Dynamic Decision Models: Identification and Estimation

Yingyao Hu and Matthew Shum

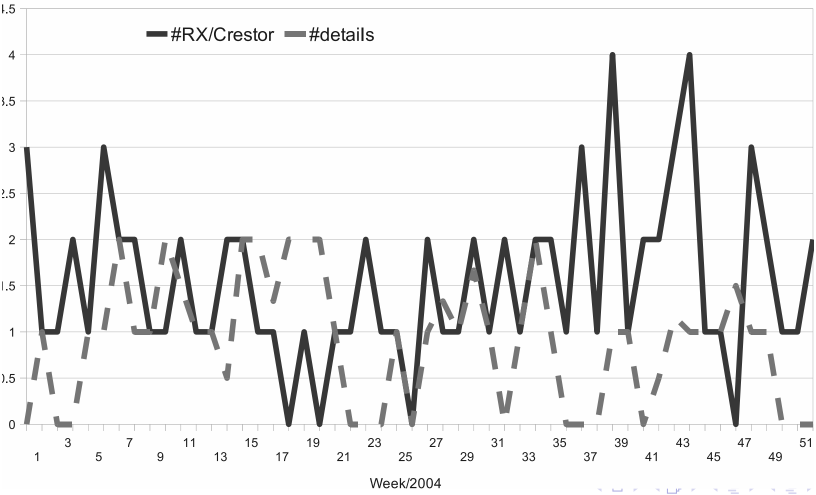
Johns Hopkins University & Caltech

April 15, 2012

Dynamic decision models

"Doctor #1022"'s weekly prescriptions of *Crestor*, 2004

$\Rightarrow$ can learning model explain?



Examples: Introduce some notation

- **Example 1: Dynamic learning**
  - Y_t: which brand is consumed
  - M_t: # advertisements seen
  - X_t^*: current beliefs (posterior mean) about each brand
- **Example 2: Dynamic Investment Model**
  - Y_t: firm investment
  - M_t: capital stock
  - X_t^*: firm-level productivity
- Examples of **Markov dynamic choice models** with *serially correlated unobserved state variables*

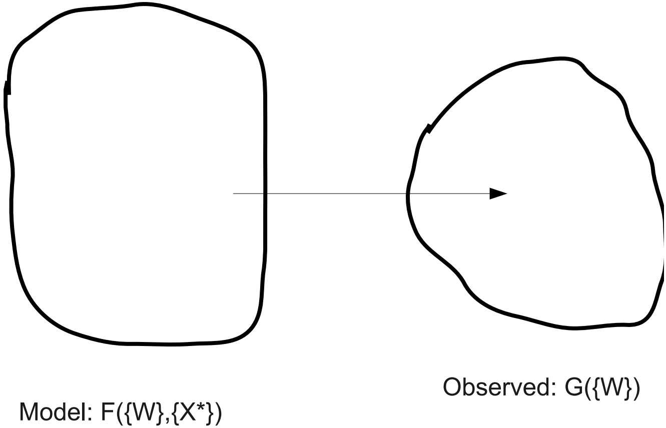
Introduction

All these model reduce to the following "data problem":

- Model follows first-order Markov process $\{W_t, X_t^*\}_{t=1}^T$
- But only $\{W_t\}$ for $t = 1, 2, ..., T$ is observed
  - In many empirical dynamic models, $W_t = (Y_t, M_t)$:
    - ★ Y_t is choice variable: agent's action in period t
    - ★ M_t is observed state variable
  - X_t^* is persistent (serially-correlated) unobserved state variable
- Focus on nonparametric identification of Markov law of motion $Pr(W_t, X_t^* | W_{t-1}, X_{t-1}^*)$.
  For estimation of models: structural components fully summarized by this law of motion

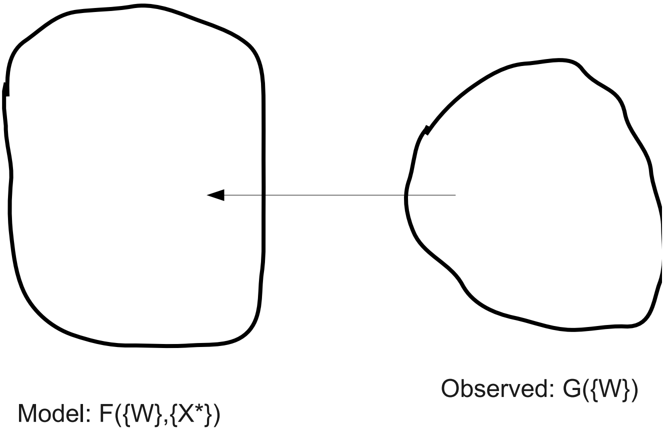
Identification: rough schematics

Model $\Rightarrow$ empirical implications on data



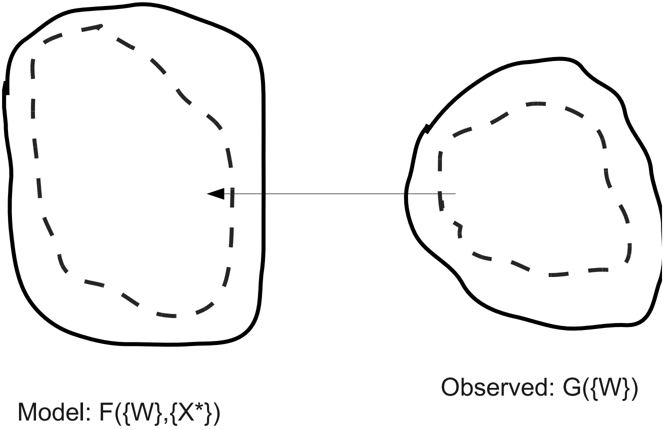
Identification: rough schematics

Identification: data $\Rightarrow$ model



Identification: rough schematics

Identification result: typically involve restrictions on both model and observables



What we do

- New features of our results (relative to literature):
  - X_t^* serially correlated: *unobserved state variable*
  - X_t^* can be continuous or discrete
  - Feedback: evolution of X_t^* can depend on W_{t-1}, X_{t-1}^*
  - Novel identification approach: use recent findings from nonclassical measurement error econometrics: Hu (2008), Hu-Schennach (2007), Carroll, Chen, and Hu (2008). Details
- Relation to *Hidden State Markov* models (hmm)

Basic setup: conditions for identification

- Consider dynamic processes $\{(W_T, X_T^*), ..., (W_t, X_t^*), ..., (W_1, X_1^*)\}_i$, i.i.d across agents $i \in \{1, 2, \ldots, n\}$.
- The researcher has panel data: $\{W_{t+1}, W_t, W_{t-1}, W_{t-2}, W_{t-3}\}_i$ for many agents i (5 obs)
- Assumption: (Dimension-reduction) The process (W_t, X_t^*) satisfies

(i) *First-order Markov*: $f_{W_t, X_t^* | W_{t-1}, ..., W_1, X_{t-1}^*, ..., X_1^*} = f_{W_t, X_t^* | W_{t-1}, X_{t-1}^*}$

- Standard in most applications of DDC models

(ii) *Limited feedback:* $f_{W_t | W_{t-1}, X_t^*, X_{t-1}^*} = f_{W_t | W_{t-1}, X_t^*}$. Picture

- Satisfied in many empirical applications
- think of as timing restriction (X_t^* occurs before M_t). Details

Special case: Discrete X_t^*

- Main result for case of continuous X_t^*
- Build intuition by considering discrete case:

$$\forall t, \; X_t^* \in \mathcal{X}^* \equiv \{1, 2, \ldots, J\}.$$

- For convenience, assume W_t also discrete, with same support $\mathcal{W}_t = \mathcal{X}_t^*$.
- In what follows:
  - "L" denotes J-square matrix
  - "D" denotes J-diagonal matrix.

Backbone of argument

- BROWN: elements identified from data
- PURPLE: elements identified in proof

For fixed (w_t, w_{t-1}), in matrix notation: here

- **Lemma 2:** Markov law of motion $L_{w_t,X_t^*|w_{t-1},X_{t-1}^*}$ Proof

$$= L^{-1}_{W_{t+1}|w_t,X_t^*} L_{W_{t+1},w_t|w_{t-1},W_{t-2}} L^{-1}_{W_t|w_{t-1},W_{t-2}} L_{W_t|w_{t-1},X_{t-1}^*}$$

  Hence, all we must identify are $L_{W_{t+1}|w_t,X_t^*}$ and $L_{W_t|w_{t-1},X_{t-1}^*}$.

- **Lemma 3:** From $f_{W_{t+1},w_t|w_{t-1},W_{t-2}}$, identify $L_{W_{t+1}|w_t,X_t^*}$.
- Stationary case: $L_{W_{t+1}|w_t,X_t^*} = L_{W_t|w_{t-1},X_{t-1}^*}$, so Lemma 3 implies identification (4 obs)
- Non-stationary case: apply Lemma 3 in turn to $f_{W_{t+1},w_t|w_{t-1},W_{t-2}}$ and $f_{W_t,w_{t-1}|w_{t-2},W_{t-3}}$ (5 obs)

Lemma 3: proof

- Similar to Carroll, Chen, and Hu (2008)
- Key factorization: $f_{W_{t+1},W_t,W_{t-1},W_{t-2}}$

$$
\begin{aligned}
&= \int\int f_{W_{t+1},W_t,W_{t-1},W_{t-2},X_t^*,X_{t-1}^*} dx_t^* dx_{t-1}^* \\
&= \int\int f_{W_{t+1}|W_t,X_t^*} \cdot f_{W_t,X_t^*|W_{t-1},X_{t-1}^*} \cdot f_{W_{t-1},W_{t-2},X_{t-1}^*} dx_t^* dx_{t-1}^* \\
&= \int\int f_{W_{t+1}|W_t,X_t^*} \cdot f_{W_t|W_{t-1},X_t^*,X_{t-1}^*} \cdot f_{X_t^*,X_{t-1}^*,W_{t-1},W_{t-2}} dx_t^* dx_{t-1}^* \\
&= \int f_{W_{t+1}|W_t,X_t^*} f_{W_t|W_{t-1},X_t^*} \cdot f_{X_t^*,W_{t-1},W_{t-2}} dx_t^*
\end{aligned}
$$

- Discrete-case, matrix notation (for any fixed w_t, w_{t-1}) details:

$$
L_{W_{t+1},w_t|w_{t-1},W_{t-2}} = L_{W_{t+1}|w_t,X_t^*} D_{w_t|w_{t-1},X_t^*} L_{X_t^*|w_{t-1},W_{t-2}}
$$

Lemma 3: Proof (cont'd)

- Important feature: for (w_t, w_{t-1}),

$$L_{W_{t+1},w_t|w_{t-1},W_{t-2}} = \underbrace{L_{W_{t+1}|w_t,X_t^*}}_{\text{no } w_{t-1}} \underbrace{D_{w_t|w_{t-1},X_t^*}}_{\text{only } J \text{ unkwns.}} \underbrace{L_{X_t^*|w_{t-1},W_{t-2}}}_{\text{no } w_t}$$

- for (w_t, w_{t-1}), $(\overline{w}_t, w_{t-1})$, $(\overline{w}_t, \overline{w}_{t-1})$ $(w_t, \overline{w}_{t-1})$,

$$\begin{aligned}
L_{W_{t+1},w_t|w_{t-1},W_{t-2}} &= L_{W_{t+1}|w_t,X_t^*} \quad D_{w_t|w_{t-1},X_t^*} \quad \underbrace{L_{X_t^*|w_{t-1},W_{t-2}}} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| \\
L_{W_{t+1},\overline{w}_t|w_{t-1},W_{t-2}} &= \underbrace{L_{W_{t+1}|\overline{w}_t,X_t^*}} \quad D_{\overline{w}_t|w_{t-1},X_t^*} \quad \overbrace{L_{X_t^*|w_{t-1},W_{t-2}}} \\
&\qquad\quad \| \\
L_{W_{t+1},\overline{w}_t|\overline{w}_{t-1},W_{t-2}} &= \overbrace{L_{W_{t+1}|\overline{w}_t,X_t^*}} \quad D_{\overline{w}_t|\overline{w}_{t-1},X_t^*} \quad \underbrace{L_{X_t^*|\overline{w}_{t-1},W_{t-2}}} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| \\
L_{W_{t+1},w_t|\overline{w}_{t-1},W_{t-2}} &= L_{W_{t+1}|w_t,X_t^*} \quad D_{w_t|\overline{w}_{t-1},X_t^*} \quad \overbrace{L_{X_t^*|\overline{w}_{t-1},W_{t-2}}}
\end{aligned}$$

Lemma 3: Proof (cont'd)

- Assume: (Invertibility) LHS invertible, which is testable
- eliminate $L_{X_t^*|w_{t-1},W_{t-2}}$ using first two equations

$$\begin{aligned}\mathbf{A} &\equiv L_{W_{t+1},w_t|w_{t-1},W_{t-2}} L^{-1}_{W_{t+1},\overline{w}_t|w_{t-1},W_{t-2}} \\ &= L_{W_{t+1}|w_t,X_t^*} D_{w_t|w_{t-1},X_t^*} D^{-1}_{\overline{w}_t|w_{t-1},X_t^*} L^{-1}_{W_{t+1}|\overline{w}_t,X_t^*}\end{aligned}$$

- eliminate $L_{X_t^*|\overline{w}_{t-1},W_{t-2}}$ using last two equations

$$\begin{aligned}\mathbf{B} &\equiv L_{W_{t+1},w_t|\overline{w}_{t-1},W_{t-2}} L^{-1}_{W_{t+1},\overline{w}_t|\overline{w}_{t-1},W_{t-2}} \\ &= L_{W_{t+1}|w_t,X_t^*} D_{w_t|\overline{w}_{t-1},X_t^*} D^{-1}_{\overline{w}_t|\overline{w}_{t-1},X_t^*} L^{-1}_{W_{t+1}|\overline{w}_t,X_t^*}\end{aligned}$$

- eliminate $L^{-1}_{W_{t+1}|\overline{w}_t,X_t^*}$

$$\mathbf{AB}^{-1} = L_{W_{t+1}|w_t,X_t^*} D_{w_t,\overline{w}_t,w_{t-1},\overline{w}_{t-1},X_t^*} L^{-1}_{W_{t+1}|w_t,X_t^*}$$

with diagonal matrix

$$D_{w_t,\overline{w}_t,w_{t-1},\overline{w}_{t-1},X_t^*} = D_{w_t|w_{t-1},X_t^*} D^{-1}_{\overline{w}_t|w_{t-1},X_t^*} D_{\overline{w}_t|\overline{w}_{t-1},X_t^*} D^{-1}_{w_t|\overline{w}_{t-1},X_t^*}$$

Lemma 3: Proof (cont'd)

Eigenvalue-eigenvector decomposition of observed $\mathbf{AB}^{-1}$

$$\mathbf{AB}^{-1} = L_{W_{t+1}|w_t,X_t^*} D_{w_t,\overline{w}_t,w_{t-1},\overline{w}_{t-1},X_t^*} L^{-1}_{W_{t+1}|w_t,X_t^*}$$

- eigenvalues: diagonal entry in $D_{w_t,\overline{w}_t,w_{t-1},\overline{w}_{t-1},X_t^*}$

$$\left(D_{w_t,\overline{w}_t,w_{t-1},\overline{w}_{t-1},X_t^*}\right)_{j,j} = \frac{f_{W_t|W_{t-1},X_t^*}(w_t|w_{t-1},j) f_{W_t|W_{t-1},X_t^*}(\overline{w}_t|\overline{w}_{t-1},j)}{f_{W_t|W_{t-1},X_t^*}(\overline{w}_t|w_{t-1},j) f_{W_t|W_{t-1},X_t^*}(w_t|\overline{w}_{t-1},j)}$$

Assume: (Unique eigendecomposition) $\left(D_{w_t,\overline{w}_t,w_{t-1},\overline{w}_{t-1},X_t^*}\right)_{j,j}$ are distinctive

- eigenvector: column in $L_{W_{t+1}|w_t,X_t^*}$, (normalized because sums to 1)

Hence, $L_{W_{t+1}|w_t,X_t^*}$ is identified (up to the value of x_t^*). Any permutation of eigenvectors yields same decomposition.

Lemma 3: Proof (cont'd)

To pin-down the value of x_t^*: need to "order" eigenvectors

- $f_{W_{t+1}|W_t,X_t^*}(\cdot|w_t,x_t^*)$ for any w_t is identified up to value of x_t^*. So– to pin-down:
- Assume: (Normalization) there is *known* functional

$$h(w_t,x_t^*) \equiv G\left[f_{W_{t+1}|W_t,X_t^*}(\cdot|w_t,\cdot)\right] \text{ is monotonic in } x_t^*.$$

  $G[f]$ may be mean, mode, median, other quantile of f.
  Then set $x_t^* = G\left[f_{W_{t+1}|W_t,X_t^*}(\cdot|w_t,\cdot)\right]$
- Note: in unobserved heterogeneity case ($X_t^* = X^*, \ \forall t$), it is enough to identify $f_{W_{t+1}|W_t,X_t^*}$.

Continuous case

- generalize the results in discrete case

discrete X_t^*	$\Rightarrow$	continuous X_t^*
matrix	$\Rightarrow$	linear operator here
invertible	$\Rightarrow$	one-to-one, "injective"
matrix diagonalization	$\Rightarrow$	spectral decomposition
eigenvector	$\Rightarrow$	eigenfunction

- $W_t = \mathcal{W}_t \subseteq \mathbb{R}^d$, $X_t^* \in \mathcal{X}_t^* \subseteq \mathbb{R}$, for all t. Details

Assumptions for continuous case

[1] (Dimension-reduction) (i) First-order Markov; (ii) Limited feedback

[2] (Invertibility) There exists variable(s) $V \subseteq W$ such that
(i) for any $w_t \in \mathcal{W}_t$, there exists a $w_{t-1} \in \mathcal{W}_{t-1}$ and a neighborhood $\mathcal{N}^r$ around (w_t, w_{t-1}) such that, for any $(\overline{w}_t, \overline{w}_{t-1}) \in \mathcal{N}^r$, $L_{V_{t-2},\overline{w}_{t-1},\overline{w}_t,V_{t+1}}$ is one-to-one;
(ii) for any $w_t \in \mathcal{W}_t$, $L_{V_{t+1}|w_t,X_t^*}$ is one-to-one;
(iii) for any $w_{t-1} \in \mathcal{W}_{t-1}$, L_{V_{t-2},w_{t-1},V_t} is one-to-one.

Assumptions (cont'd)

[3] (unique eigendecomposition) For any $w_t \in \mathcal{W}_t$ and any $\overline{x}_t^* \neq \widetilde{x}_t^* \in \mathcal{X}_t^*$, there exists a $w_{t-1} \in \mathcal{W}_{t-1}$ and corresponding neighborhood $\mathcal{N}^r$ satisfying Assumption 2.i such that, for some $(\overline{w}_t, \overline{w}_{t-1}) \in \mathcal{N}^r$ with $\overline{w}_t \neq w_t,\ \overline{w}_{t-1} \neq w_{t-1}$:
(i) $0 < k\left(w_t, \overline{w}_t, w_{t-1}, \overline{w}_{t-1}, x_t^*\right) < C < \infty$ for any $x_t^* \in \mathcal{X}_t^*$ and some constant C;
(ii) $k(w_t, \overline{w}_t, w_{t-1}, \overline{w}_{t-1}, \overline{x}_t^*) \neq k(w_t, \overline{w}_t, w_{t-1}, \overline{w}_{t-1}, \widetilde{x}_t^*)$, where

$$k\left(w_t, \overline{w}_t, w_{t-1}, \overline{w}_{t-1}, x_t^*\right) = \frac{f_{W_t|W_{t-1},X_t^*}(w_t|w_{t-1}, x_t^*) f_{W_t|W_{t-1},X_t^*}(\overline{w}_t|\overline{w}_{t-1}, x_t^*)}{f_{W_t|W_{t-1},X_t^*}(\overline{w}_t|w_{t-1}, x_t^*) f_{W_t|W_{t-1},X_t^*}(w_t|\overline{w}_{t-1}, x_t^*)}.$$

[4] (normalization) For any $w_t \in \mathcal{W}_t$, there exists a known functional G such that $G\left[f_{V_{t+1}|W_t,X_t^*}(\cdot|w_t, x_t^*)\right]$ is monotonic in x_t^*. We normalize $x_t^* = G\left[f_{V_{t+1}|W_t,X_t^*}(\cdot|w_t, x_t^*)\right]$.

Main results

- **Theorem 1**: Under assumptions above, the density $f_{W_{t+1},W_t,W_{t-1},W_{t-2},W_{t-3}}$ uniquely determines $f_{W_t,X_t^*|W_{t-1},X_{t-1}^*}$
- **Corollary 1**: With stationarity, the density $f_{W_{t+1},W_t,W_{t-1},W_{t-2}}$ uniquely determines $f_{W_2,X_2^*|W_1,X_1^*}$
- For specific *dynamic discrete-choice models* (IO, labor, health), we can use existing arguments from Magnac-Thesmar (2002), Bajari-Chernozhukov-Hong-Nekipelov to argue identification of utility functions, once $W_t, X_t^*|W_{t-1}, X_{t-1}^*$ known here

Estimation

- Identification results are *population* statements: what is, in principle, possible to recover when "data are unlimited"
- With finite samples, estimation issues are at forefront.
- Two cases:
  1. X^* continuous: approximate infinite-dimensional functions as finite-order polynomials. "Spectral decomposition" estimator (Still working...)
  2. X^* discrete: identification argument can be directly mimicked for estimation. Application to learning in experimental data.
- Virtue of both approaches: estimator is *non-iterative* (ie. not cast as optimizer of objective function). Computationally convenient.

Examples

- Empirical IO example: advertising and pharmaceuticals here
- Experimental example: learning model here
- Deeper details on assumptions for continuous case: John Rust bus-engine replacement model here

Conclusion

Empirical application: *Crestor* RX in early 2004

- Dataset: from *ImpactRX*. Sample period: 2004.
- With Wei Tan (SUNY-Stony Brook), Tim Derdenger (CMU)
  - One day a week sample of doctors' prescriptions of statin drugs
  - Complete log of detailing visits.
  - Focus on prescriptions of *Crestor* (entered October 2003)
  - Unique for (i) doctor-lvl advertising; (ii) time series of prescriptions
- Estimate binary discrete choice model. Allow detailing to be endogenous, due to unobserved X^*.

Persuasive vs. Informative Advertising

- Crestor contraindicated for Asian patients:

  > *Pharmacokinetic studies show an approximate 2-fold elevation in median exposure in Japanese subjects residing in Japan and in Chinese subjects residing in Singapore when compared with Caucasians residing in North America and Europe.*

- Does detailing increase doctors' prescriptions to Asians?
- If detailing is *informative*, then "no"; if it is *persuasive*, then "yes"

Reduced-form evidence indicates: Yes

$M =$	$Y =$	$P(Y=1\|M)$		$P(M'=1\|Y,M)$	
		Asians	non-Asians	Asians	non-Asians
0		0.1086	0.2452		
0	0			0.1731	0.1816
0	1			0.5147	0.5780
1		0.2000	0.3489		
1	0			0.3158	0.2396
1	1			0.5882	0.6505
# obs		260	22928		
$P(Y=1)$		0.1308	0.2811		
$P(M=1)$		0.3000	0.3323		

How robust? Use framework to control for *endogeneity of advertising:* time-varying unobservables X^* which may be related to detailing incidence (beliefs?)

Estimates for CCP $P(Y = 1|X^*, M)$

$M =$	$X^* =$	Asians	non-Asians	Asians	non-Asians
0	0	0.0913 (0.1104)	0.1114 (0.0058)	0.1086	0.2452
0	1	0.6806 (0.2302)	0.6318 (0.0174)		
1	0	0.1420 (0.0983)	0.1474 (0.0109)	0.2000	0.3489
1	1	0.9208 (0.2496)	0.6568 (0.0171)		

- "Causal effect" of M strikingly different depending on X^*
- Interpreting X^* as "optimism" of doctors about Crestor, then detailing especially leads optimistic doctors to prescribe Crestor.
- Result also robust to number of "less structural" checks.

Estimates: Laws of motion for detailing $P(M' = 1|M, Y, X^*)$

$X^* =$	$M =$	$Y =$	Asians	non-Asians
0	0	0	0.1727 (0.0884)	0.1754 (0.0052)
0	0	1	0.2481 (0.2272)	0.2022 (0.0220)
0	1	0	0.5062 (0.1388)	0.5669 (0.0107)
0	1	1	0.1629 (0.2586)	0.6527 (0.0324)
1	0	0	0.9168 (0.3143)	0.2538 (0.0204)
1	0	1	0.5037 (0.3135)	0.2676 (0.0119)
1	1	0	0.4198 (0.2935)	0.6319 (0.0276)
1	1	1	0.0000 (0.3935)	0.6532 (0.0141)

Estimates for law of motion $P(X^{*\prime} = 1 | M', M, X^*)$

Go back

$M' =$	$M =$	$X^* =$	Asians	non-Asians
0	0	0	0.0322 (0.1868)	0.0010 (0.0072)
0	0	1	0.9933 (0.4641)	0.9546 (0.0357)
0	1	0	0.0115 (0.1954)	0.0140 (0.0215)
0	1	1	0.9933 (0.2047)	0.9580 (0.0498)
1	0	0	0.0804 (0.2360)	0.0095 (0.0213)
1	0	1	0.7223 (0.2610)	1.0000 (0.0124)
1	1	0	0.0067 (0.2236)	0.0097 (0.0200)
1	1	1	0.3849 (0.3191)	1.0000 (0.0090)

Discuss assumptions: example from Rust (1987)

Consider particular version of Rust (1987): $W_t = (Y_t, M_t)$:

- $Y_t \in \{0, 1\}$ (don't replace, replace)
- M_t is mileage
- X_t^* denotes weather shocks, driver health shocks.
  Assume X_t^* is trunc. normal process w/ bounded support $[L, U]$:

$$X_t^* = 0.8X_{t-1}^* + 0.2\nu_t, \quad \nu_t \sim N(0, 1), \text{ i.i.d.}$$

- wlog, focus on observations $t = 1, 2, 3, 4$
- Very brief discussion, gory details in paper/online appendix
- Construct example where conditions directly verifiable: sufficient conditions

Two different specifications:

Specification A	Specification B
$u_t = \begin{cases} -c(M_t) + X_t^* + \epsilon_{0t}, & Y_t = 0 \\ -RC + \epsilon_{1t}, & Y_t = 1. \end{cases}$ $c(\cdot)$ bounded away from $0, +\infty$	$u_t = \begin{cases} -c(M_t) + \epsilon_{0t} \\ -RC + \epsilon_{1t} \end{cases}$ ...
$M_{t+1} = \begin{cases} M_t + \eta_{t+1}, & Y_t = 0 \\ \eta_{t+1}, & Y_t = 1 \end{cases}$ η_t are distd extreme-value, i.i.d.	$M_{t+1} = \begin{cases} M_t + \exp(\eta_{t+1} + X_{t+1}^*) \\ \exp(\eta_{t+1} + X_{t+1}^*). \end{cases}$ ...

- Specifications differ in where X_t^* enters.
- Discuss each assumption in turn
- Assumption 1 (Markov, LF) satisfied

Assumption 2: invertibility assumptions

Use $V_t = M_t$ (continuous element of W_t).

- Analogous to completeness conditions in econometric IV literature:

$$\int f(x,y)h(y)dy = 0 \ \forall x \ \Rightarrow \ h(y) = 0 \ a.s.$$

- Direct verification possible because of convolution structure to laws of motion for M, X^*.
- **Model A:** Distribution of M_{t+1} does not depend on (w_{t-1}, M_{t-2}): $L_{M_{t+1},w_t|w_{t-1},M_{t-2}}$ cannot be invertible.
  NB: when M_{t+1} depends just on w_t, but not on X^*_{t+1}, then cannot use $V_t = M_t$: "too little feedback".
- **Model B:** Invertibility of required operators obtains, under additional restrictions on initial conditions:
  - M_0, X_0^* are independent.
  - $Y_1 = 0$ with probability 1, exogenous.

Assumption 3: Bounded, distinct eigenvalues

Eigenvalues are $\frac{f_{W_3|W_2,X_3^*}(w_3|w_2,j)f_{W_3|W_2,X_3^*}(\overline{w}_3|\overline{w}_2,j)}{f_{W_3|W_2,X_3^*}(\overline{w}_3|w_2,j)f_{W_3|W_2,X_3^*}(w_3|\overline{w}_2,j)}$.

Must be (i) $\in (0,1)$; (ii) distinct for all j.

- Eigenvalues depend on behavior of joint density:

$$f_{W_3|W_2,X_3^*} = f_{Y_3|M_3,X_3^*} \cdot f_{M_3|Y_3,M_2,X_3^*}$$

- First term (CCP) doesn't depend on W_2: cancels out from eigenvalues
- **Model A:** M_3 doesn't depend on X_3^*, never distinct.
- **Model B:** by judicious choice of $(w_2, \bar{w}_3, \bar{w}_2)$ for any w_3, can show that eigenvalue is (i) bounded; and (ii) monotonic in x_3^*. Thus satisfying assumption.

Assumption 4

Appropriate normalization to pin down unobserved X_t^*

- For Spec. B, median of $f_{M_4|M_3,Y_3,X_3^*}(\cdot|m_3, y_3, z)$ is

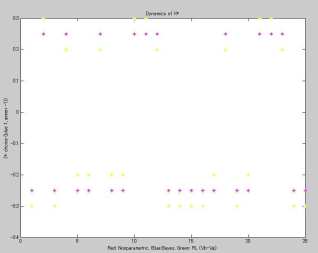
$$(1-y_3)m_3 + (m_3)^{1-y_3} C_{med} \cdot \exp(0.8z)$$

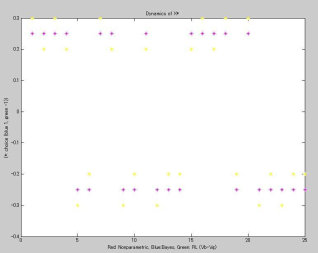
where $C_{med} = \text{med}\left[\exp(\eta_4 + 0.2\nu_4)\right]$ (fixed).

- This is monotonic in z
- So pin down $x_t^* = med\left[f_{M_{t+1}|M_t,Y_t,X_t^*}(\cdot|m_t, y_t, x_t^*)\right]$

Go back

"The Eyes Have It" (with Yutaka Kayaba)





- Can learning model explain these choices?
- Nonparametric estimates of learning rules (beliefs) and choice rules
- Auxiliary data on eye movements ("noisy" measures of beliefs)
- NP approach appears new relative to both behavioral/experimental, and empirical IO/marketing lits
  - No functional form restrictions: "what the subjects actually think"
  - Take auxiliary measures seriously: cannot test

Experimental setup: probabilistic reversal learning

- Follows Hampton, Bossaerts, O'Doherty (2006)
- Two-armed bandit: BLUE or GREEN
- Each period: one arm is "good", one is "bad". This state variable not observed by subjkects:

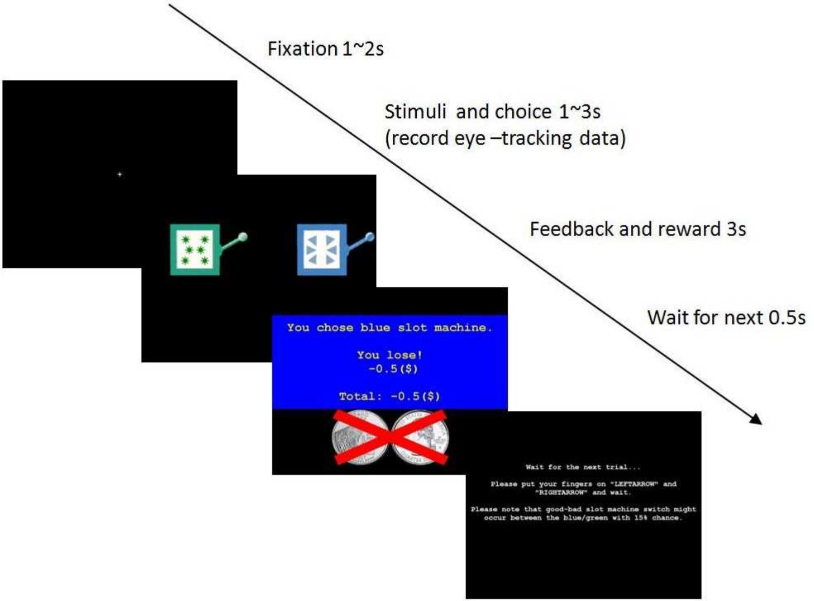
$$S_t = \begin{cases} 1 & \text{GREEN good} \\ 2 & \text{BLUE good} \end{cases} \qquad P(S'|S) = \begin{bmatrix} 0.85 & 0.15 \\ 0.15 & 0.85 \end{bmatrix}$$

  "Probabilistic reversal": good arm becomes bad with probability 0.15 each period. S_t serially correlated: opportunity for subjects to learn.
- Good (bad) arm: yields reward

$$R_t = \begin{cases} 2(= \$0.50) & \text{with prob } 0.7\ (0.4) \\ 1(= -\$0.50) & \text{with prob } 0.3\ (0.6) \end{cases}$$

Data

- 21 subjects: from CIT Social Science Experimental Laboratory pool (Caltech undergrads, postdocs, community members)
- November-December 2009
- Each subject plays for 200 periods (broken in 8 blocks of 25 trials). Choice $Y_t \in \{1(\text{green}), 2(\text{blue})\}$
- Attached to eye-tracker; eye-movements recorded. Z_t


Fixation 1~2s
Stimuli and choice 1~3s
(record eye –tracking data)
Feedback and reward 3s
Wait for next 0.5s
You chose blue slot machine.
You lose!
-0.5($)
Total: -0.5($)
Wait for the next trial...
Please put your fingers on "LEFTARROW" and "RIGHTARROW" and wait.
Please note that good-bad slot machine switch might occur between the blue/green with 15% chance.

Nonparametric Learning model

- Map experimental task into learning model (much simpler than previous framework)
- Decision variable $Y_t = 1$ (green) or 2 (blue).
- R_t: reward $\in \{\text{"1"}, \text{"2"}\}$
- Unobserved state X_t^*; summarizes beliefs about state S_t (higher value more favorable towards blue).
  Assume *beliefs take three values:*
  $X_t^* \in \{1(\text{green}), 2(\text{not sure}), 3(\text{blue})\}$.
- Noisy measure Z_t: Fraction of reaction time spent gazing at blue (from eye-tracker).
  Also discretize Z_t into 3 regions.

Identification and Estimation

- From observed data $\{Y_t, R_t, Z_t\}$, we want to identify/estimate
  1. choice probabilities $P(Y_t|X_t^*)$;
  2. learning rule $P(X^*{}_{t+1}|X_t^*, R_t, Y_t)$;
  3. measurement rule $P(Z_t|X_t^*)$
- Identification follows from previous argument (much simpler, in fact)
- Mimick identification argument directly for estimation
- NB: Z_t is "instrument" in the sense of being excluded from choice and learning rules.
- Use both time-series (across trials) and cross-sectional (across agents) variation. Standard errors via bootstrap.

Summary statistics for Y, R, Z_p, RT, Z

	gr	bl
Y	2108	2092

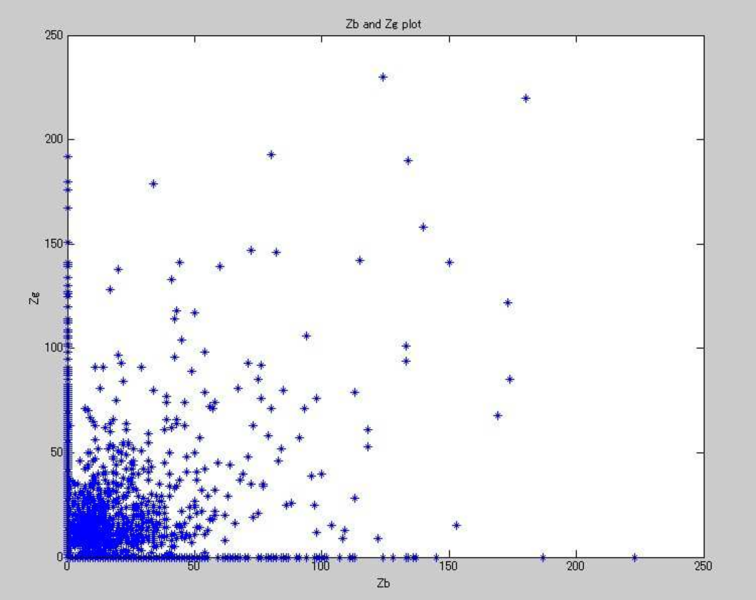
	mean	medi	upper 5%	lower 5%
Z_p	-0.0309	0	1.3987	-1.4091
RT	88.22	59.3	212.2	36.8

	hi	lo
R	2398	1802

Sample size	21 subj	168 bl	4200 tr
Corr.(Y,Z_p)			0.7647

Z (after discretization with three values)

sid.	1(gr, $Z_t <$ -sid.)	2(not sure)	3(bl, $Z_t >$ sid.)
0.05	2015	255	1930
(baseline) 0.20	1887	540	1773
0.40	1725	869	1606



Scatter plot of Z_b (fixation on blue) and Z_g(fixation on green)

Three-value estimates: choice and measurement probs

Choice probabilities $P(Y_t|X_t^*)$

X_t^*	1(green)	2(not sure)	3(blue)
$Y_t = 1$	0.9866	0.4421	0.0064
(green)	(0.0561)	(0.1274)	(0.0146)
2	0.0134	0.5579	0.9936
(blue)			

Choice rule: ϵ-greedy?

Measurement probabilities $P(Z_t|X_t^*)$

X_t^*	1(green)	2(not sure)	3(blue)
$Z_t = 1$	0.8639	0.2189	0.0599
(green)	(0.0468)	(0.1039)	(0.0218)
2	0.0815	0.6311	0.0980
(middle)	(0.0972)	(0.1410)	(0.0369)
3	0.0546	0.1499	0.8421
(blue)	(0.0581)	(0.1206)	(0.0529)

Results (2): learning rules $X'|X, R, Y$

$P(X^*_{t+1}\|X^*_t, y, r)$, r = 1, y = 1(green)				$P(X^*_{t+1}\|X^*_t, y, r)$, r = 2, y = 1(green)		
X^*_t	1(green)	2 (unsure)	3(blue)	1(green)	2 (unsure)	3(blue)
$X^*_{t+1} = 1$	0.5724	0.3075	0.1779	0.8889	0.6621	0.8242
(green)	(0.0694)	(0.0881)	(0.2257)	(0.0894)	(0.1309)	(0.2734)
2	0.0000	0.3138	0.4002	0.0000	0.2702	0.1758
(unsure)	(0.0662)	(0.1042)	(0.2284)	(0.0911)	(0.1297)	(0.1981)
3	0.4276	0.3787	0.4219	0.1111	0.0678	0.0000
(blue)	(0.0624)	(0.0945)	(0.2195)	(0.0340)	(0.0485)	(0.1876)

Focus on belief updating after green choices (results analogous)

Results (2): learning rules $X'|X, R, Y$

X_t^*	$P(X_{t+1}^*\|X_t^*, y, r)$, r = 1, y = 1(green): 1(green)	2 (unsure)	3(blue)	$P(X_{t+1}^*\|X_t^*, y, r)$, r = 2, y = 1(green): 1(green)	2 (unsure)	3(blue)
$X_{t+1}^* = 1$	0.5724	0.3075	0.1779	0.8889	0.6621	0.8242
(green)	(0.0694)	(0.0881)	(0.2257)	(0.0894)	(0.1309)	(0.2734)
2	0.0000	0.3138	0.4002	0.0000	0.2702	0.1758
(unsure)	(0.0662)	(0.1042)	(0.2284)	(0.0911)	(0.1297)	(0.1981)
3	0.4276	0.3787	0.4219	0.1111	0.0678	0.0000
(blue)	(0.0624)	(0.0945)	(0.2195)	(0.0340)	(0.0485)	(0.1876)

First columns show learning after "exploitative" (belief-congruent) choices: given $X_t^* = 1$ (positive to green), choosing "green" and getting high reward triggers more updating towards $X_{t+1}^* = 1$ (green): 89% vs. 57%

Results (2): learning rules $X'|X, R, Y$

$P(X^*_{t+1}|X^*_t, y, r)$, r = 1, y = 1(green)				$P(X^*_{t+1}|X^*_t, y, r)$, r = 2, y = 1(green)		
X^*_t	1(green)	2 (unsure)	3(blue)	1(green)	2 (unsure)	3(blue)
$X^*_{t+1} = 1$	0.5724	0.3075	0.1779	0.8889	0.6621	0.8242
(green)	(0.0694)	(0.0881)	(0.2257)	(0.0894)	(0.1309)	(0.2734)
2	0.0000	0.3138	0.4002	0.0000	0.2702	0.1758
(unsure)	(0.0662)	(0.1042)	(0.2284)	(0.0911)	(0.1297)	(0.1981)
3	0.4276	0.3787	0.4219	0.1111	0.0678	0.0000
(blue)	(0.0624)	(0.0945)	(0.2195)	(0.0340)	(0.0485)	(0.1876)

OTOH, non-small probability that beliefs jump to "blue": 11% even after positive reward. Optimal belief-updating in probabilistic reversal context?

Results (2): learning rules $X'|X, R, Y$

$P(X^*_{t+1}\|X^*_t, y, r)$, r = 1, y = 1(green)				$P(X^*_{t+1}\|X^*_t, y, r)$, r = 2, y = 1(green)		
X^*_t	1(green)	2 (unsure)	3(blue)	1(green)	2 (unsure)	3(blue)
$X^*_{t+1} = 1$	0.5724	0.3075	0.1779	0.8889	0.6621	0.8242
(green)	(0.0694)	(0.0881)	(0.2257)	(0.0894)	(0.1309)	(0.2734)
2	0.0000	0.3138	0.4002	0.0000	0.2702	0.1758
(unsure)	(0.0662)	(0.1042)	(0.2284)	(0.0911)	(0.1297)	(0.1981)
3	0.4276	0.3787	0.4219	0.1111	0.0678	0.0000
(blue)	(0.0624)	(0.0945)	(0.2195)	(0.0340)	(0.0485)	(0.1876)

Third columns show learning after "exploratory" (low probability contrarian) choices.

Given $X^*_t = 3$ (positive to blue), and choosing "green", stronger updating towards green ($X^*_{t+1} = 1$) after low reward (82% vs. 18%)

Results (2): learning rules $X'|X, R, Y$

$P(X^*_{t+1}\|X^*_t, y, r)$, r = 1, y = 1(green)				$P(X^*_{t+1}\|X^*_t, y, r)$, r = 2, y = 1(green)		
X^*_t	1(green)	2 (unsure)	3(blue)	1(green)	2 (unsure)	3(blue)
$X^*_{t+1} = 1$	0.5724	0.3075	0.1779	0.8889	0.6621	0.8242
(green)	(0.0694)	(0.0881)	(0.2257)	(0.0894)	(0.1309)	(0.2734)
2	0.0000	0.3138	0.4002	0.0000	0.2702	0.1758
(unsure)	(0.0662)	(0.1042)	(0.2284)	(0.0911)	(0.1297)	(0.1981)
3	0.4276	0.3787	0.4219	0.1111	0.0678	0.0000
(blue)	(0.0624)	(0.0945)	(0.2195)	(0.0340)	(0.0485)	(0.1876)

Second column show learning following (almost-) random choices: stronger updating towards "green" when green choice yielded higher reward (66% vs. 31%)

Results (2): learning rules $X'|X, R, Y$

$P(X^*_{t+1}\|X^*_t, y, r)$, r = 1, y = 1(green)				$P(X^*_{t+1}\|X^*_t, y, r)$, r = 2, y = 1(green)		
X^*_t	1(green)	2 (unsure)	3(blue)	1(green)	2 (unsure)	3(blue)
$X^*_{t+1} = 1$	0.5724	0.3075	0.1779	0.8889	0.6621	0.8242
(green)	(0.0694)	(0.0881)	(0.2257)	(0.0894)	(0.1309)	(0.2734)
2	0.0000	0.3138	0.4002	0.0000	0.2702	0.1758
(unsure)	(0.0662)	(0.1042)	(0.2284)	(0.0911)	(0.1297)	(0.1981)
3	0.4276	0.3787	0.4219	0.1111	0.0678	0.0000
(blue)	(0.0624)	(0.0945)	(0.2195)	(0.0340)	(0.0485)	(0.1876)

Overall, results pretty sensible. Considerable "stickiness" in beliefs.

Results (2): learning rules $X'|X, R, Y$

$P(X^*_{t+1}|X^*_t, y, r)$, r = 1, y = 1(green) | $P(X^*_{t+1}|X^*_t, y, r)$, r = 2, y = 1(green)

X^*_t	1(green)	2 (unsure)	3(blue)	1(green)	2 (unsure)	3(blue)
$X^*_{t+1} = 1$	0.5724	0.3075	0.1779	0.8889	0.6621	0.8242
(green)	(0.0694)	(0.0881)	(0.2257)	(0.0894)	(0.1309)	(0.2734)
2	0.0000	0.3138	0.4002	0.0000	0.2702	0.1758
(unsure)	(0.0662)	(0.1042)	(0.2284)	(0.0911)	(0.1297)	(0.1981)
3	0.4276	0.3787	0.4219	0.1111	0.0678	0.0000
(blue)	(0.0624)	(0.0945)	(0.2195)	(0.0340)	(0.0485)	(0.1876)

$P(X^*_{t+1}|X^*_t, y, r)$, r = 1, y = 2(blue) | $P(X^*_{t+1}|X^*_t, y, r)$, r = 2, y = 2(blue)

X^*_t	3(blue)	2 (unsure)	1(green)	3(blue)	2 (unsure)	1(green)
$X^*_{t+1} = 3$	0.5376	0.2297	0.2123	0.8845	0.6163	0.6319
(blue)	(0.0890)	(0.0731)	(0.1436)	(0.1000)	(0.1136)	(0.1647)
2	0.0458	0.2096	0.1086	0.0000	0.3558	0.3566
(unsure)	(0.0732)	(0.0958)	(0.1524)	(0.0968)	(0.1160)	(0.1637)
1	0.4166	0.5607	0.6792	0.1155	0.0279	0.0116
(green)	(0.0874)	(0.0968)	(0.1881)	(0.0499)	(0.0373)	(0.0679)

Comparing NP to prespecified learning models

Compare predictive ability of NP model to other models

- Bayesian model
- Reinforcement learning (temporal difference learning with softmax choice probabilities)

NB:

- NP model estimated from data, so should perform better.
- But NP model use eye-movement data; other models don't. Can do poorly is Z not related to choices (from above, this isn't the case)

Comparison of 3 learning models

- $\{X_t^*\}$: beliefs from NP model which maximize conditional ("posterior") probability $P(\{X_t^*\} \mid \{Y_t, R_t, Z_t\})$
- Bayesian beliefs $\{B_t^*\}$ (directly computed given true data generating process of experiment). Myopic (not Gittins-type)
- RL valuation $\left\{V_t^* = V_b^t - V_g^t\right\}$: calibrated

Panel 1:

X^*	1(green)	2(unsure)	3(blue)
	1878	366	1956

Panel 2:

	mean	median	std.	1/3 quantile	2/3 quantile
B^*	0.4960	0.5000	0.1433	0.4201	0.5644
V^*	-0.0035	0	1.1152	-0.6588	0.6068

Corr.(Y, X^*)	0.7552
Corr.(Y, B^*)	0.5560
Corr.(Y, V^*)	0.5175

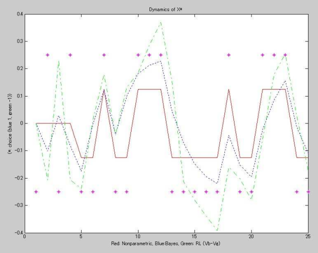
	last 10	first 5
Corr.(Y, X^*)	0.7474	0.6908
Corr.(Y, B^*)	0.5582	0.5201
Corr.(Y, V^*)	0.5267	0.4678

Corr.(X^*, B^*)	0.5274
Corr.(X^*, V^*)	0.5874
Corr.(B^*, V^*)	0.8271

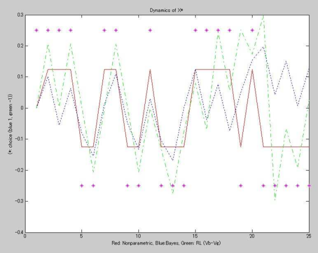
Corr.(Z, X^*)	0.8575
Corr.(Z, B^*)	0.4296
Corr.(Z, V^*)	0.4717

Incidence of "exploratory" (contrarian) choices

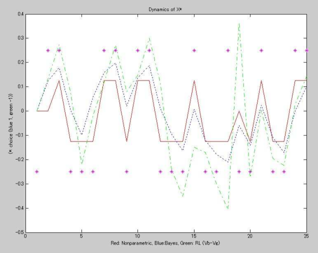
Nonparametric	402
Reinforcement Learning	455
Bayesian	543



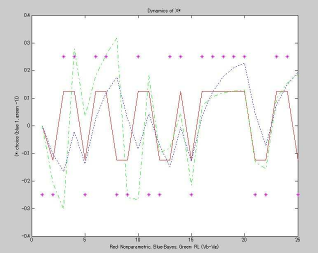
Subject 6, block 4



Subject 4, block 6



Subject 5, block 8



Subject 1, block 3

Concluding remarks

- Identification of Markov process $f_{W_t, X_t^* | W_{t-1}, X_{t-1}^*}$, where X_t^* is unobserved state variable
  1. nonstationary: law of motion $f_{W_t, X_t^* | W_{t-1}, X_{t-1}^*}$ identified from $f_{W_{t+1}, W_t, W_{t-1}, W_{t-2}, W_{t-3}}$ (5 obs.)
  2. stationary: law of motion $f_{W_2, X_2^* | W_1, X_1^*}$ identified from $f_{W_{t+1}, W_t, W_{t-1}, W_{t-2}}$ (4 obs.)
- Learning application: can try other types of data with both choice and non-choice variables (fMRI, primates?)
- Other applications:
  - Dynamic games of incomplete information Details
  - Auction models: (i) unobserved # bidders; (ii) unobserved heterogeneity (with David McAdams, Yonghong An)
  - Multiple equilibria in empirical games models (with Emerson Melo)
  - Mergers in online markets, where number of firms not observed (only prices) (with Mike Baye & John Morgan)

Details on limited feedback

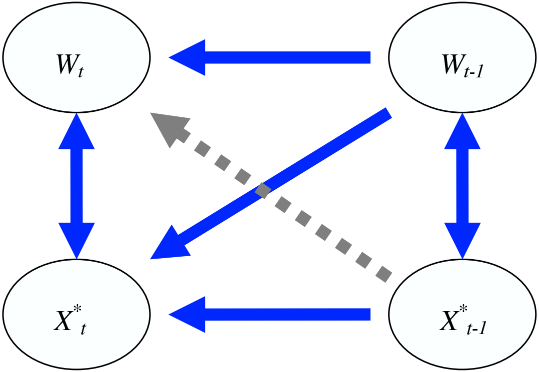
1. Learning model (Crawford-Shum; Ching; Erdem-Keane)
   - Y_t: choice of drug treatment
   - M_t: # times drug has been tried
   - X_t^*: current beliefs ("posterior mean") regarding drug effectiveness
2. Dynamic stockpiling model (Hendel-Nevo)
   - Y_t: brand of detergent purchased
   - M_t: inclusive values from each detergent brand
   - X_t^*: inventory of detergent

- In both these models, evolution of M_t depends just on (Y_{t-1}, M_{t-1}), not on X_t^* or X_{t-1}^*.
- Note: little restriction on evolution of X_{t+1}^*, can depend on $X_{t-1}^*, Y_{t-1}, M_{t-1}$.

Return

Flowchart

Return



Identifying utility functions (sketch)

Assumptions:

1. Action set: $\mathcal{Y} = \{0, 1, \ldots, K\}$.
2. State variables are $S \equiv (M, X^*)$.
3. Per-period utility from choosing $y \in \mathcal{Y}$:

$$u_y(S_t) + \epsilon_{y,t}, \;\; \forall y \in \mathcal{Y}, \;\; \epsilon \sim F(\epsilon), \;\; i.i.d.$$

4. From data, the CCP's $P_y(S) \equiv \text{Prob}(Y = 1|S)$ and state transitions $p(S'|Y, S)$ are identified. (Main Theorem)
5. $u_0(S) = 0$, for all S
6. Discount factor β is known.

Goal: From $W', X^{*\prime}|W, X^*$, identify $u_y(\cdot), \;\; y = 1, \ldots, K$

Identifying utility functions

- From HM, MT: $\exists$ *known* one-to-one mapping $q(S) : \mathbb{R}^K \to \mathbb{R}^K$, which maps $(p_1(S), \ldots, p_K(S))$ to $(\Delta_1(S), \ldots, \Delta_K(S))$, where

  $\Delta_y(S) \equiv V_y(S) - V_0(S)$ diff. in choice-specific value functions.

- "Bellman" equation for zero choice:

  $$V_0(S) = \beta E_{S'|S,Y}\left[G(\Delta_1(S'), \ldots, \Delta_K(S')) + V_0(S')\right].$$

  Hence, can recover $V_0(\cdot)$ function. G is "social-surplus" function (known).

- Hence, utilities identified from

  $$u_y(S) = V_y(S) - \beta E_{S'|S,Y}\left[G(\Delta_1(S'), \ldots, \Delta_K(S')) + V_0(S')\right], \ \forall y \in \mathcal{Y}$$

Logit case Return

Linear operators

- for example, for given w_t, w_{t-1}

$$\left(L_{W_{t+1}|w_t,X_t^*}h\right)(x) = \int f_{W_{t+1}|W_t,X_t^*}(x|w_t,x_t^*)h(x_t^*)dx_t^*$$

$$\left(L_{W_{t+1},w_t|w_{t-1},X_{t-2}}h\right)(x) = \int f_{W_{t+1},W_t|W_{t-1},X_{t-2}}\left(x,w_t|w_{t-1},z\right)h(z)dz.$$

- Matrix is linear operator in finite-dimensional space

Return

Continuous case: Step 1

Return

- The key equation is

$$f_{V_{t+1},W_t|W_{t-1},V_{t-2}} = \int f_{V_{t+1}|W_t,X_t^*} f_{W_t|W_{t-1},X_t^*} f_{X_t^*|W_{t-1},V_{t-2}} dx_t^*.$$

- decomposition of an observed operator

$$L_{V_{t+1},w_t|w_{t-1},V_{t-2}} L^{-1}_{V_{t+1},\overline{w}_t|w_{t-1},V_{t-2}} \left(L_{V_{t+1},w_t|\overline{w}_{t-1},V_{t-2}} L^{-1}_{V_{t+1},\overline{w}_t|\overline{w}_{t-1},V_{t-2}} \right)^{-1}$$
$$= L_{V_{t+1}|w_t,X_t^*} D_{w_t,\overline{w}_t,w_{t-1},\overline{w}_{t-1},X_t^*} L^{-1}_{V_{t+1}|w_t,X_t^*}$$

where a diagonal operator $D_{w_t,\overline{w}_t,w_{t-1},\overline{w}_{t-1},X_t^*}$:

$$\left(D_{w_t,\overline{w}_t,w_{t-1},\overline{w}_{t-1},X_t^*} g \right)(x_t^*) = k\left(w_t, \overline{w}_t, w_{t-1}, \overline{w}_{t-1}, x_t^*\right) g(x_t^*).$$

- eigenvalue for index x_t^*

$$k\left(\ldots, x_t^*\right) = \frac{f_{W_t|W_{t-1},X_t^*}(w_t|w_{t-1}, x_t^*) f_{W_t|W_{t-1},X_t^*}(\overline{w}_t|\overline{w}_{t-1}, x_t^*)}{f_{W_t|W_{t-1},X_t^*}(\overline{w}_t|w_{t-1}, x_t^*) f_{W_t|W_{t-1},X_t^*}(w_t|\overline{w}_{t-1}, x_t^*)}.$$

Matrix definitions

- $L_{w_t,X_t^*,w_{t-1},W_{t-2}} = \left[f_{W_t,X_t^*,W_{t-1},W_{t-2}}(w_t,i|w_{t-1},j) \right]_{i,j}$ Return 2
-

$$
\begin{aligned}
L_{w_t,X_t^*|w_{t-1},W_{t-2}} &= \left[f_{W_t,X_t^*|W_{t-1},W_{t-2}}(w_t,i|w_{t-1},j) \right]_{i,j} \\
L_{w_t,X_t^*|w_{t-1},X_{t-1}^*} &= \left[f_{W_t,X_t^*|W_{t-1},X_{t-1}^*}(w_t,i|w_{t-1},j) \right]_{i,j} \\
L_{X_{t-1}^*|w_{t-1},W_{t-2}} &= \left[f_{X_{t-1}^*|W_{t-1},W_{t-2}}(i|w_{t-1},j) \right]_{i,j}
\end{aligned}
$$

Return 3

Logit case

$$G(\Delta_1(S), \ldots, \Delta_K(S)) = \log \left[1 + \sum_{y=1}^{K} \exp(\Delta_y(S)) \right]$$

$$q_y(S) = \Delta_y(S) = \log(p_y(S)) - \log(p_0(S)), \ \forall y = 1, \ldots K,$$

where $p_0(S) \equiv 1 - \sum_{y=1}^{K} p_y(S)$.

Return

Matrix notation

- Define the J-by-J matrices (fix w_t and w_{t-1})

$$
\begin{aligned}
L_{W_{t+1},w_t|w_{t-1},W_{t-2}} &= \left[f_{W_{t+1},W_t|W_{t-1},W_{t-2}}(i, w_t|w_{t-1}, j)\right]_{i,j} \\
L_{W_{t+1}|w_t,X_t^*} &= \left[f_{W_{t+1}|W_t,X_t^*}(i|w_t, j)\right]_{i,j} \\
L_{X_t^*|w_{t-1},W_{t-2}} &= \left[f_{X_t^*|W_{t-1},W_{t-2}}(i|w_{t-1}, j)\right]_{i,j}
\end{aligned}
$$

$$
D_{w_t|w_{t-1},X_t^*} = \begin{bmatrix} f_{W_t|W_{t-1},X_t^*}(w_t|w_{t-1},1) & 0 & 0 \\ 0 & \ldots & 0 \\ 0 & 0 & f_{W_t|W_{t-1},X_t^*}(w_t|w_{t-1},J) \end{bmatrix}
$$

Return1

Return2

Extensions

1. Companion work on dynamic games Return
   - X_t^* is multivariate (X_t^* includes USV's for each player).
   - For example, dynamic capacity investment
     - $Y_t = (Y_{1t}, Y_{2t})$: each firm's capacity investment
     - $M_t = (M_{1t}, M_{2t})$: each firm's total capacity
     - $X_t^* = (X_{1t}^*, X_{2t}^*)$: each firm's productivity
   - Consider alternatives to LF:

$$f_{W_t, X_t^* | W_{t-1}, X_{t-1}^*} = \underbrace{f_{Y_t | M_t, X_t^*}}_{\text{CCP}} \cdot \underbrace{f_{X_t^* | M_t, M_{t-1}, X_{t-1}^*}}_{X \text{ transition}} \cdot \underbrace{f_{M_t | Y_{t-1}, M_{t-1}, X_{t-1}^*}}_{M \text{ transition}}$$

   - Can apply arguments in Hu-Schennach (2008):

$$f_{Y_t, M_t, Y_{t-1} | M_{t-1}, Y_{t-2}} = \int f_{Y_t | M_t, M_{t-1}, X_{t-1}^*} f_{M_t, Y_{t-1} | M_{t-1}, X_{t-1}^*} f_{X_{t-1}^* | M_{t-1}, Y_{t-2}} dx_{t-1}^*$$

   - Assumption 4 more complicated: monotonicity not enough.
2. Two-step estimation (as in HM, BBL):
   - Estimate CCP, LOM by sieve MLE
   - Estimate structural parameters from optimality conditions

Relation to literature: nonclassical measurement errors

"Message": in X-section context, three "observations" (x, y, z) of latent x^* enough to identify (x, y, z, x^*)

- Hu (2008, JOE): X^*-discrete latent variable

$$f_{X,Y,Z} = \sum_{x^*} f_{X|X^*} f_{Y|X^*} f_{X^*,Z}$$

- Hu and Schennach (2008, ECMA): X^*:continuous latent variable

$$f_{X,Y,Z} = \int f_{X|X^*} f_{Y|X^*} f_{X^*,Z} dx^*$$

- Carroll, Chen and Hu (2008): $S-$sample indicator (*this paper*)

$$f_{X,Y,Z,S} = \int f_{X|X^*,S} f_{Y|X^*,Z} f_{X^*,Z,S} dx^*$$

Back

Comments on limited feedback

- LF rules out direct effects from previous X^*_{t-1} to W_t:

$$\begin{aligned} f_{W_t|W_{t-1},X^*_t,X^*_{t-1}} &= f_{Y_t,M_t|Y_{t-1},M_{t-1},X^*_t,X^*_{t-1}} \\ &= f_{Y_t|M_t,Y_{t-1},M_{t-1},X^*_t,X^*_{t-1}} \cdot f_{M_t|Y_{t-1},M_{t-1},X^*_t,X^*_{t-1}} \\ &= \underbrace{f_{Y_t|M_t,Y_{t-1},M_{t-1},X^*_t}}_{\text{CCP}} \cdot \underbrace{f_{M_t|Y_{t-1},M_{t-1},X^*_t}}_{M_t \text{ law of motion}} . \end{aligned}$$

- LF restricts M_t law of motion. Can think of as timing restriction: X^*_t occurs before M_t.
  Satisfied by many empirical applications Details
- Could relax: (i) impose additional restrictions on CCP; (ii) identify higher-order Markov models; (iii) exploit "static" variables

Back

Therefore, my dear friend and companion,

if you should think me somewhat sparing of my narrative on my first setting out–bear with me,–and let me go on, and tell my story my own way:–Or, if I should seem now and then to trifle upon the road,–or should sometimes put on a fool's cap with a bell to it, for a moment or two as we pass along,–don't fly off,–but rather courteously give me credit for a little more wisdom than appears upon my outside;–and as we jog on, either laugh with me, or at me, or in short do any thing,–only keep your temper.

L. Sterne, *Tristram Shandy*, 1:VI Back

Lemma 2: representation of $f_{W_t,X_t^*|W_{t-1},X_{t-1}^*}$

- Main equation: for any (w_t, w_{t-1}) here here

$$
\begin{aligned}
L_{W_{t+1},w_t|w_{t-1},W_{t-2}} &= L_{W_{t+1}|w_t,X^*} L_{w_t,X_t^*|w_{t-1},W_{t-2}} \\
&= L_{W_{t+1}|w_t,X^*} L_{w_t,X_t^*|w_{t-1},X_{t-1}^*} L_{X_{t-1}^*|w_{t-1},W_{t-2}}
\end{aligned}
$$

- Similarly: $L_{W_t|w_{t-1},W_{t-2}} = L_{W_t|w_{t-1},X_{t-1}^*} L_{X_{t-1}^*|w_{t-1},W_{t-2}}$
- Manipulating above two equations: $L_{w_t,X_t^*|w_{t-1},X_{t-1}^*}$

$$
\begin{aligned}
&= L^{-1}_{W_{t+1}|w_t,X^*} L_{W_{t+1},w_t|w_{t-1},W_{t-2}} L^{-1}_{X_{t-1}^*|w_{t-1},W_{t-2}} \\
&= L^{-1}_{W_{t+1}|w_t,X_t^*} L_{W_{t+1},w_t|w_{t-1},W_{t-2}} L^{-1}_{W_t|w_{t-1},W_{t-2}} L_{W_t|w_{t-1},X_{t-1}^*}
\end{aligned}
$$

- Identification of $L_{w_t,X_t^*|w_{t-1},X_{t-1}^*}$ boils down to that of $L_{W_{t+1}|w_t,X_t^*}$ for t & $t-1$ (Lemma 3)

back

Hidden State Markov Models

- Models considered here fall under "HSM" rubric, with X^* denoting the unobserved "state"
- How do estimation approaches compare?
- X^* discrete: "nonparametric" is essentially "parametric" (finite-dimensional multinomial distribution). Estimation via EM-algorithm.
  Computationally difficult; iterative
- X^* continuous: focus on linear Kalman filter.
  Heavy parameterization: linear relationships between observables and X^*; Gaussian shocks.
- Nonparametric identification results for discrete case: Allman, Matias, Rhodes (2009), Kruskal (1970's). Not constructive.

back