Modern Particle Physics
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Selected Solutions

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1 Introduction

Problem 1.2
Draw the Feynman diagram for $\tau^- \to \pi^- \nu_\tau$. (The π^- is the lightest $d\bar{u}$ meson)

Solution:

Problem 1.8
Tugsten has a radiation length of $X_0 = 0.35$ cm and a critical energy of $E_c - 7.97$ MeV. Roughly what thickness of tungsten is required to fully contain a 500 GeV electromagnetic shower from an electron?

Solution:

Problem 1.10
In a fixed-target pp experiment, what proton energy would be required to achieve the same centre-of-mass energy as the LHC, which will ultimately operate at 14 TeV.

Solution:

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2 Underlying Concepts

Problem 2.1
When expressed in natural units the lifetime of the W boson is approximately $\tau \approx 0.5$ GeV^{-1}. What is the corresponding value in S.I. units?

Solution:

Problem 2.2
A cross section is measured to be 1 pb; convert this to natural units.

Solution: Taking note that $\hbar c = 0.197$ GeV $\cdot$ fm

$$1 \text{ pb} = 100 \text{ fm}^2 = 100 \times \left(\frac{\hbar c}{0.197}\right)^2 \text{ GeV}^{-2} = 100 \times \left(\frac{1.055 \times 10^{-34} \times 2.998 \times 10^8}{0.197}\right) \text{ GeV}^{-2}$$

Problem 2.3
Show that the process $\gamma \to e^+ e^-$ can not occur in vacuum.

Solution:

Problem 2.4
A particle of mass 3 GeV is travelling in the positive z-direction with momentum 4 GeV. What are its energy and velocity?

Solution: Using the relation of $m^2 = E^2 - |\mathbf{p}|^2$, one gets $E^2 = 25$ GeV^2 thus the energy is $E = 5$ GeV. Now considering the relation of $|\mathbf{p}| = E\beta$, it is seen that $\beta = |\mathbf{p}|E^{-1} = 0.8$ thus the velocity is $0.8c$.

Problem 2.5
In the laboratory frame, denoted Σ, a particle travelling in the z-direction has momentum $\mathbf{p} = p_z\hat{\mathbf{z}}$ and energy E.

(a) Use the Lorentz transformation to find expressions for the momentum p'_z and energy E' of the particle in a frame Σ' which is moving in a velopcity $\mathbf{v} = +v\hat{\mathbf{z}}$ relative to Σ, and show that $E^2 - p_z^2 = (E')^2 - (p'_z)^2$.

(b) For a system of particles, prove that the total four-momentum squared,

$$p^\mu p_\mu \equiv \left(\sum_i E_i\right)^2 - \left(\sum_i \mathbf{p}_i\right)^2$$

is invariant under Lorentz transformations.

Solution:

(a) Let the four-momentum of the given particle in the frame Σ and Σ' as $p = (E, 0, 0, p_z), p' = (E', \mathbf{p}')$ respectively. Denoting the corresponding matrix representation of the given Lorentz transformation as $\mathbf{\Lambda}$, one could write down the transformation of p as,

$$\begin{aligned} p' = \mathbf{\Lambda}p \implies p'^\mu &= \Lambda^\mu_\nu p^\nu \\ &= \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} E \\ 0 \\ 0 \\ p_z \end{pmatrix} = \gamma \begin{pmatrix} E - \beta p_z \\ 0 \\ 0 \\ -E\beta + p_z \end{pmatrix} \end{aligned}$$

which implies that $E' = \gamma\left(E - \beta p_z\right)$ and $p'_z = -\gamma(E\beta - p_z)$. Using such expression of p', one could show that :

$$\begin{aligned} (E')^2 - (p'_z)^2 &= \gamma^2(E - \beta p_z)^2 - \gamma^2(E\beta - p_z)^2 \\ &= \gamma^2\left[(E - \beta p_z)^2 - (E\beta - p_z)^2\right] \\ &= \gamma^2\left[(E - \beta p_z + E\beta - p_z)\left(E - \beta p_z - E\beta + p_z\right)\right] \\ &= \gamma^2(1+\beta)(1-\beta)(E - p_z)(E + p_z) = E^2 - p_z^2 \quad \square \end{aligned}$$

(b)

Problem 2.6

For the decay $a \to 1 + 2$, show that the mass of the particle a can be expressed as

$$m_a^2 = m_1^2 + m_2^2 + 2E_1E_2\,(1 - \beta_1\beta_2\cos\theta)$$

where β_1 and β_2 are the velocities of the daughter particles and θ is the angle between them.

Solution: Let the four-momenta of the daughters as $p_i = (E_i, \mathbf{p}_i)$ for $i = 1, 2$. Momentum conservation states that $p_a = p_1 + p_2$ where p_a is the four-momentum of the mother particle. Squaring both sides, one obtains

$$\begin{aligned}
p_a \cdot p_a = m_a^2 = (p_1 + p_2)^2 \\
&= p_1 \cdot p_1 + p_2 \cdot p_2 + 2p_1 \cdot p_2 \\
&= m_1^2 + m_2^2 + 2(E_1E_2 - \mathbf{p}_1 \cdot \mathbf{p}_2) \\
&= m_1^2 + m_2^2 + 2(E_1E_2 - |\mathbf{p}_1||\mathbf{p}_2|\cos\theta) \\
&= m_1^2 + m_2^2 + 2(E_1E_2 - E_1\beta_1E_2\beta_2\cos\theta) \\
&= m_1^2 + m_2^2 + 2E_1E_2\,(1 - \beta_1\beta_2\cos\theta) \quad \square
\end{aligned}$$

Problem 2.7
In a collider experiment, Λ baryons can be identified from the decay $\Lambda \to \pi^- p$, which gives rise to a displaced vertex in a tracking detector. In a particular decay, the momenta of the π^+ and p are measured to be 0.75 GeV and 4.25 GeV respectively, and the opening angle between the tracks is 9°. The masses of the pion and proton are 189.6 MeV and 938.3 MeV.

(a) Calculate the mass of the Λ baryon.

(b) On average, Λ baryons of this energy are observed to decay at a distance of 0.35 m from the point of production. Calculate the lifetime of the Λ.

Solution:

Problem 2.8
In the laboratory frame, a proton with total energy E collides with proton at rest. Find the minimum proton energy such that process

$$p + p \to p + p + \bar{p} + \bar{p}$$

is kinematically allowed.

Solution:

Problem 2.9
Find the maximum opening angle between the photons produced in the decay $\pi^0 \to \gamma\gamma$ if the energy of the neutral pion is 10 GeV, given that $m_{\pi^0} = 135$ MeV.

Solution: Using the results derived in Problem 2 and taking account on the fact that photons are massless, one could write down

$$m_{\pi_0^2} = 2E_1E_2\,(1 - \beta_1\beta_2\cos\theta) = 2E_1E_2\,(1 - \cos\theta) \implies \cos\theta = \frac{m_{\pi_0}^2}{2E_1E_2} - 1$$

Taking account that $E_1 + E_2 = 10$ GeV, let $E_1 = E$ and express θ in terms of E as,

$$\cos\theta = \frac{m_{\pi_0}^2}{2E(10-E)} - 1$$

In the range of $E \in [0, 10]$ GeV the RHS of the above identity will take a local minimum when $E = 5$ GeV which will give the maximum value of θ, which will be denoted as θ^*. One could get θ^* as,

$$\cos\theta^* = \frac{\left(1.35\times 10^{-1}\ \text{GeV}\right)^2}{100\ \text{GeV}^2} - 1 =$$

Problem 2.10
The maximum of the $\pi^- p$ cross section, which occurs at $p_\pi = 300$ MeV, corresponds to the resonant production of the Δ^0 baryon (i.e. $\sqrt{s} = m_\Delta$). What is the mass of the Δ?

Solution:

Problem 2.11
Tau-leptons are produced in the process $e^+e^- \to \tau^+\tau^-$ at a centre-of-mass energy of 91.2 GeV. The angular distribution of the π^- from the decay $\tau^- \to \pi^-\nu_\tau$ is

$$\frac{\mathrm{d}N}{\mathrm{d}\left(\cos\theta^*\right)} \propto 1 + \cos\theta^*$$

where θ^* is the polar angle of the π^- in the tau-lepton rest frame, relative to the direction defined by the τ spin. Determine the laboratory frame energy distribution of the π^- for the cases where the tau lepton spin is (i) *aligned with* or (ii) *opposite* to its direction of flight.

Solution:

Problem 2.12
For the process 1+2 $\to$ 3+4, the Mandelstam variables s, t and u are defined as $s = (p_1+p_2)^2$, $t = (p_1 - p_3)^2$ and $u = (p_1 - p_4)^2$. Show that

$$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2.$$

Solution: By definition of the Mandelstam variables, one could express $(s + t + u)$ as

$$\begin{aligned} s + t + u &= (p_1 + p_2)^2 + (p_1 - p_3)^2 + (p_1 - p_4)^2 \\ &= \sum_i p_i \cdot p_i + 2p_1 \cdot p_1 + 2p_1 \cdot p_2 - 2p_1 \cdot p_3 - 2p_1 \cdot p_4 \\ &= \sum_i m_i^2 + 2p_1 \cdot (p_1 + p_2 - p_3 - p_4) \\ &= \sum_i m_i^2 \quad \square \end{aligned}$$

The fact that in any frame $p^\mu p_\mu = m^2$ for a particle with mass m is used in the third identity, and in the last step the conservation of momentum $p_1 + p_2 = p_3 + p_4$ is used.

Problem 2.13
At the HERA collider, 27.5 GeV electrons were collided head-on with 820 GeV protons. Calculate the centre-of-mass energy.

Solution: Let the four-momentum of the electron and proton as $p_e = (E_e, \mathbf{p}_e)$, $p_p = (E_p, \mathbf{p}_p)$ respectively. The centre-of-mass energy $\sqrt{s}$ can be expressed as,

$$\begin{aligned} s &= (p_e + p_p)^2 = p_e \cdot p_e + p_p \cdot p_p + 2p_e \cdot p_p \\ &= m_e^2 + m_p^2 + 2\left(E_e E_p - \mathbf{p}_e \cdot \mathbf{p}_p\right) \\ &= m_e^2 + m_p^2 + 2\left(E_e E_p + |\mathbf{p}_e|\,|\mathbf{p}_p|\right) \simeq 4E_e E_p \qquad \left(|\mathbf{p}_i|^2 = E_i^2 - m_i^2 \sim E_i^2\right) \end{aligned}$$

As the collision is occuring head-on, one could say that $\mathbf{p}_e \cdot \mathbf{p}_p = -|\mathbf{p}_e|\,|\mathbf{p}_p|$ which was used in the last identity. Looking upon the order of the variables, $m_e \simeq 0.5$ MeV, $m_p \simeq 93.8$ MeV and $E_e = 27.5$ GeV, $E_p = 820$ GeV for an approximation it is okay to consider $m_e, m_p \sim 0$. Thus the centre-of-mass energy $\sqrt{s} \simeq 300$ GeV when all the needed values are plugged in.

Problem 2.14

Solution:

Problem 2.15

Solution:

Problem 2.16

Solution:

Problem 2.17
Find the third-order term in the transition matrix element of Fermi's golden rule.

Solution:

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3 Decay Rates and Cross Sections

Problem 3.1
Calculate the energy of the μ^- produced in the decay at rest $\pi^- \to \mu\bar{\nu}_\mu$. Assume $m_\pi = 140$ MeV, $m_\mu = 106$ MeV and take $m_\nu \sim 0$.

Solution: Let the four-momenta of the muon and the neutrino to be $p_1 = (E_1, 0, 0, E_2)$ and $p_2 = (E_2, 0, 0, -E_2)$. In the pion rest frame, $E_1 + E_2 = m_\pi$ and from the muon mass constraint $m_\mu^2 = E_1^2 - E_2^2$. Solving these equation gives

$$E_1 = \frac{m_\pi^2 + m_\mu^2}{2m_\pi} = 110.13 \text{ GeV}$$

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Problem 3.2
For the decay $a \to 1 + 2$, show that the momenta of both daughter particles in the centre-of mass frame p^* are

$$p^* = \frac{1}{2m_a}\sqrt{[m_a^2 - (m_1^2 + m_2^2)][m_a^2 - (m_1^2 - m_2^2)]}$$

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Solution: Let the four-momenta of the mother particle and the daughter particles to be $p_a = (m_a, 0, 0, 0)$, $p_1 = (E_1, 0, 0, p^*)$, $p_2 = (E_2, 0, 0, -p^*)$ From the mass constraints, we get $E_1 + E_2 = m_a$, $E_1^2 - p^{*2} = m_1^2$, and $E_2^2 - p^{*2} = m_2^2$.

Since we have three unknown variables E_1, E_2, p^* and three equations, it is possible to get p^* in terms of m_a, m_1 and m_2, which gives the desired solution.