Noise-tolerant Wireless Sensor Networks Localization via Multi-norms Regularized Matrix Completion

***Abstract*—Accurate and sufficient location information is the prerequisite for most Wireless Sensor Networks (WSNs) applications. Existing range-based localization approaches often suffer from incomplete and corrupted range measurements. Recently, approaches based on matrix completion (MC) are proposed. However corrupted range measurements are simply modeled as Gaussian noise and outlier, in spite of structural noise resulting in inferior localization accuracy and terrible outlier recognition accuracy is rarely considered over there. To address this challenge, a noise-tolerant localization algorithm called LMRMC is presented in this paper. By utilizing the natural low rank property of the Euclidean distance matrix (EDM), the reconstructing incomplete and noisy distance matrix is formulated as a Multi-norms Regularized Matrix Completion (MRMC) problem, where the outlier and structural noise are smoothed by L_1 norm and $L_{1,2}$ norm respectively, while the Gaussian noise is sifted implicitly. As far as we are aware of, this is the first scheme capable of recovering the missing range measurements and explicitly consider the complex mixed noise simultaneously. Extensive simulation results demonstrate that LMRMC can achieve better localization accuracy and stabilized outlier recognition accuracy compared with traditional algorithms. In addition, LMRMC provides an accurate prediction of structural noise positions, is a powerful aid for node fault diagnosis, scheduling policy and topology adjustment in WSNs.**

I. INTRODUCTION

As a bridge between the physical world and the digital world, WSNs are widely used to deal with sensing information in many fields, such as environmental monitoring [1], smart home, and data gathering, etc. Localization is one of the basic technologies in WSNs as the accurate location information of nodes is the prerequisite for most WSNs applications. Generally, the most existing localization algorithms of WSNs are classified into two major categories: range-based and range-free [2]. Range-based algorithms obtain Euclidean distance or angle information in their locations estimations by using ranging techniques, e.g., Radio Signal Strength Indicator (RSSI) and Time Difference of Arrival (TDOA), while range-free algorithms only use connectivity information between unknown nodes and anchor nodes [3]. The former category results in higher localization accuracy at the cost of additional hardware supplement, large communication traffic, and high computation burden, while the latter one is cost-effective but only suitable for coarse-grained localization.

In this paper, we focus on the range-based localization algorithms. The main idea in range-based localization methods is that some anchor nodes (by equipping GPS devices) with known their global coordinates assist unknown nodes in localizing themselves based on inter-node EDM and the global locations of those anchor nodes. In general, the range-based localization techniques in WSNs applications are confronted with two challenges:

Data missing of the inter-node EDM. The existing localization algorithms require an ideal wireless communication environments and reliable network node devices. However, in practical situations [4], due to the restrictions of node hardware conditions, node energy and deployment conditions, communication range of nodes are limited, thus only a small fraction of inter-node distance information is transmitted to sink node while the remaining distances are missing. That is to say, we can only obtain incomplete inter-node EDM in WSNs localization.

Complex mixed noise in EDM. Noise is inevitable in distance ranging, complex mixed noise in EDM usually including Gaussian noise, outlier [5] and structural in practical localization. Coming from the limitations of hardware and computation precision, Gaussian noise tending to be moderate and following the Gaussian distribution has existed. By contrast, outlier means anomalous distance measurements far beyond the normal range, which leads to severe localization error. Outlier may be caused by hardware malfunction, multipath effect, or malicious attacks, and thus it appears randomly and is difficult to predict. Ignoring the existence of ranging noises is not a wise choice, and it has been demonstrated that even a small number of outlier can degrade localization accuracy drastically [6]. However, In the process of distance measurements, due to the malfunction caused by uncertainty environment and other factors of the receiving module or the sending module of the nodes, that leads to consecutive error in EDM and we call it structural noise which take the form of row error or column error in EDM. More seriously, ignoring the existence of structural noise not only leads to low localization accuracy, but also leads to terrible recognition accuracy of outlier.

Through the above analysis we can see that only a subset of inter-node distances, corrupted by mixed random noise, can be used by localization algorithms. As a result, if we only use the sensor node to build the inter-node EDM, then it is incomplete and noisy. Therefore, many MC-based algorithms [6] [7] appeared. However, due to the randomness of outlier and structural noise, these existing algorithms often ignore to deal with them, thus they always suffer from largely reduced localization accuracy for corrupted distance measurements.

All of these challenges have motivated us to propose a noise-tolerant algorithm for wireless sensor networks localization from incomplete and noisy distance measurements. It works as followed. First, reconstructing EDM is formulated as a Multi-norms Regularized Matrix Completion (MRMC) problem, i.e., utilizing the natural low rank property of EDM and introducing norms regularized technique in the field of machine learning, where outlier and structural noise are smoothed by L_1 norm and $L_{1,2}$ norm respectively. Then we use Alternating Direction Method of Multiplier (ADMM) to solve the MRMC problem, where Gaussian noise is sifted implicitly. Finally, Multidimensional Scaling (MDS) [8] is applied to localize unknown node after obtaining complete and pure inter-node EDM. Simulation results demonstrate that compared with traditional algorithms, LMRMC achieves better localization performance with small fraction of noisy range measurements. The main contributions of our work are summarized as follows:

- We propose a novel localization algorithm based on multi-norms regularized matrix completion. To the best of our knowledge, this is the first scheme that can reconstruct node distance information and smooth Gaussian noise, outlier and structural noise simultaneously. Simulation results demonstrate our algorithm is proved with high localization accuracy.
- Utilizing the separable structure of MRMC problem, ADMM is applied to our algorithm, thus we can sift Gaussian noise implicitly. In order to enhance the stability of the solution of the MRMC problem, we combine Frobenius-norm and nuclear norm of the matrix to be completed, which leads to a robust localization.
- In addition to localize sensor nodes, our algorithm is able to recognize the position of both outlier and structural noise, which provides the basis for node fault diagnosis, scheduling policy and topology adjustment in WSNs.

The rest of this paper is organized as follows. Related work is summarized in Section II. Section III gives the problem statement and problem formulation. In Section IV, Optimizing MRMC using TNNR-ADMM and robust localization based on the reconstructed EDM are introduced. Section V evaluates the performance of the proposed algorithm. Finally, we conclude this paper in Section VI.

II. RELATED WORK

A. Review

In recent years, localization has been a subject of intense study and many efforts have been made, such as fingerprinting-based localization [9], MDS-MAP [10], SDP [11] [12] and Maximum Likelihood (ML) approaches [13]. In [9], the temporal correlation of the RSS is used to improve reliability of location estimation, which is based on a newly proposed radio propagation model considering the time-varying property of signals from a given Wi-Fi AP. MDS-MAP [10] consists of three parts. First, an all-pairs shortest-paths algorithm is used to toughly estimate the distance between each possible pair of nodes. Then, node locations that fit those estimated distances is derived using MDS technique. Finally, normalizing the resulting coordinates to take account any nodes whose positions are known. However, this algorithm requires sufficient and relatively accurate distance measurements. With respect to [11] [12], sensor network localization problem is converted to a Semi-Definite Programming problem (SDP) which is an optimization problem. This method uses the distance measurements as constraints and reduces the quadratic constraints in liner forms by introducing slack variables. It is proven that the converted problem by SDP relaxation produces a solution to the original localization problem. Regrettably, SDP can only solve small-scale problem due to its terrible computational complexity. In [13], a constrained ML approach for the problem of positioning nodes based on only connectivity information is proposed, where obtaining the node positions matrix is formulated as a probabilistic problem. Due to over-reliance on distance information of inter-nodes, these techniques are not suitable for WSNs localization under the hypothesis of incomplete and noisy EDM.

In practical situations, the existence of mixed ranging noise including Gaussian noise, outlier and structural noise is a fact that cannot be ignored for localization algorithms. In view of the above problems, a localization approach with outlier detection is proposed in [14]. By defining verifiable edges and deriving the conditions for an edge to be verifiable, this approach can explicitly detect and eliminate outlier before location computation. Based on graph rigidity theory, Xiao [15] proposes a robust patch merging operation that rejects outlier for both multilateration and patch merging, and a robust network localization algorithm is further developed. However, the above studies are based on how to reduce the noise during the process of locating but not on how to separate and recognize the noise after distances information transmitted to sink node. To the best of our knowledge, Feng et al. [7] first introduce the matrix completion theory into location finding problem in WSNs. In [7], localization from a small fraction of random entries of EDM is formulated as a low rank matrix recovery problem. Regrettably, this algorithm simply treats the ranging noises as Gaussian random noise, while outlier and structural noise are ignored. This approach suffers high level localization error from the outlier and structural noise of range measurements. In [6], Gaussian noise and outlier are considered simultaneously, while the structural noise is ignored which leads to a low localization accuracy and terrible outlier recognition accuracy. Motivated by this limitations, we design an accurate and robust approach based on multi-norms regularized matrix completion to eliminate the impact of the above three kinds of noise.

B. *MDS-based Method*

In localization algorithms, once we get reconstructed inter-nodes EDM (we call it $D, D \in R^{n\times n}$, and n is the number of nodes.), the relative positions, i.e., relative coordinates of all nodes are generated. Then position alignment is needed to map the relative coordinates to absolute ones based on three or more anchor nodes [10]. Let $W, T \in R^{d\times n}$ denotes the relative coordinates and absolute coordinates of n nodes in d dimensional space, respectively. Then map W to T through a liner transformation including translation, rotation and reflection. Assuming nodes $1,2,...,k(k\geq 3)$ are anchor nodes. The general principle of the method is as follow. First, we have:

$$\begin{aligned}&\left[T_2-T_1,T_3-T_1,...,T_k-T_1\right]\\&\quad=Q_1Q_2\left[W_2-W_1,W_3-W_1,...,W_k-W_1\right]\end{aligned} \tag{1}$$

where Q_1, Q_2 denotes the rotation matrix and reflection matrix, respectively.

Then, we have coordinate-transform matrix Q :

$$Q=Q_1Q_2=\frac{\left[T_2-T_1,T_3-T_1,...,T_k-T_1\right]}{\left[W_2-W_1,W_3-W_1,...,W_k-W_1\right]} \tag{2}$$

Finally, we obtain the absolute coordinates of all the unknown nodes by:

$$\left\{T_i=Q\cdot\left(W_i-W_1\right)+T_1, i=k+1,2,...,n\right\} \tag{3}$$

C. Matrix Completion Theory

The matrix completion problem is to recover a low-rank matrix from a subset of its entries, and it utilizes the low rank property of matrix, which is equivalent to the sparsity of singular values, to reconstruct the original matrix from a fraction of all entries. A standard matrix completion problem [7] can be formulated as follows:

$$\min_{X\in R^{m\times n}} rank(X) \quad s.t.\ P_\Omega(M)=P_\Omega(X) \tag{4}$$

where $X, M \in R^{m\times n}$ denotes the objective matrix and sampled matrix Respectively, $rank(\cdot)$ is rank function, $\Omega \subseteq \{1,2,...,m\}\times\{1,2,...,n\}$ represents the support set of subscripts of sampled entries, and $P_\Omega(\cdot)$ is the orthogonal projection operator defined as:

$$[P_\Omega(M)]_{ij} = \begin{cases} M_{ij}, if\ (i,j)\in\Omega \\ 0, \quad if\ (i,j)\notin\Omega \end{cases} \tag{5}$$

Unfortunately, the rank minimization problem (4) is NP-hard due to the combinational nature of the rank function. To simplify the problem, Candes and Recht [16] introduce the nuclear norm, i.e. the sum of singular values of a matrix, to replace the rank function. So the problem (4) can be reformulated as:

$$\min_{X\in R^{m\times n}} \|X\|_* \quad s.t.\ P_\Omega(M)=P_\Omega(X) \tag{6}$$

where $\|X\|_* = \sum_{i=1}^{r}\sigma_i(X)$ denotes the nuclear norm of the matrix X, r is the rank of X, $\sigma_i(X)$ is the ith largest singular value of matrix X.

In real application, the sampled entries might be noisy and equality constraint in (6) will be too strict, resulting in over-fitting. Therefore, the following relaxed form of (6) is often considered for matrix completion with noise:

$$\min_{X\in R^{m\times n}} \lambda\|X\|_* + \frac{1}{2}\|P_\Omega(M-X)\|_F^2 \tag{7}$$

where $\|K\|_F = \sqrt{\sum_{i=1}^{m}\sum_{j}^{n}|x_{ij}|^2}$ denotes the Frobenius-norm of matrix $K = P_\Omega(M-X)$. The parameter λ controls the rank of X and the selection of λ should depend on the noise level.

A variety of algorithms have been proposed to solve problem (7) and its variants. Such as Singular Value Thresholding (SVT) algorithm [17], Fixed Point Continuation with Approximate SVD (FPCA) algorithm [18], the classical alternating direction augmented Lagrangian method in [19] and the closely related sparse and low-rank matrix decomposition in [20].

III. PROBLEM STATEMENT AND PROBLEM FORMULATION

A. Problem Statement

We consider a wireless sensor network for environmental monitoring or event detections, where nodes are deployed randomly to meet the requirement. Among them, a few nodes are anchor nodes and have their own absolute coordinates. The range-based algorithm is usually based on the distances between the anchor nodes and the unknown nodes to locate the unknown nodes [21]. Generally, the localization procedure in WSNs is divided into three steps as Fig. 1.

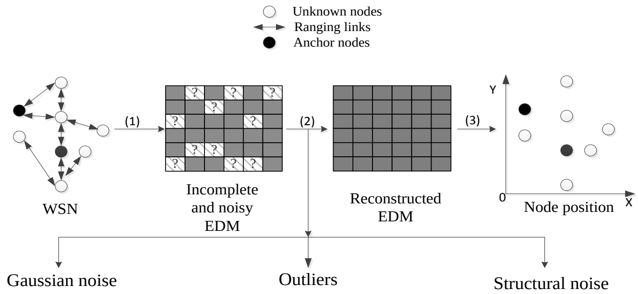


Fig. 1. Localization procedure in WSN

***Step (1)* Distance measurements.** A small fraction of inter-node distances contaminated by Gaussian noise, outlier and structural noise are collected.

***Step (2) EDM* reconstruction and noise recognition.** Reconstructing EDM and sifting/recognizing mixed noise are formulated as a MRMC problem.

***Step (3)* Obtaining absolute positions.** MDS-based method is used for obtaining absolute position of nodes.

Coming from the limitations of hardware and computation precision, the Gaussian noise tends to be moderate and follows the Gaussian distribution. Outlier may be caused by hardware malfunction, multipath effect, or malicious attacks [5], and thus it appears randomly and is difficult to predict. In this paper we analyze the reason for structural noise in localization as Fig. 2. In a typical wireless sensor network, the node hardware generally contains the signal sending and receiving module, but since the node hardware failure or environmental impact, causing them to not work properly, sending data or receiving data among nodes will be accompanied by persistent error. The receiving module is responsible for measuring the distance. The value of the row (i) in the EDM is the relative distance between the ith node and all other nodes. Therefore, row-structured noise generates when the ith node receiving module is not working properly. Correspondingly, when the jth node sending module malfunction, column-structured noise appears as Fig. 2.

B. Problem Formulation

We focus on the localization of WSNs in 2D space in this paper, and it can be easily extended to 3D space. Suppose that a set of wireless sensor nodes are deployed randomly to a certain area as Fig.1 shows. Let $\{x_1, x_2, ..., x_n\}\in R^d$ denotes the coordinate of n nodes in d dimensional space. $X=[x_1,x_2,...,x_n]\in R^{d\times n}$ denotes the matrix of all node locations. We further define Euclidean distance matrix $D\in R^{n\times n}$, where $D_{ij}=\|x_i-x_j\|^2, i,j\in\{1,2,...,n\}$. According to the step (1) in Fig. 1, only an incomplete and noisy EDM can be measured, and we call it sampled matrix M. The relationship between D and M can be given by:

$$P_\Omega(M)=P_\Omega(D+G+Z+A+B) \tag{8}$$

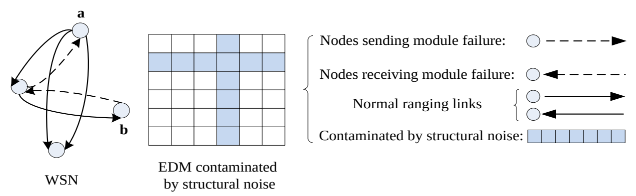


Fig. 2. The model of structural noise in localization

where G,Z,A,B denotes the Gaussian noise, the sparse outlier noise, the row-structured noise, and the col-structured noise, respectively. The relative positions of all nodes in WSNs cannot be estimated accurately simply by applying MDS-based methods due to the incomplete and noisy EDM. So the goal of this paper is to give accurate algorithms for reconstructing the entire distance matrix and sifting the various noises. In literature [7], the authors use matrix completion methods as a means to reconstruct the distance matrix only with the Gaussian noise. But a crucial question naturally arises. Can matrix completion apply to reconstructing distance matrix? The fact that Euclidean distance matrix D is of rank at most $d+2$ [6]. Therefore, for the sampled matrix M without noise or only with moderate Gaussian noise, we can utilize the existing matrix completion methods to reconstruct the Euclidean distance matrix D. Further, Literature [6] proposed a method called NLIRM which can reconstruct distance matrix D from incomplete matrix contaminated by outlier and Gaussian noise. However, the sampled matrix M usually involves the structural noise. So the issue of reconstructing distance matrix in this paper can be modeled as the multi-norms regularized matrix completion (MRMC) problem:

$$\begin{aligned}\min_{D,Z,G,A,B\in R^{n\times n}} & \; rank(D)+\mu\|Z\|_1+\varsigma\|G\|_F^2+\lambda_1\|A\|_{1,2}+\lambda_2\|B^T\|_{1,2} \\ s.t.\; & P_\Omega(M)=P_\Omega(D+Z+G+A+B)\end{aligned} \tag{9}$$

where $\|Z\|_1=\sum_{i=1}^n\sum_{j=1}^n|Z_{ij}|$, $\|X\|_{1,2}=\sum_i\left(\sum_j x_{ij}^2\right)^{1/2}$, and $\mu,\varsigma,\lambda_1,\lambda_2$ are penalty factor. Unfortunately, the above problem is NP-hard due to the non-convexity of rank function. So the complexity of the solution grows exponentially with the dimension of the matrix. Following [22], rank function can be relaxed to truncated nuclear norm, we rewrite problem (9) as:

$$\begin{aligned}\min_{D,Z,G,A,B\in R^{n\times n}} & \; \|D\|_l+\mu\|Z\|_1+\varsigma\|G\|_F^2+\lambda_1\|A\|_{1,2}+\lambda_2\|B^T\|_{1,2} \\ s.t.\; & P_\Omega(M)=P_\Omega(D+Z+G+A+B)\end{aligned} \tag{10}$$

where l is the real/estimated rank of the object matrix, and $\|D\|_l=\sum_{i=l+1}^n\sigma_i(D)$ denotes the truncated nuclear norm of the matrix D, $\sigma_i(D)$ is the ith largest singular value of matrix D. Since $\|D\|_l$ is nonconvex, it is not easy to solve (10) directly, we have the following theorem.

Theorem 1 [22]. Given a matrix $X\in R^{m\times n}$, let $X=U\Sigma V^T$ is the Singular Value Decomposition (SVD) of matrix X. Where $U=(u_1,u_2,...,u_m)\in R^{m\times m}$, $V=(v_1,v_2,...,v_n)\in R^{n\times n}$, $\sum\in R^{m\times n}$. If let $\tilde{S}=(u_1,u_2,...,u_l)^T$, $\tilde{P}=(v_1,v_2,...,v_l)^T$ then we have:

$$(\tilde{S},\tilde{P})=\arg\max_{SS^T=I_{l\times l},PP^T=I_{l\times l}} Tr(SXP^T) \tag{11}$$

And

$$\max_{SS^T=I_{l\times l},PP^T=I_{l\times l}} Tr(SXP^T)=\sum_{i=1}^l\sigma_i(X) \tag{12}$$

Therefore, for any matrix $X\in R^{m\times n}$ and each nonnegative integer $l\le\min(m,n)$ we have:

$$\begin{aligned}\|X\|_l &=\|X\|_*-\max_{SS^T=I_{l\times l},PP^T=I_{l\times l}} Tr(SXP^T) \\ Tr(SXP^T) &=\langle X,S_k^TB_k\rangle\end{aligned} \tag{13}$$

Thus, the optimization problem (10) can be rewritten as follows:

$$\begin{aligned}\min_{D,Z,G,A,B\in R^{n\times n}} & \; \|D\|_*-\max_{SS^T=I_{l\times l},PP^T=I_{l\times l}} Tr(SDP^T)+\mu\|Z\|_1 \\ & +\varsigma\|G\|_F^2+\lambda_1\|A\|_{1,2}+\lambda_2\|B^T\|_{1,2} \\ s.t.\; & P_\Omega(M)=P_\Omega(D+Z+G+A+B)\end{aligned} \tag{14}$$

The key to the proposed method is the use of the truncated nuclear norm, which achieves a better approximation of the rank function than the nuclear norm. According to theorem 1, we can design a simple but efficient iterative scheme to solve the problem (14). Specifically, we initiate $D_0=P_\Omega(M)$, and $Z_0=G_0=A_0=B_0=0,k=0$ in the kth iteration, we first fix D_k and compute S_k and P_k according to (11) based on the SVD of D_k, and then we fix S_k and P_k to update D_{k+1}. Therefore, the key to solve the problem (14) is how to solve the following problem:

$$\begin{aligned}&(D_{k+1},Z_{k+1},G_{k+1},A_{k+1},B_{k+1})= \\ &\arg\min_{D,Z,G,A,B\in R^{n\times n}}\left\{\begin{array}{l}\|D\|_*-Tr(S_kDP_k^T)+\mu\|Z\|_1 \\ +\varsigma\|G\|_F^2+\lambda_1\|A\|_{1,2}+\lambda_2\|B^T\|_{1,2} \\ s.t.\; P_\Omega(M)=P_\Omega(D+Z+G+A+B)\end{array}\right\}\end{aligned} \tag{15}$$

IV. ROBUST LOCALIZATION WITH INCOMPLETE AND NOISY DISTANCE INFORMATION

A. *Optimizing MRMC Using TNNR-ADMM*

Alternating Direction Method of Multipliers (ADMM) is a popular algorithm for solving convex optimization problem, which can be viewed as an attempt to blend the benefits of dual decomposition and augmented Lagrangian methods. Since we need to solve (15) using ADMM, we have some theorems as follows.

Theorem 2 [23]. For any $\tau>0$ and $Y\in R^{m\times n}$, the solution of the following problem:

$$\arg\min_{X\in R^{m\times n}}\left\{\tau\|X\|_1+\frac{1}{2}\|X-Y\|_F^2\right\} \tag{16}$$

is given by the shrinkage operator $\boldsymbol{S}_\tau(Y)\in R^{m\times n}$, which is defined component-wisely as:

$$[\boldsymbol{S}_\tau(Y)]_{ij}=sign(Y_{ij})\cdot\max(0,|Y_{ij}|-\tau)$$

where $sign(\cdot)$ is the sign function.

Theorem 3 [17]. For any $\tau>0$ and $Y\in R^{m\times n}$ with rank r, the solution of the following problem:

$$\arg\min_{X\in R^{m\times n}}\left\{\tau\|X\|_*+\frac{1}{2}\|X-Y\|_F^2\right\} \tag{17}$$

is given by the soft-thresholding operator $\boldsymbol{D}_\tau(Y)\in R^{m\times n}$, which is defined as:

$$\boldsymbol{D}_\tau(Y)=U\boldsymbol{S}_\tau(\Sigma)V^T$$

where $U\in R^{m\times r}$, $V\in R^{n\times r}$ and $\Sigma\in R^{r\times r}$ are obtained by the SVD of $Y=U\Sigma V^T$.

Theorem 4 [24]. For any $\tau>0$ and $Y\in R^{m\times n}$, the solution of the problem $\arg\min_{X\in R^{m\times n}}\left\{\tau\|X\|_{1,2}+\frac{\mu}{2}\|X-Y\|_F^2\right\}$ is:

$$\left(X^*\right)^{\bar{k}}=\max\left\{\left\|Y^{\bar{k}}\right\|_2-\tau/\mu,0\right\}\cdot Y^{\bar{k}}/\left\|Y^{\bar{k}}\right\|_2,i=1,2,...,m \quad (18)$$

where $(\cdot)^{\bar{k}}$ denotes the kth row of matrix, and vector $\mathbf{x}\in R^m$, $\|\mathbf{x}\|_2=\sqrt{\sum_{i=1}^m x_i^2}$.

Theorem 5 [24]. For any $\tau>0$ and $Y\in R^{m\times n}$, the solution of the problem $\arg\min_{X\in R^{m\times n}}\left\{\tau\|X^T\|_{1,2}+\frac{\mu}{2}\|X-Y\|_F^2\right\}$ is:

$$\left(X^*\right)_{-}^{k}=\max\left\{\left\|Y_{-}^{k}\right\|_2-\tau/\mu,0\right\}\cdot Y_{-}^{k}/\left\|Y_{-}^{k}\right\|_2,i=1,2,...,n \quad (19)$$

where $(\cdot)_{-}^{k}$ denotes the kth col of matrix, and vector $\mathbf{x}\in R^n$, $\|\mathbf{x}\|_2=\sqrt{\sum_{i=1}^n x_i^2}$.

We may formulate the problem (15) as follows:

$$\begin{aligned}&\left(D_{k+1},Z_{k+1},G_{k+1},A_{k+1},B_{k+1}\right)=\\&\min_{\substack{D,Z,G,A,B\in R^{n\times n}\\P_\Omega(E)=0}}\left\{\begin{array}{c}\|D\|_*-Tr(S_kDP_k^T)+\mu\|Z\|_1\\+\varsigma\|G\|_F^2+\lambda_1\|A\|_{1,2}+\lambda_2\|B^T\|_{1,2}\\s.t.\ D+Z+G+A+B+E-M=0\end{array}\right\}\end{aligned} \quad (20)$$

As E will compensate for the unknown entries of M, the unknown entries of M are simply set as zeros. Then the augmented Lagrangian function corresponding to problem (20) is:

$$\begin{aligned}L_\rho\left(D,Z,G,A,B,E,Y\right)&=\|D\|_*-Tr(S_kDP_k^T)\\&+\mu\|Z\|_1+\varsigma\|G\|_F^2+\lambda_1\|A\|_{1,2}+\lambda_2\|B^T\|_{1,2}\\&+\left\langle Y,D+Z+G+A+B+E-M\right\rangle\\&+\frac{\rho}{2}\|D+Z+G+A+B+E-M\|_F^2\end{aligned} \quad (21)$$

where Y is the Lagrange multiplier of the linear constraint, $\rho>0$ is the penalty parameter for the violation of the linear constraint, and $\langle\cdot\rangle$ denotes the standard trace inner product.

Remark 1. We found that problem (21) and problem (22) are equivalence relation. Gaussian noise can smoothed by the penalty ρ. When the variance of Gaussian noise is large, ρ is set to a smaller value. Conversely, ρ set to a larger value.

$$\begin{aligned}L_\rho\left(D,Z,A,B,E,Y\right)&=\|D\|_*-Tr(S_kDP_k^T)\\&+\mu\|Z\|_1+\lambda_1\|A\|_{1,2}+\lambda_2\|B^T\|_{1,2}\\&+\left\langle Y,D+Z+A+B+E-M\right\rangle\\&+\frac{\rho}{2}\|D+Z+A+B+E-M\|_F^2\end{aligned} \quad (22)$$

Remark 2 In order to enhance the stability of the solution of the MRMC problem, we adopt Elastic-net model [25] which combines Frobenius-norm and nuclear norm of the matrix D. Problem (21) can be rewritten as follows.

$$\begin{aligned}L_\rho\left(D,Z,A,B,E,Y\right)&=\|D\|_*-Tr(S_kDP_k^T)\\&+\gamma\|D\|_F^2+\mu\|Z\|_1+\lambda_1\|A\|_{1,2}+\lambda_2\|B^T\|_{1,2}\\&+\left\langle Y,D+Z+A+B+E-M\right\rangle\\&+\frac{\rho}{2}\|D+Z+A+B+E-M\|_F^2\end{aligned} \quad (23)$$

Applying ADMM to (23), given the initial setting $Z_{k+1}^0=Z_k,A_{k+1}^0=A_k,B_{k+1}^0=B_k,E_{k+1}^0=E_k,Y_{k+1}^0=Y_k$, $i=0$ and ignoring constant terms at each iteration, we have the following iterative scheme:

Step (1). Update D

$$\begin{aligned}D_{k+1}^{i+1}=\arg\min_{D\in R^{n\times n}}&\|D\|_*-Tr(S_kXP_k^T)+\gamma\|D\|_F^2+\left\langle Y_{k+1}^i,D\right\rangle\\&+\frac{\rho}{2}\|D+Z_{k+1}^i+A_{k+1}^i+B_{k+1}^i+E_{k+1}^i-M\|_F^2\end{aligned} \quad (24)$$

Node that $Tr(S_kDP_k^T)=\left\langle D,S_k^TP_k\right\rangle$, we have:

$$\begin{aligned}&D_{k+1}^{i+1}=\arg\min_{D\in R^{n\times n}}\|D\|_*+(\gamma+\frac{\rho}{2})\|D+\Delta\|_F^2\\&\Delta=\frac{\rho}{2\gamma+\rho}(Z_{k+1}^i+A_{k+1}^i+B_{k+1}^i+E_{k+1}^i-M+\frac{Y_{k+1}^i-S_k^TP_k}{\rho})\end{aligned} \quad (25)$$

Based on theorem 3, D_{k+1}^{i+1} can be calculated as follows:

$$D_{k+1}^{i+1}=\boldsymbol{D}_{\frac{1}{2\gamma+\rho}}\left(\frac{\rho}{2\gamma+\rho}(M-Z_{k+1}^i-A_{k+1}^i-B_{k+1}^i-E_{k+1}^i-\frac{Y_{k+1}^i-S_k^TP_k}{\rho})\right) \quad (26)$$

Step (2). Update Z

$$\begin{aligned}Z_{k+1}^{i+1}=\arg\min_{Z\in R^{n\times n}}&\mu\|Z\|_1\\&+\frac{\rho}{2}\left\|Z+D_{k+1}^{i+1}+A_{k+1}^i+B_{k+1}^i+E_{k+1}^i-M+\frac{Y_{k+1}^i}{\rho}\right\|_F^2\end{aligned} \quad (27)$$

Based on theorem 2, Z_{k+1}^{i+1} can be calculated as follows:

$$Z_{k+1}^{i+1}=\boldsymbol{S}_{\frac{\mu}{\rho}}(M-D_{k+1}^{i+1}-A_{k+1}^i-B_{k+1}^i-E_{k+1}^i-\frac{Y_{k+1}^i}{\rho}) \quad (28)$$

Step (3). Update A

$$\begin{aligned}A_{k+1}^{i+1}=\arg\min_{A\in R^{n\times n}}&\lambda_1\|A\|_{1,2}\\&+\frac{\rho}{2}\left\|D_{k+1}^{i+1}+Z_{k+1}^{i+1}+A+B_{k+1}^i+E_{k+1}^i-M+\frac{Y_{k+1}^i}{\rho}\right\|_F^2\end{aligned} \quad (29)$$

Based on theorem 4, A_{k+1}^{i+1} can be calculated as follows:

Let $V_*=M-D_{k+1}^{i+1}-Z_{k+1}^{i+1}-B_{k+1}^i-E_{k+1}^i-\frac{Y_{k+1}^i}{\rho}$, then

$$\left(A_{k+1}^{i+1}\right)^{\bar{m}}=\max\left\{0,1-\frac{\lambda_1}{\rho}/\left\|\left(V_*\right)^{\bar{m}}\right\|_2\right\}\times\left(V_*\right)^{\bar{m}} \quad (30)$$

Step (4). Update B

$$\begin{aligned}B_{k+1}^{i+1}=\arg\min_{B\in R^{n\times n}}&\lambda_2\|B^T\|_{1,2}\\&+\frac{\rho}{2}\left\|D_{k+1}^{i+1}+Z_{k+1}^{i+1}+A_{k+1}^{i+1}+B+E_{k+1}^i-M+\frac{Y_{k+1}^i}{\rho}\right\|_F^2\end{aligned} \quad (31)$$

Based on theorem 5, B_{k+1}^{i+1} can be calculated as follows:

Let $U_*=M-D_{k+1}^{i+1}-Z_{k+1}^{i+1}-A_{k+1}^{i+1}-E_{k+1}^i-\frac{Y_{k+1}^i}{\rho}$, then

$$\left(B_{k+1}^{i+1}\right)_{-}^{m}=\max\left\{0,1-\frac{\lambda_2}{\rho}/\left\|\left(U_*\right)_{-}^{m}\right\|_2\right\}\times\left(U_*\right)_{-}^{m} \quad (32)$$

Step (5). Update E

$$E_{k+1}^{i+1}=\arg\min_{P_{\Omega}(E)=0}\left\langle Y_{k+1}^{i},E\right\rangle$$
$$+\frac{\rho}{2}\left\|D_{k+1}^{i+1}+Z_{k+1}^{i+1}+E+A_{k+1}^{i+1}+B_{k+1}^{i+1}-M\right\|_F^2 \quad (33)$$
$$=P_{\bar{\Omega}}(M-D_{k+1}^{i+1}-Z_{k+1}^{i+1}-A_{k+1}^{i+1}-B_{k+1}^{i+1}-\frac{Y_{k+1}^{i}}{\rho})$$

where $\bar{\Omega}$ denotes the complementary set of Ω .

Step (6). Update Y

$$Y_{k+1}^{i+1}=Y_{k+1}^{i}+\rho(D_{k+1}^{i+1}+Z_{k+1}^{i+1}+A_{k+1}^{i+1}+B_{k+1}^{i+1}+E_{k+1}^{i+1}-M) \quad (34)$$

Based on the above analysis, we summarize the whole procedure in Algorithm 1 named as Multi-Norm Regularized Matrix Completion using ADMM (MRMC-ADMM).

Algorithm 1:MRMC-ADMM

Input: Sampled matrix M , the support set Ω , the estimated rank l of objective matrix, and the parameters $\rho,\lambda_1,\lambda_2,\mu,\gamma$.

Output: Objective matrix D^{opt} , row-structured noise matrix A^{opt} , outlier matrix Z^{opt} , row-structured noise matrix B^{opt} .

1.**Initialize** $D_0=P_{\Omega}\left(M\right),Z_0=A_0=B_0=E_0=Y_0,k=0$;
2.**while not converged do**
3. $[U,\Sigma,V]=svd\left(D_k\right),S_k=U\left(:,1:l\right),P_k=V\left(:,1:l\right)^T$;
4. $Z_{k+1}^{0}=Z_k,A_{k+1}^{0}=A_k,B_{k+1}^{0}=B_k$;
5. $E_{k+1}^{0}=E_k,Y_{k+1}^{0}=Y_k,i=0$;
6. **while not converged do**
7. $ADMM\left(D_{k+1}^{i+1},Z_{k+1}^{i+1},A_{k+1}^{i+1},B_{k+1}^{i+1},E_{k+1}^{i+1},Y_{k+1}^{i+1},i=i+1\right)$;
8. **end while**
9. $D_{k+1}=D_{k+1}^{i+1},Z_{k+1}=Z_{k+1}^{i+1},A_{k+1}=A_{k+1}^{i+1},B_{k+1}=B_{k+1}^{i+1}$;
10. $E_{k+1}=E_{k+1}^{i+1},Y_{k+1}=Y_{k+1}^{i+1},k=k+1$;
11.**end while**
12.**Output:** $D^{opt}\leftarrow D_k,Z^{opt}\leftarrow Z_k,A^{opt}\leftarrow A_k,B^{opt}\leftarrow B_k$.

In Algorithm 1, the step (7) attempts to find the exact solution of problem (23) which is dispensable. In fact, the convergence of Algorithm 2 for problem (23) is guaranteed by the ADMM. So the updating of D,Z,A,B,E,Y only once is adequate. This leads to Algorithm 2 which called inexact-MRMC-ADMM.

Algorithm 2: inexact-MRMC-ADMM

Input: Sampled matrix M , the support set Ω , the estimated rank l of objective matrix, and the parameters $\rho,\lambda_1,\lambda_2,\mu,\gamma$.

Output: Objective matrix D^{opt} , row-structured noise matrix A^{otp} outlier matrix Z^{otp} , row-structured noise matrix B^{otp} .

1.**Initialize** $D_0=P_{\Omega}(M),Z_0=A_0=B_0=E_0=Y_0,k=0;$
2.**while not converged do**
3. $\left[U,\Sigma,V\right]=svd(D_k),S_k=U\left(:,1:l\right)^T,P_k=V\left(:,1:l\right)^T$;
4. $E_{k+1}=P_{\bar{\Omega}}(M-D_k-Z_k-A_k-B_k-\frac{Y_k}{\rho})$;
5. $Z_{k+1}=\mathbf{S}_{\frac{\mu}{\rho}}(M-D_k-A_k-B_k-E_{k+1}-\frac{Y_k}{\rho})$;
6. $\left(A_{k+1}\right)_{-}^{m}=\max\left\{0,1-\frac{\lambda_1}{\rho}/\left\|\left(V_*\right)_{-}^{m}\right\|_2\right\}\times\left(V_*\right)_{-}^{m}$;
7. $\left(B_{k+1}\right)_{-}^{m}=\max\left\{0,1-\frac{\lambda_2}{\rho}/\left\|\left(U_*\right)_{-}^{m}\right\|_2\right\}\times\left(U_*\right)_{-}^{m}$;
8. $D_{k+1}=\boldsymbol{D}_{\frac{1}{2\gamma+\rho}}\left(\frac{\rho}{2\gamma+\rho}(M-Z_{k+1}-A_{k+1}-B_{k+1}-E_{k+1}-\frac{Y_k-S_k^TP_k}{\rho})\right)$;
9. $Y_{k+1}=Y_k+\rho(D_{k+1}+Z_{k+1}+A_{k+1}+B_{k+1}+E_{k+1}-M)$;
10. $k=k+1$;
11.**end while**
12.**Output:** $D^{opt}\leftarrow D_k,Z^{opt}\leftarrow Z_k,A^{opt}\leftarrow A_k,B^{opt}\leftarrow B_k$.

In Algorithm 2, the main computational cost in each iteration is the SVD in step3 and step8. For large-size problems, we use the well-known PROPACK package (http://sun.stanford.edu/~rmunk/PROPACK/) to accomplish the SVD.

B. Robust Localization Based on the Reconstructed EDM

We can reconstruct EDM by utilizing Algorithm 2, which means that all pair-wise range measurements between sensor nodes are available. So MDS-based method descripted in Section II can be applied to derive node locations that fit those range measurements. Based on this method, the proposed Noise-tolerant Wireless Sensor Networks Localization via Multi-norms Regularized Matrix Completion (LMRMC) is summarized in Algorithm 3.

Algorithm 3: LMRMC

Input: Sampled matrix M ,the support set Ω , the estimated rank l of objective matrix, and the positions of anchor nodes $\left\{T_1,T_2,...,T_k\right\}\left(k>3\right)$.

Output: The absolute positions of unknown nodes: $\left\{T_i \mid i=k+1,k+2,...,n\right\}$.

#Reconstructing EDM#

1. **Reconstruct EDM** from incomplete sampled matrix M by utilizing inexact-MRMC-TNNR-ADMM.

$$\min_{D,Z,A,B\in R^{n\times n}}\left\|D\right\|_*-Tr(S_kDP_k^T)+\gamma\left\|D\right\|_F^2+\mu\left\|Z\right\|_1+\lambda_1\left\|A\right\|_{1,2}+\lambda_2\left\|B^T\right\|_{1,2}$$
$$+\left\langle Y,D+Z+A+B+E-M\right\rangle+\frac{\rho}{2}\left\|D+Z+A+B+E-M\right\|_F^2 ;$$

 # Calculating locations by MDS-based method#

2. **Double center** the reconstructed matrix D :

$$Temp=-\frac{1}{2}JDJ$$

Where $J=I-\frac{1}{n}\mathbf{1}\cdot\mathbf{1}^T$ and I is the identity matrix;

3. **Decompose** matrix $Temp$:

$$\left[U,\Lambda,V\right]=svd\left(Temp\right);$$

4. **Compute** the coordinate-transform matrix:

$$W=\left[W_1,W_2,...,W_3\right]=\Lambda_2^{1/2}U_2^T$$

Where:

$W_i \in R^{2\times 1}, \Lambda_2 = \Lambda(1:2,1:2)$ and $U_2 = U(:,1:2)$;

5. **Build** the coordinate-transform matrix:

$$Q = Q_1 Q_2 = \frac{[T_2 - T_1, T_3 - T_1, ..., T_k - T_1]}{[W_2 - W_1, W_3 - W_1, ..., W_k - W_1]};$$

6. **Map** the relative coordinates to absolute ones:

$$\{T_i = Q \cdot (W_i - W_1) + T_i, i = k+1, k+2, ..., n\};$$

7. **Output:** $\{T_i, i = k+1, k+2, ..., n\}$.

V. PERFORMANCE EVALUATION

In order to evaluate the performance of the proposed LMRMC algorithm, EDM reconstruction error, localization error, outlier recognition accuracy and structural noise recognition accuracy are observed by comparing with the SVT-based localization algorithm [7] and NLIRM method [6]. We consider a WSN consisting of 200 nodes which randomly distributed in a 100×100 unit square area with 6 of them are anchor node, and then generate random coordinates and EDM of these nodes. Finally, after adding noise to original EDM, we sample the EDM. Four contrast experiments are designed to evaluate the performance of our algorithm under different noise conditions and sample rate, i.e. Integrity degree of the EDM.

A. Evaluation Metrics

To evaluate the performance of Algorithm 3, some evaluation metrics are defined as follows. Let $X \in R^{2\times 200}, D \in R^{200\times 200}$ denotes the coordinate matrix of all nodes and EDM respectively.

1) EDM reconstruction error:

$$e_c = \|D^* - D\|_F / \|D\|_F,$$

where D^* denotes the reconstructed EDM;

2) Localization error:

$$e_l = \|X^* - X\|_F / n,$$

where X^* denotes the absolute coordinate matrix;

3) Outlier recognition accuracy:

$$o_l = 2\cdot\frac{l_1 \cdot l_2}{l_1 + l_2},\ l_1 = \frac{o\_true}{o\_all}, l_2 = \frac{o\_true}{o\_act},$$

where o_all and o_true denote the number of outlier recognized by algorithm and the number of true outlier among them, respectively. o_act is the actual number of outlier;

4) Row-structured noise recognition accuracy:

$$s_r = 2\cdot\frac{r_1 \cdot r_2}{r_1 + r_2},\ r_1 = \frac{r\_true}{r\_all}, r_2 = \frac{r\_true}{r\_act},$$

where r_all and r_true denote the number of row-structured noise recognized by LMRMC and the number of true row-structured noise among them, respectively. r_act is the actual number of row-structured noise;

5) Col-structured noise recognition accuracy:

$$s_c = 2\cdot\frac{c_1 \cdot c_2}{c_1 + c_2},\ c_1 = \frac{c\_true}{c\_all}, c_2 = \frac{c\_true}{c\_act},$$

where c_all and c_true denote the number of col-structured noise recognized by LMRMC and the number of true col-structured noise among them, respectively. c_act is the actual number of col-structured noise.

B. Contrast Experiment

Four categories contrast experiments are designed to evaluate the performance of LMRMC under different noise conditions, and Gaussian noise and outlier are called common noise in this paper.

- **NCNS: no common noise, no structural noise.**
- **WCNS: with common noise, no structural noise.**
- **NCWS: no common noise, with structural noise.**
- **WCWS: with common noise, with structural noise.**

1) Localization under NCNS

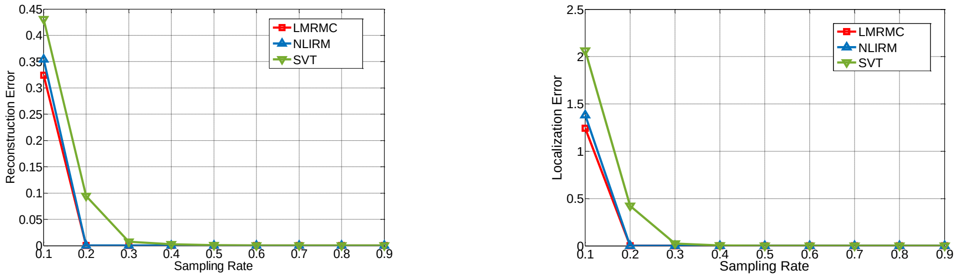
This set of experiments assumes that the sampled distance measurements are credible and deterministic. Reconstruction error and localization error are showed in (a) and (b) respectively in Fig. 3. It is obvious that both reconstruction error and localization error of each algorithms decline rapidly and maintain at a same and low level as the increase of sampling rate. The performance of SVT is inferior to the other two algorithms in terms of both localization error and reconstruction error when the sampling rate is below 30% as it introduces nuclear norm rather than truncated nuclear norm to approximate the rank function of EDM. Further, we can see that LMRMC and NLIRM achieve a similar performance in both reconstruction error and localization error. Most strikingly, in the absence of noise, our algorithm can achieve a close to zero localization error when the sampling rate reaching 20%.

2) Localization under WCNS

This set of experiments assumes that the sampled measurements are contaminated by both Gaussian noise with zero mean and variance of 100 and outlier with value from 5000 to 10000. The ratio of outlier in LMRMC and NLIRM is set to be 5%, while the ratio in SVT algorithm is only 1%. In Fig. 4 (a) (b), LMRMC and NLIRM handle the Gaussian noise and outlier quite well, and they can acquire a close to zero reconstruction error and localization error when the sampling rate reaching 30%. However the SVT-based method does not seem to work well under the condition of WCNS even the sampling rate reaching to 90%, so we can conclude that the SVT-based method cannot recognize the position of outlier. In Fig. 4 (c), we can see that both LMRMC and NLIRM are able to recognize the positions of outlier accurately when the sampling rate reaches a certain bound value. Benefited by this ability, node fault diagnosis, scheduling policy and topology adjustment can be carried out more easily.

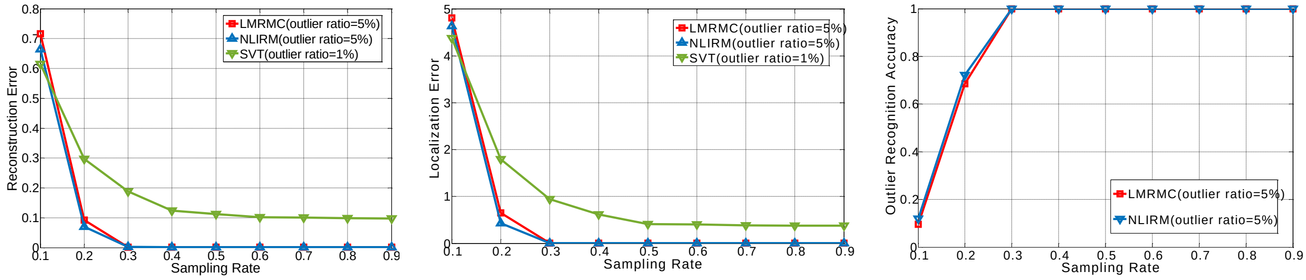
3) Localization under NCWS

This set of experiments assumes that the sampled distance measurements are affected by structural noise. The value of each element in structural noise is randomly generated from -500 to 500. The ratio of this noise in LMRMC is 5% (2.5% row-structured noise and 2.5% col-structured noise), while in SVT-based method and NLIRM is 1% (0.5% row-structured noise and 0.5%col-structured noise). From Fig. 5 (a) (b), we



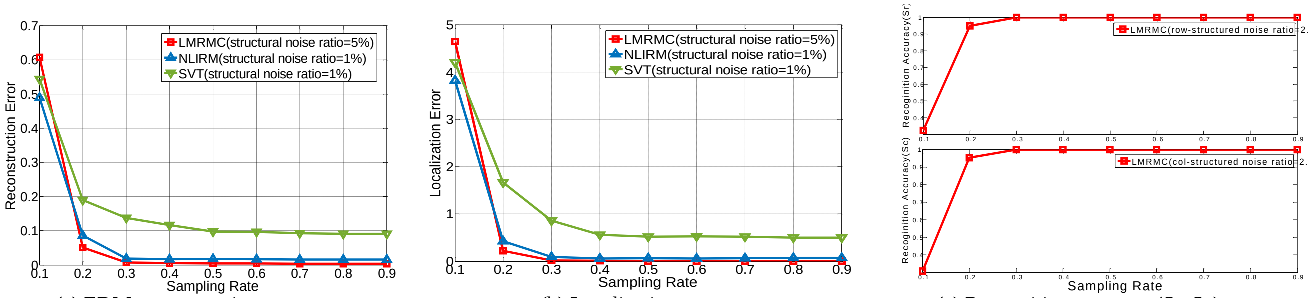
(a) EDM reconstruction error (b) Localization error

Fig. 3. Performance evaluation under the condition of NCNS



(a) EDM reconstruction error (b) Localization error (c) Outlier recognition accuracy

Fig. 4. Performance evaluation under the condition of WCNS



(a) EDM reconstruction error (b) Localization error (c) Recognition accuracy (Sr, Sc)

Fig. 5. Performance evaluation under the condition of NCWS

can see that LMRMC is still a little bit better than NLIRM despite only contaminated by a 1% of structural noise in terms of reconstruction accuracy and localization accuracy, and SVT-based method is at a complete loss in the face of structural noise. Furthermore, in Fig. 5 (c), we can find that LMRMC is able to recognize the positions of structural noise, that is to say, it can estimate the positions of node receiving/sending module failure. These abilities are also beneficial to node fault diagnosis.

4) *Localization under WCWS*

This set of experiments assumes that the distance measurements are contaminated by a mixed noise (Gaussian noise, outlier and structural noise). The Gaussian noise owns a mean value of zero and variance of 100, the value of each outlier is generated from 5000 to 10000 randomly and the value of each element of structural noise is from -500 to 500. The ratio of outlier and structural noise can be seen in Fig. 6. Reconstruction error and localization error are shown in Fig. 6 (a) (b), similarly, LMRMC is still a little bit better than NLIRM despite contaminated by a 1% of structural noise in terms of reconstruction accuracy and localization accuracy, and performance of SVT-based method is worse. Furthermore, in Fig. 6 (c), the outlier recognition accuracy of NLIRM is not so optimistic due to the impact of structural noise. Most strikingly, in Fig. 6 (c) (d), we can see that LMRMC obtains a close to 100% outlier recognition accuracy and structural noise recognition accuracy when the sampling rate reaching to 30%.

VI. CONCLUSION

This study explored the data missing and noise problem in wireless sensor networks localization. Accurate and sufficient location information of nodes is the prerequisite for most WSNs applications, but noise and data missing are inevitable distance ranging. Then a noise-tolerant WSNs localization method via multi-norms matrix completion (LMRMC) is proposed in this paper, in which simulations suggest that LMRMC achieves a satisfactory performance on dealing with data missing and noise compared with traditional algorithms during the distance measurement in WSNs.

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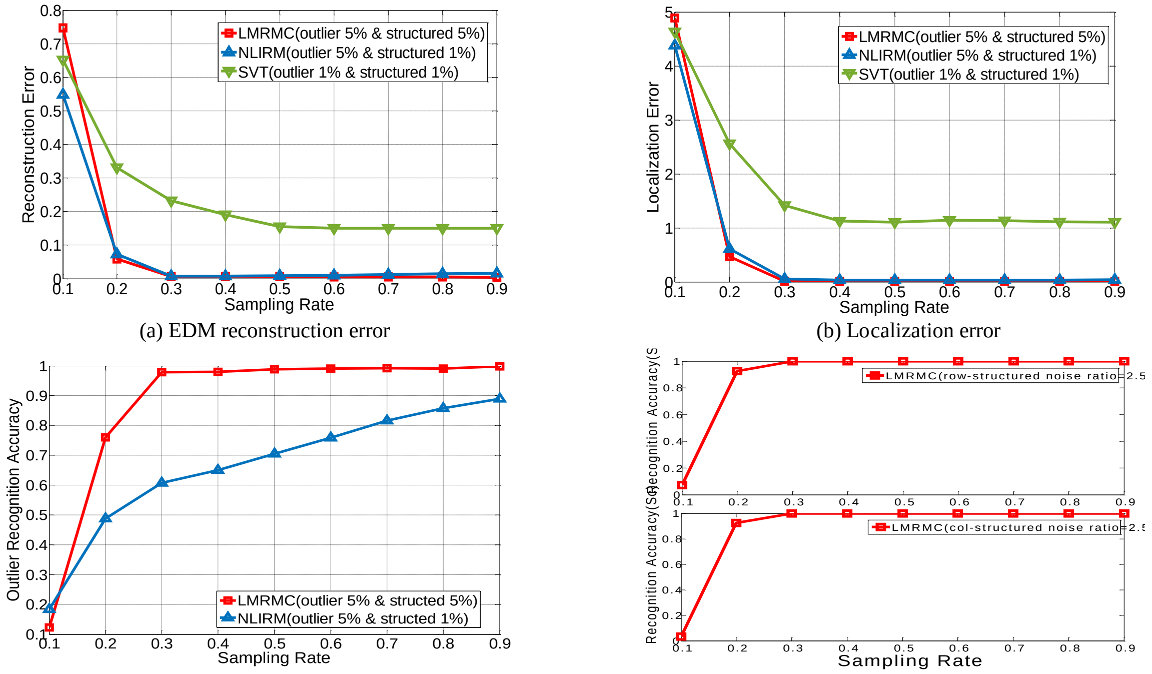

(a) EDM reconstruction error

(b) Localization error

(c) Outlier recognition accuracy

(d) Recognition accuracy (Sr, Sc)

Fig. 6. Performance evaluation under the condition of WCWS

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