# 신경망 네트워크와 수학적 기반

## Linear functions

# Superposition and linear functions

- $f : \mathbf{R}^n \rightarrow \mathbf{R}$ means $f$ is a function mapping $n$-vectors to numbers
- $f$ satisfies the ***superposition property*** if

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$$

  holds for all numbers $\alpha$, $\beta$, and all $n$-vectors $x$, $y$

- a function that satisfies superposition is called ***linear***

# The inner product function

- with $a$ an $n$-vector, the function

$$f(x) = a^T x = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$$

  is the ***inner product function***

- $f(x)$ is a weighted sum of the entries of $x$
- the inner product function is linear:

$$\begin{aligned} f(\alpha x + \beta y) &= a^T(\alpha x + \beta y) \\ &= a^T(\alpha x) + a^T(\beta y) \\ &= \alpha(a^T x) + \beta(a^T y) \\ &= \alpha f(x) + \beta f(y) \end{aligned}$$

## Linear functions

# all linear functions are inner products

- suppose $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is linear
- then it can be expressed as $f(x) = a^T x$ for some $a$
- specifically: $a_i = f(e_i)$
- follows from

$$\begin{aligned} f(x) &= f(x_1 e_1 + x_2 e_2 + \cdots + x_n e_n) \\ &= x_1 f(e_1) + x_2 f(e_2) + \cdots + x_n f(e_n) \end{aligned}$$

# Affine functions

- a function that is linear plus a constant is called ***affine***
- general form is $f(x) = a^T x + b$, with $a$ an $n$-vector and $b$ a scalar
- a function $f : \mathbf{R}^n \to \mathbf{R}$ is affine if and only if

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$$

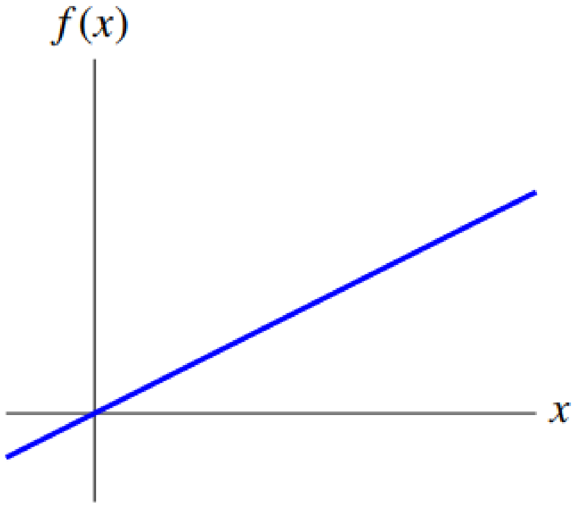
  holds for all $\alpha, \beta$ with $\alpha + \beta = 1$, and all $n$-vectors $x, y$

- sometimes people refer to affine functions as linear

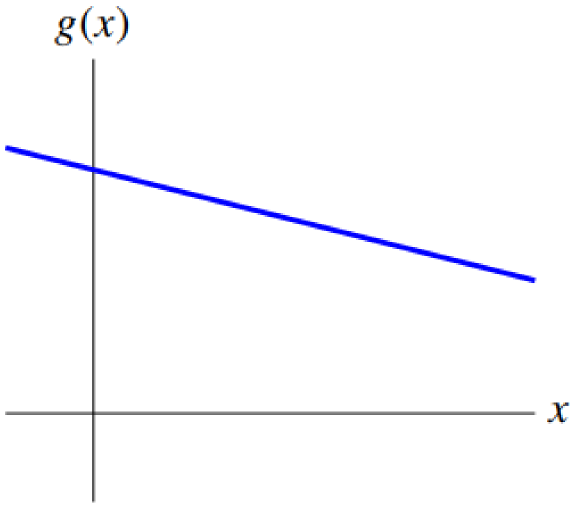
## Linear functions

# Linear versus affine functions

$f$ is linear




$g$ is affine, not linear



## Linear functions

# First-order Taylor approximation

- suppose $f : \mathbf{R}^n \to \mathbf{R}$
- ***first-order Taylor approximation*** of $f$, near point $z$:

$$\hat{f}(x) = f(z) + \frac{\partial f}{\partial x_1}(z)(x_1 - z_1) + \cdots + \frac{\partial f}{\partial x_n}(z)(x_n - z_n)$$

- $\hat{f}(x)$ is very close to $f(x)$ when $x_i$ are all near $z_i$
- $\hat{f}$ is an **affine function** of $x$
- can write using inner product as

$$\hat{f}(x) = f(z) + \nabla f(z)^T(x - z)$$

- where $n$-vector $\nabla f(z)$ is the ***gradient*** of $f$ at $z$

$$\nabla f(z) = \left(\frac{\partial f}{\partial x_1}(z), \ldots, \frac{\partial f}{\partial x_n}(z)\right)$$

## Linear functions
# Example