.3, non-zero values of T result in amplitudes that can exceed the unit circle. To keep the waveforms in arbitrary but strict amplitude limits, clipping or compression must be applied to the output signals. A simple hard clipper with a variable input attenuator is applied to the oversampled signal in our implementation to keep the audio signals within [-1,+1] limits.

4.4 Filtering

A traditional sculpting lowpass filter as known from subtractive synthesis is employed to further shape the pro-

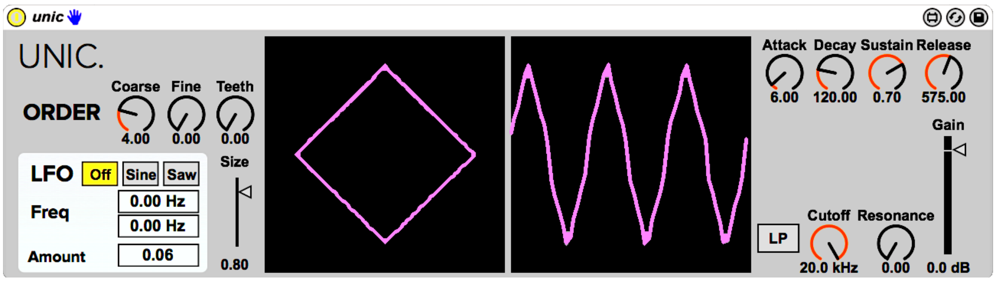


Figure 7: Screenshot of the proof-of-concept device in Max-For-Live.

duced waveforms. As expected, this introduces rounded vertices in the polar domain and overshoot depending on the resonance setting.

Any polygon with a geometric centroid different from 0 additionally introduces a dc offset when projected into the time domain. While this can be done deliberately to produce modulation signals, it should be avoided in the audio domain. A DC blocker [15] is implemented at the output of the synth.

4.5 Phase Modulation

A simple phase modulation scheme based on an LFO was implemented, which can be toggled between sinusoidal and linear waveforms. Very slow LFO frequencies allow for ever changing soundscapes, whereas modulation frequencies in the audible range allow for FM-esque sounds [16].

5. CONCLUSIONS

The proposed continuous-order polygonal-waveform synthesis is able to generate a wide variety of timbres, ranging from more traditional waveforms such as square and triangle to harsh digital sounds. The unusual control parameters give a new approach to modulation and our test implementation shows that interacting with it is quite rewarding.

Future work should include the effect of adding additional voices which can be synced, detuned and cross modulated. LFO tracking and more sophisticated filter topologies would also open further sonic sculpting capabilities. Because of its immediate visual appeal, an implementation with a larger display surface should be employed, which could happen both in the virtual as well as the real world.

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6. REFERENCES

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