2D Taylor-Melcher Rough Draft

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Contents

1 Introduction

In this paper, we study a simple two-dimensional model that describes the dynamics of two immiscible fluids subject to surface tension along their interface. To be more specific, let Γ be a time-dependent simple closed curve in $\mathbb{R}^2$ that represents the interface between two immiscible fluids. Then the model is given by

$$\mu\Delta\boldsymbol{u} - \nabla p = \boldsymbol{0} \quad \text{on } \mathbb{R}^2 \setminus \Gamma, \tag{1}$$
$$\nabla \cdot \boldsymbol{u} = 0 \quad \text{on } \mathbb{R}^2 \setminus \Gamma, \tag{2}$$
$$[\boldsymbol{u}] = \boldsymbol{0}, \tag{3}$$
$$[\Sigma(\boldsymbol{u}, p)\boldsymbol{n}] = -\gamma\kappa\boldsymbol{n}, \tag{4}$$

where $\boldsymbol{u}$ and p denote the velocity and pressure of the fluid, respectively; μ denotes the fluid viscosity, which is a constant within each fluid but may differ across the two fluids; $\Sigma(\boldsymbol{u}, p)$ denotes the stress tensor for the Newtonian fluid of viscosity μ; $\boldsymbol{n}$ is the outward-pointing unit normal vector to the interface Γ; γ denotes the surface tension coefficient which is a constant; κ denotes the signed curvature of the interface; and the notation $[\cdot]$ means the interior value minus the exterior value. We assume that the two fluids have the same viscosity μ, which we normalize to 1.

In words, this model states that the interior and exterior fluids are both incompressible Stokes fluids with no interfacial jump in the fluid velocity and they are driven by a stress imbalance along the interface given by $-\gamma\kappa\boldsymbol{n}$. The observation that the interfacial force depends exclusively on the geometry of the interface via curvature κ plays a pivotal role in our analysis as it enables us to introduce a convenient parametrization of the interface without affecting the physical dynamics of the system.

In this model, there are two unknown variables to solve for: the two-dimensional fluid velocity $\boldsymbol{u}$ and the scalar pressure p. In the remainder of this paper, we study the well-posedness of this model with respect to the fluid velocity by imposing a certain ansatz on the fluid velocity satisfying the specified model. The use of the ansatz reduces the original problem to the well-posedness of the dynamics of the interface, which is the contents of the main theorem in our paper stated in Section 6. Throughout the rest of the paper, we may suppress certain expressions' dependence on time t for readability.

1.1 Key Function Spaces

In an analytical study of well-posedness, function spaces provide an essential framework to formulate results and can often induce interesting properties of solutions by imposing sufficiently strong constraints on its functions, such as analyticity. To introduce the function spaces adopted for our analysis, we first define the Fourier transform of a periodic function defined on $[-\pi, \pi)$. For a periodic function f defined on $[-\pi, \pi)$, its Fourier transform is defined as

$$\mathcal{F}(f)(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\alpha)e^{-ik\alpha}d\alpha. \tag{5}$$

We may sometimes write $\hat{f}(k)$ to denote the Fourier transform of f with no difference in meaning. The corresponding Fourier series is given as

$$f(\alpha) = \sum_{k\in\mathbb{Z}} \hat{f}(k)e^{ik\alpha}. \tag{6}$$

For our study, we use a family of Banach spaces $\mathcal{F}^{0,1}_{\nu}$ and $\dot{\mathcal{F}}^{s,1}_{\nu}$, $s \geq 0$, equipped respectively with norms

$$\|f\|_{\mathcal{F}^{0,1}_{\nu}} = \sum_{k\in\mathbb{Z}} e^{\nu(t)|k|}\left|\hat{f}(k)\right|, \tag{7}$$

$$\|f\|_{\dot{\mathcal{F}}^{s,1}_{\nu}} = \sum_{k\neq 0} e^{\nu(t)|k|}\,|k|^{s}\left|\hat{f}(k)\right|, \tag{8}$$

where

$$\nu(t) = \frac{t}{1+t}\nu_0. \tag{9}$$

We note that if $\nu_0 > 0$, then $0 < \nu'(t) \leq \nu_0$. We also use a family of Banach spaces $\mathcal{F}^{0,1}$ and $\dot{\mathcal{F}}^{s,1}$, $s \geq 0$, equipped respectively with norms

$$\|f\|_{\mathcal{F}^{0,1}} = \sum_{k\in\mathbb{Z}} \left|\hat{f}(k)\right|, \tag{10}$$

$$\|f\|_{\dot{\mathcal{F}}^{s,1}} = \sum_{k\neq 0} |k|^s \left|\hat{f}(k)\right|. \tag{11}$$

The space $\mathcal{F}^{0,1}$ equipped with the norm (10) is the classical Wiener algebra, i.e., the space of absolutely convergent Fourier series. Below are some useful properties of these function spaces.

Proposition 1. *(Embeddings.) For $0 < s_1 \leq s_2$, the following norm inequality is satisfied.*

$$\|f\|_{\dot{\mathcal{F}}_\nu^{s_1,1}} \leq \|f\|_{\dot{\mathcal{F}}_\nu^{s_2,1}}. \tag{12}$$

Proposition 2. *(Estimates.) Let $n \geq 1$. Then*

$$\|f_1 f_2 \cdots f_n\|_{\mathcal{F}_\nu^{0,1}} \leq \Pi_{k=1}^n \|f_k\|_{\mathcal{F}_\nu^{0,1}}. \tag{13}$$

For $s > 0$,

$$\|f_1 f_2 \cdots f_n\|_{\dot{\mathcal{F}}_\nu^{s,1}} \leq b(n,s) \sum_{j=1}^n \|f_j\|_{\dot{\mathcal{F}}_\nu^{s,1}} \Pi_{k=1,k\neq j}^n \|f_k\|_{\mathcal{F}_\nu^{0,1}}, \tag{14}$$

where

$$b(n,s) = \begin{cases} 1 & 0 \leq s \leq 1, \\ n^{s-1} & s > 1. \end{cases} \tag{15}$$

Remark. *The estimates in Proposition 2 hold with $\mathcal{F}_\nu^{0,1}$ and $\dot{\mathcal{F}}_\nu^{s,1}$ replaced by $\mathcal{F}^{0,1}$ and $\dot{\mathcal{F}}^{s,1}$, respectively. For proof of Proposition 2, see Lemma 5.1 of [1].*

Proposition 3. *For $s \geq 0$, we have the estimate*

$$\|g_1 g_2\|_{\dot{\mathcal{F}}_\nu^{s,1}} \leq b(2,s)\left(\|g_1\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|g_2\|_{\mathcal{F}_\nu^{0,1}} + \|g_1\|_{\mathcal{F}_\nu^{0,1}} \|g_2\|_{\dot{\mathcal{F}}_\nu^{s,1}}\right), \tag{16}$$

where

$$b(n,s) = \begin{cases} 1 & \text{if } 0 \leq s \leq 1, \\ n^{s-1} & \text{if } s > 1. \end{cases} \tag{17}$$

Proof. The case in which $s > 0$ follows from Proposition 2. Let us consider the case $s = 0$.

$$\begin{aligned}
\|g_1 g_2\|_{\dot{\mathcal{F}}_\nu^{0,1}} &= \sum_{k\neq 0} e^{\nu(t)|k|} \left|\mathcal{F}(g_1 g_2)(k)\right| && (18)\\
&= \sum_{k\neq 0} e^{\nu(t)|k|} \left|\sum_{j\in\mathbb{Z}} \hat{g}_1(k-j)\hat{g}_2(j)\right| && (19)\\
&\leq \sum_{k\neq 0}\sum_{j\in\mathbb{Z}} e^{\nu(t)|k|} \left|\hat{g}_1(k-j)\right| \left|\hat{g}_2(j)\right| && (20)\\
&\leq \sum_{k\neq 0}\sum_{j\in\mathbb{Z}} e^{\nu(t)|k-j|} e^{\nu(t)|j|} \left|\hat{g}_1(k-j)\right| \left|\hat{g}_2(j)\right| && (21)\\
&\leq \|g_1\|_{\dot{\mathcal{F}}_\nu^{0,1}} \|g_2\|_{\mathcal{F}_\nu^{0,1}} + \|g_1\|_{\mathcal{F}_\nu^{0,1}} \|g_2\|_{\dot{\mathcal{F}}_\nu^{0,1}}. && (22)
\end{aligned}$$

The last inequality holds because

$$\begin{aligned}
& \|g_1\|_{\dot{\mathcal{F}}_\nu^{0,1}} \|g_2\|_{\mathcal{F}_\nu^{0,1}} + \|g_1\|_{\mathcal{F}_\nu^{0,1}} \|g_2\|_{\dot{\mathcal{F}}_\nu^{0,1}} && (23)\\
=& \sum_{k\neq 0} e^{\nu(t)|k|} \left|\hat{g}_1(k)\right| \cdot \sum_{j\in\mathbb{Z}} e^{\nu(t)|j|} \left|\hat{g}_2(j)\right| + \sum_{k\in\mathbb{Z}} e^{\nu(t)|k|} \left|\hat{g}_1(k)\right| \cdot \sum_{j\neq 0} e^{\nu(t)|j|} \left|\hat{g}_2(j)\right| && (24)\\
=& \sum_{k\neq 0}\sum_{j\in\mathbb{Z}} e^{\nu(t)|k|} e^{\nu(t)|j|} \left|\hat{g}_1(k)\right| \left|\hat{g}_2(j)\right| + \sum_{k\in\mathbb{Z}}\sum_{j\neq 0} e^{\nu(t)|k|} e^{\nu(t)|j|} \left|\hat{g}_1(k)\right| \left|\hat{g}_2(j)\right| && (25)\\
=& \sum_{k\neq j}\sum_{j\in\mathbb{Z}} e^{\nu(t)|k-j|} e^{\nu(t)|j|} \left|\hat{g}_1(k-j)\right| \left|\hat{g}_2(j)\right| + \sum_{k\in\mathbb{Z}}\sum_{j\neq k} e^{\nu(t)|k|} e^{\nu(t)|k-j|} \left|\hat{g}_1(k)\right| \left|\hat{g}_2(k-j)\right|. && (26)
\end{aligned}$$

The first term in (26) contains all but terms of the form

$$e^{\nu(t)|j|} \left|\hat{g}_1(0)\right| \left|\hat{g}_2(j)\right|,\ j \in \mathbb{Z} \tag{27}$$

while the second term in (26) contains terms of the form

$$e^{\nu(t)|-j|} \left|\hat{g}_1(0)\right| \left|\hat{g}_2(-j)\right|,\ j \neq 0. \tag{28}$$

The only term that is not covered between these two terms is $|\hat{g}_1(0)|\,|\hat{g}_2(0)|$. However, this term is not covered by the sum in (21), either. This completes the proof. ■

We define the following frequently used operator

$$\mathcal{M}(f)(\alpha) = \int_0^\alpha f(\eta)d\eta - \frac{\alpha}{2\pi}\int_{-\pi}^{\pi} f(\eta)d\eta. \tag{29}$$

We note that

$$\mathcal{F}(\mathcal{M}(f))(k) = \begin{cases} -\frac{i}{k}\hat{f}(k) & k \neq 0 \\ \sum_{j\neq 0}\frac{i}{j}\hat{f}(j) & k = 0. \end{cases} \tag{30}$$

For $N \geq 0$, we also define high frequency cut-off operators $\mathcal{J}_N$ and $\mathcal{J}_N^1$ as

$$\begin{aligned}
\mathcal{F}(\mathcal{J}_N f)(k) &= 1_{|k|\leq N}\mathcal{F}(f)(k), && (31)\\
\mathcal{F}(\mathcal{J}_N^1 f)(k) &= 1_{|k|\neq 1}1_{|k|\leq N}\mathcal{F}(f)(k). && (32)
\end{aligned}$$

2 Boundary Integral Formulation

Section 1 has mentioned that for our study a certain ansatz is imposed on the fluid velocity which satisfies our model. We now introduce the ansatz. For the fluid velocity, we adopt

$$u_j(\boldsymbol{x}) = \frac{1}{4\pi}\int_\Gamma (-\gamma\kappa(s)\boldsymbol{n}(s))_i G_{ij}(\boldsymbol{x}-\boldsymbol{y}(s))ds, \quad \boldsymbol{x}\in\mathbb{R}^2, \tag{33}$$

where $\boldsymbol{u}(\boldsymbol{x}) = (u_1(\boldsymbol{x}), u_2(\boldsymbol{x}))$ and $G = (G_{ij})$ given by

$$G_{ij}(\boldsymbol{w}) = -\delta_{ij}\log|\boldsymbol{w}| + \frac{w_i w_j}{|\boldsymbol{w}|^2} \tag{34}$$

is the Green's function for two-dimensional infinite unbounded incompressible Stokes flow [5]. Being a Green's function, G can be used to represent a solution of two-dimensional incompressible Stokes flow driven by a concentrated point force of some strength in the plane. In our model, there is a force density $-\gamma\kappa\boldsymbol{n}$ along the interface as opposed to a concentrated force at a single point. In this case, the solution to infinite unbounded Stokes flow can be represented via (33), which will henceforth be referred to as the single-layer potential. In general, the two-dimensional Green's function for infinite unbounded flow suffers from the so-called Stokes' paradox of logarithmic growth of the fluid velocity $\boldsymbol{u}$ at infinity. However, the fluid velocity in our model does not suffer from this paradox because the force density along the interface integrates to 0. The single-layer potential ensures that the fluid velocity satisfies equations (1) through (4). In particular, its analytical form guarantees continuity across the interface. The representation of the fluid velocity as a single-layer potential provides a convenient framework to study well-posedness of the model under study both analytically and numerically.

3 Interface Parametrization

In our model, the fluids are driven exclusively by a stress imbalance along the interface given by $-\gamma\kappa\boldsymbol{n}$, which can be derived explicitly from first principles of physics by assuming that surface tension along the interface be proportional to the unit tangent vector to the interface. The fact that this force differential depends exclusively on the geometry of the interface, via curvature κ, ensures that whatever parametrization we choose for the interface will have no bearing on the physical dynamics of the system. For our interface parametrization, we will adopt a frame in which the interface's tangent angle and length are the independent dynamical variables, as opposed to the interface's x- and y-positions. A detailed derivation of our parametrization is in order.

Due to the continuity of the fluid velocity across the interface as stated in (3), the fluid velocity along the interface is well-defined. We note that the shape of the interface is determined entirely by its normal velocity and the tangential velocity of the interface can only alter the frame of the interface parametrization. This means that the tangential velocity can be entered into the equations without affecting the interface shape. Let us write the interfacial fluid velocity as

$$\boldsymbol{u} = -U\boldsymbol{n} + T\boldsymbol{\tau}, \tag{35}$$

where $\boldsymbol{\tau}$ is the unit tangent vector. There is a minus sign in front of the normal term, because $\boldsymbol{n}$ is by definition the outward-pointing unit normal vector to the interface. We first represent the interface with some parametrization $z(\alpha, t)$ where $\alpha \in [-\pi, \pi)$. Let us define a tangential angle variable θ by writing the tangent vector $z_\alpha(\alpha, t)$ in complex variable notation

$$z_\alpha(\alpha, t) = |z_\alpha(\alpha, t)|\, e^{i(\alpha+\theta(\alpha,t))}. \tag{36}$$

Using the parametrization, we can rewrite (35) as

$$z_t(\alpha, t) = -U(\alpha, t)\boldsymbol{n}(\alpha, t) + T(\alpha, t)\boldsymbol{\tau}(\alpha, t), \tag{37}$$

which in complex variable notation can be written as

$$z_t(\alpha, t) = U(\alpha, t) \cdot ie^{i(\alpha+\theta(\alpha,t))} + T(\alpha, t) \cdot e^{i(\alpha+\theta(\alpha,t))}, \tag{38}$$

keeping in mind that in complex variable notation

$$\boldsymbol{\tau}(\alpha, t) = e^{i(\alpha+\theta(\alpha,t))}, \tag{39}$$

$$\boldsymbol{n}(\alpha, t) = -ie^{i(\alpha+\theta(\alpha,t))}. \tag{40}$$

By differentiating (36) with respect to t and (38) with respect to α and then setting them equal to each other, we can obtain evolution equations for the interface in terms of θ and $|z_\alpha(\alpha, t)|$ from the real and imaginary parts of the equation, i.e.,

$$|z_\alpha(\alpha, t)|_t = -U(\alpha, t) - U(\alpha, t)\theta_\alpha(\alpha, t) + T_\alpha(\alpha, t), \tag{41}$$

$$\theta_t(\alpha, t) = \frac{1}{|z_\alpha(\alpha, t)|}\Big(U_\alpha(\alpha, t) + T(\alpha, t) + T(\alpha, t)\theta_\alpha(\alpha, t)\Big). \tag{42}$$

Of all possible frames of the interface parametrization, a particularly useful one for our analysis can be selected by requiring the tangential speed $T(\alpha, t)$ to be of the form

$$T(\alpha, t) = \int_0^\alpha (1 + \theta_\eta(\eta, t))U(\eta, t)d\eta - \frac{\alpha}{2\pi}\int_{-\pi}^{\pi} (1 + \theta_\eta(\eta, t))U(\eta, t)d\eta + T(0, t), \tag{43}$$

where $T(0, t)$ is a number that depends on t, which allows for a change of frame. The frame that is chosen by the imposition of (43) ensures that $|z_\alpha(\alpha, t)|$ is independent of α, i.e.,

$$|z_\alpha(\alpha, t)| = \frac{1}{2\pi}\int_{-\pi}^{\pi} |z_\alpha(\eta, t)|\, d\eta = \frac{L(t)}{2\pi}, \tag{44}$$

where $L(t)$ is the length of the interface at time t. This can be checked by integrating (41) with respect to time from 0 to t and then differentiating with respect to α. Using this tangential speed formula, (41) and (42) can be rewritten as

$$L_t(t) = -\int_{-\pi}^{\pi} (1 + \theta_\alpha(\alpha))U(\alpha)d\alpha \tag{45}$$

$$\theta_t(\alpha, t) = \frac{2\pi}{L(t)}U_\alpha(\alpha) + \frac{2\pi}{L(t)}T(\alpha)(1 + \theta_\alpha(\alpha)). \tag{46}$$

The use of this particular parametrization of a fluid interface has been pioneered by [2] in the context of removing numerical stiffness from interfacial flows with surface tension. From now on, we will refer to this parametrization as Hou-Lowengrub-Shelley (HLS) parametrization in honor of its authors. For the purposes of our analysis, the HLS parametrization of the interface is useful because it provides a natural basis for a powerful analytical and numerical principle for solving interfacial fluid problems called *small-scale decomposition*. Under this principle, a principal linear operator of the evolution equation of θ is extracted and the remainder terms are shown to be of lower order in some sense under the choice of an appropriate function space [4]. [1] contains an application of this principle for an analytical study of the two-dimensional Muskat equation with two immiscible fluids under gravity in which one fluid is completely surrounded by the other. While [4] does not use the HLS parametrization, it employs small-scale decomposition to address the well-posedness of the Peskin problem in which the model is set up identically to our own except the force differential driving the system is of elastic nature, not surface tension.

4 Estimates for $L(t)$

We can derive a certain analytical expression for $L(t)$ from the incompressibility of the internal fluid. In fact, this analytical expression and (45) are equivalent provided that $L(t) > 0$ for all time t. The following proposition, whose proof can be garnered from [1], summarizes these observations. As we shall see later in the paper, a careful estimate of the analytical expression itself shows that it is bounded above and below by certain expressions in terms of an appropriate norm of a tangential angle variable, which becomes useful to derive a key *a priori* estimate for it.

Proposition 4. *Let $V_0 = \pi R^2$ be the initial volume of the internal fluid. For any $t \geq 0$ such that $L(t) > 0$,*

$$\left(\frac{L(t)}{2\pi}\right)^2 = R^2\left(1 + \frac{1}{2\pi} Im \int_{-\pi}^{\pi}\int_0^{\alpha} e^{i(\alpha-\eta)} \sum_{n\geq 1} \frac{i^n}{n!}(\theta(\alpha)-\theta(\eta))^n d\eta d\alpha\right)^{-1} \tag{47}$$

implies

$$L_t(t) = -\int_{-\pi}^{\pi}(1+\theta_\alpha(\alpha))U(\alpha)d\alpha. \tag{48}$$

The converse holds without the assumption that $L(t) > 0$.

Remark. *That $V_0 = \pi R^2$ is not to say that the internal fluid must initially be a circle of radius R. We set $V_0 = \pi R^2$ because if the second term inside the parentheses on the right hand side of (47) is sufficiently small in magnitude, then (47) can be taken to mean that $L(t)$ is the length at time t of the interface perturbed about a circle of radius R, i.e., $L(t)$ is roughly equal to $2\pi R$. As far as our analysis is concerned, this is a desirable formulation of $L(t)$ because we are interested in whether there exist nontrivial solutions to (45) and (46) given an initial perturbation about a circular interface.*

Before we commence the proof of Proposition 4, let us first derive (47) from the incompressibility condition on the internal fluid. Let $\mathcal{D}$ be the region enclosed by the fluid boundary Γ. Then the volume of the region $\mathcal{D}$ is given by

$$V = \int_{\mathcal{D}} dx \wedge dy \tag{49}$$
$$= \frac{1}{2} \int_{\mathcal{D}} d(-ydx + xdy) \tag{50}$$
$$= \frac{1}{2} \int_{\Gamma} -ydx + xdy \tag{51}$$
$$= \frac{1}{2} \int_{-\pi}^{\pi} (-z_2(\alpha), z_1(\alpha)) \cdot z_\alpha(\alpha) d\alpha, \tag{52}$$

where $\wedge$ in (49) denotes the wedge product of differential forms; (50) results from the exterior derivative of the differential form; (51) is a consequence of the generalized Stokes' theorem; and (52) follows from the definition of the line integral. Taking $z(\alpha)$ and $z_\alpha(\alpha)$ to be complex numbers instead of vectors, we can express the volume in complex-variable notation

$$V = \frac{1}{2} \int_{-\pi}^{\pi} \mathrm{Im}\left(\overline{z(\alpha)} z_\alpha(\alpha) \right) d\alpha = \frac{1}{2} \mathrm{Im} \int_{-\pi}^{\pi} \overline{z(\alpha)} z_\alpha(\alpha) d\alpha. \tag{53}$$

Using that

$$z_\alpha(\alpha) = \frac{L(t)}{2\pi} e^{i(\alpha + \theta(\alpha))} \tag{54}$$
$$z(\alpha) = z(0) + \int_0^\alpha z_\eta(\eta) d\eta, \tag{55}$$

we can write

$$V = \frac{1}{2} \mathrm{Im} \int_{-\pi}^{\pi} \overline{z(\alpha)} z_\alpha(\alpha) d\alpha \tag{56}$$
$$= \frac{1}{2} \left(\frac{L(t)}{2\pi} \right)^2 \mathrm{Im} \int_{-\pi}^{\pi} \int_0^\alpha e^{i(\alpha-\eta)} e^{i(\theta(\alpha)-\theta(\eta))} d\eta d\alpha \tag{57}$$
$$= \frac{1}{2} \left(\frac{L(t)}{2\pi} \right)^2 2\pi \cdot \mathrm{Im}\left(i + \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_0^\alpha e^{i(\alpha-\eta)} \sum_{n=1}^\infty \frac{i^n}{n!} (\theta(\alpha) - \theta(\eta))^n d\eta d\alpha \right) \tag{58}$$
$$= \pi \left(\frac{L(t)}{2\pi} \right)^2 \left(1 + \frac{1}{2\pi} \mathrm{Im} \int_{-\pi}^{\pi} \int_0^\alpha e^{i(\alpha-\eta)} \sum_{n=1}^\infty \frac{i^n}{n!} (\theta(\alpha) - \theta(\eta))^n d\eta d\alpha \right). \tag{59}$$

Since the internal fluid is incompressible,

$$V_0 = \pi R^2 = V, \tag{60}$$

which implies

$$\left(\frac{L(t)}{2\pi} \right)^2 = R^2 \left(1 + \frac{1}{2\pi} \mathrm{Im} \int_{-\pi}^{\pi} \int_0^\alpha e^{i(\alpha-\eta)} \sum_{n=1}^\infty \frac{i^n}{n!} (\theta(\alpha) - \theta(\eta))^n d\eta d\alpha \right)^{-1}. \tag{61}$$

This reveals that the converse to Proposition 4 holds without the condition $L(t) > 0$. Now, we prove Proposition 4.

Proof. Setting (57) and (60) equal to each other, we obtain

$$\pi R^2 = \frac{1}{2}\left(\frac{L(t)}{2\pi}\right)^2 \mathrm{Im}\int_{-\pi}^{\pi}\int_0^{\alpha} e^{i(\alpha-\eta)}e^{i(\theta(\alpha)-\theta(\eta))}d\eta d\alpha. \tag{62}$$

After differentiating this equation with respect to t and then using $L(t) > 0$, we can rearrange the equation to obtain

$$\begin{aligned}
L'(t) =& -\frac{1}{2R^2}\left(\frac{L(t)}{2\pi}\right)^3 \mathrm{Im}\left(\int_{-\pi}^{\pi}\int_0^{\alpha} ie^{i(\alpha-\eta)}e^{i(\theta(\alpha)-\theta(\eta))}(\theta_t(\alpha)-\theta_t(\eta))d\eta d\alpha\right) && (63)\\
=& -\frac{1}{2R^2}\left(\frac{L(t)}{2\pi}\right)^3 \left(\mathrm{Im}\left(\int_{-\pi}^{\pi} ie^{i\alpha}e^{i\theta(\alpha)}\theta_t(\alpha)\int_0^{\alpha} e^{-i\eta}e^{-i\theta(\eta)}d\eta d\alpha\right)\right. && (64)\\
&\left. - \mathrm{Im}\left(\int_{-\pi}^{\pi} ie^{i\alpha}e^{i\theta(\alpha)}\int_0^{\alpha} e^{-i\eta}e^{-i\theta(\eta)}\theta_t(\eta)d\eta d\alpha\right)\right). && (65)
\end{aligned}$$

Observe that

$$\begin{aligned}
&\int_{-\pi}^{\pi} ie^{i\alpha}e^{i\theta(\alpha)}\theta_t(\alpha)\int_0^{\alpha} e^{-i\eta}e^{-i\theta(\eta)}d\eta d\alpha && (66)\\
=&i\int_{-\pi}^{\pi} e^{i\alpha}e^{i\theta(\alpha)}\theta_t(\alpha)\int_0^{\alpha} e^{-i\eta}e^{-i\theta(\eta)}d\eta d\alpha && (67)\\
=&i\int_{-\pi}^{\pi} \frac{\partial}{\partial\alpha}\int_0^{\alpha} e^{i\eta}e^{i\theta(\eta)}\theta_t(\eta)d\eta \cdot \int_0^{\alpha} e^{-i\eta}e^{-i\theta(\eta)}d\eta d\alpha && (68)\\
=&i\int_{-\pi}^{\pi}\left(\frac{\partial}{\partial\alpha}\left(\int_0^{\alpha} e^{i\eta}e^{i\theta(\eta)}\theta_t(\eta)d\eta \cdot \int_0^{\alpha} e^{-i\eta}e^{-i\theta(\eta)}d\eta\right)\right. && (69)\\
&\left. - \int_0^{\alpha} e^{i\eta}e^{i\theta(\eta)}\theta_t(\eta)d\eta \cdot \frac{\partial}{\partial\alpha}\int_0^{\alpha} e^{-i\eta}e^{-i\theta(\eta)}d\eta\right)d\alpha && (70)\\
=&i\left(\int_0^{\pi} e^{i\eta}e^{i\theta(\eta)}\theta_t(\eta)d\eta \cdot \int_0^{\pi} e^{-i\eta}e^{-i\theta(\eta)}d\eta - \int_0^{-\pi} e^{i\eta}e^{i\theta(\eta)}\theta_t(\eta)d\eta \cdot \int_0^{-\pi} e^{-i\eta}e^{-i\theta(\eta)}d\eta\right. && (71)\\
&\left. - \int_{-\pi}^{\pi}\int_0^{\alpha} e^{i\eta}e^{i\theta(\eta)}\theta_t(\eta)d\eta \cdot e^{-i\alpha}e^{-i\theta(\alpha)}d\alpha\right). && (72)
\end{aligned}$$

Using that

$$\int_{-\pi}^{\pi} e^{i(\eta+\theta(\eta))}d\eta = 0, \tag{73}$$

we can write

$$\begin{aligned}
&\int_{-\pi}^{\pi} ie^{i\alpha}e^{i\theta(\alpha)}\theta_t(\alpha)\int_0^{\alpha} e^{-i\eta}e^{-i\theta(\eta)}d\eta d\alpha && (74)\\
=&i\left(\int_0^{\pi} e^{i\eta}e^{i\theta(\eta)}\theta_t(\eta)d\eta \cdot \int_0^{\pi} e^{-i\eta}e^{-i\theta(\eta)}d\eta + \int_{-\pi}^{0} e^{i\eta}e^{i\theta(\eta)}\theta_t(\eta)d\eta \cdot \int_0^{\pi} e^{-i\eta}e^{-i\theta(\eta)}d\eta\right. && (75)\\
&\left. - \int_{-\pi}^{\pi}\int_0^{\alpha} e^{i\eta}e^{i\theta(\eta)}\theta_t(\eta)d\eta \cdot e^{-i\alpha}e^{-i\theta(\alpha)}d\alpha\right) && (76)\\
=&i\left(\int_0^{\pi} e^{-i\eta}e^{-i\theta(\eta)}d\eta \cdot \int_{-\pi}^{\pi} e^{i\eta}e^{i\theta(\eta)}\theta_t(\eta)d\eta - \int_{-\pi}^{\pi}\int_0^{\alpha} e^{i\eta}e^{i\theta(\eta)}\theta_t(\eta)d\eta \cdot e^{-i\alpha}e^{-i\theta(\alpha)}d\alpha\right). && (77)
\end{aligned}$$

Due to (73),

$$\int_{-\pi}^{\pi} e^{i\eta}e^{i\theta(\eta)}\theta_t(\eta)d\eta = -i\int_{-\pi}^{\pi} e^{i\eta}\frac{\partial}{\partial t}e^{i\theta(\eta)}d\eta = -i\frac{\partial}{\partial t}\int_{-\pi}^{\pi} e^{i\eta}e^{i\theta(\eta)}d\eta = 0. \tag{78}$$

Hence,

$$\int_{-\pi}^{\pi} ie^{i\alpha}e^{i\theta(\alpha)}\theta_t(\alpha)\int_0^{\alpha} e^{-i\eta}e^{-i\theta(\eta)}d\eta d\alpha = -i\int_{-\pi}^{\pi}\int_0^{\alpha} e^{i\eta}e^{i\theta(\eta)}\theta_t(\eta)d\eta\cdot e^{-i\alpha}e^{-i\theta(\alpha)}d\alpha. \tag{79}$$

Therefore,

$$L'(t) = \frac{1}{R^2}\left(\frac{L(t)}{2\pi}\right)^3\mathrm{Im}\left(\int_{-\pi}^{\pi} ie^{i\alpha}e^{i\theta(\alpha)}\int_0^{\alpha} e^{-i\eta}e^{-i\theta(\eta)}\theta_t(\eta)d\eta d\alpha\right). \tag{80}$$

Using (46), we obtain

$$\frac{L(t)}{2\pi}\int_0^{\alpha} e^{-i\eta}e^{-i\theta(\eta)}\theta_t(\eta)d\eta \tag{81}$$

$$=\int_0^{\alpha} e^{-i\eta}e^{-i\theta(\eta)}\Big(U_\eta(\eta)+T(\eta)(1+\theta_\eta(\eta))\Big)d\eta \tag{82}$$

$$=\int_0^{\alpha} e^{-i\eta}e^{-i\theta(\eta)}U_\eta(\eta)d\eta+\int_0^{\alpha} e^{-i\eta}e^{-i\theta(\eta)}T(\eta)(1+\theta_\eta(\eta))d\eta \tag{83}$$

$$=\int_0^{\alpha}\frac{\partial}{\partial\eta}\Big(e^{-i\eta}e^{-i\theta(\eta)}U(\eta)\Big)-\frac{\partial}{\partial\eta}\Big(e^{-i\eta}e^{-i\theta(\eta)}\Big)U(\eta)d\eta+i\int_0^{\alpha}T(\eta)\frac{\partial}{\partial\eta}\Big(e^{-i(\eta+\theta(\eta))}\Big)d\eta \tag{84}$$

$$=e^{-i\alpha}e^{-i\theta(\alpha)}U(\alpha)-e^{-i\theta(0)}U(0)+i\int_0^{\alpha} e^{-i(\eta+\theta(\eta))}(1+\theta_\eta(\eta))U(\eta)d\eta \tag{85}$$

$$+i\int_0^{\alpha}T(\eta)\frac{\partial}{\partial\eta}\Big(e^{-i(\eta+\theta(\eta))}\Big)d\eta. \tag{86}$$

Using that

$$T_\eta(\eta) = (1+\theta_\eta(\eta))U(\eta)-\frac{1}{2\pi}\int_{-\pi}^{\pi}(1+\theta_\eta(\xi))U(\xi)d\xi, \tag{87}$$

we obtain

$$\int_0^{\alpha}T(\eta)\frac{\partial}{\partial\eta}\Big(e^{-i(\eta+\theta(\eta))}\Big)d\eta \tag{88}$$

$$=\int_0^{\alpha}\frac{\partial}{\partial\eta}\Big(T(\eta)e^{-i(\eta+\theta(\eta))}\Big)-T_\eta(\eta)e^{-i(\eta+\theta(\eta))}d\eta \tag{89}$$

$$=T(\alpha)e^{-i(\alpha+\theta(\alpha))}-T(0)e^{-i\theta(0)}-\int_0^{\alpha}T_\eta(\eta)e^{-i(\eta+\theta(\eta))}d\eta \tag{90}$$

$$=T(\alpha)e^{-i(\alpha+\theta(\alpha))}-T(0)e^{-i\theta(0)} \tag{91}$$

$$-\int_0^{\alpha}\left((1+\theta_\eta(\eta))U(\eta)-\frac{1}{2\pi}\int_{-\pi}^{\pi}(1+\theta_\xi(\xi))U(\xi)d\xi\right)e^{-i(\eta+\theta(\eta))}d\eta. \tag{92}$$

Therefore,

$$\int_0^\alpha T(\eta)\frac{\partial}{\partial\eta}\left(e^{-i(\eta+\theta(\eta))}\right)d\eta \tag{93}$$

$$=T(\alpha)e^{-i(\alpha+\theta(\alpha))}-T(0)e^{-i\theta(0)}-\int_0^\alpha(1+\theta_\eta(\eta))U(\eta)e^{-i(\eta+\theta(\eta))}d\eta \tag{94}$$

$$+\frac{1}{2\pi}\int_{-\pi}^{\pi}(1+\theta_\eta(\eta))U(\eta)d\eta\int_0^\alpha e^{-i(\eta+\theta(\eta))}d\eta. \tag{95}$$

Hence,

$$\frac{L(t)}{2\pi}\int_0^\alpha e^{-i\eta}e^{-i\theta(\eta)}\theta_t(\eta)d\eta \tag{96}$$

$$=e^{-i\alpha}e^{-i\theta(\alpha)}U(\alpha)-e^{-i\theta(0)}U(0)+T(\alpha)ie^{-i(\alpha+\theta(\alpha))}-T(0)ie^{-i\theta(0)} \tag{97}$$

$$+\frac{i}{2\pi}\int_{-\pi}^{\pi}(1+\theta_\eta(\eta))U(\eta)d\eta\cdot\int_0^\alpha e^{-i(\eta+\theta(\eta))}d\eta. \tag{98}$$

Then, using (73), we obtain

$$L'(t)=\frac{1}{R^2}\left(\frac{L(t)}{2\pi}\right)^3\mathrm{Im}\left(\int_{-\pi}^{\pi}ie^{i\alpha}e^{i\theta(\alpha)}\int_0^\alpha e^{-i\eta}e^{-i\theta(\eta)}\theta_t(\eta)d\eta d\alpha\right) \tag{99}$$

$$=\frac{1}{R^2}\left(\frac{L(t)}{2\pi}\right)^2\mathrm{Im}\left(\int_{-\pi}^{\pi}ie^{i\alpha}e^{i\theta(\alpha)}\left(e^{-i\alpha}e^{-i\theta(\alpha)}U(\alpha)-e^{-i\theta(0)}U(0)\right.\right. \tag{100}$$

$$+T(\alpha)ie^{-i(\alpha+\theta(\alpha))}-T(0)ie^{-i\theta(0)} \tag{101}$$

$$\left.\left.+\frac{i}{2\pi}\int_{-\pi}^{\pi}(1+\theta_\eta(\eta))U(\eta)d\eta\cdot\int_0^\alpha e^{-i(\eta+\theta(\eta))}d\eta\right)d\alpha\right) \tag{102}$$

$$=\frac{1}{R^2}\left(\frac{L(t)}{2\pi}\right)^2\mathrm{Im}\left(i\int_{-\pi}^{\pi}U(\alpha)d\alpha-ie^{-i\theta(0)}U(0)\int_{-\pi}^{\pi}e^{i\alpha}e^{i\theta(\alpha)}d\alpha-\int_{-\pi}^{\pi}T(\alpha)d\alpha\right. \tag{103}$$

$$+T(0)e^{-i\theta(0)}\int_{-\pi}^{\pi}e^{i\alpha}e^{i\theta(\alpha)}d\alpha \tag{104}$$

$$\left.-\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{i\alpha}e^{i\theta(\alpha)}\int_{-\pi}^{\pi}(1+\theta_\eta(\eta))U(\eta)d\eta\cdot\int_0^\alpha e^{-i(\eta+\theta(\eta))}d\eta d\alpha\right) \tag{105}$$

$$=\frac{1}{R^2}\left(\frac{L(t)}{2\pi}\right)^2\mathrm{Im}\left(i\int_{-\pi}^{\pi}U(\alpha)d\alpha-\int_{-\pi}^{\pi}T(\alpha)d\alpha\right. \tag{106}$$

$$\left.-\frac{1}{2\pi}\int_{-\pi}^{\pi}(1+\theta_\eta(\eta))U(\eta)d\eta\int_{-\pi}^{\pi}e^{i\alpha}e^{i\theta(\alpha)}\int_0^\alpha e^{-i(\eta+\theta(\eta))}d\eta d\alpha\right). \tag{107}$$

By the divergence theorem,

$$\int_{\mathcal{D}}\nabla\cdot\boldsymbol{u}=\int_\Gamma\boldsymbol{u}\cdot\boldsymbol{n}=-\int_{-\pi}^{\pi}U(\alpha)\left|z_\alpha(\alpha)\right|d\alpha=-\frac{L(t)}{2\pi}\int_{-\pi}^{\pi}U(\alpha)d\alpha. \tag{108}$$

Using the incompressibility of the internal fluid, $\nabla\cdot\boldsymbol{u}=0$, we obtain

$$\int_{-\pi}^{\pi}U(\alpha)d\alpha=0. \tag{109}$$

Hence,

$$L'(t) = \frac{1}{R^2}\left(\frac{L(t)}{2\pi}\right)^2 \mathrm{Im}\Bigg(-\int_{-\pi}^{\pi} T(\alpha)d\alpha \tag{110}$$

$$-\frac{1}{2\pi}\int_{-\pi}^{\pi}(1+\theta_\eta(\eta))U(\eta)d\eta\int_{-\pi}^{\pi}e^{i\alpha}e^{i\theta(\alpha)}\int_0^\alpha e^{-i(\eta+\theta(\eta))}d\eta d\alpha\Bigg) \tag{111}$$

$$= -\frac{1}{2\pi R^2}\left(\frac{L(t)}{2\pi}\right)^2 \mathrm{Im}\left(\int_{-\pi}^{\pi}(1+\theta_\eta(\eta))U(\eta)d\eta\int_{-\pi}^{\pi}e^{i\alpha}e^{i\theta(\alpha)}\int_0^\alpha e^{-i(\eta+\theta(\eta))}d\eta d\alpha\right) \tag{112}$$

$$= -\int_{-\pi}^{\pi}(1+\theta_\eta(\eta))U(\eta)d\eta, \tag{113}$$

as needed. ■

5 The Circular Interface under HLS Parametrization

Under the HLS parametrization, the interface at time t is a circle of radius R if and only if $(\theta(\alpha,t),L(t)) = (\hat{\theta}(0,t),2\pi R)$, where $\hat{\theta}(0,t)$ is a function of time t. Combined with this observation, our study reveals that a unique global-in-time solution exists for sufficiently small initial perturbation around a circular interface, which decays exponentially fast to a circular shape. The following proposition summarizes the characterization of the circular interface under the HLS parametrization.

Proposition 5. *Let $R > 0$. The interface at time t is a circle of radius R if and only if*

$$(\theta(\alpha,t),L(t)) = (\hat{\theta}(0,t),2\pi R), \tag{114}$$

where the parametrization is HLS.

Proof. First, we check that $(\theta(\alpha,t),L(t)) = (\hat{\theta}(0,t),2\pi R)$ is a circle of radius R for fixed t. It suffices to show that the curve has a constant curvature $\left|\frac{d^2z}{ds^2}\right|$ of $1/R$. Observe that

$$\frac{d^2z}{ds^2} = \frac{d}{d\alpha}\left(\frac{dz}{d\alpha}\cdot\frac{d\alpha}{ds}\right)\frac{d\alpha}{ds} = \frac{d^2z}{d\alpha^2}\cdot|z_\alpha(\alpha,t)|^{-2} = \frac{ie^{i(\alpha+\hat{\theta}(0,t))}}{R}. \tag{115}$$

Since $\hat{\theta}(0,t)$ is a real number,

$$\left|\frac{d^2z}{ds^2}\right| = \frac{1}{R}, \tag{116}$$

as needed. To prove the converse, suppose that the interface at time t is a circle of radius R. Then $L(t) = 2\pi R$. That $\left|\frac{d^2z}{ds^2}\right| = \frac{1}{R}$ shows that $|1+\theta_\alpha(\alpha,t)| = 1$. Due to the periodicity of θ, we have $\theta_\alpha(\alpha,t) = 0$, i.e., $\theta(\alpha,t)$ depends only on time t. Then $\hat{\theta}(0,t) = \theta(\alpha,t)$, as needed. ■

In Section 8, we remark on whether circular interfaces can solve our model.

6 Statement of the Main Theorem

We are ready to state the main theorem of our study. To study the simple two-dimensional model given by (1) through (4), we have adopted the single-layer potential form (33) for the fluid velocity. As a result, anywhere in the plane the fluid velocity can be obtained by convolving the interfacial stress imbalance against the Green's function for two-dimensional infinite unbounded incompressible Stokes flow along the interface. To completely describe the dynamics of the fluid velocity, it is therefore sufficient to study the dynamics of the interface itself. To that end, we take the HLS parametrization of the interface to obtain a pair of dynamics equations, (45) and (46), for the interface. Lastly, we have reformulated the dynamics equation (45) for the length of the interface into (47). The main theorem of our study is that the equations (47) and (46) for the dynamics of the interface have a unique solution that is global in time, provided that the initial datum is sufficiently small as measured by the norm of $\dot{\mathcal{F}}^{1,1}$. The unique solution also decays exponentially in time in the norm of $\dot{\mathcal{F}}^{1,1}_{\nu}$, where ν is given in (9) and $\nu_0 > 0$ is dependent on the initial datum. In view of Proposition 5, this implies that the initial perturbed interface decays exponentially to a circular shape.

Theorem 1. *Fix $\gamma > 0$. If the initial datum $\theta^0 \in \dot{\mathcal{F}}^{1,1}$ such that $|\mathcal{F}(\theta^0)(0)|$ and $\|\theta^0\|_{\dot{\mathcal{F}}^{1,1}}$ are sufficiently small, then for any $T \in (0, \infty)$ there exists a unique solution*

$$\theta(\alpha, t) \in C([0,T]; \dot{\mathcal{F}}^{1,1}_{\nu}) \cap L^1([0,T]; \dot{\mathcal{F}}^{2,1}_{\nu}) \tag{117}$$

to the equations (47) and (46), where ν is given in (9) and $\nu_0 > 0$ is dependent on θ^0. The solution becomes instantaneously analytic. In particular, for any $t \in [0,T]$

$$\|\theta(t)\|_{\dot{\mathcal{F}}^{1,1}_{\nu}} + \left(\Lambda(\left\|\theta^0\right\|_{\dot{\mathcal{F}}^{1,1}}) - \nu_0\right) \int_0^t \|\theta(\tau)\|_{\dot{\mathcal{F}}^{2,1}_{\nu}} \, d\tau \leq \left\|\theta^0\right\|_{\dot{\mathcal{F}}^{1,1}}, \tag{118}$$

where $\Lambda(\|\theta^0\|_{\dot{\mathcal{F}}^{1,1}})$ is given in (3615). Moreover, $\|\theta(t)\|_{\dot{\mathcal{F}}^{1,1}_{\nu}}$ decays exponentially in time.

Remark. *The assumption that the initial datum be "sufficiently small" can be made explicit in the sense that for any $\gamma > 0$, there is an analytical constraint that places an upper bound on the magnitudes of $|\mathcal{F}(\theta^0)(0)|$ and $\|\theta^0\|_{\dot{\mathcal{F}}^{1,1}}$.*

7 The Interfacial Fluid Velocity

To even speak of the interfacial fluid velocity, we need to ensure that it is well-defined. Fortunately, the single-layer potential form imposed on the fluid velocity satisfies the stipulation (3) that the fluid velocity be continuous across the interface, making the interfacial fluid velocity a well-defined quantity.

7.1 Formulation in Complex Variable Notation

We set out to rewrite (33) using complex variable notation, which is more conducive to computation than vector notation. The signed curvature κ that appears in the single-layer potential is defined by, in vector notation,

$$\boldsymbol{\tau}'(s) = -\kappa(s)\boldsymbol{n}(s), \tag{119}$$

where s denotes the arclength parametrization. Letting $\boldsymbol{\tau} = (\tau_1, \tau_2)$ and $z = (z_1, z_2)$ and using the Jacobian for conversion between the arclength and HLS parametrizations, we obtain

$$\tau_i'(s) = \frac{d\tau_i}{ds} = \frac{d}{ds}\left(\frac{dz_i}{ds}\right) = \frac{d}{d\beta}\left(\frac{dz_i}{d\beta}\cdot\frac{d\beta}{ds}\right)\cdot\frac{d\beta}{ds} = \frac{d^2 z_i}{d\beta^2}\cdot|z_\beta(\beta,t)|^{-2}, \tag{120}$$

which yields, in vector notation,

$$\begin{aligned}
u_j(\boldsymbol{x}) &= \frac{1}{4\pi}\int_\Gamma(-\gamma\kappa(s)\boldsymbol{n}(s))_i G_{ij}(\boldsymbol{x}-\boldsymbol{y}(s))ds && (121)\\
&= \frac{\gamma}{4\pi}\int_\Gamma(\boldsymbol{\tau}'(s))_i G_{ij}(\boldsymbol{x}-\boldsymbol{y}(s))ds && (122)\\
&= \frac{\gamma}{4\pi}\sum_{i=1}^{2}\int_\Gamma \tau_i'(s)G_{ij}(\boldsymbol{x}-\boldsymbol{y}(s))ds && (123)\\
&= \frac{\gamma}{4\pi}\sum_{i=1}^{2}\int_{-\pi}^{\pi} z_i''(\beta)G_{ij}(\boldsymbol{x}-z(\beta))\,|z_\beta(\beta,t)|^{-1}\,d\beta && (124)\\
&= \frac{2\pi}{L(t)}\frac{\gamma}{4\pi}\sum_{i=1}^{2}\int_{-\pi}^{\pi} z_i''(\beta)G_{ij}(\boldsymbol{x}-z(\beta))d\beta && (125)\\
&= \frac{2\pi}{L(t)}\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} z''(\beta)\cdot G_{\cdot j}(\boldsymbol{x}-z(\beta))d\beta, && (126)
\end{aligned}$$

where

$$G_{\cdot j}(\boldsymbol{x}-z(\beta)) = (G_{1j}(\boldsymbol{x}-z(\beta)), G_{2j}(\boldsymbol{x}-z(\beta))). \tag{127}$$

Let $\boldsymbol{x} = z(\alpha) \in \Gamma$. To rewrite the current expression for $u_j(\boldsymbol{x}) = u_j(z(\alpha))$ in complex variable notation, we use the following complex variable expressions

$$\begin{aligned}
G_{\cdot j}(z(\alpha)-z(\beta)) &= G_{1j}(z(\alpha)-z(\beta)) + iG_{2j}(z(\alpha)-z(\beta)) && (128)\\
z'(\beta) &= \frac{L(t)}{2\pi}e^{i(\beta+\theta(\beta))}, && (129)
\end{aligned}$$

which yields, in complex variable notation,

$$\begin{aligned}
u_j(z(\alpha)) &= \frac{2\pi}{L(t)}\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathrm{Re}\Big(\overline{z''(\beta)}G_{\cdot j}(z(\alpha)-z(\beta))\Big)d\beta && (130)\\
&= \frac{2\pi}{L(t)}\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathrm{Re}\left(\frac{d}{d\beta}\left(\overline{z'(\beta)}\right)G_{\cdot j}(z(\alpha)-z(\beta))\right)d\beta && (131)\\
&= \frac{2\pi}{L(t)}\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathrm{Re}\left(\frac{d}{d\beta}\left(\overline{z'(\beta)}G_{\cdot j}(z(\alpha)-z(\beta))\right)-\overline{z'(\beta)}\frac{d}{d\beta}\Big(G_{\cdot j}(z(\alpha)-z(\beta))\Big)\right)d\beta && (132)\\
&= -\frac{2\pi}{L(t)}\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathrm{Re}\left(\overline{z'(\beta)}\frac{d}{d\beta}\Big(G_{\cdot j}(z(\alpha)-z(\beta))\Big)\right)d\beta && (133)
\end{aligned}$$

where

$$\mathrm{Re}\bigg(\overline{z'(\beta)}\frac{d}{d\beta}\Big(G_{\cdot j}(z(\alpha)-z(\beta))\Big)\bigg) \tag{134}$$

$$= \frac{L(t)}{2\pi}\bigg(\cos(\beta+\theta(\beta))\frac{d}{d\beta}\Big(G_{1j}(z(\alpha)-z(\beta))\Big) + \sin(\beta+\theta(\beta))\frac{d}{d\beta}\Big(G_{2j}(z(\alpha)-z(\beta))\Big)\bigg). \tag{135}$$

Hence,

$$u_j(z(\alpha)) = -\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\cos(\beta+\theta(\beta))\frac{d}{d\beta}\Big(G_{1j}(z(\alpha)-z(\beta))\Big) \tag{136}$$

$$+\sin(\beta+\theta(\beta))\frac{d}{d\beta}\Big(G_{2j}(z(\alpha)-z(\beta))\Big)d\beta. \tag{137}$$

By changing the variable of integration from β to $\beta' = \alpha - \beta$ and rewriting the sine and cosine in complex variable notation, we obtain

$$u_j(z(\alpha)) = \frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\cos(\alpha-\beta'+\theta(\alpha-\beta'))\frac{d}{d\beta'}\Big(G_{1j}(z(\alpha)-z(\alpha-\beta'))\Big) \tag{138}$$

$$+\sin(\alpha-\beta'+\theta(\alpha-\beta'))\frac{d}{d\beta'}\Big(G_{2j}(z(\alpha)-z(\alpha-\beta'))\Big)d\beta' \tag{139}$$

$$= \frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\frac{1}{2}\Big(e^{i(\alpha-\beta+\theta(\alpha-\beta))}+e^{-i(\alpha-\beta+\theta(\alpha-\beta))}\Big)\frac{d}{d\beta}\Big(G_{1j}(z(\alpha)-z(\alpha-\beta))\Big) \tag{140}$$

$$+\frac{1}{2i}\Big(e^{i(\alpha-\beta+\theta(\alpha-\beta))}-e^{-i(\alpha-\beta+\theta(\alpha-\beta))}\Big)\frac{d}{d\beta}\Big(G_{2j}(z(\alpha)-z(\alpha-\beta))\Big)d\beta. \tag{141}$$

7.2 The Normal Speed U

To obtain the normal speed in complex variable notation, we take the dot product of (35) and $-\boldsymbol{u}$ to get

$$U = \boldsymbol{u}\cdot(-\boldsymbol{n}), \tag{142}$$

which can be rewritten in complex variable notation as

$$U(\alpha) = \mathrm{Re}\Big((u_1(\alpha)-iu_2(\alpha))ie^{i(\alpha+\theta(\alpha))}\Big). \tag{143}$$

To obtain an analytical expression for $U(\alpha)$ in complex variable notation, we first simplify (140) and (141). We note that

$$G_{11}(z(\alpha)-z(\alpha-\beta)) = -\log|z(\alpha)-z(\alpha-\beta)| + \frac{(z_1(\alpha)-z_1(\alpha-\beta))^2}{|z(\alpha)-z(\alpha-\beta)|^2}, \tag{144}$$

$$G_{12}(z(\alpha)-z(\alpha-\beta)) = \frac{(z_1(\alpha)-z_1(\alpha-\beta))(z_2(\alpha)-z_2(\alpha-\beta))}{|z(\alpha)-z(\alpha-\beta)|^2}, \tag{145}$$

$$G_{21}(z(\alpha)-z(\alpha-\beta)) = \frac{(z_1(\alpha)-z_1(\alpha-\beta))(z_2(\alpha)-z_2(\alpha-\beta))}{|z(\alpha)-z(\alpha-\beta)|^2}, \tag{146}$$

$$G_{22}(z(\alpha)-z(\alpha-\beta)) = -\log|z(\alpha)-z(\alpha-\beta)| + \frac{(z_2(\alpha)-z_2(\alpha-\beta))^2}{|z(\alpha)-z(\alpha-\beta)|^2}. \tag{147}$$

Letting

$$w(\alpha,\beta) = \int_0^1 e^{i(\alpha+(s-1)\beta+\theta(\alpha+(s-1)\beta))}ds, \tag{148}$$

we can write

$$z(\alpha)-z(\alpha-\beta) = \beta\int_0^1 z_\alpha(\alpha+(s-1)\beta)ds = \frac{\beta L(t)}{2\pi}w(\alpha,\beta). \tag{149}$$

Denoting the complex conjugate of w by $\overline{w}$, we then obtain

$$\frac{\partial}{\partial\beta}\Big(-\log|z(\alpha)-z(\alpha-\beta)|\Big) \tag{150}$$

$$= -\frac{1}{2}\cdot\frac{\partial}{\partial\beta}\log|z(\alpha)-z(\alpha-\beta)|^2 \tag{151}$$

$$= -\frac{1}{2}\cdot\frac{1}{|z(\alpha)-z(\alpha-\beta)|^2}\cdot\frac{\partial}{\partial\beta}\Big(|z(\alpha)-z(\alpha-\beta)|^2\Big) \tag{152}$$

$$= -\frac{1}{2}\cdot\frac{1}{\frac{\beta L(t)}{2\pi}w\frac{\beta L(t)}{2\pi}\overline{w}}\cdot\frac{\partial}{\partial\beta}\left(\left(\frac{L(t)}{2\pi}\right)^2\beta^2 w\overline{w}\right) \tag{153}$$

$$= -\frac{1}{2}\cdot\frac{1}{\beta^2 w\overline{w}}\cdot\frac{\partial}{\partial\beta}\Big(\beta^2 w\overline{w}\Big) \tag{154}$$

$$= -\frac{1}{2\beta^2 w\overline{w}}\Big(2\beta w\overline{w} + \beta^2(w_\beta\overline{w} + w\overline{w}_\beta)\Big) \tag{155}$$

$$= -\frac{1}{\beta} - \frac{w_\beta\overline{w} + w\overline{w}_\beta}{2w\overline{w}} \tag{156}$$

$$= -\frac{1}{\beta} - \frac{w_\beta}{2w} - \frac{\overline{w}_\beta}{2\overline{w}}. \tag{157}$$

Moreover,

$$
\begin{aligned}
&\frac{\partial}{\partial\beta}\left(\frac{(z_1(\alpha)-z_1(\alpha-\beta))^2}{|z(\alpha)-z(\alpha-\beta)|^2}\right) && (158)\\
=&\frac{\partial}{\partial\beta}\left(\frac{\left(\frac{1}{2}\left(\frac{\beta L(t)}{2\pi}w+\frac{\beta L(t)}{2\pi}\overline{w}\right)\right)^2}{\frac{\beta L(t)}{2\pi}w\frac{\beta L(t)}{2\pi}\overline{w}}\right) && (159)\\
=&\frac{\partial}{\partial\beta}\left(\frac{(w+\overline{w})^2}{4w\overline{w}}\right) && (160)\\
=&\frac{1}{4}\cdot\frac{\partial}{\partial\beta}\left((w+\overline{w})^2(w\overline{w})^{-1}\right) && (161)\\
=&\frac{1}{4}\left(2(w+\overline{w})(w_\beta+\overline{w}_\beta)(w\overline{w})^{-1}-(w+\overline{w})^2(w\overline{w})^{-2}(w_\beta\overline{w}+w\overline{w}_\beta)\right) && (162)\\
=&\frac{(w+\overline{w})(w_\beta+\overline{w}_\beta)}{2w\overline{w}}-\frac{(w+\overline{w})^2(w_\beta\overline{w}+w\overline{w}_\beta)}{4(w\overline{w})^2} && (163)\\
=&\frac{1}{2}\cdot\left(\frac{1}{\overline{w}}+\frac{1}{w}\right)(w_\beta+\overline{w}_\beta)-\frac{1}{4}\cdot\left(\frac{1}{\overline{w}}+\frac{1}{w}\right)^2(w_\beta\overline{w}+w\overline{w}_\beta). && (164)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\frac{\partial}{\partial\beta}\left(\frac{(z_2(\alpha)-z_2(\alpha-\beta))^2}{|z(\alpha)-z(\alpha-\beta)|^2}\right) && (165)\\
=&\frac{\partial}{\partial\beta}\left(\frac{\left(\frac{1}{2i}\left(\frac{\beta L(t)}{2\pi}w-\frac{\beta L(t)}{2\pi}\overline{w}\right)\right)^2}{\frac{\beta L(t)}{2\pi}w\frac{\beta L(t)}{2\pi}\overline{w}}\right) && (166)\\
=&-\frac{1}{4}\cdot\frac{\partial}{\partial\beta}\left((w-\overline{w})^2(w\overline{w})^{-1}\right) && (167)\\
=&-\frac{1}{4}\left(2(w-\overline{w})(w_\beta-\overline{w}_\beta)(w\overline{w})^{-1}-(w-\overline{w})^2(w\overline{w})^{-2}(w_\beta\overline{w}+w\overline{w}_\beta)\right) && (168)\\
=&-\frac{1}{2}\cdot\frac{w-\overline{w}}{w\overline{w}}(w_\beta-\overline{w}_\beta)+\frac{1}{4}\left(\frac{w-\overline{w}}{w\overline{w}}\right)^2(w_\beta\overline{w}+w\overline{w}_\beta) && (169)\\
=&-\frac{1}{2}\left(\frac{1}{\overline{w}}-\frac{1}{w}\right)(w_\beta-\overline{w}_\beta)+\frac{1}{4}\left(\frac{1}{\overline{w}}-\frac{1}{w}\right)^2(w_\beta\overline{w}+w\overline{w}_\beta). && (170)
\end{aligned}
$$

Lastly,

$$
\begin{aligned}
&\frac{\partial}{\partial\beta}\left(\frac{(z_1(\alpha)-z_1(\alpha-\beta))(z_2(\alpha)-z_2(\alpha-\beta))}{|z(\alpha)-z(\alpha-\beta)|^2}\right) && (171)\\
=&\frac{\partial}{\partial\beta}\left(\frac{\frac{1}{2}\left(\frac{\beta L(t)}{2\pi}w+\frac{\beta L(t)}{2\pi}\overline{w}\right)\frac{1}{2i}\left(\frac{\beta L(t)}{2\pi}w-\frac{\beta L(t)}{2\pi}\overline{w}\right)}{\frac{\beta L(t)}{2\pi}w\frac{\beta L(t)}{2\pi}\overline{w}}\right) && (172)\\
=&\frac{1}{4i}\cdot\frac{\partial}{\partial\beta}\left(\frac{(w+\overline{w})(w-\overline{w})}{w\overline{w}}\right) && (173)\\
=&\frac{1}{4i}\cdot\frac{\partial}{\partial\beta}\left(\frac{w^2-\overline{w}^2}{w\overline{w}}\right) && (174)\\
=&\frac{1}{4i}\cdot\frac{\partial}{\partial\beta}\left(\frac{w}{\overline{w}}-\frac{\overline{w}}{w}\right) && (175)\\
=&\frac{1}{4i}\cdot\frac{\partial}{\partial\beta}\left(w\overline{w}^{-1}-\overline{w}w^{-1}\right) && (176)\\
=&\frac{1}{4i}\left(w_\beta\overline{w}^{-1}-w\overline{w}^{-2}\overline{w}_\beta-\overline{w}_\beta w^{-1}+\overline{w}w^{-2}w_\beta\right) && (177)\\
=&\frac{1}{4i}\left(\frac{w_\beta}{\overline{w}}-\frac{w\overline{w}_\beta}{\overline{w}^2}-\frac{\overline{w}_\beta}{w}+\frac{\overline{w}w_\beta}{w^2}\right) && (178)\\
=&\frac{1}{4i}\left(2\left(\frac{w_\beta}{\overline{w}}-\frac{\overline{w}_\beta}{w}\right)-\left(\frac{w}{\overline{w}}-\frac{\overline{w}}{w}\right)\left(\frac{w_\beta}{w}+\frac{\overline{w}_\beta}{\overline{w}}\right)\right) && (179)\\
=&\frac{1}{2i}\left(\frac{w_\beta}{\overline{w}}-\frac{\overline{w}_\beta}{w}\right)-\frac{1}{4i}\left(\frac{w}{\overline{w}}-\frac{\overline{w}}{w}\right)\left(\frac{w_\beta}{w}+\frac{\overline{w}_\beta}{\overline{w}}\right). && (180)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\frac{\partial}{\partial\beta}\left(G_{11}(z(\alpha)-z(\alpha-\beta))\right) && (181)\\
=&\frac{\partial}{\partial\beta}\left(-\log|z(\alpha)-z(\alpha-\beta)|\right)+\frac{\partial}{\partial\beta}\left(\frac{(z_1(\alpha)-z_1(\alpha-\beta))^2}{|z(\alpha)-z(\alpha-\beta)|^2}\right) && (182)\\
=&-\frac{1}{\beta}-\frac{w_\beta}{2w}-\frac{\overline{w}_\beta}{2\overline{w}}+\frac{1}{2}\left(\frac{1}{\overline{w}}+\frac{1}{w}\right)(w_\beta+\overline{w}_\beta)-\frac{1}{4}\left(\frac{1}{\overline{w}}+\frac{1}{w}\right)^2(w_\beta\overline{w}+w\overline{w}_\beta), && (183)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial}{\partial\beta}\left(G_{22}(z(\alpha)-z(\alpha-\beta))\right) && (184)\\
=&\frac{\partial}{\partial\beta}\left(-\log|z(\alpha)-z(\alpha-\beta)|\right)+\frac{\partial}{\partial\beta}\left(\frac{(z_2(\alpha)-z_2(\alpha-\beta))^2}{|z(\alpha)-z(\alpha-\beta)|^2}\right) && (185)\\
=&-\frac{1}{\beta}-\frac{w_\beta}{2w}-\frac{\overline{w}_\beta}{2\overline{w}}-\frac{1}{2}\left(\frac{1}{\overline{w}}-\frac{1}{w}\right)(w_\beta-\overline{w}_\beta)+\frac{1}{4}\left(\frac{1}{\overline{w}}-\frac{1}{w}\right)^2(w_\beta\overline{w}+w\overline{w}_\beta), && (186)
\end{aligned}
$$

and

$$\frac{\partial}{\partial\beta}\Big(G_{12}(z(\alpha)-z(\alpha-\beta))\Big) = \frac{\partial}{\partial\beta}\Big(G_{21}(z(\alpha)-z(\alpha-\beta))\Big) \tag{187}$$
$$=\frac{\partial}{\partial\beta}\left(\frac{(z_1(\alpha)-z_1(\alpha-\beta))(z_2(\alpha)-z_2(\alpha-\beta))}{|z(\alpha)-z(\alpha-\beta)|^2}\right) \tag{188}$$
$$=\frac{1}{2i}\left(\frac{w_\beta}{\overline{w}}-\frac{\overline{w}_\beta}{w}\right)-\frac{1}{4i}\left(\frac{w}{\overline{w}}-\frac{\overline{w}}{w}\right)\left(\frac{w_\beta}{w}+\frac{\overline{w}_\beta}{\overline{w}}\right). \tag{189}$$

For notational convenience, let us write

$$w = C_1 + L_1 + N_1, \tag{190}$$
$$w^{-1} = C_2 + L_2 + N_2, \tag{191}$$
$$w_\beta = C_\beta + L_\beta + N_\beta, \tag{192}$$

where C_1, L_1, and N_1 are the parts of w which are constant, linear, and superlinear in the variable $\phi = \theta - \hat{\theta}(0)$, respectively; C_2, L_2, and N_2 are the parts of w^{-1} which are constant, linear, and superlinear in the variable ϕ, respectively; lastly, C_β, L_β, and N_β are the parts of w_β which are constant, linear, and superlinear in the variable ϕ. We note that

$$C_1 =\frac{-e^{i(\alpha-\beta)}e^{i\hat{\theta}(0)}i(-1+e^{i\beta})}{\beta} \tag{193}$$
$$L_1 =ie^{i(\alpha-\beta)}e^{i\hat{\theta}(0)}\int_0^1 e^{is\beta}\phi(\alpha+(s-1)\beta)ds \tag{194}$$
$$N_1 =e^{i(\alpha-\beta)}e^{i\hat{\theta}(0)}\int_0^1 e^{is\beta}\sum_{n=2}^{\infty}\frac{(i\phi(\alpha+(s-1)\beta))^n}{n!}ds \tag{195}$$
$$=e^{i(\alpha-\beta)}e^{i\hat{\theta}(0)} \tag{196}$$
$$\cdot\left(\int_0^1 e^{is\beta}e^{i\phi(\alpha+(s-1)\beta)}ds - i\int_0^1 e^{is\beta}\phi(\alpha+(s-1)\beta)ds+\frac{i(-1+e^{i\beta})}{\beta}\right), \tag{197}$$

$$C_2 = \frac{e^{-i\hat{\theta}(0)}e^{-i\alpha}i\beta}{1-e^{-i\beta}} \tag{198}$$

$$L_2 = \frac{e^{-i\hat{\theta}(0)}e^{-i(\alpha+\beta)}i\beta^2}{(1-e^{-i\beta})^2}\int_0^1 e^{is\beta}\phi(\alpha+(s-1)\beta)ds \tag{199}$$

$$N_2 = \frac{e^{-i\hat{\theta}(0)}e^{-i\alpha}\beta^2}{(1-e^{-i\beta})^2}\int_0^1 e^{i(s-1)\beta}\sum_{m=2}^{\infty}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds \tag{200}$$

$$+ e^{-i\hat{\theta}(0)}e^{-i\alpha}\sum_{n=2}^{\infty}(-1)^n\frac{(i\beta)^{n+1}}{(1-e^{-i\beta})^{n+1}}\left(\int_0^1 e^{i(s-1)\beta}\sum_{m=1}^{\infty}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n \tag{201}$$

$$= \frac{e^{-i\hat{\theta}(0)}e^{-i(\alpha+\beta)}\beta^2}{(1-e^{-i\beta})^2} \tag{202}$$

$$\cdot\left(\int_0^1 e^{is\beta}e^{i\phi(\alpha+(s-1)\beta)}ds - i\int_0^1 e^{is\beta}\phi(\alpha+(s-1)\beta)ds + \frac{i(-1+e^{i\beta})}{\beta}\right) \tag{203}$$

$$+ \frac{e^{-i\hat{\theta}(0)}e^{-i\alpha}e^{-2i\beta}(i\beta)^3}{(1-e^{-i\beta})^3}\left(\int_0^1 e^{is\beta}e^{i\phi(\alpha+(s-1)\beta)}ds + \frac{i(-1+e^{i\beta})}{\beta}\right)^2 \tag{204}$$

$$\cdot\left(1 - \frac{-i\beta e^{-i\beta}}{1-e^{-i\beta}}\left(\int_0^1 e^{is\beta}e^{i\phi(\alpha+(s-1)\beta)}ds + \frac{i(-1+e^{i\beta})}{\beta}\right)\right)^{-1}, \tag{205}$$

and

$$C_\beta = \frac{ie^{i(\alpha-\beta)}e^{i\hat{\theta}(0)}(e^{i\beta}-i\beta-1)}{\beta^2} \tag{206}$$

$$L_\beta = -e^{i(\alpha-\beta)}e^{i\hat{\theta}(0)}\int_0^1 e^{is\beta}(s-1)\phi(\alpha+(s-1)\beta)ds \tag{207}$$

$$+ ie^{i(\alpha-\beta)}e^{i\hat{\theta}(0)}\int_0^1 e^{is\beta}(s-1)\phi_\alpha(\alpha+(s-1)\beta)ds \tag{208}$$

$$N_\beta = ie^{i(\alpha-\beta)}e^{i\hat{\theta}(0)}\int_0^1 e^{is\beta}(s-1)\sum_{n=2}^{\infty}\frac{(i\phi(\alpha+(s-1)\beta))^n}{n!}ds \tag{209}$$

$$+ ie^{i(\alpha-\beta)}e^{i\hat{\theta}(0)}\int_0^1 e^{is\beta}(s-1)\sum_{n=1}^{\infty}\frac{(i\phi(\alpha+(s-1)\beta))^n}{n!}\phi_\alpha(\alpha+(s-1)\beta)ds \tag{210}$$

$$= ie^{i(\alpha-\beta)}e^{i\hat{\theta}(0)}\left(\int_0^1 e^{is\beta}(s-1)e^{i\phi(\alpha+(s-1)\beta)}ds - i\int_0^1 e^{is\beta}(s-1)\phi(\alpha+(s-1)\beta)ds\right. \tag{211}$$

$$\left.-\frac{e^{i\beta}-i\beta-1}{\beta^2}\right) \tag{212}$$

$$+ ie^{i(\alpha-\beta)}e^{i\hat{\theta}(0)}\left(\int_0^1 e^{is\beta}(s-1)e^{i\phi(\alpha+(s-1)\beta)}\phi_\alpha(\alpha+(s-1)\beta)ds\right. \tag{213}$$

$$\left.-\int_0^1 e^{is\beta}(s-1)\phi_\alpha(\alpha+(s-1)\beta)ds\right). \tag{214}$$

Similarly, let us write

$$\frac{\partial}{\partial\beta}\Big(G_{11}(z(\alpha)-z(\alpha-\beta))\Big) = C_{11}+L_{11}+N_{11}, \tag{215}$$

$$\frac{\partial}{\partial\beta}\Big(G_{12}(z(\alpha)-z(\alpha-\beta))\Big) = \frac{\partial}{\partial\beta}\Big(G_{21}(z(\alpha)-z(\alpha-\beta))\Big) = C_{12}+L_{12}+N_{12}, \tag{216}$$

$$\frac{\partial}{\partial\beta}\Big(G_{22}(z(\alpha)-z(\alpha-\beta))\Big) = C_{22}+L_{22}+N_{22}, \tag{217}$$

where C_{11}, L_{11}, and N_{11} are the parts of $\frac{\partial}{\partial\beta}\Big(G_{11}(z(\alpha)-z(\alpha-\beta))\Big)$ which are constant, linear, and superlinear in the variable ϕ; C_{12}, L_{12}, and N_{12} are the parts of $\frac{\partial}{\partial\beta}\Big(G_{12}(z(\alpha)-z(\alpha-\beta))\Big)$ which are constant, linear, and superlinear in the variable ϕ; lastly, C_{22}, L_{22}, and N_{22} are the parts of $\frac{\partial}{\partial\beta}\Big(G_{22}(z(\alpha)-z(\alpha-\beta))\Big)$ which are constant, linear, and superlinear in the variable ϕ. We note that

$$C_{11} = -\frac{1}{\beta} - \frac{1}{2}C_2C_\beta - \frac{1}{2}\overline{C_2C_\beta} + \frac{1}{2}(C_2+\overline{C_2})(C_\beta+\overline{C_\beta}) \tag{218}$$

$$-\frac{1}{4}(C_2+\overline{C_2})^2(C_\beta\overline{C_1}+C_1\overline{C_\beta}), \tag{219}$$

$$L_{11} = -\frac{1}{2}(C_2L_\beta+C_\beta L_2) - \frac{1}{2}(\overline{C_2L_\beta}+\overline{L_2C_\beta}) \tag{220}$$

$$+\frac{1}{2}\Big((C_2+\overline{C_2})(L_\beta+\overline{L_\beta})+(L_2+\overline{L_2})(C_\beta+\overline{C_\beta})\Big) \tag{221}$$

$$-\frac{1}{4}\Big((C_2+\overline{C_2})^2(C_\beta\overline{L_1}+L_\beta\overline{C_1}+C_1\overline{L_\beta}+L_1\overline{C_\beta}) \tag{222}$$

$$+2(C_2+\overline{C_2})(L_2+\overline{L_2})(C_\beta\overline{C_1}+C_1\overline{C_\beta})\Big), \tag{223}$$

$$N_{11} = -\frac{1}{2}\bigg(C_2N_\beta + L_2(L_\beta + N_\beta) + N_2(C_\beta + L_\beta + N_\beta)\bigg) \tag{224}$$

$$-\frac{1}{2}\bigg(\overline{C_2N_\beta} + \overline{L_2}(\overline{L_\beta} + \overline{N_\beta}) + \overline{N_2}(\overline{C_\beta} + \overline{L_\beta} + \overline{N_\beta})\bigg) \tag{225}$$

$$+\frac{1}{2}\bigg((C_2 + \overline{C_2})(N_\beta + \overline{N_\beta}) + (L_2 + \overline{L_2})(L_\beta + \overline{L_\beta} + N_\beta + \overline{N_\beta}) \tag{226}$$

$$+ (N_2 + \overline{N_2})(C_\beta + \overline{C_\beta} + L_\beta + \overline{L_\beta} + N_\beta + \overline{N_\beta})\bigg) \tag{227}$$

$$-\frac{1}{4}\bigg((C_2 + \overline{C_2})^2\bigg(C_\beta\overline{N_1} + L_\beta(\overline{L_1} + \overline{N_1}) + N_\beta(\overline{C_1} + \overline{L_1} + \overline{N_1}) \tag{228}$$

$$+ C_1\overline{N_\beta} + L_1(\overline{L_\beta} + \overline{N_\beta}) + N_1(\overline{C_\beta} + \overline{L_\beta} + \overline{N_\beta})\bigg) \tag{229}$$

$$+ 2(C_2 + \overline{C_2})(L_2 + \overline{L_2})\bigg(C_\beta\overline{L_1} + L_\beta\overline{C_1} + C_1\overline{L_\beta} + L_1\overline{C_\beta} + C_\beta\overline{N_1} \tag{230}$$

$$+ L_\beta(\overline{L_1} + \overline{N_1}) + N_\beta(\overline{C_1} + \overline{L_1} + \overline{N_1}) + C_1\overline{N_\beta} + L_1(\overline{L_\beta} + \overline{N_\beta}) \tag{231}$$

$$+ N_1(\overline{C_\beta} + \overline{L_\beta} + \overline{N_\beta})\bigg) \tag{232}$$

$$+\bigg((C_2 + \overline{C_2})(N_2 + \overline{N_2}) + (L_2 + \overline{L_2})(L_2 + \overline{L_2} + N_2 + \overline{N_2}) \tag{233}$$

$$+ (N_2 + \overline{N_2})(C_2 + \overline{C_2} + L_2 + \overline{L_2} + N_2 + \overline{N_2})\bigg) \tag{234}$$

$$\cdot\bigg(C_\beta\overline{C_1} + C_1\overline{C_\beta} + C_\beta\overline{L_1} + L_\beta\overline{C_1} + C_1\overline{L_\beta} + L_1\overline{C_\beta} + C_\beta\overline{N_1} + L_\beta(\overline{L_1} + \overline{N_1}) \tag{235}$$

$$+ N_\beta(\overline{C_1} + \overline{L_1} + \overline{N_1}) + C_1\overline{N_\beta} + L_1(\overline{L_\beta} + \overline{N_\beta}) + N_1(\overline{C_\beta} + \overline{L_\beta} + \overline{N_\beta})\bigg)\bigg), \tag{236}$$

$$C_{12} = \frac{1}{2i}\bigg(C_\beta\overline{C_2} - \overline{C_\beta}C_2\bigg) - \frac{1}{4i}\bigg(C_1\overline{C_2} - \overline{C_1}C_2\bigg)\bigg(C_\beta C_2 + \overline{C_\beta C_2}\bigg), \tag{237}$$

$$L_{12} = \frac{1}{2i}\bigg(C_\beta\overline{L_2} + L_\beta\overline{C_2} - \overline{C_\beta}L_2 - \overline{L_\beta}C_2\bigg) - \frac{1}{4i}\bigg((C_1\overline{C_2} - \overline{C_1}C_2)(C_\beta L_2 + L_\beta C_2 + \overline{C_\beta L_2} + \overline{L_\beta C_2}) \tag{238}$$

$$+ (C_1\overline{L_2} + L_1\overline{C_2} - \overline{C_1}L_2 - \overline{L_1}C_2)(C_\beta C_2 + \overline{C_\beta C_2})\bigg), \tag{239}$$

$$N_{12} = \frac{1}{2i}\bigg(C_\beta\overline{N_2} + L_\beta(\overline{L_2} + \overline{N_2}) + N_\beta(\overline{C_2} + \overline{L_2} + \overline{N_2}) \tag{240}$$

$$- \Big(\overline{C_\beta}N_2 + \overline{L_\beta}(L_2 + N_2) + \overline{N_\beta}(C_2 + L_2 + N_2)\Big)\bigg) \tag{241}$$

$$- \frac{1}{4i}\bigg((C_1\overline{C_2} - \overline{C_1}C_2)\Big(C_\beta N_2 + L_\beta(L_2 + N_2) + N_\beta(C_2 + L_2 + N_2) \tag{242}$$

$$+ \overline{C_\beta N_2} + \overline{L_\beta}(\overline{L_2} + \overline{N_2}) + \overline{N_\beta}(\overline{C_2} + \overline{L_2} + \overline{N_2})\Big) \tag{243}$$

$$+ \Big(C_1\overline{L_2} + L_1\overline{C_2} - (\overline{C_1}L_2 + \overline{L_1}C_2)\Big) \tag{244}$$

$$\cdot \Big(C_\beta L_2 + L_\beta C_2 + \overline{C_\beta L_2} + \overline{L_\beta C_2} + C_\beta N_2 + L_\beta(L_2 + N_2) \tag{245}$$

$$+ N_\beta(C_2 + L_2 + N_2) + \overline{C_\beta N_2} + \overline{L_\beta}(\overline{L_2} + \overline{N_2}) + \overline{N_\beta}(\overline{C_2} + \overline{L_2} + \overline{N_2})\Big) \tag{246}$$

$$+ \Big(C_1\overline{N_2} + L_1(\overline{L_2} + \overline{N_2}) + N_1(\overline{C_2} + \overline{L_2} + \overline{N_2}) \tag{247}$$

$$- \Big(\overline{C_1}N_2 + \overline{L_1}(L_2 + N_2) + \overline{N_1}(C_2 + L_2 + N_2)\Big)\Big) \tag{248}$$

$$\cdot \Big(C_\beta C_2 + \overline{C_\beta C_2} + C_\beta L_2 + L_\beta C_2 + \overline{C_\beta L_2} + \overline{L_\beta C_2} + C_\beta N_2 + L_\beta(L_2 + N_2) \tag{249}$$

$$+ N_\beta(C_2 + L_2 + N_2) + \overline{C_\beta N_2} + \overline{L_\beta}(\overline{L_2} + \overline{N_2}) + \overline{N_\beta}(\overline{C_2} + \overline{L_2} + \overline{N_2})\Big)\bigg), \tag{250}$$

and

$$C_{22} = -\frac{1}{\beta} - \frac{1}{2}C_\beta C_2 - \frac{1}{2}\overline{C_\beta C_2} - \frac{1}{2}(\overline{C_2} - C_2)(C_\beta - \overline{C_\beta}) \tag{251}$$

$$+ \frac{1}{4}(\overline{C_2} - C_2)^2(C_\beta\overline{C_1} + C_1\overline{C_\beta}), \tag{252}$$

$$L_{22} = -\frac{1}{2}(C_\beta L_2 + L_\beta C_2) - \frac{1}{2}(\overline{C_\beta L_2} + \overline{L_\beta C_2}) \tag{253}$$

$$- \frac{1}{2}\Big((\overline{C_2} - C_2)(L_\beta - \overline{L_\beta}) + (\overline{L_2} - L_2)(C_\beta - \overline{C_\beta})\Big) \tag{254}$$

$$+ \frac{1}{4}\Big((\overline{C_2} - C_2)^2(C_\beta\overline{L_1} + L_\beta\overline{C_1} + C_1\overline{L_\beta} + L_1\overline{C_\beta}) \tag{255}$$

$$+ 2(\overline{C_2} - C_2)(\overline{L_2} - L_2)(C_\beta\overline{C_1} + C_1\overline{C_\beta})\Big), \tag{256}$$

$$
\begin{aligned}
N_{22} = &-\frac{1}{2}\Big(C_\beta N_2 + L_\beta(L_2+N_2) + N_\beta(C_2+L_2+N_2)\Big) && (257)\\
&-\frac{1}{2}\Big(\overline{C_\beta N_2} + \overline{L_\beta}(\overline{L_2}+\overline{N_2}) + \overline{N_\beta}(\overline{C_2}+\overline{L_2}+\overline{N_2})\Big) && (258)\\
&-\frac{1}{2}\Big((\overline{C_2}-C_2)(N_\beta-\overline{N_\beta}) + (\overline{L_2}-L_2)(L_\beta-\overline{L_\beta}+N_\beta-\overline{N_\beta}) && (259)\\
&+(\overline{N_2}-N_2)(C_\beta-\overline{C_\beta}+L_\beta-\overline{L_\beta}+N_\beta-\overline{N_\beta})\Big) && (260)\\
&+\frac{1}{4}\Big((\overline{C_2}-C_2)^2\Big(C_\beta\overline{N_1}+L_\beta(\overline{L_1}+\overline{N_1})+N_\beta(\overline{C_1}+\overline{L_1}+\overline{N_1})+C_1\overline{N_\beta} && (261)\\
&+L_1(\overline{L_\beta}+\overline{N_\beta})+N_1(\overline{C_\beta}+\overline{L_\beta}+\overline{N_\beta})\Big) && (262)\\
&+2(\overline{C_2}-C_2)(\overline{L_2}-L_2)\Big(C_\beta\overline{L_1}+L_\beta\overline{C_1}+C_1\overline{L_\beta}+L_1\overline{C_\beta}+C_\beta\overline{N_1} && (263)\\
&+L_\beta(\overline{L_1}+\overline{N_1})+N_\beta(\overline{C_1}+\overline{L_1}+\overline{N_1}) && (264)\\
&+C_1\overline{N_\beta}+L_1(\overline{L_\beta}+\overline{N_\beta})+N_1(\overline{C_\beta}+\overline{L_\beta}+\overline{N_\beta})\Big) && (265)\\
&+\Big((\overline{C_2}-C_2)(\overline{N_2}-N_2)+(\overline{L_2}-L_2)(\overline{L_2}-L_2+\overline{N_2}-N_2) && (266)\\
&+(\overline{N_2}-N_2)(\overline{C_2}-C_2+\overline{L_2}-L_2+\overline{N_2}-N_2)\Big) && (267)\\
&\cdot\Big(C_\beta\overline{C_1}+C_1\overline{C_\beta}+C_\beta\overline{L_1}+L_\beta\overline{C_1}+C_1\overline{L_\beta}+L_1\overline{C_\beta}+C_\beta\overline{N_1}+L_\beta(\overline{L_1}+\overline{N_1}) && (268)\\
&+N_\beta(\overline{C_1}+\overline{L_1}+\overline{N_1})+C_1\overline{N_\beta}+L_1(\overline{L_\beta}+\overline{N_\beta})+N_1(\overline{C_\beta}+\overline{L_\beta}+\overline{N_\beta})\Big)\Big), && (269)
\end{aligned}
$$

where $\overline{X}$ denotes the complex conjugate of X. It is clear from these expressions that C_{11}, L_{11}, N_{11}, C_{12}, L_{12}, N_{12}, C_{22}, L_{22}, and N_{22} are all real. Using these expressions, we can write

$$u_1(z(\alpha)) \tag{270}$$

$$=\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\frac{1}{2}\left(e^{i(\alpha-\beta+\theta(\alpha-\beta))}+e^{-i(\alpha-\beta+\theta(\alpha-\beta))}\right)\frac{\partial}{\partial\beta}\left(G_{11}(z(\alpha)-z(\alpha-\beta))\right) \tag{271}$$

$$+\frac{1}{2i}\left(e^{i(\alpha-\beta+\theta(\alpha-\beta))}-e^{-i(\alpha-\beta+\theta(\alpha-\beta))}\right)\frac{\partial}{\partial\beta}\left(G_{21}(z(\alpha)-z(\alpha-\beta))\right)d\beta \tag{272}$$

$$=\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\frac{1}{2}\left(e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}e^{i\phi(\alpha-\beta)}+e^{-i\hat{\theta}(0)}e^{-i(\alpha-\beta)}e^{-i\phi(\alpha-\beta)}\right) \tag{273}$$

$$\cdot\frac{\partial}{\partial\beta}\left(G_{11}(z(\alpha)-z(\alpha-\beta))\right) \tag{274}$$

$$+\frac{1}{2i}\left(e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}e^{i\phi(\alpha-\beta)}-e^{-i\hat{\theta}(0)}e^{-i(\alpha-\beta)}e^{-i\phi(\alpha-\beta)}\right) \tag{275}$$

$$\cdot\frac{\partial}{\partial\beta}\left(G_{21}(z(\alpha)-z(\alpha-\beta))\right)d\beta \tag{276}$$

$$=\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\frac{e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}e^{i\phi(\alpha-\beta)}}{2}\left(\frac{\partial}{\partial\beta}\left(G_{11}(z(\alpha)-z(\alpha-\beta))\right)\right. \tag{277}$$

$$\left.+\frac{1}{i}\frac{\partial}{\partial\beta}\left(G_{21}(z(\alpha)-z(\alpha-\beta))\right)\right) \tag{278}$$

$$+\frac{e^{-i\hat{\theta}(0)}e^{-i(\alpha-\beta)}e^{-i\phi(\alpha-\beta)}}{2}\left(\frac{\partial}{\partial\beta}\left(G_{11}(z(\alpha)-z(\alpha-\beta))\right)\right. \tag{279}$$

$$\left.-\frac{1}{i}\frac{\partial}{\partial\beta}\left(G_{21}(z(\alpha)-z(\alpha-\beta))\right)\right)d\beta \tag{280}$$

$$=\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\frac{e^{i\hat{\theta}}e^{i(\alpha-\beta)}}{2}\left(1+i\phi(\alpha-\beta)+\sum_{n=2}^{\infty}\frac{(i\phi(\alpha-\beta))^n}{n!}\right) \tag{281}$$

$$\cdot\left(C_{11}+L_{11}+N_{11}-i(C_{21}+L_{21}+N_{21})\right) \tag{282}$$

$$+\frac{e^{-i\hat{\theta}(0)}e^{-i(\alpha-\beta)}}{2}\left(1-i\phi(\alpha-\beta)+\sum_{n=2}^{\infty}\frac{(-i\phi(\alpha-\beta))^n}{n!}\right) \tag{283}$$

$$\cdot\left(C_{11}+L_{11}+N_{11}+i(C_{21}+L_{21}+N_{21})\right)d\beta, \tag{284}$$

and

$$u_2(z(\alpha)) \tag{285}$$
$$=\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\frac{1}{2}\left(e^{i(\alpha-\beta+\theta(\alpha-\beta))}+e^{-i(\alpha-\beta+\theta(\alpha-\beta))}\right)\frac{\partial}{\partial\beta}\left(G_{12}(z(\alpha)-z(\alpha-\beta))\right) \tag{286}$$
$$+\frac{1}{2i}\left(e^{i(\alpha-\beta+\theta(\alpha-\beta))}-e^{-i(\alpha-\beta+\theta(\alpha-\beta))}\right)\frac{\partial}{\partial\beta}\left(G_{22}(z(\alpha)-z(\alpha-\beta))\right)d\beta \tag{287}$$
$$=\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\frac{1}{2}\left(e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}e^{i\phi(\alpha-\beta)}+e^{-i\hat{\theta}(0)}e^{-i(\alpha-\beta)}e^{-i\phi(\alpha-\beta)}\right) \tag{288}$$
$$\cdot\frac{\partial}{\partial\beta}\left(G_{12}(z(\alpha)-z(\alpha-\beta))\right) \tag{289}$$
$$+\frac{1}{2i}\left(e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}e^{i\phi(\alpha-\beta)}-e^{-i\hat{\theta}(0)}e^{-i(\alpha-\beta)}e^{-i\phi(\alpha-\beta)}\right) \tag{290}$$
$$\cdot\frac{\partial}{\partial\beta}\left(G_{22}(z(\alpha)-z(\alpha-\beta))\right)d\beta \tag{291}$$
$$=\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\frac{e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}e^{i\phi(\alpha-\beta)}}{2}\left(\frac{\partial}{\partial\beta}\left(G_{12}(z(\alpha)-z(\alpha-\beta))\right)\right. \tag{292}$$
$$\left.+\frac{1}{i}\frac{\partial}{\partial\beta}\left(G_{22}(z(\alpha)-z(\alpha-\beta))\right)\right) \tag{293}$$
$$+\frac{e^{-i\hat{\theta}(0)}e^{-i(\alpha-\beta)}e^{-i\phi(\alpha-\beta)}}{2}\left(\frac{\partial}{\partial\beta}\left(G_{12}(z(\alpha)-z(\alpha-\beta))\right)\right. \tag{294}$$
$$\left.-\frac{1}{i}\frac{\partial}{\partial\beta}\left(G_{22}(z(\alpha)-z(\alpha-\beta))\right)\right)d\beta \tag{295}$$
$$=\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\frac{e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}\left(1+i\phi(\alpha-\beta)+\sum_{n=2}^{\infty}\frac{(i\phi(\alpha-\beta))^n}{n!}\right) \tag{296}$$
$$\cdot(C_{12}+L_{12}+N_{12}-i(C_{22}+L_{22}+N_{22})) \tag{297}$$
$$+\frac{e^{-i\hat{\theta}(0)}e^{-i(\alpha-\beta)}}{2}\left(1-i\phi(\alpha-\beta)+\sum_{n=2}^{\infty}\frac{(-i\phi(\alpha-\beta))^n}{n!}\right) \tag{298}$$
$$\cdot(C_{12}+L_{12}+N_{12}+i(C_{22}+L_{22}+N_{22}))d\beta. \tag{299}$$

Therefore,

$$u_1(z(\alpha)) - iu_2(z(\alpha)) \tag{300}$$

$$= \frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\frac{e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}\left(1 + i\phi(\alpha-\beta) + \sum_{n=2}^{\infty}\frac{(i\phi(\alpha-\beta))^n}{n!}\right) \tag{301}$$

$$\cdot\Big((C_{11}+L_{11}+N_{11}) - (C_{22}+L_{22}+N_{22}) - 2i(C_{12}+L_{12}+N_{12})\Big) \tag{302}$$

$$+ \frac{e^{-i\hat{\theta}(0)}e^{-i(\alpha-\beta)}}{2}\left(1 - i\phi(\alpha-\beta) + \sum_{n=2}^{\infty}\frac{(-i\phi(\alpha-\beta))^n}{n!}\right) \tag{303}$$

$$\cdot\Big((C_{11}+L_{11}+N_{11}) + (C_{22}+L_{22}+N_{22})\Big)d\beta. \tag{304}$$

Let

$$u_1(\alpha) - iu_2(\alpha) = \mathfrak{C}(\alpha) + \mathfrak{L}(\alpha) + \mathfrak{N}(\alpha), \tag{305}$$

where $\mathfrak{C}$, $\mathfrak{L}$, and $\mathfrak{N}$ are the parts of $u_1 - iu_2$ which are constant, linear, and superlinear in the variable $\phi = \theta - \hat{\theta}(0)$, respectively. Then

$$\mathfrak{C}(\alpha) = \frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\frac{e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}(C_{11}-C_{22}-2iC_{12}) + \frac{e^{-i\hat{\theta}(0)}e^{-i(\alpha-\beta)}}{2}(C_{11}+C_{22})d\beta, \tag{306}$$

$$\mathfrak{L}(\alpha) = \frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\left(\frac{e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}i}{2}(C_{11}-C_{22}-2iC_{12}) - \frac{e^{-i\hat{\theta}(0)}e^{-i(\alpha-\beta)}i}{2}(C_{11}+C_{22})\right) \tag{307}$$

$$\cdot\,\phi(\alpha-\beta)d\beta \tag{308}$$

$$+ \frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\left(\frac{e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}(L_{11}-L_{22}-2iL_{12}) + \frac{e^{-i\hat{\theta}(0)}e^{-i(\alpha-\beta)}}{2}(L_{11}+L_{22})\right)d\beta, \tag{309}$$

$$\mathfrak{N}(\alpha) = \frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\frac{e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}\Big((N_{11}-N_{22}-2iN_{12}) \tag{310}$$

$$+ i\phi(\alpha-\beta)(L_{11}-L_{22}-2iL_{12}+N_{11}-N_{22}-2iN_{12}) \tag{311}$$

$$+ \sum_{n=2}^{\infty}\frac{(i\phi(\alpha-\beta))^n}{n!} \tag{312}$$

$$\cdot\,(C_{11}-C_{22}-2iC_{12}+L_{11}-L_{22}-2iL_{12}+N_{11}-N_{22}-2iN_{12})\Big) \tag{313}$$

$$+ \frac{e^{-i\hat{\theta}(0)}e^{-i(\alpha-\beta)}}{2}\Big((N_{11}+N_{22}) - i\phi(\alpha-\beta)(L_{11}+L_{22}+N_{11}+N_{22}) \tag{314}$$

$$+ \sum_{n=2}^{\infty}\frac{(-i\phi(\alpha-\beta))^n}{n!}(C_{11}+C_{22}+L_{11}+L_{22}+N_{11}+N_{22})\Big)d\beta. \tag{315}$$

In particular,

$$\mathfrak{C}(\alpha) = \frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\frac{e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}(C_{11}-C_{22}-2iC_{12}) + \frac{e^{-i\hat{\theta}(0)}e^{-i(\alpha-\beta)}}{2}(C_{11}+C_{22})d\beta \tag{316}$$

$$= 0. \tag{317}$$

Let $U = U_0 + U_1 + U_{\geq 2}$, where U_0, U_1, and $U_{\geq 2}$ are the parts of U which are constant, linear, and superlinear in the variable $\phi = \theta - \hat{\theta}(0)$, respectively. Then

$$U_0(\alpha) = \mathrm{Re}\bigg(ie^{i\alpha}e^{i\hat{\theta}(0)}\mathfrak{C}(\alpha)\bigg) = 0. \tag{318}$$

To find expressions for U_1 and $U_{\geq 2}$, we rewrite

$$\begin{aligned}
U(\alpha) &= \mathrm{Re}\bigg((u_1(\alpha) - iu_2(\alpha))ie^{i(\alpha+\theta(\alpha))}\bigg) &(319)\\
&= \mathrm{Re}\bigg((\mathfrak{L}(\alpha) + \mathfrak{N}(\alpha))ie^{i(\alpha+\phi(\alpha)+\hat{\theta}(0))}\bigg) &(320)\\
&= \mathrm{Re}\bigg((\mathfrak{L}(\alpha) + \mathfrak{N}(\alpha))ie^{i\alpha}e^{i\hat{\theta}(0)}e^{i\phi(\alpha)}\bigg) &(321)\\
&= \mathrm{Re}\bigg(ie^{i\alpha}e^{i\hat{\theta}(0)}(\mathfrak{L}(\alpha) + \mathfrak{N}(\alpha))\bigg(1 + \sum_{n=1}^{\infty}\frac{(i\phi(\alpha))^n}{n!}\bigg)\bigg) &(322)\\
&= \mathrm{Re}\bigg(ie^{i\alpha}e^{i\hat{\theta}(0)}\bigg(\mathfrak{L}(\alpha) + \mathfrak{L}(\alpha)\sum_{n=1}^{\infty}\frac{(i\phi(\alpha))^n}{n!} + \mathfrak{N}(\alpha)\bigg(1 + \sum_{n=1}^{\infty}\frac{(i\phi(\alpha))^n}{n!}\bigg)\bigg)\bigg) &(323)\\
&= \mathrm{Re}\bigg(ie^{i\alpha}e^{i\hat{\theta}(0)}\bigg(\mathfrak{L}(\alpha) + \mathfrak{L}(\alpha)\sum_{n=1}^{\infty}\frac{(i\phi(\alpha))^n}{n!} + \mathfrak{N}(\alpha)e^{i\phi(\alpha)}\bigg)\bigg) &(324)\\
&= \mathrm{Re}\bigg(ie^{i\alpha}e^{i\hat{\theta}(0)}\mathfrak{L}(\alpha) + ie^{i\alpha}e^{i\hat{\theta}(0)}\bigg(\mathfrak{L}(\alpha)\sum_{n=1}^{\infty}\frac{(i\phi(\alpha))^n}{n!} + \mathfrak{N}(\alpha)e^{i\phi(\alpha)}\bigg)\bigg) &(325)\\
&= \mathrm{Re}\bigg(ie^{i\alpha}e^{i\hat{\theta}(0)}\mathfrak{L}(\alpha)\bigg) + \mathrm{Re}\bigg(ie^{i\alpha}e^{i\hat{\theta}(0)}\bigg(\mathfrak{L}(\alpha)(e^{i\phi(\alpha)} - 1) + \mathfrak{N}(\alpha)e^{i\phi(\alpha)}\bigg)\bigg). &(326)
\end{aligned}$$

Then it is clear that

$$\begin{aligned}
U_1(\alpha) &= \mathrm{Re}\bigg(ie^{i\alpha}e^{i\hat{\theta}(0)}\mathfrak{L}(\alpha)\bigg), &(327)\\
U_{\geq 2}(\alpha) &= \mathrm{Re}\bigg(ie^{i\alpha}e^{i\hat{\theta}(0)}\bigg(\mathfrak{L}(\alpha)(e^{i\phi(\alpha)} - 1) + \mathfrak{N}(\alpha)e^{i\phi(\alpha)}\bigg)\bigg). &(328)
\end{aligned}$$

7.3 The Tangential Speed T

Let us rewrite the right hand side of (46) as

$$\frac{2\pi}{L(t)}\bigg(U_\alpha(\alpha) + T(\alpha)(1 + \phi_\alpha(\alpha))\bigg) = \mathcal{C}(\alpha) + \mathcal{L}(\alpha) + \mathcal{N}(\alpha), \tag{329}$$

where $\mathcal{C}$, $\mathcal{L}$, and $\mathcal{N}$ are the parts of the right hand side of the evolution equation for θ which are constant, linear, and superlinear in the variable $\phi = \theta - \hat{\theta}(0)$, respectively. We will determine the analytical expression for $T(\alpha)$ by stipulating that $\mathcal{C} = 0$. To begin, let us rewrite the right hand side of (43) as

$$\int_0^\alpha (1+\phi_\alpha(\eta))U(\eta)d\eta - \frac{\alpha}{2\pi}\int_{-\pi}^{\pi}(1+\phi_\alpha(\eta))U(\eta)d\eta + T(0) \tag{330}$$

$$=T_0(\alpha)+T_1(\alpha)+T_{\geq 2}(\alpha), \tag{331}$$

where T_0, T_1, and $T_{\geq 2}$ are the parts of T which are constant, linear, and superlinear in the variable $\phi = \theta - \hat{\theta}(0)$, respectively. We note that

$$T_0(\alpha) = \int_0^\alpha U_0(\eta)d\eta - \frac{\alpha}{2\pi}\int_{-\pi}^{\pi} U_0(\eta)d\eta + T(0) = T(0), \tag{332}$$

$$T_1(\alpha) = \int_0^\alpha U_1(\eta)d\eta + \int_0^\alpha \phi_\alpha(\eta)U_0(\eta)d\eta \tag{333}$$

$$- \frac{\alpha}{2\pi}\int_{-\pi}^{\pi} U_1(\eta)d\eta - \frac{\alpha}{2\pi}\int_{-\pi}^{\pi}\phi_\alpha(\eta)U_0(\eta)d\eta, \tag{334}$$

$$T_{\geq 2}(\alpha) = \int_0^\alpha U_{\geq 2}(\eta)d\eta + \int_0^\alpha \phi_\alpha(\eta)U_{\geq 1}(\eta)d\eta \tag{335}$$

$$- \frac{\alpha}{2\pi}\int_{-\pi}^{\pi} U_{\geq 2}(\eta)d\eta - \frac{\alpha}{2\pi}\int_{-\pi}^{\pi}\phi_\alpha(\eta)U_{\geq 1}(\eta)d\eta, \tag{336}$$

where we define $U_{\geq 1} = U_1 + U_{\geq 2}$. Let $T(0)=0$. Then using (318), we obtain

$$\mathcal{C}(\alpha) = \frac{2\pi}{L(t)}\Big((U_0)_\alpha(\alpha) + T_0(\alpha)\Big) = \frac{2\pi}{L(t)}T_0(\alpha) = \frac{2\pi}{L(t)}T(0) = 0. \tag{337}$$

It is important for our analysis that $\mathcal{C}=0$ because we want the leading order term of the evolution equation for θ to be $\mathcal{L}$, which we show in Section 9 to be the Hilbert transform of the first derivative of θ up to the ± 1 Fourier modes.

8 Steady-State Solutions

In Section 5, we have characterized the circular interface under the HLS parametrization. In particular, we know from Proposition 5 that $(\theta(\alpha,t), L(t)) = (\hat{\theta}(0,t), 2\pi R)$ corresponds to a circle of radius R. Since

$$\int_0^\alpha z_\eta(\eta,t)d\eta = \int_0^\alpha |z_\eta(\eta,t)|\, e^{i(\eta+\theta(\eta,t))}d\eta = \frac{L(t)}{2\pi}\int_0^\alpha e^{i(\eta+\theta(\eta,t))}d\eta \tag{338}$$

under the HLS parametrization, plugging in $(\theta(\alpha,t), L(t)) = (\hat{\theta}(0,t), 2\pi R)$, we obtain

$$z(\alpha,t) - z(0,t) = R\int_0^\alpha e^{i(\eta+\hat{\theta}(0,t))}d\eta = Re^{i\hat{\theta}(0,t)}\int_0^\alpha e^{i\eta}d\eta = -iRe^{i\hat{\theta}(0,t)}(e^{i\alpha}-1). \tag{339}$$

Rearranging the equation, we obtain

$$z(\alpha,t) = Re^{i\left(\alpha+\hat{\theta}(0,t)-\frac{\pi}{2}\right)} + \left(z(0,t) + Re^{i\left(\hat{\theta}(0,t)+\frac{\pi}{2}\right)}\right). \tag{340}$$

We remark that, as expected, this expression clearly shows that for any fixed time t, the interface is a circle of radius R. That $\phi(\alpha,t) = \theta(\alpha,t) - \hat{\theta}(0,t) = 0$ implies that $\mathfrak{L}(\alpha,t) = \mathfrak{N}(\alpha,t) = 0$. Then, $U_1 = U_{\geq 2} = 0$ by (327) and (328). Combined with (318), they imply $U(\alpha,t) = 0$. Due to the analytical expression chosen for $T(\alpha)$ in Section 7.3, this implies $T(\alpha,t) = 0$. Then

$$z_t(\alpha,t) = -U(\alpha,t)\boldsymbol{n}(\alpha,t) + T(\alpha,t)\boldsymbol{\tau}(\alpha,t) = 0. \tag{341}$$

This means that $z(0,t)$ appearing in (340) is in fact a constant. We can then rewrite (340) as

$$z(\alpha,t) = Re^{i\left(\alpha+\hat{\theta}(0,t)-\frac{\pi}{2}\right)} + \left(z(0,0) + Re^{i\left(\hat{\theta}(0,t)+\frac{\pi}{2}\right)}\right). \tag{342}$$

This describes a circle of radius R whose center is bounded in time. The circular interface becomes a solution to (45) and (46) if $\hat{\theta}(0,t)$ is constant in time. In this case, the interface is stationary. The following proposition summarizes the existence of steady-state solutions to (45) and (46).

Proposition 6. *For any constant c, the circle defined by*

$$(\theta(\alpha,t), L(t)) = (c, 2\pi R) \tag{343}$$

is a time-independent solution of (45) and (46) in which $T(\alpha,t)$ is given by (43) and $U(\alpha,t)$ is given by

$$U(\alpha,t) = Re\bigg((u_1(\alpha,t) - iu_2(\alpha,t))ie^{i(\alpha+\theta(\alpha,t))}\bigg) \tag{344}$$

with $u_1(\alpha,t) - iu_2(\alpha,t)$ given by (300). This solution corresponds to a stationary circle of radius R.

Proof. Let $(\theta(\alpha,t), L(t)) = (\hat{\theta}(0,t), 2\pi R)$ be a circle of radius R such that $\hat{\theta}(0,t) = c$ for some constant c. Since $U(\alpha,t) = T(\alpha,t) = 0$, the right hand sides of (45) and (46) vanish. Since $(\theta_t(\alpha,t), L_t(t)) = (0,0)$, (45) and (46) are indeed satisfied by $(\theta(\alpha,t), L(t)) = (c, 2\pi R)$. That this solution is stationary follows from the fact that $\hat{\theta}(0,t) = c$ makes the right hand side of (342) independent of t, i.e., the circle is stationary, as needed. ■

9 The Principal Linear Operator for the θ Equation

In Section 7.3, we determined the analytical expression for $T(\alpha)$ in such a way that $\mathcal{C} = 0$ to ensure that the linear operator $\mathcal{L}$ appearing in the evolution equation for θ, which acts on $\phi = \theta - \hat{\theta}(0)$, is the Hilbert transform of the first derivative of θ up to the ± 1 Fourier modes. We prove this claim about the operator $\mathcal{L}$ through explicit computation in the Fourier space. We note that

$$\mathcal{L}(\alpha) = \frac{2\pi}{L(t)}\bigg((U_1)_\alpha(\alpha) + T_0(\alpha)\phi_\alpha(\alpha) + T_1(\alpha)\bigg) = \frac{2\pi}{L(t)}\bigg((U_1)_\alpha(\alpha) + T_1(\alpha)\bigg) \tag{345}$$

by (332). By (318),

$$T_1(\alpha) = \int_0^\alpha U_1(\eta)d\eta + \int_0^\alpha \phi_\alpha(\eta)U_0(\eta)d\eta - \frac{\alpha}{2\pi}\int_{-\pi}^{\pi} U_1(\eta)d\eta - \frac{\alpha}{2\pi}\int_{-\pi}^{\pi} \phi_\alpha(\eta)U_0(\eta)d\eta \quad (346)$$

$$= \int_0^\alpha U_1(\eta)d\eta - \frac{\alpha}{2\pi}\int_{-\pi}^{\pi} U_1(\eta)d\eta. \quad (347)$$

Using (29), we can write

$$\mathcal{L}(\alpha) = \frac{2\pi}{L(t)}\Big((U_1)_\alpha(\alpha) + \mathcal{M}(U_1)(\alpha)\Big). \quad (348)$$

9.1 The Fourier Modes of $\mathcal{L}$

Due to the complexity of the analytical expression for $\mathcal{L}$, we check that $\mathcal{L}$ is the Hilbert transform of the first derivative of θ up to the ± 1 Fourier modes by confirming that its Fourier multiplier is $|k|$ for $|k| > 1$. Ultimately, we compute $\mathcal{F}(\mathcal{L})(k)$, the kth Fourier coefficient of $\mathcal{L}(\alpha)$, for all $k \in \mathbb{Z}\setminus\{0\}$. Using (30), we obtain that for $k \neq 0$,

$$\mathcal{F}(\mathcal{L})(k) = \frac{2\pi}{L(t)}\Big(\mathcal{F}((U_1)_\alpha)(k) - \frac{i}{k}\mathcal{F}(U_1)(k)\Big). \quad (349)$$

First, we set out to find the expressions for U_1 and $(U_1)_\alpha$. From (327), we obtain

$$U_1(\alpha) = \mathrm{Re}\Big(ie^{i\alpha}e^{i\hat{\theta}(0)}\Big)\mathrm{Re}\mathfrak{L}(\alpha) - \mathrm{Im}\Big(ie^{i\alpha}e^{i\hat{\theta}(0)}\Big)\mathrm{Im}\mathfrak{L}(\alpha), \quad (350)$$

where $\mathfrak{L}(\alpha)$ is given by (308). In the expression for $\mathfrak{L}(\alpha)$, we have inside the first integral

$$\frac{e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}i}{2}(C_{11} - C_{22} - 2iC_{12}) - \frac{e^{-i\hat{\theta}(0)}e^{-i(\alpha-\beta)}i}{2}(C_{11} + C_{22}) \quad (351)$$

$$= \left(\mathrm{Re}\left(\frac{ie^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}\right) + i\mathrm{Im}\left(\frac{ie^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}\right)\right)(C_{11} - C_{22} - 2iC_{12}) \quad (352)$$

$$+ \left(\mathrm{Re}\left(\frac{-ie^{-i\hat{\theta}(0)}e^{-i(\alpha-\beta)}}{2}\right) + i\mathrm{Im}\left(\frac{-ie^{-i\hat{\theta}(0)}e^{-i(\alpha-\beta)}}{2}\right)\right)(C_{11} + C_{22}). \quad (353)$$

Since C_{11}, L_{11}, C_{12}, L_{12}, C_{22}, and L_{22} are all real, we obtain

$$\mathrm{Re}\left(\frac{e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}i}{2}(C_{11} - C_{22} - 2iC_{12}) - \frac{e^{-i\hat{\theta}(0)}e^{-i(\alpha-\beta)}i}{2}(C_{11} + C_{22})\right) \quad (354)$$

$$= \mathrm{Re}\left(\frac{ie^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}\right)(C_{11} - C_{22}) + \mathrm{Im}\left(\frac{ie^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}\right)2C_{12} \quad (355)$$

$$+ \mathrm{Re}\left(\frac{-ie^{-i\hat{\theta}(0)}e^{-i(\alpha-\beta)}}{2}\right)(C_{11} + C_{22}) \quad (356)$$

and

$$\mathrm{Im}\bigg(\frac{e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}i}{2}(C_{11}-C_{22}-2iC_{12})-\frac{e^{-i\hat{\theta}(0)}e^{-i(\alpha-\beta)}i}{2}(C_{11}+C_{22})\bigg) \tag{357}$$

$$=\mathrm{Re}\bigg(\frac{ie^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}\bigg)(-2C_{12})+\mathrm{Im}\bigg(\frac{ie^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}\bigg)(C_{11}-C_{22}) \tag{358}$$

$$+\mathrm{Im}\bigg(\frac{-ie^{-i\hat{\theta}(0)}e^{-i(\alpha-\beta)}}{2}\bigg)(C_{11}+C_{22}). \tag{359}$$

Similarly, we have inside the second integral

$$\frac{e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}(L_{11}-L_{22}-2iL_{12})+\frac{e^{-i\hat{\theta}(0)}e^{-i(\alpha-\beta)}}{2}(L_{11}+L_{22}) \tag{360}$$

$$=\bigg(\mathrm{Re}\bigg(\frac{e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}\bigg)+i\mathrm{Im}\bigg(\frac{e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}\bigg)\bigg)(L_{11}-L_{22}-2iL_{12}) \tag{361}$$

$$+\bigg(\mathrm{Re}\bigg(\frac{e^{-i\hat{\theta}(0)}e^{-i(\alpha-\beta)}}{2}\bigg)+i\mathrm{Im}\bigg(\frac{e^{-i\hat{\theta}(0)}e^{-i(\alpha-\beta)}}{2}\bigg)\bigg)(L_{11}+L_{22}). \tag{362}$$

Since C_{11}, L_{11}, C_{12}, L_{12}, C_{22}, and L_{22} are all real,

$$\mathrm{Re}\bigg(\frac{e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}(L_{11}-L_{22}-2iL_{12})+\frac{e^{-i\hat{\theta}(0)}e^{-i(\alpha-\beta)}}{2}(L_{11}+L_{22})\bigg) \tag{363}$$

$$=\mathrm{Re}\bigg(\frac{e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}\bigg)(L_{11}-L_{22})+\mathrm{Im}\bigg(\frac{e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}\bigg)2L_{12} \tag{364}$$

$$+\mathrm{Re}\bigg(\frac{e^{-i\hat{\theta}(0)}e^{-i(\alpha-\beta)}}{2}\bigg)(L_{11}+L_{22}) \tag{365}$$

and

$$\mathrm{Im}\bigg(\frac{e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}(L_{11}-L_{22}-2iL_{12})+\frac{e^{-i\hat{\theta}(0)}e^{-i(\alpha-\beta)}}{2}(L_{11}+L_{22})\bigg) \tag{366}$$

$$=\mathrm{Re}\bigg(\frac{e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}\bigg)(-2L_{12})+\mathrm{Im}\bigg(\frac{e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}\bigg)(L_{11}-L_{22}) \tag{367}$$

$$+\mathrm{Im}\bigg(\frac{e^{-i\hat{\theta}(0)}e^{-i(\alpha-\beta)}}{2}\bigg)(L_{11}+L_{22}). \tag{368}$$

Therefore,

$$\mathrm{Re}\mathfrak{L}(\alpha) \tag{369}$$

$$= \frac{\gamma}{4\pi}\int_{-\pi}^{\pi} \mathrm{Re}\bigg(\frac{e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}i}{2}(C_{11}-C_{22}-2iC_{12}) - \frac{e^{-i\hat{\theta}(0)}e^{-i(\alpha-\beta)}i}{2}(C_{11}+C_{22})\bigg) \tag{370}$$

$$\cdot\,\phi(\alpha-\beta)d\beta \tag{371}$$

$$+\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} \mathrm{Re}\bigg(\frac{e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}(L_{11}-L_{22}-2iL_{12}) + \frac{e^{-i\hat{\theta}(0)}e^{-i(\alpha-\beta)}}{2}(L_{11}+L_{22})\bigg)d\beta \tag{372}$$

$$= \frac{\gamma}{4\pi}\int_{-\pi}^{\pi} \phi(\alpha-\beta)\bigg(\mathrm{Re}\bigg(\frac{ie^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}\bigg)(C_{11}-C_{22}) + \mathrm{Im}\bigg(\frac{ie^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}\bigg)2C_{12} \tag{373}$$

$$+\mathrm{Re}\bigg(\frac{ie^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}\bigg)(C_{11}+C_{22})\bigg)d\beta \tag{374}$$

$$+\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} \bigg(\mathrm{Re}\bigg(\frac{e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}\bigg)(L_{11}-L_{22}) + \mathrm{Im}\bigg(\frac{e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}\bigg)2L_{12} \tag{375}$$

$$+\mathrm{Re}\bigg(\frac{e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}\bigg)(L_{11}+L_{22})\bigg)d\beta \tag{376}$$

$$= \frac{\gamma}{4\pi}\int_{-\pi}^{\pi} \phi(\alpha-\beta)\bigg(\mathrm{Re}\bigg(\frac{ie^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}\bigg)2C_{11} + \mathrm{Im}\bigg(\frac{ie^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}\bigg)2C_{12}\bigg)d\beta \tag{377}$$

$$+\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} \bigg(\mathrm{Re}\bigg(\frac{e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}\bigg)2L_{11} + \mathrm{Im}\bigg(\frac{e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}\bigg)2L_{12}\bigg)d\beta \tag{378}$$

and

$$\mathrm{Im}\mathfrak{L}(\alpha) \tag{379}$$
$$=\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathrm{Im}\bigg(\frac{e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}i}{2}(C_{11}-C_{22}-2iC_{12})-\frac{e^{-i\hat{\theta}(0)}e^{-i(\alpha-\beta)}i}{2}(C_{11}+C_{22})\bigg) \tag{380}$$
$$\cdot\,\phi(\alpha-\beta)d\beta \tag{381}$$
$$+\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathrm{Im}\bigg(\frac{e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}(L_{11}-L_{22}-2iL_{12})+\frac{e^{-i\hat{\theta}(0)}e^{-i(\alpha-\beta)}}{2}(L_{11}+L_{22})\bigg)d\beta \tag{382}$$
$$=\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\phi(\alpha-\beta)\bigg(\mathrm{Re}\bigg(\frac{ie^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}\bigg)(-2C_{12}) \tag{383}$$
$$+\mathrm{Im}\bigg(\frac{ie^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}\bigg)(C_{11}-C_{22})+\mathrm{Im}\bigg(\frac{-ie^{-i\hat{\theta}(0)}e^{-i(\alpha-\beta)}}{2}\bigg)(C_{11}+C_{22})\bigg)d\beta \tag{384}$$
$$+\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\bigg(\mathrm{Re}\bigg(\frac{e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}\bigg)(-2L_{12})+\mathrm{Im}\bigg(\frac{e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}\bigg)(L_{11}-L_{22}) \tag{385}$$
$$+\mathrm{Im}\bigg(\frac{e^{-i\hat{\theta}(0)}e^{-i(\alpha-\beta)}}{2}\bigg)(L_{11}+L_{22})\bigg)d\beta \tag{386}$$
$$=\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\phi(\alpha-\beta)\bigg(\mathrm{Re}\bigg(\frac{ie^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}\bigg)(-2C_{12})+\mathrm{Im}\bigg(\frac{ie^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}\bigg)(-2C_{22})\bigg)d\beta \tag{387}$$
$$+\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\bigg(\mathrm{Re}\bigg(\frac{e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}\bigg)(-2L_{12})+\mathrm{Im}\bigg(\frac{e^{i\hat{\theta}(0)}e^{i(\alpha-\beta)}}{2}\bigg)(-2L_{22})\bigg)d\beta. \tag{388}$$

Plugging (369) and (379) back into (350) and then simplifying, we obtain

$$U_1(\alpha) \tag{389}$$
$$=\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\bigg(\tag{390}$$
$$\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))ds\cdot\frac{-(-i+(i+\beta)e^{i\beta})(-1-2e^{i\beta}+e^{2i\beta})}{4(-1+e^{i\beta})^2} \tag{391}$$
$$+\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))ds\cdot\frac{e^{-i\beta}(-1+2e^{i\beta}+e^{2i\beta})(\beta+i(-1+e^{i\beta}))}{4(-1+e^{i\beta})^2} \tag{392}$$
$$+\int_0^1 e^{-i\beta s}(-1+s)\phi(\alpha+\beta(-1+s))ds\cdot\frac{-(-1+e^{i\beta})\beta(-1-2e^{i\beta}+e^{2i\beta})}{4(-1+e^{i\beta})^2} \tag{393}$$
$$+\int_0^1 e^{i\beta s}(-1+s)\phi(\alpha+\beta(-1+s))ds\cdot\frac{-e^{-i\beta}(-1+e^{i\beta})\beta(-1+2e^{i\beta}+e^{2i\beta})}{4(-1+e^{i\beta})^2} \tag{394}$$
$$+\int_0^1 e^{-i\beta s}(-1+s)\phi'(\alpha+\beta(-1+s))ds\cdot\frac{-(-1+e^{i\beta})i\beta(-1-2e^{i\beta}+e^{2i\beta})}{4(-1+e^{i\beta})^2} \tag{395}$$
$$+\int_0^1 e^{i\beta s}(-1+s)\phi'(\alpha+\beta(-1+s))ds\cdot\frac{e^{-i\beta}(-1+e^{i\beta})i\beta(-1+2e^{i\beta}+e^{2i\beta})}{4(-1+e^{i\beta})^2} \tag{396}$$
$$+\phi(\alpha-\beta)\cdot\frac{e^{-i\beta}(-1+e^{i\beta})(-i)(1+e^{i\beta}+e^{2i\beta}+e^{3i\beta})}{4(-1+e^{i\beta})^2}\bigg)d\beta. \tag{397}$$

Differentiating (389) with respect to α, we obtain

$$(U_1)_\alpha(\alpha) \tag{398}$$

$$= \frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\Bigg(\tag{399}$$

$$\int_0^1 e^{-i\beta s}\phi'(\alpha+\beta(-1+s))ds \cdot \frac{-(-i+(i+\beta)e^{i\beta})(-1-2e^{i\beta}+e^{2i\beta})}{4(-1+e^{i\beta})^2} \tag{400}$$

$$+\int_0^1 e^{i\beta s}\phi'(\alpha+\beta(-1+s))ds \cdot \frac{e^{-i\beta}(-1+2e^{i\beta}+e^{2i\beta})(\beta+i(-1+e^{i\beta}))}{4(-1+e^{i\beta})^2} \tag{401}$$

$$+\int_0^1 e^{-i\beta s}(-1+s)\phi'(\alpha+\beta(-1+s))ds \cdot \frac{-(-1+e^{i\beta})\beta(-1-2e^{i\beta}+e^{2i\beta})}{4(-1+e^{i\beta})^2} \tag{402}$$

$$+\int_0^1 e^{i\beta s}(-1+s)\phi'(\alpha+\beta(-1+s))ds \cdot \frac{-e^{-i\beta}(-1+e^{i\beta})\beta(-1+2e^{i\beta}+e^{2i\beta})}{4(-1+e^{i\beta})^2} \tag{403}$$

$$+\int_0^1 e^{-i\beta s}(-1+s)\phi''(\alpha+\beta(-1+s))ds \cdot \frac{-(-1+e^{i\beta})i\beta(-1-2e^{i\beta}+e^{2i\beta})}{4(-1+e^{i\beta})^2} \tag{404}$$

$$+\int_0^1 e^{i\beta s}(-1+s)\phi''(\alpha+\beta(-1+s))ds \tag{405}$$

$$\cdot \frac{-e^{-i\beta}(-1+e^{i\beta})i(-\beta)(-1+2e^{i\beta}+e^{2i\beta})}{4(-1+e^{i\beta})^2} \tag{406}$$

$$+\phi'(\alpha-\beta)\cdot\frac{-e^{-i\beta}(-1+e^{i\beta})i(1+e^{i\beta}+e^{2i\beta}+e^{3i\beta})}{4(-1+e^{i\beta})^2}\Bigg)d\beta. \tag{407}$$

Now, taking the Fourier modes of U_1 and $(U_1)_\alpha$ and plugging them into (349), we obtain that for $k \notin \{0, \pm 1\}$,

$$\mathcal{F}(\mathcal{L})(k) = \frac{2\pi}{L(t)}\frac{\gamma}{4\pi}\mathcal{F}(\phi)(k)\Bigg(\tag{408}$$

$$\int_{-\pi}^{\pi} \frac{(i-(i+\beta)e^{i\beta})(-1-2e^{i\beta}+e^{2i\beta})}{4(-1+e^{i\beta})^2}\frac{e^{-i\beta}-e^{-i\beta k}}{\beta}d\beta\left(\frac{k}{k-1}+\frac{1}{k(1-k)}\right) \tag{409}$$

$$+\int_{-\pi}^{\pi} \frac{e^{-i\beta}(-1+2e^{i\beta}+e^{2i\beta})(\beta+i(-1+e^{i\beta}))}{4(-1+e^{i\beta})^2}\frac{e^{i\beta}-e^{-i\beta k}}{\beta}d\beta\left(\frac{k}{1+k}-\frac{1}{k(1+k)}\right) \tag{410}$$

$$+\int_{-\pi}^{\pi} \frac{-i(-1-2e^{i\beta}+e^{2i\beta})}{4(-1+e^{i\beta})}\frac{e^{-i\beta(1+k)}(e^{i\beta k}-e^{i\beta})}{\beta}d\beta\left(\frac{-k^2}{(-1+k)^2}+\frac{1}{(-1+k)^2}\right) \tag{411}$$

$$+\int_{-\pi}^{\pi} \frac{-i(-1-2e^{i\beta}+e^{2i\beta})e^{-i\beta k}}{4(-1+e^{i\beta})}d\beta\left(\frac{ik^2}{-1+k}-\frac{i}{-1+k}\right) \tag{412}$$

$$+\int_{-\pi}^{\pi} \frac{e^{-i\beta}i(-1+2e^{i\beta}+e^{2i\beta})}{4(-1+e^{i\beta})}\frac{e^{-i\beta k}(-1+e^{i\beta(1+k)})}{\beta}d\beta\left(\frac{-k^2}{(1+k)^2}+\frac{1}{(1+k)^2}\right) \tag{413}$$

$$+\int_{-\pi}^{\pi} \frac{e^{-i\beta}(-1+2e^{i\beta}+e^{2i\beta})e^{-i\beta k}}{4(-1+e^{i\beta})}d\beta\left(\frac{-k^2}{1+k}+\frac{1}{1+k}\right) \tag{414}$$

$$+\int_{-\pi}^{\pi} \frac{1+2e^{i\beta}-e^{2i\beta}}{4(-1+e^{i\beta})}\frac{e^{-i\beta(1+k)}(e^{i\beta k}-e^{i\beta})}{\beta}d\beta\left(\frac{ik}{(-1+k)^2}-\frac{i}{k(-1+k)^2}\right) \tag{415}$$

$$+\int_{-\pi}^{\pi} \frac{(1+2e^{i\beta}-e^{2i\beta})e^{-i\beta k}}{4(-1+e^{i\beta})}d\beta\left(\frac{k}{-1+k}-\frac{1}{k(-1+k)}\right) \tag{416}$$

$$+\int_{-\pi}^{\pi} \frac{-e^{-i\beta}(-1+2e^{i\beta}+e^{2i\beta})}{4(-1+e^{i\beta})}\frac{e^{-i\beta k}(-1+e^{i\beta(1+k)})}{\beta}d\beta\left(\frac{ik}{(1+k)^2}-\frac{i}{k(1+k)^2}\right) \tag{417}$$

$$+\int_{-\pi}^{\pi} \frac{-e^{-i\beta}(-1+2e^{i\beta}+e^{2i\beta})e^{-i\beta k}}{4(-1+e^{i\beta})}d\beta\left(\frac{k}{1+k}-\frac{1}{k(1+k)}\right) \tag{418}$$

$$+\int_{-\pi}^{\pi} \frac{-e^{-i\beta}i(1+e^{i\beta}+e^{2i\beta}+e^{3i\beta})e^{-ik\beta}}{4(-1+e^{i\beta})}d\beta\left(ik-\frac{i}{k}\right)\Bigg). \tag{419}$$

For $k \notin \{0, \pm 1\}$, we define

$$J_1(k) = \tag{420}$$

$$\int_{-\pi}^{\pi} \frac{(i-(i+\beta)e^{i\beta})(-1-2e^{i\beta}+e^{2i\beta})}{4(-1+e^{i\beta})^2} \frac{e^{-i\beta}-e^{-i\beta k}}{\beta} d\beta \left(\frac{k}{k-1} + \frac{1}{k(1-k)} \right) \tag{421}$$

$$+ \int_{-\pi}^{\pi} \frac{e^{-i\beta}(-1+2e^{i\beta}+e^{2i\beta})(\beta+i(-1+e^{i\beta}))}{4(-1+e^{i\beta})^2} \frac{e^{i\beta}-e^{-i\beta k}}{\beta} d\beta \left(\frac{k}{1+k} - \frac{1}{k(1+k)} \right) \tag{422}$$

$$+ \int_{-\pi}^{\pi} \frac{-i(-1-2e^{i\beta}+e^{2i\beta})}{4(-1+e^{i\beta})} \frac{e^{-i\beta(1+k)}(e^{i\beta k}-e^{i\beta})}{\beta} d\beta \left(\frac{-k^2}{(-1+k)^2} + \frac{1}{(-1+k)^2} \right) \tag{423}$$

$$+ \int_{-\pi}^{\pi} \frac{-i(-1-2e^{i\beta}+e^{2i\beta})e^{-i\beta k}}{4(-1+e^{i\beta})} d\beta \left(\frac{ik^2}{-1+k} - \frac{i}{-1+k} \right) \tag{424}$$

$$+ \int_{-\pi}^{\pi} \frac{e^{-i\beta}i(-1+2e^{i\beta}+e^{2i\beta})}{4(-1+e^{i\beta})} \frac{e^{-i\beta k}(-1+e^{i\beta(1+k)})}{\beta} d\beta \left(\frac{-k^2}{(1+k)^2} + \frac{1}{(1+k)^2} \right) \tag{425}$$

$$+ \int_{-\pi}^{\pi} \frac{e^{-i\beta}(-1+2e^{i\beta}+e^{2i\beta})e^{-i\beta k}}{4(-1+e^{i\beta})} d\beta \left(\frac{-k^2}{1+k} + \frac{1}{1+k} \right) \tag{426}$$

$$+ \int_{-\pi}^{\pi} \frac{1+2e^{i\beta}-e^{2i\beta}}{4(-1+e^{i\beta})} \frac{e^{-i\beta(1+k)}(e^{i\beta k}-e^{i\beta})}{\beta} d\beta \left(\frac{ik}{(-1+k)^2} - \frac{i}{k(-1+k)^2} \right) \tag{427}$$

$$+ \int_{-\pi}^{\pi} \frac{(1+2e^{i\beta}-e^{2i\beta})e^{-i\beta k}}{4(-1+e^{i\beta})} d\beta \left(\frac{k}{-1+k} - \frac{1}{k(-1+k)} \right) \tag{428}$$

$$+ \int_{-\pi}^{\pi} \frac{-e^{-i\beta}(-1+2e^{i\beta}+e^{2i\beta})}{4(-1+e^{i\beta})} \frac{e^{-i\beta k}(-1+e^{i\beta(1+k)})}{\beta} d\beta \left(\frac{ik}{(1+k)^2} - \frac{i}{k(1+k)^2} \right) \tag{429}$$

$$+ \int_{-\pi}^{\pi} \frac{-e^{-i\beta}(-1+2e^{i\beta}+e^{2i\beta})e^{-i\beta k}}{4(-1+e^{i\beta})} d\beta \left(\frac{k}{1+k} - \frac{1}{k(1+k)} \right) \tag{430}$$

and

$$J_2(k) = \int_{-\pi}^{\pi} \frac{-e^{-i\beta}i(1+e^{i\beta}+e^{2i\beta}+e^{3i\beta})e^{-ik\beta}}{4(-1+e^{i\beta})} d\beta \left(ik - \frac{i}{k} \right). \tag{431}$$

Then for $|k| > 1$ we can write the kth Fourier mode of $\mathcal{L}$ as

$$\mathcal{F}(\mathcal{L})(k) = \frac{2\pi}{L(t)} \frac{\gamma}{4\pi} \mathcal{F}(\phi)(k) \Big(J_1(k) + J_2(k) \Big). \tag{432}$$

Since (348) is real, for $k \in \mathbb{Z}^+$,

$$\mathcal{F}(\mathcal{L})(-k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathcal{L}(\alpha) e^{ik\alpha} d\alpha = \overline{\frac{1}{2\pi} \int_{-\pi}^{\pi} \mathcal{L}(\alpha) e^{-ik\alpha} d\alpha} = \overline{\mathcal{F}(\mathcal{L})(k)}. \tag{433}$$

Hence, it suffices to compute $\mathcal{F}(\mathcal{L})(k)$ only for $k > 1$.

9.1.1 Computing $J_2(k)$

To compute $J_2(k)$, it suffices to compute the integral

$$ik \int_{-\pi}^{\pi} \frac{-e^{-i\beta}i(1+e^{i\beta}+e^{2i\beta}+e^{3i\beta})e^{-ik\beta}}{4(-1+e^{i\beta})} d\beta. \tag{434}$$

Using that

$$\frac{1}{-1+re^{i\beta}} = -\frac{1}{1-re^{i\beta}} = -\sum_{n=0}^{\infty}(re^{i\beta})^n, \tag{435}$$

we obtain

$$\begin{aligned}
&ik\int_{-\pi}^{\pi}\frac{-e^{-i\beta}i(1+e^{i\beta}+e^{2i\beta}+e^{3i\beta})e^{-ik\beta}}{4(-1+e^{i\beta})}d\beta && (436)\\
=&k\cdot\mathrm{pv}\int_{-\pi}^{\pi}\frac{e^{-i\beta(k+1)}(1+e^{i\beta}+e^{2i\beta}+e^{3i\beta})}{4(-1+e^{i\beta})}d\beta && (437)\\
=&k\lim_{\epsilon\to0^+}\int_{\substack{[-\pi,\pi]\\ \setminus(-\epsilon,\epsilon)}}\frac{e^{-i\beta(k+1)}(1+e^{i\beta}+e^{2i\beta}+e^{3i\beta})}{4(-1+e^{i\beta})}d\beta && (438)\\
=&\frac{k}{4}\lim_{\epsilon\to0^+}\int_{\substack{[-\pi,\pi]\\ \setminus(-\epsilon,\epsilon)}}\lim_{r\to1^-}-e^{-i\beta(k+1)}(1+e^{i\beta}+e^{2i\beta}+e^{3i\beta})\sum_{n=0}^{\infty}(re^{i\beta})^n d\beta && (439)\\
=&\frac{k}{4}\lim_{\epsilon\to0^+}\lim_{r\to1^-}\sum_{n=0}^{\infty}\int_{\substack{[-\pi,\pi]\\ \setminus(-\epsilon,\epsilon)}}-e^{-i\beta(k+1)}(1+e^{i\beta}+e^{2i\beta}+e^{3i\beta})(re^{i\beta})^n d\beta && (440)\\
=&-\frac{k}{4}\lim_{\epsilon\to0^+}\lim_{r\to1^-}\sum_{n=0}^{\infty}r^n\int_{\substack{[-\pi,\pi]\\ \setminus(-\epsilon,\epsilon)}}e^{-i\beta(k+1)}(1+e^{i\beta}+e^{2i\beta}+e^{3i\beta})e^{i\beta n}d\beta. && (441)
\end{aligned}$$

To compute the outer integral, we note that

$$\begin{aligned}
&\int_{-\pi}^{\pi}e^{-i\beta(k+1)}(1+e^{i\beta}+e^{2i\beta}+e^{3i\beta})e^{i\beta n}d\beta && (442)\\
=&\int_{-\pi}^{\pi}e^{-i\beta(k+1-n)}(1+e^{i\beta}+e^{2i\beta}+e^{3i\beta})d\beta && (443)\\
=&\begin{cases}0 & \text{if } n\notin\{k+1,k,k-1,k-2\},\\ 2\pi & \text{otherwise.}\end{cases} && (444)
\end{aligned}$$

Then

$$\begin{aligned}
&-\frac{k}{4}\lim_{\epsilon\to0^+}\lim_{r\to1^-}\sum_{n=0}^{\infty}r^n\int_{-\pi}^{\pi}e^{-i\beta(k+1)}(1+e^{i\beta}+e^{2i\beta}+e^{3i\beta})e^{i\beta n}d\beta && (445)\\
=&-\frac{k}{4}\lim_{\epsilon\to0^+}\lim_{r\to1^-}\sum_{n=0}^{\infty}r^n 2\pi 1_{\{k-2,k-1,k,k+1\}}(n) && (446)\\
=&\begin{cases}0 & \text{if } k<-1,\\ -2\pi k & \text{if } k>1.\end{cases} && (447)
\end{aligned}$$

Moreover,

$$-\frac{k}{4}\lim_{\epsilon\to0^+}\lim_{r\to1^-}\sum_{n=0}^{\infty}r^n\int_{\epsilon}^{-\epsilon}e^{-i\beta(k+1)}(1+e^{i\beta}+e^{2i\beta}+e^{3i\beta})e^{i\beta n}d\beta = k\pi. \tag{448}$$

Adding these two integrals together, we obtain

$$ik\int_{-\pi}^{\pi}\frac{-e^{-i\beta}i(1+e^{i\beta}+e^{2i\beta}+e^{3i\beta})e^{-ik\beta}}{4(-1+e^{i\beta})}d\beta=\begin{cases}\pi k & \text{if } k<-1,\\ -\pi k & \text{if } k>1.\end{cases} \tag{449}$$

Then

$$-\frac{i}{k}\cdot \mathrm{pv}\int_{-\pi}^{\pi}\frac{e^{-i\beta(k+1)}(1+e^{i\beta}+e^{2i\beta}+e^{3i\beta})}{4(-1+e^{i\beta})}d\beta=\begin{cases}-\frac{\pi}{k} & \text{if } k<-1,\\ \frac{\pi}{k} & \text{if } k>1.\end{cases} \tag{450}$$

Adding these two integrals together, we obtain

$$J_2(k)=\begin{cases}\pi\left(k-\frac{1}{k}\right) & \text{if } k<-1,\\ -\pi\left(k-\frac{1}{k}\right) & \text{if } k>1.\end{cases} \tag{451}$$

9.1.2 Computing $J_1(k)$

In view of (433), we assume that $k>1$. In our calculations, we adopt the notational convention that any summation $\sum$ in which the upper bound is strictly less than the lower bound is defined to be 0. For example, if $k=2$, then (503) vanishes. To begin, we simplify the first two integrals in (420). The first integral can be written as

$$\int_{-\pi}^{\pi}\frac{(i-(i+\beta)e^{i\beta})(-1-2e^{i\beta}+e^{2i\beta})}{4(-1+e^{i\beta})^2}\frac{e^{-i\beta}-e^{-i\beta k}}{\beta}d\beta \tag{452}$$

$$=\int_{-\pi}^{\pi}\frac{-i(-1-2e^{i\beta}+e^{2i\beta})}{4(-1+e^{i\beta})}\frac{e^{-i\beta}-e^{-i\beta k}}{\beta}d\beta \tag{453}$$

$$+\int_{-\pi}^{\pi}\frac{-\beta e^{i\beta}(-1-2e^{i\beta}+e^{2i\beta})}{4(-1+e^{i\beta})^2}\frac{e^{-i\beta}-e^{-i\beta k}}{\beta}d\beta \tag{454}$$

while the second integral can be written as

$$\int_{-\pi}^{\pi}\frac{e^{-i\beta}(-1+2e^{i\beta}+e^{2i\beta})(\beta+i(-1+e^{i\beta}))}{4(-1+e^{i\beta})^2}\frac{e^{i\beta}-e^{-i\beta k}}{\beta}d\beta \tag{455}$$

$$=\int_{-\pi}^{\pi}\frac{e^{-i\beta}(-1+2e^{i\beta}+e^{2i\beta})\beta}{4(-1+e^{i\beta})^2}\frac{e^{i\beta}-e^{-i\beta k}}{\beta}d\beta \tag{456}$$

$$+\int_{-\pi}^{\pi}\frac{(-1+2e^{i\beta}+e^{2i\beta})i}{4(-1+e^{i\beta})}\frac{1-e^{-i\beta(k+1)}}{\beta}d\beta. \tag{457}$$

For ease of notation, let us define

$$g_1(k) = \int_{-\pi}^{\pi} \frac{-i(-1-2e^{i\beta}+e^{2i\beta})}{4(-1+e^{i\beta})} \frac{e^{-i\beta}-e^{-i\beta k}}{\beta} d\beta, \tag{458}$$
$$g_2(k) = \int_{-\pi}^{\pi} \frac{-\beta e^{i\beta}(-1-2e^{i\beta}+e^{2i\beta})}{4(-1+e^{i\beta})^2} \frac{e^{-i\beta}-e^{-i\beta k}}{\beta} d\beta, \tag{459}$$
$$g_3(k) = \int_{-\pi}^{\pi} \frac{e^{-i\beta}(-1+2e^{i\beta}+e^{2i\beta})\beta}{4(-1+e^{i\beta})^2} \frac{e^{i\beta}-e^{-i\beta k}}{\beta} d\beta, \tag{460}$$
$$g_4(k) = \int_{-\pi}^{\pi} \frac{(-1+2e^{i\beta}+e^{2i\beta})i}{4(-1+e^{i\beta})} \frac{1-e^{-i\beta(k+1)}}{\beta} d\beta, \tag{461}$$
$$g_5(k) = \int_{-\pi}^{\pi} \frac{-i(-1-2e^{i\beta}+e^{2i\beta})e^{-i\beta k}}{4(-1+e^{i\beta})} d\beta, \tag{462}$$
$$g_6(k) = \int_{-\pi}^{\pi} \frac{e^{-i\beta}(-1+2e^{i\beta}+e^{2i\beta})e^{-i\beta k}}{4(-1+e^{i\beta})} d\beta, \tag{463}$$
$$g_7(k) = \int_{-\pi}^{\pi} \frac{(1+2e^{i\beta}-e^{2i\beta})e^{-i\beta k}}{4(-1+e^{i\beta})} d\beta, \tag{464}$$
$$g_8(k) = \int_{-\pi}^{\pi} \frac{-e^{-i\beta}(-1+2e^{i\beta}+e^{2i\beta})e^{-i\beta k}}{4(-1+e^{i\beta})} d\beta. \tag{465}$$

Then we can rewrite

$$J_1(k) = (g_1(k)+g_2(k))\left(\frac{k}{k-1}+\frac{1}{k(1-k)}\right) \tag{466}$$
$$+(g_3(k)+g_4(k))\left(\frac{k}{1+k}-\frac{1}{k(1+k)}\right) \tag{467}$$
$$+g_1(k)\left(\frac{-k^2}{(-1+k)^2}+\frac{1}{(-1+k)^2}\right) \tag{468}$$
$$+g_5(k)\left(\frac{ik^2}{-1+k}-\frac{i}{-1+k}\right) \tag{469}$$
$$+g_4(k)\left(\frac{-k^2}{(1+k)^2}+\frac{1}{(1+k)^2}\right) \tag{470}$$
$$+g_6(k)\left(\frac{-k^2}{1+k}+\frac{1}{1+k}\right) \tag{471}$$
$$+(-ig_1(k))\left(\frac{ik}{(-1+k)^2}-\frac{i}{k(-1+k)^2}\right) \tag{472}$$
$$+g_7(k)\left(\frac{k}{-1+k}-\frac{1}{k(-1+k)}\right) \tag{473}$$
$$+(ig_4(k))\left(\frac{ik}{(1+k)^2}-\frac{i}{k(1+k)^2}\right) \tag{474}$$
$$+g_8(k)\left(\frac{k}{1+k}-\frac{1}{k(1+k)}\right). \tag{475}$$

Simplifying this expression, we obtain

$$J_1(k) = \frac{k+1}{k} g_2(k) + \frac{k-1}{k} g_3(k) \tag{476}$$

$$+ i(k+1) g_5(k) + (1-k) g_6(k) + \frac{k+1}{k} g_7(k) + \frac{k-1}{k} g_8(k). \tag{477}$$

Let us first compute $g_2(k)$. Using that

$$1 - e^{-i\beta(k-1)} = i\beta(k-1) \int_0^1 e^{-i\beta(k-1)s} ds \tag{478}$$

$$\frac{1}{(-1+re^{i\beta})^2} = \sum_{n=0}^{\infty} (1+n)(re^{i\beta})^n, \tag{479}$$

we obtain

$$g_2(k) = \mathrm{pv} \int_{-\pi}^{\pi} \frac{1+2e^{i\beta}-e^{2i\beta}}{4(-1+e^{i\beta})^2} (1-e^{-i\beta(k-1)}) d\beta \tag{480}$$

$$= \lim_{\epsilon\to 0^+} \int_{\substack{[-\pi,\pi] \\ \setminus(-\epsilon,\epsilon)}} \frac{1+2e^{i\beta}-e^{2i\beta}}{4(-1+e^{i\beta})^2} i\beta(k-1) \int_0^1 e^{-i\beta(k-1)s} ds d\beta \tag{481}$$

$$= \frac{i(k-1)}{4} \tag{482}$$

$$\cdot \lim_{\epsilon\to 0^+} \lim_{r\to 1^-} \int_{\substack{[-\pi,\pi] \\ \setminus(-\epsilon,\epsilon)}} \int_0^1 (1+2e^{i\beta}-e^{2i\beta}) \beta e^{-i\beta(k-1)s} \sum_{n=0}^{\infty} (1+n)(re^{i\beta})^n ds d\beta \tag{483}$$

$$= \frac{i(k-1)}{4} \tag{484}$$

$$\cdot \lim_{\epsilon\to 0^+} \lim_{r\to 1^-} \sum_{n=0}^{\infty} (1+n) r^n \int_0^1 \int_{\substack{[-\pi,\pi] \\ \setminus(-\epsilon,\epsilon)}} (1+2e^{i\beta}-e^{2i\beta}) \beta e^{-i\beta(k-1)s} e^{i\beta n} d\beta ds. \tag{485}$$

To simplify this expression, we first compute the integral

$$\int_{-\pi}^{\pi} (1+2e^{i\beta}-e^{2i\beta}) \beta e^{-i\beta(k-1)s} e^{i\beta n} d\beta \tag{486}$$

$$= \frac{e^{-i\pi(n+s(1-k))}}{(n+s(1-k))^2} \Big(-1 - i\pi(n+s(1-k)) \Big) \tag{487}$$

$$+ \frac{e^{i\pi(n+s(1-k))}}{(n+s(1-k))^2} \Big(1 - i\pi(n+s(1-k)) \Big) \tag{488}$$

$$+ \frac{2e^{-i\pi(1+n+s(1-k))}}{(1+n+s(1-k))^2} \Big(-1 - i\pi(1+n+s(1-k)) \Big) \tag{489}$$

$$+ \frac{2e^{i\pi(1+n+s(1-k))}}{(1+n+s(1-k))^2} \Big(1 - i\pi(1+n+s(1-k)) \Big) \tag{490}$$

$$+ \frac{e^{-i\pi(2+n+s(1-k))}}{(2+n+s(1-k))^2} \Big(1 + i\pi(2+n+s(1-k)) \Big) \tag{491}$$

$$+ \frac{e^{i\pi(2+n+s(1-k))}}{(2+n+s(1-k))^2} \Big(-1 + i\pi(2+n+s(1-k)) \Big). \tag{492}$$

For $t \in \{1, 2\}$, we note that

$$\int_0^1 \frac{e^{i\pi(n+s(1-k))}}{(n+s(1-k))^2}(1 - i\pi(n + s(1-k)))ds = \frac{1}{k-1}\int_{n-(k-1)}^{n} \frac{e^{i\pi s}(1 - i\pi s)}{s^2}ds \tag{493}$$

$$\int_0^1 \frac{e^{i\pi(t+n+s(1-k))}}{(t+n+s(1-k))^2}(1 - i\pi(t + n + s(1-k)))ds = \frac{1}{k-1}\int_{t+n-(k-1)}^{t+n} \frac{e^{i\pi s}(1 - i\pi s)}{s^2}ds. \tag{494}$$

Using these identities, we obtain

$$\frac{i(k-1)}{4}\sum_{n=0}^{\infty}(1+n)r^n \int_0^1 \int_{-\pi}^{\pi}(1 + 2e^{i\beta} - e^{2i\beta})\beta e^{-i\beta(k-1)s}e^{i\beta n}d\beta ds \tag{495}$$

$$=\frac{i(k-1)}{4}\cdot\Bigg[\Bigg(\sum_{n=0}^{k-1}+\sum_{n=k}^{\infty}\Bigg)(1+n)r^n \int_0^1 \int_{-\pi}^{\pi} \beta e^{-i\beta(k-1)s}e^{i\beta n}d\beta ds \tag{496}$$

$$+\Bigg(\sum_{n=0}^{k-2}+\sum_{n=k-1}^{\infty}\Bigg)(1+n)r^n \int_0^1 \int_{-\pi}^{\pi} 2e^{i\beta}\beta e^{-i\beta(k-1)s}e^{i\beta n}d\beta ds \tag{497}$$

$$+\Bigg(\sum_{n=0}^{k-3}+\sum_{n=k-2}^{\infty}\Bigg)(1+n)r^n \int_0^1 \int_{-\pi}^{\pi} -e^{2i\beta}\beta e^{-i\beta(k-1)s}e^{i\beta n}d\beta ds\Bigg] \tag{498}$$

$$=\frac{i(k-1)}{4}\cdot\Bigg[\sum_{n=0}^{k-1}(1+n)r^n \int_{-\pi}^{\pi} \beta e^{i\beta n}\int_0^1 e^{-i\beta(k-1)s}dsd\beta \tag{499}$$

$$+\sum_{n=k}^{\infty}(1+n)r^n \int_0^1 \int_{-\pi}^{\pi} \beta e^{-i\beta(k-1)s}e^{i\beta n}d\beta ds \tag{500}$$

$$+\sum_{n=0}^{k-2}(1+n)r^n \int_{-\pi}^{\pi} 2e^{i\beta}\beta e^{i\beta n}\int_0^1 e^{-i\beta(k-1)s}dsd\beta \tag{501}$$

$$+\sum_{n=k-1}^{\infty}(1+n)r^n \int_0^1 \int_{-\pi}^{\pi} 2e^{i\beta}\beta e^{-i\beta(k-1)s}e^{i\beta n}d\beta ds \tag{502}$$

$$+\sum_{n=0}^{k-3}(1+n)r^n \int_{-\pi}^{\pi} -e^{2i\beta}\beta e^{i\beta n}\int_0^1 e^{-i\beta(k-1)s}dsd\beta \tag{503}$$

$$+\sum_{n=k-2}^{\infty}(1+n)r^n \int_0^1 \int_{-\pi}^{\pi} -e^{2i\beta}\beta e^{-i\beta(k-1)s}e^{i\beta n}d\beta ds\Bigg]. \tag{504}$$

After further simplification, we obtain

$$\frac{i(k-1)}{4}\sum_{n=0}^{\infty}(1+n)r^n\int_0^1\int_{-\pi}^{\pi}(1+2e^{i\beta}-e^{2i\beta})\beta e^{-i\beta(k-1)s}e^{i\beta n}d\beta ds \tag{505}$$

$$=\frac{1}{4}\Bigg(\sum_{n=0}^{k-1}(1+n)r^n\int_{-\pi}^{\pi}e^{i\beta n}(1-e^{-i\beta(k-1)})d\beta \tag{506}$$

$$+\sum_{n=0}^{k-2}(1+n)r^n\int_{-\pi}^{\pi}2e^{i\beta}e^{i\beta n}(1-e^{-i\beta(k-1)})d\beta \tag{507}$$

$$+\sum_{n=0}^{k-3}(1+n)r^n\int_{-\pi}^{\pi}-e^{2i\beta}e^{i\beta n}(1-e^{-i\beta(k-1)})d\beta\Bigg) \tag{508}$$

$$+\frac{i(k-1)}{4}\cdot\Bigg[\sum_{n=k}^{\infty}(1+n)r^n\Bigg(\frac{1}{k-1}\int_{n-(k-1)}^{n}\frac{e^{i\pi s}(1-i\pi s)}{s^2}ds \tag{509}$$

$$-\frac{1}{k-1}\int_{n-(k-1)}^{n}\frac{e^{-i\pi s}(1+i\pi s)}{s^2}ds\Bigg) \tag{510}$$

$$+\sum_{n=k-1}^{\infty}(1+n)r^n\Bigg(\frac{2}{k-1}\int_{1+n-(k-1)}^{1+n}\frac{e^{i\pi s}(1-i\pi s)}{s^2}ds \tag{511}$$

$$-\frac{2}{k-1}\int_{1+n-(k-1)}^{1+n}\frac{e^{-i\pi s}(1+i\pi s)}{s^2}ds\Bigg) \tag{512}$$

$$+\sum_{n=k-2}^{\infty}(1+n)r^n\Bigg(-\frac{1}{k-1}\int_{2+n-(k-1)}^{2+n}\frac{e^{i\pi s}(1-i\pi s)}{s^2}ds \tag{513}$$

$$+\frac{1}{k-1}\int_{2+n-(k-1)}^{2+n}\frac{e^{-i\pi s}(1+i\pi s)}{s^2}ds\Bigg)\Bigg] \tag{514}$$

$$=\frac{1}{4}\Bigg(\sum_{n=0}^{k-1}(1+n)r^n\int_{-\pi}^{\pi}e^{i\beta n}(1-e^{-i\beta(k-1)})d\beta \tag{515}$$

$$+\sum_{n=0}^{k-2}(1+n)r^n\int_{-\pi}^{\pi}2e^{i\beta}e^{i\beta n}(1-e^{-i\beta(k-1)})d\beta \tag{516}$$

$$+\sum_{n=0}^{k-3}(1+n)r^n\int_{-\pi}^{\pi}-e^{2i\beta}e^{i\beta n}(1-e^{-i\beta(k-1)})d\beta\Bigg) \tag{517}$$

$$+\frac{1}{4}\Bigg(\sum_{n=k}^{\infty}(1+n)r^n\int_{-\infty}^{\infty}i\bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\bigg)1_{[n-(k-1),n]}(s)ds \tag{518}$$

$$+\sum_{n=k-1}^{\infty}(1+n)r^n\cdot 2\int_{-\infty}^{\infty}i\bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\bigg)1_{[1+n-(k-1),1+n]}(s)ds \tag{519}$$

$$+\sum_{n=k-2}^{\infty}(1+n)r^n(-1) \tag{520}$$

$$\cdot\int_{-\infty}^{\infty}i\bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\bigg)1_{[2+n-(k-1),2+n]}(s)ds\Bigg). \tag{521}$$

We will further simplify the terms in (518), (519), and (521). The term in (518) becomes

$$\int_{-\infty}^{\infty} i\bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\bigg)\sum_{n=k}^{\infty}(1+n)r^n 1_{[n-(k-1),n]}(s)ds \tag{522}$$

$$=\bigg(\int_{-\infty}^{1}+\int_{1}^{2}+\cdots+\int_{k-2}^{k-1}+\int_{k-1}^{\infty}\bigg)i\bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\bigg) \tag{523}$$

$$\cdot\sum_{n=k}^{\infty}(1+n)r^n 1_{[n-(k-1),n]}(s)ds \tag{524}$$

$$=\int_{1}^{2} i\bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\bigg)\sum_{n=k}^{k}(1+n)r^n ds \tag{525}$$

$$+\int_{2}^{3} i\bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\bigg)\sum_{n=k}^{k+1}(1+n)r^n ds \tag{526}$$

$$+\cdots+\int_{k-2}^{k-1} i\bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\bigg)\sum_{n=k}^{2k-3}(1+n)r^n ds \tag{527}$$

$$+\int_{k-1}^{\infty} i\bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\bigg)\sum_{n=k}^{\infty}(1+n)r^n 1_{[n-(k-1),n]}(s)ds \tag{528}$$

$$=\sum_{j=1}^{k-2}\int_{j}^{1+j} i\bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\bigg)\sum_{n=k}^{k+j-1}(1+n)r^n ds \tag{529}$$

$$+\sum_{j=1}^{\infty}\int_{k+j-2}^{k+j-1} i\bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\bigg)\sum_{n=k+(j-1)}^{2k-2+(j-1)}(1+n)r^n ds. \tag{530}$$

Next, the term in (519) becomes

$$2\int_{-\infty}^{\infty} i\left(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\right)\sum_{n=k-1}^{\infty}(1+n)r^n 1_{[1+n-(k-1),1+n]}(s)ds \tag{531}$$

$$=2\left(\int_{-\infty}^{1}+\int_{1}^{2}+\cdots+\int_{k-2}^{k-1}+\int_{k-1}^{\infty}\right)i\left(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\right) \tag{532}$$

$$\cdot\sum_{n=k-1}^{\infty}(1+n)r^n 1_{[1+n-(k-1),1+n]}(s)ds \tag{533}$$

$$=2\int_{1}^{2} i\left(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\right)\sum_{n=k-1}^{k-1}(1+n)r^n ds \tag{534}$$

$$+2\int_{2}^{3} i\left(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\right)\sum_{n=k-1}^{k}(1+n)r^n ds \tag{535}$$

$$+\cdots+2\int_{k-2}^{k-1} i\left(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\right)\sum_{n=k-1}^{2k-4}(1+n)r^n ds \tag{536}$$

$$+2\int_{k-1}^{\infty} i\left(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\right)\sum_{n=k-1}^{\infty}(1+n)r^n 1_{[1+n-(k-1),1+n]}(s)ds \tag{537}$$

$$=\sum_{j=1}^{k-2}2\int_{j}^{1+j} i\left(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\right)\sum_{n=k-1}^{k+j-2}(1+n)r^n ds \tag{538}$$

$$+2\sum_{j=1}^{\infty}\int_{k+j-2}^{k+j-1} i\left(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\right)\sum_{n=k-1+(j-1)}^{2k-3+(j-1)}(1+n)r^n ds. \tag{539}$$

Lastly, the term in (521) becomes

$$-\int_{-\infty}^{\infty} i\bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\bigg)\sum_{n=k-2}^{\infty}(1+n)r^n 1_{[2+n-(k-1),2+n]}(s)ds \tag{540}$$

$$=-\bigg(\int_{-\infty}^{1}+\int_{1}^{2}+\cdots+\int_{k-2}^{k-1}+\int_{k-1}^{\infty}\bigg)i\bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\bigg) \tag{541}$$

$$\cdot\sum_{n=k-2}^{\infty}(1+n)r^n 1_{[2+n-(k-1),2+n]}(s)ds \tag{542}$$

$$=-\int_{1}^{2} i\bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\bigg)\sum_{n=k-2}^{k-2}(1+n)r^n ds \tag{543}$$

$$-\int_{2}^{3} i\bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\bigg)\sum_{n=k-2}^{k-1}(1+n)r^n ds \tag{544}$$

$$-\cdots-\int_{k-2}^{k-1} i\bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\bigg)\sum_{n=k-2}^{2k-5}(1+n)r^n ds \tag{545}$$

$$-\int_{k-1}^{\infty} i\bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\bigg)\sum_{n=k-2}^{\infty}(1+n)r^n 1_{[2+n-(k-1),2+n]}(s)ds \tag{546}$$

$$=\sum_{j=1}^{k-2}-\int_{j}^{1+j} i\bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\bigg)\sum_{n=k-2}^{k+j-3}(1+n)r^n ds \tag{547}$$

$$-\sum_{j=1}^{\infty}\int_{k+j-2}^{k+j-1} i\bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\bigg)\sum_{n=k-2+(j-1)}^{2k-4+(j-1)}(1+n)r^n ds. \tag{548}$$

Using that

$$\frac{1}{4}\bigg(\sum_{n=0}^{k-1}(1+n)\int_{-\pi}^{\pi} e^{i\beta n}(1-e^{-i\beta(k-1)})d\beta \tag{549}$$

$$+\sum_{n=0}^{k-2}(1+n)\int_{-\pi}^{\pi} 2e^{i\beta}e^{i\beta n}(1-e^{-i\beta(k-1)})d\beta \tag{550}$$

$$+\sum_{n=0}^{k-3}(1+n)\int_{-\pi}^{\pi} -e^{2i\beta}e^{i\beta n}(1-e^{-i\beta(k-1)})d\beta\bigg)=\frac{\pi}{2}-\pi k, \tag{551}$$

we obtain

$$\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\frac{i(k-1)}{4}\sum_{n=0}^{\infty}(1+n)r^n\int_0^1\int_{-\pi}^{\pi}(1+2e^{i\beta}-e^{2i\beta})\beta e^{-i\beta(k-1)s}e^{i\beta n}d\beta ds \tag{552}$$

$$=\frac{1}{4}\Bigg(\sum_{j=1}^{k-2}\int_j^{1+j}i\Bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\Bigg)\sum_{n=k}^{k+j-1}(1+n)ds \tag{553}$$

$$+\sum_{j=1}^{\infty}\int_{k+j-2}^{k+j-1}i\Bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\Bigg)\sum_{n=k+(j-1)}^{2k-2+(j-1)}(1+n)ds \tag{554}$$

$$+\sum_{j=1}^{k-2}2\int_j^{1+j}i\Bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\Bigg)\sum_{n=k-1}^{k+j-2}(1+n)ds \tag{555}$$

$$+2\sum_{j=1}^{\infty}\int_{k+j-2}^{k+j-1}i\Bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\Bigg)\sum_{n=k-1+(j-1)}^{2k-3+(j-1)}(1+n)ds \tag{556}$$

$$+\sum_{j=1}^{k-2}-\int_j^{1+j}i\Bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\Bigg)\sum_{n=k-2}^{k+j-3}(1+n)ds \tag{557}$$

$$-\sum_{j=1}^{\infty}\int_{k+j-2}^{k+j-1}i\Bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\Bigg)\sum_{n=k-2+(j-1)}^{2k-4+(j-1)}(1+n)ds\Bigg) \tag{558}$$

$$+\frac{\pi}{2}-\pi k \tag{559}$$

$$=\frac{1}{4}\Bigg((2k+1)\sum_{j=1}^{k-2}j\int_j^{1+j}i\Bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\Bigg)ds \tag{560}$$

$$+\sum_{j=1}^{k-2}j^2\int_j^{1+j}i\Bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\Bigg)ds \tag{561}$$

$$+(-1+k)(-2+3k)\sum_{j=1}^{\infty}\int_{k+j-2}^{k+j-1}i\Bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\Bigg)ds \tag{562}$$

$$+2(-1+k)\sum_{j=1}^{\infty}j\int_{k+j-2}^{k+j-1}i\Bigg(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\Bigg)ds\Bigg) \tag{563}$$

$$+\frac{\pi}{2}-\pi k. \tag{564}$$

Now, we compute the integral

$$\int_{\epsilon}^{-\epsilon}(1+2e^{i\beta}-e^{2i\beta})\beta e^{-i\beta(k-1)s}e^{i\beta n}d\beta. \tag{565}$$

We can use the same procedure to obtain that

$$\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\frac{i(k-1)}{4}\sum_{n=0}^{\infty}(1+n)r^n\int_0^1\int_{\epsilon}^{-\epsilon}(1+2e^{i\beta}-e^{2i\beta})\beta e^{-i\beta(k-1)s}e^{i\beta n}d\beta ds \tag{566}$$

$$=\frac{\pi}{2}(k-1). \tag{567}$$

Therefore,

$$g_2(k)=-\frac{\pi}{2}k+\frac{1}{4}\Bigg((2k+1)\sum_{j=1}^{k-2}j\int_j^{1+j}i\left(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\right)ds \tag{568}$$

$$+\sum_{j=1}^{k-2}j^2\int_j^{1+j}i\left(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\right)ds \tag{569}$$

$$+(-1+k)(-2+3k)\sum_{j=1}^{\infty}\int_{k+j-2}^{k+j-1}i\left(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\right)ds \tag{570}$$

$$+2(-1+k)\sum_{j=1}^{\infty}j\int_{k+j-2}^{k+j-1}i\left(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\right)ds\Bigg). \tag{571}$$

Next, we compute $g_3(k)$. Using that

$$1-e^{-i\beta(1+k)}=i\beta(1+k)\int_0^1 e^{-i\beta(1+k)s}ds, \tag{572}$$

$$\frac{1}{(-1+re^{i\beta})^2}=\sum_{n=0}^{\infty}(1+n)(re^{i\beta})^n, \tag{573}$$

we obtain

$$g_3(k)=\mathrm{pv}\int_{-\pi}^{\pi}\frac{-1+2e^{i\beta}+e^{2i\beta}}{4(-1+e^{i\beta})^2}(1-e^{-i\beta(1+k)})d\beta \tag{574}$$

$$=\lim_{\epsilon\to 0^+}\int_{\substack{[-\pi,\pi]\\ \setminus(-\epsilon,\epsilon)}}\frac{-1+2e^{i\beta}+e^{2i\beta}}{4(-1+e^{i\beta})^2}i\beta(1+k)\int_0^1 e^{-i\beta(1+k)s}dsd\beta \tag{575}$$

$$=\frac{i(1+k)}{4}\lim_{\epsilon\to 0^+}\int_{\substack{[-\pi,\pi]\\ \setminus(-\epsilon,\epsilon)}}\int_0^1\frac{(-1+2e^{i\beta}+e^{2i\beta})\beta e^{-i\beta(1+k)s}}{(-1+e^{i\beta})^2}dsd\beta \tag{576}$$

$$=\frac{i(1+k)}{4} \tag{577}$$

$$\cdot\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\int_{\substack{[-\pi,\pi]\\ \setminus(-\epsilon,\epsilon)}}\int_0^1(-1+2e^{i\beta}+e^{2i\beta})\beta e^{-i\beta(1+k)s}\sum_{n=0}^{\infty}(1+n)(re^{i\beta})^n dsd\beta \tag{578}$$

$$=\frac{i(1+k)}{4} \tag{579}$$

$$\cdot\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\sum_{n=0}^{\infty}(1+n)r^n\int_0^1\int_{\substack{[-\pi,\pi]\\ \setminus(-\epsilon,\epsilon)}}(-1+2e^{i\beta}+e^{2i\beta})\beta e^{-i\beta(1+k)s}e^{i\beta n}d\beta ds. \tag{580}$$

To simplify this expression, we first compute the expression

$$\frac{i(1+k)}{4}\sum_{n=0}^{\infty}(1+n)r^n\int_0^1\int_{-\pi}^{\pi}(-1+2e^{i\beta}+e^{2i\beta})\beta e^{-i\beta(1+k)s}e^{i\beta n}d\beta ds \tag{581}$$

$$=\frac{i(1+k)}{4}\Bigg[\Bigg(\sum_{n=0}^{1+k}+\sum_{n=2+k}^{\infty}\Bigg)(1+n)r^n\int_0^1\int_{-\pi}^{\pi}-\beta e^{-i\beta(1+k)s}e^{i\beta n}d\beta ds \tag{582}$$

$$+\Bigg(\sum_{n=0}^{k}+\sum_{n=1+k}^{\infty}\Bigg)(1+n)r^n\int_0^1\int_{-\pi}^{\pi}2e^{i\beta}\beta e^{-i\beta(1+k)s}e^{i\beta n}d\beta ds \tag{583}$$

$$+\Bigg(\sum_{n=0}^{k-1}+\sum_{n=k}^{\infty}\Bigg)(1+n)r^n\int_0^1\int_{-\pi}^{\pi}e^{2i\beta}\beta e^{-i\beta(1+k)s}e^{i\beta n}d\beta ds\Bigg] \tag{584}$$

$$=\frac{i(1+k)}{4}\Bigg[\sum_{n=0}^{1+k}(1+n)r^n\int_{-\pi}^{\pi}-\beta e^{i\beta n}\int_0^1 e^{-i\beta(1+k)s}dsd\beta \tag{585}$$

$$+\sum_{n=2+k}^{\infty}(1+n)r^n\int_0^1\int_{-\pi}^{\pi}-\beta e^{-i\beta(1+k)s}e^{i\beta n}d\beta ds \tag{586}$$

$$+\sum_{n=0}^{k}(1+n)r^n\int_{-\pi}^{\pi}2e^{i\beta}\beta e^{i\beta n}\int_0^1 e^{-i\beta(1+k)s}dsd\beta \tag{587}$$

$$+\sum_{n=1+k}^{\infty}(1+n)r^n\int_0^1\int_{-\pi}^{\pi}2e^{i\beta}\beta e^{-i\beta(1+k)s}e^{i\beta n}d\beta ds \tag{588}$$

$$+\sum_{n=0}^{k-1}(1+n)r^n\int_{-\pi}^{\pi}e^{2i\beta}\beta e^{i\beta n}\int_0^1 e^{-i\beta(1+k)s}dsd\beta \tag{589}$$

$$+\sum_{n=k}^{\infty}(1+n)r^n\int_0^1\int_{-\pi}^{\pi}e^{2i\beta}\beta e^{-i\beta(1+k)s}e^{i\beta n}d\beta ds\Bigg]. \tag{590}$$

Using the identity

$$\int_0^1 e^{-i\beta(1+k)s}ds=\frac{1-e^{-i\beta(1+k)}}{i\beta(1+k)}, \tag{591}$$

we obtain

$$\frac{i(1+k)}{4}\sum_{n=0}^{\infty}(1+n)r^n\int_0^1\int_{-\pi}^{\pi}(-1+2e^{i\beta}+e^{2i\beta})\beta e^{-i\beta(1+k)s}e^{i\beta n}d\beta ds \tag{592}$$

$$=\frac{1}{4}\Bigg[\sum_{n=0}^{1+k}(1+n)r^n\int_{-\pi}^{\pi}-e^{i\beta n}(1-e^{-i\beta(1+k)})d\beta \tag{593}$$

$$+\sum_{n=0}^{k}(1+n)r^n\int_{-\pi}^{\pi}2e^{i\beta}e^{i\beta n}(1-e^{-i\beta(1+k)})d\beta \tag{594}$$

$$+\sum_{n=0}^{k-1}(1+n)r^n\int_{-\pi}^{\pi}e^{2i\beta}e^{i\beta n}(1-e^{-i\beta(1+k)})d\beta\Bigg] \tag{595}$$

$$+\frac{1}{4}\Bigg[\sum_{n=2+k}^{\infty}(1+n)r^n\Bigg(-\pi\int_{-\infty}^{\infty}\left(\frac{e^{i\pi s}}{s}+\frac{e^{-i\pi s}}{s}\right)1_{[n-(1+k),n]}(s)ds \tag{596}$$

$$-i\int_{-\infty}^{\infty}\left(\frac{e^{i\pi s}}{s^2}-\frac{e^{-i\pi s}}{s^2}\right)1_{[n-(1+k),n]}(s)ds\Bigg) \tag{597}$$

$$+\sum_{n=1+k}^{\infty}(1+n)r^n\left(2i\int_{-\infty}^{\infty}\left(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\right)1_{[1+n-(1+k),1+n]}(s)ds\right) \tag{598}$$

$$+\sum_{n=k}^{\infty}(1+n)r^n\left(i\int_{-\infty}^{\infty}\left(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\right)1_{[2+n-(1+k),2+n]}(s)ds\right)\Bigg]. \tag{599}$$

We will further simplify the terms in (596), (597), (598), and (599). The term in (596) becomes

$$-\pi\int_{-\infty}^{\infty}\left(\frac{e^{i\pi s}}{s}+\frac{e^{-i\pi s}}{s}\right)\sum_{n=2+k}^{\infty}(1+n)r^n 1_{[n-(1+k),n]}(s)ds \tag{600}$$

$$=-\pi\left(\int_{-\infty}^{1}+\int_{1}^{2}+\cdots+\int_{k}^{1+k}+\int_{1+k}^{\infty}\right)\left(\frac{e^{i\pi s}}{s}+\frac{e^{-i\pi s}}{s}\right) \tag{601}$$

$$\cdot\sum_{n=2+k}^{\infty}(1+n)r^n 1_{[n-(1+k),n]}(s)ds \tag{602}$$

$$=-\pi\int_{1}^{2}\left(\frac{e^{i\pi s}}{s}+\frac{e^{-i\pi s}}{s}\right)\sum_{n=2+k}^{2+k}(1+n)r^n ds-\pi\int_{2}^{3}\left(\frac{e^{i\pi s}}{s}+\frac{e^{-i\pi s}}{s}\right) \tag{603}$$

$$\cdot\sum_{n=2+k}^{3+k}(1+n)r^n ds \tag{604}$$

$$-\cdots-\pi\int_{k}^{1+k}\left(\frac{e^{i\pi s}}{s}+\frac{e^{-i\pi s}}{s}\right)\sum_{n=2+k}^{1+2k}(1+n)r^n ds \tag{605}$$

$$-\pi\int_{1+k}^{\infty}\left(\frac{e^{i\pi s}}{s}+\frac{e^{-i\pi s}}{s}\right)\sum_{n=2+k}^{\infty}(1+n)r^n 1_{[n-(1+k),n]}(s)ds \tag{606}$$

$$=\sum_{n=1}^{k}-\pi\int_{j}^{1+j}\left(\frac{e^{i\pi s}}{s}+\frac{e^{-i\pi s}}{s}\right)\sum_{n=2+k}^{1+k+j}(1+n)r^n ds \tag{607}$$

$$-\pi\sum_{j=1}^{\infty}\int_{j+k}^{j+1+k}\left(\frac{e^{i\pi s}}{s}+\frac{e^{-i\pi s}}{s}\right)\sum_{n=2+k}^{\infty}(1+n)r^n 1_{[n-(1+k),n]}(s)ds \tag{608}$$

$$=\sum_{j=1}^{k}-\pi\int_{j}^{1+j}\left(\frac{e^{i\pi s}}{s}+\frac{e^{-i\pi s}}{s}\right)\sum_{n=2+k}^{1+k+j}(1+n)r^n ds \tag{609}$$

$$-\pi\sum_{j=1}^{\infty}\int_{j+k}^{j+1+k}\left(\frac{e^{i\pi s}}{s}+\frac{e^{-i\pi s}}{s}\right)\sum_{n=(j+1)+k}^{(j+1)+2k}(1+n)r^n ds. \tag{610}$$

Next, the term in (597) becomes

$$-i\int_{-\infty}^{\infty}\left(\frac{e^{i\pi s}}{s^2}-\frac{e^{-i\pi s}}{s^2}\right)\sum_{n=2+k}^{\infty}(1+n)r^n 1_{[n-(1+k),n]}(s)ds \tag{611}$$

$$=-i\left(\int_{-\infty}^{1}+\int_{1}^{2}+\cdots+\int_{k}^{1+k}+\int_{1+k}^{\infty}\right)\left(\frac{e^{i\pi s}}{s^2}-\frac{e^{-i\pi s}}{s^2}\right) \tag{612}$$

$$\cdot\sum_{n=2+k}^{\infty}(1+n)r^n 1_{[n-(1+k),n]}(s)ds \tag{613}$$

$$=-i\int_{1}^{2}\left(\frac{e^{i\pi s}}{s^2}-\frac{e^{-i\pi s}}{s^2}\right)\sum_{n=2+k}^{2+k}(1+n)r^n ds-i\int_{2}^{3}\left(\frac{e^{i\pi s}}{s^2}-\frac{e^{-i\pi s}}{s^2}\right)\sum_{n=2+k}^{3+k}(1+n)r^n ds \tag{614}$$

$$-\cdots-i\int_{k}^{1+k}\left(\frac{e^{i\pi s}}{s^2}-\frac{e^{-i\pi s}}{s^2}\right)\sum_{n=2+k}^{1+2k}(1+n)r^n ds \tag{615}$$

$$-i\int_{1+k}^{\infty}\left(\frac{e^{i\pi s}}{s^2}-\frac{e^{-i\pi s}}{s^2}\right)\sum_{n=2+k}^{\infty}(1+n)r^n 1_{[n-(1+k),n]}(s)ds \tag{616}$$

$$=\sum_{j=1}^{k}-i\int_{j}^{1+j}\left(\frac{e^{i\pi s}}{s^2}-\frac{e^{-i\pi s}}{s^2}\right)\sum_{n=2+k}^{1+k+j}(1+n)r^n ds \tag{617}$$

$$-i\sum_{j=1}^{\infty}\int_{j+k}^{j+1+k}\left(\frac{e^{i\pi s}}{s^2}-\frac{e^{-i\pi s}}{s^2}\right)\sum_{n=(j+1)+k}^{(j+1)+2k}(1+n)r^n ds. \tag{618}$$

Next, the term in (598) becomes

$$2i\int_{-\infty}^{\infty}\left(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\right)\sum_{n=1+k}^{\infty}(1+n)r^n 1_{[1+n-(1+k),1+n]}(s)ds \tag{619}$$

$$=2i\left(\int_{-\infty}^{1}+\int_{1}^{2}+\cdots+\int_{k}^{1+k}+\int_{1+k}^{\infty}\right)\left(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\right) \tag{620}$$

$$\cdot\sum_{n=1+k}^{\infty}(1+n)r^n 1_{[1+n-(1+k),1+n]}(s)ds \tag{621}$$

$$=2i\int_{1}^{2}\left(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\right)\sum_{n=1+k}^{1+k}(1+n)r^n ds \tag{622}$$

$$+2i\int_{2}^{3}\left(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\right)\sum_{n=1+k}^{2+k}(1+n)r^n ds \tag{623}$$

$$+\cdots+2i\int_{k}^{1+k}\left(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\right)\sum_{n=1+k}^{2k}(1+n)r^n ds \tag{624}$$

$$+2i\int_{1+k}^{\infty}\left(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\right)\sum_{n=1+k}^{\infty}(1+n)r^n 1_{[1+n-(1+k),1+n]}(s)ds \tag{625}$$

$$=\sum_{j=1}^{k}2i\int_{j}^{1+j}\left(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\right)\sum_{n=1+k}^{k+j}(1+n)r^n ds \tag{626}$$

$$+2i\sum_{j=1}^{\infty}\int_{j+k}^{j+1+k}\left(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\right)\sum_{n=j+k}^{j+2k}(1+n)r^n ds. \tag{627}$$

Lastly, the term in (599) becomes

$$i\int_{-\infty}^{\infty}\left(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\right)\sum_{n=k}^{\infty}(1+n)r^n 1_{[2+n-(1+k),2+n]}(s)ds \tag{628}$$

$$=i\left(\int_{-\infty}^{1}+\int_{1}^{2}+\cdots+\int_{k}^{1+k}+\int_{1+k}^{\infty}\right)\left(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\right) \tag{629}$$

$$\cdot\sum_{n=k}^{\infty}(1+n)r^n 1_{[2+n-(1+k),2+n]}(s)ds \tag{630}$$

$$=i\int_{1}^{2}\left(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\right)\sum_{n=k}^{k}(1+n)r^n ds \tag{631}$$

$$+i\int_{2}^{3}\left(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\right)\sum_{n=k}^{k+1}(1+n)r^n ds \tag{632}$$

$$+\cdots+i\int_{k}^{1+k}\left(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\right)\sum_{n=k}^{2k-1}(1+n)r^n ds \tag{633}$$

$$+i\int_{1+k}^{\infty}\left(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\right)\sum_{n=k}^{\infty}(1+n)r^n 1_{[2+n-(1+k),2+n]}(s)ds \tag{634}$$

$$=\sum_{j=1}^{k} i\int_{j}^{1+j}\left(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\right)\sum_{n=k}^{j-1+k}(1+n)r^n ds \tag{635}$$

$$+i\sum_{j=1}^{\infty}\int_{j+k}^{j+1+k}\left(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\right)\sum_{n=(j-1)+k}^{(j-1)+2k}(1+n)r^n ds. \tag{636}$$

Therefore,

$$\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\frac{i(1+k)}{4}\sum_{n=0}^{\infty}(1+n)r^n\int_0^1\int_{-\pi}^{\pi}(-1+2e^{i\beta}+e^{2i\beta})\beta e^{-i\beta(1+k)s}e^{i\beta n}d\beta ds \tag{637}$$

$$=\frac{1}{4}\Bigg[\sum_{n=0}^{1+k}(1+n)r^n\int_{-\pi}^{\pi}-e^{i\beta n}(1-e^{-i\beta(1+k)})d\beta \tag{638}$$

$$+\sum_{n=0}^{k}(1+n)r^n\int_{-\pi}^{\pi}2e^{i\beta}e^{i\beta n}(1-e^{-i\beta(1+k)})d\beta \tag{639}$$

$$+\sum_{n=0}^{k-1}(1+n)r^n\int_{-\pi}^{\pi}e^{2i\beta}e^{i\beta n}(1-e^{-i\beta(1+k)})d\beta\Bigg] \tag{640}$$

$$+\frac{1}{4}\Bigg[\sum_{j=1}^{k}-\pi\int_j^{1+j}\left(\frac{e^{i\pi s}}{s}+\frac{e^{-i\pi s}}{s}\right)\sum_{n=2+k}^{1+k+j}(1+n)r^n ds \tag{641}$$

$$-\pi\sum_{j=1}^{\infty}\int_{j+k}^{j+1+k}\left(\frac{e^{i\pi s}}{s}+\frac{e^{-i\pi s}}{s}\right)\sum_{n=(j+1)+k}^{(j+1)+2k}(1+n)r^n ds \tag{642}$$

$$+\sum_{j=1}^{k}-i\int_j^{1+j}\left(\frac{e^{i\pi s}}{s^2}-\frac{e^{-i\pi s}}{s^2}\right)\sum_{n=2+k}^{1+k+j}(1+n)r^n ds \tag{643}$$

$$-i\sum_{j=1}^{\infty}\int_{j+k}^{j+1+k}\left(\frac{e^{i\pi s}}{s^2}-\frac{e^{-i\pi s}}{s^2}\right)\sum_{n=(j+1)+k}^{(j+1)+2k}(1+n)r^n ds \tag{644}$$

$$+\sum_{j=1}^{k}2i\int_j^{1+j}\left(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\right)\sum_{n=1+k}^{k+j}(1+n)r^n ds \tag{645}$$

$$+2i\sum_{j=1}^{\infty}\int_{j+k}^{j+1+k}\left(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\right)\sum_{n=j+k}^{j+2k}(1+n)r^n ds \tag{646}$$

$$+\sum_{j=1}^{k}i\int_j^{1+j}\left(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\right)\sum_{n=k}^{j-1+k}(1+n)r^n ds \tag{647}$$

$$+i\sum_{j=1}^{\infty}\int_{j+k}^{j+1+k}\left(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\right)\sum_{n=(j-1)+k}^{(j-1)+2k}(1+n)r^n ds\Bigg]. \tag{648}$$

Using that

$$\frac{1}{4}\Bigg[\sum_{n=0}^{1+k}(1+n)r^n\int_{-\pi}^{\pi}-e^{i\beta n}(1-e^{-i\beta(1+k)})d\beta \tag{649}$$

$$+\sum_{n=0}^{k}(1+n)r^n\int_{-\pi}^{\pi}2e^{i\beta}e^{i\beta n}(1-e^{-i\beta(1+k)})d\beta \tag{650}$$

$$+\sum_{n=0}^{k-1}(1+n)r^n\int_{-\pi}^{\pi}e^{2i\beta}e^{i\beta n}(1-e^{-i\beta(1+k)})d\beta\Bigg]=-\frac{\pi}{2}-\pi k, \tag{651}$$

we obtain

$$\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\frac{i(1+k)}{4}\sum_{n=0}^{\infty}(1+n)r^n\int_0^1\int_{-\pi}^{\pi}(-1+2e^{i\beta}+e^{2i\beta})\beta e^{-i\beta(1+k)s}e^{i\beta n}d\beta ds \tag{652}$$

$$=-\frac{\pi}{2}-\pi k+\frac{1}{4}\Bigg[(2k+1)\sum_{j=1}^{k}j\int_j^{1+j}i\left(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\right)ds \tag{653}$$

$$+\sum_{j=1}^{k}j^2\int_j^{1+j}i\left(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\right)ds \tag{654}$$

$$+3k(1+k)\sum_{j=1}^{\infty}\int_{j+k}^{j+1+k}i\left(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\right)ds \tag{655}$$

$$+2(1+k)\sum_{j=1}^{\infty}j\int_{j+k}^{j+1+k}i\left(\frac{e^{i\pi s}(1-i\pi s)}{s^2}-\frac{e^{-i\pi s}(1+i\pi s)}{s^2}\right)ds\Bigg]. \tag{656}$$

Now, we compute the expression

$$\frac{i(1+k)}{4}\sum_{n=0}^{\infty}(1+n)r^n\int_0^1\int_{\epsilon}^{-\epsilon}(-1+2e^{i\beta}+e^{2i\beta})\beta e^{-i\beta(1+k)s}e^{i\beta n}d\beta ds \tag{657}$$

$$=\frac{i(1+k)}{4}\Bigg[\Bigg(\sum_{n=0}^{1+k}+\sum_{n=2+k}^{\infty}\Bigg)(1+n)r^n\int_0^1\int_{\epsilon}^{-\epsilon}-\beta e^{-i\beta(1+k)s}e^{i\beta n}d\beta ds \tag{658}$$

$$+\Bigg(\sum_{n=0}^{k}+\sum_{n=1+k}^{\infty}\Bigg)(1+n)r^n\int_0^1\int_{\epsilon}^{-\epsilon}2e^{i\beta}\beta e^{-i\beta(1+k)s}e^{i\beta n}d\beta ds \tag{659}$$

$$+\Bigg(\sum_{n=0}^{k-1}+\sum_{n=k}^{\infty}\Bigg)(1+n)r^n\int_0^1\int_{\epsilon}^{-\epsilon}e^{2i\beta}\beta e^{-i\beta(1+k)s}e^{i\beta n}d\beta ds\Bigg] \tag{660}$$

$$=\frac{1}{4}\Bigg[\sum_{n=0}^{1+k}(1+n)r^n\int_{\epsilon}^{-\epsilon}-e^{i\beta n}(1-e^{-i\beta(1+k)})d\beta \tag{661}$$

$$+\sum_{n=0}^{k}(1+n)r^n\int_{\epsilon}^{-\epsilon}2e^{i\beta}e^{i\beta n}(1-e^{-i\beta(1+k)})d\beta \tag{662}$$

$$+\sum_{n=0}^{k-1}(1+n)r^n\int_{\epsilon}^{-\epsilon}e^{2i\beta}e^{i\beta n}(1-e^{-i\beta(1+k)})d\beta\Bigg] \tag{663}$$

$$+\frac{1}{4}\Bigg[\sum_{n=2+k}^{\infty}(1+n)r^n\Bigg(i\int_{-\infty}^{\infty}\Bigg(\frac{e^{is\epsilon}}{s^2}-\frac{e^{-is\epsilon}}{s^2}\Bigg)1_{[n-(1+k),n]}(s)ds \tag{664}$$

$$+\epsilon\int_{-\infty}^{\infty}\Bigg(\frac{e^{is\epsilon}}{s}+\frac{e^{-is\epsilon}}{s}\Bigg)1_{[n-(1+k),n]}(s)ds\Bigg) \tag{665}$$

$$+\sum_{n=1+k}^{\infty}(1+n)r^n\Bigg(-2\int_{-\infty}^{\infty}\Bigg(\frac{e^{is\epsilon}(i+s\epsilon)}{s^2}+\frac{e^{-is\epsilon}(-i+s\epsilon)}{s^2}\Bigg)1_{[1+n-(1+k),1+n]}(s)ds\Bigg) \tag{666}$$

$$+\sum_{n=k}^{\infty}(1+n)r^n\Bigg(-\int_{-\infty}^{\infty}\Bigg(\frac{e^{is\epsilon}(i+s\epsilon)}{s^2}+\frac{e^{-is\epsilon}(-i+s\epsilon)}{s^2}\Bigg)1_{[2+n-(1+k),2+n]}(s)ds\Bigg)\Bigg]. \tag{667}$$

We are interested in how the terms in (664), (665), (666), and (667) behave as first r goes to 1 from below and then ϵ goes to 0 from above. For the term in (664),

$$i\int_{-\infty}^{\infty}\left(\frac{e^{is\epsilon}}{s^2}-\frac{e^{-is\epsilon}}{s^2}\right)\sum_{n=2+k}^{\infty}(1+n)r^n 1_{[n-(1+k),n]}(s)ds \tag{668}$$

$$=\sum_{j=1}^{k} i\int_{j}^{1+j}\left(\frac{e^{is\epsilon}}{s^2}-\frac{e^{-is\epsilon}}{s^2}\right)\sum_{n=2+k}^{1+k+j}(1+n)r^n ds \tag{669}$$

$$+\,i\sum_{j=1}^{\infty}\int_{j+k}^{j+1+k}\left(\frac{e^{is\epsilon}}{s^2}-\frac{e^{-is\epsilon}}{s^2}\right)\sum_{n=(j+1)+k}^{(j+1)+2k}(1+n)r^n ds \tag{670}$$

$$\xrightarrow{r\to1^-}\sum_{j=1}^{k} i\int_{j}^{1+j}\left(\frac{e^{is\epsilon}}{s^2}-\frac{e^{-is\epsilon}}{s^2}\right)\frac{j(5+j+2k)}{2}ds \tag{671}$$

$$+\,i\sum_{j=1}^{\infty}\int_{j+k}^{j+1+k}\left(\frac{e^{is\epsilon}}{s^2}-\frac{e^{-is\epsilon}}{s^2}\right)\frac{(1+k)(4+2j+3k)}{2}ds \tag{672}$$

$$=\sum_{j=1}^{k}\frac{j(5+j+2k)}{2}\int_{j}^{1+j} i\left(\frac{e^{is\epsilon}}{s^2}-\frac{e^{-is\epsilon}}{s^2}\right)ds \tag{673}$$

$$+\,\frac{(1+k)(4+3k)}{2}\sum_{j=1}^{\infty}\int_{j+k}^{j+1+k} i\left(\frac{e^{is\epsilon}}{s^2}-\frac{e^{-is\epsilon}}{s^2}\right)ds \tag{674}$$

$$+\,(1+k)\sum_{j=1}^{\infty} j\int_{j+k}^{j+1+k} i\left(\frac{e^{is\epsilon}}{s^2}-\frac{e^{-is\epsilon}}{s^2}\right)ds \tag{675}$$

$$\xrightarrow{\epsilon\to0^+}(1+k)(-\pi). \tag{676}$$

For the term in (665),

$$\epsilon \int_{-\infty}^{\infty} \left(\frac{e^{is\epsilon}}{s} + \frac{e^{-is\epsilon}}{s} \right) \sum_{n=2+k}^{\infty} (1+n) r^n 1_{[n-(1+k),n]}(s) ds \tag{677}$$

$$= \sum_{j=1}^{k} \epsilon \int_{j}^{1+j} \left(\frac{e^{is\epsilon}}{s} + \frac{e^{-is\epsilon}}{s} \right) \sum_{n=2+k}^{1+k+j} (1+n) r^n ds \tag{678}$$

$$+ \epsilon \sum_{j=1}^{\infty} \int_{j+k}^{j+1+k} \left(\frac{e^{is\epsilon}}{s} + \frac{e^{-is\epsilon}}{s} \right) \sum_{n=(j+1)+k}^{(j+1)+2k} (1+n) r^n ds \tag{679}$$

$$\xrightarrow{r \to 1^-} \epsilon \sum_{j=1}^{k} \frac{j(5+j+2k)}{2} \int_{j}^{1+j} \left(\frac{e^{is\epsilon}}{s} + \frac{e^{-is\epsilon}}{s} \right) ds \tag{680}$$

$$+ \epsilon \frac{(1+k)(4+3k)}{2} \sum_{j=1}^{\infty} \int_{j+k}^{j+1+k} \left(\frac{e^{is\epsilon}}{s} + \frac{e^{-is\epsilon}}{s} \right) ds \tag{681}$$

$$+ \epsilon (1+k) \sum_{j=1}^{\infty} j \int_{j+k}^{j+1+k} \left(\frac{e^{is\epsilon}}{s} + \frac{e^{-is\epsilon}}{s} \right) ds \tag{682}$$

$$\xrightarrow{\epsilon \to 0^+} 0. \tag{683}$$

For the term in (666),

$$
\begin{aligned}
& -2\int_{-\infty}^{\infty}\left(\frac{e^{is\epsilon}(i+s\epsilon)}{s^2}+\frac{e^{-is\epsilon}(-i+s\epsilon)}{s^2}\right)\sum_{n=1+k}^{\infty}(1+n)r^n 1_{[1+n-(1+k),1+n]}(s)ds && (684)\\
= & -2i\int_{-\infty}^{\infty}\left(\frac{e^{is\epsilon}(1-is\epsilon)}{s^2}-\frac{e^{-is\epsilon}(1+is\epsilon)}{s^2}\right)\sum_{n=1+k}^{\infty}(1+n)r^n 1_{[1+n-(1+k),1+n]}(s)ds && (685)\\
= & -\sum_{j=1}^{k}2i\int_{j}^{1+j}\left(\frac{e^{is\epsilon}(1-is\epsilon)}{s^2}-\frac{e^{-is\epsilon}(1+is\epsilon)}{s^2}\right)\sum_{n=1+k}^{k+j}(1+n)r^n ds && (686)\\
& -2i\sum_{j=1}^{\infty}\int_{j+k}^{j+1+k}\left(\frac{e^{is\epsilon}(1-is\epsilon)}{s^2}-\frac{e^{-is\epsilon}(1+is\epsilon)}{s^2}\right)\sum_{n=j+k}^{j+2k}(1+n)r^n ds && (687)\\
\xrightarrow{r\to 1^-} & -\sum_{j=1}^{k}j(3+j+2k)\int_{j}^{1+j}i\left(\frac{e^{is\epsilon}}{s^2}-\frac{e^{-is\epsilon}}{s^2}\right)ds && (688)\\
& -\epsilon\sum_{j=1}^{k}j(3+j+2k)\int_{j}^{1+j}\left(\frac{e^{is\epsilon}}{s}+\frac{e^{-is\epsilon}}{s}\right)ds && (689)\\
& -(1+k)(2+3k)\sum_{j=1}^{\infty}\int_{j+k}^{j+1+k}i\left(\frac{e^{is\epsilon}}{s^2}-\frac{e^{-is\epsilon}}{s^2}\right)ds && (690)\\
& -2(1+k)\sum_{j=1}^{\infty}j\int_{j+k}^{j+1+k}i\left(\frac{e^{is\epsilon}}{s^2}-\frac{e^{-is\epsilon}}{s^2}\right)ds && (691)\\
& -\epsilon(1+k)(2+3k)\sum_{j=1}^{\infty}\int_{j+k}^{j+1+k}\left(\frac{e^{is\epsilon}}{s}+\frac{e^{-is\epsilon}}{s}\right)ds && (692)\\
& -2(1+k)\epsilon\sum_{j=1}^{\infty}j\int_{j+k}^{j+1+k}\left(\frac{e^{is\epsilon}}{s}+\frac{e^{-is\epsilon}}{s}\right)ds && (693)\\
\xrightarrow{\epsilon\to 0^+} & -2(1+k)(-\pi). && (694)
\end{aligned}
$$

Lastly, for the term in (667),

$$
\begin{aligned}
&-\int_{-\infty}^{\infty}\left(\frac{e^{is\epsilon}(i+s\epsilon)}{s^2}+\frac{e^{-is\epsilon}(-i+s\epsilon)}{s^2}\right)\sum_{n=k}^{\infty}(1+n)r^n 1_{[2+n-(1+k),2+n]}(s)ds && (695)\\
=&-\sum_{j=1}^{k} i\int_{j}^{1+j}\left(\frac{e^{is\epsilon}(1-is\epsilon)}{s^2}-\frac{e^{-is\epsilon}(1+is\epsilon)}{s^2}\right)\sum_{n=k}^{j-1+k}(1+n)r^n ds && (696)\\
&-i\sum_{j=1}^{\infty}\int_{j+k}^{j+1+k}\left(\frac{e^{is\epsilon}(1-is\epsilon)}{s^2}-\frac{e^{-is\epsilon}(1+is\epsilon)}{s^2}\right)\sum_{n=(j-1)+k}^{(j-1)+2k}(1+n)r^n ds && (697)\\
\xrightarrow{r\to1^-}&-\sum_{j=1}^{k}\frac{j(1+j+2k)}{2}\int_{j}^{1+j} i\left(\frac{e^{is\epsilon}}{s^2}-\frac{e^{-is\epsilon}}{s^2}\right)ds && (698)\\
&-\epsilon\sum_{j=1}^{k}\frac{j(1+j+2k)}{2}\int_{j}^{1+j}\left(\frac{e^{is\epsilon}}{s}+\frac{e^{-is\epsilon}}{s}\right)ds && (699)\\
&-\frac{(1+k)3k}{2}\sum_{j=1}^{\infty}\int_{j+k}^{j+1+k} i\left(\frac{e^{is\epsilon}}{s^2}-\frac{e^{-is\epsilon}}{s^2}\right)ds && (700)\\
&-\epsilon\frac{(1+k)3k}{2}\sum_{j=1}^{\infty}\int_{j+k}^{j+1+k}\left(\frac{e^{is\epsilon}}{s}+\frac{e^{-is\epsilon}}{s}\right)ds && (701)\\
&-(1+k)\sum_{j=1}^{\infty} j\int_{j+k}^{j+1+k} i\left(\frac{e^{is\epsilon}}{s^2}-\frac{e^{-is\epsilon}}{s^2}\right)ds && (702)\\
&-\epsilon(1+k)\sum_{j=1}^{\infty} j\int_{j+k}^{j+1+k}\left(\frac{e^{is\epsilon}}{s}+\frac{e^{-is\epsilon}}{s}\right)ds && (703)\\
\xrightarrow{\epsilon\to0^+}&-(1+k)(-\pi). && (704)
\end{aligned}
$$

Using that

$$
\begin{aligned}
&\lim_{\epsilon\to0^+}\lim_{r\to1^-}\frac{1}{4}\Bigg[\sum_{n=0}^{1+k}(1+n)r^n\int_{\epsilon}^{-\epsilon}-e^{i\beta n}(1-e^{-i\beta(1+k)})d\beta && (705)\\
&+\sum_{n=0}^{k}(1+n)r^n\int_{\epsilon}^{-\epsilon}2e^{i\beta}e^{i\beta n}(1-e^{-i\beta(1+k)})d\beta && (706)\\
&+\sum_{n=0}^{k-1}(1+n)r^n\int_{\epsilon}^{-\epsilon}e^{2i\beta}e^{i\beta n}(1-e^{-i\beta(1+k)})d\beta\Bigg]=0, && (707)
\end{aligned}
$$

we obtain

$$
\begin{aligned}
&\lim_{\epsilon\to0^+}\lim_{r\to1^-}\frac{i(1+k)}{4}\sum_{n=0}^{\infty}(1+n)r^n\int_0^1\int_{\epsilon}^{-\epsilon}(-1+2e^{i\beta}+e^{2i\beta})\beta e^{-i\beta(1+k)s}e^{i\beta n}d\beta ds && (708)\\
=&\frac{\pi}{2}(1+k). && (709)
\end{aligned}
$$

Therefore,

$$g_3(k) = -\frac{\pi}{2}k + \frac{1}{4}\Bigg[(2k+1)\sum_{j=1}^{k} j\int_j^{1+j} i\left(\frac{e^{i\pi s}(1-i\pi s)}{s^2} - \frac{e^{-i\pi s}(1+i\pi s)}{s^2}\right)ds \tag{710}$$

$$+\sum_{j=1}^{k} j^2\int_j^{1+j} i\left(\frac{e^{i\pi s}(1-i\pi s)}{s^2} - \frac{e^{-i\pi s}(1+i\pi s)}{s^2}\right)ds \tag{711}$$

$$+3k(1+k)\sum_{j=1}^{\infty}\int_{j+k}^{j+1+k} i\left(\frac{e^{i\pi s}(1-i\pi s)}{s^2} - \frac{e^{-i\pi s}(1+i\pi s)}{s^2}\right)ds \tag{712}$$

$$+2(1+k)\sum_{j=1}^{\infty} j\int_{j+k}^{j+1+k} i\left(\frac{e^{i\pi s}(1-i\pi s)}{s^2} - \frac{e^{-i\pi s}(1+i\pi s)}{s^2}\right)ds\Bigg]. \tag{713}$$

To simplify the expressions for $g_2(k)$ and $g_3(k)$, we note that for $j \neq \{-1, 0\}$,

$$\int_j^{j+1} i\left(\frac{e^{i\pi s}(1-i\pi s)}{s^2} - \frac{e^{-i\pi s}(1+i\pi s)}{s^2}\right)ds \tag{714}$$

$$= i\int_j^{j+1} \frac{e^{i\pi s}}{s^2} - \frac{e^{-i\pi s}}{s^2} - i\pi\left(\frac{e^{i\pi s}}{s} + \frac{e^{-i\pi s}}{s}\right)ds \tag{715}$$

$$= i\int_j^{j+1} \frac{2i\sin(\pi s)}{s^2} - i\pi\frac{2\cos(\pi s)}{s}ds \tag{716}$$

$$= -2\int_j^{j+1} \frac{\sin(\pi s)}{s^2}ds + 2\pi\int_j^{j+1}\frac{\cos(\pi s)}{s}ds \tag{717}$$

$$= -2\pi\int_j^{j+1} \frac{\cos(\pi s)}{s}ds + 2\pi\int_j^{j+1}\frac{\cos(\pi s)}{s}ds \tag{718}$$

$$= 0. \tag{719}$$

Using this simplification, we obtain

$$g_2(k) = -\frac{\pi}{2}k, \tag{720}$$

$$g_3(k) = -\frac{\pi}{2}k. \tag{721}$$

Next, let us compute $g_5(k)$. We have

$$g_5(k) = \mathrm{pv}\int_{-\pi}^{\pi} \frac{-i(-1-2e^{i\beta}+e^{2i\beta})e^{-i\beta k}}{4(-1+e^{i\beta})}d\beta \tag{722}$$

$$= \frac{i}{4}\lim_{\epsilon\to 0^+}\int_{\substack{[-\pi,\pi]\\ \setminus(-\epsilon,\epsilon)}} (-1-2e^{i\beta}+e^{2i\beta})e^{-i\beta k}\lim_{r\to 1^-}\sum_{n=0}^{\infty}(re^{i\beta})^n d\beta \tag{723}$$

$$= \frac{i}{4}\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\sum_{n=0}^{\infty} r^n\int_{\substack{[-\pi,\pi]\\ \setminus(-\epsilon,\epsilon)}} (-1-2e^{i\beta}+e^{2i\beta})e^{-i\beta k}e^{i\beta n}d\beta. \tag{724}$$

To simplify this expression, we first compute the expression

$$\frac{i}{4}\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\sum_{n=0}^{\infty} r^n \int_{-\pi}^{\pi}(-1-2e^{i\beta}+e^{2i\beta})e^{-i\beta k}e^{i\beta n}d\beta \tag{725}$$

$$=\frac{i}{4}\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\Bigg(\sum_{n=0}^{\infty} r^n \int_{-\pi}^{\pi} -e^{i\beta(n-k)}d\beta + \sum_{n=0}^{\infty} r^n \int_{-\pi}^{\pi} -2e^{i\beta}e^{i\beta(n-k)}d\beta \tag{726}$$

$$+\sum_{n=0}^{\infty} r^n \int_{-\pi}^{\pi} e^{2i\beta}e^{i\beta(n-k)}d\beta\Bigg) \tag{727}$$

$$=\frac{i}{4}\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\Bigg(\sum_{n=0}^{k-1} r^n \int_{-\pi}^{\pi} -e^{i\beta(n-k)}d\beta + \sum_{n=k+1}^{\infty} r^n \int_{-\pi}^{\pi} -e^{i\beta(n-k)}d\beta + r^k(-2\pi) \tag{728}$$

$$+\sum_{n=0}^{k-2} r^n \int_{-\pi}^{\pi} -2e^{i\beta(n-k+1)}d\beta + \sum_{n=k}^{\infty} r^n \int_{-\pi}^{\pi} -2e^{i\beta(n-k+1)}d\beta + r^{k-1}(-4\pi) \tag{729}$$

$$+\sum_{n=0}^{k-3} r^n \int_{-\pi}^{\pi} e^{i\beta(n-k+2)}d\beta + \sum_{n=k-1}^{\infty} r^n \int_{-\pi}^{\pi} e^{i\beta(n-k+2)}d\beta + r^{k-2}(2\pi)\Bigg) \tag{730}$$

$$=\frac{i}{4}\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\Bigg(\sum_{n=0}^{k-1} r^n \frac{i}{k-n}(e^{i(k-n)\pi}-e^{-i(k-n)\pi}) \tag{731}$$

$$+\sum_{n=k+1}^{\infty} r^n \frac{i}{k-n}(e^{i(k-n)\pi}-e^{-i(k-n)\pi}) + r^k(-2\pi) \tag{732}$$

$$+\sum_{n=0}^{k-2} r^n \frac{-2i}{k-n-1}(e^{i(k-n)\pi}-e^{-i(k-n)\pi}) \tag{733}$$

$$+\sum_{n=k}^{\infty} r^n \frac{-2i}{k-n-1}(e^{i(k-n)\pi}-e^{-i(k-n)\pi}) + r^{k-1}(-4\pi) \tag{734}$$

$$+\sum_{n=0}^{k-3} r^n \frac{-i}{k-n-2}(e^{i(k-n)\pi}-e^{-i(k-n)\pi}) \tag{735}$$

$$+\sum_{n=k-1}^{\infty} r^n \frac{-i}{k-n-2}(e^{i(k-n)\pi}-e^{-i(k-n)\pi}) + r^{k-2}(2\pi)\Bigg). \tag{736}$$

Using that

$$\sum_{n=0}^{k-1} r^n \frac{i}{k-n}(e^{i(k-n)\pi}-e^{-i(k-n)\pi}) = 0 \tag{737}$$

and

$$\sum_{n=k+1}^{\infty} r^n \frac{i}{k-n}\left(e^{i(k-n)\pi}-e^{-i(k-n)\pi}\right) = 0, \tag{738}$$

we obtain

$$\frac{i}{4}\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\sum_{n=0}^{\infty} r^n \int_{-\pi}^{\pi}(-1-2e^{i\beta}+e^{2i\beta})e^{-i\beta k}e^{i\beta n}d\beta = -i\pi. \tag{739}$$

Next, we compute the expression

$$\frac{i}{4}\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\sum_{n=0}^{\infty} r^n \int_{\epsilon}^{-\epsilon}(-1-2e^{i\beta}+e^{2i\beta})e^{-i\beta k}e^{i\beta n}d\beta \tag{740}$$

$$=\frac{i}{4}\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\Bigg(\sum_{n=0}^{\infty} r^n \int_{\epsilon}^{-\epsilon} -e^{i\beta(n-k)}d\beta + \sum_{n=0}^{\infty} r^n \int_{\epsilon}^{-\epsilon} -2e^{i\beta}e^{i\beta(n-k)}d\beta \tag{741}$$

$$+\sum_{n=0}^{\infty} r^n \int_{\epsilon}^{-\epsilon} e^{2i\beta}e^{i\beta(n-k)}d\beta\Bigg) \tag{742}$$

$$=\frac{i}{4}\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\Bigg(\sum_{n=0}^{k-1} r^n \int_{\epsilon}^{-\epsilon} -e^{i\beta(n-k)}d\beta + \sum_{n=k+1}^{\infty} r^n \int_{\epsilon}^{-\epsilon} -e^{i\beta(n-k)}d\beta + r^k\cdot 2\epsilon \tag{743}$$

$$+\sum_{n=0}^{k-2} r^n \int_{\epsilon}^{-\epsilon} -2e^{i\beta(n-k+1)}d\beta + \sum_{n=k}^{\infty} r^n \int_{\epsilon}^{-\epsilon} -2e^{i\beta(n-k+1)}d\beta + r^{k-1}4\epsilon \tag{744}$$

$$+\sum_{n=0}^{k-3} r^n \int_{\epsilon}^{-\epsilon} e^{i\beta(n-k+2)}d\beta + \sum_{n=k-1}^{\infty} r^n \int_{\epsilon}^{-\epsilon} e^{i\beta(n-k+2)}d\beta + r^{k-2}(-2\epsilon)\Bigg) \tag{745}$$

$$=\frac{i}{4}\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\Bigg(\sum_{n=0}^{k-1} r^n \frac{-i}{k-n}(e^{i(k-n)\epsilon}-e^{-i(k-n)\epsilon}) \tag{746}$$

$$+\sum_{n=k+1}^{\infty} r^n \frac{-i}{k-n}(e^{i(k-n)\epsilon}-e^{-i(k-n)\epsilon}) + r^k\cdot 2\epsilon \tag{747}$$

$$+\sum_{n=0}^{k-2} r^n \frac{2i}{k-n-1}(e^{-i(k-n-1)\epsilon}-e^{i(k-n-1)\epsilon}) \tag{748}$$

$$+\sum_{n=k}^{\infty} r^n \frac{2i}{k-n-1}(e^{-i(k-n-1)\epsilon}-e^{i(k-n-1)\epsilon}) + r^{k-1}4\epsilon \tag{749}$$

$$+\sum_{n=0}^{k-3} r^n \frac{i}{k-n-2}(e^{i(k-n-2)\epsilon}-e^{-i(k-n-2)\epsilon}) \tag{750}$$

$$+\sum_{n=k-1}^{\infty} r^n \frac{i}{k-n-2}(e^{i(k-n-2)\epsilon}-e^{-i(k-n-2)\epsilon}) \tag{751}$$

$$+r^{k-2}(-2\epsilon)\Bigg). \tag{752}$$

Using that

$$\sum_{n=0}^{k-1} r^n \frac{-i}{k-n}(e^{i(k-n)\epsilon} - e^{-i(k-n)\epsilon}) \xrightarrow{r\to 1^-} -i\sum_{n=1}^{k} \frac{(e^{-i\epsilon})^{-n}}{n} + i\sum_{n=1}^{k} \frac{(e^{i\epsilon})^{-n}}{n} \tag{753}$$

$$\xrightarrow{\epsilon\to 0^+} 0 \tag{754}$$

and

$$\sum_{n=k+1}^{\infty} r^n \frac{-i}{k-n}(e^{i(k-n)\epsilon} - e^{-i(k-n)\epsilon}) \tag{755}$$

$$= ie^{ik\epsilon}(re^{-i\epsilon})^k(-\mathrm{Log}(1-re^{-i\epsilon})) - ie^{-ik\epsilon}(re^{i\epsilon})^k(-\mathrm{Log}(1-re^{i\epsilon})) \tag{756}$$

$$\xrightarrow{r\to 1^-} -i\mathrm{Log}(1-e^{-i\epsilon}) + i\mathrm{Log}(1-e^{i\epsilon}) \tag{757}$$

$$= -i(\log\left|1-e^{-i\epsilon}\right| + i\mathrm{Arg}(1-e^{-i\epsilon})) + i(\log\left|1-e^{i\epsilon}\right| + i\mathrm{Arg}(1-e^{i\epsilon})) \tag{758}$$

$$= \mathrm{Arg}(1-e^{-i\epsilon}) - \mathrm{Arg}(1-e^{i\epsilon}) \tag{759}$$

$$\xrightarrow{\epsilon\to 0^+} \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi, \tag{760}$$

where Log denotes the principal branch of the complex logarithm, we obtain

$$\frac{i}{4}\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\sum_{n=0}^{\infty} r^n \int_{\epsilon}^{-\epsilon} (-1-2e^{i\beta}+e^{2i\beta})e^{-i\beta k}e^{i\beta n}d\beta = \frac{i\pi}{2}. \tag{761}$$

Therefore,

$$g_5(k) = -\frac{i\pi}{2}. \tag{762}$$

Next, let us compute $g_6(k)$. We have

$$g_6(k) = \mathrm{pv}\int_{-\pi}^{\pi} \frac{e^{-i\beta}(-1+2e^{i\beta}+e^{2i\beta})e^{-i\beta k}}{4(-1+e^{i\beta})}d\beta \tag{763}$$

$$= -\frac{1}{4}\lim_{\epsilon\to 0^+}\int_{\substack{[-\pi,\pi]\\ \setminus(-\epsilon,\epsilon)}} e^{-i\beta}(-1+2e^{i\beta}+e^{2i\beta})e^{-i\beta k}\lim_{r\to 1^-}\sum_{n=0}^{\infty}(re^{i\beta})^n d\beta \tag{764}$$

$$= -\frac{1}{4}\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\sum_{n=0}^{\infty} r^n \int_{\substack{[-\pi,\pi]\\ \setminus(-\epsilon,\epsilon)}} (2-e^{-i\beta}+e^{i\beta})e^{-i\beta k}e^{i\beta n}d\beta. \tag{765}$$

To simplify this expression, we first compute the expression

$$-\frac{1}{4}\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\sum_{n=0}^{\infty} r^n \int_{-\pi}^{\pi}(2-e^{-i\beta}+e^{i\beta})e^{-i\beta k}e^{i\beta n}d\beta \tag{766}$$

$$= -\frac{1}{4}\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\left(\sum_{n=0}^{\infty} r^n \int_{-\pi}^{\pi} 2e^{i\beta(n-k)}d\beta + \sum_{n=0}^{\infty} r^n \int_{-\pi}^{\pi}(-e^{-i\beta})e^{i\beta(n-k)}d\beta + \sum_{n=0}^{\infty} r^n \int_{-\pi}^{\pi} e^{i\beta}e^{i\beta(n-k)}d\beta\right) \tag{767}$$

$$= -\frac{1}{4}\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\left(\sum_{n=0}^{k-1} r^n \int_{-\pi}^{\pi} 2e^{i\beta(n-k)}d\beta + \sum_{n=k+1}^{\infty} r^n \int_{-\pi}^{\pi} 2e^{i\beta(n-k)}d\beta + r^k(4\pi)\right. \tag{768}$$

$$+\sum_{n=0}^{k} r^n \int_{-\pi}^{\pi} -e^{i\beta(n-k-1)}d\beta + \sum_{n=k+2}^{\infty} r^n \int_{-\pi}^{\pi} -e^{i\beta(n-k-1)}d\beta + r^{k+1}(-2\pi) \tag{769}$$

$$\left.+\sum_{n=0}^{k-2} r^n \int_{-\pi}^{\pi} e^{i\beta(n-k+1)}d\beta + \sum_{n=k}^{\infty} r^n \int_{-\pi}^{\pi} e^{i\beta(n-k+1)}d\beta + r^{k-2}(2\pi)\right) \tag{770}$$

$$= -\pi. \tag{771}$$

Next, we compute the expression

$$-\frac{1}{4}\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\sum_{n=0}^{\infty} r^n \int_{\epsilon}^{-\epsilon}(2-e^{-i\beta}+e^{i\beta})e^{-i\beta k}e^{i\beta n}d\beta \tag{772}$$

$$= -\frac{1}{4}\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\left(\sum_{n=0}^{\infty} r^n \int_{\epsilon}^{-\epsilon} 2e^{i\beta(n-k)}d\beta + \sum_{n=0}^{\infty} r^n \int_{\epsilon}^{-\epsilon} -e^{i\beta(n-k-1)}d\beta + \sum_{n=0}^{\infty} r^n \int_{\epsilon}^{-\epsilon} e^{i\beta(n-k+1)}d\beta\right) \tag{773}$$

$$= -\frac{1}{4}\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\left(\sum_{n=0}^{k-1} r^n \int_{\epsilon}^{-\epsilon} 2e^{i\beta(n-k)}d\beta + \sum_{n=k+1}^{\infty} r^n \int_{\epsilon}^{-\epsilon} 2e^{i\beta(n-k)}d\beta + r^k\cdot(-4\epsilon)\right. \tag{774}$$

$$+\sum_{n=0}^{k} r^n \int_{\epsilon}^{-\epsilon} -e^{i\beta(n-k-1)}d\beta + \sum_{n=k+2}^{\infty} r^n \int_{\epsilon}^{-\epsilon} -e^{i\beta(n-k-1)}d\beta + r^{k+1}2\epsilon \tag{775}$$

$$\left.+\sum_{n=0}^{k-2} r^n \int_{\epsilon}^{-\epsilon} e^{i\beta(n-k+1)}d\beta + \sum_{n=k}^{\infty} r^n \int_{\epsilon}^{-\epsilon} e^{i\beta(n-k+1)}d\beta + r^{k-1}(-2\epsilon)\right) \tag{776}$$

$$=\frac{\pi}{2}. \tag{777}$$

Therefore,

$$g_6(k) = -\frac{\pi}{2}. \tag{778}$$

Next, let us compute $g_7(k)$. We have

$$g_7(k) = \text{pv}\int_{-\pi}^{\pi} \frac{(1+2e^{i\beta}-e^{2i\beta})e^{-i\beta k}}{4(-1+e^{i\beta})}d\beta \tag{779}$$

$$= -\frac{1}{4}\lim_{\epsilon\to 0^+}\int_{\substack{[-\pi,\pi]\\ \setminus(-\epsilon,\epsilon)}}(1+2e^{i\beta}-e^{2i\beta})e^{-i\beta k}\lim_{r\to 1^-}\sum_{n=0}^{\infty}(re^{i\beta})^n d\beta \tag{780}$$

$$= -\frac{1}{4}\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\sum_{n=0}^{\infty}r^n\int_{\substack{[-\pi,\pi]\\ \setminus(-\epsilon,\epsilon)}}(1+2e^{i\beta}-e^{2i\beta})e^{i\beta(n-k)}d\beta. \tag{781}$$

To simplify this expression, we first compute the expression

$$-\frac{1}{4}\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\sum_{n=0}^{\infty}r^n\int_{-\pi}^{\pi}(1+2e^{i\beta}-e^{2i\beta})e^{i\beta(n-k)}d\beta \tag{782}$$

$$= -\frac{1}{4}\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\left(\sum_{n=0}^{\infty}r^n\int_{-\pi}^{\pi}e^{i\beta(n-k)}d\beta+\sum_{n=0}^{\infty}r^n\int_{-\pi}^{\pi}2e^{i\beta(n-k+1)}d\beta+\sum_{n=0}^{\infty}r^n\int_{-\pi}^{\pi}-e^{i\beta(n-k+2)}d\beta\right) \tag{783}$$

$$= -\frac{1}{4}\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\left(\sum_{n=0}^{k-1}r^n\int_{-\pi}^{\pi}e^{i\beta(n-k)}d\beta+\sum_{n=k+1}^{\infty}r^n\int_{-\pi}^{\pi}e^{i\beta(n-k)}d\beta+r^k(2\pi)\right. \tag{784}$$

$$+\sum_{n=0}^{k-2}r^n\int_{-\pi}^{\pi}2e^{i\beta(n-k+1)}d\beta+\sum_{n=k}^{\infty}r^n\int_{-\pi}^{\pi}2e^{i\beta(n-k+1)}d\beta+r^{k-1}(4\pi) \tag{785}$$

$$\left.+\sum_{n=0}^{k-3}r^n\int_{-\pi}^{\pi}-e^{i\beta(n-k+2)}d\beta+\sum_{n=k-1}^{\infty}r^n\int_{-\pi}^{\pi}-e^{i\beta(n-k+2)}d\beta+r^{k-2}(-2\pi)\right) \tag{786}$$

$$= -\pi. \tag{787}$$

Next, we compute the expression

$$-\frac{1}{4}\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\sum_{n=0}^{\infty}r^n\int_{\epsilon}^{-\epsilon}(1+2e^{i\beta}-e^{2i\beta})e^{i\beta(n-k)}d\beta \tag{788}$$

$$= -\frac{1}{4}\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\left(\sum_{n=0}^{\infty}r^n\int_{\epsilon}^{-\epsilon}e^{i\beta(n-k)}d\beta+\sum_{n=0}^{\infty}r^n\int_{\epsilon}^{-\epsilon}2e^{i\beta}e^{i\beta(n-k)}d\beta+\sum_{n=0}^{\infty}r^n\int_{\epsilon}^{-\epsilon}-e^{2i\beta}e^{i\beta(n-k)}d\beta\right) \tag{789}$$

$$= -\frac{1}{4}\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\left(\sum_{n=0}^{k-1}r^n\int_{\epsilon}^{-\epsilon}e^{i\beta(n-k)}d\beta+\sum_{n=k+1}^{\infty}r^n\int_{\epsilon}^{-\epsilon}e^{i\beta(n-k)}d\beta+r^k\cdot(-2\epsilon)\right. \tag{790}$$

$$+\sum_{n=0}^{k-2}r^n\int_{\epsilon}^{-\epsilon}2e^{i\beta(n-k+1)}d\beta+\sum_{n=k}^{\infty}r^n\int_{\epsilon}^{-\epsilon}2e^{i\beta(n-k+1)}d\beta+r^{k-1}(-4\epsilon) \tag{791}$$

$$\left.+\sum_{n=0}^{k-3}r^n\int_{\epsilon}^{-\epsilon}-e^{i\beta(n-k+2)}d\beta+\sum_{n=k-1}^{\infty}r^n\int_{\epsilon}^{-\epsilon}-e^{i\beta(n-k+2)}d\beta+r^{k-2}(2\epsilon)\right) \tag{792}$$

$$=\frac{\pi}{2}. \tag{793}$$

Therefore,

$$g_7(k) = -\frac{\pi}{2}. \tag{794}$$

Lastly, let us compute $g_8(k)$. We have

$$g_8(k) = \mathrm{pv}\int_{-\pi}^{\pi} \frac{-e^{-i\beta}(-1+2e^{i\beta}+e^{2i\beta})e^{-i\beta k}}{4(-1+e^{i\beta})} d\beta \tag{795}$$

$$= \frac{1}{4}\lim_{\epsilon\to 0^+}\int_{\substack{[-\pi,\pi]\\ \setminus(-\epsilon,\epsilon)}} e^{-i\beta}(-1+2e^{i\beta}+e^{2i\beta})e^{-i\beta k}\lim_{r\to 1^-}\sum_{n=0}^{\infty}(re^{i\beta})^n d\beta \tag{796}$$

$$= \frac{1}{4}\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\sum_{n=0}^{\infty} r^n \int_{\substack{[-\pi,\pi]\\ \setminus(-\epsilon,\epsilon)}} (2-e^{-i\beta}+e^{i\beta})e^{i\beta(n-k)} d\beta. \tag{797}$$

To simplify this expression, we first compute the expression

$$\frac{1}{4}\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\sum_{n=0}^{\infty} r^n \int_{-\pi}^{\pi}(2-e^{-i\beta}+e^{i\beta})e^{i\beta(n-k)} d\beta \tag{798}$$

$$=\frac{1}{4}\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\left(\sum_{n=0}^{\infty} r^n \int_{-\pi}^{\pi} 2e^{i\beta(n-k)} d\beta + \sum_{n=0}^{\infty} r^n \int_{-\pi}^{\pi} -e^{i\beta(n-k-1)} d\beta + \sum_{n=0}^{\infty} r^n \int_{-\pi}^{\pi} e^{i\beta(n-k+1)} d\beta\right) \tag{799}$$

$$=\frac{1}{4}\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\Bigg(\sum_{n=0}^{k-1} r^n \int_{-\pi}^{\pi} 2e^{i\beta(n-k)} d\beta + \sum_{n=k+1}^{\infty} r^n \int_{-\pi}^{\pi} 2e^{i\beta(n-k)} d\beta + r^k(4\pi) \tag{800}$$

$$+\sum_{n=0}^{k} r^n \int_{-\pi}^{\pi} -e^{i\beta(n-k-1)} d\beta + \sum_{n=k+2}^{\infty} r^n \int_{-\pi}^{\pi} -e^{i\beta(n-k-1)} d\beta + r^{k+1}(-2\pi) \tag{801}$$

$$+\sum_{n=0}^{k-2} r^n \int_{-\pi}^{\pi} e^{i\beta(n-k+1)} d\beta + \sum_{n=k}^{\infty} r^n \int_{-\pi}^{\pi} e^{i\beta(n-k+1)} d\beta + r^{k-1}(2\pi)\Bigg) \tag{802}$$

$$=\pi. \tag{803}$$

Next, we compute the expression

$$\frac{1}{4}\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\sum_{n=0}^{\infty} r^n \int_{\epsilon}^{-\epsilon}(2-e^{-i\beta}+e^{i\beta})e^{i\beta(n-k)}d\beta \tag{804}$$

$$=\frac{1}{4}\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\left(\sum_{n=0}^{\infty} r^n \int_{\epsilon}^{-\epsilon} 2e^{i\beta(n-k)}d\beta+\sum_{n=0}^{\infty} r^n \int_{\epsilon}^{-\epsilon} -e^{i\beta(n-k-1)}d\beta+\sum_{n=0}^{\infty} r^n \int_{\epsilon}^{-\epsilon} e^{i\beta(n-k+1)}d\beta\right) \tag{805}$$

$$=\frac{1}{4}\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\Bigg(\sum_{n=0}^{k-1} r^n \int_{\epsilon}^{-\epsilon} 2e^{i\beta(n-k)}d\beta+\sum_{n=k+1}^{\infty} r^n \int_{\epsilon}^{-\epsilon} 2e^{i\beta(n-k)}d\beta+r^k\cdot(-4\epsilon) \tag{806}$$

$$+\sum_{n=0}^{k} r^n \int_{\epsilon}^{-\epsilon} -e^{i\beta(n-k-1)}d\beta+\sum_{n=k+2}^{\infty} r^n \int_{\epsilon}^{-\epsilon} -e^{i\beta(n-k-1)}d\beta+r^{k+1}(2\epsilon) \tag{807}$$

$$+\sum_{n=0}^{k-2} r^n \int_{\epsilon}^{-\epsilon} e^{i\beta(n-k+1)}d\beta+\sum_{n=k}^{\infty} r^n \int_{\epsilon}^{-\epsilon} e^{i\beta(n-k+1)}d\beta+r^{k-1}(-2\epsilon)\Bigg) \tag{808}$$

$$=-\frac{\pi}{2}. \tag{809}$$

Therefore,

$$g_8(k)=\frac{\pi}{2}. \tag{810}$$

Plugging the calculated values of $g_2(k)$, $g_3(k)$, $g_5(k)$, $g_6(k)$, $g_7(k)$, and $g_8(k)$ into (476), we obtain

$$J_1(k)=-\frac{\pi}{k}. \tag{811}$$

Using (433) and (451), we deduce that

$$J_1(k)=\begin{cases}-\frac{\pi}{k} & k>1,\\ \frac{\pi}{k} & k<-1.\end{cases} \tag{812}$$

9.2 Summary

Plugging the results of Sections 9.1.1 and 9.1.2 into (432), we obtain that for $k>1$,

$$\mathcal{F}(\mathcal{L})(k)=-\frac{2\pi}{L(t)}\frac{\gamma}{4\pi}\mathcal{F}(\phi)(k)\pi k=-\frac{2\pi}{L(t)}\frac{\gamma}{4\pi}\mathcal{F}(\phi)(k)\pi\left|k\right|. \tag{813}$$

Since for $k>1$

$$\mathcal{F}(\mathcal{L})(-k)=\overline{\mathcal{F}(\mathcal{L})(k)} \tag{814}$$

$$=-\frac{2\pi}{L(t)}\frac{\gamma}{4\pi}\overline{\mathcal{F}(\phi)(k)}\pi k \tag{815}$$

$$=-\frac{2\pi}{L(t)}\frac{\gamma}{4\pi}\mathcal{F}(\phi)(-k)\pi\left|k\right|, \tag{816}$$

we conclude that for $|k| > 1$,

$$\mathcal{F}(\mathcal{L})(k) = -\frac{2\pi}{L(t)}\frac{\gamma}{4\pi}\mathcal{F}(\phi)(k)\pi\,|k|\,. \tag{817}$$

This concludes that proof that $\mathcal{L}$ is the Hilbert transform of the first derivative of θ up to the ± 1 Fourier modes. To compute $\mathcal{F}(\mathcal{L})(k)$ for $|k| = 1$, we use that for $k \in \mathbb{Z}$

$$\mathcal{F}((U_1)_\alpha)(k) = ik\mathcal{F}(U_1)(k) \tag{818}$$

to rewrite (349) as

$$\mathcal{F}(\mathcal{L})(k) = \frac{2\pi}{L(t)}\left(ik\mathcal{F}(U_1)(k) - \frac{i}{k}\mathcal{F}(U_1)(k)\right), \tag{819}$$

where $k \neq 0$. Then $\mathcal{F}(\mathcal{L})(\pm 1) = 0$. That the ± 1 Fourier modes of $\mathcal{L}$ are zero poses a technical challenge in dealing with the term in the evolution equation for θ that induces an exponential decay in time of initial perturbation of the interface. This challenge can be resolved by observing that the identity

$$\int_{-\pi}^{\pi} z_\alpha(\alpha, t)d\alpha = 0 \tag{820}$$

provides a means to control the ± 1 Fourier modes of $\mathcal{L}$ using the other nonzero Fourier modes.

10 Derivation of an *a priori* Estimate

Before embarking on the derivation of a key *a priori* estimate for $\phi = \theta - \hat{\theta}(0)$, we first derive crucial estimates for $L(t)$. Let us derive certain upper and lower bounds of $L(t)$ which tightly control it as long as $\|\phi(t)\|_{\mathcal{F}^{0,1}}$ is sufficiently small for all $t \geq 0$.

Proposition 7. *If $\|\phi(t)\|_{\mathcal{F}^{0,1}}$ is sufficiently small for all $t \geq 0$, then*

$$\frac{R^2}{1 + \frac{\pi}{2}(e^{2\|\phi(t)\|_{\mathcal{F}^{0,1}}} - 1)} \leq \left(\frac{L(t)}{2\pi}\right)^2 \leq \frac{R^2}{1 - \frac{\pi}{2}(e^{2\|\phi(t)\|_{\mathcal{F}^{0,1}}} - 1)}. \tag{821}$$

Proof. By the definition of the Fourier transform,

$$\mathcal{F}\left(\int_0^\alpha e^{-i\eta}(\phi(\alpha) - \phi(\eta))^n d\eta\right)(-1) \tag{822}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_0^\alpha e^{-i\eta}(\phi(\alpha) - \phi(\eta))^n d\eta \cdot e^{i\alpha}d\alpha \tag{823}$$

$$= \frac{1}{i}\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_0^\alpha e^{-i\eta}(\phi(\alpha) - \phi(\eta))^n d\eta \cdot \frac{\partial}{\partial\alpha}e^{i\alpha}d\alpha. \tag{824}$$

Integration by parts yields

$$\frac{1}{i}\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{0}^{\alpha}e^{-i\eta}(\phi(\alpha)-\phi(\eta))^n d\eta\cdot\frac{\partial}{\partial\alpha}e^{i\alpha}d\alpha \tag{825}$$

$$=\frac{1}{2\pi i}\int_{-\pi}^{\pi}\frac{\partial}{\partial\alpha}\left(\int_0^{\alpha}e^{-i\eta}(\phi(\alpha)-\phi(\eta))^n d\eta\cdot e^{i\alpha}\right)d\alpha \tag{826}$$

$$=\frac{1}{2\pi i}\left(-\int_0^{\pi}e^{-i\eta}(\phi(\pi)-\phi(\eta))^n d\eta-\int_{-\pi}^{0}e^{-i\eta}(\phi(\pi)-\phi(\eta))^n d\eta\right) \tag{827}$$

$$=-\frac{1}{2\pi i}\int_{-\pi}^{\pi}e^{-i\eta}(\phi(\pi)-\phi(\eta))^n d\eta \tag{828}$$

$$=i\mathcal{F}((\phi(\pi)-\phi(\eta))^n)(1). \tag{829}$$

Then

$$\int_{-\pi}^{\pi}\int_0^{\alpha}e^{i(\alpha-\eta)}\sum_{n\geq 1}\frac{i^n}{n!}(\phi(\alpha)-\phi(\eta))^n d\eta d\alpha \tag{830}$$

$$=\sum_{n\geq 1}\frac{i^n}{n!}\int_{-\pi}^{\pi}\int_0^{\alpha}e^{i(\alpha-\eta)}(\phi(\alpha)-\phi(\eta))^n d\eta d\alpha \tag{831}$$

$$=2\pi i\sum_{n\geq 1}\frac{i^n}{n!}\mathcal{F}((\phi(\pi)-\phi(\eta))^n)(1). \tag{832}$$

Hence,

$$\mathrm{Im}\left(\int_{-\pi}^{\pi}\int_0^{\alpha}e^{i(\alpha-\eta)}\sum_{n\geq 1}\frac{i^n}{n!}(\phi(\alpha)-\phi(\eta))^n d\eta d\alpha\right) \tag{833}$$

$$=\frac{1}{2i}\left(2\pi i\sum_{n\geq 1}\frac{i^n}{n!}\mathcal{F}((\phi(\pi)-\phi(\eta))^n)(1)+2\pi i\sum_{n\geq 1}\frac{(-i)^n}{n!}\overline{\mathcal{F}((\phi(\pi)-\phi(\eta))^n)(1)}\right) \tag{834}$$

$$=\pi\left(\sum_{n\geq 1}\frac{i^n}{n!}\mathcal{F}((\phi(\pi)-\phi(\eta))^n)(1)+\sum_{n\geq 1}\frac{(-i)^n}{n!}\overline{\mathcal{F}((\phi(\pi)-\phi(\eta))^n)(1)}\right). \tag{835}$$

It follows that

$$\left|\mathrm{Im}\left(\int_{-\pi}^{\pi}\int_0^{\alpha}e^{i(\alpha-\eta)}\sum_{n\geq 1}\frac{i^n}{n!}(\phi(\alpha)-\phi(\eta))^n d\eta d\alpha\right)\right| \tag{836}$$

$$\leq 2\pi\sum_{n\geq 1}\frac{|\mathcal{F}((\phi(\pi)-\phi(\eta))^n)(1)|}{n!} \tag{837}$$

$$\leq 2\pi\sum_{n\geq 1}\frac{\|(\phi(\pi)-\phi(\cdot))^n\|_{\mathcal{F}^{0,1}}}{n!}. \tag{838}$$

By Proposition 2,

$$\|(\phi(\pi)-\phi(\cdot))^n\|_{\mathcal{F}^{0,1}}\leq\|\phi(\pi)-\phi(\cdot)\|_{\mathcal{F}^{0,1}}^n. \tag{839}$$

Then

$$\left|\mathrm{Im}\left(\int_{-\pi}^{\pi}\int_{0}^{\alpha} e^{i(\alpha-\eta)}\sum_{n\geq 1}\frac{i^n}{n!}(\phi(\alpha)-\phi(\eta))^n d\eta d\alpha\right)\right| \leq 2\pi\sum_{n\geq 1}\frac{\|\phi(\pi)-\phi(\cdot)\|_{\mathcal{F}^{0,1}}^n}{n!} \tag{840}$$

$$= 2\pi\left(e^{\|\phi(\pi)-\phi(\cdot)\|_{\mathcal{F}^{0,1}}}-1\right) \tag{841}$$

$$\leq 2\pi(e^{\|\phi(\pi)\|_{\mathcal{F}^{0,1}}}e^{\|\phi\|_{\mathcal{F}^{0,1}}}-1) \tag{842}$$

$$= 2\pi(e^{\phi(\pi)}e^{\|\phi\|_{\mathcal{F}^{0,1}}}-1). \tag{843}$$

By (6),

$$|\phi(\pi)| \leq \sum_{k\in\mathbb{Z}}\left|\hat{\phi}(k)\right| = \|\phi\|_{\mathcal{F}^{0,1}}. \tag{844}$$

Therefore,

$$\left|\mathrm{Im}\left(\int_{-\pi}^{\pi}\int_{0}^{\alpha} e^{i(\alpha-\eta)}\sum_{n\geq 1}\frac{i^n}{n!}(\phi(\alpha)-\phi(\eta))^n d\eta d\alpha\right)\right| \leq \pi^2(e^{2\|\phi\|_{\mathcal{F}^{0,1}}}-1). \tag{845}$$

This estimate shows that

$$\frac{R^2}{1+\frac{\pi}{2}(e^{2\|\phi\|_{\mathcal{F}^{0,1}}}-1)} \leq \left(\frac{L(t)}{2\pi}\right)^2 \leq \frac{R^2}{1-\frac{\pi}{2}(e^{2\|\phi\|_{\mathcal{F}^{0,1}}}-1)}, \tag{846}$$

as needed. ∎

Remark. *If $\|\phi(t)\|_{\mathcal{F}^{0,1}}$ is sufficiently small for all $t \geq 0$, then Proposition 7 ensures that $L(t) > 0$ for all $t \geq 0$, making (45) and (47) equivalent.*

Using Proposition 7, we can also prove the following useful estimate.

Proposition 8. *For sufficiently small $\|\phi\|_{\mathcal{F}^{0,1}}$,*

$$\left|R\frac{2\pi}{L(t)}-1\right| \leq 1-\sqrt{1-\frac{\pi}{2}(e^{2\|\phi\|_{\mathcal{F}^{0,1}}}-1)}. \tag{847}$$

Proof. From Proposition 7, we obtain

$$\sqrt{1-\frac{\pi}{2}(e^{2\|\phi\|_{\mathcal{F}^{0,1}}}-1)}-1 \leq \frac{2\pi R}{L(t)}-1 \leq \sqrt{1+\frac{\pi}{2}(e^{2\|\phi\|_{\mathcal{F}^{0,1}}}-1)}-1. \tag{848}$$

Then

$$\left|\frac{2\pi R}{L(t)}-1\right| \leq \max\left\{\left|\sqrt{1-\frac{\pi}{2}(e^{2\|\phi\|_{\mathcal{F}^{0,1}}}-1)}-1\right|, \left|\sqrt{1+\frac{\pi}{2}(e^{2\|\phi\|_{\mathcal{F}^{0,1}}}-1)}-1\right|\right\} \tag{849}$$

$$= \left|\sqrt{1-\frac{\pi}{2}(e^{2\|\phi\|_{\mathcal{F}^{0,1}}}-1)}-1\right| \tag{850}$$

$$= 1-\sqrt{1-\frac{\pi}{2}(e^{2\|\phi\|_{\mathcal{F}^{0,1}}}-1)}, \tag{851}$$

as needed. ∎

We now derive a key *a priori* estimate for $\phi = \theta - \hat{\theta}(0)$. In Section 9, we have shown that

$$\mathcal{F}(\mathcal{L})(k) = \begin{cases} 0 & \text{if } |k| = 1, \\ \frac{2\pi}{L(t)}\frac{\gamma}{4\pi}\mathcal{F}(\phi)(k)(J_1(k) + J_2(k)) & \text{if } |k| > 1, \end{cases} \tag{852}$$

where J_1 and J_2 are given in (812) and (451). Let

$$\widetilde{\mathcal{L}}(\alpha) = \frac{L(t)}{2\pi}\mathcal{L}(\alpha), \tag{853}$$

$$\widetilde{\mathcal{N}}(\alpha) = \frac{L(t)}{2\pi}\mathcal{N}(\alpha). \tag{854}$$

Then for $|k| \geq 1$,

$$\frac{\partial}{\partial t}\mathcal{F}(\phi)(k) = \frac{2\pi}{L(t)}\left(\mathcal{F}(\widetilde{\mathcal{L}})(k) + \mathcal{F}(\widetilde{\mathcal{N}})(k)\right) \tag{855}$$

$$= \frac{2\pi}{L(t)}\left(\frac{L(t)}{2\pi}\mathcal{F}(\mathcal{L})(k) + \mathcal{F}(\widetilde{\mathcal{N}})(k)\right) \tag{856}$$

$$= \mathcal{F}(\mathcal{L})(k) + \frac{2\pi}{L(t)}\mathcal{F}(\widetilde{\mathcal{N}})(k) \tag{857}$$

$$= \begin{cases} \frac{2\pi}{L(t)}\mathcal{F}(\widetilde{\mathcal{N}})(k) & \text{if } |k| = 1, \\ \frac{2\pi}{L(t)}\frac{\gamma}{4\pi}\mathcal{F}(\phi)(k)(J_1(k) + J_2(k)) + \frac{2\pi}{L(t)}\mathcal{F}(\widetilde{\mathcal{N}})(k) & \text{if } |k| > 1. \end{cases} \tag{858}$$

For convenience of notation, define $J_1(k) = J_2(k) = 0$ for $|k| = 1$ so that for $k \in \mathbb{Z} \setminus \{0\}$,

$$\frac{\partial}{\partial t}\mathcal{F}(\phi)(k) = \frac{2\pi}{L(t)}\frac{\gamma}{4\pi}\mathcal{F}(\phi)(k)(J_1(k) + J_2(k)) + \frac{2\pi}{L(t)}\mathcal{F}(\widetilde{\mathcal{N}})(k). \tag{859}$$

We observe that the principal linear term, i.e., the first term on the right hand side, has a time-dependent coefficient. This dependence occurs, however, only through $L(t)$. If we chose an initial circular interface of radius R to perturb around, then it is natural to make the principal linear term independent of time by replacing $L(t)$ with $2\pi R$ and keeping an error term. That is, we rewrite (859) as

$$\frac{\partial}{\partial t}\mathcal{F}(\phi)(k) = \frac{1}{R}\cdot\frac{\gamma}{4\pi}\mathcal{F}(\phi)(k)(J_1(k) + J_2(k)) + \frac{2\pi}{L(t)}\mathcal{F}(\widetilde{\mathcal{N}})(k) \tag{860}$$

$$+ \frac{\gamma}{4\pi}\mathcal{F}(\phi)(k)(J_1(k) + J_2(k))\left(-\frac{1}{R} + \frac{2\pi}{L(t)}\right). \tag{861}$$

We note that for $k > 0$,

$$\left|\hat{\phi}(-k)\right| = \left|\overline{\hat{\phi}(k)}\right| = \left|\hat{\phi}(k)\right| \tag{862}$$

since ϕ is real-valued. Then for $s > 0$,

$$\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} = \sum_{k\neq 0} e^{\nu(t)|k|}|k|^s\left|\hat{\phi}(k)\right| = 2\sum_{k\geq 1} e^{\nu(t)k}k^s\left|\hat{\phi}(k)\right|. \tag{863}$$

Differentiating this equation with respect to t, we obtain

$$\begin{aligned}
&\frac{d}{dt}\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} && (864)\\
=&2\sum_{k\geq 1} e^{\nu(t)k}\nu'(t)k\cdot k^s\left|\hat{\phi}(k)\right| + e^{\nu(t)k}k^s\frac{\partial}{\partial t}\left|\hat{\phi}(k)\right| && (865)\\
=&2\sum_{k\geq 1} e^{\nu(t)k}\nu'(t)k^{s+1}\left|\hat{\phi}(k)\right| + e^{\nu(t)k}k^s\frac{1}{\left|\hat{\phi}(k)\right|}\frac{1}{2}\left(\hat{\phi}(k)\overline{\frac{\partial}{\partial t}\hat{\phi}(k)} + \overline{\hat{\phi}(k)}\frac{\partial}{\partial t}\hat{\phi}(k)\right) && (866)\\
=&2\sum_{k\geq 1} e^{\nu(t)k}\nu'(t)k^{s+1}\left|\hat{\phi}(k)\right| + 2\sum_{k\geq 1} e^{\nu(t)k}k^s\frac{\hat{\phi}(k)\overline{\frac{\partial}{\partial t}\hat{\phi}(k)} + \overline{\hat{\phi}(k)}\frac{\partial}{\partial t}\hat{\phi}(k)}{2\left|\hat{\phi}(k)\right|}. && (867)
\end{aligned}$$

Let us simplify the second term. Using (860) and that J_1 and J_2 are real for $k \geq 1$, we obtain

$$\begin{aligned}
&\hat{\phi}(k)\overline{\frac{\partial}{\partial t}\hat{\phi}(k)} + \overline{\hat{\phi}(k)}\frac{\partial}{\partial t}\hat{\phi}(k) && (868)\\
=&\frac{1}{R}\frac{\gamma}{4\pi}(J_1+J_2)(k)\left|\hat{\phi}(k)\right|^2 + \frac{2\pi}{L(t)}\overline{\mathcal{F}(\widetilde{\mathcal{N}})(k)}\hat{\phi}(k) && (869)\\
&+\frac{\gamma}{4\pi}(J_1+J_2)(k)\left(-\frac{1}{R}+\frac{2\pi}{L(t)}\right)\left|\hat{\phi}(k)\right|^2 && (870)\\
&+\frac{1}{R}\frac{\gamma}{4\pi}(J_1+J_2)(k)\left|\hat{\phi}(k)\right|^2 + \frac{2\pi}{L(t)}\mathcal{F}(\widetilde{\mathcal{N}})(k)\overline{\hat{\phi}(k)} && (871)\\
&+\frac{\gamma}{4\pi}(J_1+J_2)(k)\left(-\frac{1}{R}+\frac{2\pi}{L(t)}\right)\left|\hat{\phi}(k)\right|^2. && (872)
\end{aligned}$$

Then

$$2\sum_{k\geq 1} e^{\nu(t)k}k^s \frac{\hat{\phi}(k)\overline{\frac{\partial}{\partial t}\hat{\phi}(k)}+\overline{\hat{\phi}(k)}\frac{\partial}{\partial t}\hat{\phi}(k)}{2\left|\hat{\phi}(k)\right|} \tag{873}$$

$$=\sum_{k\geq 1} e^{\nu(t)k}k^s \frac{\hat{\phi}(k)\overline{\frac{\partial}{\partial t}\hat{\phi}(k)}+\overline{\hat{\phi}(k)}\frac{\partial}{\partial t}\hat{\phi}(k)}{\left|\hat{\phi}(k)\right|} \tag{874}$$

$$=\sum_{k\geq 1} e^{\nu(t)k}k^s\Bigg(\frac{2}{R}\frac{\gamma}{4\pi}(J_1+J_2)(k)\left|\hat{\phi}(k)\right| \tag{875}$$

$$+\frac{2\pi}{L(t)}\frac{\mathcal{F}(\widetilde{\mathcal{N}})(k)\overline{\hat{\phi}(k)}+\overline{\mathcal{F}(\widetilde{\mathcal{N}})(k)}\hat{\phi}(k)}{\left|\hat{\phi}(k)\right|} \tag{876}$$

$$+2\frac{\gamma}{4\pi}(J_1+J_2)(k)\left(-\frac{1}{R}+\frac{2\pi}{L(t)}\right)\left|\hat{\phi}(k)\right|\Bigg) \tag{877}$$

$$=\frac{2}{R}\frac{\gamma}{4\pi}\sum_{k\geq 1} e^{\nu(t)k}k^s(J_1+J_2)(k)\left|\hat{\phi}(k)\right| \tag{878}$$

$$+\frac{2\pi}{L(t)}\sum_{k\geq 1} e^{\nu(t)k}k^s\frac{\mathcal{F}(\widetilde{\mathcal{N}})(k)\overline{\hat{\theta}(k)}+\overline{\mathcal{F}(\widetilde{\mathcal{N}})(k)}\hat{\phi}(k)}{\left|\hat{\phi}(k)\right|} \tag{879}$$

$$+2\frac{\gamma}{4\pi}\left(-\frac{1}{R}+\frac{2\pi}{L(t)}\right)\sum_{k\geq 1} e^{\nu(t)k}k^s(J_1+J_2)(k)\left|\hat{\phi}(k)\right|. \tag{880}$$

Therefore,

$$\frac{d}{dt}\|\phi\|_{\dot{\mathcal{F}}^{s,1}_{\nu}}=2\sum_{k\geq 1} e^{\nu(t)k}\nu'(t)k^{s+1}\left|\hat{\phi}(k)\right|+\frac{2}{R}\frac{\gamma}{4\pi}\sum_{k\geq 1} e^{\nu(t)k}k^s(J_1+J_2)(k)\left|\hat{\phi}(k)\right| \tag{881}$$

$$+\frac{2\pi}{L(t)}\sum_{k\geq 1} e^{\nu(t)k}k^s\frac{\mathcal{F}(\widetilde{\mathcal{N}})(k)\overline{\hat{\phi}(k)}+\overline{\mathcal{F}(\widetilde{\mathcal{N}})(k)}\hat{\phi}(k)}{\left|\hat{\phi}(k)\right|} \tag{882}$$

$$+2\frac{\gamma}{4\pi}\left(-\frac{1}{R}+\frac{2\pi}{L(t)}\right)\sum_{k\geq 1} e^{\nu(t)k}k^s(J_1+J_2)(k)\left|\hat{\phi}(k)\right|. \tag{883}$$

First, let us estimate (883). Observe that

$$2\frac{\gamma}{4\pi}\left(-\frac{1}{R}+\frac{2\pi}{L(t)}\right)\sum_{k\geq 1} e^{\nu(t)k}k^s(J_1+J_2)(k)\left|\hat{\phi}(k)\right| \tag{884}$$

$$=2\frac{\gamma}{4\pi}\frac{1}{R}\left(-1+R\frac{2\pi}{L(t)}\right)\sum_{k\geq 2} e^{\nu(t)k}k^s(J_1+J_2)(k)\left|\hat{\phi}(k)\right| \tag{885}$$

$$=-\pi\cdot 2\frac{\gamma}{4\pi}\frac{1}{R}\left(R\frac{2\pi}{L(t)}-1\right)\sum_{k\geq 2} e^{\nu(t)k}k^{s+1}\left|\hat{\phi}(k)\right|. \tag{886}$$

Using Proposition 8, we obtain

$$\left| R\frac{2\pi}{L(t)} - 1 \right| \leq A \left\| \phi \right\|_{\mathcal{F}^{0,1}}, \tag{887}$$

where we define

$$A = A(\left\| \phi \right\|_{\mathcal{F}^{0,1}}) = \frac{1 - \sqrt{1 - \frac{\pi}{2}(e^{2\left\| \phi \right\|_{\mathcal{F}^{0,1}}} - 1)}}{\left\| \phi \right\|_{\mathcal{F}^{0,1}}}. \tag{888}$$

Then

$$\left| 2\frac{\gamma}{4\pi}\left(-\frac{1}{R} + \frac{2\pi}{L(t)} \right) \sum_{k \geq 1} e^{\nu(t)k} k^s (J_1 + J_2)(k) \left| \hat{\phi}(k) \right| \right| \tag{889}$$

$$= \left| -\pi \cdot 2\frac{\gamma}{4\pi}\frac{1}{R}\left(R\frac{2\pi}{L(t)} - 1 \right) \sum_{k \geq 2} e^{\nu(t)k} k^{s+1} \left| \hat{\phi}(k) \right| \right| \tag{890}$$

$$\leq 2\pi\frac{\gamma}{4\pi}\frac{1}{R}\left| R\frac{2\pi}{L(t)} - 1 \right| \sum_{k \geq 2} e^{\nu(t)k} k^{s+1} \left| \hat{\phi}(k) \right| \tag{891}$$

$$\leq 2\pi\frac{\gamma}{4\pi}\frac{1}{R} A \left\| \phi \right\|_{\mathcal{F}^{0,1}} \sum_{k \geq 2} e^{\nu(t)k} k^{s+1} \left| \hat{\phi}(k) \right|. \tag{892}$$

Next, let us estimate (881) and (882).

$$2\sum_{k \geq 1} e^{\nu(t)k} \nu'(t) k^{s+1} \left| \hat{\phi}(k) \right| + \frac{2}{R}\frac{\gamma}{4\pi} \sum_{k \geq 1} e^{\nu(t)k} k^s (J_1 + J_2)(k) \left| \hat{\phi}(k) \right| \tag{893}$$

$$+ \frac{2\pi}{L(t)} \sum_{k \geq 1} e^{\nu(t)k} k^s \frac{\mathcal{F}(\widetilde{\mathcal{N}})(k)\overline{\hat{\phi}(k)} + \overline{\mathcal{F}(\widetilde{\mathcal{N}})(k)}\hat{\phi}(k)}{\left| \hat{\phi}(k) \right|} \tag{894}$$

$$= 2\sum_{k \geq 1} e^{\nu(t)k} \nu'(t) k^{s+1} \left| \hat{\phi}(k) \right| - \pi\frac{2}{R}\frac{\gamma}{4\pi} \sum_{k \geq 2} e^{\nu(t)k} k^{s+1} \left| \hat{\phi}(k) \right| \tag{895}$$

$$+ \frac{2\pi}{L(t)} \sum_{k \geq 1} e^{\nu(t)k} k^s \frac{\mathcal{F}(\widetilde{\mathcal{N}})(k)\overline{\hat{\phi}(k)} + \overline{\mathcal{F}(\widetilde{\mathcal{N}})(k)}\hat{\phi}(k)}{\left| \hat{\phi}(k) \right|} \tag{896}$$

$$\leq \nu'(t) \left\| \phi \right\|_{\dot{\mathcal{F}}_\nu^{s+1,1}} - \pi\frac{2}{R}\frac{\gamma}{4\pi} \sum_{k \geq 2} e^{\nu(t)k} k^{s+1} \left| \hat{\phi}(k) \right| + \frac{2\pi}{L(t)} \sum_{k \geq 1} e^{\nu(t)k} k^s 2 \left| \mathcal{F}(\widetilde{\mathcal{N}})(k) \right| \tag{897}$$

$$\leq \nu'(t) \left\| \phi \right\|_{\dot{\mathcal{F}}_\nu^{s+1,1}} - \pi\frac{2}{R}\frac{\gamma}{4\pi} \sum_{k \geq 2} e^{\nu(t)k} k^{s+1} \left| \hat{\phi}(k) \right| + \frac{2\pi}{L(t)} \left\| \widetilde{\mathcal{N}} \right\|_{\dot{\mathcal{F}}_\nu^{s,1}}. \tag{898}$$

Plugging (892) and (898) into (881), we obtain

$$\frac{d}{dt}\|\phi\|_{\dot{\mathcal{F}}^{s,1}_{\nu}} = 2\sum_{k\geq 1} e^{\nu(t)k}\nu'(t)k^{s+1}\left|\hat{\phi}(k)\right| + \frac{2}{R}\frac{\gamma}{4\pi}\sum_{k\geq 1} e^{\nu(t)k}k^{s}(J_1+J_2)(k)\left|\hat{\phi}(k)\right| \tag{899}$$

$$+ \frac{2\pi}{L(t)}\sum_{k\geq 1} e^{\nu(t)k}k^{s}\frac{\mathcal{F}(\widetilde{N})(k)\overline{\hat{\phi}(k)} + \overline{\mathcal{F}(\widetilde{N})(k)}\hat{\phi}(k)}{\left|\hat{\phi}(k)\right|} \tag{900}$$

$$+ 2\frac{\gamma}{4\pi}\left(-\frac{1}{R}+\frac{2\pi}{L(t)}\right)\sum_{k\geq 1} e^{\nu(t)k}k^{s}(J_1+J_2)(k)\left|\hat{\phi}(k)\right| \tag{901}$$

$$\leq \nu'(t)\|\phi\|_{\dot{\mathcal{F}}^{s+1,1}_{\nu}} - \pi\frac{2}{R}\frac{\gamma}{4\pi}\sum_{k\geq 2} e^{\nu(t)k}k^{s+1}\left|\hat{\phi}(k)\right| + \frac{2\pi}{L(t)}\left\|\widetilde{\mathcal{N}}\right\|_{\dot{\mathcal{F}}^{s,1}_{\nu}} \tag{902}$$

$$+ 2\frac{\gamma}{4\pi}\frac{1}{R}A\|\phi\|_{\mathcal{F}^{0,1}}\sum_{k\geq 2} e^{\nu(t)k}k^{s+1}\left|\hat{\phi}(k)\right|. \tag{903}$$

With the minus sign in front, the second term in (902) is associated with dissipation of the initial interfacial perturbation. It is clear that the ± 1 Fourier modes of ϕ play no part in dissipation. This presents a technical difficulty because the norm of the function space that we intend to use involves all nonzero Fourier modes of ϕ. To resolve this issue, we note that (820) and $\hat{\phi}(0) = 0$ imply

$$0 = \int_{-\pi}^{\pi} e^{i(\alpha + \hat{\phi}(1)e^{i\alpha} + \hat{\phi}(-1)e^{-i\alpha} + \sum_{|k|>1}\hat{\phi}(k)e^{ik\alpha})}d\alpha. \tag{904}$$

This identity provides an implicit relation between the ± 1 Fourier modes and the other nonzero Fourier modes of ϕ, which allows us to control the former in terms of the latter. This observation is summarized in Proposition 4.1 of [1]. In particular, we use the following result contained in the proposition.

Proposition 9. *Let $r \in (0, \frac{1}{2}\log\frac{5}{4})$. Consider $\|\phi\|_{\mathcal{F}^{0,1}} < r$. Then*

$$\left|\hat{\phi}(1)\right| + \left|\hat{\phi}(-1)\right| \leq C_I(r)r\sum_{|k|\geq 2}\left|\hat{\phi}(k)\right|, \tag{905}$$

where

$$C_I(r) = \frac{1}{r}\cdot\frac{2e^r(e^r-1)}{1-4(e^{2r}-1)}. \tag{906}$$

Here, $C_I(r) > 0$ is a strictly increasing function of r where

$$\lim_{r\to 0^+} C_I(r) = 2, \tag{907}$$

$$\lim_{r\to \log\frac{5}{4}^-} C_I(r) = \infty. \tag{908}$$

Suppose that $\|\phi\|_{\mathcal{F}^{0,1}} \in (0, \frac{1}{2}\log\frac{5}{4})$. By Proposition 9, for all $r \in (\|\phi\|_{\mathcal{F}^{0,1}}, \frac{1}{2}\log\frac{5}{4})$,

$$\left|\hat{\phi}(1)\right| + \left|\hat{\phi}(-1)\right| \leq C_I(r) r \sum_{|k|\geq 2} \left|\hat{\phi}(k)\right|. \tag{909}$$

Then

$$\left|\hat{\phi}(1)\right| + \left|\hat{\phi}(-1)\right| \leq C_I(\|\phi\|_{\mathcal{F}^{0,1}}) \|\phi\|_{\mathcal{F}^{0,1}} \sum_{|k|\geq 2} \left|\hat{\phi}(k)\right|. \tag{910}$$

By (862), this simplifies to

$$2\left|\hat{\phi}(1)\right| \leq 2C_I(\|\phi\|_{\mathcal{F}^{0,1}}) \|\phi\|_{\mathcal{F}^{0,1}} \sum_{k\geq 2} \left|\hat{\phi}(k)\right|. \tag{911}$$

Hence, for $s > 0$,

$$\begin{aligned}
\|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu} &= 2\sum_{k\geq 1} e^{\nu(t)k} k^s \left|\hat{\phi}(k)\right| && (912)\\
&= 2\Bigg(e^{\nu(t)}\left|\hat{\phi}(1)\right| + \sum_{k\geq 2} e^{\nu(t)k} k^s \left|\hat{\phi}(k)\right|\Bigg) && (913)\\
&\leq 2C_I(\|\phi\|_{\mathcal{F}^{0,1}}) \|\phi\|_{\mathcal{F}^{0,1}} \sum_{k\geq 2} \left|\hat{\phi}(k)\right| e^{\nu(t)} + 2\sum_{k\geq 2} e^{\nu(t)k} k^s \left|\hat{\phi}(k)\right| && (914)\\
&\leq 2\Bigg(C_I(\|\phi\|_{\mathcal{F}^{0,1}}) \|\phi\|_{\mathcal{F}^{0,1}} + 1\Bigg) \sum_{k\geq 2} e^{\nu(t)k} k^s \left|\hat{\phi}(k)\right|. && (915)
\end{aligned}$$

Replacing s with $s+1$, we obtain

$$\|\phi\|_{\dot{\mathcal{F}}^{s+1,1}_\nu} \leq 2\Bigg(C_I(\|\phi\|_{\mathcal{F}^{0,1}}) \|\phi\|_{\mathcal{F}^{0,1}} + 1\Bigg) \sum_{k\geq 2} e^{\nu(t)k} k^{s+1} \left|\hat{\phi}(k)\right|, \tag{916}$$

which, when rearranged, yields

$$-\sum_{k\geq 2} e^{\nu(t)k} k^{s+1} \left|\hat{\phi}(k)\right| \leq -\frac{1}{2\Bigg(C_I(\|\phi\|_{\mathcal{F}^{0,1}}) \|\phi\|_{\mathcal{F}^{0,1}} + 1\Bigg)} \|\phi\|_{\dot{\mathcal{F}}^{s+1,1}_\nu}. \tag{917}$$

Using this estimate in (899), we obtain

$$\begin{aligned}
\frac{d}{dt}\|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu} \leq{}& \nu'(t)\|\phi\|_{\dot{\mathcal{F}}^{s+1,1}_\nu} - \frac{1}{2\Bigg(C_I(\|\phi\|_{\mathcal{F}^{0,1}}) \|\phi\|_{\mathcal{F}^{0,1}} + 1\Bigg)} \|\phi\|_{\dot{\mathcal{F}}^{s+1,1}_\nu} \pi \frac{2}{R}\frac{\gamma}{4\pi} && (918)\\
&+ \frac{2\pi}{L(t)} \left\|\widetilde{\mathcal{N}}\right\|_{\dot{\mathcal{F}}^{s,1}_\nu} + 2\frac{\gamma}{4\pi}\frac{1}{R} A \|\phi\|_{\mathcal{F}^{0,1}} \sum_{k\geq 2} e^{\nu(t)k} k^{s+1} \left|\hat{\phi}(k)\right|. && (919)
\end{aligned}$$

From Proposition 7, we have

$$2\pi R A_1 \leq L(t) \leq 2\pi R A_2, \tag{920}$$

where we define

$$A_1 = \frac{1}{\sqrt{1 + \frac{\pi}{2}(e^{2\|\phi\|_{\mathcal{F}^{0,1}}} - 1)}}, \tag{921}$$

$$A_2 = \frac{1}{\sqrt{1 - \frac{\pi}{2}(e^{2\|\phi\|_{\mathcal{F}^{0,1}}} - 1)}}. \tag{922}$$

Using this estimate in (918), we obtain

$$\frac{d}{dt}\|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu} \leq \nu'(t)\|\phi\|_{\dot{\mathcal{F}}^{s+1,1}_\nu} - \frac{1}{2\Big(C_I(\|\phi\|_{\mathcal{F}^{0,1}})\|\phi\|_{\mathcal{F}^{0,1}} + 1\Big)}\|\phi\|_{\dot{\mathcal{F}}^{s+1,1}_\nu}\pi\frac{2}{R}\frac{\gamma}{4\pi} \tag{923}$$

$$+ \frac{1}{R}\frac{1}{A_1}\left\|\widetilde{\mathcal{N}}\right\|_{\dot{\mathcal{F}}^{s,1}_\nu} + 2\frac{\gamma}{4\pi}\frac{1}{R}A\|\phi\|_{\mathcal{F}^{0,1}}\sum_{k\geq 2} e^{\nu(t)k}k^{s+1}\left|\hat{\phi}(k)\right|. \tag{924}$$

By Proposition 1,

$$\frac{d}{dt}\|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu} \tag{925}$$

$$\leq\left(\nu'(t) - \frac{1}{2\Big(C_I(\|\phi\|_{\mathcal{F}^{0,1}})\|\phi\|_{\mathcal{F}^{0,1}} + 1\Big)}\pi\frac{2}{R}\frac{\gamma}{4\pi} + \frac{\gamma}{4\pi}\frac{1}{R}A\|\phi\|_{\mathcal{F}^{0,1}}\right)\|\phi\|_{\dot{\mathcal{F}}^{s+1,1}_\nu} \tag{926}$$

$$+ \frac{1}{R}\frac{1}{A_1}\left\|\widetilde{\mathcal{N}}\right\|_{\dot{\mathcal{F}}^{s,1}_\nu}. \tag{927}$$

11 Estimating $\widetilde{\mathcal{N}}$

In Section 10, we derived an *a priori* estimate containing the $\dot{\mathcal{F}}^{s,1}_\nu$ norm of $\widetilde{\mathcal{N}}$, where

$$\widetilde{\mathcal{N}}(\alpha) = (U_{\geq 2})_\alpha(\alpha) + T_{\geq 2}(\alpha)(1 + \phi_\alpha(\alpha)) + T_1(\alpha)\phi_\alpha(\alpha). \tag{928}$$

We consider the each of the three terms separately. In Sections 11.1 and 11.2, we will see that the bounds for the second the third terms depend on the $\dot{\mathcal{F}}^{s,1}_\nu$ and $\mathcal{F}^{0,1}_\nu$ norms of U_1 and $U_{\geq 2}$. In Sections 12 and 13, respectively, we will estimate these norms in terms of the corresponding norms of ϕ. Although the first term in (928) can be bounded above by the $\dot{\mathcal{F}}^{s+1,1}_\nu$ norm of $U_{\geq 2}$, the resulting estimate is not strong enough for the purposes of our study. For this reason, we will estimate it more carefully in Section 14.

11.1 Estimating $T_{\geq 2}(\alpha)(1+\phi_\alpha(\alpha))$

We prove the following estimate.

Lemma 1. *For $s \geq 1$,*

$$
\begin{aligned}
&\|T_{\geq 2}(1+\phi_\alpha)\|_{\dot{\mathcal{F}}^{s,1}_\nu} && (929)\\
\leq &\left(1+b(2,s)\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu}\right) && (930)\\
&\cdot\left(\|U_{\geq 2}\|_{\dot{\mathcal{F}}^{s-1,1}_\nu}+b(2,s-1)\left(\|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu}\|U_{\geq 1}\|_{\mathcal{F}^{0,1}_\nu}+\|U_{\geq 1}\|_{\dot{\mathcal{F}}^{s-1,1}_\nu}\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu}\right)\right) && (931)\\
&+b(2,s)\|\phi\|_{\dot{\mathcal{F}}^{s+1,1}_\nu}\left(2\|U_{\geq 2}\|_{\dot{\mathcal{F}}^{0,1}_\nu}+2\left(\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu}\|U_{\geq 1}\|_{\mathcal{F}^{0,1}_\nu}+\|\phi\|_{\mathcal{F}^{1,1}_\nu}\|U_{\geq 1}\|_{\dot{\mathcal{F}}^{0,1}_\nu}\right)\right). && (932)
\end{aligned}
$$

For $0 \leq s < 1$,

$$
\begin{aligned}
&\|T_{\geq 2}(1+\phi_\alpha)\|_{\dot{\mathcal{F}}^{s,1}_\nu} && (933)\\
\leq &\left(1+b(2,s)\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu}\right) && (934)\\
&\cdot\left(\|U_{\geq 2}\|_{\dot{\mathcal{F}}^{0,1}_\nu}+b(2,s)\left(\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu}\|U_{\geq 1}\|_{\mathcal{F}^{0,1}_\nu}+\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu}\|U_{\geq 1}\|_{\dot{\mathcal{F}}^{0,1}_\nu}\right)\right) && (935)\\
&+b(2,s)\|\phi\|_{\dot{\mathcal{F}}^{s+1,1}_\nu}\left(2\|U_{\geq 2}\|_{\dot{\mathcal{F}}^{0,1}_\nu}+2\left(\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu}\|U_{\geq 1}\|_{\mathcal{F}^{0,1}_\nu}+\|\phi\|_{\mathcal{F}^{1,1}_\nu}\|U_{\geq 1}\|_{\dot{\mathcal{F}}^{0,1}_\nu}\right)\right). && (936)
\end{aligned}
$$

Proof. Using Proposition 3, we obtain that for $s \geq 0$,

$$
\begin{aligned}
\|T_{\geq 2}(1+\phi_\alpha)\|_{\dot{\mathcal{F}}^{s,1}_\nu} &\leq \|T_{\geq 2}\|_{\dot{\mathcal{F}}^{s,1}_\nu}+\|T_{\geq 2}\phi_\alpha\|_{\dot{\mathcal{F}}^{s,1}_\nu} && (937)\\
&\leq \|T_{\geq 2}\|_{\dot{\mathcal{F}}^{s,1}_\nu}+b(2,s)\left(\|T_{\geq 2}\|_{\dot{\mathcal{F}}^{s,1}_\nu}\|\phi_\alpha\|_{\mathcal{F}^{0,1}_\nu}+\|\phi_\alpha\|_{\dot{\mathcal{F}}^{s,1}_\nu}\|T_{\geq 2}\|_{\mathcal{F}^{0,1}_\nu}\right) && (938)\\
&\leq \|T_{\geq 2}\|_{\dot{\mathcal{F}}^{s,1}_\nu}+b(2,s)\left(\|T_{\geq 2}\|_{\dot{\mathcal{F}}^{s,1}_\nu}\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu}+\|\phi\|_{\dot{\mathcal{F}}^{s+1,1}_\nu}\|T_{\geq 2}\|_{\mathcal{F}^{0,1}_\nu}\right). && (939)
\end{aligned}
$$

Note that

$$
\begin{aligned}
T_{\geq 2}(\alpha) &= \int_0^\alpha U_{\geq 2}(\eta)d\eta+\int_0^\alpha \phi_\alpha(\eta)U_{\geq 1}(\eta)d\eta && (940)\\
&\quad -\frac{\alpha}{2\pi}\int_{-\pi}^{\pi}U_{\geq 2}(\eta)d\eta-\frac{\alpha}{2\pi}\int_{-\pi}^{\pi}\phi_\alpha(\eta)U_{\geq 1}(\eta)d\eta && (941)\\
&= \mathcal{M}(U_{\geq 2})(\alpha)+\mathcal{M}(\phi_\alpha U_{\geq 1})(\alpha). && (942)
\end{aligned}
$$

Hence for $s \geq 1$, using Proposition 3, we obtain

$$\begin{aligned}
\|T_{\geq 2}\|_{\dot{\mathcal{F}}^{s,1}_\nu} &= \sum_{k\neq 0} e^{\nu(t)|k|}|k|^s |\mathcal{F}(T_{\geq 2})(k)| && (943)\\
&\leq \sum_{k\neq 0} e^{\nu(t)|k|}|k|^s |\mathcal{F}(\mathcal{M}(U_{\geq 2}))(k)| && (944)\\
&\quad + \sum_{k\neq 0} e^{\nu(t)|k|}|k|^s |\mathcal{F}(\mathcal{M}(\phi_\alpha U_{\geq 1}))(k)| && (945)\\
&= \sum_{k\neq 0} e^{\nu(t)|k|}|k|^{s-1} |\mathcal{F}(U_{\geq 2})(k)| && (946)\\
&\quad + \sum_{k\neq 0} e^{\nu(t)|k|}|k|^{s-1} |\mathcal{F}(\phi_\alpha U_{\geq 1})(k)| && (947)\\
&= \|U_{\geq 2}\|_{\dot{\mathcal{F}}^{s-1,1}_\nu} + \|\phi_\alpha U_{\geq 1}\|_{\dot{\mathcal{F}}^{s-1,1}_\nu} && (948)\\
&\leq \|U_{\geq 2}\|_{\dot{\mathcal{F}}^{s-1,1}_\nu} + b(2,s-1)(\|\phi_\alpha\|_{\dot{\mathcal{F}}^{s-1,1}_\nu}\|U_{\geq 1}\|_{\mathcal{F}^{0,1}_\nu} + \|U_{\geq 1}\|_{\dot{\mathcal{F}}^{s-1,1}_\nu}\|\phi_\alpha\|_{\mathcal{F}^{0,1}_\nu}) && (949)\\
&= \|U_{\geq 2}\|_{\dot{\mathcal{F}}^{s-1,1}_\nu} + b(2,s-1)(\|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu}\|U_{\geq 1}\|_{\mathcal{F}^{0,1}_\nu} + \|U_{\geq 1}\|_{\dot{\mathcal{F}}^{s-1,1}_\nu}\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu}). && (950)
\end{aligned}$$

Moreover,

$$\begin{aligned}
\|T_{\geq 2}\|_{\mathcal{F}^{0,1}_\nu} &= \sum_{k\in\mathbb{Z}} e^{\nu(t)|k|} |\mathcal{F}(T_{\geq 2})(k)| && (951)\\
&= \sum_{k\in\mathbb{Z}} e^{\nu(t)|k|} |\mathcal{F}(\mathcal{M}(U_{\geq 2}))(k)| + \sum_{k\in\mathbb{Z}} e^{\nu(t)|k|} |\mathcal{F}(\mathcal{M}(\phi_\alpha U_{\geq 1}))(k)| && (952)\\
&= |\mathcal{F}(\mathcal{M}(U_{\geq 2}))(0)| + |\mathcal{F}(\mathcal{M}(\phi_\alpha U_{\geq 1}))(0)| && (953)\\
&\quad + \sum_{k\neq 0} e^{\nu(t)|k|} |\mathcal{F}(\mathcal{M}(U_{\geq 2}))(k)| + \sum_{k\neq 0} e^{\nu(t)|k|} |\mathcal{F}(\mathcal{M}(\phi_\alpha U_{\geq 1}))(k)| && (954)\\
&\leq |\mathcal{F}(\mathcal{M}(U_{\geq 2}))(0)| + |\mathcal{F}(\mathcal{M}(\phi_\alpha U_{\geq 1}))(0)| && (955)\\
&\quad + \sum_{k\neq 0} e^{\nu(t)|k|} |\mathcal{F}(U_{\geq 2})(k)| + \sum_{k\neq 0} e^{\nu(t)|k|} |\mathcal{F}(\phi_\alpha U_{\geq 1})(k)| && (956)\\
&\leq \sum_{j\neq 0} |j|^{-1} |\mathcal{F}(U_{\geq 2})(j)| + \sum_{j\neq 0} |j|^{-1} |\mathcal{F}(\phi_\alpha U_{\geq 1})(j)| && (957)\\
&\quad + \sum_{k\neq 0} e^{\nu(t)|k|} |\mathcal{F}(U_{\geq 2})(k)| + \sum_{k\neq 0} e^{\nu(t)|k|} |\mathcal{F}(\phi_\alpha U_{\geq 1})(k)| && (958)\\
&\leq \sum_{j\neq 0} e^{\nu(t)|j|} |\mathcal{F}(U_{\geq 2})(j)| + \sum_{j\neq 0} e^{\nu(t)|j|} |\mathcal{F}(\phi_\alpha U_{\geq 1})(j)| && (959)\\
&\quad + \sum_{k\neq 0} e^{\nu(t)|k|} |\mathcal{F}(U_{\geq 2})(k)| + \sum_{k\neq 0} e^{\nu(t)|k|} |\mathcal{F}(\phi_\alpha U_{\geq 1})(k)| && (960)\\
&= 2\|U_{\geq 2}\|_{\dot{\mathcal{F}}^{0,1}_\nu} + 2\|\phi_\alpha U_{\geq 1}\|_{\dot{\mathcal{F}}^{0,1}_\nu}. && (961)
\end{aligned}$$

Using Proposition 3, we obtain that

$$\|T_{\geq 2}\|_{\mathcal{F}^{0,1}_\nu} \leq 2\|U_{\geq 2}\|_{\dot{\mathcal{F}}^{0,1}_\nu} + 2\|\phi_\alpha U_{\geq 1}\|_{\dot{\mathcal{F}}^{0,1}_\nu} \tag{962}$$
$$\leq 2\|U_{\geq 2}\|_{\dot{\mathcal{F}}^{0,1}_\nu} + 2(\|\phi_\alpha\|_{\dot{\mathcal{F}}^{0,1}_\nu}\|U_{\geq 1}\|_{\mathcal{F}^{0,1}_\nu} + \|\phi_\alpha\|_{\mathcal{F}^{0,1}_\nu}\|U_{\geq 1}\|_{\dot{\mathcal{F}}^{0,1}_\nu}) \tag{963}$$
$$= 2\|U_{\geq 2}\|_{\dot{\mathcal{F}}^{0,1}_\nu} + 2(\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu}\|U_{\geq 1}\|_{\mathcal{F}^{0,1}_\nu} + \|\phi\|_{\mathcal{F}^{1,1}_\nu}\|U_{\geq 1}\|_{\dot{\mathcal{F}}^{0,1}_\nu}). \tag{964}$$

Now, let us consider the case in which $0 \leq s < 1$. Then

$$\|T_{\geq 2}\|_{\dot{\mathcal{F}}^{s,1}_\nu} = \sum_{k\neq 0} e^{\nu(t)|k|}|k|^s|\mathcal{F}(T_{\geq 2})(k)| \tag{965}$$
$$\leq \sum_{k\neq 0} e^{\nu(t)|k|}|k|^s|\mathcal{F}(\mathcal{M}(U_{\geq 2}))(k)| \tag{966}$$
$$+ \sum_{k\neq 0} e^{\nu(t)|k|}|k|^s|\mathcal{F}(\mathcal{M}(\phi_\alpha U_{\geq 1}))(k)| \tag{967}$$
$$= \sum_{k\neq 0} e^{\nu(t)|k|}|k|^{s-1}|\mathcal{F}(U_{\geq 2})(k)| \tag{968}$$
$$+ \sum_{k\neq 0} e^{\nu(t)|k|}|k|^{s-1}|\mathcal{F}(\phi_\alpha U_{\geq 1})(k)| \tag{969}$$
$$\leq \sum_{k\neq 0} e^{\nu(t)|k|}|\mathcal{F}(U_{\geq 2})(k)| \tag{970}$$
$$+ \sum_{k\neq 0} e^{\nu(t)|k|}|\mathcal{F}(\phi_\alpha U_{\geq 1})(k)| \tag{971}$$
$$= \|U_{\geq 2}\|_{\dot{\mathcal{F}}^{0,1}_\nu} + \|\phi_\alpha U_{\geq 1}\|_{\dot{\mathcal{F}}^{0,1}_\nu}. \tag{972}$$

Using Proposition 3, we obtain that

$$\|T_{\geq 2}\|_{\dot{\mathcal{F}}^{s,1}_\nu} \leq \|U_{\geq 2}\|_{\dot{\mathcal{F}}^{0,1}_\nu} + \|\phi_\alpha U_{\geq 1}\|_{\dot{\mathcal{F}}^{0,1}_\nu} \tag{973}$$
$$\leq \|U_{\geq 2}\|_{\dot{\mathcal{F}}^{0,1}_\nu} + b(2,s)\left(\|\phi_\alpha\|_{\dot{\mathcal{F}}^{0,1}_\nu}\|U_{\geq 1}\|_{\mathcal{F}^{0,1}_\nu} + \|\phi_\alpha\|_{\mathcal{F}^{0,1}_\nu}\|U_{\geq 1}\|_{\dot{\mathcal{F}}^{0,1}_\nu}\right) \tag{974}$$
$$\leq \|U_{\geq 2}\|_{\dot{\mathcal{F}}^{0,1}_\nu} + b(2,s)\left(\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu}\|U_{\geq 1}\|_{\mathcal{F}^{0,1}_\nu} + \|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu}\|U_{\geq 1}\|_{\dot{\mathcal{F}}^{0,1}_\nu}\right). \tag{975}$$

■

11.2 Estimating $T_1(\alpha)\phi_\alpha(\alpha)$

We prove the following estimate.

Lemma 2. *For $s \geq 1$,*

$$\|T_1\phi_\alpha\|_{\dot{\mathcal{F}}^{s,1}_\nu} \leq b(2,s)\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu}\|U_1\|_{\dot{\mathcal{F}}^{s-1,1}_\nu} + b(2,s)\|\phi\|_{\dot{\mathcal{F}}^{s+1,1}_\nu} 2\|U_1\|_{\dot{\mathcal{F}}^{0,1}_\nu}. \tag{976}$$

For $0 \leq s < 1$,

$$\|T_1\phi_\alpha\|_{\dot{\mathcal{F}}^{s,1}_\nu} \leq b(2,s)\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu}\|U_1\|_{\dot{\mathcal{F}}^{0,1}_\nu} + b(2,s)\|\phi\|_{\dot{\mathcal{F}}^{s+1,1}_\nu} 2\|U_1\|_{\dot{\mathcal{F}}^{0,1}_\nu}. \tag{977}$$

Proof. Using Proposition 3, we obtain that for $s \geq 0$,

$$\|T_1\phi_\alpha\|_{\dot{\mathcal{F}}_\nu^{s,1}} \leq b(2,s)\bigg(\|T_1\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi_\alpha\|_{\mathcal{F}_\nu^{0,1}} + \|\phi_\alpha\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|T_1\|_{\mathcal{F}_\nu^{0,1}}\bigg) \tag{978}$$

$$= b(2,s)\bigg(\|T_1\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}} + \|\phi\|_{\dot{\mathcal{F}}_\nu^{s+1,1}}\|T_1\|_{\mathcal{F}_\nu^{0,1}}\bigg). \tag{979}$$

Recall that $T_1(\alpha) = \mathcal{M}(U_1)(\alpha)$. Then for $s \geq 1$,

$$\|T_1\|_{\dot{\mathcal{F}}_\nu^{s,1}} = \sum_{k\neq 0} e^{\nu(t)|k|}|k|^s|\mathcal{F}(T_1)(k)| \tag{980}$$

$$= \sum_{k\neq 0} e^{\nu(t)|k|}|k|^s|\mathcal{F}(\mathcal{M}(U_1))(k)| \tag{981}$$

$$= \sum_{k\neq 0} e^{\nu(t)|k|}|k|^{s-1}|\mathcal{F}(U_1)(k)| \tag{982}$$

$$= \|U_1\|_{\dot{\mathcal{F}}_\nu^{s-1,1}}. \tag{983}$$

Moreover,

$$\|T_1\|_{\mathcal{F}_\nu^{0,1}} = \sum_{k\in\mathbb{Z}} e^{\nu(t)|k|}|\mathcal{F}(T_1)(k)| \tag{984}$$

$$= \sum_{k\in\mathbb{Z}} e^{\nu(t)|k|}|\mathcal{F}(\mathcal{M}(U_1))(k)| \tag{985}$$

$$= |\mathcal{F}(\mathcal{M}(U_1))(0)| + \sum_{k\neq 0} e^{\nu(t)|k|}|\mathcal{F}(\mathcal{M}(U_1))(k)| \tag{986}$$

$$= \left|\sum_{j\neq 0}\frac{i}{j}\mathcal{F}(U_1)(j)\right| + \sum_{k\neq 0} e^{\nu(t)|k|}|k|^{-1}|\mathcal{F}(U_1)(k)| \tag{987}$$

$$\leq \sum_{j\neq 0} e^{\nu(t)|j|}|\mathcal{F}(U_1)(j)| + \sum_{k\neq 0} e^{\nu(t)|k|}|\mathcal{F}(U_1)(k)| \tag{988}$$

$$= 2\|U_1\|_{\dot{\mathcal{F}}_\nu^{0,1}}. \tag{989}$$

Now, let us consider the case in which $0 \leq s < 1$. Then

$$\|T_1\|_{\dot{\mathcal{F}}_\nu^{s,1}} = \sum_{k\neq 0} e^{\nu(t)|k|}|k|^s|\mathcal{F}(T_1)(k)| \tag{990}$$

$$= \sum_{k\neq 0} e^{\nu(t)|k|}|k|^s|\mathcal{F}(\mathcal{M}(U_1))(k)| \tag{991}$$

$$= \sum_{k\neq 0} e^{\nu(t)|k|}|k|^{s-1}|\mathcal{F}(U_1)(k)| \tag{992}$$

$$\leq \|U_1\|_{\dot{\mathcal{F}}_\nu^{0,1}}. \tag{993}$$

$\blacksquare$

12 Estimating U_1

To estimate the $\dot{\mathcal{F}}_{\nu}^{s,1}$ and $\mathcal{F}_{\nu}^{0,1}$ norms of U_1, we first estimate the Fourier modes of U_1.

12.1 Estimating Fourier Modes of U_1

For any norm $\|\cdot\|$, we can estimate (327) as

$$\|U_1\| \leq \left\| ie^{i\alpha}e^{i\hat{\theta}(0)}\mathfrak{L}(\alpha)\right\|. \tag{994}$$

To estimate the $\dot{\mathcal{F}}_{\nu}^{s,1}$ and $\mathcal{F}_{\nu}^{0,1}$ norms of (994), we can write

$$ie^{i\alpha}e^{i\hat{\theta}(0)}\mathfrak{L}(\alpha) = \sum_{j=1}^{7} \frac{\gamma}{4\pi}\int_{-\pi}^{\pi} E_j(\alpha,\beta)d\beta, \tag{995}$$

where

$$E_1(\alpha,\beta) = \frac{-e^{i\beta}(-1+e^{i\beta})(i(-1+e^{i\beta})+\beta(1+e^{i\beta}))}{2(-1+e^{i\beta})^2}\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))ds, \tag{996}$$

$$E_2(\alpha,\beta) = \frac{i(-1-2i\beta+e^{2i\beta})}{2(-1+e^{i\beta})^2}\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))ds, \tag{997}$$

$$E_3(\alpha,\beta) = \frac{-(-1+e^{i\beta})\beta e^{i\beta}(-1+e^{i\beta})}{2(-1+e^{i\beta})^2}\int_0^1 e^{-i\beta s}(-1+s)\phi(\alpha+\beta(-1+s))ds, \tag{998}$$

$$E_4(\alpha,\beta) = \frac{-(-1+e^{i\beta})\beta(1+e^{i\beta})}{2(-1+e^{i\beta})^2}\int_0^1 e^{i\beta s}(-1+s)\phi(\alpha+\beta(-1+s))ds, \tag{999}$$

$$E_5(\alpha,\beta) = \frac{-(-1+e^{i\beta})i\beta e^{i\beta}(-1+e^{i\beta})}{2(-1+e^{i\beta})^2}\int_0^1 e^{-i\beta s}(-1+s)\phi'(\alpha+\beta(-1+s))ds, \tag{1000}$$

$$E_6(\alpha,\beta) = \frac{-(-1+e^{i\beta})i(-\beta)(1+e^{i\beta})}{2(-1+e^{i\beta})^2}\int_0^1 e^{i\beta s}(-1+s)\phi'(\alpha+\beta(-1+s))ds, \tag{1001}$$

$$E_7(\alpha,\beta) = \frac{-(-1+e^{i\beta})i(-1+2e^{i\beta}+e^{2i\beta})}{2(-1+e^{i\beta})^2}\phi(\alpha-\beta). \tag{1002}$$

First, we compute the Fourier modes of $E_1(\alpha,\beta)$.

$$\mathcal{F}(E_1)(k,\beta) = \frac{-e^{i\beta}(i(-1+e^{i\beta})+\beta(1+e^{i\beta}))}{2(-1+e^{i\beta})}\cdot\int_0^1 e^{-i\beta s}e^{ik\beta(-1+s)}ds\cdot\mathcal{F}(\phi)(k). \tag{1003}$$

Since

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\frac{-e^{i\beta}}{2(-1+e^{i\beta})}(i(-1+e^{i\beta})+\beta(1+e^{i\beta}))\int_0^1 e^{-i\beta s}e^{ik\beta(-1+s)}dsd\beta\right| \tag{1004}$$

$$=\left|\frac{\gamma}{4\pi}\left(\int_{-\pi}^{\pi}\frac{-ie^{i\beta}}{2}\int_0^1 e^{-i\beta s}e^{ik\beta(-1+s)}dsd\beta\right.\right. \tag{1005}$$

$$\left.\left.+\int_{-\pi}^{\pi}\frac{-\beta e^{i\beta}(1+e^{i\beta})}{2(-1+e^{i\beta})}\int_0^1 e^{-i\beta s}e^{ik\beta(-1+s)}dsd\beta\right)\right| \tag{1006}$$

$$=\left|\frac{\gamma}{4\pi}\left(\int_0^1\int_{-\pi}^{\pi}\frac{-ie^{i\beta}}{2}e^{-i\beta s}e^{ik\beta(-1+s)}d\beta ds\right.\right. \tag{1007}$$

$$\left.\left.+\int_0^1\int_{-\pi}^{\pi}\frac{-\beta e^{i\beta}(1+e^{i\beta})}{2(-1+e^{i\beta})}e^{-i\beta s}e^{ik\beta(-1+s)}d\beta ds\right)\right| \tag{1008}$$

$$\leq\frac{\gamma}{4\pi}\left(\int_0^1\int_{-\pi}^{\pi}\frac{1}{2}d\beta ds\right. \tag{1009}$$

$$\left.+\left|\int_0^1\int_{-\pi}^{\pi}\left(\frac{i\beta}{-1+e^{i\beta}}-1+1\right)\frac{ie^{i\beta}(1+e^{i\beta})}{2}e^{-i\beta s}e^{ik\beta(-1+s)}d\beta ds\right|\right) \tag{1010}$$

$$\leq\frac{\gamma}{4\pi}\left(\pi+\int_0^1\int_{-\pi}^{\pi}\left|\frac{i\beta}{1-e^{i\beta}}-1\right|\cdot\frac{1}{2}d\beta ds+\int_0^1\int_{-\pi}^{\pi}\frac{1}{2}d\beta ds\right) \tag{1011}$$

$$=\frac{\gamma}{4\pi}\left(\pi+\int_0^1\int_{-\pi}^{\pi}|\beta|\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}\cdot\frac{1}{2}d\beta ds+\pi\right) \tag{1012}$$

$$=\frac{\gamma}{4\pi}\left(2\pi+\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}\cdot\frac{1}{2}\pi^2\right), \tag{1013}$$

we obtain

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(E_1)(k,\beta)d\beta\right|\leq\frac{\gamma}{4\pi}\left(2\pi+\frac{\pi^2}{4}\sqrt{1+\frac{\pi^2}{4}}\right)|\mathcal{F}(\phi)(k)|\,. \tag{1014}$$

Next, we compute the Fourier modes of $E_2(\alpha,\beta)$.

$$\mathcal{F}(E_2)(k,\beta)=\frac{i(-1-2i\beta+e^{2i\beta})}{2(-1+e^{i\beta})^2}\mathcal{F}(\phi)(k)\int_0^1 e^{i\beta s}e^{ik\beta(-1+s)}ds. \tag{1015}$$

Since

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\frac{i(-1-2i\beta+e^{2i\beta})}{2(-1+e^{i\beta})^2}\int_0^1 e^{i\beta s}e^{ik\beta(-1+s)}dsd\beta\right| \tag{1016}$$

$$=\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\frac{i(-1+e^{2i\beta})}{2(-1+e^{i\beta})^2}\int_0^1 e^{i\beta s}e^{ik\beta(-1+s)}dsd\beta\right. \tag{1017}$$

$$\left.+\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\frac{i(-2i\beta)}{2(-1+e^{i\beta})^2}\int_0^1 e^{i\beta s}e^{ik\beta(-1+s)}dsd\beta\right| \tag{1018}$$

$$=\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\frac{i(e^{i\beta}+1)}{2(-1+e^{i\beta})}\int_0^1 e^{i\beta s}e^{ik\beta(-1+s)}dsd\beta\right. \tag{1019}$$

$$\left.+\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\frac{i(-2i\beta)\beta}{2(-1+e^{i\beta})^2}\frac{1}{\beta}\int_0^1 e^{i\beta s}e^{ik\beta(-1+s)}dsd\beta\right| \tag{1020}$$

$$=\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\frac{i\beta(e^{i\beta}+1)}{2(-1+e^{i\beta})}\frac{1}{\beta}\int_0^1 e^{i\beta s}e^{ik\beta(-1+s)}dsd\beta\right. \tag{1021}$$

$$\left.+\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\left(\frac{i\beta}{1-e^{i\beta}}\right)^2\frac{-1}{\beta}\int_0^1 e^{i\beta s}e^{ik\beta(-1+s)}dsd\beta\right| \tag{1022}$$

$$\leq\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\left|\frac{i\beta}{1-e^{i\beta}}-1+1\right|\left|\frac{e^{i\beta}+1}{-2}\right|\frac{1}{|\beta|}d\beta+\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\left|\left(\frac{i\beta}{1-e^{i\beta}}\right)^2-1+1\right|\frac{1}{|\beta|}d\beta \tag{1023}$$

$$\leq\frac{\gamma}{4\pi}\left(\int_{-\pi}^{\pi}\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}d\beta+\int_{-\pi}^{\pi}\frac{1}{|\beta|}d\beta\right) \tag{1024}$$

$$+\frac{\gamma}{4\pi}\left(\int_{-\pi}^{\pi}2\cdot\frac{\pi}{2}\cdot\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}d\beta+\int_{-\pi}^{\pi}\frac{1}{|\beta|}d\beta\right) \tag{1025}$$

$$=\frac{\gamma}{4\pi}\left(\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}\cdot 2\pi+\pi^2\right)+\frac{\gamma}{4\pi}\left(2\cdot\frac{\pi}{2}\cdot\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}\cdot 2\pi+\pi^2\right), \tag{1026}$$

we obtain

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(E_2)(k,\beta)d\beta\right| \tag{1027}$$

$$\leq\left(\frac{\gamma}{4\pi}\left(\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}\cdot 2\pi+\pi^2\right)+\frac{\gamma}{4\pi}\left(2\cdot\frac{\pi}{2}\cdot\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}\cdot 2\pi+\pi^2\right)\right)|\mathcal{F}(\phi)(k)|. \tag{1028}$$

Next, we compute the Fourier modes of $E_3(\alpha,\beta)$.

$$\mathcal{F}(E_3)(k,\beta)=\frac{-(-1+e^{i\beta})\beta e^{i\beta}(-1+e^{i\beta})}{2(-1+e^{i\beta})^2}\cdot\int_0^1 e^{-i\beta s}(-1+s)e^{ik\beta(-1+s)}ds\cdot\mathcal{F}(\phi)(k). \tag{1029}$$

Since

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\frac{-(-1+e^{i\beta})\beta e^{i\beta}(-1+e^{i\beta})}{2(-1+e^{i\beta})^2}\cdot\int_0^1 e^{-i\beta s}(-1+s)e^{ik\beta(-1+s)}dsd\beta\right| \tag{1030}$$

$$\leq\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\frac{|\beta|}{2}d\beta=\frac{\gamma}{4\pi}\cdot\frac{\pi^2}{2}, \tag{1031}$$

we obtain

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(E_3)(k,\beta)d\beta\right| \leq \frac{\gamma}{4\pi}\cdot\frac{\pi^2}{2}\left|\mathcal{F}(\phi)(k)\right|. \tag{1032}$$

Next, we compute the Fourier modes of $E_4(\alpha,\beta)$.

$$\mathcal{F}(E_4)(k,\beta) = \frac{-(-1+e^{i\beta})\beta(1+e^{i\beta})}{2(-1+e^{i\beta})^2}\cdot\int_0^1 e^{i\beta s}(-1+s)e^{ik\beta(-1+s)}ds\cdot\mathcal{F}(\phi)(k). \tag{1033}$$

Since

$$\begin{aligned}
&\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\frac{-(-1+e^{i\beta})\beta(1+e^{i\beta})}{2(-1+e^{i\beta})^2}\cdot\int_0^1 e^{i\beta s}(-1+s)e^{ik\beta(-1+s)}dsd\beta\right| &(1034)\\
\leq&\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\left|\frac{i\beta}{1-e^{i\beta}}-1+1\right|\left|\frac{i(1+e^{i\beta})}{-2}\right|d\beta &(1035)\\
\leq&\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}|\beta|\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}d\beta+\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}d\beta &(1036)\\
=&\frac{\gamma}{4\pi}\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}\cdot\pi^2+\frac{\gamma}{4\pi}\cdot 2\pi, &(1037)
\end{aligned}$$

we obtain

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(E_4)(k,\beta)d\beta\right| \leq \frac{\gamma}{4\pi}\left(\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}\cdot\pi^2+2\pi\right)\left|\mathcal{F}(\phi)(k)\right|. \tag{1038}$$

Next, we compute the Fourier modes of $E_5(\alpha,\beta)$.

$$\begin{aligned}
&\mathcal{F}(E_5)(k,\beta) &(1039)\\
=&\frac{-(-1+e^{i\beta})i\beta e^{i\beta}(-1+e^{i\beta})}{2(-1+e^{i\beta})^2}\cdot\int_0^1 e^{-i\beta s}(-1+s)e^{ik\beta(-1+s)}ds\cdot ik\mathcal{F}(\phi)(k). &(1040)
\end{aligned}$$

Since

$$\begin{aligned}
&\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\frac{-(-1+e^{i\beta})i\beta e^{i\beta}(-1+e^{i\beta})}{2(-1+e^{i\beta})^2}\cdot\int_0^1 e^{-i\beta s}(-1+s)e^{ik\beta(-1+s)}dsd\beta\right| &(1041)\\
\leq&\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\frac{|\beta|}{2}d\beta=\frac{\gamma}{4\pi}\cdot\frac{\pi^2}{2}, &(1042)
\end{aligned}$$

we obtain

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(E_5)(k,\beta)d\beta\right| \leq \frac{\gamma}{4\pi}\cdot\frac{\pi^2}{2}\left|k\right|\cdot\left|\mathcal{F}(\phi)(k)\right|. \tag{1043}$$

Next, we compute the Fourier modes of $E_6(\alpha,\beta)$.

$$\begin{aligned}
&\mathcal{F}(E_6)(k,\beta) &(1044)\\
=&\frac{-(-1+e^{i\beta})i(-\beta)(1+e^{i\beta})}{2(-1+e^{i\beta})^2}\cdot\int_0^1 e^{i\beta s}(-1+s)e^{ik\beta(-1+s)}ds\cdot ik\mathcal{F}(\phi)(k). &(1045)
\end{aligned}$$

Since

$$\begin{aligned}
&\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\frac{-(-1+e^{i\beta})i(-\beta)(1+e^{i\beta})}{2(-1+e^{i\beta})^2}\cdot\int_0^1 e^{i\beta s}(-1+s)e^{ik\beta(-1+s)}dsd\beta\right| && (1046)\\
=&\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\left(\frac{i\beta}{1-e^{i\beta}}-1+1\right)\frac{1+e^{i\beta}}{-2}\int_0^1 e^{i\beta s}(-1+s)e^{ik\beta(-1+s)}dsd\beta\right| && (1047)\\
\leq&\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\left|\frac{i\beta}{1-e^{i\beta}}-1+1\right|d\beta && (1048)\\
\leq&\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}|\beta|\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}d\beta+\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}d\beta && (1049)\\
=&\frac{\gamma}{4\pi}\left(\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}\cdot\pi^2+2\pi\right), && (1050)
\end{aligned}$$

we obtain

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(E_6)(k,\beta)d\beta\right|\leq\frac{\gamma}{4\pi}\left(\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}\cdot\pi^2+2\pi\right)|k|\left|\mathcal{F}(\phi)(k)\right|. \tag{1051}$$

Lastly, we compute the Fourier modes of $E_7(\alpha,\beta)$.

$$\mathcal{F}(E_7)(k,\beta)=\frac{-(-1+e^{i\beta})i(-1+2e^{i\beta}+e^{2i\beta})}{2(-1+e^{i\beta})^2}e^{-ik\beta}\mathcal{F}(\phi)(k). \tag{1052}$$

Since

$$\begin{aligned}
&\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\frac{-(-1+e^{i\beta})i(-1+2e^{i\beta}+e^{2i\beta})}{2(-1+e^{i\beta})^2}e^{-ik\beta}d\beta\right| && (1053)\\
=&\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\left(\frac{i\beta}{1-e^{i\beta}}-1+1\right)\frac{(-1+2e^{i\beta}+e^{2i\beta})e^{-ik\beta}}{2\beta}d\beta\right| && (1054)\\
\leq&\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}|\beta|\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}\cdot\frac{3}{2|\beta|}d\beta+\frac{\gamma}{4\pi}\left|\int_{-\pi}^{\pi}\frac{e^{-ik\beta}(-1+2e^{i\beta}+e^{2i\beta})}{2\beta}d\beta\right| && (1055)\\
\leq&\frac{\gamma}{4\pi}\cdot\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}\cdot\frac{3}{2}\cdot2\pi+\frac{\gamma}{4\pi}\cdot\frac{1}{2}\left|\int_{-\pi}^{\pi}\frac{e^{-ik\beta}(-1+2e^{i\beta}+e^{2i\beta})}{\beta}d\beta\right| && (1056)\\
\leq&\frac{\gamma}{4\pi}\cdot\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}\cdot\frac{3}{2}\cdot2\pi+\frac{\gamma}{4\pi}\cdot\frac{1}{2}\cdot4\cdot5, && (1057)
\end{aligned}$$

we obtain

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(E_7)(k,\beta)d\beta\right|\leq\frac{\gamma}{4\pi}\left(\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}\cdot\frac{3}{2}\cdot2\pi+\frac{1}{2}\cdot4\cdot5\right)\left|\mathcal{F}(\phi)(k)\right|. \tag{1058}$$

12.2 Estimating $\|U_1\|_{\mathcal{F}_\nu^{0,1}}$

In Section 12.1, we observed that

$$\|U_1\|_{\mathcal{F}_\nu^{0,1}}\leq\sum_{j=1}^{7}\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}E_j(\alpha,\beta)d\beta\right\|_{\mathcal{F}_\nu^{0,1}}. \tag{1059}$$

Since

$$\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}E_1(\alpha,\beta)d\beta\right\|_{\mathcal{F}_\nu^{0,1}} =\sum_{k\in\mathbb{Z}}e^{\nu(t)|k|}\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(E_1)(k,\beta)d\beta\right| \tag{1060}$$

$$\leq\frac{\gamma}{4\pi}\left(2\pi+\frac{\pi^2}{4}\sqrt{1+\frac{\pi^2}{4}}\right)\|\phi\|_{\mathcal{F}_\nu^{0,1}} \tag{1061}$$

$$\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}E_2(\alpha,\beta)d\beta\right\|_{\mathcal{F}_\nu^{0,1}} =\sum_{k\in\mathbb{Z}}e^{\nu(t)|k|}\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(E_2)(k,\beta)d\beta\right| \tag{1062}$$

$$\leq\left(\frac{\gamma}{4\pi}\left(\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}\cdot 2\pi+\pi^2\right)\right. \tag{1063}$$

$$\left.+\frac{\gamma}{4\pi}\left(2\cdot\frac{\pi}{2}\cdot\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}\cdot 2\pi+\pi^2\right)\right)\|\phi\|_{\mathcal{F}_\nu^{0,1}} \tag{1064}$$

$$\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}E_3(\alpha,\beta)d\beta\right\|_{\mathcal{F}_\nu^{0,1}} =\sum_{k\in\mathbb{Z}}e^{\nu(t)|k|}\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(E_3)(k,\beta)d\beta\right| \tag{1065}$$

$$\leq\frac{\gamma}{4\pi}\cdot\frac{\pi^2}{2}\|\phi\|_{\mathcal{F}_\nu^{0,1}} \tag{1066}$$

$$\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}E_4(\alpha,\beta)d\beta\right\|_{\mathcal{F}_\nu^{0,1}} =\sum_{k\in\mathbb{Z}}e^{\nu(t)|k|}\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(E_4)(k,\beta)d\beta\right| \tag{1067}$$

$$\leq\frac{\gamma}{4\pi}\left(\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}\cdot\pi^2+2\pi\right)\|\phi\|_{\mathcal{F}_\nu^{0,1}} \tag{1068}$$

$$\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}E_5(\alpha,\beta)d\beta\right\|_{\mathcal{F}_\nu^{0,1}} =\sum_{k\in\mathbb{Z}}e^{\nu(t)|k|}\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(E_5)(k,\beta)d\beta\right| \tag{1069}$$

$$\leq\frac{\gamma}{4\pi}\cdot\frac{\pi^2}{2}\|\phi\|_{\mathcal{F}_\nu^{1,1}} \tag{1070}$$

$$\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}E_6(\alpha,\beta)d\beta\right\|_{\mathcal{F}_\nu^{0,1}} =\sum_{k\in\mathbb{Z}}e^{\nu(t)|k|}\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(E_6)(k,\beta)d\beta\right| \tag{1071}$$

$$\leq\frac{\gamma}{4\pi}\left(\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}\cdot\pi^2+2\pi\right)\|\phi\|_{\mathcal{F}_\nu^{1,1}} \tag{1072}$$

$$\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}E_7(\alpha,\beta)d\beta\right\|_{\mathcal{F}_\nu^{0,1}} =\sum_{k\in\mathbb{Z}}e^{\nu(t)|k|}\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(E_7)(k,\beta)d\beta\right| \tag{1073}$$

$$\leq\frac{\gamma}{4\pi}\left(\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}\cdot\frac{3}{2}\cdot 2\pi+\frac{1}{2}\cdot 4\cdot 5\right)\|\phi\|_{\mathcal{F}_\nu^{0,1}}, \tag{1074}$$

we obtain

$$\|U_1\|_{\mathcal{F}_\nu^{0,1}}\leq H_3\|\phi\|_{\mathcal{F}_\nu^{0,1}}+H_4\|\phi\|_{\mathcal{F}_\nu^{1,1}}, \tag{1075}$$

where H_3 and H_4 are constants.

12.3 Estimating $\|U_1\|_{\dot{\mathcal{F}}_\nu^{s,1}}$

In Section 12.1, we observed that

$$\|U_1\|_{\dot{\mathcal{F}}_\nu^{s,1}} \leq \sum_{j=1}^{7} \left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} E_j(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}}. \tag{1076}$$

Since

$$\begin{aligned}
\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} E_1(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \leq & \frac{\gamma}{4\pi}\left(2\pi + \frac{\pi^2}{4}\sqrt{1+\frac{\pi^2}{4}}\right) \sum_{k\neq 0} e^{\nu(t)|k|} |k|^s |\mathcal{F}(\phi)(k)| && (1077)\\
\leq & \frac{\gamma}{4\pi}\left(2\pi + \frac{\pi^2}{4}\sqrt{1+\frac{\pi^2}{4}}\right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} && (1078)\\
\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} E_2(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \leq & \Bigg(\frac{\gamma}{4\pi}\left(\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}} \cdot 2\pi + \pi^2\right) && (1079)\\
& + \frac{\gamma}{4\pi}\left(2\cdot\frac{\pi}{2}\cdot\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}} \cdot 2\pi + \pi^2\right)\Bigg) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} && (1080)\\
\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} E_3(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \leq & \frac{\gamma}{4\pi}\cdot\frac{\pi^2}{2} \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} && (1081)\\
\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} E_4(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \leq & \frac{\gamma}{4\pi}\left(\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}} \cdot \pi^2 + 2\pi\right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} && (1082)\\
\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} E_5(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \leq & \frac{\gamma}{4\pi}\cdot\frac{\pi^2}{2} \|\phi\|_{\dot{\mathcal{F}}_\nu^{s+1,1}} && (1083)\\
\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} E_6(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \leq & \frac{\gamma}{4\pi}\left(\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}} \cdot \pi^2 + 2\pi\right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s+1,1}} && (1084)\\
\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} E_7(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \leq & \frac{\gamma}{4\pi}\left(\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}} \cdot \frac{3}{2} \cdot 2\pi + \frac{1}{2}\cdot 4 \cdot 5\right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}, && (1085)
\end{aligned}$$

we obtain

$$\|U_1\|_{\dot{\mathcal{F}}_\nu^{s,1}} \leq H_1 \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} + H_2 \|\phi\|_{\dot{\mathcal{F}}_\nu^{s+1,1}}, \tag{1086}$$

where H_1 and H_2 are constants.

13 Estimating $U_{\geq 2}$

For any norm $\|\cdot\|$, we can estimate (328) as

$$\begin{aligned}
\|U_{\geq 2}\| &\leq \left\| ie^{i\alpha} e^{i\hat{\theta}(0)} \left(\mathfrak{L}(\alpha)(e^{i\phi(\alpha)} - 1) + \mathfrak{N}(\alpha) e^{i\phi(\alpha)} \right) \right\| && (1087)\\
&\leq \left\| ie^{i\alpha} e^{i\hat{\theta}(0)} \left(e^{i\phi(\alpha)} (\mathfrak{L}(\alpha) + \mathfrak{N}(\alpha)) - \mathfrak{L}(\alpha) \right) \right\|. && (1088)
\end{aligned}$$

To estimate the $\dot{\mathcal{F}}_{\nu}^{s,1}$ and $\mathcal{F}_{\nu}^{0,1}$ norms of (1088), we can write

$$ie^{i\alpha}e^{i\hat{\theta}(0)}\left(e^{i\phi(\alpha)}(\mathfrak{L}(\alpha)+\mathfrak{N}(\alpha))-\mathfrak{L}(\alpha)\right)=\sum_{j=1}^{16}\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}B_j(\alpha,\beta)d\beta, \tag{1089}$$

where

$$B_1(\alpha,\beta)=-\frac{e^{i(\beta+\phi(\alpha))}e^{-i\phi(\alpha-\beta)}}{2\int_0^1 e^{-i(\beta s+\phi(\alpha+\beta(-1+s)))}ds}\cdot\int_0^1 e^{-i(\beta s+\phi(\alpha+\beta(-1+s)))}(-1+s)ds \tag{1090}$$

$$B_2(\alpha,\beta)=-\frac{e^{i(\beta+\phi(\alpha))}e^{-i\phi(\alpha-\beta)}}{2\int_0^1 e^{-i(\beta s+\phi(\alpha+\beta(-1+s)))}ds} \tag{1091}$$

$$\cdot\int_0^1 e^{-i(\beta s+\phi(\alpha+\beta(-1+s)))}(-1+s)\phi'(\alpha+\beta(-1+s))ds \tag{1092}$$

$$B_3(\alpha,\beta)=\frac{e^{i(\beta+\phi(\alpha)+\phi(\alpha-\beta))}}{2\left(\int_0^1 e^{i(\beta s+\phi(\alpha+\beta(-1+s)))}ds\right)^2}\int_0^1 e^{-i(\beta s+\phi(\alpha+\beta(-1+s)))}ds \tag{1093}$$

$$\cdot\int_0^1 e^{i(\beta s+\phi(\alpha+\beta(-1+s)))}(-1+s)ds \tag{1094}$$

$$B_4(\alpha,\beta)=\frac{e^{i(\beta+\phi(\alpha)+\phi(\alpha-\beta))}}{2\left(\int_0^1 e^{i(\beta s+\phi(\alpha+\beta(-1+s)))}ds\right)^2}\int_0^1 e^{-i(\beta s+\phi(\alpha+\beta(-1+s)))}ds \tag{1095}$$

$$\cdot\int_0^1 e^{i(\beta s+\phi(\alpha+\beta(-1+s)))}(-1+s)\phi'(\alpha+\beta(-1+s))ds \tag{1096}$$

$$B_5(\alpha,\beta)=\frac{e^{i(\beta+\phi(\alpha))}e^{i\phi(\alpha-\beta)}}{2\int_0^1 e^{i(\beta s+\phi(\alpha+\beta(-1+s)))}ds}\int_0^1 e^{-i(\beta s+\phi(\alpha+\beta(-1+s)))}(-1+s)ds \tag{1097}$$

$$B_6(\alpha,\beta) = \frac{e^{i(\beta+\phi(\alpha))}e^{-i\phi(\alpha-\beta)}}{2\int_0^1 e^{i(\beta s+\phi(\alpha+\beta(-1+s)))}ds}\int_0^1 e^{i(\beta s+\phi(\alpha+\beta(-1+s)))}(-1+s)ds \tag{1098}$$

$$B_7(\alpha,\beta) = \frac{e^{i(\beta+\phi(\alpha))}e^{i\phi(\alpha-\beta)}}{2\int_0^1 e^{i(\beta s+\phi(\alpha+\beta(-1+s)))}ds} \tag{1099}$$

$$\cdot \int_0^1 e^{-i(\beta s+\phi(\alpha+\beta(-1+s)))}(-1+s)\phi'(\alpha+\beta(-1+s))ds \tag{1100}$$

$$B_8(\alpha,\beta) = \frac{e^{i(\beta+\phi(\alpha))}e^{-i\phi(\alpha-\beta)}}{2\int_0^1 e^{i(\beta s+\phi(\alpha+\beta(-1+s)))}ds} \tag{1101}$$

$$\cdot \int_0^1 e^{i(\beta s+\phi(\alpha+\beta(-1+s)))}(-1+s)\phi'(\alpha+\beta(-1+s))ds \tag{1102}$$

$$B_9(\alpha,\beta) = \frac{e^{i\beta}}{2(-1+e^{i\beta})}\cdot(i(-1+e^{i\beta})+\beta(1+e^{i\beta}))\cdot\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))ds \tag{1103}$$

$$B_{10}(\alpha,\beta) = -\frac{2\beta+i(-1+e^{2i\beta})}{2(-1+e^{i\beta})^2}\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))ds \tag{1104}$$

$$B_{11}(\alpha,\beta) = \frac{\beta e^{i\beta}}{2}\int_0^1 e^{-i\beta s}(-1+s)\phi(\alpha+\beta(-1+s))ds \tag{1105}$$

$$B_{12}(\alpha,\beta) = \frac{\beta(1+e^{i\beta})}{2(-1+e^{i\beta})}\int_0^1 e^{i\beta s}(-1+s)\phi(\alpha+\beta(-1+s))ds \tag{1106}$$

$$B_{13}(\alpha,\beta) = \frac{-i(-2e^{i\beta}+2e^{2i\beta})e^{i\phi(\alpha)}e^{-i\phi(\alpha-\beta)}}{2\beta(-1+e^{i\beta})} \tag{1107}$$

$$B_{14}(\alpha,\beta) = \frac{i\beta e^{i\beta}}{2}\int_0^1 e^{-i\beta s}(-1+s)\phi'(\alpha+\beta(-1+s))ds \tag{1108}$$

$$B_{15}(\alpha,\beta) = \frac{-i\beta(1+e^{i\beta})}{2(-1+e^{i\beta})}\int_0^1 e^{i\beta s}(-1+s)\phi'(\alpha+\beta(-1+s))ds \tag{1109}$$

$$B_{16}(\alpha,\beta) = \frac{-i(\beta-2\beta e^{i\beta}-\beta e^{2i\beta})}{2\beta(-1+e^{i\beta})}\phi(\alpha-\beta). \tag{1110}$$

Using the Taylor expansion, we write

$$B_1(\alpha,\beta) = -\sum_{j_1,j_2,j_3,n\geq 0}\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}\frac{(-1)^{j_1+j_3}i^{j_1+j_2+j_3}}{2j_1!j_2!j_3!}\phi(\alpha-\beta)^{j_1}\phi(\alpha)^{j_2} \tag{1111}$$

$$\cdot\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds \tag{1112}$$

$$\cdot\left(\sum_{m=1}^{\infty}\frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\frac{(-i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n \tag{1113}$$

$$B_2(\alpha,\beta) = \tag{1114}$$

$$-\frac{1}{2}\sum_{j_1,j_2,j_3,n\geq 0}\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}\frac{(-1)^{j_1+j_3}i^{j_1+j_2+j_3}}{j_1!j_2!j_3!}\phi(\alpha-\beta)^{j_1}\phi(\alpha)^{j_2} \tag{1115}$$

$$\cdot\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)\phi'(\alpha+\beta(-1+s))ds \tag{1116}$$

$$\cdot\left(\sum_{m=1}^{\infty}\frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\frac{(-i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n \tag{1117}$$

$$B_3(\alpha,\beta) = \tag{1118}$$

$$\frac{1}{2}\sum_{j_1,j_2,j_3,j_4,n\geq 0}(n+1)\frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2}\frac{i^{j_1+j_2+j_3+j_4}(-1)^{j_3}}{j_1!j_2!j_3!j_4!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2} \tag{1119}$$

$$\cdot\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}ds\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_4}(-1+s)ds \tag{1120}$$

$$\cdot\left(\sum_{m=1}^{\infty}\frac{-i^m}{m!}e^{-i\alpha}\frac{i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(\alpha+(s-1)\beta)}\phi(\alpha+(s-1)\beta)^m ds\right)^n \tag{1121}$$

$$B_4(\alpha,\beta) = \tag{1122}$$

$$\frac{1}{2}\sum_{j_1,j_2,j_3,j_4,n\geq 0}(n+1)\frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2}\frac{i^{j_1+j_2+j_3+j_4}(-1)^{j_3}}{j_1!j_2!j_3!j_4!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2} \tag{1123}$$

$$\cdot\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}ds \tag{1124}$$

$$\cdot\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_4}(-1+s)\phi'(\alpha+\beta(-1+s))ds \tag{1125}$$

$$\cdot\left(\sum_{m=1}^{\infty}\frac{-i^m}{m!}e^{-i\alpha}\frac{i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(\alpha+(s-1)\beta)}\phi(\alpha+(s-1)\beta)^m ds\right)^n \tag{1126}$$

$$B_5(\alpha,\beta) = \tag{1127}$$

$$\frac{1}{2}\sum_{j_1,j_2,j_3,n\geq 0}\frac{i\beta}{1-e^{-i\beta}}\frac{i^{j_1+j_2+j_3}(-1)^{j_3}}{j_1!j_2!j_3!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2} \tag{1128}$$

$$\cdot\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds \tag{1129}$$

$$\cdot\left(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n \tag{1130}$$

$$B_6(\alpha,\beta) = \tag{1131}$$
$$\frac{1}{2}\sum_{j_1,j_2,j_3,n\geq 0}\frac{i\beta}{1-e^{-i\beta}}\frac{i^{j_1+j_2+j_3}(-1)^{j_2}}{j_1!j_2!j_3!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2} \tag{1132}$$
$$\cdot\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds0 \tag{1133}$$
$$\cdot\left(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n \tag{1134}$$
$$B_7(\alpha,\beta) = \tag{1135}$$
$$\frac{1}{2}\sum_{j_1,j_2,j_3,n\geq 0}\frac{i\beta}{1-e^{-i\beta}}\frac{i^{j_1+j_2+j_3}(-1)^{j_3}}{j_1!j_2!j_3!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2}0 \tag{1136}$$
$$\cdot\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)\phi'(\alpha+\beta(-1+s))ds \tag{1137}$$
$$\cdot\left(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n \tag{1138}$$
$$B_8(\alpha,\beta) = \tag{1139}$$
$$\frac{1}{2}\sum_{j_1,j_2,j_3,n\geq 0}\frac{i\beta}{1-e^{-i\beta}}\frac{i^{j_1+j_2+j_3}(-1)^{j_2}}{j_1!j_2!j_3!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2} \tag{1140}$$
$$\cdot\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)\phi'(\alpha+\beta(-1+s))ds \tag{1141}$$
$$\cdot\left(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n \tag{1142}$$
$$B_{13}(\alpha,\beta) = \tag{1143}$$
$$\frac{-i(-2e^{i\beta}+2e^{2i\beta})}{2\beta(-1+e^{i\beta})}\sum_{j_1,j_2\geq 0}\frac{i^{j_1+j_2}(-1)^{j_2}}{j_1!j_2!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2}. \tag{1144}$$

For ease of notation, let $B(\alpha,\beta)=\sum_{j=1}^{16}B_j(\alpha,\beta)$. We show that the parts of $B(\alpha,\beta)$ which are constant or linear in ϕ are both zero. We observe that for $i\in\{9,10,11,12,14,15,16\}$, $B_i(\alpha,\beta)$ is an expression linear in ϕ. To prove that $B(\alpha,\beta)$ has no part linear in ϕ, we first extract terms from $B_i(\alpha,\beta)$ for $i\in\{1,2,3,4,5,6,7,8,13\}$ which contain the integrals that appear in $B_i(\alpha,\beta)$ for $i\in\{9,10,11,12,14,15\}$. We first collect all terms containing $\int_0^1 e^{i\beta s}(-1+s)\phi'(\alpha+\beta(-1+s))ds$. In B_4, when $j_1=j_2=j_3=j_4=n=0$, we have

$$\frac{1}{2}\frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2}\int_0^1 e^{-i\beta s}ds\int_0^1 e^{i\beta s}(-1+s)\phi'(\alpha+\beta(-1+s))ds. \tag{1145}$$

In B_8, when $j_1=j_2=j_3=n=0$, we have

$$\frac{1}{2}\frac{i\beta}{1-e^{-i\beta}}\int_0^1 e^{i\beta s}(-1+s)\phi'(\alpha+\beta(-1+s))ds. \tag{1146}$$

Next, we collect all terms containing $\int_0^1 e^{-i\beta s}(-1+s)\phi'(\alpha+\beta(-1+s))ds$. In B_2, when $j_1 = j_2 = j_3 = n = 0$, we have

$$-\frac{1}{2}\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}\int_0^1 e^{-i\beta s}(-1+s)\phi'(\alpha+\beta(-1+s))ds. \tag{1147}$$

In B_7, when $j_1 = j_2 = j_3 = n = 0$, we have

$$\frac{1}{2}\frac{i\beta}{1-e^{-i\beta}}\int_0^1 e^{-i\beta s}(-1+s)\phi'(\alpha+\beta(-1+s))ds. \tag{1148}$$

Next, we collect all terms containing $\int_0^1 e^{i\beta s}(-1+s)\phi(\alpha+\beta(-1+s))ds$. In B_3, when $j_1 = j_2 = j_3 = n = 0$ and $j_4 = 1$, we have

$$\frac{1}{2}\frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2}\frac{i}{1}\int_0^1\int_0^1 e^{-i\beta s}ds\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))(-1+s)ds. \tag{1149}$$

In B_6, when $j_1 = j_2 = n = 0$ and $j_3 = 1$, we have

$$\frac{1}{2}\frac{i\beta}{1-e^{-i\beta}}\frac{i}{1}\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))ds. \tag{1150}$$

Next, we collect all terms containing $\int_0^1 e^{-i\beta s}(-1+s)\phi(\alpha+\beta(-1+s))ds$. In B_1, when $j_1 = j_2 = n = 0$ and $j_3 = 1$, we have

$$-\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}\frac{(-1)i}{2}\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))(-1+s)ds. \tag{1151}$$

Inside B_5, when $j_1 = j_2 = n = 0$ and $j_3 = 1$, we have

$$\frac{1}{2}\frac{i\beta}{1-e^{-i\beta}}\frac{i(-1)}{1}\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))(-1+s)ds. \tag{1152}$$

Next, we collect all terms containing $\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))ds$. In B_3, when $j_1 = j_2 = j_3 = j_4 = 0$ and $n = 1$, we have

$$\frac{1}{2}\cdot 2\frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2}\int_0^1 e^{-i\beta s}ds\int_0^1 e^{i\beta s}(-1+s)ds\cdot \tag{1153}$$

$$\Bigg(-ie^{-i\alpha}\frac{i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(\alpha+(s-1)\beta)}\phi(\alpha+(s-1)\beta)ds \tag{1154}$$

$$+\sum_{m=2}^{\infty}\frac{-i^m}{m!}e^{-i\alpha}\frac{i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(\alpha+(s-1)\beta)}\phi(\alpha+(s-1)\beta)^m ds\Bigg). \tag{1155}$$

In B_5, when $j_1 = j_2 = j_3 = 0$ and $n = 1$, we have

$$\frac{1}{2}\frac{i\beta}{1-e^{-i\beta}}\int_0^1 e^{-i\beta s}(-1+s)ds\Bigg(\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}i\phi(\alpha+(s-1)\beta)ds \tag{1156}$$

$$+\sum_{m=2}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\Bigg), \tag{1157}$$

Inside B_6, when $j_1 = j_2 = j_3 = 0$ and $n = 1$, we have (1158)

$$\frac{1}{2}\frac{i\beta}{1-e^{-i\beta}}\int_0^1 e^{i\beta s}(-1+s)ds\Bigg(\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}i\phi(\alpha+(s-1)\beta)ds \tag{1159}$$

$$+\sum_{m=2}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s+1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\Bigg). \tag{1160}$$

Lastly, we collect all terms containing $\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))ds$. In B_1, when $j_1 = j_2 = j_3 = 0$ and $n = 1$, we have

$$-\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}\cdot\frac{1}{2}\int_0^1 e^{-i\beta s}(-1+s)ds\Bigg(\frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}(-i)\phi(\alpha+(s-1)\beta)d\beta \tag{1161}$$

$$+\sum_{m=2}^{\infty}\frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\frac{(-i\phi(\alpha+(s-1)\beta))^m}{m!}ds\Bigg). \tag{1162}$$

In B_3, when $j_1 = j_2 = j_4 = n = 0$ and $j_3 = 1$, we have

$$\frac{1}{2}\frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2}\frac{i(-1)}{1}\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))ds\int_0^1 e^{i\beta s}(-1+s)ds. \tag{1163}$$

When added with $B_i(\alpha,\beta)$ for $i \in \{9, 10, 11, 12, 14, 15\}$, the terms extracted above containing $\int_0^1 e^{i\beta s}(-1+s)\phi'(\alpha+\beta(-1+s))ds$, $\int_0^1 e^{-i\beta s}(-1+s)\phi'(\alpha+\beta(-1+s))ds$, $\int_0^1 e^{i\beta s}(-1+s)\phi(\alpha+\beta(-1+s))ds$, $\int_0^1 e^{-i\beta s}(-1+s)\phi(\alpha+\beta(-1+s))ds$, $\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))ds$, and $\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))ds$ vanish. To complete the proof that $B(\alpha,\beta)$ has no parts that are constant or linear in ϕ, we collect all terms constant or linear in $\phi(\alpha)$ or $\phi(\alpha-\beta)$. From B_1, we have

$$-\Bigg(\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}\cdot\frac{1}{2}\int_0^1 e^{-i\beta s}(-1+s)ds \tag{1164}$$

$$+\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}\cdot\frac{(-1)i}{2}\cdot\phi(\alpha-\beta)\int_0^1 e^{-i\beta s}(-1+s)ds+\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}\cdot\frac{i}{2}\cdot\phi(\alpha)\cdot \tag{1165}$$

$$\int_0^1 e^{-i\beta s}(-1+s)ds\Bigg). \tag{1166}$$

From B_3, we have

$$\frac{1}{2}\bigg(\frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2}\int_0^1 e^{-i\beta s}ds\int_0^1 e^{i\beta s}(-1+s)ds+\frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2}\cdot\frac{i}{1}\cdot\phi(\alpha)\cdot \tag{1167}$$

$$\int_0^1 e^{-i\beta s}ds\int_0^1 e^{i\beta s}(-1+s)ds \tag{1168}$$

$$+\frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2}\cdot\frac{i}{1}\cdot\phi(\alpha-\beta)\int_0^1 e^{-i\beta s}ds\int_0^1 e^{i\beta s}(-1+s)ds\bigg). \tag{1169}$$

From B_5, we have

$$\frac{1}{2}\bigg(\frac{i\beta}{1-e^{-i\beta}}\int_0^1 e^{-i\beta s}(-1+s)ds+\frac{i\beta}{1-e^{-i\beta}}\cdot\frac{i}{1}\cdot\phi(\alpha)\int_0^1 e^{-i\beta s}(-1+s)ds \tag{1170}$$

$$+\frac{i\beta}{1-e^{-i\beta}}\cdot\frac{i}{1}\cdot\phi(\alpha-\beta)\int_0^1 e^{-i\beta s}(-1+s)ds\bigg). \tag{1171}$$

From B_6, we have

$$\frac{1}{2}\bigg(\frac{i\beta}{1-e^{-i\beta}}\int_0^1 e^{i\beta s}(-1+s)ds+\frac{i\beta}{1-e^{-i\beta}}\cdot\frac{i}{1}\cdot\phi(\alpha)\int_0^1 e^{i\beta s}(-1+s)ds \tag{1172}$$

$$+\frac{i\beta}{1-e^{-i\beta}}\cdot\frac{i(-1)}{1}\cdot\phi(\alpha-\beta)\int_0^1 e^{i\beta s}(-1+s)ds\bigg). \tag{1173}$$

From B_{13}, we have

$$\frac{-i(-2e^{i\beta}+2e^{2i\beta})}{2\beta(-1+e^{i\beta})}\bigg(1+\frac{i}{1}\phi(\alpha)+\frac{i(-1)}{1}\phi(\alpha-\beta)\bigg). \tag{1174}$$

From B_{16}, we have

$$\frac{-i(\beta-2\beta e^{i\beta}-\beta e^{2i\beta})}{2\beta(-1+e^{i\beta})}\cdot\phi(\alpha-\beta). \tag{1175}$$

Of these terms, those linear in $\phi(\alpha-\beta)$ add up to 0. When integrated with respect to β, the terms which are constant and linear in $\phi(\alpha)$ become 0. Setting zero all but summation variables j_1 and j_2 in $B_i(\alpha,\beta)$ for $i\in\{1,3,5,6,13,16\}$, we obtain a smaller sum $\sum_{j_1+j_2\geq 0}$. Each of the above terms that are constant or linear in $\phi(\alpha)$ or $\phi(\alpha-\beta)$ belongs to one of these smaller sums. From these smaller sums, we take out these terms and add them up to obtain

$$\sum_{j_1+j_2\geq 2}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2}\frac{i^{j_1+j_2}}{j_1!j_2!}\bigg((-1)^{j_2}\cdot\frac{e^{i\beta}(1+e^{i\beta})}{2(-1+e^{i\beta})}-\frac{1}{2}\bigg). \tag{1176}$$

Observe that

$$B_1=-\sum_{j_1,j_2,j_3,n\geq 0}=-\sum_{\substack{j_1,j_2\geq 0\\ j_3=n=0}}-\sum_{\substack{j_1,j_2\geq 0\\ j_3+n\geq 1}}=-\sum_{\substack{j_1,j_2\geq 0\\ j_3=n=0}}-\sum_{\substack{j_1,j_2\geq 0\\ j_3=1\\ n\geq 0}}-\sum_{\substack{j_1,j_2\geq 0\\ n=1\\ j_3\geq 0}} \tag{1177}$$

$$=-\sum_{\substack{j_1,j_2\geq 0\\ j_3=n=0}}-\sum_{\substack{j_1=j_2=n=0\\ j_3=1}}-\sum_{\substack{j_1+j_2+n\geq 1\\ j_3=1}}-\sum_{\substack{j_1=j_2=j_3=0\\ n=1}}-\sum_{\substack{j_1+j_2+j_3\geq 1\\ n=1}}, \tag{1178}$$

$$B_2 = -\frac{1}{2}\sum_{j_1,j_2,j_3,n\geq 0} = -\frac{1}{2}\sum_{j_1=j_2=j_3=n=0} -\frac{1}{2}\sum_{j_1+j_2+j_3+n\geq 1}, \tag{1179}$$

$$B_3 = \frac{1}{2}\sum_{j_1,j_2,j_3,j_4,n\geq 0} \tag{1180}$$

$$= \frac{1}{2}\sum_{\substack{j_1,j_2\geq 0\\ j_3=j_4=n=0}} + \frac{1}{2}\sum_{\substack{j_1,j_2\geq 0\\ j_3+j_4+n\geq 1}} = \frac{1}{2}\sum_{\substack{j_1,j_2\geq 0\\ j_3=j_4=n=0}} + \frac{1}{2}\sum_{\substack{j_1=j_2=0\\ j_3+j_4+n\geq 1}} + \frac{1}{2}\sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}} \tag{1181}$$

$$= \frac{1}{2}\sum_{\substack{j_1,j_2\geq 0\\ j_3=j_4=n=0}} + \frac{1}{2}\sum_{\substack{j_1=j_2=0\\ j_3+j_4+n=1}} + \frac{1}{2}\sum_{\substack{j_1=j_2=0\\ j_3+j_4+n\geq 2}} + \frac{1}{2}\sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}}, \tag{1182}$$

$$B_4 = \frac{1}{2}\sum_{j_1,j_2,j_3,j_4,n\geq 0} = \frac{1}{2}\sum_{j_1=j_2=j_3=j_4=n=0} + \frac{1}{2}\sum_{j_1+j_2+j_3+j_4+n\geq 1}, \tag{1183}$$

$$B_5 = \frac{1}{2}\sum_{j_1,j_2,j_3,n\geq 0} = \frac{1}{2}\sum_{\substack{j_1,j_2\geq 0\\ j_3=n=0}} + \frac{1}{2}\sum_{\substack{j_1,j_2\geq 0\\ j_3+n\geq 1}} = \frac{1}{2}\sum_{\substack{j_1,j_2\geq 0\\ j_3=n=0}} + \frac{1}{2}\sum_{\substack{j_1=j_2=0\\ j_3+n\geq 1}} + \frac{1}{2}\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \tag{1184}$$

$$= \frac{1}{2}\sum_{\substack{j_1,j_2\geq 0\\ j_3=n=0}} + \frac{1}{2}\sum_{\substack{j_1=j_2=0\\ j_3+n=1}} + \frac{1}{2}\sum_{\substack{j_1=j_2=0\\ j_3+n\geq 2}} + \frac{1}{2}\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}}, \tag{1185}$$

$$B_6 = \frac{1}{2}\sum_{j_1,j_2,j_3,n\geq 0} = \frac{1}{2}\sum_{\substack{j_1,j_2\geq 0\\ j_3=n=0}} + \frac{1}{2}\sum_{\substack{j_1,j_2\geq 0\\ j_3+n\geq 1}} = \frac{1}{2}\sum_{\substack{j_1,j_2\geq 0\\ j_3=n=0}} + \frac{1}{2}\sum_{\substack{j_1=j_2=0\\ j_3+n\geq 1}} + \frac{1}{2}\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \tag{1186}$$

$$= \frac{1}{2}\sum_{\substack{j_1,j_2\geq 0\\ j_3=n=0}} + \frac{1}{2}\sum_{\substack{j_1=j_2=0\\ j_3+n=1}} + \frac{1}{2}\sum_{\substack{j_1=j_2=0\\ j_3+n\geq 2}} + \frac{1}{2}\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}}, \tag{1187}$$

$$B_7 = \frac{1}{2}\sum_{j_1,j_2,j_3,n\geq 0} = \frac{1}{2}\sum_{j_1=j_2=j_3=n=0} + \frac{1}{2}\sum_{j_1+j_2+j_3+n\geq 1}, \tag{1188}$$

$$B_8 = \frac{1}{2}\sum_{j_1,j_2,j_3,n\geq 0} = \frac{1}{2}\sum_{j_1=j_2=j_3=n=0} + \frac{1}{2}\sum_{j_1+j_2+j_3+n\geq 1}, \tag{1189}$$

$$B_{13} = \sum_{j_1,j_2\geq 0} = \sum_{j_1+j_2=0} + \sum_{j_1+j_2=1} + \sum_{j_1+j_2\geq 2}. \tag{1190}$$

From these expressions for B_i for $i \in \{1,2,3,4,5,6,7,8,13\}$, we take out all the terms that are either constant or linear in $\phi(\alpha)$ or $\phi(\alpha-\beta)$, or contain the integrals involving ϕ that had been identified earlier because they have been shown to vanish. Once they are taken out, we add all the smaller sums of the form $\sum_{j_1+j_2\geq 2}$ with that of B_{13}, which is equal to (1176). Then we can write

$$ie^{i\alpha}e^{i\hat{\theta}(0)}\left(e^{i\phi(\alpha)}(\mathfrak{L}(\alpha)+\mathfrak{N}(\alpha)) - \mathfrak{L}(\alpha)\right) = \frac{\gamma}{4\pi}\int_{-\pi}^{\pi} B(\alpha,\beta)d\beta, \tag{1191}$$

where $B(\alpha,\beta)=\sum_{j=1}^{8}\widetilde{B_j}(\alpha,\beta)+\widetilde{B_{13}}(\alpha,\beta)$, in which

$$\widetilde{B_1}(\alpha,\beta)=-\sum_{\substack{j_1+j_2+n\geq 1\\ j_3=1}}\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}\frac{(-1)^{j_1+j_3}i^{j_1+j_2+j_3}}{2j_1!j_2!j_3!}\phi(\alpha-\beta)^{j_1}\phi(\alpha)^{j_2} \tag{1192}$$

$$\cdot\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds \tag{1193}$$

$$\cdot\left(\sum_{m=1}^{\infty}\frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\frac{(-i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n \tag{1194}$$

$$-\sum_{\substack{j_1+j_2+j_3\geq 1\\ n=1}}\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}\frac{(-1)^{j_1+j_3}i^{j_1+j_2+j_3}}{2j_1!j_2!j_3!}\phi(\alpha-\beta)^{j_1}\phi(\alpha)^{j_2} \tag{1195}$$

$$\cdot\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds \tag{1196}$$

$$\cdot\left(\sum_{m=1}^{\infty}\frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\frac{(-i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n \tag{1197}$$

$$-\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}\cdot\frac{1}{2}\int_0^1 e^{-i\beta s}(-1+s)ds \tag{1198}$$

$$\cdot\sum_{m=2}^{\infty}\frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\frac{(-i\phi(\alpha+(s-1)\beta))^m}{m!}ds \tag{1199}$$

$$\widetilde{B_2}(\alpha,\beta)=-\frac{1}{2}\sum_{j_1+j_2+j_3+n\geq 1}\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}\frac{(-1)^{j_1+j_3}i^{j_1+j_2+j_3}}{j_1!j_2!j_3!}\phi(\alpha-\beta)^{j_1}\phi(\alpha)^{j_2} \tag{1200}$$

$$\cdot\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)\phi'(\alpha+\beta(-1+s))ds \tag{1201}$$

$$\cdot\left(\sum_{m=1}^{\infty}\frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\frac{(-i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n \tag{1202}$$

$$\widetilde{B_3}(\alpha,\beta) = \frac{1}{2}\sum_{\substack{j_1=j_2=0\\ j_3+j_4+n\geq 2}}(n+1)\frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2}\frac{i^{j_1+j_2+j_3+j_4}(-1)^{j_3}}{j_1!j_2!j_3!j_4!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2} \tag{1203}$$

$$\cdot\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}ds\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_4}(-1+s)ds \tag{1204}$$

$$\cdot\left(\sum_{m=1}^{\infty}\frac{-i^m}{m!}e^{-i\alpha}\frac{i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(\alpha+(s-1)\beta)}\phi(\alpha+(s-1)\beta)^m ds\right)^n \tag{1205}$$

$$+\frac{1}{2}\sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}}(n+1)\frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2}\frac{i^{j_1+j_2+j_3+j_4}(-1)^{j_3}}{j_1!j_2!j_3!j_4!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2} \tag{1206}$$

$$\cdot\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}ds\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_4}(-1+s)ds \tag{1207}$$

$$\cdot\left(\sum_{m=1}^{\infty}\frac{-i^m}{m!}e^{-i\alpha}\frac{i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(\alpha+(s-1)\beta)}\phi(\alpha+(s-1)\beta)^m ds\right)^n \tag{1208}$$

$$+\frac{1}{2}\cdot 2\cdot\frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2}\int_0^1 e^{-i\beta s}ds\int_0^1 e^{i\beta s}(-1+s)ds\sum_{m=2}^{\infty}\frac{-i^m}{m!}e^{-i\alpha}\frac{i\beta}{1-e^{-i\beta}} \tag{1209}$$

$$\cdot\int_0^1 e^{i(\alpha+(s-1)\beta)}\phi(\alpha+(s-1)\beta)^m ds \tag{1210}$$

$$\widetilde{B_4}(\alpha,\beta) = \frac{1}{2}\sum_{j_1+j_2+j_3+j_4+n\geq 1}(n+1)\frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2}\frac{i^{j_1+j_2+j_3+j_4}(-1)^{j_3}}{j_1!j_2!j_3!j_4!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2} \tag{1211}$$

$$\cdot\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}ds \tag{1212}$$

$$\cdot\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_4}(-1+s)\phi'(\alpha+\beta(-1+s))ds \tag{1213}$$

$$\cdot\left(\sum_{m=1}^{\infty}\frac{-i^m}{m!}e^{-i\alpha}\frac{i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(\alpha+(s-1)\beta)}\phi(\alpha+(s-1)\beta)^m ds\right)^n \tag{1214}$$

$$\widetilde{B_5}(\alpha,\beta) = \frac{1}{2}\sum_{\substack{j_1=j_2=0\\ j_3+n\geq 2}} \frac{i\beta}{1-e^{-i\beta}} \frac{i^{j_1+j_2+j_3}(-1)^{j_3}}{j_1!j_2!j_3!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2} \tag{1215}$$

$$\cdot \int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds \tag{1216}$$

$$\cdot \left(\sum_{m=1}^{\infty} \frac{-i\beta}{1-e^{-i\beta}} \int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n \tag{1217}$$

$$+ \frac{1}{2}\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{i\beta}{1-e^{-i\beta}} \frac{i^{j_1+j_2+j_3}(-1)^{j_3}}{j_1!j_2!j_3!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2} \tag{1218}$$

$$\cdot \int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds \tag{1219}$$

$$\cdot \left(\sum_{m=1}^{\infty} \frac{-i\beta}{1-e^{-i\beta}} \int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n \tag{1220}$$

$$+ \frac{1}{2}\cdot \frac{i\beta}{1-e^{-i\beta}} \int_0^1 e^{-i\beta s}(-1+s)ds \tag{1221}$$

$$\cdot \sum_{m=2}^{\infty} \frac{-i\beta}{1-e^{-i\beta}} \int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds \tag{1222}$$

$$\widetilde{B_6}(\alpha,\beta) = \frac{1}{2}\sum_{\substack{j_1=j_2=0\\ j_3+n\geq 2}} \frac{i\beta}{1-e^{-i\beta}} \frac{i^{j_1+j_2+j_3}(-1)^{j_2}}{j_1!j_2!j_3!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2} \tag{1223}$$

$$\cdot \int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds \tag{1224}$$

$$\cdot \left(\sum_{m=1}^{\infty} \frac{-i\beta}{1-e^{-i\beta}} \int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n \tag{1225}$$

$$+ \frac{1}{2}\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{i\beta}{1-e^{-i\beta}} \frac{i^{j_1+j_2+j_3}(-1)^{j_2}}{j_1!j_2!j_3!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2} \tag{1226}$$

$$\cdot \int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds \tag{1227}$$

$$\cdot \left(\sum_{m=1}^{\infty} \frac{-i\beta}{1-e^{-i\beta}} \int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n \tag{1228}$$

$$+ \frac{1}{2}\cdot \frac{i\beta}{1-e^{-i\beta}} \int_0^1 e^{i\beta s}(-1+s)ds \tag{1229}$$

$$\cdot \sum_{m=2}^{\infty} \frac{-i\beta}{1-e^{-i\beta}} \int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds \tag{1230}$$

$$\widetilde{B_7}(\alpha,\beta) = \frac{1}{2}\sum_{j_1+j_2+j_3+n\geq 1}\frac{i\beta}{1-e^{-i\beta}}\frac{i^{j_1+j_2+j_3}(-1)^{j_3}}{j_1!j_2!j_3!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2} \tag{1231}$$

$$\cdot\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)\phi'(\alpha+\beta(-1+s))ds \tag{1232}$$

$$\cdot\left(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n \tag{1233}$$

$$\widetilde{B_8}(\alpha,\beta) = \frac{1}{2}\sum_{j_1+j_2+j_3+n\geq 1}\frac{i\beta}{1-e^{-i\beta}}\frac{i^{j_1+j_2+j_3}(-1)^{j_2}}{j_1!j_2!j_3!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2} \tag{1234}$$

$$\cdot\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)\phi'(\alpha+\beta(-1+s))ds \tag{1235}$$

$$\cdot\left(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n \tag{1236}$$

$$\widetilde{B_{13}}(\alpha,\beta) = \sum_{j_1+j_2\geq 2}\frac{i^{j_1+j_2}}{j_1!j_2!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2}\left((-1)^{j_2}\frac{e^{i\beta}(1+e^{i\beta})}{2(-1+e^{i\beta})}-\frac{1}{2}\right). \tag{1237}$$

13.1 Estimating Fourier Modes of $U_{\geq 2}$

In our calculations, we adopt the notational convention that any product $\prod$ in which the upper bound is strictly less than the lower bound is defined to be 1. To compute the Fourier modes of $U_{\geq 2}$, we frequently use the identity

$$\mathcal{F}(g_1g_2\cdots g_n)(k_1) = \sum_{k_2,\dots,k_n\in\mathbb{Z}}\left(\prod_{d=1}^{n-1}\mathcal{F}(g_d)(k_d-k_{d+1})\right)\mathcal{F}(g_n)(k_n). \tag{1238}$$

We define

$$P(k) = \sum_{m=1}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}(\phi^m)(k), \tag{1239}$$

$$\widetilde{P}(k) = \sum_{m=2}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}(\phi^m)(k), \tag{1240}$$

$$Q(k) = \sum_{m=1}^{\infty}\frac{i^m}{m!}\mathcal{F}(\phi^m)(k), \tag{1241}$$

$$\widetilde{Q}(k) = \sum_{k=2}^{\infty}\frac{i^m}{m!}\mathcal{F}(\phi^m)(k). \tag{1242}$$

For $n \geq 0$, let

$$I_n(k_1, k_{j_1+1}, k_{j_1+j_2+1}, \ldots, k_{j_1+j_2+n}, k_{j_1+j_2+n+1}, \beta) \tag{1243}$$

$$= \prod_{d=1}^{n} \left(\frac{i\beta e^{i\beta}}{1-e^{i\beta}} \int_0^1 e^{-is\beta} e^{i(s-1)\beta(k_{j_1+j_2+d} - k_{j_1+j_2+d+1})} ds \right) \cdot e^{-i\beta(k_1 - k_{j_1+1})} \tag{1244}$$

$$\cdot \int_0^1 e^{-i\beta s} e^{i\beta(-1+s)k_{j_1+j_2+n+1}} (-1+s) ds \tag{1245}$$

and

$$C_n = \frac{\gamma}{4\pi} \left((n+1) \cdot \left(\frac{\pi}{2}\right)^n \cdot \frac{1}{2}\sqrt{1+\frac{\pi^2}{4}} \cdot \pi^2 + 2\pi \right). \tag{1246}$$

The following estimate is used frequently.

Lemma 3. *For $n \geq 0$,*

$$\left| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} I_n(k_1, k_{j_1+1}, k_{j_1+j_2+1}, \ldots, k_{j_1+j_2+n}, k_{j_1+j_2+n+1}, \beta) \cdot \frac{-i\beta e^{2i\beta}}{1-e^{i\beta}} d\beta \right| \leq C_n. \tag{1247}$$

Proof. We note that

$$\int_0^1 e^{-is\beta} e^{i(s-1)\beta k} ds = \begin{cases} \frac{i(e^{-i\beta} - e^{-i\beta k})}{\beta(1-k)} & \text{if } k \neq 1, \\ e^{-i\beta} & \text{if } k = 1. \end{cases} \tag{1248}$$

First let $n \geq 1$. Suppose that $0 \leq l \leq n$ and l elements of $\{k_{j_1+j_2+d} - k_{j_1+j_2+d+1}\}_{d=1}^n$ satisfy $k_{j_1+j_2+d} - k_{j_1+j_2+d+1} = 1$. Reordering the subscripts such that $k_{j_1+j_2+d} - k_{j_1+j_2+d+1} \neq 1$ for $d = 1, \ldots, n-l$, we obtain

$$I_n = e^{-i\beta(k_1 - k_{j_1+1})} \prod_{d=1}^{n-l} \frac{-(1 - e^{-i\beta(-1+k_{j_1+j_2+d} - k_{j_1+j_2+d+1})})}{(1-e^{i\beta})(1 - k_{j_1+j_2+d} + k_{j_1+j_2+d+1})} \left(\frac{i\beta}{1-e^{i\beta}} \right)^l \tag{1249}$$

$$\cdot \int_0^1 e^{-i\beta s} e^{i\beta(-1+s)k_{j_1+j_2+n+1}} (-1+s) ds. \tag{1250}$$

If $k_{j_1+j_2+d} - k_{j_1+j_2+d+1} > 1$, then

$$\frac{-(1 - e^{-i\beta(-1+k_{j_1+j_2+d} - k_{j_1+j_2+d+1})})}{1-e^{i\beta}} = e^{-i\beta} \sum_{r_d=0}^{-2+k_{j_1+j_2+d} - k_{j_1+j_2+d+1}} (e^{-i\beta})^{r_d}. \tag{1251}$$

If $k_{j_1+j_2+d} - k_{j_1+j_2+d+1} < 1$, then

$$\frac{-(1 - e^{-i\beta(-1+k_{j_1+j_2+d} - k_{j_1+j_2+d+1})})}{1-e^{i\beta}} = -\sum_{r_d=0}^{-(k_{j_1+j_2+d} - k_{j_1+j_2+d+1})} (e^{i\beta})^{r_d}. \tag{1252}$$

Suppose that $k_{j_1+j_2+d} - k_{j_1+j_2+d+1} < 1$ only for $d = w, \dots, n-l$. Then

$$\prod_{d=1}^{n-l} \frac{-(1-e^{-i\beta(-1+k_{j_1+j_2+d}-k_{j_1+j_2+d+1})})}{1-e^{i\beta}} \tag{1253}$$

$$= e^{-i\beta} \sum_{r_1=0}^{-2+k_{j_1+j_2+1}-k_{j_1+j_2+2}} (e^{-i\beta})^{r_1} \cdot \dots \cdot e^{-i\beta} \sum_{r_{w-1}=0}^{-2+k_{j_1+j_2+w-1}-k_{j_1+j_2+w}} (e^{-i\beta})^{r_{w-1}} \tag{1254}$$

$$\cdot (-1) \sum_{r_w=0}^{-(k_{j_1+j_2+w}-k_{j_1+j_2+w+1})} (e^{i\beta})^{r_w} \cdot \dots \cdot (-1) \sum_{r_{n-l}=0}^{-(k_{j_1+j_2+n-l}-k_{j_1+j_2+n-l+1})} (e^{i\beta})^{r_{n-l}} \tag{1255}$$

$$= \sum_{r_1=0}^{-2+k_{j_1+j_2+1}-k_{j_1+j_2+2}} \cdots \sum_{r_{w-1}=0}^{-2+k_{j_1+j_2+w-1}-k_{j_1+j_2+w}} \tag{1256}$$

$$\sum_{r_w=0}^{-(k_{j_1+j_2+w}-k_{j_1+j_2+w+1})} \cdots \sum_{r_{n-l}=0}^{-(k_{j_1+j_2+n-l}-k_{j_1+j_2+n-l+1})} \tag{1257}$$

$$(e^{-i\beta})^{w-1}(-1)^{n-l-w+1}(e^{-i\beta})^{r_1+\dots+r_{w-1}}(e^{i\beta})^{r_w+\dots+r_{n-l}}. \tag{1258}$$

Hence,

$$I_n = \prod_{d=1}^{n-l} \frac{1}{1-k_{j_1+j_2+d}+k_{j_1+j_2+d+1}} \cdot \tag{1259}$$

$$\sum_{r_1=0}^{-2+k_{j_1+j_2+1}-k_{j_1+j_2+2}} \cdots \sum_{r_{w-1}=0}^{-2+k_{j_1+j_2+w-1}-k_{j_1+j_2+w}} \tag{1260}$$

$$\sum_{r_w=0}^{-(k_{j_1+j_2+w}-k_{j_1+j_2+w+1})} \cdots \sum_{r_{n-l}=0}^{-(k_{j_1+j_2+n-l}-k_{j_1+j_2+n-l+1})} \tag{1261}$$

$$(-1)^{n-l-w+1}(e^{-i\beta})^{w-1}(e^{-i\beta})^{r_1+\dots+r_{w-1}}(e^{i\beta})^{r_w+\dots+r_{n-l}} \tag{1262}$$

$$\cdot \left(\frac{i\beta}{1-e^{i\beta}}\right)^l \cdot \int_0^1 e^{-i\beta s} e^{i\beta(-1+s)k_{j_1+j_2+n+1}}(-1+s)ds \cdot e^{-i\beta(k_1-k_{j_1+1})}. \tag{1263}$$

Let

$$J_n = \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} (e^{-i\beta})^{w-1+r_1+\dots+r_{w-1}-(r_w+\dots+r_{n-l})} \left(\frac{i\beta}{1-e^{i\beta}}\right)^l e^{-i\beta(k_1-k_{j_1+1})} \tag{1264}$$

$$\cdot \int_0^1 e^{-i\beta s} e^{i\beta(-1+s)k_{j_1+j_2+n+1}}(-1+s)ds \cdot \frac{-i\beta e^{2i\beta}}{1-e^{i\beta}} d\beta. \tag{1265}$$

For all $l \geq 0$,

$$\left| \left(\frac{i\beta}{1-e^{i\beta}}\right)^l - 1 \right| \leq |\beta| \cdot l \cdot \left(\frac{\pi}{2}\right)^{l-1} \cdot \frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}. \tag{1266}$$

Then

$$|J_n| \leq \frac{\gamma}{4\pi}\left(\int_{-\pi}^{\pi}\left|\left(\frac{i\beta}{1-e^{i\beta}}\right)^{l+1}-1\right|d\beta+2\pi\right) \tag{1267}$$
$$\leq \frac{\gamma}{4\pi}\left((l+1)\cdot\left(\frac{\pi}{2}\right)^{l}\cdot\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}\cdot\pi^2+2\pi\right) \tag{1268}$$
$$\leq \frac{\gamma}{4\pi}\left((n+1)\cdot\left(\frac{\pi}{2}\right)^{n}\cdot\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}\cdot\pi^2+2\pi\right) \tag{1269}$$
$$= C_n. \tag{1270}$$

Thus,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} I_n(k_1,k_{j_1+1},k_{j_1+j_2+1},\ldots,k_{j_1+j_2+n},k_{j_1+j_2+n+1},\beta)\cdot\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}d\beta\right| \tag{1271}$$
$$\leq \prod_{d=1}^{n-l}\frac{1}{|1-k_{j_1+j_2+d}+k_{j_1+j_2+d+1}|}\cdot\sum_{r_1=0}^{-2+k_{j_1+j_2+1}-k_{j_1+j_2+2}}\cdots\sum_{r_{w-1}=0}^{-2+k_{j_1+j_2+w-1}-k_{j_1+j_2+w}} \tag{1272}$$
$$\cdot\sum_{r_w=0}^{-(k_{j_1+j_2+w}-k_{j_1+j_2+w+1})}\cdots\sum_{r_{n-l}=0}^{-(k_{j_1+j_2+n-l}-k_{j_1+j_2+n-l+1})}|J_n| \tag{1273}$$
$$\leq C_n. \tag{1274}$$

If $n=0$, then

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} I_0(k_1,k_{j_1+1},k_{j_1+j_2+1},\beta)\cdot\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}d\beta\right| \tag{1275}$$
$$\leq\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} e^{-i\beta(k_1-k_{j_1+1})}\int_0^1 e^{-i\beta s}e^{i\beta(-1+s)k_{j_1+j_2+n+1}}(-1+s)ds\cdot(-e^{2i\beta})\left(\frac{i\beta}{1-e^{i\beta}}-1+1\right)d\beta\right| \tag{1276}$$
$$\leq\frac{\gamma}{4\pi}\left(\int_{-\pi}^{\pi}|\beta|\cdot\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}d\beta+2\pi\right) \tag{1277}$$
$$=C_0, \tag{1278}$$

where

$$C_0=\frac{\gamma}{4\pi}\left(\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}+2\pi\right). \tag{1279}$$

■

Let us compute the Fourier modes of $\widetilde{B_1}(\alpha, \beta)$. Let $\widetilde{B_1} = \sum_{j=1}^{3} \widetilde{B_{1,j}}$, where

$$\widetilde{B_{1,1}}(\alpha, \beta) = -\sum_{\substack{j_1+j_2+n\geq 1\\ j_3=1}} \frac{-i\beta e^{2i\beta}}{1-e^{i\beta}} \frac{(-1)^{j_1+j_3} i^{j_1+j_2+j_3}}{2j_1!j_2!j_3!} \phi(\alpha-\beta)^{j_1}\phi(\alpha)^{j_2} \tag{1280}$$

$$\cdot \int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds \tag{1281}$$

$$\cdot \left(\sum_{m=1}^{\infty} \frac{i\beta}{1-e^{i\beta}} \int_0^1 e^{-i(s-1)\beta} \frac{(-i\phi(\alpha+(s-1)\beta))^m}{m!} ds\right)^n \tag{1282}$$

$$\widetilde{B_{1,2}}(\alpha, \beta) = -\sum_{\substack{j_1+j_2+j_3\geq 1\\ n=1}} \frac{-i\beta e^{2i\beta}}{1-e^{i\beta}} \frac{(-1)^{j_1+j_3} i^{j_1+j_2+j_3}}{2j_1!j_2!j_3!} \phi(\alpha-\beta)^{j_1}\phi(\alpha)^{j_2} \tag{1283}$$

$$\cdot \int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds \tag{1284}$$

$$\cdot \left(\sum_{m=1}^{\infty} \frac{i\beta}{1-e^{i\beta}} \int_0^1 e^{-i(s-1)\beta} \frac{(-i\phi(\alpha+(s-1)\beta))^m}{m!} ds\right)^n \tag{1285}$$

$$\widetilde{B_{1,3}}(\alpha, \beta) = -\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}} \cdot \frac{1}{2} \int_0^1 e^{-i\beta s}(-1+s)ds \tag{1286}$$

$$\cdot \sum_{m=2}^{\infty} \frac{i\beta}{1-e^{i\beta}} \int_0^1 e^{-i(s-1)\beta} \frac{(-i\phi(\alpha+(s-1)\beta))^m}{m!} ds. \tag{1287}$$

First, we compute the Fourier modes of $\widetilde{B_{1,1}}(\alpha, \beta)$.

$$\mathcal{F}(\widetilde{B_{1,1}})(k_1, \beta) = -\sum_{j_1+j_2+n\geq 1} \frac{-i\beta e^{2i\beta}}{1-e^{i\beta}} \frac{(-1)^{j_1+1} i^{j_1+j_2+1}}{2j_1!j_2!} \cdot \tag{1288}$$

$$\mathcal{F}\Bigg(\phi(\alpha-\beta)^{j_1}\phi(\alpha)^{j_2} \int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))(-1+s)ds \cdot \tag{1289}$$

$$\left(\sum_{m=1}^{\infty} \frac{i\beta}{1-e^{i\beta}} \int_0^1 e^{-i(s-1)\beta} \frac{(-i\phi(\alpha+(s-1)\beta))^m}{m!} ds\right)^n\Bigg)(k_1). \tag{1290}$$

We can write

$$\mathcal{F}\bigg(\phi(\alpha-\beta)^{j_1}\phi(\alpha)^{j_2}\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))(-1+s)ds \tag{1291}$$

$$\cdot\bigg(\sum_{m=1}^{\infty}\frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\frac{(-i\phi(\alpha+(s-1)\beta))^m}{m!}ds\bigg)^n\bigg)(k_1) \tag{1292}$$

$$=\sum_{k_2,\dots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\bigg(\prod_{d=1}^{j_1}\mathcal{F}(\phi(\alpha-\beta))(k_d-k_{d+1})\prod_{d=1}^{j_2}\mathcal{F}(\phi)(k_{j_1+d}-k_{j_1+d+1}) \tag{1293}$$

$$\cdot\prod_{d=1}^{n}\mathcal{F}\bigg(\sum_{m=1}^{\infty}\frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\frac{(-i\phi(\alpha+(s-1)\beta))^m}{m!}ds\bigg) \tag{1294}$$

$$(k_{j_1+j_2+d}-k_{j_1+j_2+d+1})\bigg) \tag{1295}$$

$$\cdot\mathcal{F}\bigg(\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))(-1+s)ds\bigg)(k_{j_1+j_2+n+1}) \tag{1296}$$

$$=\sum_{k_2,\dots,k_{j_1+j_2+n+1}\in\mathbb{Z}} e^{-i\beta(k_1-k_{j_1+1})}\prod_{d=1}^{j_1+j_2}\mathcal{F}(\phi)(k_d-k_{d+1})\prod_{d=1}^{n}\bigg(\frac{i\beta e^{i\beta}}{1-e^{i\beta}} \tag{1297}$$

$$\cdot\int_0^1 e^{-is\beta}e^{i(s-1)\beta(k_{j_1+j_2+d}-k_{j_1+j_2+d+1})}ds\cdot\sum_{m=1}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}(\phi^m)(k_{j_1+j_2+d}-k_{j_1+j_2+d+1})\bigg) \tag{1298}$$

$$\cdot\mathcal{F}(\phi)(k_{j_1+j_2+n+1})\int_0^1 e^{-i\beta s}(-1+s)e^{ik_{j_1+j_2+n+1}\beta(-1+s)}ds \tag{1299}$$

$$=\sum_{k_2,\dots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}\mathcal{F}(\phi)(k_d-k_{d+1})\mathcal{F}(\phi)(k_{j_1+j_2+n+1}) \tag{1300}$$

$$\cdot\prod_{d=1}^{n}P(k_{j_1+j_2+d}-k_{j_1+j_2+d+1})\cdot I_n(k_1,k_{j_1+1},k_{j_1+j_2+1},\dots,k_{j_1+j_2+n},k_{j_1+j_2+n+1},\beta). \tag{1301}$$

Then

$$\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{1,1}})(k_1,\beta)d\beta \tag{1302}$$

$$=-\sum_{j_1+j_2+n\geq 1}\frac{(-1)^{j_1+1}i^{j_1+j_2+1}}{2j_1!j_2!} \tag{1303}$$

$$\cdot\sum_{k_2,\dots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}\mathcal{F}(\phi)(k_d-k_{d+1})\mathcal{F}(\phi)(k_{j_1+j_2+n+1}) \tag{1304}$$

$$\cdot\prod_{d=1}^{n}P(k_{j_1+j_2+d}-k_{j_1+j_2+d+1})\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}} \tag{1305}$$

$$\cdot I_n(k_1,k_{j_1+1},k_{j_1+j_2+1},\dots,k_{j_1+j_2+n},k_{j_1+j_2+n+1},\beta)d\beta. \tag{1306}$$

By Lemma 3,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{1,1}})(k_1,\beta)d\beta\right| \tag{1307}$$

$$\leq \sum_{j_1+j_2+n\geq 1}\frac{C_n}{2j_1!j_2!}\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})|\,|\mathcal{F}(\phi)(k_{j_1+j_2+n+1})| \tag{1308}$$

$$\cdot\prod_{d=1}^{n}|P(k_{j_1+j_2+d}-k_{j_1+j_2+d+1})|. \tag{1309}$$

Next, let us compute the Fourier coefficient of $\widetilde{B_{1,2}}(\alpha,\beta)$.

$$\mathcal{F}(\widetilde{B_{1,2}})(k_1,\beta) \tag{1310}$$

$$= -\sum_{j_1+j_2+j_3\geq 1}\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}\frac{(-1)^{j_1+j_3}i^{j_1+j_2+j_3}}{2j_1!j_2!j_3!} \tag{1311}$$

$$\cdot\mathcal{F}\Bigg(\phi(\alpha-\beta)^{j_1}\phi(\alpha)^{j_2}\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds \tag{1312}$$

$$\cdot\sum_{m=1}^{\infty}\frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\frac{(-i\phi(\alpha+(s-1)\beta))^m}{m!}ds\Bigg)(k_1). \tag{1313}$$

We can write

$$\mathcal{F}\Bigg(\phi(\alpha-\beta)^{j_1}\phi(\alpha)^{j_2}\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds \tag{1314}$$

$$\cdot\sum_{m=1}^{\infty}\frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\frac{(-i\phi(\alpha+(s-1)\beta))^m}{m!}ds\Bigg)(k_1) \tag{1315}$$

$$= \sum_{k_2,\ldots,k_{j_1+j_2+2}\in\mathbb{Z}}\Bigg(\prod_{d=1}^{j_1}\mathcal{F}(\phi(\alpha-\beta))(k_d-k_{d+1})\prod_{d=1}^{j_2}\mathcal{F}(\phi)(k_{j_1+d}-k_{j_1+d+1}) \tag{1316}$$

$$\cdot\mathcal{F}\Bigg(\sum_{m=1}^{\infty}\frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\frac{(-i\phi(\alpha+(s-1)\beta))^m}{m!}ds\Bigg)(k_{j_1+j_2+1}-k_{j_1+j_2+2})\Bigg) \tag{1317}$$

$$\cdot\mathcal{F}\Bigg(\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds\Bigg)(k_{j_1+j_2+2}) \tag{1318}$$

$$= \sum_{k_2,\ldots,k_{j_1+j_2+2}\in\mathbb{Z}} I_1(k_1,k_{j_1+1},k_{j_1+j_2+1},k_{j_1+j_2+2},\beta) \tag{1319}$$

$$\cdot\prod_{d=1}^{j_1+j_2}\mathcal{F}(\phi)(k_d-k_{d+1})\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+2})P(k_{j_1+j_2+1}-k_{j_1+j_2+2}). \tag{1320}$$

Then

$$\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{1,2}})(k_1,\beta)d\beta \tag{1321}$$
$$=-\sum_{j_1+j_2+j_3\geq 1}\frac{(-1)^{j_1+j_3}i^{j_1+j_2+j_3}}{2j_1!j_2!j_3!} \tag{1322}$$
$$\cdot\sum_{k_2,\ldots,k_{j_1+j_2+2}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}\mathcal{F}(\phi)(k_d-k_{d+1})\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+2})P(k_{j_1+j_2+1}-k_{j_1+j_2+2}) \tag{1323}$$
$$\cdot\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}I_1(k_1,k_{j_1+1},k_{j_1+j_2+1,k_{j_1+j_2+2}},\beta)d\beta. \tag{1324}$$

By Lemma 3,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{1,2}})(k_1,\beta)d\beta\right| \tag{1325}$$
$$\leq\sum_{j_1+j_2+j_3\geq 1}\frac{C_1}{2j_1!j_2!j_3!}\sum_{k_2,\ldots,k_{j_1+j_2+2}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}\left|\mathcal{F}(\phi)(k_d-k_{d+1})\right|\left|\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+2})\right| \tag{1326}$$
$$\cdot\left|P(k_{j_1+j_2+1}-k_{j_1+j_2+2})\right|. \tag{1327}$$

Next, let us compute the Fourier coefficient of $\widetilde{B_{1,3}}(\alpha,\beta)$. We can write

$$\mathcal{F}(\widetilde{B_{1,3}})(k_1,\beta) \tag{1328}$$
$$=\frac{i\beta e^{2i\beta}}{1-e^{i\beta}}\cdot\frac{1}{2}\int_0^1 e^{-i\beta s}(-1+s)ds \tag{1329}$$
$$\cdot\mathcal{F}\left(\sum_{m=2}^{\infty}\frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\frac{(-i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)(k_1) \tag{1330}$$
$$=\frac{i\beta e^{2i\beta}}{1-e^{i\beta}}\cdot\frac{1}{2}\int_0^1 e^{-i\beta s}(-1+s)ds\cdot\sum_{m=2}^{\infty}\frac{i\beta e^{i\beta}}{1-e^{i\beta}}\frac{(-i)^m}{m!}\mathcal{F}(\phi^m)(k_1)\cdot\int_0^1 e^{-is\beta}e^{i(s-1)\beta k_1}ds \tag{1331}$$
$$=\frac{i\beta e^{2i\beta}}{1-e^{i\beta}}\cdot\frac{1}{2}\int_0^1 e^{-i\beta s}(-1+s)ds\cdot\frac{i\beta e^{i\beta}}{1-e^{i\beta}}\cdot\int_0^1 e^{-is\beta}e^{i(s-1)\beta k_1}ds\cdot\widetilde{P}(k_1). \tag{1332}$$

Then

$$\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{1,3}})(k_1,\beta)d\beta \tag{1333}$$
$$=\widetilde{P}(k_1)\cdot\frac{1}{2}\cdot\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}e^{3i\beta}\left(\frac{i\beta}{1-e^{i\beta}}\right)^2\cdot\int_0^1 e^{-i\beta s}(-1+s)ds\cdot\int_0^1 e^{-is\beta}e^{i(s-1)\beta k_1}dsd\beta. \tag{1334}$$

It follows that

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{1,3}})(k_1,\beta)d\beta\right| \tag{1335}$$

$$\leq\frac{\left|\widetilde{P}(k_1)\right|}{2}\cdot\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\left|\left(\frac{i\beta}{1-e^{i\beta}}\right)^2-1+1\right|d\beta \tag{1336}$$

$$\leq\frac{1}{2}\cdot\frac{\gamma}{4\pi}\left(\frac{\pi}{2}\sqrt{1+\frac{\pi^2}{4}}\cdot\pi^2+2\pi\right)\left|\widetilde{P}(k_1)\right|. \tag{1337}$$

Next, let us compute the Fourier coefficient of $\widetilde{B}_2(\alpha,\beta)$.

$$\mathcal{F}(\widetilde{B_2})(k_1,\beta) \tag{1338}$$

$$=-\frac{1}{2}\sum_{j_1+j_2+j_3+n\geq 1}\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}\frac{(-1)^{j_1+j_3}i^{j_1+j_2+j_3}}{j_1!j_2!j_3!} \tag{1339}$$

$$\cdot\mathcal{F}\Bigg(\phi(\alpha-\beta)^{j_1}\phi(\alpha)^{j_2}\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)\phi'(\alpha+\beta(-1+s))ds \tag{1340}$$

$$\cdot\left(\sum_{m=1}^{\infty}\frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\frac{(-i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n\Bigg)(k_1) \tag{1341}$$

$$=-\frac{1}{2}\sum_{j_1+j_2+j_3+n\geq 1}\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}\frac{(-1)^{j_1+j_3}i^{j_1+j_2+j_3}}{j_1!j_2!j_3!} \tag{1342}$$

$$\cdot\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1}\mathcal{F}(\phi(\alpha-\beta))(k_d-k_{d+1}) \tag{1343}$$

$$\cdot\prod_{d=1}^{j_2}\mathcal{F}(\phi)(k_{j_1+d}-k_{j_1+d+1}) \tag{1344}$$

$$\cdot\prod_{d=1}^{n}\mathcal{F}\left(\sum_{m=1}^{\infty}\frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\frac{(-i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)(k_{j_1+j_2+d}-k_{j_1+j_2+d+1}) \tag{1345}$$

$$\cdot\mathcal{F}\left(\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)\phi'(\alpha+\beta(-1+s))ds\right)(k_{j_1+j_2+n+1}). \tag{1346}$$

We can write

$$\mathcal{F}(\widetilde{B_2})(k_1,\beta) \tag{1347}$$
$$= -\frac{1}{2}\sum_{j_1+j_2+j_3+n\geq 1}\frac{(-1)^{j_1+j_3}i^{j_1+j_2+j_3}}{j_1!j_2!j_3!} \tag{1348}$$
$$\cdot \sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}\mathcal{F}(\phi)(k_d-k_{d+1}) \tag{1349}$$
$$\cdot \prod_{d=1}^{n}\left(\frac{i\beta e^{i\beta}}{1-e^{i\beta}}\int_0^1 e^{-is\beta}e^{i(s-1)\beta(k_{j_1+j_2+d}-k_{j_1+j_2+d+1})}ds\right) \tag{1350}$$
$$\cdot \prod_{d=1}^{n}P(k_{j_1+j_2+d}-k_{j_1+j_2+d+1})\cdot\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}\cdot e^{-i\beta(k_1-k_{j_1+1})} \tag{1351}$$
$$\cdot \int_0^1 e^{-i\beta s}e^{i\beta(-1+s)k_{j_1+j_2+n+1}}(-1+s)ds\cdot\mathcal{F}(\phi^{j_3}\phi')(k_{j_1+j_2+n+1}). \tag{1352}$$

Then

$$\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_2})(k_1,\beta)d\beta \tag{1353}$$
$$= -\frac{1}{2}\sum_{j_1+j_2+j_3+n\geq 1}\frac{(-1)^{j_1+j_3}i^{j_1+j_2+j_3}}{j_1!j_2!j_3!}\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}\mathcal{F}(\phi)(k_d-k_{d+1}) \tag{1354}$$
$$\cdot \prod_{d=1}^{n}P(k_{j_1+j_2+d}-k_{j_1+j_2+d+1})\mathcal{F}(\phi^{j_3}\phi')(k_{j_1+j_2+n+1}) \tag{1355}$$
$$\cdot \frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\prod_{d=1}^{n}\left(\frac{i\beta e^{i\beta}}{1-e^{i\beta}}\int_0^1 e^{-is\beta}e^{i(s-1)\beta(k_{j_1+j_2+d}-k_{j_1+j_2+d+1})}ds\right)\cdot\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}e^{-i\beta(k_1-k_{j_1+1})} \tag{1356}$$
$$\cdot \int_0^1 e^{-i\beta s}e^{i\beta(-1+s)k_{j_1+j_2+n+1}}(-1+s)dsd\beta. \tag{1357}$$

By Lemma 3,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_2})(k_1,\beta)d\beta\right| \tag{1358}$$
$$\leq \frac{1}{2}\sum_{j_1+j_2+j_3+n\geq 1}\frac{C_n}{j_1!j_2!j_3!}\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})| \tag{1359}$$
$$\cdot \prod_{d=1}^{n}|P(k_{j_1+j_2+d}-k_{j_1+j_2+d+1})|\cdot\left|\mathcal{F}(\phi^{j_3}\phi')(k_{j_1+j_2+n+1})\right|. \tag{1360}$$

Next, let us compute the Fourier coefficient of $\widetilde{B_3}(\alpha,\beta)$. Let $\widetilde{B_3} = \sum_{j=1}^{3} \widetilde{B_{3,j}}$, where

$$\widetilde{B_{3,1}}(\alpha,\beta) = \frac{1}{2}\sum_{\substack{j_1=j_2=0\\ j_3+j_4+n\geq 2}} (n+1)\frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2}\frac{i^{j_1+j_2+j_3+j_4}(-1)^{j_3}}{j_1!j_2!j_3!j_4!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2} \tag{1361}$$

$$\cdot \int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}ds \int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_4}(-1+s)ds \tag{1362}$$

$$\cdot \left(\sum_{m=1}^{\infty}\frac{-i^m}{m!}e^{-i\alpha}\frac{i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(\alpha+(s-1)\beta)}\phi(\alpha+(s-1)\beta)^m ds\right)^n \tag{1363}$$

$$\widetilde{B_{3,2}}(\alpha,\beta) = \frac{1}{2}\sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}} (n+1)\frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2}\frac{i^{j_1+j_2+j_3+j_4}(-1)^{j_3}}{j_1!j_2!j_3!j_4!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2} \tag{1364}$$

$$\cdot \int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}ds \int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_4}(-1+s)ds \tag{1365}$$

$$\cdot \left(\sum_{m=1}^{\infty}\frac{-i^m}{m!}e^{-i\alpha}\frac{i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(\alpha+(s-1)\beta)}\phi(\alpha+(s-1)\beta)^m ds\right)^n \tag{1366}$$

$$\widetilde{B_{3,3}}(\alpha,\beta) = \frac{1}{2}\cdot 2\cdot\frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2}\int_0^1 e^{-i\beta s}ds\int_0^1 e^{i\beta s}(-1+s)ds\sum_{m=2}^{\infty}\frac{-i^m}{m!}e^{-i\alpha}\frac{i\beta}{1-e^{-i\beta}} \tag{1367}$$

$$\cdot \int_0^1 e^{i(\alpha+(s-1)\beta)}\phi(\alpha+(s-1)\beta)^m ds. \tag{1368}$$

First, let us compute the Fourier coefficient of $\widetilde{B_{3,1}}(\alpha,\beta)$.

$$\mathcal{F}(\widetilde{B_{3,1}})(k_1,\beta) = \frac{1}{2}\sum_{j_3+j_4+n\geq 2}(n+1)\frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2}\frac{i^{j_3+j_4}(-1)^{j_3}}{j_3!j_4!} \tag{1369}$$

$$\cdot \mathcal{F}\Bigg(\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}ds\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_4}(-1+s)ds \tag{1370}$$

$$\cdot \left(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i\beta(s-1)}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n\Bigg)(k_1). \tag{1371}$$

We can write

$$\mathcal{F}\Big(\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}ds\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_4}(-1+s)ds \tag{1372}$$

$$\cdot\Big(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i\beta(s-1)}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\Big)^n\Big)(k_1) \tag{1373}$$

$$=\sum_{k_2,\dots,k_{n+2}\in\mathbb{Z}}\prod_{d=1}^n\mathcal{F}\Big(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i\beta(s-1)}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\Big)(k_d-k_{d+1}) \tag{1374}$$

$$\cdot\mathcal{F}\Big(\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}ds\Big)(k_{n+1}-k_{n+2}) \tag{1375}$$

$$\cdot\mathcal{F}\Big(\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_4}(-1+s)ds\Big)(k_{n+2}) \tag{1376}$$

$$=\sum_{k_2,\dots,k_{n+2}\in\mathbb{Z}}\prod_{d=1}^n\Big(\frac{-i\beta e^{-i\beta}}{1-e^{-i\beta}}\int_0^1 e^{is\beta}e^{i(s-1)\beta(k_d-k_{d+1})}ds\sum_{m=1}^{\infty}\frac{i^m}{m!}\mathcal{F}(\phi^m)(k_d-k_{d+1})\Big) \tag{1377}$$

$$\cdot\mathcal{F}(\phi^{j_3})(k_{n+1}-k_{n+2})\int_0^1 e^{-i\beta s}e^{i(k_{n+1}-k_{n+2})\beta(-1+s)}ds \tag{1378}$$

$$\cdot\mathcal{F}(\phi^{j_4})(k_{n+2})\int_0^1 e^{i\beta s}(-1+s)e^{ik_{n+2}\beta(-1+s)}ds \tag{1379}$$

$$=\sum_{k_2,\dots,k_{n+2}\in\mathbb{Z}}\prod_{d=1}^n\Big(\frac{-i\beta e^{-i\beta}}{1-e^{-i\beta}}\int_0^1 e^{is\beta}e^{i(s-1)\beta(k_d-k_{d+1})}ds\Big)\prod_{d=1}^n Q(k_d-k_{d+1}) \tag{1380}$$

$$\cdot\mathcal{F}(\phi^{j_3})(k_{n+1}-k_{n+2})\mathcal{F}(\phi^{j_4})(k_{n+2})\cdot\int_0^1 e^{-i\beta s}e^{i(k_{n+1}-k_{n+2})\beta(-1+s)}ds \tag{1381}$$

$$\cdot\int_0^1 e^{i\beta s}(-1+s)e^{ik_{n+2}\beta(-1+s)}ds \tag{1382}$$

$$=\sum_{k_2,\dots,k_{n+2}\in\mathbb{Z}}\widetilde{I}_n(k_1,\dots,k_{n+2},\beta)\prod_{d=1}^n Q(k_d-k_{d+1}) \tag{1383}$$

$$\cdot\mathcal{F}(\phi^{j_3})(k_{n+1}-k_{n+2})\mathcal{F}(\phi^{j_4})(k_{n+2}), \tag{1384}$$

where

$$\widetilde{I}_n(k_1,\dots,k_{n+2},\beta) \tag{1385}$$

$$=\prod_{d=1}^n\Big(\frac{-i\beta e^{-i\beta}}{1-e^{-i\beta}}\int_0^1 e^{is\beta}e^{i(s-1)\beta(k_d-k_{d+1})}ds\Big)\cdot e^{i\beta p} \tag{1386}$$

$$\cdot\int_0^1 e^{-i\beta s}e^{i\beta(-1+s)(k_{n+1}-k_{n+2})}ds\int_0^1 e^{i\beta s}e^{ik_{n+2}\beta(-1+s)}(-1+s)ds. \tag{1387}$$

Then

$$\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{3,1}})(k_1,\beta)d\beta \tag{1388}$$

$$=\frac{1}{2}\sum_{j_3+j_4+n\geq 2}(n+1)\frac{i^{j_3+j_4}(-1)^{j_3}}{j_3!j_4!}\sum_{k_2,\dots,k_{n+2}\in\mathbb{Z}}\prod_{d=1}^{n}Q(k_d-k_{d+1})\mathcal{F}(\phi^{j_3})(k_{n+1}-k_{n+2}) \tag{1389}$$

$$\cdot\mathcal{F}(\phi^{j_4})(k_{n+2})\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\widetilde{I}_n(k_1,\dots,k_{n+2},\beta)\frac{-\beta^2e^{-i\beta}}{(1-e^{-i\beta})^2}d\beta. \tag{1390}$$

Using an argument similar to Lemma 3, we obtain

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{3,1}})(k_1,\beta)d\beta\right|\leq\frac{1}{2}\sum_{j_3+j_4+n\geq 2}(n+1)\frac{C_{n+1}}{j_3!j_4!} \tag{1391}$$

$$\cdot\sum_{k_2,\dots,k_{n+2}\in\mathbb{Z}}\prod_{d=1}^{n}|Q(k_d-k_{d+1})|\left|\mathcal{F}(\phi^{j_3})(k_{n+1}-k_{n+2})\right|\left|\mathcal{F}(\phi^{j_4})(k_{n+2})\right|. \tag{1392}$$

Next, let us compute the Fourier coefficient of $\widetilde{B_{3,2}}(\alpha,\beta)$.

$$\mathcal{F}(\widetilde{B_{3,2}})(k_1,\beta)=\frac{1}{2}\sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}}(n+1)\frac{-\beta^2e^{-i\beta}}{(1-e^{-i\beta})^2}\frac{i^{j_1+j_2+j_3+j_4}(-1)^{j_3}}{j_1!j_2!j_3!j_4!} \tag{1393}$$

$$\cdot\mathcal{F}\Bigg(\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2}\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}ds \tag{1394}$$

$$\cdot\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_4}(-1+s)ds \tag{1395}$$

$$\cdot\left(\sum_{m=1}^{\infty}\frac{-i^m}{m!}e^{-i\alpha}\frac{i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(\alpha+(s-1)\beta)}\phi(\alpha+(s-1)\beta)^m ds\right)^n\Bigg)(k_1). \tag{1396}$$

We can write

$$
\begin{aligned}
&\mathcal{F}\Bigg(\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2} && (1397)\\
&\cdot \int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}ds \int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_4}(-1+s)ds && (1398)\\
&\cdot \Bigg(\sum_{m=1}^{\infty}\frac{-i^m}{m!}e^{-i\alpha}\frac{i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(\alpha+(s-1)\beta)}\phi(\alpha+(s-1)\beta)^m ds\Bigg)^n\Bigg)(k_1) && (1399)\\
=&\sum_{k_2,\ldots,k_{j_1+j_2+n+2}\in\mathbb{Z}}\prod_{d=1}^{j_1}\mathcal{F}(\phi)(k_d-k_{d+1})\prod_{d=1}^{j_2}\mathcal{F}(\phi(\alpha-\beta))(k_{j_1+d}-k_{j_1+d+1}) && (1400)\\
&\cdot \prod_{d=1}^{n}\mathcal{F}\Bigg(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\Bigg)(k_{j_1+j_2+d}-k_{j_1+j_2+d+1}) && (1401)\\
&\cdot \mathcal{F}\Bigg(\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}ds\Bigg)(k_{j_1+j_2+n+1}-k_{j_1+j_2+n+2}) && (1402)\\
&\cdot \mathcal{F}\Bigg(\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_4}(-1+s)ds\Bigg)(k_{j_1+j_2+n+2}) && (1403)\\
=&\sum_{k_2,\ldots,k_{j_1+j_2+n+2}\in\mathbb{Z}} e^{-i\beta(k_{j_1+1}-k_{j_1+j_2+1})}\prod_{d=1}^{j_1+j_2}\mathcal{F}(\phi)(k_d-k_{d+1})\prod_{d=1}^{n}\Bigg(\frac{-i\beta e^{-i\beta}}{1-e^{-i\beta}} && (1404)\\
&\cdot \sum_{m=1}^{\infty}\frac{i^m}{m!}\mathcal{F}(\phi^m)(k_{j_1+j_2+d}-k_{j_1+j_2+d+1})\int_0^1 e^{is\beta}e^{i(s-1)\beta(k_{j_1+j_2+d}-k_{j_1+j_2+d+1})}ds\Bigg) && (1405)\\
&\cdot \mathcal{F}(\phi^{j_3})(k_{j_1+j_2+n+1}-k_{j_1+j_2+n+2})\int_0^1 e^{-i\beta s}e^{i\beta(-1+s)(k_{j_1+j_2+n+1}-k_{j_1+j_2+n+2})}ds && (1406)\\
&\cdot \int_0^1 e^{i\beta s}e^{i\beta(-1+s)k_{j_1+j_2+n+2}}(-1+s)ds\cdot\mathcal{F}(\phi^{j_4})(k_{j_1+j_2+n+2}) && (1407)\\
=&\sum_{k_2,\ldots,k_{j_1+j_2+n+2}\in\mathbb{Z}}\widetilde{I_n}(k_1,\ldots,k_{n+2},\beta)\prod_{d=1}^{j_1+j_2}\mathcal{F}(\phi)(k_d-k_{d+1})\prod_{d=1}^{n}Q(k_{j_1+j_2+d}-k_{j_1+j_2+d}) && (1408)\\
&\cdot \mathcal{F}(\phi^{j_3})(k_{j_1+j_2+n+1}-k_{j_1+j_2+n+2})\cdot\mathcal{F}(\phi^{j_4})(k_{j_1+j_2+n+2}), && (1409)
\end{aligned}
$$

where $\widetilde{I_n}$ is defined in (1385). Then

$$
\begin{aligned}
&\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{3,2}})(k_1,\beta)d\beta=\frac{1}{2}\sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}}(n+1)\frac{i^{j_1+j_2+j_3+j_4}(-1)^{j_3}}{j_1!j_2!j_3!j_4!} && (1410)\\
&\cdot \sum_{k_2,\ldots,k_{j_1+j_2+n+2}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}\mathcal{F}(\phi)(k_d-k_{d+1})\prod_{d=1}^{n}Q(k_{j_1+j_2+d}-k_{j_1+j_2+d}) && (1411)\\
&\cdot \mathcal{F}(\phi^{j_3})(k_{j_1+j_2+n+1}-k_{j_1+j_2+n+2})\cdot\mathcal{F}(\phi^{j_4})(k_{j_1+j_2+n+2}) && (1412)\\
&\cdot \frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\frac{-\beta^2e^{-i\beta}}{(1-e^{-i\beta})^2}\widetilde{I_n}(k_1,\ldots,k_{n+2},\beta)d\beta. && (1413)
\end{aligned}
$$

Using an argument similar to Lemma 3, we obtain

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{3,2}})(k_1,\beta)d\beta\right| \leq \frac{1}{2}\sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}}(n+1)\frac{C_{n+1}}{j_1!j_2!j_3!j_4!} \tag{1414}$$

$$\cdot\sum_{k_2,\dots,k_{j_1+j_2+n+2}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})|\prod_{d=1}^{n}|Q(k_{j_1+j_2+d}-k_{j_1+j_2+d})| \tag{1415}$$

$$\cdot\left|\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+n+1}-k_{j_1+j_2+n+2})\right|\cdot\left|\mathcal{F}(\phi^{j_4})(k_{j_1+j_2+n+2})\right|. \tag{1416}$$

Next, let us compute the Fourier coefficient of $\widetilde{B_{3,3}}(\alpha,\beta)$.

$$\begin{aligned}\mathcal{F}(\widetilde{B_{3,3}})(k_1,\beta) =&\frac{-\beta^2e^{-i\beta}}{(1-e^{-i\beta})^2}\int_0^1 e^{-i\beta s}ds\int_0^1 e^{i\beta s}(-1+s)ds &(1417)\\ &\cdot\mathcal{F}\left(\sum_{k=2}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i\beta(s-1)}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)(k_1) &(1418)\\ =&\frac{-\beta^2e^{-i\beta}}{(1-e^{-i\beta})^2}\int_0^1 e^{-i\beta s}ds\int_0^1 e^{i\beta s}(-1+s)ds &(1419)\\ &\cdot\frac{-i\beta e^{-i\beta}}{1-e^{-i\beta}}\int_0^1 e^{is\beta}e^{i(s-1)\beta k_1}ds\cdot\sum_{m=2}^{\infty}\frac{i^m}{m!}\mathcal{F}(\phi^m)(k_1). &(1420)\end{aligned}$$

Then

$$\begin{aligned}\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{3,3}})d\beta =&\widetilde{Q}(k_1)\cdot\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}e^{-2i\beta}\left(\frac{-i\beta}{1-e^{-i\beta}}\right)^3\int_0^1 e^{-i\beta s}ds &(1421)\\ &\cdot\int_0^1 e^{i\beta s}(-1+s)ds\int_0^1 e^{is\beta}e^{i(s-1)\beta k_1}dsd\beta. &(1422)\end{aligned}$$

It follows that

$$\begin{aligned}\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{3,3}})d\beta\right| \leq&\left|\widetilde{Q}(k_1)\right|\cdot\frac{\gamma}{4\pi}\left(\int_{-\pi}^{\pi}\left|\left(\frac{-i\beta}{1-e^{-i\beta}}\right)^3-1\right|d\beta+2\pi\right) &(1423)\\ \leq&\left|\widetilde{Q}(k_1)\right|\cdot\frac{\gamma}{4\pi}\left(3\left(\frac{\pi}{2}\right)^2\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}\cdot\pi^2+2\pi\right). &(1424)\end{aligned}$$

Next, let us compute the Fourier coefficient of $\widetilde{B_4}(\alpha,\beta)$.

$$\mathcal{F}(\widetilde{B_4})(k_1,\beta) = \frac{1}{2}\sum_{j_1+j_2+j_3+j_4+n\geq 1}(n+1)\frac{-\beta^2e^{-i\beta}}{(1-e^{-i\beta})^2}\frac{i^{j_1+j_2+j_3+j_4}(-1)^{j_3}}{j_1!j_2!j_3!j_4!} \tag{1425}$$

$$\cdot\mathcal{F}\Bigg(\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2}\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}ds \tag{1426}$$

$$\cdot\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_4}(-1+s)\phi'(\alpha+\beta(-1+s))ds \tag{1427}$$

$$\cdot\left(\sum_{m=1}^{\infty}\frac{-i^m}{m!}\frac{i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\phi(\alpha+(s-1)\beta)^m ds\right)^n\Bigg)(k_1). \tag{1428}$$

We can write

$$\mathcal{F}\Bigg(\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2}\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}ds \tag{1429}$$

$$\cdot\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_4}(-1+s)\phi'(\alpha+\beta(-1+s))ds \tag{1430}$$

$$\cdot\Bigg(\sum_{m=1}^{\infty}\frac{-i^m}{m!}\frac{i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\phi(\alpha+(s-1)\beta)^m ds\Bigg)^n\Bigg)(k_1) \tag{1431}$$

$$=\sum_{k_2,\ldots,k_{j_1+j_2+n+2}\in\mathbb{Z}}\prod_{d=1}^{j_1}\mathcal{F}(\phi)(k_d-k_{d+1})\prod_{d=1}^{j_2}\mathcal{F}(\phi(\alpha-\beta))(k_{j_1+d}-k_{j_1+d+1}) \tag{1432}$$

$$\cdot\prod_{d=1}^{n}\mathcal{F}\Bigg(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\Bigg)(k_{j_1+j_2+d}-k_{j_1+j_2+d+1}) \tag{1433}$$

$$\cdot\mathcal{F}\Bigg(\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}ds\Bigg)(k_{j_1+j_2+n+1}-k_{j_1+j_2+n+2}) \tag{1434}$$

$$\cdot\mathcal{F}\Bigg(\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_4}(-1+s)\phi'(\alpha+\beta(-1+s))ds\Bigg)(k_{j_1+j_2+n+2}) \tag{1435}$$

$$=\sum_{k_2,\ldots,k_{j_1+j_2+n+2}\in\mathbb{Z}}e^{-i\beta(k_{j_1+1}-k_{j_1+j_2+1})} \tag{1436}$$

$$\cdot\prod_{d=1}^{j_1+j_2}\mathcal{F}(\phi)(k_d-k_{d+1})\prod_{d=1}^{n}\Bigg(\sum_{m=1}^{\infty}\frac{-i\beta e^{-i\beta}}{1-e^{-i\beta}}\frac{i^m}{m!}\mathcal{F}(\phi^m)(k_{j_1+j_2+d}-k_{j_1+j_2+d+1}) \tag{1437}$$

$$\cdot\int_0^1 e^{is\beta}e^{i(s-1)\beta(k_{j_1+j_2+d}-k_{j_1+j_2+d+1})}ds\Bigg)\cdot\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+n+1}-k_{j_1+j_2+n+2}) \tag{1438}$$

$$\cdot\int_0^1 e^{-i\beta s}e^{i(k_{j_1+j_2+n+1}-k_{j_1+j_2+n+2})\beta(-1+s)}ds\int_0^1 e^{i\beta s}e^{i\beta(-1+s)k_{j_1+j_2+n+2}}(-1+s)ds \tag{1439}$$

$$\cdot\mathcal{F}(\phi^{j_4}\phi')(k_{j_1+j_2+n+2}) \tag{1440}$$

$$=\sum_{k_2,\ldots,k_{j_1+j_2+n+2}\in\mathbb{Z}}\widetilde{I}_n(k_1,\ldots,k_{n+2},\beta)\cdot\prod_{d=1}^{j_1+j_2}\mathcal{F}(\phi)(k_d-k_{d+1}) \tag{1441}$$

$$\cdot\prod_{d=1}^{n}Q(k_{j_1+j_2+d}-k_{j_1+j_2+d+1})\cdot\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+n+1}-k_{j_1+j_2+n+2}) \tag{1442}$$

$$\cdot\mathcal{F}(\phi^{j_4}\phi')(k_{j_1+j_2+n+2}). \tag{1443}$$

Then

$$\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_4})(k_1,\beta)d\beta = \frac{1}{2}\sum_{j_1+j_2+j_3+j_4+n\geq 1}(n+1)\frac{i^{j_1+j_2+j_3+j_4}(-1)^{j_3}}{j_1!j_2!j_3!j_4!} \tag{1444}$$

$$\cdot \sum_{k_2,\ldots,k_{j_1+j_2+n+2}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}\mathcal{F}(\phi)(k_d-k_{d+1}) \tag{1445}$$

$$\cdot \prod_{d=1}^{n}Q(k_{j_1+j_2+d}-k_{j_1+j_2+d+1})\cdot\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+n+1}-k_{j_1+j_2+n+2}) \tag{1446}$$

$$\cdot \mathcal{F}(\phi^{j_4}\phi')(k_{j_1+j_2+n+2})\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\frac{-\beta^2e^{-i\beta}}{(1-e^{-i\beta})^2}\widetilde{I_n}(k_1,\ldots,k_{n+2},\beta)d\beta, \tag{1447}$$

where $\widetilde{I_n}$ is defined in (1385). Using an argument similar to Lemma 3, we obtain

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_4})(k_1,\beta)d\beta\right| \leq \frac{1}{2}\sum_{j_1+j_2+j_3+j_4+n\geq 1}(n+1)\frac{C_{n+1}}{j_1!j_2!j_3!j_4!} \tag{1448}$$

$$\cdot \sum_{k_2,\ldots,k_{j_1+j_2+n+2}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})|\cdot\prod_{d=1}^{n}|Q(k_{j_1+j_2+d}-k_{j_1+j_2+d+1})| \tag{1449}$$

$$\cdot \left|\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+n+1}-k_{j_1+j_2+n+2})\right|\cdot\left|\mathcal{F}(\phi^{j_4}\phi')(k_{j_1+j_2+n+2})\right|. \tag{1450}$$

Next, let us compute the Fourier coefficient of $\widetilde{B_5}(\alpha,\beta)$. We will write $\widetilde{B_5}=\sum_{j=1}^{3}\widetilde{B_{5,j}}$, where

$$\widetilde{B_{5,1}}(\alpha,\beta) = \frac{1}{2}\sum_{\substack{j_1=j_2=0\\ j_3+n\geq 2}}\frac{i\beta}{1-e^{-i\beta}}\frac{i^{j_1+j_2+j_3}(-1)^{j_3}}{j_1!j_2!j_3!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2} \tag{1451}$$

$$\cdot \int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds \tag{1452}$$

$$\cdot \left(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n \tag{1453}$$

$$\widetilde{B_{5,2}}(\alpha,\beta) = \frac{1}{2}\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}}\frac{i\beta}{1-e^{-i\beta}}\frac{i^{j_1+j_2+j_3}(-1)^{j_3}}{j_1!j_2!j_3!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2} \tag{1454}$$

$$\cdot \int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds \tag{1455}$$

$$\cdot \left(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n \tag{1456}$$

$$\widetilde{B_{5,3}}(\alpha,\beta) = \frac{1}{2}\cdot\frac{i\beta}{1-e^{-i\beta}}\int_0^1 e^{-i\beta s}(-1+s)ds \tag{1457}$$

$$\cdot \sum_{m=2}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds. \tag{1458}$$

First, let us compute the Fourier coefficient of $\widetilde{B_{5,1}}(\alpha, \beta)$.

$$\mathcal{F}(\widetilde{B_{5,1}})(k_1, \beta) = \frac{1}{2} \sum_{j_3+n\geq 2} \frac{i\beta}{1-e^{-i\beta}} \frac{i^{j_3}(-1)^{j_3}}{j_3!} \tag{1459}$$

$$\cdot \mathcal{F}\bigg(\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds \tag{1460}$$

$$\cdot \bigg(\sum_{m=1}^{\infty} \frac{-i\beta}{1-e^{-i\beta}} \int_0^1 e^{i(s-1)\beta} \frac{(i\phi(\alpha+(s-1)\beta))^m}{m!} ds\bigg)^n\bigg)(k_1). \tag{1461}$$

We can write

$$\mathcal{F}\bigg(\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds \tag{1462}$$

$$\cdot \bigg(\sum_{m=1}^{\infty} \frac{-i\beta}{1-e^{-i\beta}} \int_0^1 e^{i(s-1)\beta} \frac{(i\phi(\alpha+(s-1)\beta))^m}{m!} ds\bigg)^n\bigg)(k_1) \tag{1463}$$

$$= \sum_{k_2,\ldots,k_{n+1}\in\mathbb{Z}} \prod_{d=1}^{n} \mathcal{F}\bigg(\sum_{m=1}^{\infty} \frac{-i\beta}{1-e^{-i\beta}} \int_0^1 e^{i(s-1)\beta} \frac{(i\phi(\alpha+(s-1)\beta))^m}{m!} ds\bigg)(k_d-k_{d+1}) \tag{1464}$$

$$\cdot \mathcal{F}\bigg(\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds\bigg)(k_{n+1}) \tag{1465}$$

$$= \sum_{k_2,\ldots,k_{n+1}\in\mathbb{Z}} \prod_{d=1}^{n} \bigg(\sum_{m=1}^{\infty} \frac{-i\beta e^{-i\beta}}{1-e^{-i\beta}} \frac{i^m}{m!} \mathcal{F}(\phi^m)(k_d-k_{d+1}) \int_0^1 e^{is\beta} e^{i(s-1)\beta(k_d-k_{d+1})} ds\bigg) \tag{1466}$$

$$\cdot \int_0^1 e^{-i\beta s} e^{i\beta(-1+s)k_{n+1}}(-1+s)ds \cdot \mathcal{F}(\phi^{j_3})(k_{n+1}) \tag{1467}$$

$$= \sum_{k_2,\ldots,k_{n+1}\in\mathbb{Z}} \prod_{d=1}^{n} Q(k_d-k_{d+1}) \mathcal{F}(\phi^{j_3})(k_{n+1}) \prod_{d=1}^{n} \bigg(\frac{-i\beta e^{-i\beta}}{1-e^{-i\beta}} \int_0^1 e^{is\beta} e^{i(s-1)\beta(k_d-k_{d+1})} ds\bigg) \tag{1468}$$

$$\cdot \int_0^1 e^{-i\beta s} e^{i\beta(-1+s)k_{n+1}}(-1+s)ds \tag{1469}$$

$$= \sum_{k_2,\ldots,k_{n+1}\in\mathbb{Z}} I_{n,1}(k_1,\ldots,k_{n+1},\beta) \cdot \prod_{d=1}^{n} Q(k_d-k_{d+1}) \cdot \mathcal{F}(\phi^{j_3})(k_{n+1}) \tag{1470}$$

$$\tag{1471}$$

where

$$I_{n,1}(k_1,\ldots,k_{n+1},\beta) = \prod_{d=1}^{n} \bigg(\frac{-i\beta e^{-i\beta}}{1-e^{-i\beta}} \int_0^1 e^{is\beta} e^{i(s-1)\beta(k_d-k_{d+1})} ds\bigg) \tag{1472}$$

$$\cdot \int_0^1 e^{-i\beta s} e^{i\beta(-1+s)k_{n+1}}(-1+s)ds. \tag{1473}$$

Then

$$\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{5,1}})(k_1,\beta)d\beta = \frac{1}{2}\sum_{j_3+n\geq 2}\frac{i^{j_3}(-1)^{j_3}}{j_3!} \tag{1474}$$

$$\cdot \sum_{k_2,\ldots,k_{n+1}\in\mathbb{Z}}\prod_{d=1}^{n}Q(k_d-k_{d+1})\cdot\mathcal{F}(\phi^{j_3})(k_{n+1})\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}I_{n,1}(k_1,\ldots,k_{n+1},\beta)\frac{i\beta}{1-e^{-i\beta}}d\beta. \tag{1475}$$

Using an argument similar to Lemma 3, we obtain

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}I_{n,1}(k_1,\ldots,k_{n+1},\beta)\frac{i\beta}{1-e^{-i\beta}}d\beta\right| \tag{1476}$$

$$\leq\frac{\gamma}{4\pi}\left((n+1)\cdot\left(\frac{\pi}{2}\right)^n\cdot\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}\cdot\pi^2+2\pi\right) \tag{1477}$$

$$=C_n. \tag{1478}$$

It follows that

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{5,1}})(k_1,\beta)d\beta\right| \tag{1479}$$

$$\leq\frac{1}{2}\sum_{j_3+n\geq 2}\frac{C_n}{j_1!j_2!j_3!}\sum_{k_2,\ldots,k_{n+1}\in\mathbb{Z}}\prod_{d=1}^{n}\left|Q(k_d-k_{d+1})\right|\left|\mathcal{F}(\phi^{j_3})(k_{n+1})\right|. \tag{1480}$$

Next, let us compute the Fourier coefficient of $\widetilde{B_{5,2}}(\alpha,\beta)$.

$$\mathcal{F}(\widetilde{B_{5,2}})(k_1,\beta) = \frac{1}{2}\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}}\frac{i\beta}{1-e^{-i\beta}}\frac{i^{j_1+j_2+j_3}(-1)^{j_3}}{j_1!j_2!j_3!} \tag{1481}$$

$$\cdot\mathcal{F}\Bigg(\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2}\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds \tag{1482}$$

$$\cdot\left(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n\Bigg)(k_1). \tag{1483}$$

We can write

$$\mathcal{F}\Bigg(\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2}\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds \tag{1484}$$

$$\cdot\Bigg(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\Bigg)^n\Bigg)(k_1) \tag{1485}$$

$$=\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1}\mathcal{F}(\phi)(k_d-k_{d+1})\prod_{d=1}^{j_2}\mathcal{F}(\phi(\alpha-\beta))(k_{j_1+d}-k_{j_1+d+1}) \tag{1486}$$

$$\cdot\prod_{d=1}^{n}\mathcal{F}\Bigg(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\Bigg)(k_{j_1+j_2+d}-k_{j_1+j_2+d+1}) \tag{1487}$$

$$\cdot\mathcal{F}\Bigg(\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds\Bigg)(k_{j_1+j_2+n+1}) \tag{1488}$$

$$=\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}e^{-i\beta(k_{j_1+1}-k_{j_1+j_2+1})}\prod_{d=1}^{j_1+j_2}\mathcal{F}(\phi)(k_d-k_{d+1}) \tag{1489}$$

$$\cdot\prod_{d=1}^{n}\Bigg(\sum_{m=1}^{\infty}\frac{-i\beta e^{-i\beta}}{1-e^{-i\beta}}\frac{i^m}{m!}\mathcal{F}(\phi^m)(k_{j_1+j_2+d}-k_{j_1+j_2+d+1}) \tag{1490}$$

$$\cdot\int_0^1 e^{is\beta}e^{i(s-1)\beta(k_{j_1+j_2+d}-k_{k_1+k_2+d+1})}ds\Bigg) \tag{1491}$$

$$\cdot\int_0^1 e^{-i\beta s}e^{i\beta(-1+s)k_{j_1+j_2+n+1}}(-1+s)ds\cdot\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+n+1}) \tag{1492}$$

$$=\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}\mathcal{F}(\phi)(k_d-k_{d+1})\prod_{d=1}^{n}Q(k_{j_1+j_2+d}-k_{j_1+j_2+d+1}) \tag{1493}$$

$$\cdot\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+n+1}) \tag{1494}$$

$$\cdot e^{-i\beta(k_{j_1+1}-k_{j_1+j_2+1})}\prod_{d=1}^{n}\Bigg(\frac{-i\beta e^{-i\beta}}{1-e^{-i\beta}}\int_0^1 e^{is\beta}e^{i(s-1)\beta(k_{j_1+j_2+d}-k_{j_1+j_2+d+1})}ds\Bigg) \tag{1495}$$

$$\cdot\int_0^1 e^{-i\beta s}e^{i\beta(-1+s)k_{j_1+j_2+n+1}}(-1+s)ds \tag{1496}$$

$$=\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}\mathcal{F}(\phi)(k_d-k_{d+1})\prod_{d=1}^{n}Q(k_{j_1+j_2+d}-k_{j_1+j_2+d+1}) \tag{1497}$$

$$\cdot\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+n+1})\cdot I_{n,2}(k_{j_1+1},k_{j_1+j_2+1},\ldots,k_{j_1+j_2+n+1},\beta), \tag{1498}$$

where

$$I_{n,2}(k_{j_1+1}, k_{j_1+j_2+1}, \ldots, k_{j_1+j_2+n+1}, \beta) \tag{1499}$$

$$= e^{-i\beta(k_{j_1+1}-k_{j_1+j_2+1})} \prod_{d=1}^{n} \left(\frac{-i\beta e^{-i\beta}}{1-e^{-i\beta}} \int_0^1 e^{is\beta} e^{i(s-1)\beta(k_{j_1+j_2+d}-k_{j_1+j_2+d+1})} ds \right) \tag{1500}$$

$$\cdot \int_0^1 e^{-i\beta s} e^{i\beta(-1+s)k_{j_1+j_2+n+1}}(-1+s)ds. \tag{1501}$$

Then

$$\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} \mathcal{F}(\widetilde{B_{5,2}})(k_1,\beta)d\beta = \frac{1}{2}\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{i^{j_1+j_2+j_3}(-1)^{j_3}}{j_1!j_2!j_3!} \tag{1502}$$

$$\cdot \sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}} \prod_{d=1}^{j_1+j_2} \mathcal{F}(\phi)(k_d-k_{d+1}) \prod_{d=1}^{n} Q(k_{j_1+j_2+d}-k_{j_1+j_2+d+1}) \tag{1503}$$

$$\cdot \mathcal{F}(\phi^{j_3})(k_{j_1+j_2+n+1}) \tag{1504}$$

$$\cdot \frac{\gamma}{4\pi}\int_{-\pi}^{\pi} I_{n,2}(k_{j_1+1}, k_{j_1+j_2+1}, \ldots, k_{j_1+j_2+n+1}, \beta)\frac{i\beta}{1-e^{-i\beta}}d\beta. \tag{1505}$$

Using an argument similar to Lemma 3, we obtain

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} I_{n,2}(k_{j_1+1}, k_{j_1+j_2+1}, \ldots, k_{j_1+j_2+n+1}, \beta)\frac{i\beta}{1-e^{-i\beta}}d\beta\right| \tag{1506}$$

$$\leq \frac{\gamma}{4\pi}\left((n+1)\left(\frac{\pi}{2}\right)^n \cdot \frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}\cdot\pi^2 + 2\pi\right) \tag{1507}$$

$$= C_n. \tag{1508}$$

It follows that

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} \mathcal{F}(\widetilde{B_{5,2}})(k_1,\beta)d\beta\right| \leq \frac{1}{2}\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{C_n}{j_1!j_2!j_3!} \tag{1509}$$

$$\cdot \sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}} \prod_{d=1}^{j_1+j_2} |\mathcal{F}(\phi)(k_d-k_{d+1})| \prod_{d=1}^{n} |Q(k_{j_1+j_2+d}-k_{j_1+j_2+d+1})| \tag{1510}$$

$$\cdot \left|\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+n+1})\right|. \tag{1511}$$

Next, let us compute the Fourier coefficient of $\widetilde{B_{5,3}}(\alpha,\beta)$. We can write

$$\mathcal{F}(\widetilde{B_{5,3}})(k_1,\beta) = \frac{1}{2}\cdot\frac{i\beta}{1-e^{-i\beta}}\int_0^1 e^{-i\beta s}(-1+s)ds \tag{1512}$$

$$\cdot \mathcal{F}\left(\sum_{m=2}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)(k_1) \tag{1513}$$

$$= \frac{1}{2}\cdot\frac{i\beta}{1-e^{-i\beta}}\int_0^1 e^{-i\beta s}(-1+s)ds \cdot \frac{-i\beta e^{-i\beta}}{1-e^{-i\beta}}\cdot\int_0^1 e^{is\beta}e^{i(s-1)\beta k_1}ds\cdot\widetilde{Q}(k_1). \tag{1514}$$

Then

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{5,3}})(k_1,\beta)d\beta\right| \leq \frac{1}{2}\cdot\left|\widetilde{Q}(k_1)\right|\cdot\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\left|\left(\frac{-i\beta}{1-e^{-i\beta}}\right)^2-1+1\right|d\beta \tag{1515}$$

$$\leq \frac{1}{2}\cdot\left|\widetilde{Q}(k_1)\right|\frac{\gamma}{4\pi}\left(2\left(\frac{\pi}{2}\right)\cdot\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}\cdot\pi^2+2\pi\right). \tag{1516}$$

Next, let us compute the Fourier coefficient of $\widetilde{B_6}(\alpha,\beta)$. We will write $\widetilde{B_6}=\sum_{j=1}^{3}\widetilde{B_{6,j}}$, where

$$\widetilde{B_{6,1}}(\alpha,\beta) = \frac{1}{2}\sum_{\substack{j_1=j_2=0\\ j_3+n\geq 2}}\frac{i\beta}{1-e^{-i\beta}}\frac{i^{j_1+j_2+j_3}(-1)^{j_2}}{j_1!j_2!j_3!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2} \tag{1517}$$

$$\cdot\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds \tag{1518}$$

$$\cdot\left(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n \tag{1519}$$

$$\widetilde{B_{6,2}}(\alpha,\beta) = \frac{1}{2}\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}}\frac{i\beta}{1-e^{-i\beta}}\frac{i^{j_1+j_2+j_3}(-1)^{j_2}}{j_1!j_2!j_3!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2} \tag{1520}$$

$$\cdot\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds \tag{1521}$$

$$\cdot\left(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n \tag{1522}$$

$$\widetilde{B_{6,3}}(\alpha,\beta) = \frac{1}{2}\cdot\frac{i\beta}{1-e^{-i\beta}}\int_0^1 e^{i\beta s}(-1+s)ds \tag{1523}$$

$$\cdot\sum_{m=2}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds. \tag{1524}$$

First, let us compute the Fourier coefficient of $\widetilde{B_{6,1}}(\alpha,\beta)$.

$$\mathcal{F}(\widetilde{B_{6,1}})(k_1,\beta) = \frac{1}{2}\sum_{j_3+n\geq 2}\frac{i\beta}{1-e^{-i\beta}}\frac{i^{j_3}}{j_3!}\mathcal{F}\left(\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds\right. \tag{1525}$$

$$\left.\cdot\left(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n\right)(k_1). \tag{1526}$$

We can write

$$\mathcal{F}\Bigg(\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds \tag{1527}$$

$$\cdot\Bigg(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\Bigg)^n\Bigg)(k_1) \tag{1528}$$

$$=\sum_{k_2,\ldots,k_{n+1}\in\mathbb{Z}}\prod_{d=1}^{n}\mathcal{F}\Bigg(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\Bigg)(k_d-k_{d+1}) \tag{1529}$$

$$\cdot\mathcal{F}\Bigg(\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds\Bigg)(k_{n+1}) \tag{1530}$$

$$=\sum_{k_2,\ldots,k_{n+1}\in\mathbb{Z}}\prod_{d=1}^{n}\Bigg(\sum_{m=1}^{\infty}\frac{-i\beta e^{-i\beta}}{1-e^{-i\beta}}\cdot\frac{i^m}{m!}\cdot\mathcal{F}(\phi^m)(k_d-k_{d+1})\int_0^1 e^{is\beta}e^{i(s-1)\beta(k_d-k_{d+1})}ds\Bigg) \tag{1531}$$

$$\cdot\int_0^1 e^{i\beta s}e^{i\beta(-1+s)k_{n+1}}(-1+s)ds\cdot\mathcal{F}(\phi^{j_3})(k_{n+1}) \tag{1532}$$

$$=\sum_{k_2,\ldots,k_{n+1}\in\mathbb{Z}}\prod_{d=1}^{n}\Bigg(\frac{-i\beta e^{-i\beta}}{1-e^{-i\beta}}Q(k_d-k_{d+1})\int_0^1 e^{is\beta}e^{i(s-1)\beta(k_d-k_{d+1})}ds\Bigg) \tag{1533}$$

$$\cdot\int_0^1 e^{i\beta s}e^{i\beta(-1+s)k_{n+1}}(-1+s)ds\cdot\mathcal{F}(\phi^{j_3})(k_{n+1}) \tag{1534}$$

$$=\sum_{k_2,\ldots,k_{n+1}\in\mathbb{Z}}\prod_{d=1}^{n}Q(k_d-k_{d+1})\mathcal{F}(\phi^{j_3})(k_{n+1})I_{n,3}(k_1\ldots,k_{n+1},\beta), \tag{1535}$$

where

$$I_{n,3}(k_1\ldots,k_{n+1},\beta) \tag{1536}$$

$$=\prod_{d=1}^{n}\Bigg(\frac{-i\beta e^{-i\beta}}{1-e^{-i\beta}}\int_0^1 e^{is\beta}e^{i(s-1)\beta(k_d-k_{d+1})}ds\Bigg)\int_0^1 e^{i\beta s}e^{i\beta(-1+s)k_{n+1}}(-1+s)ds. \tag{1537}$$

Then

$$\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{6,1}})(k_1,\beta)d\beta=\frac{1}{2}\sum_{j_3+n\geq 2}\frac{i^{j_3}}{j_3!} \tag{1538}$$

$$\cdot\sum_{k_2,\ldots,k_{n+1}\in\mathbb{Z}}\prod_{d=1}^{n}Q(k_d-k_{d+1})\mathcal{F}(\phi^{j_3})(k_{n+1})\cdot\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}I_{n,3}(k_1\ldots,k_{n+1},\beta)\frac{i\beta}{1-e^{-i\beta}}d\beta. \tag{1539}$$

Using an argument similar to Lemma 3, we obtain

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}I_{n,3}(k_1\ldots,k_{n+1},\beta)\frac{i\beta}{1-e^{-i\beta}}d\beta\right| \tag{1540}$$

$$\leq\frac{\gamma}{4\pi}\Bigg((n+1)\cdot\left(\frac{\pi}{2}\right)^n\cdot\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}\cdot\pi^2+2\pi\Bigg) \tag{1541}$$

$$=C_n. \tag{1542}$$

It follows that

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{6,1}})(k_1,\beta)d\beta\right| \leq \frac{1}{2}\sum_{j_3+n\geq 2}\frac{C_n}{j_3!} \tag{1543}$$

$$\cdot \sum_{k_2,\ldots,k_{n+1}\in\mathbb{Z}}\prod_{d=1}^{n}|Q(k_d-k_{d+1})|\left|\mathcal{F}(\phi^{j_3})(k_{n+1})\right|. \tag{1544}$$

Next, let us compute the Fourier coefficient of $\widetilde{B_{6,2}}(\alpha,\beta)$.

$$\mathcal{F}(\widetilde{B_{6,2}})(k_1,\beta) = \frac{1}{2}\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}}\frac{i\beta}{1-e^{-i\beta}}\frac{i^{j_1+j_2+j_3}(-1)^{j_2}}{j_1!j_2!j_3!} \tag{1545}$$

$$\cdot\,\mathcal{F}\Bigg(\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2}\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds \tag{1546}$$

$$\cdot\left(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n\Bigg)(k_1). \tag{1547}$$

We can write

$$\mathcal{F}\bigg(\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2}\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds \tag{1548}$$

$$\cdot\bigg(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\bigg)^n\bigg)(k_1) \tag{1549}$$

$$=\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1}\mathcal{F}(\phi)(k_d-k_{d+1})\prod_{d=1}^{j_2}\mathcal{F}(\phi(\alpha-\beta))(k_{j_1+d}-k_{j_1+d+1}) \tag{1550}$$

$$\cdot\prod_{d=1}^{n}\mathcal{F}\bigg(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\bigg)(k_{j_1+j_2+d}-k_{j_1+j_2+d+1}) \tag{1551}$$

$$\cdot\mathcal{F}\bigg(\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds\bigg)(k_{j_1+j_2+n+1}) \tag{1552}$$

$$=\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}e^{-i\beta(k_{j_1+1}-k_{j_1+j_2+1})}\prod_{d=1}^{j_1+j_2}\mathcal{F}(\phi)(k_d-k_{d+1}) \tag{1553}$$

$$\cdot\prod_{d=1}^{n}\bigg(\frac{-i\beta e^{-i\beta}}{1-e^{-i\beta}}\int_0^1 e^{is\beta}e^{i(s-1)\beta(k_{j_1+j_2+d}-k_{j_1+j_2+d+1})}ds \tag{1554}$$

$$\cdot\sum_{d=1}^{\infty}\frac{i^m}{m!}\mathcal{F}(\phi^m)(k_{j_1+j_2+d}-k_{j_1+j_2+d+1})\bigg) \tag{1555}$$

$$\cdot\int_0^1 e^{i\beta s}e^{i\beta(-1+s)k_{j_1+j_2+n+1}}(-1+s)ds\cdot\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+n+1}) \tag{1556}$$

$$=\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}\mathcal{F}(\phi)(k_d-k_{d+1})\prod_{d=1}^{n}Q(k_{j_1+j_2+d}-k_{j_1+j_2+d+1}) \tag{1557}$$

$$\cdot\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+n+1})\cdot I_{n,4}(k_{j_1+1},k_{j_1+j_2+1},\ldots,k_{j_1+j_2+n+1},\beta), \tag{1558}$$

where

$$I_{n,4}(k_{j_1+1},k_{j_1+j_2+1},\ldots,k_{j_1+j_2+n+1},\beta)=e^{-i\beta(k_{j_1+1}-k_{j_1+j_2+1})} \tag{1559}$$

$$\cdot\prod_{d=1}^{n}\bigg(\frac{-i\beta e^{-i\beta}}{1-e^{-i\beta}}\int_0^1 e^{is\beta}e^{i(s-1)\beta(k_{j_1+j_2+d}-k_{j_1+j_2+d+1})}ds\bigg) \tag{1560}$$

$$\cdot\int_0^1 e^{i\beta s}e^{i\beta(-1+s)k_{j_1+j_2+n+1}}(-1+s)ds. \tag{1561}$$

Then

$$\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{6,2}})(k_1,\beta)d\beta=\frac{1}{2}\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}}\frac{i^{j_1+j_2+j_3}(-1)^{j_2}}{j_1!j_2!j_3!} \tag{1562}$$

$$\cdot\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}\mathcal{F}(\phi)(k_d-k_{d+1})\prod_{d=1}^{n}Q(k_{j_1+j_2+d}-k_{j_1+j_2+d+1}) \tag{1563}$$

$$\cdot\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+n+1})\cdot\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}I_{n,4}(k_{j_1+1},k_{j_1+j_2+1},\ldots,k_{j_1+j_2+n+1},\beta)\frac{i\beta}{1-e^{-i\beta}}d\beta. \tag{1564}$$

Using an argument similar to Lemma 3, we obtain

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}I_{n,4}(k_{j_1+1},k_{j_1+j_2+1},\ldots,k_{j_1+j_2+n+1},\beta)\frac{i\beta}{1-e^{-i\beta}}d\beta\right| \tag{1565}$$

$$\leq\frac{\gamma}{4\pi}\left((n+1)\cdot\left(\frac{\pi}{2}\right)^{n}\cdot\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}\cdot\pi^2+2\pi\right) \tag{1566}$$

$$=C_n. \tag{1567}$$

It follows that

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{6,2}})(k_1,\beta)d\beta\right|\leq\frac{1}{2}\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}}\frac{C_n}{j_1!j_2!j_3!} \tag{1568}$$

$$\cdot\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})|\prod_{d=1}^{n}|Q(k_{j_1+j_2+d}-k_{j_1+j_2+d+1})| \tag{1569}$$

$$\cdot\left|\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+n+1})\right|. \tag{1570}$$

Next, let us compute the Fourier coefficient of $\widetilde{B_{6,3}}(\alpha,\beta)$.

$$\mathcal{F}(\widetilde{B_{6,3}})(k_1,\beta)=\frac{1}{2}\cdot\frac{i\beta}{1-e^{-i\beta}}\int_0^1 e^{i\beta s}(-1+s)ds \tag{1571}$$

$$\cdot\mathcal{F}\left(\sum_{m=2}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)(k_1). \tag{1572}$$

We can write

$$\mathcal{F}\left(\sum_{m=2}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)(k_1) \tag{1573}$$

$$=\frac{-i\beta e^{-i\beta}}{1-e^{-i\beta}}\int_0^1 e^{is\beta}e^{i(s-1)\beta k_1}ds\sum_{m=2}^{\infty}\frac{i^m}{m!}\mathcal{F}(\phi^m)(k_1) \tag{1574}$$

$$=\frac{-i\beta e^{-i\beta}}{1-e^{-i\beta}}\int_0^1 e^{is\beta}e^{i(s-1)\beta k_1}ds\cdot\widetilde{Q}(k_1). \tag{1575}$$

It follows that

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{6,3}})(k_1,\beta)d\beta\right| \leq \frac{1}{2}\cdot\left|\widetilde{Q}(k_1)\right|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\left|\left(\frac{-i\beta}{1-e^{-i\beta}}\right)^2-1+1\right|d\beta \tag{1576}$$

$$\leq \frac{1}{2}\cdot\left|\widetilde{Q}(k_1)\right|\frac{\gamma}{4\pi}\left(2\cdot\left(\frac{\pi}{2}\right)\cdot\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}\cdot\pi^2+2\pi\right). \tag{1577}$$

Next, let us compute the Fourier coefficient of $\widetilde{B_7}(\alpha,\beta)$.

$$\mathcal{F}(\widetilde{B_7})(k_1,\beta) = \frac{1}{2}\sum_{j_1+j_2+j_3+n\geq 1}\frac{i\beta}{1-e^{-i\beta}}\frac{i^{j_1+j_2+j_3}(-1)^{j_3}}{j_1!j_2!j_3!} \tag{1578}$$

$$\cdot\mathcal{F}\Bigg(\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2}\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)\phi'(\alpha+\beta(-1+s))ds \tag{1579}$$

$$\cdot\left(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n\Bigg)(k_1). \tag{1580}$$

We can write

$$\mathcal{F}\Bigg(\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2}\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)\phi'(\alpha+\beta(-1+s))ds \tag{1581}$$

$$\cdot\left(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n\Bigg)(k_1) \tag{1582}$$

$$= \sum_{k_2,\dots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1}\mathcal{F}(\phi)(k_d-k_{d+1})\prod_{d=1}^{j_2}\mathcal{F}(\phi(\alpha-\beta))(k_{j_1+d}-k_{j_1+d+1}) \tag{1583}$$

$$\cdot\prod_{d=1}^{n}\mathcal{F}\left(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)(k_{j_1+j_2+d}-k_{j_1+j_2+d+1}) \tag{1584}$$

$$\cdot\mathcal{F}\left(\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)\phi'(\alpha+\beta(-1+s))ds\right)(k_{j_1+j_2+n+1}) \tag{1585}$$

$$= \sum_{k_2,\dots,k_{j_1+j_2+n+1}\in\mathbb{Z}} e^{-i\beta(k_{j_1+1}-k_{j_1+j_2+1})} \tag{1586}$$

$$\cdot\prod_{d=1}^{j_1+j_2}\mathcal{F}(\phi)(k_d-k_{d+1})\prod_{d=1}^{n}\Bigg(\sum_{m=1}^{\infty}\frac{-i\beta e^{-i\beta}}{1-e^{-i\beta}}\cdot\frac{i^m}{m!}\cdot\mathcal{F}(\phi^m)(k_{j_1+j_2+d}-k_{j_1+j_2+d+1}) \tag{1587}$$

$$\cdot\int_0^1 e^{is\beta}e^{i(s-1)\beta(k_{j_1+j_2+d}-k_{j_1+j_2+d+1})}ds\Bigg) \tag{1588}$$

$$\cdot\int_0^1 e^{-i\beta s}e^{i\beta(-1+s)k_{j_1+j_2+n+1}}(-1+s)ds\cdot\mathcal{F}(\phi^{j_3}\phi')(k_{j_1+j_2+n+1}) \tag{1589}$$

$$= \sum_{k_2,\dots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}\mathcal{F}(\phi)(k_d-k_{d+1})\prod_{d=1}^{n}Q(k_{j_1+j_2+d}-k_{j_1+j_2+d+1}) \tag{1590}$$

$$\cdot\mathcal{F}(\phi^{j_3}\phi')(k_{j_1+j_2+n+1})I_{n,5}(k_{j_1+1},k_{j_1+j_2+1},\dots,k_{j_1+j_2+n+1},\beta), \tag{1591}$$

where

$$
\begin{aligned}
&I_{n,5}(k_{j_1+1}, k_{j_1+j_2+1}, \dots, k_{j_1+j_2+n+1}, \beta) && (1592)\\
=&e^{-i\beta(k_{j_1+1}-k_{j_1+j_2+1})} \prod_{d=1}^{n} \left(\frac{-i\beta e^{-i\beta}}{1-e^{-i\beta}} \int_0^1 e^{is\beta} e^{i(s-1)\beta(k_{j_1+j_2+d}-k_{j_1+j_2+d+1})} ds \right) && (1593)\\
&\cdot \int_0^1 e^{-i\beta s} e^{i\beta(-1+s)k_{j_1+j_2+n+1}}(-1+s) ds. && (1594)
\end{aligned}
$$

Then

$$
\begin{aligned}
&\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} \mathcal{F}(\widetilde{B_7})(k_1,\beta) d\beta = \frac{1}{2} \sum_{j_1+j_2+j_3+n\geq 1} \frac{i^{j_1+j_2+j_3}(-1)^{j_3}}{j_1! j_2! j_3!} && (1595)\\
&\cdot \sum_{k_2,\dots,k_{j_1+j_2+n+1}\in\mathbb{Z}} \prod_{d=1}^{j_1+j_2} \mathcal{F}(\phi)(k_d - k_{d+1}) \prod_{d=1}^{n} Q(k_{j_1+j_2+d} - k_{j_1+j_2+d+1}) && (1596)\\
&\cdot \mathcal{F}(\phi^{j_3}\phi')(k_{j_1+j_2+n+1}) \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} I_{n,5}(k_{j_1+1}, k_{j_1+j_2+1}, \dots, k_{j_1+j_2+n+1}, \beta) \frac{i\beta}{1-e^{-i\beta}} d\beta. && (1597)
\end{aligned}
$$

Using an argument similar to Lemma 3, we obtain

$$
\begin{aligned}
&\left| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} I_{n,5}(k_{j_1+1}, k_{j_1+j_2+1}, \dots, k_{j_1+j_2+n+1}, \beta) \frac{i\beta}{1-e^{-i\beta}} d\beta \right| && (1598)\\
\leq& \frac{\gamma}{4\pi} \left((n+1) \cdot \left(\frac{\pi}{2}\right)^n \cdot \frac{1}{2}\sqrt{1+\frac{\pi^2}{4}} \cdot \pi^2 + 2\pi \right) && (1599)\\
=&C_n. && (1600)
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\left| \frac{\gamma}{4\pi}\int_{-\pi}^{\pi} \mathcal{F}(\widetilde{B_7})(k_1,\beta) d\beta \right| \leq \frac{1}{2} \sum_{j_1+j_2+j_3+n\geq 1} \frac{C_n}{j_1! j_2! j_3!} && (1601)\\
&\cdot \sum_{k_2,\dots,k_{j_1+j_2+n+1}\in\mathbb{Z}} \prod_{d=1}^{j_1+j_2} \left|\mathcal{F}(\phi)(k_d - k_{d+1})\right| \prod_{d=1}^{n} \left|Q(k_{j_1+j_2+d} - k_{j_1+j_2+d+1})\right| && (1602)\\
&\cdot \left|\mathcal{F}(\phi^{j_3}\phi')(k_{j_1+j_2+n+1})\right|. && (1603)
\end{aligned}
$$

Next, let us compute the Fourier coefficient of $\widetilde{B_8}(\alpha,\beta)$.

$$
\begin{aligned}
&\mathcal{F}(\widetilde{B_8})(k_1,\beta) = \frac{1}{2} \sum_{j_1+j_2+j_3+n\geq 1} \frac{i\beta}{1-e^{-i\beta}} \frac{i^{j_1+j_2+j_3}(-1)^{j_2}}{j_1! j_2! j_3!} && (1604)\\
&\cdot \mathcal{F}\Bigg(\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2} \int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)\phi'(\alpha+\beta(-1+s)) ds && (1605)\\
&\cdot \left(\sum_{m=1}^{\infty} \frac{-i\beta}{1-e^{-i\beta}} \int_0^1 e^{i(s-1)\beta} \frac{(i\phi(\alpha+(s-1)\beta))^m}{m!} ds \right)^n \Bigg)(k_1). && (1606)
\end{aligned}
$$

We can write

$$\mathcal{F}\Bigg(\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2}\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)\phi'(\alpha+\beta(-1+s))ds \tag{1607}$$

$$\cdot\Bigg(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\Bigg)^n\Bigg)(k_1) \tag{1608}$$

$$=\sum_{k_2,\dots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1}\mathcal{F}(\phi)(k_d-k_{d+1})\prod_{d=1}^{j_2}\mathcal{F}(\phi(\alpha-\beta))(k_{j_1+d}-k_{j_1+d+1}) \tag{1609}$$

$$\cdot\prod_{d=1}^{n}\mathcal{F}\Bigg(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\Bigg)(k_{j_1+j_2+d}-k_{j_1+j_2+d+1}) \tag{1610}$$

$$\cdot\mathcal{F}\Bigg(\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)\phi'(\alpha+\beta(-1+s))ds\Bigg)(k_{j_1+j_2+n+1}) \tag{1611}$$

$$=\sum_{k_2,\dots,k_{j_1+j_2+n+1}\in\mathbb{Z}}e^{-i\beta(k_{j_1+1}-k_{j_1+j_2+1})}\prod_{d=1}^{j_1+j_2}\mathcal{F}(\phi)(k_d-k_{d+1}) \tag{1612}$$

$$\cdot\prod_{d=1}^{n}\Bigg(\sum_{m=1}^{\infty}\frac{-i\beta e^{-i\beta}}{1-e^{-i\beta}}\cdot\frac{i^m}{m!}\cdot\mathcal{F}(\phi^m)(k_{j_1+j_2+d}-k_{j_1+j_2+d+1})\int_0^1 e^{is\beta}e^{i(s-1)\beta(k_{j_1+j_2+d}-k_{j_1+j_2+d+1})}ds\Bigg) \tag{1613}$$

$$\cdot\int_0^1 e^{i\beta s}e^{i\beta(-1+s)k_{j_1+j_2+n+1}}(-1+s)ds\cdot\mathcal{F}(\phi^{j_3}\phi')(k_{j_1+j_2+n+1}) \tag{1614}$$

$$=\sum_{k_2,\dots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}\mathcal{F}(\phi)(k_d-k_{d+1})\prod_{d=1}^{n}Q(k_{j_1+j_2+d}-k_{j_1+j_2+d+1}) \tag{1615}$$

$$\cdot\mathcal{F}(\phi^{j_3}\phi')(k_{j_1+j_2+n+1})\cdot I_{n,6}(k_{j_1+1},k_{j_1+j_2+1},k_{j_1+j_2+n+1},\beta), \tag{1616}$$

where

$$I_{n,6}(k_{j_1+1},k_{j_1+j_2+1},k_{j_1+j_2+n+1},\beta)=e^{-i\beta(k_{j_1+1}-k_{j_1+j_2+1})} \tag{1617}$$

$$\cdot\prod_{d=1}^{n}\Bigg(\frac{-i\beta e^{-i\beta}}{1-e^{-i\beta}}\int_0^1 e^{is\beta}e^{i(s-1)\beta(k_{j_1+j_2+d}-k_{j_1+j_2+d+1})}ds\Bigg) \tag{1618}$$

$$\cdot\int_0^1 e^{i\beta s}e^{i\beta(-1+s)k_{j_1+j_2+n+1}}(-1+s)ds. \tag{1619}$$

Then

$$\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_8})(k_1,\beta)d\beta=\frac{1}{2}\sum_{j_1+j_2+j_3+n\geq 1}\frac{i^{j_1+j_2+j_3}(-1)^{j_2}}{j_1!j_2!j_3!} \tag{1620}$$

$$\cdot\sum_{k_2,\dots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}\mathcal{F}(\phi)(k_d-k_{d+1})\prod_{d=1}^{n}Q(k_{j_1+j_2+d}-k_{j_1+j_2+d+1}) \tag{1621}$$

$$\cdot\mathcal{F}(\phi^{j_3}\phi')(k_{j_1+j_2+n+1})\cdot\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}I_{n,6}(k_{j_1+1},k_{j_1+j_2+1},k_{j_1+j_2+n+1},\beta)\frac{i\beta}{1-e^{-i\beta}}d\beta. \tag{1622}$$

Using an argument similar to Lemma 3, we obtain

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} I_{n,6}(k_{j_1+1},k_{j_1+j_2+1},k_{j_1+j_2+n+1},\beta)\frac{i\beta}{1-e^{-i\beta}}d\beta\right| \tag{1623}$$

$$\leq\frac{\gamma}{4\pi}\left((n+1)\cdot\left(\frac{\pi}{2}\right)^n\cdot\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}\cdot\pi^2+2\pi\right) \tag{1624}$$

$$=C_n. \tag{1625}$$

It follows that

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_8})(k_1,\beta)d\beta\right|\leq\frac{1}{2}\sum_{j_1+j_2+j_3+n\geq 1}\frac{C_n}{j_1!j_2!j_3!} \tag{1626}$$

$$\cdot\sum_{k_2,\dots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})|\prod_{d=1}^{n}|Q(k_{j_1+j_2+d}-k_{j_1+j_2+d+1})| \tag{1627}$$

$$\cdot\left|\mathcal{F}(\phi^{j_3}\phi')(k_{j_1+j_2+n+1})\right|. \tag{1628}$$

Next, let us compute the Fourier coefficient of $\widetilde{B_{13}}(\alpha,\beta)$. We can write

$$\mathcal{F}(\widetilde{B_{13}})(k_1,\beta)=\sum_{j_1+j_2\geq 2}\frac{i^{j_1+j_2}}{j_1!j_2!}\left((-1)^{j_2}\frac{e^{i\beta}(1+e^{i\beta})}{2(-1+e^{i\beta})}-\frac{1}{2}\right)\cdot\mathcal{F}(\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2})(k_1) \tag{1629}$$

$$=\sum_{j_1+j_2\geq 2}\frac{i^{j_1+j_2}}{j_1!j_2!}\left((-1)^{j_2}\frac{e^{i\beta}(1+e^{i\beta})}{2(-1+e^{i\beta})}-\frac{1}{2}\right) \tag{1630}$$

$$\cdot\sum_{k_2,\dots,k_{j_1+j_2}\in\mathbb{Z}}\prod_{d=1}^{j_1}\mathcal{F}(\phi)(k_d-k_{d+1})\prod_{d=1}^{j_2-1}\mathcal{F}(\phi(\alpha-\beta))(k_{j_1+d}-k_{j_1+d+1}) \tag{1631}$$

$$\mathcal{F}(\phi(\alpha-\beta))(k_{j_1+j_2}) \tag{1632}$$

$$=\sum_{j_1+j_2\geq 2}\frac{i^{j_1+j_2}}{j_1!j_2!}\sum_{k_2,\dots,k_{j_1+j_2}\in\mathbb{Z}}\left((-1)^{j_2}\frac{e^{i\beta}(1+e^{i\beta})}{2(-1+e^{i\beta})}-\frac{1}{2}\right)e^{-i\beta(k_{j_1+1}-k_{j_1+j_2})}e^{-i\beta k_{j_1+j_2}} \tag{1633}$$

$$\cdot\prod_{d=1}^{j_1+j_2-1}\mathcal{F}(\phi)(k_d-k_{d+1})\mathcal{F}(\phi)(k_{j_1+j_2}) \tag{1634}$$

$$=\sum_{j_1+j_2\geq 2}\frac{i^{j_1+j_2}}{j_1!j_2!}\sum_{k_2,\dots,k_{j_1+j_2}\in\mathbb{Z}}I_{n,7}(k_{j_1+1},\dots,k_{j_1+j_2},\beta)\cdot\prod_{d=1}^{j_1+j_2-1}\mathcal{F}(\phi)(k_d-k_{d+1}) \tag{1635}$$

$$\cdot\mathcal{F}(\phi)(k_{j_1+j_2}), \tag{1636}$$

where

$$I_{n,7}(k_{j_1+1},\dots,k_{j_1+j_2},\beta)=\left((-1)^{j_2}\frac{e^{i\beta}(1+e^{i\beta})}{2(-1+e^{i\beta})}-\frac{1}{2}\right)e^{-i\beta(k_{j_1+1}-k_{j_1+j_2})}e^{-i\beta k_{j_1+j_2}}. \tag{1637}$$

Then

$$\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{13}})(k_1,\beta)d\beta=\sum_{j_1+j_2\geq 2}\frac{i^{j_1+j_2}}{j_1!j_2!} \tag{1638}$$

$$\cdot\sum_{k_2,\ldots,k_{j_1+j_2}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2-1}\mathcal{F}(\phi)(k_d-k_{d+1})\mathcal{F}(\phi)(k_{j_1+j_2})\cdot\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}I_{n,7}(k_{j_1+1},\ldots,k_{j_1+j_2},\beta)d\beta. \tag{1639}$$

We note that for $l\in\mathbb{Z}$,

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{e^{-i\beta\cdot l}}{1-e^{-i\beta}}d\beta=1_{l\leq 0}(l)-1_{l\geq 1}(l). \tag{1640}$$

For proof, see (5.9) in [1]. Then

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}I_{n,7}(k_{j_1+1},\ldots,k_{j_1+j_2},\beta)d\beta\right| \tag{1641}$$

$$\leq\frac{\gamma}{4\pi}\left|\int_{-\pi}^{\pi}\frac{(-1)^{j_2}e^{i\beta}e^{-i\beta(k_{j_1+1}-k_{j_1+j_2})}e^{-i\beta k_{j_1+j_2}}(1+e^{i\beta})}{2(-1+e^{i\beta})}d\beta\right| \tag{1642}$$

$$+\frac{1}{2}\frac{\gamma}{4\pi}\left|\int_{-\pi}^{\pi}e^{-i\beta(k_{j_1+1}-k_{j_1+j_2})}e^{-i\beta k_{j_1+j_2}}d\beta\right| \tag{1643}$$

$$\leq\frac{\gamma}{4\pi}\cdot\frac{1}{2}\left|\int_{-\pi}^{\pi}\frac{e^{i\beta(1-(k_{j_1+1}-k_{j_1+j_2})-k_{j_1+j_2})}}{1-e^{i\beta}}d\beta+\int_{-\pi}^{\pi}\frac{e^{i\beta(2-(k_{j_1+1}-k_{j_1+j_2})-k_{j_1+j_2})}}{1-e^{i\beta}}d\beta\right| \tag{1644}$$

$$+\frac{\gamma}{4\pi}\cdot\pi \tag{1645}$$

$$\leq\frac{\gamma}{4\pi}(1+\pi). \tag{1646}$$

It follows that

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{13}})(k_1,\beta)d\beta\right| \tag{1647}$$

$$\leq\frac{\gamma}{4\pi}(1+\pi)\sum_{j_1+j_2\geq 2}\frac{1}{j_1!j_2!}\sum_{k_2,\ldots,k_{j_1+j_2}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2-1}\left|\mathcal{F}(\phi)(k_d-k_{d+1})\right|\left|\mathcal{F}(\phi)(k_{j_1+j_2})\right|. \tag{1648}$$

13.2 Estimating $\|U_{\geq 2}\|_{\mathcal{F}_\nu^{0,1}}$

We prove the following estimate for $\|U_{\geq 2}\|_{\mathcal{F}_\nu^{0,1}}$.

Lemma 4.

$$\|U_{\geq 2}\|_{\mathcal{F}_\nu^{0,1}}\leq D_1(\|\phi\|_{\mathcal{F}_\nu^{0,1}})\|\phi\|_{\mathcal{F}_\nu^{0,1}}^2+D_2(\|\phi\|_{\mathcal{F}_\nu^{0,1}})\|\phi\|_{\mathcal{F}_\nu^{0,1}}\|\phi'\|_{\mathcal{F}_\nu^{0,1}}, \tag{1649}$$

where D_1 and D_2 are monotone increasing functions of $\|\phi\|_{\mathcal{F}_\nu^{0,1}}$.

Before commencing the proof of Lemma 4, let us introduce the setup for the proof. For ease of notation, we define the l^1_ν norm of a sequence $a = a(k)$ defined on $\mathbb{Z}$ by

$$\|a\|_{l^1_\nu} = \sum_{k\in\mathbb{Z}} e^{\nu(t)|k|} |a(k)| . \tag{1650}$$

The following estimate of the l^1_ν norm of the convolution is frequently used.

Proposition 10. *If $a_1, \dots, a_n$ are sequences on $\mathbb{Z}$ whose l^1_ν norms are finite, then*

$$\|a_1 * \cdots * a_n\|_{l^1_\nu} \leq \prod_{j=1}^{n} \|a_j\|_{l^1_\nu} . \tag{1651}$$

Proof. It suffices to show the case of $n = 2$ because the general case follows from repeated applications of this case. Indeed, we have

$$\begin{aligned}
\|a * b\|_{l^1_\nu} &= \sum_{k\in\mathbb{Z}} e^{\nu(t)|k|} |(a * b)(k)| &(1652)\\
&\leq \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} e^{\nu(t)|k-j|} e^{\nu(t)|j|} |a(k-j)| |b(j)| &(1653)\\
&= \sum_{j\in\mathbb{Z}} e^{\nu(t)|j|} |b(j)| \sum_{k\in\mathbb{Z}} e^{\nu(t)|k-j|} |a(k-j)| &(1654)\\
&= \|a\|_{l^1_\nu} \|b\|_{l^1_\nu} , &(1655)
\end{aligned}$$

as needed. ■

We note that

$$\begin{aligned}
\|P\|_{l^1_\nu} &\leq \sum_{m=1}^{\infty} \frac{\|\phi\|^m_{\mathcal{F}^{0,1}_\nu}}{m!} = e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}} - 1, &(1656)\\
\left\|\widetilde{P}\right\|_{l^1_\nu} &\leq \sum_{m=2}^{\infty} \frac{\|\phi\|^m_{\mathcal{F}^{0,1}_\nu}}{m!} = e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}} - \|\phi\|_{\mathcal{F}^{0,1}_\nu} - 1, &(1657)\\
\|Q\|_{l^1_\nu} &\leq \sum_{m=1}^{\infty} \frac{\|\phi\|^m_{\mathcal{F}^{0,1}_\nu}}{m!} = e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}} - 1, &(1658)\\
\left\|\widetilde{Q}\right\|_{l^1_\nu} &\leq \sum_{m=2}^{\infty} \frac{\|\phi\|^m_{\mathcal{F}^{0,1}_\nu}}{m!} = e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}} - \|\phi\|_{\mathcal{F}^{0,1}_\nu} - 1. &(1659)
\end{aligned}$$

To begin the proof of Lemma 4, we observe that

$$\|U_{\geq 2}\|_{\mathcal{F}^{0,1}_\nu} \leq \sum_{j=1}^{8} \left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} \widetilde{B_j}(\alpha,\beta) d\beta \right\|_{\mathcal{F}^{0,1}_\nu} + \left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} \widetilde{B_{13}}(\alpha,\beta) d\beta \right\|_{\mathcal{F}^{0,1}_\nu} . \tag{1660}$$

This means that it suffices to estimate each of the $\mathcal{F}_\nu^{0,1}$ norms on the right hand side. By Proposition 10 and (1656), we obtain

$$\sum_{k_1\in\mathbb{Z}} e^{\nu(t)|k_1|}\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{1,1}})(k_1,\beta)d\beta\right| \tag{1661}$$

$$\leq \sum_{k_1\in\mathbb{Z}} e^{\nu(t)|k_1|}\sum_{j_1+j_2+n\geq 1}\frac{C_n}{2j_1!j_2!}(|\mathcal{F}(\phi)|*\cdots*|\mathcal{F}(\phi)|*|P|*\cdots*|P|*|\mathcal{F}(\phi)|)(k_1) \tag{1662}$$

$$= \sum_{j_1+j_2+n\geq 1}\frac{C_n}{2j_1!j_2!}\left\||\mathcal{F}(\phi)|*\cdots*|\mathcal{F}(\phi)|*|P|*\cdots*|P|*|\mathcal{F}(\phi)|\right\|_{l^1_\nu} \tag{1663}$$

$$\leq \sum_{j_1+j_2+n\geq 1}\frac{C_n}{2j_1!j_2!}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+1}\|P\|_{l^1_\nu}^n \tag{1664}$$

$$\leq \sum_{j_1+j_2+n\geq 1}\frac{C_n}{2j_1!j_2!}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+1}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n. \tag{1665}$$

By Propositions 2 and 10 and (1656), we obtain

$$\sum_{k_1\in\mathbb{Z}} e^{\nu(t)|k_1|}\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{1,2}})(k_1,\beta)d\beta\right| \tag{1666}$$

$$\leq \sum_{j_1+j_2+j_3\geq 1}\frac{C_1}{2j_1!j_2!j_3!}\sum_{k_1\in\mathbb{Z}} e^{\nu(t)|k_1|}(|\mathcal{F}(\phi)|*\cdots*|\mathcal{F}(\phi)|*|P|*\left|\mathcal{F}(\phi^{j_3})\right|)(k_1) \tag{1667}$$

$$= \sum_{j_1+j_2+j_3\geq 1}\frac{C_1}{2j_1!j_2!j_3!}\left\||\mathcal{F}(\phi)|*\cdots*|\mathcal{F}(\phi)|*|P|*\left|\mathcal{F}(\phi^{j_3})\right|\right\|_{l^1_\nu} \tag{1668}$$

$$\leq \sum_{j_1+j_2+j_3\geq 1}\frac{C_1}{2j_1j_2!j_3!}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1). \tag{1669}$$

By (1657), we obtain

$$\sum_{k_1\in\mathbb{Z}} e^{\nu(t)|k_1|}\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{1,3}})(k_1,\beta)d\beta\right| \leq \frac{C_2}{2}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-\|\phi\|_{\mathcal{F}_\nu^{0,1}}-1). \tag{1670}$$

By Propositions 2 and 10 and (1656), we obtain

$$\sum_{k_1\in\mathbb{Z}} e^{\nu(t)|k_1|}\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_2})(k_1,\beta)d\beta\right| \tag{1671}$$

$$\leq \frac{1}{2}\sum_{j_1+j_2+j_3+n\geq 1}\frac{C_n}{j_1!j_2!j_3!} \tag{1672}$$

$$\cdot \sum_{k_1\in\mathbb{Z}} e^{\nu(t)|k_1|}(|\mathcal{F}(\phi)|*\cdots*|\mathcal{F}(\phi)|*|P|*\cdots*|P|*\left|\mathcal{F}(\phi^{j_3}\phi')\right|)(k_1) \tag{1673}$$

$$\leq \frac{1}{2}\sum_{j_1+j_2+j_3+n\geq 1}\frac{C_n}{j_1!j_2!j_3!}\left\||\mathcal{F}(\phi)|*\cdots*|\mathcal{F}(\phi)|*|P|*\cdots*|P|*\left|\mathcal{F}(\phi^{j_3}\phi')\right|\right\|_{l^1_\nu} \tag{1674}$$

$$\leq \frac{1}{2}\sum_{j_1+j_2+j_3+n\geq 1}\frac{C_n}{j_1!j_2!j_3!}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n\|\phi'\|_{\mathcal{F}_\nu^{0,1}}. \tag{1675}$$

By Propositions 2 and 10 and (1658), we obtain

$$\sum_{k_1\in\mathbb{Z}} e^{\nu(t)|k_1|}\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{3,1}})(k_1,\beta)d\beta\right| \tag{1676}$$

$$\leq \sum_{k_1\in\mathbb{Z}} e^{\nu(t)|k_1|}\cdot\frac{1}{2}\sum_{j_3+j_4+n\geq 2}(n+1)\frac{C_{n+1}}{j_3!j_4!}(|Q|*\cdots*|Q|*\left|\mathcal{F}(\phi^{j_3})\right|*\left|\mathcal{F}(\phi^{j_4})\right|)(k_1) \tag{1677}$$

$$\leq \frac{1}{2}\sum_{j_3+j_4+n\geq 2}(n+1)\frac{C_{n+1}}{j_3!j_4!}\|Q\|_{l^1_\nu}^n\|\phi\|_{\mathcal{F}^{0,1}_\nu}^{j_3+j_4} \tag{1678}$$

$$\leq \frac{1}{2}\sum_{j_3+j_4+n\geq 2}(n+1)\frac{C_{n+1}}{j_3!j_4!}(e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1)^n\|\phi\|_{\mathcal{F}^{0,1}_\nu}^{j_3+j_4}. \tag{1679}$$

Next, recalling that

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{3,2}})(k_1,\beta)d\beta\right| \leq \frac{1}{2}\sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}}(n+1)\frac{\widetilde{C_n}}{j_1!j_2!j_3!j_4!} \tag{1680}$$

$$\cdot\sum_{k_2,\dots,k_{j_1+j_2+n+2}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})|\prod_{d=1}^{n}|Q(k_{j_1+j_2+d}-k_{j_1+j_2+d+1})| \tag{1681}$$

$$\cdot\left|\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+n+1}-k_{j_1+j_2+n+2})\right|\cdot\left|\mathcal{F}(\phi^{j_4})(k_{j_1+j_2+n+2})\right|, \tag{1682}$$

By Propositions 2 and 10 and (1658), we obtain

$$\sum_{k_1\in\mathbb{Z}} e^{\nu(t)|k_1|}\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{3,2}})(k_1,\beta)d\beta\right| \tag{1683}$$

$$\leq \sum_{k_1\in\mathbb{Z}} e^{\nu(t)|k_1|}\cdot\frac{1}{2}\sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}}(n+1)\frac{C_{n+1}}{j_1!j_2!j_3!j_4!} \tag{1684}$$

$$\cdot(|\mathcal{F}(\phi)|*\cdots*|\mathcal{F}(\phi)|*|Q|*\cdots*|Q|*\left|\mathcal{F}(\phi^{j_3})\right|*\left|\mathcal{F}(\phi^{j_4})\right|)(k_1) \tag{1685}$$

$$\leq \frac{1}{2}\sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}}(n+1)\frac{C_{n+1}}{j_1!j_2!j_3!j_4!} \tag{1686}$$

$$\cdot\left\||\mathcal{F}(\phi)|*\cdots*|\mathcal{F}(\phi)|*|Q|*\cdots*|Q|*\left|\mathcal{F}(\phi^{j_3})\right|*\left|\mathcal{F}(\phi^{j_4})\right|\right\|_{l^1_\nu} \tag{1687}$$

$$\leq \frac{1}{2}\sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}}(n+1)\frac{C_{n+1}}{j_1!j_2!j_3!j_4!}\|\phi\|_{\mathcal{F}^{0,1}_\nu}^{j_1+j_2+j_3+j_4}\|Q\|_{l^1_\nu}^n \tag{1688}$$

$$\leq \frac{1}{2}\sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}}\frac{(n+1)C_{n+1}}{j_1!j_2!j_3!j_4!}\|\phi\|_{\mathcal{F}^{0,1}_\nu}^{j_1+j_2+j_3+j_4}(e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1)^n. \tag{1689}$$

By (1659), we have

$$\sum_{k_1\in\mathbb{Z}} e^{\nu(t)|k_1|}\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{3,3}})(k_1,\beta)d\beta\right| \leq C_3(e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-\|\phi\|_{\mathcal{F}^{0,1}_\nu}-1). \tag{1690}$$

By Propositions 2 and 10 and (1658), we obtain

$$\sum_{k_1\in\mathbb{Z}} e^{\nu(t)|k_1|}\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_4})(k_1,\beta)d\beta\right| \tag{1691}$$

$$\leq \frac{1}{2}\sum_{j_1+j_2+j_3+j_4+n\geq 1}(n+1)\frac{C_{n+1}}{j_1!j_2!j_3!j_4!}\sum_{k_1\in\mathbb{Z}} e^{\nu(t)|k_1|} \tag{1692}$$

$$\cdot\left(|\mathcal{F}(\phi)|*\cdots*|\mathcal{F}(\phi)|*|Q|*\cdots*|Q|*\left|\mathcal{F}(\phi^{j_3})\right|*\left|\mathcal{F}(\phi^{j_4}\phi')\right|\right)(k_1) \tag{1693}$$

$$=\frac{1}{2}\sum_{j_1+j_2+j_3+j_4+n\geq 1}(n+1)\frac{C_{n+1}}{j_1!j_2!j_3!j_4!} \tag{1694}$$

$$\cdot\left\|\left|\mathcal{F}(\phi)\right|*\cdots*|\mathcal{F}(\phi)|*|Q|*\cdots*|Q|*\left|\mathcal{F}(\phi^{j_3})\right|*\left|\mathcal{F}(\phi^{j_4}\phi')\right|\right\|_{l^1_\nu} \tag{1695}$$

$$\leq \frac{1}{2}\sum_{j_1+j_2+j_3+j_4+n\geq 1}(n+1)\frac{C_{n+1}}{j_1!j_2!j_3!j_4!}\left\|\phi\right\|_{\mathcal{F}^{0,1}_\nu}^{j_1+j_2+j_3+j_4}\left\|\phi'\right\|_{\mathcal{F}^{0,1}_\nu}\left\|Q\right\|_{l^1_\nu}^{n} \tag{1696}$$

$$\leq \frac{1}{2}\sum_{j_1+j_2+j_3+j_4+n\geq 1}(n+1)\frac{C_{n+1}}{j_1!j_2!j_3!j_4!}\left\|\phi\right\|_{\mathcal{F}^{0,1}_\nu}^{j_1+j_2+j_3+j_4}\left\|\phi'\right\|_{\mathcal{F}^{0,1}_\nu}\left(e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1\right)^{n}. \tag{1697}$$

By Propositions 2 and 10 and (1658), we obtain

$$\sum_{k_1\in\mathbb{Z}} e^{\nu(t)|k_1|}\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{5,1}})(k_1,\beta)d\beta\right| \tag{1698}$$

$$\leq \frac{1}{2}\sum_{j_3+n\geq 2}\frac{C_n}{j_3!}\sum_{k_1\in\mathbb{Z}} e^{\nu(t)|k_1|}\left(|Q|*\cdots*|Q|*\left|\mathcal{F}(\phi^{j_3})\right|\right)(k_1) \tag{1699}$$

$$\leq \frac{1}{2}\sum_{j_3+n\geq 2}\frac{C_n}{j_3!}\left\|Q\right\|_{l^1_\nu}^{n}\left\|\phi\right\|_{\mathcal{F}^{0,1}_\nu}^{j_3} \tag{1700}$$

$$\leq \frac{1}{2}\sum_{j_3+n\geq 2}\frac{C_n}{j_3!}\left(e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1\right)^{n}\left\|\phi\right\|_{\mathcal{F}^{0,1}_\nu}^{j_3}. \tag{1701}$$

By Propositions 2 and 10 and (1658), we obtain

$$\sum_{k_1\in\mathbb{Z}} e^{\nu(t)|k_1|}\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{5,2}})(k_1,\beta)d\beta\right| \tag{1702}$$

$$\leq \frac{1}{2}\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}}\frac{C_n}{j_1!j_2!j_3!} \tag{1703}$$

$$\cdot\sum_{k_1\in\mathbb{Z}} e^{\nu(t)|k_1|}\cdot\left(|\mathcal{F}(\phi)|*\cdots*|\mathcal{F}(\phi)|*|Q|*\cdots*|Q|*\left|\mathcal{F}(\phi^{j_3})\right|\right)(k_1) \tag{1704}$$

$$\leq \frac{1}{2}\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}}\frac{C_n}{j_1!j_2!j_3!}\left\|Q\right\|_{l^1_\nu}^{n}\left\|\phi\right\|_{\mathcal{F}^{0,1}_\nu}^{j_1+j_2+j_3} \tag{1705}$$

$$\leq \frac{1}{2}\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}}\frac{C_n}{j_1!j_2!j_3!}\left(e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1\right)^{n}\left\|\phi\right\|_{\mathcal{F}^{0,1}_\nu}^{j_1+j_2+j_3}. \tag{1706}$$

By (1659), we obtain

$$\sum_{k_1\in\mathbb{Z}} e^{\nu(t)|k_1|}\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{5,3}})(k_1,\beta)d\beta\right| \leq \frac{C_1}{2}(e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}} - \|\phi\|_{\mathcal{F}^{0,1}_\nu} - 1). \tag{1707}$$

By Propositions 2 and 10 and (1658), we have

$$\sum_{k_1\in\mathbb{Z}} e^{\nu(t)|k_1|}\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{6,1}})(k_1,\beta)d\beta\right| \tag{1708}$$

$$\leq \frac{1}{2}\sum_{j_3+n\geq 2}\frac{C_n}{j_3!}\sum_{k_1\in\mathbb{Z}} e^{\nu(t)|k_1|}(|Q| * \cdots * |Q| * \left|\mathcal{F}(\phi^{j_3})\right|)(k_1) \tag{1709}$$

$$\leq \frac{1}{2}\sum_{j_3+n\geq 2}\frac{C_n}{j_3!}\|Q\|^n_{l^1_\nu}\|\phi\|^{j_3}_{\mathcal{F}^{0,1}_\nu} \tag{1710}$$

$$\leq \frac{1}{2}\sum_{j_3+n\geq 2}\frac{C_n}{j_3!}(e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}} - 1)^n\|\phi\|^{j_3}_{\mathcal{F}^{0,1}_\nu}. \tag{1711}$$

By Propositions 2 and 10 and (1658), we obtain

$$\sum_{k_1\in\mathbb{Z}} e^{\nu(t)|k_1|}\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{6,2}})(k_1,\beta)d\beta\right| \tag{1712}$$

$$\leq \frac{1}{2}\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}}\frac{C_n}{j_1!j_2!j_3!}\sum_{k_1\in\mathbb{Z}} e^{\nu(t)|k_1|}(|\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |Q| * \cdots * |Q| * \left|\mathcal{F}(\phi^{j_3})\right|)(k_1) \tag{1713}$$

$$\leq \frac{1}{2}\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}}\frac{C_n}{j_1!j_2!j_3!}\|\phi\|^{j_1+j_2+j_3}_{\mathcal{F}^{0,1}_\nu}\|Q\|^n_{l^1_\nu} \tag{1714}$$

$$\leq \frac{1}{2}\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}}\frac{C_n}{j_1!j_2!j_3!}\|\phi\|^{j_1+j_2+j_3}_{\mathcal{F}^{0,1}_\nu}(e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}} - 1)^n. \tag{1715}$$

By (1659), we obtain

$$\sum_{k_1\in\mathbb{Z}} e^{\nu(t)|k_1|}\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{6,3}})(k_1,\beta)d\beta\right| \leq \frac{C_1}{2}(e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}} - \|\phi\|_{\mathcal{F}^{0,1}_\nu} - 1). \tag{1716}$$

By Propositions 2 and 10 and (1658), we obtain

$$\sum_{k_1\in\mathbb{Z}} e^{\nu(t)|k_1|} \left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} \mathcal{F}(\widetilde{B_7})(k_1,\beta)d\beta\right| \tag{1717}$$

$$\leq \frac{1}{2}\sum_{j_1+j_2+j_3+n\geq 1} \frac{C_n}{j_1!j_2!j_3!} \tag{1718}$$

$$\cdot \sum_{k_1\in\mathbb{Z}} e^{\nu(t)|k_1|}(|\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |Q| * \cdots * |Q| * \left|\mathcal{F}(\phi^{j_3}\phi')\right|)(k_1) \tag{1719}$$

$$\leq \frac{1}{2}\sum_{j_1+j_2+j_3+n\geq 1} \frac{C_n}{j_1!j_2!j_3!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3} \|Q\|_{l_\nu^1}^n \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \tag{1720}$$

$$\leq \frac{1}{2}\sum_{j_1+j_2+j_3+n\geq 1} \frac{C_n}{j_1!j_2!j_3!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n \|\phi'\|_{\mathcal{F}_\nu^{0,1}}\,. \tag{1721}$$

By Propositions 2 and 10 and (1658), we obtain

$$\sum_{k_1\in\mathbb{Z}} e^{\nu(t)|k_1|} \left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} \mathcal{F}(\widetilde{B_8})(k_1,\beta)d\beta\right| \tag{1722}$$

$$\leq \frac{1}{2}\sum_{j_1+j_2+j_3+n\geq 1} \frac{C_n}{j_1!j_2!j_3!} \tag{1723}$$

$$\cdot \sum_{k_1\in\mathbb{Z}} e^{\nu(t)|k_1|}(|\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |Q| * \cdots * |Q| * \left|\mathcal{F}(\phi^{j_3}\phi')\right|)(k_1) \tag{1724}$$

$$\leq \frac{1}{2}\sum_{j_1+j_2+j_3+n\geq 1} \frac{C_n}{j_1!j_2!j_3!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3} \|Q\|_{l_\nu^1}^n \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \tag{1725}$$

$$\leq \frac{1}{2}\sum_{j_1+j_2+j_3+n\geq 1} \frac{C_n}{j_1!j_2!j_3!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n \|\phi'\|_{\mathcal{F}_\nu^{0,1}}\,. \tag{1726}$$

Lastly, recalling that

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} \mathcal{F}(\widetilde{B_{13}})(k_1,\beta)d\beta\right| \tag{1727}$$

$$\leq \frac{\gamma}{4\pi}(1+\pi)\sum_{j_1+j_2\geq 2}\frac{1}{j_1!j_2!}\sum_{k_2,\ldots,k_{j_1+j_2}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2-1}|\mathcal{F}(\phi)(k_d-k_{d+1})|\,|\mathcal{F}(\phi)(k_{j_1+j_2})|\,, \tag{1728}$$

we have

$$\sum_{k_1\in\mathbb{Z}} e^{\nu(t)|k_1|} \left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} \mathcal{F}(\widetilde{B_{13}})(k_1,\beta)d\beta\right| \tag{1729}$$

$$\leq \frac{\gamma}{4\pi}(1+\pi)\sum_{j_1+j_2\geq 2}\frac{1}{j_1!j_2!}\sum_{k_1\in\mathbb{Z}} e^{\nu(t)|k_1|}(|\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)|)(k_1) \tag{1730}$$

$$\leq \frac{\gamma}{4\pi}(1+\pi)\sum_{j_1+j_2\geq 2}\frac{1}{j_1!j_2!}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2}\,. \tag{1731}$$

This completes the proof of Lemma 4.

13.3 Estimating $\|U_{\geq 2}\|_{\dot{\mathcal{F}}^{s,1}_\nu}$

We prove the following estimate for $\|U_{\geq 2}\|_{\dot{\mathcal{F}}^{s,1}_\nu}$, $s > 0$.

Lemma 5. *For $s > 0$,*

$$\|U_{\geq 2}\|_{\dot{\mathcal{F}}^{s,1}_\nu} \leq F_1(\|\phi\|_{\mathcal{F}^{0,1}_\nu}) \|\phi\|_{\mathcal{F}^{0,1}_\nu} \|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu} + F_2(\|\phi\|_{\mathcal{F}^{0,1}_\nu}) \|\phi\|^2_{\mathcal{F}^{0,1}_\nu} \|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu} \tag{1732}$$

$$+ F_3(\|\phi\|_{\mathcal{F}^{0,1}_\nu}) \|\phi'\|_{\mathcal{F}^{0,1}_\nu} \|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu} + F_4(\|\phi\|_{\mathcal{F}^{0,1}_\nu}) \|\phi\|_{\mathcal{F}^{0,1}_\nu} \|\phi'\|_{\dot{\mathcal{F}}^{s,1}_\nu}, \tag{1733}$$

where F_1, F_2, F_3, and F_4 are monotone increasing functions of $\|\phi\|_{\mathcal{F}^{0,1}_\nu}$.

Before commencing the proof of Lemma 5, let us introduce the setup for the proof. For ease of notation, we define the l^s_ν norm of a sequence $a = a(k)$ defined on $\mathbb{Z}$ by

$$\|a\|_{l^s_\nu} = \sum_{k \neq 0} e^{\nu(t)|k|} |k|^s |a_k|. \tag{1734}$$

The following estimate of the l^s_ν norm of the convolution is frequently used.

Proposition 11. *Let $s > 0$. If $a_1, \ldots, a_n$ are sequences on $\mathbb{Z}$ whose l^s_ν norms are finite, then*

$$\|a_1 * \cdots * a_n\|_{l^s_\nu} \leq b(n,s) \sum_{j=1}^{n} \|a_j\|_{l^s_\nu} \prod_{\substack{k=1 \\ k \neq j}}^{n} \|a_k\|_{l^1_\nu}. \tag{1735}$$

Proof. We note that for any $k_1, \ldots, k_n \in \mathbb{Z}$,

$$|k_1|^s \leq b(n,s)(|k_1 - k_2|^s + |k_2 - k_3|^s + \cdots + |k_{n-1} - k_n|^s + |k_n|^s), \tag{1736}$$

which follows from convexity of the function $|\cdot|^s$ for $s \geq 1$ and the triangle inequality for $0 < s < 1$. Then, using (1238), we obtain

$$\|a_1 * \cdots * a_n\|_{l^s_\nu} = \sum_{k_1 \in \mathbb{Z}} e^{\nu(t)|k_1|} |k_1|^s |(a_1 * \cdots * a_n)(k_1)| \tag{1737}$$

$$\leq \sum_{k_1 \in \mathbb{Z}} \sum_{k_2, \ldots, k_n \in \mathbb{Z}} e^{\nu(t)|k_1|} |k_1|^s \prod_{d=1}^{n-1} |a_d(k_d - k_{d+1})| |a_n(k_n)| \tag{1738}$$

$$\leq \sum_{j=2}^{n} \sum_{k_1 \in \mathbb{Z}} \sum_{k_2, \ldots, k_n \in \mathbb{Z}} e^{\nu(t)|k_1|} b(n,s) |k_{j-1} - k_j|^s \tag{1739}$$

$$\cdot \prod_{d=1}^{n-1} |a_d(k_d - k_{d+1})| |a_n(k_n)| \tag{1740}$$

$$+ \sum_{k_1 \in \mathbb{Z}} \sum_{k_2, \ldots, k_n \in \mathbb{Z}} e^{\nu(t)|k_1|} b(n,s) |k_n|^s \prod_{d=1}^{n-1} |a_d(k_d - k_{d+1})| |a_n(k_n)|. \tag{1741}$$

For $j \in \{2, \dots, n\}$, we have

$$\sum_{k_1\in\mathbb{Z}} \sum_{k_2,\dots,k_n\in\mathbb{Z}} e^{\nu(t)|k_1|} b(n,s)\left|k_{j-1}-k_j\right|^s \prod_{d=1}^{n-1}\left|a_d(k_d-k_{d+1})\right|\left|a_n(k_n)\right| \tag{1742}$$

$$\leq b(n,s)\sum_{k_n\in\mathbb{Z}}\left|a_n(k_n)\right| e^{\nu(t)|k_n|}\sum_{k_{n-1}\in\mathbb{Z}}\left|a_{n-1}(k_{n-1}-k_n)\right| e^{\nu(t)|k_{n-1}-k_n|} \tag{1743}$$

$$\cdots\cdots \sum_{k_{j-1}\in\mathbb{Z}}\left|k_{j-1}-k_j\right|^s\left|a_{j-1}(k_{j-1}-k_j)\right| e^{\nu(t)|k_{j-1}-k_j|} \tag{1744}$$

$$\cdots\cdots \sum_{k_2\in\mathbb{Z}}\left|a_2(k_2-k_3)\right| e^{\nu(t)|k_2-k_3|}\sum_{k_1\in\mathbb{Z}}\left|a_1(k_1-k_2)\right| e^{\nu(t)|k_1-k_2|}. \tag{1745}$$

Changing the summation variables

$$k_1' = k_1 - k_2 \tag{1746}$$

$$k_2' = k_2 - k_3 \tag{1747}$$

$$\vdots \tag{1748}$$

$$k_{n-1}' = k_{n-1} - k_n \tag{1749}$$

in that order, we obtain

$$\sum_{k_1\in\mathbb{Z}} \sum_{k_2,\dots,k_n\in\mathbb{Z}} e^{\nu(t)|k_1|} b(n,s)\left|k_{j-1}-k_j\right|^s \prod_{d=1}^{n-1}\left|a_d(k_d-k_{d+1})\right|\left|a_n(k_n)\right| \tag{1750}$$

$$\leq b(n,s)\left\|a_{j-1}\right\|_{l^s_\nu}\prod_{\substack{k=1\\k\neq j}}^{n}\left\|a_k\right\|_{l^1_\nu}. \tag{1751}$$

Similarly,

$$\sum_{k_1\in\mathbb{Z}} \sum_{k_2,\dots,k_n\in\mathbb{Z}} e^{\nu(t)|k_1|} b(n,s)\left|k_n\right|^s \prod_{d=1}^{n-1}\left|a_d(k_d-k_{d+1})\right|\left|a_n(k_n)\right| \tag{1752}$$

$$\leq b(n,s)\left\|a_n\right\|_{l^s_\nu}\prod_{k=1}^{n-1}\left\|a_k\right\|_{l^1_\nu}. \tag{1753}$$

This completes the proof. ∎

We note that

$$\|P\|_{l^s_\nu} = \sum_{k_1\neq 0} e^{\nu(t)|k_1|}\,|k_1|^s\,|P(k_1)| \leq \sum_{k_1\neq 0} e^{\nu(t)|k_1|}\,|k_1|^s \sum_{m=1}^{\infty} \frac{|\mathcal{F}(\phi^m)(k_1)|}{m!} \tag{1754}$$

$$= \sum_{m=1}^{\infty} \frac{1}{m!}\,\|\phi^m\|_{\dot{\mathcal{F}}^{s,1}_\nu} \leq \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!}\,\|\phi\|^{m-1}_{\mathcal{F}^{0,1}_\nu}\right)\|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu}\,, \tag{1755}$$

$$\left\|\widetilde{P}\right\|_{l^s_\nu} = \sum_{k_1\neq 0} e^{\nu(t)|k_1|}\,|k_1|^s\left|\widetilde{P}(k_1)\right| \leq \sum_{k_1\neq 0} e^{\nu(t)|k_1|}\,|k_1|^s \sum_{m=2}^{\infty} \frac{|\mathcal{F}(\phi^m)(k_1)|}{m!} \tag{1756}$$

$$= \sum_{m=2}^{\infty} \frac{1}{m!}\,\|\phi^m\|_{\dot{\mathcal{F}}^{s,1}_\nu} \leq \left(\sum_{m=2}^{\infty} \frac{b(m,s)}{(m-1)!}\,\|\phi\|^{m-1}_{\mathcal{F}^{0,1}_\nu}\right)\|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu}\,, \tag{1757}$$

$$\|Q\|_{l^s_\nu} = \sum_{k_1\neq 0} e^{\nu(t)|k_1|}\,|k_1|^s\,|Q(k_1)| \leq \sum_{k_1\neq 0} e^{\nu(t)|k_1|}\,|k_1|^s \sum_{m=1}^{\infty} \frac{|\mathcal{F}(\phi^m)(k_1)|}{m!} \tag{1758}$$

$$= \sum_{m=1}^{\infty} \frac{1}{m!}\,\|\phi^m\|_{\dot{\mathcal{F}}^{s,1}_\nu} \leq \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!}\,\|\phi\|^{m-1}_{\mathcal{F}^{0,1}_\nu}\right)\|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu}\,, \tag{1759}$$

$$\left\|\widetilde{Q}\right\|_{l^s_\nu} = \sum_{k_1\neq 0} e^{\nu(t)|k_1|}\,|k_1|^s\left|\widetilde{Q}(k_1)\right| \leq \sum_{k_1\neq 0} e^{\nu(t)|k_1|}\,|k_1|^s \sum_{m=2}^{\infty} \frac{|\mathcal{F}(\phi^m)(k_1)|}{m!} \tag{1760}$$

$$= \sum_{m=2}^{\infty} \frac{1}{m!}\,\|\phi^m\|_{\dot{\mathcal{F}}^{s,1}_\nu} \leq \left(\sum_{m=2}^{\infty} \frac{b(m,s)}{(m-1)!}\,\|\phi\|^{m-1}_{\mathcal{F}^{0,1}_\nu}\right)\|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu}\,. \tag{1761}$$

To begin the proof of Lemma 5, we observe that

$$\|U_{\geq 2}\|_{\dot{\mathcal{F}}^{s,1}_\nu} \leq \sum_{j=1}^{8} \left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} \widetilde{B_j}(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}^{s,1}_\nu} + \left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} \widetilde{B_{13}}(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}^{s,1}_\nu}. \tag{1762}$$

This means that it suffices to estimate each of the $\dot{\mathcal{F}}^{s,1}_\nu$ norms on the right hand side. By

Proposition 11 and (1755), we obtain

$$\sum_{k_1\neq 0} e^{\nu(t)|k_1|}|k_1|^s \left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{1,1}})(k_1,\beta)d\beta\right| \tag{1763}$$

$$\leq \sum_{k_1\neq 0} e^{\nu(t)|k_1|}|k_1|^s \sum_{j_1+j_2+n\geq 1}\frac{C_n}{2j_1!j_2!}(|\mathcal{F}(\phi)|*\cdots*|\mathcal{F}(\phi)|*|P|*\cdots*|P|)(k_1) \tag{1764}$$

$$\leq \sum_{j_1+j_2+n\geq 1}\frac{C_n}{2j_1!j_2!}\left\||\mathcal{F}(\phi)|*\cdots*|\mathcal{F}(\phi)|*|P|*\cdots*|P|\right\|_{l^s_\nu} \tag{1765}$$

$$\leq \sum_{j_1+j_2+n\geq 1}\frac{C_n}{2j_1!j_2!}b(j_1+j_2+n+1,s) \tag{1766}$$

$$\cdot\left(\|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu}\|\phi\|^{j_1+j_2}_{\mathcal{F}^{0,1}_\nu}\|P\|^n_{l^1_\nu}(j_1+j_2+1)+\|P\|_{l^s_\nu}\|\phi\|^{j_1+j_2+1}_{\mathcal{F}^{0,1}_\nu}\|P\|^{n-1}_{l^1_\nu}\cdot n\right) \tag{1767}$$

$$\leq \sum_{j_1+j_2+n\geq 1}\frac{C_n}{2j_1!j_2!}b(j_1+j_2+n+1,s)\cdot \tag{1768}$$

$$\Bigg(\|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu}\|\phi\|^{j_1+j_2}_{\mathcal{F}^{0,1}_\nu}(e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1)^n(j_1+j_2+1) \tag{1769}$$

$$+\left(\sum_{m=1}^{\infty}\frac{b(m,s)}{(m-1)!}\|\phi\|^{m-1}_{\mathcal{F}^{0,1}_\nu}\right)\|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu}\|\phi\|^{j_1+j_2+1}_{\mathcal{F}^{0,1}_\nu}(e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1)^{n-1}\cdot n\Bigg). \tag{1770}$$

By Proposition 11 and (1755), we obtain

$$\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\widetilde{B_{1,2}}(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}^{s,1}_\nu}=\sum_{k_1\neq 0} e^{\nu(t)|k_1|}|k_1|^s\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{1,2}})(k_1,\beta)d\beta\right| \tag{1771}$$

$$\leq \sum_{k_1\neq 0} e^{\nu(t)|k_1|}|k_1|^s\sum_{j_1+j_2+j_3\geq 1}\frac{C_1}{2j_1!j_2!j_3!}(|\mathcal{F}(\phi)|*\cdots*|\mathcal{F}(\phi)|*|P|*\left|\mathcal{F}(\phi^{j_3})\right|)(k_1) \tag{1772}$$

$$\leq \sum_{j_1+j_2+j_3\geq 1}\frac{C_1}{2j_1!j_2!j_3!}b(j_1+j_2+2,s)\Bigg(\|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu}\|\phi\|^{j_1+j_2-1}_{\mathcal{F}^{0,1}_\nu}\left\|\phi^{j_3}\right\|_{\mathcal{F}^{0,1}_\nu}\|P\|_{l^1_\nu}\cdot(j_1+j_2) \tag{1773}$$

$$+\left\|\phi^{j_3}\right\|_{\dot{\mathcal{F}}^{s,1}_\nu}\|\phi\|^{j_1+j_2}_{\mathcal{F}^{0,1}_\nu}\|P\|_{l^1_\nu}+\|P\|_{l^s_\nu}\|\phi\|^{j_1+j_2}_{\mathcal{F}^{0,1}_\nu}\left\|\phi^{j_3}\right\|_{\mathcal{F}^{0,1}_\nu}\Bigg) \tag{1774}$$

$$\leq \sum_{j_1+j_2+j_3\geq 1}\frac{C_1}{2j_1!j_2!j_3!}b(j_1+j_2+2,s)\Bigg(\|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu}\|\phi\|^{j_1+j_2+j_3-1}_{\mathcal{F}^{0,1}_\nu}(e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1)\cdot(j_1+j_2) \tag{1775}$$

$$+b(j_3,s)\|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu}\|\phi\|_{\mathcal{F}^{0,1}_\nu}\cdot j_3\cdot\|\phi\|^{j_1+j_2}_{\mathcal{F}^{0,1}_\nu}(e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1) \tag{1776}$$

$$+\left(\sum_{m=1}^{\infty}\frac{b(m,s)}{(m-1)!}\|\phi\|^{m-1}_{\mathcal{F}^{0,1}_\nu}\right)\|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu}\|\phi\|^{j_1+j_2+j_3}_{\mathcal{F}^{0,1}_\nu}\Bigg). \tag{1777}$$

By (1757), we obtain

$$\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} \widetilde{B_{1,3}}(\alpha, \beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} = \sum_{k_1 \neq 0} e^{\nu(t)|k_1|} |k_1|^s \left| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} \mathcal{F}(\widetilde{B_{1,3}})(k_1, \beta) d\beta \right| \tag{1778}$$

$$\leq \frac{C_1}{2} \left(\sum_{m=2}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1} \right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} . \tag{1779}$$

By Proposition 11 and (1755), we obtain

$$\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} \widetilde{B_2}(\alpha, \beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} = \sum_{k_1 \neq 0} e^{\nu(t)|k_1|} |k_1|^s \left| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} \mathcal{F}(\widetilde{B_2})(k_1, \beta) d\beta \right| \tag{1780}$$

$$\leq \sum_{k_1 \neq 0} e^{\nu(t)|k_1|} |k_1|^s \frac{1}{2} \sum_{j_1+j_2+j_3+n \geq 1} \frac{C_n}{j_1! j_2! j_3!} \sum_{k_2, \ldots, k_{j_1+j_2+n+1} \in \mathbb{Z}} \tag{1781}$$

$$\prod_{d=1}^{j_1+j_2} |\mathcal{F}(\phi)(k_d - k_{d+1})| \cdot \prod_{d=1}^{n} |P(k_{j_1+j_2+d} - k_{j_1+j_2+d+1})| \cdot \left| \mathcal{F}(\phi^{j_3} \phi')(k_{j_1+j_2+n+1}) \right| \tag{1782}$$

$$\leq \frac{1}{2} \sum_{j_1+j_2+j_3+n \geq 1} \frac{C_n}{j_1! j_2! j_3!} \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |P| * \cdots * |P| * \left| \mathcal{F}(\phi^{j_3} \phi') \right| \right\|_{l_\nu^s} \tag{1783}$$

$$\leq \frac{1}{2} \sum_{j_1+j_2+j_3+n \geq 1} \frac{C_n}{j_1! j_2! j_3!} b(j_1 + j_2 + n + 1, s) \tag{1784}$$

$$\cdot \Bigg(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1} \|P\|_{l_\nu^1}^n \left\| \phi^{j_3} \phi' \right\|_{\mathcal{F}_\nu^{0,1}} (j_1 + j_2) \tag{1785}$$

$$+ \|P\|_{l_\nu^s} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2} \left\| \phi^{j_3} \phi' \right\|_{\mathcal{F}_\nu^{0,1}} \|P\|_{l_\nu^1}^{n-1} \cdot n \tag{1786}$$

$$+ \left\| \phi^{j_3} \phi' \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2} \|P\|_{l_\nu^1}^n \Bigg) \tag{1787}$$

$$\leq \frac{1}{2} \sum_{j_1+j_2+j_3+n \geq 1} \frac{C_n}{j_1! j_2! j_3!} b(j_1 + j_2 + n + 1, s) \tag{1788}$$

$$\cdot \Bigg(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n \|\phi'\|_{\mathcal{F}_\nu^{0,1}} (j_1 + j_2) \tag{1789}$$

$$+ \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1} \right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^{n-1} \cdot n \tag{1790}$$

$$+ b(j_3 + 1, s) (\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \cdot j_3 + \|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3}) \tag{1791}$$

$$\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n \Bigg). \tag{1792}$$

By Proposition 11 and (1759), we obtain

$$\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} \widetilde{B_{3,1}}(\alpha, \beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \tag{1793}$$

$$\leq \sum_{k_1 \neq 0} e^{\nu(t)|k_1|} |k_1|^s \left| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} \mathcal{F}(\widetilde{B_{3,1}})(k_1, \beta) d\beta \right| \tag{1794}$$

$$\leq \sum_{k_1 \neq 0} e^{\nu(t)|k_1|} |k_1|^s \cdot \frac{1}{2} \sum_{j_3 + j_4 + n \geq 2} (n+1) \frac{C_{n+1}}{j_3! j_4!} \tag{1795}$$

$$\cdot (|Q| * \cdots * |Q| * \left|\mathcal{F}(\phi^{j_3})\right| * \left|\mathcal{F}(\phi^{j_4})\right|)(k_1) \tag{1796}$$

$$\leq \frac{1}{2} \sum_{j_3 + j_4 + n \geq 2} (n+1) \frac{C_{n+1}}{j_3! j_4!} \left\| |Q| * \cdots * |Q| * \left|\mathcal{F}(\phi^{j_3})\right| * \left|\mathcal{F}(\phi^{j_4})\right| \right\|_{l_\nu^s} \tag{1797}$$

$$\leq \frac{1}{2} \sum_{j_3 + j_4 + n \geq 2} (n+1) \frac{C_{n+1}}{j_3! j_4!} b(n+2, s) \Bigg(\|Q\|_{l_\nu^s} \|Q\|_{l_\nu^1}^{n-1} \left\|\phi^{j_3}\right\|_{\mathcal{F}_\nu^{0,1}} \left\|\phi^{j_4}\right\|_{\mathcal{F}_\nu^{0,1}} \cdot n \tag{1798}$$

$$+ \left\|\phi^{j_3}\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|Q\|_{l_\nu^1}^n \left\|\phi^{j_4}\right\|_{\mathcal{F}_\nu^{0,1}} + \left\|\phi^{j_4}\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|Q\|_{l_\nu^1}^n \left\|\phi^{j_3}\right\|_{\mathcal{F}_\nu^{0,1}} \Bigg) \tag{1799}$$

$$\leq \frac{1}{2} \sum_{j_3 + j_4 + n \geq 2} (n+1) \frac{C_{n+1}}{j_3! j_4!} b(n+2, s) \tag{1800}$$

$$\cdot \left(\left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1} \right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^{n-1} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3 + j_4} \cdot n \right. \tag{1801}$$

$$+ b(j_3, s) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3 - 1} \cdot j_3 \cdot (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_4} \tag{1802}$$

$$\left. + b(j_4, s) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_4 - 1} \cdot j_4 \cdot (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3} \right). \tag{1803}$$

By Proposition 11 and (1759), we obtain

$$
\begin{aligned}
&\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} \widetilde{B_{3,2}}(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} && (1804)\\
\leq & \sum_{k_1\neq 0} e^{\nu(t)|k_1|} |k_1|^s \left| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} \mathcal{F}(\widetilde{B_{3,2}})(k_1,\beta) d\beta \right| && (1805)\\
\leq & \sum_{k_1\neq 0} e^{\nu(t)|k_1|} |k_1|^s \cdot \frac{1}{2} \sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}} (n+1) \frac{C_{n+1}}{j_1!j_2!j_3!j_4!} && (1806)\\
& \cdot (|\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |Q| * \cdots * |Q| * \left|\mathcal{F}(\phi^{j_3})\right| * \left|\mathcal{F}(\phi^{j_4})\right|)(k_1) && (1807)\\
\leq & \frac{1}{2} \sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}} \frac{(n+1)C_{n+1}}{j_1!j_2!j_3!j_4!} && (1808)\\
& \cdot \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |Q| * \cdots * |Q| * \left|\mathcal{F}(\phi^{j_3})\right| * \left|\mathcal{F}(\phi^{j_4})\right| \right\|_{l_\nu^s} && (1809)\\
\leq & \frac{1}{2} \sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}} \frac{(n+1)C_{n+1}}{j_1!j_2!j_3!j_4!} \cdot b(j_1+j_2+n+2,s) && (1810)\\
& \cdot \Bigg(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1} \|Q\|_{l_\nu^1}^n \left\|\phi^{j_3}\right\|_{\mathcal{F}_\nu^{0,1}} \left\|\phi^{j_4}\right\|_{\mathcal{F}_\nu^{0,1}} (j_1+j_2) && (1811)\\
& + \|Q\|_{l_\nu^s} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2} \left\|\phi^{j_3}\right\|_{\mathcal{F}_\nu^{0,1}} \left\|\phi^{j_4}\right\|_{\mathcal{F}_\nu^{0,1}} \|Q\|_{l_\nu^1}^{n-1} \cdot n && (1812)\\
& + \left\|\phi^{j_3}\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2} \|Q\|_{l_\nu^1}^n \left\|\phi^{j_4}\right\|_{\mathcal{F}_\nu^{0,1}} && (1813)\\
& + \left\|\phi^{j_4}\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2} \|Q\|_{l_\nu^1}^n \left\|\phi^{j_3}\right\|_{\mathcal{F}_\nu^{0,1}} \Bigg) && (1814)\\
\leq & \frac{1}{2} \sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}} \frac{(n+1)C_{n+1}}{j_1!j_2!j_3!j_4!} \cdot b(j_1+j_2+n+2,s) && (1815)\\
& \cdot \Bigg(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3+j_4-1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n (j_1+j_2) && (1816)\\
& + \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1} \right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3+j_4} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-1} \cdot n && (1817)\\
& + b(j_3,s) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1} \cdot j_3 \cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_4} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n && (1818)\\
& + b(j_4,s) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_4-1} \cdot j_4 \cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n \Bigg). && (1819)
\end{aligned}
$$

By (1761), we obtain

$$\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\widetilde{B_{3,3}}(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}_{\nu}^{s,1}} \tag{1820}$$

$$=\sum_{k_1\neq 0}e^{\nu(t)|k_1|}\left|k_1\right|^{s}\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{3,3}})(k_1,\beta)d\beta\right| \tag{1821}$$

$$\leq C_2\left(\sum_{m=2}^{\infty}\frac{b(m,s)}{(m-1)!}\left\|\phi\right\|_{\mathcal{F}_{\nu}^{0,1}}^{m-1}\right)\left\|\phi\right\|_{\dot{\mathcal{F}}_{\nu}^{s,1}}. \tag{1822}$$

By Proposition 11 and (1759), we obtain

$$\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} \widetilde{B_4}(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \tag{1823}$$

$$= \sum_{k_1 \neq 0} e^{\nu(t)|k_1|} |k_1|^s \left| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} \mathcal{F}(\widetilde{B_4})(k_1,\beta) d\beta \right| \tag{1824}$$

$$\leq \sum_{k_1 \neq 0} e^{\nu(t)|k_1|} |k_1|^s \cdot \frac{1}{2} \sum_{j_1+j_2+j_3+j_4+n \geq 1} \frac{(n+1)C_{n+1}}{j_1!j_2!j_3!j_4!} \tag{1825}$$

$$\cdot \left(|\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |Q| * \cdots * |Q| * \left|\mathcal{F}(\phi^{j_3})\right| * \left|\mathcal{F}(\phi^{j_4}\phi')\right| \right)(k_1) \tag{1826}$$

$$\leq \frac{1}{2} \sum_{j_1+j_2+j_3+j_4+n \geq 1} \frac{(n+1)C_{n+1}}{j_1!j_2!j_3!j_4!} \tag{1827}$$

$$\cdot \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |Q| * \cdots * |Q| * \left|\mathcal{F}(\phi^{j_3})\right| * \left|\mathcal{F}(\phi^{j_4}\phi')\right| \right\|_{l^s_\nu} \tag{1828}$$

$$\leq \frac{1}{2} \sum_{j_1+j_2+j_3+j_4+n \geq 1} \frac{(n+1)C_{n+1}}{j_1!j_2!j_3!j_4!} b(j_1+j_2+n+2, s) \tag{1829}$$

$$\cdot \Bigg(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1} \|Q\|_{l^1_\nu}^n \left\|\phi^{j_3}\right\|_{\mathcal{F}_\nu^{0,1}} \left\|\phi^{j_4}\phi'\right\|_{\mathcal{F}_\nu^{0,1}} (j_1+j_2) \tag{1830}$$

$$+ \|Q\|_{l^s_\nu} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2} \|Q\|_{l^1_\nu}^{n-1} \left\|\phi^{j_3}\right\|_{\mathcal{F}_\nu^{0,1}} \left\|\phi^{j_4}\phi'\right\|_{\mathcal{F}_\nu^{0,1}} \cdot n \tag{1831}$$

$$+ \left\|\phi^{j_3}\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2} \|Q\|_{l^1_\nu}^n \left\|\phi^{j_4}\phi'\right\|_{\mathcal{F}_\nu^{0,1}} \tag{1832}$$

$$+ \left\|\phi^{j_4}\phi'\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2} \|Q\|_{l^1_\nu}^n \left\|\phi^{j_3}\right\|_{\mathcal{F}_\nu^{0,1}} \Bigg) \tag{1833}$$

$$\leq \frac{1}{2} \sum_{j_1+j_2+j_3+j_4+n \geq 1} \frac{(n+1)C_{n+1}}{j_1!j_2!j_3!j_4!} b(j_1+j_2+n+2, s) \tag{1834}$$

$$\cdot \Bigg(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1} \left(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1\right)^n \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3+j_4} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \cdot (j_1+j_2) \tag{1835}$$

$$+ \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1} \right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3+j_4} \left(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1\right)^{n-1} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \cdot n \tag{1836}$$

$$+ b(j_3,s) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1} \cdot j_3 \cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_4} \left(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1\right)^n \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \tag{1837}$$

$$+ b(j_4+1,s) \left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_4-1} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \cdot j_4 + \|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_4} \right) \tag{1838}$$

$$\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3} \left(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1\right)^n \Bigg). \tag{1839}$$

By Proposition 11 and (1759), we obtain

$$\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\widetilde{B_{5,1}}(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}_{\nu}^{s,1}} \tag{1840}$$

$$\leq \sum_{k_1\neq 0} e^{\nu(t)|k_1|}|k_1|^s\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{5,1}})(k_1,\beta)d\beta\right| \tag{1841}$$

$$\leq \sum_{k_1\neq 0} e^{\nu(t)|k_1|}|k_1|^s\cdot\frac{1}{2}\sum_{j_3+n\geq 2}\frac{C_n}{j_3!}\sum_{k_2,\ldots,k_{n+1}\in\mathbb{Z}}\prod_{d=1}^{n}|Q(k_d-k_{d+1})|\left|\mathcal{F}(\phi^{j_3})(k_{n+1})\right| \tag{1842}$$

$$\leq\frac{1}{2}\sum_{j_3+n\geq 2}\frac{C_n}{j_3!}\left\||Q|*\cdots*|Q|*\left|\mathcal{F}(\phi^{j_3})\right|\right\|_{l_\nu^s} \tag{1843}$$

$$\leq\frac{1}{2}\sum_{j_3+n\geq 2}\frac{C_n}{j_3!}b(n+1,s)\left(\|Q\|_{l_\nu^s}\|Q\|_{l_\nu^1}^{n-1}\left\|\phi^{j_3}\right\|_{\mathcal{F}_\nu^{0,1}}\cdot n+\left\|\phi^{j_3}\right\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|Q\|_{l_\nu^1}^{n}\right) \tag{1844}$$

$$\leq\frac{1}{2}\sum_{j_3+n\geq 2}\frac{C_n}{j_3!}b(n+1,s) \tag{1845}$$

$$\cdot\left(\left(\sum_{m=1}^{\infty}\frac{b(m,s)}{(m-1)!}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1}\right)\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-1}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3}\cdot n\right. \tag{1846}$$

$$\left.+\,b(j_3,s)\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1}\cdot j_3(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n}\right). \tag{1847}$$

By Proposition 11 and (1759), we obtain

$$\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} \widetilde{B_{5,2}}(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \tag{1848}$$

$$= \sum_{k_1 \neq 0} e^{\nu(t)|k_1|} |k_1|^s \left| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} \mathcal{F}(\widetilde{B_{5,2}})(k_1,\beta) d\beta \right| \tag{1849}$$

$$\leq \sum_{k_1 \neq 0} e^{\nu(t)|k_1|} |k_1|^s \cdot \frac{1}{2} \sum_{\substack{j_1+j_2 \geq 1 \\ j_3+n \geq 1}} \frac{C_n}{j_1! j_2! j_3!} \tag{1850}$$

$$\cdot \left(|\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |Q| * \cdots * |Q| * \left|\mathcal{F}(\phi^{j_3})\right| \right)(k_1) \tag{1851}$$

$$\leq \frac{1}{2} \sum_{\substack{j_1+j_2 \geq 1 \\ j_3+n \geq 1}} \frac{C_n}{j_1! j_2! j_3!} \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |Q| * \cdots * |Q| * \left|\mathcal{F}(\phi^{j_3})\right| \right\|_{l_\nu^s} \tag{1852}$$

$$\leq \frac{1}{2} \sum_{\substack{j_1+j_2 \geq 1 \\ j_3+n \geq 1}} \frac{C_n}{j_1! j_2! j_3!} b(j_1+j_2+n+1,s) \tag{1853}$$

$$\cdot \Bigg(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1} \|Q\|_{l_\nu^1}^n \left\|\phi^{j_3}\right\|_{\mathcal{F}_\nu^{0,1}} \cdot (j_1+j_2) \tag{1854}$$

$$+ \|Q\|_{l_\nu^s} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2} \|Q\|_{l_\nu^1}^{n-1} \left\|\phi^{j_3}\right\|_{\mathcal{F}_\nu^{0,1}} \cdot n \tag{1855}$$

$$+ \left\|\phi^{j_3}\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|Q\|_{l_\nu^1}^n \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2} \Bigg) \tag{1856}$$

$$\leq \frac{1}{2} \sum_{\substack{j_1+j_2 \geq 1 \\ j_3+n \geq 1}} \frac{C_n}{j_1! j_2! j_3!} b(j_1+j_2+n+1,s) \tag{1857}$$

$$\cdot \Bigg(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n \cdot (j_1+j_2) \tag{1858}$$

$$+ \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1} \right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^{n-1} \cdot n \tag{1859}$$

$$+ b(j_3,s) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1} \cdot j_3 \cdot (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2} \Bigg). \tag{1860}$$

By (1761), we obtain

$$\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} \widetilde{B_{5,3}}(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \tag{1861}$$

$$= \sum_{k_1 \neq 0} e^{\nu(t)|k_1|} |k_1|^s \left| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} \mathcal{F}(\widetilde{B_{5,3}})(k_1,\beta) d\beta \right| \tag{1862}$$

$$\leq \frac{C_1}{2} \left(\sum_{m=2}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1} \right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}. \tag{1863}$$

By Proposition 11 and (1759), we obtain

$$\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} \widetilde{B_{6,1}}(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \tag{1864}$$

$$= \sum_{k_1 \neq 0} e^{\nu(t)|k_1|} |k_1|^s \left| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} \mathcal{F}(\widetilde{B}_{6,1})(k_1,\beta) d\beta \right| \tag{1865}$$

$$\leq \sum_{k_1 \neq 0} e^{\nu(t)|k_1|} |k_1|^s \frac{1}{2} \sum_{j_3+n \geq 2} \frac{C_n}{j_3!} \sum_{k_2,\ldots,k_{n+1} \in \mathbb{Z}} \prod_{d=1}^{n} |Q(k_d - k_{d+1})| \left|\mathcal{F}(\phi^{j_3})(k_{n+1})\right| \tag{1866}$$

$$\leq \frac{1}{2} \sum_{j_3+n \geq 2} \frac{C_n}{j_3!} \sum_{k_1 \neq 0} e^{\nu(t)|k_1|} |k_1|^s \left(|Q| * \cdots * |Q| * \left|\mathcal{F}(\phi^{j_3})\right|\right)(k_1) \tag{1867}$$

$$\leq \frac{1}{2} \sum_{j_3+n \geq 2} \frac{C_n}{j_3!} \left\| |Q| * \cdots * |Q| * \left|\mathcal{F}(\phi^{j_3})\right| \right\|_{l_\nu^s} \tag{1868}$$

$$\leq \frac{1}{2} \sum_{j_3+n \geq 2} \frac{C_n}{j_3!} b(n+1,s) \left(\|Q\|_{l_\nu^s} \|Q\|_{l_\nu^1}^{n-1} \left\|\phi^{j_3}\right\|_{\mathcal{F}_\nu^{0,1}} \cdot n + \left\|\phi^{j_3}\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|Q\|_{l_\nu^1}^{n} \right) \tag{1869}$$

$$\leq \frac{1}{2} \sum_{j_3+n \geq 2} \frac{C_n}{j_3!} b(n+1,s) \tag{1870}$$

$$\cdot \left(\left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1} \right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^{n-1} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3} \cdot n \right. \tag{1871}$$

$$\left. + b(j_3,s) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1} \cdot j_3 \cdot (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n \right). \tag{1872}$$

By Proposition 11 and (1759), we obtain

$$\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} \widetilde{B_{6,2}}(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_{\nu}^{s,1}} \tag{1873}$$

$$\leq \sum_{k_1 \neq 0} e^{\nu(t)|k_1|} |k_1|^s \left| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} \mathcal{F}(\widetilde{B_{6,2}})(k_1,\beta) d\beta \right| \tag{1874}$$

$$\leq \sum_{k_1 \neq 0} e^{\nu(t)|k_1|} |k_1|^s \cdot \frac{1}{2} \sum_{\substack{j_1+j_2 \geq 1 \\ j_3+n \geq 1}} \frac{C_n}{j_1! j_2! j_3!} \tag{1875}$$

$$\cdot (|\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |Q| * \cdots * |Q| * \left|\mathcal{F}(\phi^{j_3})\right|)(k_1) \tag{1876}$$

$$\leq \frac{1}{2} \sum_{\substack{j_1+j_2 \geq 1 \\ j_3+n \geq 1}} \frac{C_n}{j_1! j_2! j_3!} \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |Q| * \cdots * |Q| * \left|\mathcal{F}(\phi^{j_3})\right| \right\|_{l_{\nu}^s} \tag{1877}$$

$$\leq \frac{1}{2} \sum_{\substack{j_1+j_2 \geq 1 \\ j_3+n \geq 1}} \frac{C_n}{j_1 j_2! j_3!} b(j_1+j_2+n+1,s) \tag{1878}$$

$$\cdot \Bigg(\|\phi\|_{\dot{\mathcal{F}}_{\nu}^{s,1}} \|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_1+j_2-1} \|Q\|_{l_{\nu}^1}^n \left\|\phi^{j_3}\right\|_{\mathcal{F}_{\nu}^{0,1}} \cdot (j_1+j_2) \tag{1879}$$

$$+ \|Q\|_{l_{\nu}^s} \|Q\|_{l_{\nu}^1}^{n-1} \|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_1+j_2} \left\|\phi^{j_3}\right\|_{\mathcal{F}_{\nu}^{0,1}} \cdot n \tag{1880}$$

$$+ \left\|\phi^{j_3}\right\|_{\dot{\mathcal{F}}_{\nu}^{s,1}} \|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_1+j_2} \|Q\|_{l_{\nu}^1}^n \Bigg) \tag{1881}$$

$$\leq \frac{1}{2} \sum_{\substack{j_1+j_2 \geq 1 \\ j_3+n \geq 1}} \frac{C_n}{j_1 j_2! j_3!} b(j_1+j_2+n+1,s) \tag{1882}$$

$$\cdot \Bigg(\|\phi\|_{\dot{\mathcal{F}}_{\nu}^{s,1}} \|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_1+j_2+j_3-1} (e^{\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}} - 1)^n \cdot (j_1+j_2) \tag{1883}$$

$$+ \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{m-1} \right) \|\phi\|_{\dot{\mathcal{F}}_{\nu}^{s,1}} (e^{\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}} - 1)^{n-1} \|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_1+j_2+j_3} \cdot n \tag{1884}$$

$$+ b(j_3,s) \|\phi\|_{\dot{\mathcal{F}}_{\nu}^{s,1}} \|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_3-1} \cdot j_3 \cdot \|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{j_1+j_2} (e^{\|\phi\|_{\mathcal{F}_{\nu}^{0,1}}} - 1)^n \Bigg). \tag{1885}$$

By (1761), we obtain

$$\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} \widetilde{B_{6,3}}(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_{\nu}^{s,1}} \tag{1886}$$

$$\leq \sum_{k_1 \neq 0} e^{\nu(t)|k_1|} |k_1|^s \left| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} \mathcal{F}(\widetilde{B_{6,3}})(k_1,\beta) d\beta \right| \tag{1887}$$

$$\leq \frac{C_1}{2} \left(\sum_{m=2}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_{\nu}^{0,1}}^{m-1} \right) \|\phi\|_{\dot{\mathcal{F}}_{\nu}^{s,1}}. \tag{1888}$$

By Proposition 11 and (1759), we obtain

$$\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} \widetilde{B_7}(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \tag{1889}$$

$$\leq \sum_{k_1 \neq 0} e^{\nu(t)|k_1|} |k_1|^s \left| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} \mathcal{F}(\widetilde{B_7})(k_1,\beta) d\beta \right| \tag{1890}$$

$$\leq \sum_{k_1 \neq 0} e^{\nu(t)|k_1|} |k_1|^s \cdot \frac{1}{2} \sum_{j_1+j_2+j_3+n \geq 1} \frac{C_n}{j_1! j_2! j_3!} \tag{1891}$$

$$\cdot (|\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |Q| \cdots * |Q| * \left|\mathcal{F}(\phi^{j_3}\phi')\right|)(k_1) \tag{1892}$$

$$\leq \frac{1}{2} \sum_{j_1+j_2+j_3+n \geq 1} \frac{C_n}{j_1! j_2! j_3!} \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |Q| \cdots * |Q| * \left|\mathcal{F}(\phi^{j_3}\phi')\right| \right\|_{l_\nu^s} \tag{1893}$$

$$\leq \frac{1}{2} \sum_{j_1+j_2+j_3+n \geq 1} \frac{C_n}{j_1! j_2! j_3!} b(j_1+j_2+n+1,s) \tag{1894}$$

$$\cdot \Bigg(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1} \|Q\|_{l_\nu^1}^n \left\|\phi^{j_3}\phi'\right\|_{\mathcal{F}_\nu^{0,1}} \cdot (j_1+j_2) \tag{1895}$$

$$+ \|Q\|_{l_\nu^s} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2} \|Q\|_{l_\nu^1}^{n-1} \left\|\phi^{j_3}\phi'\right\|_{\mathcal{F}_\nu^{0,1}} \cdot n + \left\|\phi^{j_3}\phi'\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2} \|Q\|_{l_\nu^1}^n \Bigg) \tag{1896}$$

$$\leq \frac{1}{2} \sum_{j_1+j_2+j_3+n \geq 1} \frac{C_n}{j_1! j_2! j_3!} b(j_1+j_2+n+1,s) \tag{1897}$$

$$\cdot \Bigg(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \cdot (j_1+j_2) \tag{1898}$$

$$+ \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1} \right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^{n-1} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \cdot n \tag{1899}$$

$$+ b(j_3+1,s)(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \cdot j_3 + \|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3}) \tag{1900}$$

$$\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n \Bigg). \tag{1901}$$

By Proposition 11 and (1759), we obtain

$$\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} \widetilde{B_8}(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \tag{1902}$$

$$\leq \sum_{k_1 \neq 0} e^{\nu(t)|k_1|} |k_1|^s \left| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} \mathcal{F}(\widetilde{B_8})(k_1,\beta) d\beta \right| \tag{1903}$$

$$\leq \sum_{k_1 \neq 0} e^{\nu(t)|k_1|} |k_1|^s \cdot \frac{1}{2} \sum_{j_1+j_2+j_3+n \geq 1} \frac{C_n}{j_1! j_2! j_3!} \tag{1904}$$

$$\cdot (|\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |Q| \cdots * |Q| * \left|\mathcal{F}(\phi^{j_3}\phi')\right|)(k_1) \tag{1905}$$

$$\leq \frac{1}{2} \sum_{j_1+j_2+j_3+n \geq 1} \frac{C_n}{j_1! j_2! j_3!} \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |Q| \cdots * |Q| * \left|\mathcal{F}(\phi^{j_3}\phi')\right| \right\|_{l_\nu^s} \tag{1906}$$

$$\leq \frac{1}{2} \sum_{j_1+j_2+j_3+n \geq 1} \frac{C_n}{j_1! j_2! j_3!} b(j_1+j_2+n+1,s) \tag{1907}$$

$$\cdot \Bigg(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1} \|Q\|_{l_\nu^1}^n \left\|\phi^{j_3}\phi'\right\|_{\mathcal{F}_\nu^{0,1}} \cdot (j_1+j_2) \tag{1908}$$

$$+ \|Q\|_{l_\nu^s} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2} \|Q\|_{l_\nu^1}^{n-1} \left\|\phi^{j_3}\phi'\right\|_{\mathcal{F}_\nu^{0,1}} \cdot n + \left\|\phi^{j_3}\phi'\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2} \|Q\|_{l_\nu^1}^n \Bigg) \tag{1909}$$

$$\leq \frac{1}{2} \sum_{j_1+j_2+j_3+n \geq 1} \frac{C_n}{j_1! j_2! j_3!} b(j_1+j_2+n+1,s) \tag{1910}$$

$$\cdot \Bigg(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \cdot (j_1+j_2) \tag{1911}$$

$$+ \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1} \right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^{n-1} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \cdot n \tag{1912}$$

$$+ b(j_3+1,s)(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \cdot j_3 + \|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3}) \tag{1913}$$

$$\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n \Bigg). \tag{1914}$$

Lastly,

$$\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\widetilde{B_{13}}(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \tag{1915}$$

$$\leq\sum_{k_1\neq 0}e^{\nu(t)|k_1|}\left|k_1\right|^s\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(\widetilde{B_{13}})(k_1,\beta)d\beta\right| \tag{1916}$$

$$\leq\frac{\gamma}{4\pi}(1+\pi)\sum_{k_1\neq 0}e^{\nu(t)|k_1|}\left|k_1\right|^s\sum_{j_1+j_2\geq 2}\frac{1}{j_1!j_2!}(|\mathcal{F}(\phi)|*\cdots*|\mathcal{F}(\phi)|)(k_1) \tag{1917}$$

$$\leq\frac{\gamma}{4\pi}(1+\pi)\sum_{j_1+j_2\geq 2}\frac{1}{j_1!j_2!}\left\||\mathcal{F}(\phi)|*\cdots*|\mathcal{F}(\phi)|\right\|_{l^s_\nu} \tag{1918}$$

$$\leq\frac{\gamma}{4\pi}(1+\pi)\sum_{j_1+j_2\geq 2}\frac{1}{j_1!j_2!}b(j_1+j_2,s)\left\|\phi\right\|_{\dot{\mathcal{F}}_\nu^{s,1}}\left\|\phi\right\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1}\cdot(j_1+j_2) \tag{1919}$$

$$\leq\frac{\gamma}{4\pi}(1+\pi)\left\|\phi\right\|_{\dot{\mathcal{F}}_\nu^{s,1}}\sum_{j_1+j_2\geq 2}\frac{b(j_1+j_2,s)}{j_1!j_2!}\left\|\phi\right\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1}\cdot(j_1+j_2). \tag{1920}$$

This completes the proof of Lemma 5.

14 Estimating $(U_{\geq 2})_\alpha$

In Section 13, we derived that

$$U_{\geq 2}(\alpha)=\mathrm{Re}\left(\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}B(\alpha,\beta)d\beta\right), \tag{1921}$$

where

$$B(\alpha,\beta)=\sum_{j=1}^{8}\widetilde{B_j}(\alpha,\beta)+\widetilde{B_{13}}(\alpha,\beta). \tag{1922}$$

To estimate the $\dot{\mathcal{F}}_\nu^{s,1}$ norm of $(U_{\geq 2})_\alpha$, we differentiate the right hand side with respect to α. Recalling that

$$\widetilde{B_1}(\alpha,\beta)=\widetilde{B_{1,1}}(\alpha,\beta)+\widetilde{B_{1,2}}(\alpha,\beta)+\widetilde{B_{1,3}}(\alpha,\beta), \tag{1923}$$

we note that

$$(\widetilde{B_{1,1}})_\alpha(\alpha,\beta)=\sum_{j=1}^{4}B_{1,1}^j(\alpha,\beta), \tag{1924}$$

$$(\widetilde{B_{1,2}})_\alpha(\alpha,\beta)=\sum_{j=1}^{4}B_{1,2}^j(\alpha,\beta), \tag{1925}$$

where

$$B^1_{1,1}(\alpha,\beta) = -\sum_{j_1+j_2+n\geq 1} \frac{-i\beta e^{2i\beta}}{1-e^{i\beta}} \cdot \frac{(-1)^{j_1+1} i^{j_1+j_2+1}}{2j_1!j_2!} j_1\phi(\alpha-\beta)^{j_1-1}\phi_\alpha(\alpha-\beta)\phi(\alpha)^{j_2} \tag{1926}$$

$$\cdot \int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))(-1+s)ds \tag{1927}$$

$$\cdot \left(\sum_{m=1}^\infty \frac{i\beta}{1-e^{i\beta}} \int_0^1 e^{-i(s-1)\beta} \frac{(-i\phi(\alpha+(s-1)\beta))^m}{m!} ds\right)^n, \tag{1928}$$

$$B^2_{1,1}(\alpha,\beta) = -\sum_{j_1+j_2+n\geq 1} \frac{-i\beta e^{2i\beta}}{1-e^{i\beta}} \cdot \frac{(-1)^{j_1+1} i^{j_1+j_2+1}}{2j_1!j_2!} \phi(\alpha-\beta)^{j_1} j_2\phi(\alpha)^{j_2-1}\phi_\alpha(\alpha) \tag{1929}$$

$$\cdot \int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))(-1+s)ds \tag{1930}$$

$$\cdot \left(\sum_{m=1}^\infty \frac{i\beta}{1-e^{i\beta}} \int_0^1 e^{-i(s-1)\beta} \frac{(-i\phi(\alpha+(s-1)\beta))^m}{m!} ds\right)^n \tag{1931}$$

$$B^3_{1,1}(\alpha,\beta) = -\sum_{j_1+j_2+n\geq 1} \frac{-i\beta e^{2i\beta}}{1-e^{i\beta}} \cdot \frac{(-1)^{j_1+1} i^{j_1+j_2+1}}{2j_1!j_2!} \phi(\alpha-\beta)^{j_1}\phi(\alpha)^{j_2} \tag{1932}$$

$$\cdot \int_0^1 e^{-i\beta s}\phi_\alpha(\alpha+\beta(-1+s))(-1+s)ds \tag{1933}$$

$$\cdot \left(\sum_{m=1}^\infty \frac{i\beta}{1-e^{i\beta}} \int_0^1 e^{-i(s-1)\beta} \frac{(-i\phi(\alpha+(s-1)\beta))^m}{m!} ds\right)^n, \tag{1934}$$

$$B^4_{1,1}(\alpha,\beta) = -\sum_{j_1+j_2+n\geq 1} \frac{-i\beta e^{2i\beta}}{1-e^{i\beta}} \cdot \frac{(-1)^{j_1+1} i^{j_1+j_2+1}}{2j_1!j_2!} \phi(\alpha-\beta)^{j_1}\phi(\alpha)^{j_2} \tag{1935}$$

$$\cdot \int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))(-1+s)ds \tag{1936}$$

$$\cdot\, n\left(\sum_{m=1}^\infty \frac{i\beta}{1-e^{i\beta}} \int_0^1 e^{-i(s-1)\beta} \frac{(-i\phi(\alpha+(s-1)\beta))^m}{m!} ds\right)^{n-1} \tag{1937}$$

$$\cdot \left(\sum_{m=1}^\infty \frac{i\beta}{1-e^{i\beta}}\right. \tag{1938}$$

$$\left.\cdot \int_0^1 e^{-i(s-1)\beta} \frac{m(-i\phi(\alpha+(s-1)\beta))^{m-1}(-i)\phi_\alpha(\alpha+(s-1)\beta)}{m!} ds\right) \tag{1939}$$

and

$$B^1_{1,2}(\alpha,\beta) = -\sum_{j_1+j_2+j_3\geq 1} \frac{-i\beta e^{2i\beta}}{1-e^{i\beta}} \cdot \frac{(-1)^{j_1+j_3} i^{j_1+j_2+j_3}}{2 j_1! j_2! j_3!} j_1 \phi(\alpha-\beta)^{j_1-1}\phi_\alpha(\alpha-\beta)\phi(\alpha)^{j_2} \tag{1940}$$

$$\cdot \int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds \tag{1941}$$

$$\cdot \left(\sum_{m=1}^\infty \frac{i\beta}{1-e^{i\beta}} \int_0^1 e^{-i(s-1)\beta} \frac{(-i\phi(\alpha+(s-1)\beta))^m}{m!} ds\right), \tag{1942}$$

$$B^2_{1,2}(\alpha,\beta) = -\sum_{j_1+j_2+j_3\geq 1} \frac{-i\beta e^{2i\beta}}{1-e^{i\beta}} \cdot \frac{(-1)^{j_1+j_3} i^{j_1+j_2+j_3}}{2 j_1! j_2! j_3!} \phi(\alpha-\beta)^{j_1} j_2 \phi(\alpha)^{j_2-1}\phi_\alpha(\alpha) \tag{1943}$$

$$\cdot \int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds \tag{1944}$$

$$\cdot \left(\sum_{m=1}^\infty \frac{i\beta}{1-e^{i\beta}} \int_0^1 e^{-i(s-1)\beta} \frac{(-i\phi(\alpha+(s-1)\beta))^m}{m!} ds\right), \tag{1945}$$

$$B^3_{1,2}(\alpha,\beta) = -\sum_{j_1+j_2+j_3\geq 1} \frac{-i\beta e^{2i\beta}}{1-e^{i\beta}} \cdot \frac{(-1)^{j_1+j_3} i^{j_1+j_2+j_3}}{2 j_1! j_2! j_3!} \phi(\alpha-\beta)^{j_1} \phi(\alpha)^{j_2} \tag{1946}$$

$$\cdot \int_0^1 e^{-i\beta s} j_3 \phi(\alpha+\beta(-1+s))^{j_3-1}\phi_\alpha(\alpha+\beta(-1+s))(-1+s)ds \tag{1947}$$

$$\cdot \left(\sum_{m=1}^\infty \frac{i\beta}{1-e^{i\beta}} \int_0^1 e^{-i(s-1)\beta} \frac{(-i\phi(\alpha+(s-1)\beta))^m}{m!} ds\right), \tag{1948}$$

$$B^4_{1,2}(\alpha,\beta) = -\sum_{j_1+j_2+j_3\geq 1} \frac{-i\beta e^{2i\beta}}{1-e^{i\beta}} \cdot \frac{(-1)^{j_1+j_3} i^{j_1+j_2+j_3}}{2 j_1! j_2! j_3!} \phi(\alpha-\beta)^{j_1} \phi(\alpha)^{j_2} \tag{1949}$$

$$\cdot \int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds \tag{1950}$$

$$\cdot \left(\sum_{m=1}^\infty \frac{i\beta}{1-e^{i\beta}} \right. \tag{1951}$$

$$\left. \cdot \int_0^1 e^{-i(s-1)\beta} \frac{m(-i\phi(\alpha+(s-1)\beta))^{m-1}(-i)\phi_\alpha(\alpha+(s-1)\beta)}{m!} ds\right). \tag{1952}$$

Moreover,

$$(\widetilde{B_{1,3}})_\alpha(\alpha,\beta) = -\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}} \cdot \frac{1}{2}\int_0^1 e^{-i\beta s}(-1+s)ds \tag{1953}$$

$$\cdot \left(\sum_{m=2}^\infty \frac{i\beta}{1-e^{i\beta}} \int_0^1 e^{-i(s-1)\beta} \frac{m(-i\phi(\alpha+(s-1)\beta))^{m-1}(-i)\phi_\alpha(\alpha+(s-1)\beta)}{m!} ds\right). \tag{1954}$$

We note that

$$(\widetilde{B_2})_\alpha(\alpha,\beta) = \sum_{j=1}^5 B_2^j(\alpha,\beta), \tag{1955}$$

where

$$B_2^1(\alpha,\beta) = -\sum_{j_1+j_2+j_3+n\geq 1} \frac{-i\beta e^{2i\beta}}{1-e^{i\beta}} \cdot \frac{(-1)^{j_1+j_3} i^{j_1+j_2+j_3}}{2j_1!j_2!j_3!} j_1\phi(\alpha-\beta)^{j_1-1}\phi_\alpha(\alpha-\beta) \tag{1956}$$

$$\cdot \phi(\alpha)^{j_2} \int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)\phi'(\alpha+\beta(-1+s))ds \tag{1957}$$

$$\cdot \left(\sum_{m=1}^{\infty} \frac{i\beta}{1-e^{i\beta}} \int_0^1 e^{-i(s-1)\beta} \frac{(-i\phi(\alpha+(s-1)\beta))^m}{m!} ds\right)^n, \tag{1958}$$

$$B_2^2(\alpha,\beta) = -\sum_{j_1+j_2+j_3+n\geq 1} \frac{-i\beta e^{2i\beta}}{1-e^{i\beta}} \cdot \frac{(-1)^{j_1+j_3} i^{j_1+j_2+j_3}}{2j_1!j_2!j_3!} \phi(\alpha-\beta)^{j_1} j_2\phi(\alpha)^{j_2-1}\phi_\alpha(\alpha) \tag{1959}$$

$$\cdot \int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)\phi'(\alpha+\beta(-1+s))ds \tag{1960}$$

$$\cdot \left(\sum_{m=1}^{\infty} \frac{i\beta}{1-e^{i\beta}} \int_0^1 e^{-i(s-1)\beta} \frac{(-i\phi(\alpha+(s-1)\beta))^m}{m!} ds\right)^n, \tag{1961}$$

$$B_2^3(\alpha,\beta) = -\sum_{j_1+j_2+j_3+n\geq 1} \frac{-i\beta e^{2i\beta}}{1-e^{i\beta}} \cdot \frac{(-1)^{j_1+j_3} i^{j_1+j_2+j_3}}{2j_1!j_2!j_3!} \phi(\alpha-\beta)^{j_1}\phi(\alpha)^{j_2} \tag{1962}$$

$$\cdot \int_0^1 e^{-i\beta s} j_3\phi(\alpha+\beta(-1+s))^{j_3-1}\phi_\alpha(\alpha+\beta(-1+s)) \tag{1963}$$

$$\cdot (-1+s)\phi'(\alpha+\beta(-1+s))ds \tag{1964}$$

$$\cdot \left(\sum_{m=1}^{\infty} \frac{i\beta}{1-e^{i\beta}} \int_0^1 e^{-i(s-1)\beta} \frac{(-i\phi(\alpha+(s-1)\beta))^m}{m!} ds\right)^n, \tag{1965}$$

$$B_2^4(\alpha,\beta) = -\sum_{j_1+j_2+j_3+n\geq 1} \frac{-i\beta e^{2i\beta}}{1-e^{i\beta}} \cdot \frac{(-1)^{j_1+j_3} i^{j_1+j_2+j_3}}{2j_1!j_2!j_3!} \phi(\alpha-\beta)^{j_1}\phi(\alpha)^{j_2} \tag{1966}$$

$$\cdot \int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)\phi''(\alpha+\beta(-1+s))ds \tag{1967}$$

$$\cdot \left(\sum_{m=1}^{\infty} \frac{i\beta}{1-e^{i\beta}} \int_0^1 e^{-i(s-1)\beta} \frac{(-i\phi(\alpha+(s-1)\beta))^m}{m!} ds\right)^n, \tag{1968}$$

$$B_2^5(\alpha,\beta) = -\sum_{j_1+j_2+j_3+n\geq 1} \frac{-i\beta e^{2i\beta}}{1-e^{i\beta}} \cdot \frac{(-1)^{j_1+j_3} i^{j_1+j_2+j_3}}{2j_1!j_2!j_3!} \phi(\alpha-\beta)^{j_1}\phi(\alpha)^{j_2} \tag{1969}$$

$$\cdot \int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)\phi'(\alpha+\beta(-1+s))ds \tag{1970}$$

$$\cdot\, n\Big(\sum_{m=1}^{\infty} \frac{i\beta}{1-e^{i\beta}} \int_0^1 e^{-i(s-1)\beta} \frac{(-i\phi(\alpha+(s-1)\beta))^m}{m!} ds\Big)^{n-1} \tag{1971}$$

$$\cdot \Big(\sum_{m=1}^{\infty} \frac{i\beta}{1-e^{i\beta}} \tag{1972}$$

$$\cdot \int_0^1 e^{-i(s-1)\beta} \frac{m(-i\phi(\alpha+(s-1)\beta))^{m-1}(-i)\phi_\alpha(\alpha+(s-1)\beta)}{m!} ds\Big). \tag{1973}$$

Recalling that

$$\widetilde{B_3}(\alpha,\beta) = \widetilde{B_{3,1}}(\alpha,\beta) + \widetilde{B_{3,2}}(\alpha,\beta) + \widetilde{B_{3,3}}(\alpha,\beta), \tag{1974}$$

we note that

$$(\widetilde{B_{3,1}})_\alpha(\alpha,\beta) = \sum_{j=1}^{3} B_{3,1}^j(\alpha,\beta), \tag{1975}$$

$$(\widetilde{B_{3,2}})_\alpha(\alpha,\beta) = \sum_{j=1}^{5} B_{3,2}^j(\alpha,\beta), \tag{1976}$$

where

$$B^1_{3,1}(\alpha,\beta) = \sum_{j_3+j_4+n\geq 2} (n+1)\frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2}\cdot\frac{i^{j_3+j_4}(-1)^{j_3}}{2j_3!j_4!} \tag{1977}$$
$$\cdot\int_0^1 e^{-i\beta s} j_3\phi(\alpha+\beta(-1+s))^{j_3-1}\phi_\alpha(\alpha+\beta(-1+s))ds \tag{1978}$$
$$\cdot\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_4}(-1+s)ds \tag{1979}$$
$$\cdot\Big(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\Big)^n, \tag{1980}$$
$$B^2_{3,1}(\alpha,\beta) = \sum_{j_3+j_4+n\geq 2} (n+1)\frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2}\cdot\frac{i^{j_3+j_4}(-1)^{j_3}}{2j_3!j_4!} \tag{1981}$$
$$\cdot\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}ds \tag{1982}$$
$$\cdot\int_0^1 e^{i\beta s} j_4\phi(\alpha+\beta(-1+s))^{j_4-1}\phi_\alpha(\alpha+\beta(-1+s))(-1+s)ds \tag{1983}$$
$$\cdot\Big(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\Big)^n, \tag{1984}$$
$$B^3_{3,1}(\alpha,\beta) = \sum_{j_3+j_4+n\geq 2} (n+1)\frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2}\cdot\frac{i^{j_3+j_4}(-1)^{j_3}}{2j_3!j_4!} \tag{1985}$$
$$\cdot\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}ds \tag{1986}$$
$$\cdot\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_4}(-1+s)ds \tag{1987}$$
$$\cdot n\Big(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\Big)^{n-1} \tag{1988}$$
$$\cdot\Big(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}} \tag{1989}$$
$$\cdot\int_0^1 e^{i(s-1)\beta}\frac{i^m\cdot m\phi(\alpha+(s-1)\beta)^{m-1}}{m!}\phi_\alpha(\alpha+(s-1)\beta)ds\Big), \tag{1990}$$

and

$$
\begin{aligned}
B^1_{3,2}(\alpha,\beta) = \sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}} & (n+1)\frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2}\cdot\frac{i^{j_1+j_2+j_3+j_4}(-1)^{j_3}}{2j_1!j_2!j_3!j_4!} && (1991)\\
& \cdot j_1\phi(\alpha)^{j_1-1}\phi_\alpha(\alpha)\phi(\alpha-\beta)^{j_2} && (1992)\\
& \cdot\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}ds && (1993)\\
& \cdot\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_4}(-1+s)ds && (1994)\\
& \cdot\left(\sum_{m=1}^\infty\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n, && (1995)\\
B^2_{3,2}(\alpha,\beta) = \sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}} & (n+1)\frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2}\cdot\frac{i^{j_1+j_2+j_3+j_4}(-1)^{j_3}}{2j_1!j_2!j_3!j_4!}\phi(\alpha)^{j_1} && (1996)\\
& \cdot j_2\phi(\alpha-\beta)^{j_2-1}\phi_\alpha(\alpha-\beta) && (1997)\\
& \cdot\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}ds && (1998)\\
& \cdot\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_4}(-1+s)ds && (1999)\\
& \cdot\left(\sum_{m=1}^\infty\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n, && (2000)\\
B^3_{3,2}(\alpha,\beta) = \sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}} & (n+1)\frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2}\cdot\frac{i^{j_1+j_2+j_3+j_4}(-1)^{j_3}}{2j_1!j_2!j_3!j_4!} && (2001)\\
& \cdot\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2} && (2002)\\
& \cdot\int_0^1 e^{-i\beta s}j_3\phi(\alpha+\beta(-1+s))^{j_3-1}\phi_\alpha(\alpha+\beta(-1+s))ds && (2003)\\
& \cdot\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_4}(-1+s)ds && (2004)\\
& \cdot\left(\sum_{m=1}^\infty\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n, && (2005)
\end{aligned}
$$

$$B^4_{3,2}(\alpha,\beta) = \sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}} (n+1)\frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2}\cdot\frac{i^{j_1+j_2+j_3+j_4}(-1)^{j_3}}{2j_1!j_2!j_3!j_4!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2}\cdot \tag{2006}$$

$$\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}ds\cdot \tag{2007}$$

$$\int_0^1 e^{i\beta s}j_4\phi(\alpha+\beta(-1+s))^{j_4-1}\phi_\alpha(\alpha+\beta(-1+s))(-1+s)ds\cdot \tag{2008}$$

$$\left(\sum_{m=1}^\infty\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n, \tag{2009}$$

$$B^5_{3,2}(\alpha,\beta) = \sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}} (n+1)\frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2}\cdot\frac{i^{j_1+j_2+j_3+j_4}(-1)^{j_3}}{2j_1!j_2!j_3!j_4!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2} \tag{2010}$$

$$\cdot\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}ds\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_4}(-1+s)ds \tag{2011}$$

$$\cdot\, n\left(\sum_{m=1}^\infty\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^{n-1} \tag{2012}$$

$$\cdot\left(\sum_{m=1}^\infty\frac{-i\beta}{1-e^{-i\beta}}\right. \tag{2013}$$

$$\left.\cdot\int_0^1 e^{i(s-1)\beta}\frac{i^m\cdot m\phi(\alpha+(s-1)\beta)^{m-1}}{m!}\phi_\alpha(\alpha+(s-1)\beta)ds\right). \tag{2014}$$

Moreover,

$$(\widetilde{B_{3,3}})_\alpha(\alpha,\beta) = \frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2}\cdot\int_0^1 e^{-i\beta s}ds\int_0^1 e^{i\beta s}(-1+s)ds \tag{2015}$$

$$\cdot\left(\sum_{m=2}^\infty\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{i^m\cdot m\phi(\alpha+(s-1)\beta)^{m-1}\phi_\alpha(\alpha+(s-1)\beta)}{m!}ds\right). \tag{2016}$$

We note that

$$(\widetilde{B_4})_\alpha(\alpha,\beta) = \sum_{j=1}^6 B_4^j(\alpha,\beta), \tag{2017}$$

where

$$B_4^1(\alpha,\beta) = \sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1)\frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2} \cdot \frac{i^{j_1+j_2+j_3+j_4}(-1)^{j_3}}{2j_1!j_2!j_3!j_4!} \tag{2018}$$

$$\cdot\, j_1\phi(\alpha)^{j_1-1}\phi_\alpha(\alpha)\phi(\alpha-\beta)^{j_2} \tag{2019}$$

$$\cdot \int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}ds \tag{2020}$$

$$\cdot \int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_4}(-1+s)\phi'(\alpha+\beta(-1+s))ds \tag{2021}$$

$$\cdot \left(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n, \tag{2022}$$

$$B_4^2(\alpha,\beta) = \sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1)\frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2} \cdot \frac{i^{j_1+j_2+j_3+j_4}(-1)^{j_3}}{2j_1!j_2!j_3!j_4!}\phi(\alpha)^{j_1} \tag{2023}$$

$$\cdot\, j_2\phi(\alpha-\beta)^{j_2-1}\phi_\alpha(\alpha-\beta) \tag{2024}$$

$$\cdot \int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}ds \tag{2025}$$

$$\cdot \int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_4}(-1+s)\phi'(\alpha+\beta(-1+s))ds \tag{2026}$$

$$\cdot \left(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n, \tag{2027}$$

$$B_4^3(\alpha,\beta) = \sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1)\frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2} \cdot \frac{i^{j_1+j_2+j_3+j_4}(-1)^{j_3}}{2j_1!j_2!j_3!j_4!} \tag{2028}$$

$$\cdot\, \phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2} \tag{2029}$$

$$\cdot \int_0^1 e^{-i\beta s}j_3\phi(\alpha+\beta(-1+s))^{j_3-1}\phi_\alpha(\alpha+\beta(-1+s))ds \tag{2030}$$

$$\cdot \int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_4}(-1+s)\phi'(\alpha+\beta(-1+s))ds\cdot \tag{2031}$$

$$\left(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n, \tag{2032}$$

$$B_4^4(\alpha,\beta) = \sum_{j_1+j_2+j_3+j_4+n\geq 1}(n+1)\frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2}\cdot\frac{i^{j_1+j_2+j_3+j_4}(-1)^{j_3}}{2j_1!j_2!j_3!j_4!} \tag{2033}$$

$$\cdot\,\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2}\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}ds \tag{2034}$$

$$\cdot\int_0^1 e^{i\beta s}j_4\phi(\alpha+\beta(-1+s))^{j_4-1}\phi_\alpha(\alpha+\beta(-1+s)) \tag{2035}$$

$$\cdot\,(-1+s)\phi'(\alpha+\beta(-1+s))ds \tag{2036}$$

$$\cdot\left(\sum_{m=1}^\infty\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n, \tag{2037}$$

$$B_4^5(\alpha,\beta) = \sum_{j_1+j_2+j_3+j_4+n\geq 1}(n+1)\frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2}\cdot\frac{i^{j_1+j_2+j_3+j_4}(-1)^{j_3}}{2j_1!j_2!j_3!j_4!} \tag{2038}$$

$$\cdot\,\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2}\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}ds \tag{2039}$$

$$\cdot\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_4}(-1+s)\phi''(\alpha+\beta(-1+s))ds \tag{2040}$$

$$\cdot\left(\sum_{m=1}^\infty\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n, \tag{2041}$$

$$B_4^6(\alpha,\beta) = \sum_{j_1+j_2+j_3+j_4+n\geq 1}(n+1)\frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2}\cdot\frac{i^{j_1+j_2+j_3+j_4}(-1)^{j_3}}{2j_1!j_2!j_3!j_4!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2} \tag{2042}$$

$$\cdot\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}ds \tag{2043}$$

$$\cdot\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_4}(-1+s)\phi'(\alpha+\beta(-1+s))ds \tag{2044}$$

$$\cdot\, n\left(\sum_{m=1}^\infty\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^{n-1} \tag{2045}$$

$$\cdot\left(\sum_{m=1}^\infty\frac{-i\beta}{1-e^{-i\beta}}\right. \tag{2046}$$

$$\left.\cdot\int_0^1 e^{i(s-1)\beta}\frac{i^m\cdot m\phi(\alpha+(s-1)\beta)^{m-1}\phi_\alpha(\alpha+(s-1)\beta)}{m!}ds\right). \tag{2047}$$

Recalling that

$$\widetilde{B_5}(\alpha,\beta) = \widetilde{B_{5,1}}(\alpha,\beta)+\widetilde{B_{5,2}}(\alpha,\beta)+\widetilde{B_{5,3}}(\alpha,\beta), \tag{2048}$$

we note that

$$(\widetilde{B_{5,1}})_\alpha(\alpha,\beta) = \sum_{j=1}^{2} B_{5,1}^j(\alpha,\beta), \tag{2049}$$

$$(\widetilde{B_{5,2}})_\alpha(\alpha,\beta) = \sum_{j=1}^{4} B_{5,2}^j(\alpha,\beta), \tag{2050}$$

where

$$B_{5,1}^1(\alpha,\beta) = \sum_{j_3+n\geq 2} \frac{i\beta}{1-e^{-i\beta}} \cdot \frac{i^{j_3}(-1)^{j_3}}{2j_3!} \tag{2051}$$

$$\cdot \int_0^1 e^{-i\beta s} j_3 \phi(\alpha+\beta(-1+s))^{j_3-1} \phi_\alpha(\alpha+\beta(-1+s))(-1+s)ds \tag{2052}$$

$$\cdot \left(\sum_{m=1}^{\infty} \frac{-i\beta}{1-e^{-i\beta}} \int_0^1 e^{i(s-1)\beta} \frac{(i\phi(\alpha+(s-1)\beta))^m}{m!} ds\right)^n, \tag{2053}$$

$$B_{5,1}^2(\alpha,\beta) = \sum_{j_3+n\geq 2} \frac{i\beta}{1-e^{-i\beta}} \cdot \frac{i^{j_3}(-1)^{j_3}}{2j_3!} \tag{2054}$$

$$\cdot \int_0^1 e^{-i\beta s} \phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds \tag{2055}$$

$$\cdot\, n\left(\sum_{m=1}^{\infty} \frac{-i\beta}{1-e^{-i\beta}} \int_0^1 e^{i(s-1)\beta} \frac{(i\phi(\alpha+(s-1)\beta))^m}{m!} ds\right)^{n-1} \tag{2056}$$

$$\cdot \left(\sum_{m=1}^{\infty} \frac{-i\beta}{1-e^{-i\beta}}\right. \tag{2057}$$

$$\left.\cdot \int_0^1 e^{i(s-1)\beta} \frac{i^m \cdot m\phi(\alpha+(s-1)\beta)^{m-1}}{m!} \phi_\alpha(\alpha+(s-1)\beta)ds\right), \tag{2058}$$

and

$$B^1_{5,2}(\alpha,\beta) = \sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{i\beta}{1-e^{-i\beta}} \cdot \frac{i^{j_1+j_2+j_3}(-1)^{j_3}}{2j_1!j_2!j_3!} j_1\phi(\alpha)^{j_1-1}\phi_\alpha(\alpha)\phi(\alpha-\beta)^{j_2} \tag{2059}$$

$$\cdot \int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds \tag{2060}$$

$$\cdot \left(\sum_{m=1}^\infty \frac{-i\beta}{1-e^{-i\beta}} \int_0^1 e^{i(s-1)\beta} \frac{(i\phi(\alpha+(s-1)\beta))^m}{m!} ds\right)^n, \tag{2061}$$

$$B^2_{5,2}(\alpha,\beta) = \sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{i\beta}{1-e^{-i\beta}} \cdot \frac{i^{j_1+j_2+j_3}(-1)^{j_3}}{2j_1!j_2!j_3!} \phi(\alpha)^{j_1} j_2\phi(\alpha-\beta)^{j_2-1}\phi_\alpha(\alpha-\beta) \tag{2062}$$

$$\cdot \int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds \tag{2063}$$

$$\cdot \left(\sum_{m=1}^\infty \frac{-i\beta}{1-e^{-i\beta}} \int_0^1 e^{i(s-1)\beta} \frac{(i\phi(\alpha+(s-1)\beta))^m}{m!} ds\right)^n, \tag{2064}$$

$$B^3_{5,2}(\alpha,\beta) = \sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{i\beta}{1-e^{-i\beta}} \cdot \frac{i^{j_1+j_2+j_3}(-1)^{j_3}}{2j_1!j_2!j_3!} \phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2} \tag{2065}$$

$$\cdot \int_0^1 e^{-i\beta s} j_3\phi(\alpha+\beta(-1+s))^{j_3-1}\phi_\alpha(\alpha+\beta(-1+s))(-1+s)ds \tag{2066}$$

$$\cdot \left(\sum_{m=1}^\infty \frac{-i\beta}{1-e^{-i\beta}} \int_0^1 e^{i(s-1)\beta} \frac{(i\phi(\alpha+(s-1)\beta))^m}{m!} ds\right)^n, \tag{2067}$$

$$B^4_{5,2}(\alpha,\beta) = \sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{i\beta}{1-e^{-i\beta}} \cdot \frac{i^{j_1+j_2+j_3}(-1)^{j_3}}{2j_1!j_2!j_3!} \phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2} \tag{2068}$$

$$\cdot \int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds \tag{2069}$$

$$\cdot n\left(\sum_{m=1}^\infty \frac{-i\beta}{1-e^{-i\beta}} \int_0^1 e^{i(s-1)\beta} \frac{(i\phi(\alpha+(s-1)\beta))^m}{m!} ds\right)^{n-1} \tag{2070}$$

$$\cdot \left(\sum_{m=1}^\infty \frac{-i\beta}{1-e^{-i\beta}}\right. \tag{2071}$$

$$\left.\cdot \int_0^1 e^{i(s-1)\beta} \frac{i^m \cdot m\phi(\alpha+(s-1)\beta)^{m-1}}{m!}\phi_\alpha(\alpha+(s-1)\beta)ds\right). \tag{2072}$$

Moreover,

$$(\widetilde{B_{5,3}})_\alpha(\alpha,\beta) = \frac{1}{2}\cdot\frac{i\beta}{1-e^{-i\beta}} \int_0^1 e^{-i\beta s}(-1+s)ds \tag{2073}$$

$$\cdot \left(\sum_{m=2}^\infty \frac{-i\beta}{1-e^{-i\beta}} \int_0^1 e^{i(s-1)\beta} \frac{i^m \cdot m\phi(\alpha+(s-1)\beta)^{m-1}}{m!}\phi_\alpha(\alpha+(s-1)\beta)ds\right). \tag{2074}$$

Recalling that

$$\widetilde{B_6}(\alpha,\beta) = \widetilde{B_{6,1}}(\alpha,\beta) + \widetilde{B_{6,2}}(\alpha,\beta) + \widetilde{B_{6,3}}(\alpha,\beta), \tag{2075}$$

we note that

$$(\widetilde{B_{6,1}})_\alpha(\alpha,\beta) = \sum_{j=1}^{2} B_{6,1}^{j}(\alpha,\beta), \tag{2076}$$

$$(\widetilde{B_{6,2}})_\alpha(\alpha,\beta) = \sum_{j=1}^{4} B_{6,2}^{j}(\alpha,\beta), \tag{2077}$$

where

$$B_{6,1}^1(\alpha,\beta) = \sum_{j_3+n\geq 2} \frac{i\beta}{1-e^{-i\beta}} \cdot \frac{i^{j_3}}{2j_3!} \tag{2078}$$

$$\cdot \int_0^1 e^{i\beta s} j_3 \phi(\alpha+\beta(-1+s))^{j_3-1}\phi_\alpha(\alpha+\beta(-1+s))(-1+s)ds \tag{2079}$$

$$\cdot \left(\sum_{m=1}^{\infty} \frac{-i\beta}{1-e^{-i\beta}} \int_0^1 e^{i(s-1)\beta} \frac{(i\phi(\alpha+(s-1)\beta))^m}{m!} ds\right)^n, \tag{2080}$$

$$B_{6,1}^2(\alpha,\beta) = \sum_{j_3+n\geq 2} \frac{i\beta}{1-e^{-i\beta}} \cdot \frac{i^{j_3}}{2j_3!} \tag{2081}$$

$$\cdot \int_0^1 e^{i\beta s} \phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds \tag{2082}$$

$$\cdot\, n\left(\sum_{m=1}^{\infty} \frac{-i\beta}{1-e^{-i\beta}} \int_0^1 e^{i(s-1)\beta} \frac{(i\phi(\alpha+(s-1)\beta))^m}{m!} ds\right)^{n-1} \tag{2083}$$

$$\cdot \left(\sum_{m=1}^{\infty} \frac{-i\beta}{1-e^{-i\beta}}\right. \tag{2084}$$

$$\left.\cdot \int_0^1 e^{i(s-1)\beta} \frac{i^m \cdot m\phi(\alpha+(s-1)\beta)^{m-1}}{m!} \phi_\alpha(\alpha+(s-1)\beta)ds\right), \tag{2085}$$

and

$$B^1_{6,2}(\alpha,\beta) = \sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{i\beta}{1-e^{-i\beta}} \cdot \frac{i^{j_1+j_2+j_3}(-1)^{j_2}}{2j_1!j_2!j_3!} j_1\phi(\alpha)^{j_1-1}\phi_\alpha(\alpha)\phi(\alpha-\beta)^{j_2} \tag{2086}$$

$$\cdot \int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds \tag{2087}$$

$$\cdot \left(\sum_{m=1}^\infty \frac{-i\beta}{1-e^{-i\beta}} \int_0^1 e^{i(s-1)\beta} \frac{(i\phi(\alpha+(s-1)\beta))^m}{m!} ds\right)^n, \tag{2088}$$

$$B^2_{6,2}(\alpha,\beta) = \sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{i\beta}{1-e^{-i\beta}} \cdot \frac{i^{j_1+j_2+j_3}(-1)^{j_2}}{2j_1!j_2!j_3!} \phi(\alpha)^{j_1} j_2\phi(\alpha-\beta)^{j_2-1}\phi_\alpha(\alpha-\beta) \tag{2089}$$

$$\cdot \int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds \tag{2090}$$

$$\cdot \left(\sum_{m=1}^\infty \frac{-i\beta}{1-e^{-i\beta}} \int_0^1 e^{i(s-1)\beta} \frac{(i\phi(\alpha+(s-1)\beta))^m}{m!} ds\right)^n, \tag{2091}$$

$$B^3_{6,2}(\alpha,\beta) = \sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{i\beta}{1-e^{-i\beta}} \cdot \frac{i^{j_1+j_2+j_3}(-1)^{j_2}}{2j_1!j_2!j_3!} \phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2} \tag{2092}$$

$$\cdot \int_0^1 e^{i\beta s} j_3\phi(\alpha+\beta(-1+s))^{j_3-1}\phi_\alpha(\alpha+\beta(-1+s))(-1+s)ds \tag{2093}$$

$$\cdot \left(\sum_{m=1}^\infty \frac{-i\beta}{1-e^{-i\beta}} \int_0^1 e^{i(s-1)\beta} \frac{(i\phi(\alpha+(s-1)\beta))^m}{m!} ds\right)^n, \tag{2094}$$

$$B^4_{6,2}(\alpha,\beta) = \sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{i\beta}{1-e^{-i\beta}} \cdot \frac{i^{j_1+j_2+j_3}(-1)^{j_2}}{2j_1!j_2!j_3!} \phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2} \tag{2095}$$

$$\cdot \int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)ds \tag{2096}$$

$$\cdot\, n\left(\sum_{m=1}^\infty \frac{-i\beta}{1-e^{-i\beta}} \int_0^1 e^{i(s-1)\beta} \frac{(i\phi(\alpha+(s-1)\beta))^m}{m!} ds\right)^{n-1} \tag{2097}$$

$$\cdot \left(\sum_{m=1}^\infty \frac{-i\beta}{1-e^{-i\beta}}\right. \tag{2098}$$

$$\left.\cdot \int_0^1 e^{i(s-1)\beta} \frac{i^m \cdot m\phi(\alpha+(s-1)\beta)^{m-1}}{m!}\phi_\alpha(\alpha+(s-1)\beta)ds\right). \tag{2099}$$

Moreover,

$$(\widetilde{B_{6,3}})_\alpha(\alpha,\beta) = \frac{1}{2}\cdot\frac{i\beta}{1-e^{-i\beta}}\int_0^1 e^{i\beta s}(-1+s)ds \tag{2100}$$

$$\cdot \left(\sum_{m=2}^\infty \frac{-i\beta}{1-e^{-i\beta}} \int_0^1 e^{i(s-1)\beta} \frac{i^m \cdot m\phi(\alpha+(s-1)\beta)^{m-1}}{m!}\phi_\alpha(\alpha+(s-1)\beta)ds\right). \tag{2101}$$

We note that

$$(\widetilde{B_7})_\alpha(\alpha,\beta) = \sum_{j=1}^{5} B_7^j(\alpha,\beta), \tag{2102}$$

where

$$B_7^1(\alpha,\beta) = \sum_{j_1+j_2+j_3+n\geq 1} \frac{i\beta}{1-e^{-i\beta}} \cdot \frac{i^{j_1+j_2+j_3}(-1)^{j_3}}{2j_1!j_2!j_3!} j_1\phi(\alpha)^{j_1-1}\phi_\alpha(\alpha)\phi(\alpha-\beta)^{j_2} \tag{2103}$$

$$\cdot \int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)\phi'(\alpha+\beta(-1+s))ds \tag{2104}$$

$$\cdot \left(\sum_{m=1}^{\infty} \frac{-i\beta}{1-e^{-i\beta}} \int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n, \tag{2105}$$

$$B_7^2(\alpha,\beta) = \sum_{j_1+j_2+j_3+n\geq 1} \frac{i\beta}{1-e^{-i\beta}} \cdot \frac{i^{j_1+j_2+j_3}(-1)^{j_3}}{2j_1!j_2!j_3!} \phi(\alpha)^{j_1} j_2\phi(\alpha-\beta)^{j_2-1}\phi_\alpha(\alpha-\beta) \tag{2106}$$

$$\cdot \int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)\phi'(\alpha+\beta(-1+s))ds \tag{2107}$$

$$\cdot \left(\sum_{m=1}^{\infty} \frac{-i\beta}{1-e^{-i\beta}} \int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n, \tag{2108}$$

$$B_7^3(\alpha,\beta) = \sum_{j_1+j_2+j_3+n\geq 1} \frac{i\beta}{1-e^{-i\beta}} \cdot \frac{i^{j_1+j_2+j_3}(-1)^{j_3}}{2j_1!j_2!j_3!} \phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2} \tag{2109}$$

$$\cdot \int_0^1 e^{-i\beta s} j_3\phi(\alpha+\beta(-1+s))^{j_3-1}\phi_\alpha(\alpha+\beta(-1+s)) \tag{2110}$$

$$\cdot (-1+s)\phi'(\alpha+\beta(-1+s))ds \tag{2111}$$

$$\cdot \left(\sum_{m=1}^{\infty} \frac{-i\beta}{1-e^{-i\beta}} \int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n, \tag{2112}$$

$$B_7^4(\alpha,\beta) = \sum_{j_1+j_2+j_3+n\geq 1} \frac{i\beta}{1-e^{-i\beta}} \cdot \frac{i^{j_1+j_2+j_3}(-1)^{j_3}}{2j_1!j_2!j_3!} \phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2} \tag{2113}$$

$$\cdot \int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)\phi''(\alpha+\beta(-1+s))ds \tag{2114}$$

$$\cdot \left(\sum_{m=1}^{\infty} \frac{-i\beta}{1-e^{-i\beta}} \int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n, \tag{2115}$$

$$B_7^5(\alpha,\beta) = \sum_{j_1+j_2+j_3+n\geq 1} \frac{i\beta}{1-e^{-i\beta}} \cdot \frac{i^{j_1+j_2+j_3}(-1)^{j_3}}{2j_1!j_2!j_3!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2} \tag{2116}$$

$$\cdot \int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)\phi'(\alpha+\beta(-1+s))ds \tag{2117}$$

$$\cdot\, n\Bigg(\sum_{m=1}^{\infty} \frac{-i\beta}{1-e^{-i\beta}} \int_0^1 e^{i(s-1)\beta} \frac{(i\phi(\alpha+(s-1)\beta))^m}{m!} ds\Bigg)^{n-1} \tag{2118}$$

$$\cdot \Bigg(\sum_{m=1}^{\infty} \frac{-i\beta}{1-e^{-i\beta}} \tag{2119}$$

$$\cdot \int_0^1 e^{i(s-1)\beta} \frac{i^m \cdot m\phi(\alpha+(s-1)\beta)^{m-1}\phi_\alpha(\alpha+(s-1)\beta)}{m!} ds\Bigg). \tag{2120}$$

We note that

$$(\widetilde{B_8})_\alpha(\alpha,\beta) = \sum_{j=1}^{5} B_8^j(\alpha,\beta), \tag{2121}$$

where

$$B_8^1(\alpha,\beta)=\sum_{j_1+j_2+j_3+n\geq 1}\frac{i\beta}{1-e^{-i\beta}}\cdot\frac{i^{j_1+j_2+j_3}(-1)^{j_2}}{2j_1!j_2!j_3!}j_1\phi(\alpha)^{j_1-1}\phi_\alpha(\alpha)\phi(\alpha-\beta)^{j_2} \tag{2122}$$

$$\cdot\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)\phi'(\alpha+\beta(-1+s))ds \tag{2123}$$

$$\cdot\left(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n, \tag{2124}$$

$$B_8^2(\alpha,\beta)=\sum_{j_1+j_2+j_3+n\geq 1}\frac{i\beta}{1-e^{-i\beta}}\cdot\frac{i^{j_1+j_2+j_3}(-1)^{j_2}}{2j_1!j_2!j_3!}\phi(\alpha)^{j_1}j_2\phi(\alpha-\beta)^{j_2-1}\phi_\alpha(\alpha-\beta) \tag{2125}$$

$$\cdot\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)\phi'(\alpha+\beta(-1+s))ds \tag{2126}$$

$$\cdot\left(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n, \tag{2127}$$

$$B_8^3(\alpha,\beta)=\sum_{j_1+j_2+j_3+n\geq 1}\frac{i\beta}{1-e^{-i\beta}}\cdot\frac{i^{j_1+j_2+j_3}(-1)^{j_2}}{2j_1!j_2!j_3!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2} \tag{2128}$$

$$\cdot\int_0^1 e^{i\beta s}j_3\phi(\alpha+\beta(-1+s))^{j_3-1}\phi_\alpha(\alpha+\beta(-1+s)) \tag{2129}$$

$$\cdot(-1+s)\phi'(\alpha+\beta(-1+s))ds \tag{2130}$$

$$\cdot\left(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n, \tag{2131}$$

$$B_8^4(\alpha,\beta)=\sum_{j_1+j_2+j_3+n\geq 1}\frac{i\beta}{1-e^{-i\beta}}\cdot\frac{i^{j_1+j_2+j_3}(-1)^{j_2}}{2j_1!j_2!j_3!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2} \tag{2132}$$

$$\cdot\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)\phi''(\alpha+\beta(-1+s))ds \tag{2133}$$

$$\cdot\left(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n, \tag{2134}$$

$$B_8^5(\alpha,\beta)=\sum_{j_1+j_2+j_3+n\geq 1}\frac{i\beta}{1-e^{-i\beta}}\cdot\frac{i^{j_1+j_2+j_3}(-1)^{j_2}}{2j_1!j_2!j_3!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2} \tag{2135}$$

$$\cdot\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)\phi'(\alpha+\beta(-1+s))ds \tag{2136}$$

$$\cdot n\left(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^{n-1} \tag{2137}$$

$$\cdot\left(\sum_{m=1}^{\infty}\frac{-i\beta}{1-e^{-i\beta}}\right. \tag{2138}$$

$$\left.\cdot\int_0^1 e^{i(s-1)\beta}\frac{i^m\cdot m\phi(\alpha+(s-1)\beta)^{m-1}\phi_\alpha(\alpha+(s-1)\beta)}{m!}ds\right). \tag{2139}$$

Lastly, we note that

$$(\widetilde{B_{13}})_\alpha(\alpha,\beta) = \sum_{j=1}^{2} B_{13}^j(\alpha,\beta), \tag{2140}$$

where

$$B_{13}^1(\alpha,\beta) = \sum_{j_1+j_2\geq 2} \frac{i^{j_1+j_2}}{j_1!j_2!} j_1\phi(\alpha)^{j_1-1}\phi_\alpha(\alpha)\phi(\alpha-\beta)^{j_2}\left((-1)^{j_2}\frac{e^{i\beta}(1+e^{i\beta})}{2(-1+e^{i\beta})} - \frac{1}{2}\right), \tag{2141}$$

$$B_{13}^2(\alpha,\beta) = \sum_{j_1+j_2\geq 2} \frac{i^{j_1+j_2}}{j_1!j_2!} \phi(\alpha)^{j_1} j_2\phi(\alpha-\beta)^{j_2-1}\phi_\alpha(\alpha-\beta)\left((-1)^{j_2}\frac{e^{i\beta}(1+e^{i\beta})}{2(-1+e^{i\beta})} - \frac{1}{2}\right). \tag{2142}$$

Of these terms, B_2^4, B_4^5, B_7^4, and B_8^4 contain the second derivative of ϕ, which need to be re-expressed in terms of lower-order derivatives of ϕ for the resulting estimate of the $\dot{\mathcal{F}}_\nu^{s,1}$ norm of $(U_{\geq 2})_\alpha$ to be useful. To re-express these terms, we use integration by parts to obtain

$$\int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)\phi''(\alpha+\beta(-1+s))ds \tag{2143}$$

$$= \int_0^1 e^{-i\beta s}(-1+s)\left(\frac{\partial}{\partial s}\left(\phi(\alpha+\beta(-1+s))^{j_3}\frac{\phi'(\alpha+\beta(-1+s))}{\beta}\right)\right. \tag{2144}$$

$$\left. - j_3\phi(\alpha+\beta(-1+s))^{j_3-1}\phi'(\alpha+\beta(-1+s))^2\right)ds \tag{2145}$$

$$= \int_0^1 \left(\frac{\partial}{\partial s}\left(e^{-i\beta s}(-1+s)\phi(\alpha+\beta(-1+s))^{j_3}\frac{\phi'(\alpha+\beta(-1+s))}{\beta}\right)\right. \tag{2146}$$

$$\left. - \frac{\partial}{\partial s}\left(e^{-i\beta s}(-1+s)\right)\phi(\alpha+\beta(-1+s))^{j_3}\frac{\phi'(\alpha+\beta(-1+s))}{\beta}\right)ds \tag{2147}$$

$$- \int_0^1 e^{-i\beta s}(-1+s)j_3\phi(\alpha+\beta(-1+s))^{j_3-1}\phi'(\alpha+\beta(-1+s))^2ds \tag{2148}$$

$$= \frac{\phi(\alpha-\beta)^{j_3}\phi'(\alpha-\beta)}{\beta} + i\int_0^1 e^{-i\beta s}(-1+s)\phi(\alpha+\beta(-1+s))^{j_3}\phi'(\alpha+\beta(-1+s))ds \tag{2149}$$

$$- \int_0^1 \frac{e^{-i\beta s}}{\beta}\phi(\alpha+\beta(-1+s))^{j_3}\phi'(\alpha+\beta(-1+s))ds \tag{2150}$$

$$- \int_0^1 e^{-i\beta s}(-1+s)j_3\phi(\alpha+\beta(-1+s))^{j_3-1}\phi'(\alpha+\beta(-1+s))^2ds \tag{2151}$$

and

$$\int_0^1 e^{i\beta s}\phi(\alpha+\beta(-1+s))^{j_3}(-1+s)\phi''(\alpha+\beta(-1+s))ds \tag{2152}$$

$$=\int_0^1 e^{i\beta s}(-1+s)\bigg(\frac{\partial}{\partial s}\bigg(\phi(\alpha+\beta(-1+s))^{j_3}\frac{\phi'(\alpha+\beta(-1+s))}{\beta}\bigg) \tag{2153}$$

$$-j_3\phi(\alpha+\beta(-1+s))^{j_3-1}\phi'(\alpha+\beta(-1+s))^2\bigg)ds \tag{2154}$$

$$=\int_0^1\bigg(\frac{\partial}{\partial s}\bigg(e^{i\beta s}(-1+s)\phi(\alpha+\beta(-1+s))^{j_3}\frac{\phi'(\alpha+\beta(-1+s))}{\beta}\bigg) \tag{2155}$$

$$-\frac{\partial}{\partial s}\bigg(e^{i\beta s}(-1+s)\bigg)\phi(\alpha+\beta(-1+s))^{j_3}\frac{\phi'(\alpha+\beta(-1+s))}{\beta}\bigg)ds \tag{2156}$$

$$-\int_0^1 e^{i\beta s}(-1+s)j_3\phi(\alpha+\beta(-1+s))^{j_3-1}\phi'(\alpha+\beta(-1+s))^2ds \tag{2157}$$

$$=\frac{\phi(\alpha-\beta)^{j_3}\phi'(\alpha-\beta)}{\beta}-i\int_0^1 e^{i\beta s}(-1+s)\phi(\alpha+\beta(-1+s))^{j_3}\phi'(\alpha+\beta(-1+s))ds \tag{2158}$$

$$-\int_0^1\frac{e^{i\beta s}}{\beta}\phi(\alpha+\beta(-1+s))^{j_3}\phi'(\alpha+\beta(-1+s))ds \tag{2159}$$

$$-\int_0^1 e^{i\beta s}(-1+s)j_3\phi(\alpha+\beta(-1+s))^{j_3-1}\phi'(\alpha+\beta(-1+s))^2ds. \tag{2160}$$

Then we can write $B_2^4(\alpha,\beta) = \sum_{j=1}^4 B_2^{4,j}(\alpha,\beta)$, where

$$B_2^{4,1}(\alpha,\beta) = -\sum_{j_1+j_2+j_3+n\geq 1} \frac{-i\beta e^{2i\beta}}{1-e^{i\beta}} \cdot \frac{(-1)^{j_1+j_3} i^{j_1+j_2+j_3}}{2j_1!j_2!j_3!} \phi(\alpha-\beta)^{j_1}\phi(\alpha)^{j_2} \tag{2161}$$

$$\cdot \frac{\phi(\alpha-\beta)^{j_3}\phi'(\alpha-\beta)}{\beta} \tag{2162}$$

$$\cdot \left(\sum_{m=1}^{\infty} \frac{i\beta}{1-e^{i\beta}} \int_0^1 e^{-i(s-1)\beta} \frac{(-i\phi(\alpha+(s-1)\beta))^m}{m!} ds\right)^n, \tag{2163}$$

$$B_2^{4,2}(\alpha,\beta) = -\sum_{j_1+j_2+j_3+n\geq 1} \frac{-i\beta e^{2i\beta}}{1-e^{i\beta}} \cdot \frac{(-1)^{j_1+j_3} i^{j_1+j_2+j_3}}{2j_1!j_2!j_3!} \phi(\alpha-\beta)^{j_1}\phi(\alpha)^{j_2} \tag{2164}$$

$$\cdot\, i \int_0^1 e^{-i\beta s}(-1+s)\phi(\alpha+\beta(-1+s))^{j_3}\phi'(\alpha+\beta(-1+s))ds \tag{2165}$$

$$\cdot \left(\sum_{m=1}^{\infty} \frac{i\beta}{1-e^{i\beta}} \int_0^1 e^{-i(s-1)\beta} \frac{(-i\phi(\alpha+(s-1)\beta))^m}{m!} ds\right)^n, \tag{2166}$$

$$B_2^{4,3}(\alpha,\beta) = -\sum_{j_1+j_2+j_3+n\geq 1} \frac{-i\beta e^{2i\beta}}{1-e^{i\beta}} \cdot \frac{(-1)^{j_1+j_3} i^{j_1+j_2+j_3}}{2j_1!j_2!j_3!} \phi(\alpha-\beta)^{j_1}\phi(\alpha)^{j_2} \tag{2167}$$

$$\cdot \int_0^1 \frac{-e^{-i\beta s}}{\beta} \phi(\alpha+\beta(-1+s))^{j_3}\phi'(\alpha+\beta(-1+s))ds \tag{2168}$$

$$\cdot \left(\sum_{m=1}^{\infty} \frac{i\beta}{1-e^{i\beta}} \int_0^1 e^{-i(s-1)\beta} \frac{(-i\phi(\alpha+(s-1)\beta))^m}{m!} ds\right)^n, \tag{2169}$$

$$B_2^{4,4}(\alpha,\beta) = -\sum_{j_1+j_2+j_3+n\geq 1} \frac{-i\beta e^{2i\beta}}{1-e^{i\beta}} \cdot \frac{(-1)^{j_1+j_3} i^{j_1+j_2+j_3}}{2j_1!j_2!j_3!} \phi(\alpha-\beta)^{j_1}\phi(\alpha)^{j_2} \tag{2170}$$

$$\cdot \int_0^1 -e^{-i\beta s}(-1+s)j_3\phi(\alpha+\beta(-1+s))^{j_3-1}\phi'(\alpha+\beta(-1+s))^2 ds \tag{2171}$$

$$\cdot \left(\sum_{m=1}^{\infty} \frac{i\beta}{1-e^{i\beta}} \int_0^1 e^{-i(s-1)\beta} \frac{(-i\phi(\alpha+(s-1)\beta))^m}{m!} ds\right)^n. \tag{2172}$$

Moreover, we can write $B_4^5(\alpha,\beta) = \sum_{j=1}^4 B_4^{5,j}(\alpha,\beta)$, where

$$B_4^{5,1}(\alpha,\beta) = \sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1)\frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2} \cdot \frac{i^{j_1+j_2+j_3+j_4}(-1)^{j_3}}{2j_1!j_2!j_3!j_4!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2} \tag{2173}$$

$$\cdot \int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3} ds \frac{\phi(\alpha-\beta)^{j_4}\phi'(\alpha-\beta)}{\beta} \tag{2174}$$

$$\cdot \left(\sum_{m=1}^\infty \frac{-i\beta}{1-e^{-i\beta}} \int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!} ds\right)^n, \tag{2175}$$

$$B_4^{5,2}(\alpha,\beta) = \sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1)\frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2} \cdot \frac{i^{j_1+j_2+j_3+j_4}(-1)^{j_3}}{2j_1!j_2!j_3!j_4!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2} \tag{2176}$$

$$\cdot \int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3} ds \tag{2177}$$

$$\cdot \int_0^1 -ie^{i\beta s}(-1+s)\phi(\alpha+\beta(-1+s))^{j_4}\phi'(\alpha+\beta(-1+s)) ds \tag{2178}$$

$$\cdot \left(\sum_{m=1}^\infty \frac{-i\beta}{1-e^{-i\beta}} \int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!} ds\right)^n, \tag{2179}$$

$$B_4^{5,3}(\alpha,\beta) = \sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1)\frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2} \cdot \frac{i^{j_1+j_2+j_3+j_4}(-1)^{j_3}}{2j_1!j_2!j_3!j_4!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2} \tag{2180}$$

$$\cdot \int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3} ds \tag{2181}$$

$$\cdot \int_0^1 \frac{-e^{i\beta s}}{\beta}\phi(\alpha+\beta(-1+s))^{j_4}\phi'(\alpha+\beta(-1+s)) ds \tag{2182}$$

$$\cdot \left(\sum_{m=1}^\infty \frac{-i\beta}{1-e^{-i\beta}} \int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!} ds\right)^n, \tag{2183}$$

$$B_4^{5,4}(\alpha,\beta) = \sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1)\frac{-\beta^2 e^{-i\beta}}{(1-e^{-i\beta})^2} \cdot \frac{i^{j_1+j_2+j_3+j_4}(-1)^{j_3}}{2j_1!j_2!j_3!j_4!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2} \tag{2184}$$

$$\cdot \int_0^1 e^{-i\beta s}\phi(\alpha+\beta(-1+s))^{j_3} ds \tag{2185}$$

$$\cdot \int_0^1 -e^{i\beta s}(-1+s)j_4\phi(\alpha+\beta(-1+s))^{j_4-1}\phi'(\alpha+\beta(-1+s))^2 ds \cdot \tag{2186}$$

$$\left(\sum_{m=1}^\infty \frac{-i\beta}{1-e^{-i\beta}} \int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!} ds\right)^n. \tag{2187}$$

Moreover, we can write $B_7^4(\alpha,\beta) = \sum_{j=1}^4 B_7^{4,j}(\alpha,\beta)$, where

$$B_7^{4,1}(\alpha,\beta) = \sum_{j_1+j_2+j_3+n\geq 1} \frac{i\beta}{1-e^{-i\beta}} \cdot \frac{i^{j_1+j_2+j_3}(-1)^{j_3}}{2j_1!j_2!j_3!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2} \tag{2188}$$

$$\cdot \frac{\phi(\alpha-\beta)^{j_3}\phi'(\alpha-\beta)}{\beta} \tag{2189}$$

$$\cdot \left(\sum_{m=1}^{\infty} \frac{-i\beta}{1-e^{-i\beta}} \int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n, \tag{2190}$$

$$B_7^{4,2}(\alpha,\beta) = \sum_{j_1+j_2+j_3+n\geq 1} \frac{i\beta}{1-e^{-i\beta}} \cdot \frac{i^{j_1+j_2+j_3}(-1)^{j_3}}{2j_1!j_2!j_3!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2} \tag{2191}$$

$$\cdot \int_0^1 ie^{-i\beta s}(-1+s)\phi(\alpha+\beta(-1+s))^{j_3}\phi'(\alpha+\beta(-1+s))ds \tag{2192}$$

$$\cdot \left(\sum_{m=1}^{\infty} \frac{-i\beta}{1-e^{-i\beta}} \int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n, \tag{2193}$$

$$B_7^{4,3}(\alpha,\beta) = \sum_{j_1+j_2+j_3+n\geq 1} \frac{i\beta}{1-e^{-i\beta}} \cdot \frac{i^{j_1+j_2+j_3}(-1)^{j_3}}{2j_1!j_2!j_3!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2} \tag{2194}$$

$$\cdot \int_0^1 \frac{-e^{-i\beta s}}{\beta}\phi(\alpha+\beta(-1+s))^{j_3}\phi'(\alpha+\beta(-1+s))ds \tag{2195}$$

$$\cdot \left(\sum_{m=1}^{\infty} \frac{-i\beta}{1-e^{-i\beta}} \int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n, \tag{2196}$$

$$B_7^{4,4}(\alpha,\beta) = \sum_{j_1+j_2+j_3+n\geq 1} \frac{i\beta}{1-e^{-i\beta}} \cdot \frac{i^{j_1+j_2+j_3}(-1)^{j_3}}{2j_1!j_2!j_3!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2} \tag{2197}$$

$$\cdot \int_0^1 -e^{-i\beta s}(-1+s)j_3\phi(\alpha+\beta(-1+s))^{j_3-1}\phi'(\alpha+\beta(-1+s))^2ds \tag{2198}$$

$$\cdot \left(\sum_{m=1}^{\infty} \frac{-i\beta}{1-e^{-i\beta}} \int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n. \tag{2199}$$

Lastly, we can write $B_8^4(\alpha,\beta) = \sum_{j=1}^4 B_8^{4,j}(\alpha,\beta)$, where

$$B_8^{4,1}(\alpha,\beta) = \sum_{j_1+j_2+j_3+n\geq 1} \frac{i\beta}{1-e^{-i\beta}} \cdot \frac{i^{j_1+j_2+j_3}(-1)^{j_2}}{2j_1!j_2!j_3!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2} \tag{2200}$$

$$\cdot \frac{\phi(\alpha-\beta)^{j_3}\phi'(\alpha-\beta)}{\beta} \tag{2201}$$

$$\cdot \left(\sum_{m=1}^{\infty} \frac{-i\beta}{1-e^{-i\beta}} \int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n, \tag{2202}$$

$$B_8^{4,2}(\alpha,\beta) = \sum_{j_1+j_2+j_3+n\geq 1} \frac{i\beta}{1-e^{-i\beta}} \cdot \frac{i^{j_1+j_2+j_3}(-1)^{j_2}}{2j_1!j_2!j_3!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2} \tag{2203}$$

$$\cdot \int_0^1 -ie^{i\beta s}(-1+s)\phi(\alpha+\beta(-1+s))^{j_3}\phi'(\alpha+\beta(-1+s))ds \tag{2204}$$

$$\cdot \left(\sum_{m=1}^{\infty} \frac{-i\beta}{1-e^{-i\beta}} \int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n, \tag{2205}$$

$$B_8^{4,3}(\alpha,\beta) = \sum_{j_1+j_2+j_3+n\geq 1} \frac{i\beta}{1-e^{-i\beta}} \cdot \frac{i^{j_1+j_2+j_3}(-1)^{j_2}}{2j_1!j_2!j_3!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2} \tag{2206}$$

$$\cdot \int_0^1 \frac{-e^{i\beta s}}{\beta}\phi(\alpha+\beta(-1+s))^{j_3}\phi'(\alpha+\beta(-1+s))ds \tag{2207}$$

$$\cdot \left(\sum_{m=1}^{\infty} \frac{-i\beta}{1-e^{-i\beta}} \int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n, \tag{2208}$$

$$B_8^{4,4}(\alpha,\beta) = \sum_{j_1+j_2+j_3+n\geq 1} \frac{i\beta}{1-e^{-i\beta}} \cdot \frac{i^{j_1+j_2+j_3}(-1)^{j_2}}{2j_1!j_2!j_3!}\phi(\alpha)^{j_1}\phi(\alpha-\beta)^{j_2} \tag{2209}$$

$$\cdot \int_0^1 -e^{i\beta s}(-1+s)j_3\phi(\alpha+\beta(-1+s))^{j_3-1}\phi'(\alpha+\beta(-1+s))^2ds \tag{2210}$$

$$\cdot \left(\sum_{m=1}^{\infty} \frac{-i\beta}{1-e^{-i\beta}} \int_0^1 e^{i(s-1)\beta}\frac{(i\phi(\alpha+(s-1)\beta))^m}{m!}ds\right)^n. \tag{2211}$$

14.1 Estimating Fourier Modes of $(U_{\geq 2})_\alpha$

We use arguments as in Section 13.1 to estimate the Fourier modes of $(U_{\geq 2})_\alpha$. First,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} \mathcal{F}(B_{1,1}^1)(k_1,\beta)d\beta\right| \leq \tag{2212}$$

$$\sum_{j_1+j_2+n\geq 1} \frac{j_1 C_n}{2j_1!j_2!} \sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}} \prod_{d=1}^{j_1-1} |\mathcal{F}(\phi)(k_d-k_{d+1})| \cdot |\mathcal{F}(\phi')(k_{j_1}-k_{j_1+1})| \tag{2213}$$

$$\cdot \prod_{d=j_1+1}^{j_1+j_2} |\mathcal{F}(\phi)(k_d-k_{d+1})| \cdot \prod_{d=j_1+j_2+1}^{j_1+j_2+n} |P(k_d-k_{d+1})| \cdot |\mathcal{F}(\phi)(k_{j_1+j_2+n+1})|. \tag{2214}$$

Moreover,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_{1,1}^2)(k_1,\beta)d\beta\right| \leq \tag{2215}$$

$$\sum_{j_1+j_2+n\geq 1}\frac{j_2C_n}{2j_1!j_2!}\sum_{k_2,\dots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2-1}|\mathcal{F}(\phi)(k_d-k_{d+1})|\,|\mathcal{F}(\phi')(k_{j_1+j_2}-k_{j_1+j_2+1})| \tag{2216}$$

$$\cdot\prod_{d=j_1+j_2+1}^{j_1+j_2+n}|P(k_d-k_{d+1})|\cdot|\mathcal{F}(\phi)(k_{j_1+j_2+n+1})|. \tag{2217}$$

Moreover,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_{1,1}^3)(k_1,\beta)d\beta\right| \leq \tag{2218}$$

$$\sum_{j_1+j_2+n\geq 1}\frac{C_n}{2j_1!j_2!}\sum_{k_2,\dots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})| \tag{2219}$$

$$\cdot\prod_{d=j_1+j_2+1}^{j_1+j_2+n}|P(k_d-k_{d+1})|\cdot|\mathcal{F}(\phi')(k_{j_1+j_2+n+1})|. \tag{2220}$$

Lastly,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_{1,1}^4)(k_1,\beta)d\beta\right| \leq \tag{2221}$$

$$\sum_{j_1+j_2+n\geq 1}\frac{nC_n}{2j_1!j_2!}\sum_{k_2,\dots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})| \tag{2222}$$

$$\cdot\prod_{d=j_1+j_2+1}^{j_1+j_2+n-1}|P(k_d-k_{d+1})|\cdot\left|\sum_{m=1}^{\infty}\frac{m(-i)^m}{m!}\mathcal{F}(\phi^{m-1}\phi')(k_{j_1+j_2+n}-k_{j_1+j_2+n+1})\right| \tag{2223}$$

$$\cdot|\mathcal{F}(\phi)(k_{j_1+j_2+n+1})|. \tag{2224}$$

Next,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_{1,2}^1)(k_1,\beta)d\beta\right| \leq \tag{2225}$$

$$\sum_{j_1+j_2+j_3\geq 1}\frac{j_1C_1}{2j_1!j_2!j_3!}\sum_{k_2,\dots,k_{j_1+j_2+2}\in\mathbb{Z}}\prod_{d=1}^{j_1-1}|\mathcal{F}(\phi)(k_d-k_{d+1})|\cdot|\mathcal{F}(\phi')(k_{j_1}-k_{j_1+1})| \tag{2226}$$

$$\cdot\prod_{d=j_1+1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})|\cdot|P(k_{j_1+j_2+1}-k_{j_1+j_2+2})|\cdot\left|\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+2})\right|. \tag{2227}$$

Moreover,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_{1,2}^2)(k_1,\beta)d\beta\right| \leq \tag{2228}$$

$$\sum_{j_1+j_2+j_3\geq 1}\frac{j_2 C_1}{2j_1!j_2!j_3!}\sum_{k_2,\ldots,k_{j_1+j_2+2}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2-1}|\mathcal{F}(\phi)(k_d-k_{d+1})|\,|\mathcal{F}(\phi')(k_{j_1+j_2}-k_{j_1+j_2+1})| \tag{2229}$$

$$\cdot|P(k_{j_1+j_2+1}-k_{j_1+j_2+2})|\cdot\left|\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+2})\right|. \tag{2230}$$

Moreover,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_{1,2}^3)(k_1,\beta)d\beta\right| \leq \tag{2231}$$

$$\sum_{j_1+j_2+j_3\geq 1}\frac{j_3 C_1}{2j_1!j_2!j_3!}\sum_{k_2,\ldots,k_{j_1+j_2+2}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})| \tag{2232}$$

$$\cdot|P(k_{j_1+j_2+1}-k_{j_1+j_2+2})|\cdot\left|\mathcal{F}(\phi^{j_3-1}\phi')(k_{j_1+j_2+2})\right|. \tag{2233}$$

Lastly,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_{1,2}^4)(k_1,\beta)d\beta\right| \leq \tag{2234}$$

$$\sum_{j_1+j_2+j_3\geq 1}\frac{C_1}{2j_1!j_2!j_3!}\sum_{k_2,\ldots,k_{j_1+j_2+2}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})| \tag{2235}$$

$$\cdot\left|\sum_{m=1}^{\infty}\frac{m(-i)^m}{m!}\mathcal{F}(\phi^{m-1}\phi')(k_{j_1+j_2+1}-k_{j_1+j_2+2})\right|\cdot\left|\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+2})\right|. \tag{2236}$$

Next,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}((\widetilde{B_{1,3}})_\alpha)(k_1,\beta)d\beta\right| \leq \frac{C_1}{2}\left|\sum_{m=2}^{\infty}\frac{m(-i)^{m-1}}{m!}\mathcal{F}(\phi^{m-1}\phi')(k_1)\right|. \tag{2237}$$

Next,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_2^1)(k_1,\beta)d\beta\right| \leq \tag{2238}$$

$$\sum_{j_1+j_2+j_3+n\geq 1}\frac{j_1 C_n}{2j_1!j_2!j_3!}\cdot\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1-1}|\mathcal{F}(\phi)(k_d-k_{d+1})|\cdot|\mathcal{F}(\phi')(k_{j_1}-k_{j_1+1})| \tag{2239}$$

$$\cdot\prod_{d=j_1+1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})|\cdot\prod_{d=j_1+j_2+1}^{j_1+j_2+n}|P(k_d-k_{d+1})|\cdot\left|\mathcal{F}(\phi^{j_3}\phi')(k_{j_1+j_2+n+1})\right|. \tag{2240}$$

Moreover,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_2^2)(k_1,\beta)d\beta\right| \leq \tag{2241}$$

$$\sum_{j_1+j_2+j_3+n\geq 1}\frac{j_2 C_n}{2j_1!j_2!j_3!}\cdot\sum_{k_2,\dots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2-1}|\mathcal{F}(\phi)(k_d-k_{d+1})| \tag{2242}$$

$$\cdot|\mathcal{F}(\phi')(k_{j_1+j_2}-k_{j_1+j_2+1})| \tag{2243}$$

$$\cdot\prod_{d=j_1+j_2+1}^{j_1+j_2+n}|P(k_d-k_{d+1})|\cdot\left|\mathcal{F}(\phi^{j_3}\phi')(k_{j_1+j_2+n+1})\right|. \tag{2244}$$

Moreover,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_2^3)(k_1,\beta)d\beta\right| \leq \tag{2245}$$

$$\sum_{j_1+j_2+j_3+n\geq 1}\frac{j_3 C_n}{2j_1!j_2!j_3!}\cdot\sum_{k_2,\dots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})| \tag{2246}$$

$$\cdot\prod_{d=j_1+j_2+1}^{j_1+j_2+n}|P(k_d-k_{d+1})|\cdot\left|\mathcal{F}(\phi^{j_3-1}\phi'^2)(k_{j_1+j_2+n+1})\right|. \tag{2247}$$

Lastly,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_2^5)(k_1,\beta)d\beta\right| \leq \tag{2248}$$

$$\sum_{j_1+j_2+j_3+n\geq 1}\frac{n C_n}{2j_1!j_2!j_3!}\cdot\sum_{k_2,\dots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})| \tag{2249}$$

$$\cdot\prod_{d=j_1+j_2+1}^{j_1+j_2+n-1}|P(k_d-k_{d+1})|\cdot\left|\sum_{m=1}^{\infty}\frac{m(-i)^m}{m!}\mathcal{F}(\phi^{m-1}\phi')(k_{j_1+j_2+n}-k_{j_1+j_2+n+1})\right| \tag{2250}$$

$$\cdot\left|\mathcal{F}(\phi^{j_3}\phi')(k_{j_1+j_2+n+1})\right|. \tag{2251}$$

Next,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_{3,1}^1)(k_1,\beta)d\beta\right| \leq \tag{2252}$$

$$\sum_{j_3+j_4+n\geq 2}(n+1)\frac{j_3 C_{n+1}}{2j_3!j_4!}\sum_{k_2,\dots,k_{n+2}\in\mathbb{Z}}\prod_{d=1}^{n}|Q(k_d-k_{d+1})|\cdot\left|\mathcal{F}(\phi^{j_3-1}\phi')(k_{n+1}-k_{n+2})\right| \tag{2253}$$

$$\cdot\left|\mathcal{F}(\phi^{j_4})(k_{n+2})\right|. \tag{2254}$$

Moreover,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_{3,1}^2)(k_1,\beta)d\beta\right| \leq \tag{2255}$$

$$\sum_{j_3+j_4+n\geq 2}(n+1)\frac{j_4 C_{n+1}}{2j_3!j_4!}\sum_{k_2,\dots,k_{n+2}\in\mathbb{Z}}\prod_{d=1}^{n}|Q(k_d-k_{d+1})|\cdot\left|\mathcal{F}(\phi^{j_4-1}\phi')(k_{n+1}-k_{n+2})\right| \tag{2256}$$

$$\cdot\left|\mathcal{F}(\phi^{j_3})(k_{n+2})\right|. \tag{2257}$$

Lastly,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_{3,1}^3)(k_1,\beta)d\beta\right| \leq \tag{2258}$$

$$\sum_{j_3+j_4+n\geq 2}(n+1)\frac{nC_{n+1}}{2j_3!j_4!}\sum_{k_2,\dots,k_{n+2}\in\mathbb{Z}}\prod_{d=1}^{n-1}|Q(k_d-k_{d+1})| \tag{2259}$$

$$\cdot\left|\sum_{m=1}^{\infty}\frac{mi^{m-1}}{m!}\mathcal{F}(\phi^{m-1}\phi')(k_n-k_{n+1})\right| \tag{2260}$$

$$\cdot\left|\mathcal{F}(\phi^{j_4})(k_{n+1}-k_{n+2})\right|\cdot\left|\mathcal{F}(\phi^{j_3})(k_{n+2})\right|. \tag{2261}$$

Next,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_{3,2}^1)(k_1,\beta)d\beta\right| \leq \tag{2262}$$

$$\sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}}(n+1)\frac{j_1 C_{n+1}}{2j_1!j_2!j_3!j_4!}\sum_{k_2,\dots,k_{j_1+j_2+n+2}\in\mathbb{Z}}\prod_{d=1}^{j_1-1}|\mathcal{F}(\phi)(k_d-k_{d+1})| \tag{2263}$$

$$\cdot|\mathcal{F}(\phi')(k_{j_1}-k_{j_1+1})| \tag{2264}$$

$$\cdot\prod_{d=j_1+1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})|\cdot\prod_{d=j_1+j_2+1}^{j_1+j_2+n}|Q(k_d-k_{d+1})| \tag{2265}$$

$$\cdot\left|\mathcal{F}(\phi^{j_4})(k_{j_1+j_2+n+1}-k_{j_1+j_2+n+2})\right|\left|\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+n+2})\right|. \tag{2266}$$

Moreover,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_{3,2}^2)(k_1,\beta)d\beta\right| \leq \tag{2267}$$

$$\sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}}(n+1)\frac{j_2 C_{n+1}}{2j_1!j_2!j_3!j_4!}\sum_{k_2,\dots,k_{j_1+j_2+n+2}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2-1}|\mathcal{F}(\phi)(k_d-k_{d+1})| \tag{2268}$$

$$\cdot|\mathcal{F}(\phi')(k_{j_1+j_2}-k_{j_1+j_2+1})|\cdot \tag{2269}$$

$$\prod_{d=j_1+j_2+1}^{j_1+j_2+n}|Q(k_d-k_{d+1})|\cdot\left|\mathcal{F}(\phi^{j_4})(k_{j_1+j_2+n+1}-k_{j_1+j_2+n+2})\right|\left|\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+n+2})\right|. \tag{2270}$$

Moreover,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_{3,2}^3)(k_1,\beta)d\beta\right| \leq \tag{2271}$$

$$\sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}}(n+1)\frac{j_3 C_{n+1}}{2j_1!j_2!j_3!j_4!}\sum_{k_2,\ldots,k_{j_1+j_2+n+2}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})| \tag{2272}$$

$$\cdot\prod_{d=j_1+j_2+1}^{j_1+j_2+n}|Q(k_d-k_{d+1})|\cdot\left|\mathcal{F}(\phi^{j_4})(k_{j_1+j_2+n+1}-k_{j_1+j_2+n+2})\right| \tag{2273}$$

$$\cdot\left|\mathcal{F}(\phi^{j_3-1}\phi')(k_{j_1+j_2+n+2})\right|. \tag{2274}$$

Moreover,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_{3,2}^4)(k_1,\beta)d\beta\right| \leq \tag{2275}$$

$$\sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}}(n+1)\frac{j_4 C_{n+1}}{2j_1!j_2!j_3!j_4!}\sum_{k_2,\ldots,k_{j_1+j_2+n+2}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})| \tag{2276}$$

$$\cdot\prod_{d=j_1+j_2+1}^{j_1+j_2+n}|Q(k_d-k_{d+1})|\cdot\left|\mathcal{F}(\phi^{j_4-1}\phi')(k_{j_1+j_2+n+1}-k_{j_1+j_2+n+2})\right| \tag{2277}$$

$$\cdot\left|\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+n+2})\right|. \tag{2278}$$

Lastly,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_{3,2}^5)(k_1,\beta)d\beta\right| \leq \tag{2279}$$

$$\sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}}(n+1)\frac{n C_{n+1}}{2j_1!j_2!j_3!j_4!}\sum_{k_2,\ldots,k_{j_1+j_2+n+2}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})| \tag{2280}$$

$$\cdot\prod_{d=j_1+j_2+1}^{j_1+j_2+n-1}|Q(k_d-k_{d+1})|\cdot\left|\sum_{m=1}^{\infty}\frac{mi^{m-1}}{m!}\mathcal{F}(\phi^{m-1}\phi')(k_{j_1+j_2+n}-k_{j_1+j_2+n+1})\right| \tag{2281}$$

$$\cdot\left|\mathcal{F}(\phi^{j_4})(k_{j_1+j_2+n+1}-k_{j_1+j_2+n+2})\right|\cdot\left|\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+n+2})\right|. \tag{2282}$$

Next,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}((\widetilde{B_{3,3}})_\alpha)(k_1,\beta)d\beta\right| \leq C_2\left|\sum_{m=2}^{\infty}\frac{mi^{m-1}}{m!}\mathcal{F}(\phi^{m-1}\phi')(k_1)\right|. \tag{2283}$$

Next,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_4^1)(k_1,\beta)d\beta\right| \leq \tag{2284}$$

$$\sum_{j_1+j_2+j_3+j_4+n\geq 1}(n+1)\frac{j_1 C_{n+1}}{2j_1!j_2!j_3!j_4!}\sum_{k_2,\ldots,k_{j_1+j_2+n+2}\in\mathbb{Z}}\prod_{d=1}^{j_1-1}|\mathcal{F}(\phi)(k_d-k_{d+1})| \tag{2285}$$

$$\cdot|\mathcal{F}(\phi')(k_{j_1}-k_{j_1+1})| \tag{2286}$$

$$\cdot\prod_{d=j_1+1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})|\cdot\prod_{d=j_1+j_2+1}^{j_1+j_2+n}|Q(k_d-k_{d+1})| \tag{2287}$$

$$\cdot\left|\mathcal{F}(\phi^{j_4}\phi')(k_{j_1+j_2+n+1}-k_{j_1+j_2+n+2})\right|\cdot\left|\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+n+2})\right|. \tag{2288}$$

Moreover,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_4^2)(k_1,\beta)d\beta\right| \leq \tag{2289}$$

$$\sum_{j_1+j_2+j_3+j_4+n\geq 1}(n+1)\frac{j_2 C_{n+1}}{2j_1!j_2!j_3!j_4!}\sum_{k_2,\ldots,k_{j_1+j_2+n+2}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2-1}|\mathcal{F}(\phi)(k_d-k_{d+1})| \tag{2290}$$

$$\cdot|\mathcal{F}(\phi')(k_{j_1+j_2}-k_{j_1+j_2+1})|\cdot\prod_{d=j_1+j_2+1}^{j_1+j_2+n}|Q(k_d-k_{d+1})| \tag{2291}$$

$$\cdot\left|\mathcal{F}(\phi^{j_4}\phi')(k_{j_1+j_2+n+1}-k_{j_1+j_2+n+2})\right|\cdot\left|\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+n+2})\right|. \tag{2292}$$

Moreover,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_4^3)(k_1,\beta)d\beta\right| \leq \tag{2293}$$

$$\sum_{j_1+j_2+j_3+j_4+n\geq 1}(n+1)\frac{j_3 C_{n+1}}{2j_1!j_2!j_3!j_4!}\sum_{k_2,\ldots,k_{j_1+j_2+n+2}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})| \tag{2294}$$

$$\cdot\prod_{d=j_1+j_2+1}^{j_1+j_2+n}|Q(k_d-k_{d+1})| \tag{2295}$$

$$\cdot\left|\mathcal{F}(\phi^{j_4}\phi')(k_{j_1+j_2+n+1}-k_{j_1+j_2+n+2})\right|\cdot\left|\mathcal{F}(\phi^{j_3-1}\phi')(k_{j_1+j_2+n+2})\right|. \tag{2296}$$

Moreover,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_4^4)(k_1,\beta)d\beta\right| \leq \tag{2297}$$

$$\sum_{j_1+j_2+j_3+j_4+n\geq 1}(n+1)\frac{j_4 C_{n+1}}{2j_1!j_2!j_3!j_4!}\sum_{k_2,\ldots,k_{j_1+j_2+n+2}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})| \tag{2298}$$

$$\cdot\prod_{d=j_1+j_2+1}^{j_1+j_2+n}|Q(k_d-k_{d+1})| \tag{2299}$$

$$\cdot\left|\mathcal{F}(\phi^{j_4-1}\phi'^2)(k_{j_1+j_2+n+1}-k_{j_1+j_2+n+2})\right|\cdot\left|\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+n+2})\right|. \tag{2300}$$

Lastly,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_4^6)(k_1,\beta)d\beta\right| \leq \tag{2301}$$

$$\sum_{j_1+j_2+j_3+j_4+n\geq 1}(n+1)\frac{nC_{n+1}}{2j_1!j_2!j_3!j_4!}\sum_{k_2,\dots,k_{j_1+j_2+n+2}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})| \tag{2302}$$

$$\cdot\prod_{d=j_1+j_2+1}^{j_1+j_2+n}|Q(k_d-k_{d+1})|\cdot\left|\sum_{m=1}^{\infty}\frac{mi^{m-1}}{m!}\mathcal{F}(\phi^{m-1}\phi')(k_{j_1+j_2+n}-k_{j_1+j_2+n+1})\right| \tag{2303}$$

$$\cdot\left|\mathcal{F}(\phi^{j_4}\phi')(k_{j_1+j_2+n+1}-k_{j_1+j_2+n+2})\right|\cdot\left|\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+n+2})\right|. \tag{2304}$$

Next,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_{5,1}^1)(k_1,\beta)d\beta\right| \leq \sum_{j_3+n\geq 2}\frac{j_3C_n}{2j_3!}\sum_{k_2,\dots,k_{n+1}\in\mathbb{Z}}\prod_{d=1}^{n}|Q(k_d-k_{d+1})| \tag{2305}$$

$$\cdot\left|\mathcal{F}(\phi^{j_3-1}\phi')(k_{n+1})\right|. \tag{2306}$$

Lastly,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_{5,1}^2)(k_1,\beta)d\beta\right| \leq \sum_{j_3+n\geq 2}\frac{nC_n}{2j_3!}\sum_{k_2,\dots,k_{n+1}\in\mathbb{Z}}\prod_{d=1}^{n-1}|Q(k_d-k_{d+1})| \tag{2307}$$

$$\cdot\left|\sum_{m=1}^{\infty}\frac{mi^{m-1}}{m!}\mathcal{F}(\phi^{m-1}\phi')(k_n-k_{n+1})\right|\left|\mathcal{F}(\phi^{j_3})(k_{n+1})\right|. \tag{2308}$$

Next,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_{5,2}^1)(k_1,\beta)d\beta\right| \leq \tag{2309}$$

$$\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}}\frac{j_1C_n}{2j_1!j_2!j_3!}\sum_{k_2,\dots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1-1}|\mathcal{F}(\phi)(k_d-k_{d+1})|\,|\mathcal{F}(\phi')(k_{j_1}-k_{j_1+1})| \tag{2310}$$

$$\cdot\prod_{d=j_1+1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})|\prod_{d=j_1+j_2+1}^{j_1+j_2+n}|Q(k_d-k_{d+1})|\left|\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+n+1})\right|. \tag{2311}$$

Moreover,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_{5,2}^2)(k_1,\beta)d\beta\right| \leq \tag{2312}$$

$$\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}}\frac{j_2C_n}{2j_1!j_2!j_3!}\sum_{k_2,\dots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2-1}|\mathcal{F}(\phi)(k_d-k_{d+1})|\,|\mathcal{F}(\phi')(k_{j_1+j_2}-k_{j_1+j_2+1})| \tag{2313}$$

$$\cdot\prod_{d=j_1+j_2+1}^{j_1+j_2+n}|Q(k_d-k_{d+1})|\left|\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+n+1})\right|. \tag{2314}$$

Moreover,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_{5,2}^3)(k_1,\beta)d\beta\right| \leq \tag{2315}$$

$$\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}}\frac{j_3C_n}{2j_1!j_2!j_3!}\sum_{k_2,\dots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})| \tag{2316}$$

$$\cdot\prod_{d=j_1+j_2+1}^{j_1+j_2+n}|Q(k_d-k_{d+1})|\left|\mathcal{F}(\phi^{j_3-1}\phi')(k_{j_1+j_2+n+1})\right|. \tag{2317}$$

Lastly,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_{5,2}^4)(k_1,\beta)d\beta\right| \leq \tag{2318}$$

$$\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}}\frac{nC_n}{2j_1!j_2!j_3!}\sum_{k_2,\dots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})| \tag{2319}$$

$$\cdot\prod_{d=j_1+j_2+1}^{j_1+j_2+n-1}|Q(k_d-k_{d+1})|\left|\sum_{m=1}^{\infty}\frac{mi^{m-1}}{m!}\mathcal{F}(\phi^{m-1}\phi')(k_{j_1+j_2+n}-k_{j_1+j_2+n+1})\right| \tag{2320}$$

$$\cdot\left|\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+n+1})\right|. \tag{2321}$$

Next,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}((\widetilde{B_{5,3}})_\alpha)(k_1,\beta)d\beta\right| \leq \frac{1}{2}C_1\left|\sum_{m=2}^{\infty}\frac{mi^{m-1}}{m!}\mathcal{F}(\phi^{m-1}\phi')(k_1)\right|. \tag{2322}$$

Next,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_{6,1}^1)(k_1,\beta)d\beta\right| \leq \sum_{j_3+n\geq 2}\frac{j_3C_n}{2j_3!}\sum_{k_2,\dots,k_{n+1}\in\mathbb{Z}}\prod_{d=1}^{n}|Q(k_d-k_{d+1})| \tag{2323}$$

$$\cdot\left|\mathcal{F}(\phi^{j_3-1}\phi')(k_{n+1})\right|. \tag{2324}$$

Lastly,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_{6,1}^2)(k_1,\beta)d\beta\right| \leq \sum_{j_3+n\geq 2}\frac{nC_n}{2j_3!}\sum_{k_2,\dots,k_{n+1}\in\mathbb{Z}}\prod_{d=1}^{n-1}|Q(k_d-k_{d+1})| \tag{2325}$$

$$\cdot\left|\sum_{m=1}^{\infty}\frac{mi^{m-1}}{m!}\mathcal{F}(\phi^{m-1}\phi')(k_n-k_{n+1})\right|\left|\mathcal{F}(\phi^{j_3})(k_{n+1})\right|. \tag{2326}$$

Next,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_{6,2}^{1})(k_1,\beta)d\beta\right| \leq \tag{2327}$$

$$\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}}\frac{j_1C_n}{2j_1!j_2!j_3!}\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1-1}|\mathcal{F}(\phi)(k_d-k_{d+1})|\,|\mathcal{F}(\phi')(k_{j_1}-k_{j_1+1})| \tag{2328}$$

$$\cdot\prod_{d=j_1+1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})|\prod_{d=j_1+j_2+1}^{j_1+j_2+n}|Q(k_d-k_{d+1})|\left|\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+n+1})\right|. \tag{2329}$$

Moreover,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_{6,2}^{2})(k_1,\beta)d\beta\right| \leq \tag{2330}$$

$$\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}}\frac{j_2C_n}{2j_1!j_2!j_3!}\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2-1}|\mathcal{F}(\phi)(k_d-k_{d+1})|\,|\mathcal{F}(\phi')(k_{j_1+j_2}-k_{j_1+j_2+1})| \tag{2331}$$

$$\cdot\prod_{d=j_1+j_2+1}^{j_1+j_2+n}|Q(k_d-k_{d+1})|\left|\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+n+1})\right|. \tag{2332}$$

Moreover,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_{6,2}^{3})(k_1,\beta)d\beta\right| \leq \tag{2333}$$

$$\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}}\frac{j_3C_n}{2j_1!j_2!j_3!}\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})| \tag{2334}$$

$$\cdot\prod_{d=j_1+j_2+1}^{j_1+j_2+n}|Q(k_d-k_{d+1})|\left|\mathcal{F}(\phi^{j_3-1}\phi')(k_{j_1+j_2+n+1})\right|. \tag{2335}$$

Lastly,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_{6,2}^{4})(k_1,\beta)d\beta\right| \leq \tag{2336}$$

$$\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}}\frac{nC_n}{2j_1!j_2!j_3!}\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})| \tag{2337}$$

$$\cdot\prod_{d=j_1+j_2+1}^{j_1+j_2+n-1}|Q(k_d-k_{d+1})|\left|\sum_{m=1}^{\infty}\frac{mi^{m-1}}{m!}\mathcal{F}(\phi^{m-1}\phi')(k_{j_1+j_2+n}-k_{j_1+j_2+n+1})\right| \tag{2338}$$

$$\cdot\left|\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+n+1})\right|. \tag{2339}$$

Next,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}((\widetilde{B_{6,3}})_\alpha)(k_1,\beta)d\beta\right| \leq \frac{1}{2}C_1\left|\sum_{m=2}^{\infty}\frac{mi^{m-1}}{m!}\mathcal{F}(\phi^{m-1}\phi')(k_1)\right| \tag{2340}$$

Next,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_7^1)(k_1,\beta)d\beta\right| \leq \tag{2341}$$

$$\sum_{j_1+j_2+j_3+n\geq 1}\frac{j_1C_n}{2j_1!j_2!j_3!}\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1-1}\left|\mathcal{F}(\phi)(k_d-k_{d+1})\right|\left|\mathcal{F}(\phi')(k_{j_1}-k_{j_1+1})\right| \tag{2342}$$

$$\cdot\prod_{d=j_1+1}^{j_1+j_2}\left|\mathcal{F}(\phi)(k_d-k_{d+1})\right|\prod_{d=j_1+j_2+1}^{j_1+j_2+n}\left|Q(k_d-k_{d+1})\right|\left|\mathcal{F}(\phi^{j_3}\phi')(k_{j_1+j_2+n+1})\right|. \tag{2343}$$

Moreover,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_7^2)(k_1,\beta)d\beta\right| \leq \tag{2344}$$

$$\sum_{j_1+j_2+j_3+n\geq 1}\frac{j_2C_n}{2j_1!j_2!j_3!}\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2-1}\left|\mathcal{F}(\phi)(k_d-k_{d+1})\right| \tag{2345}$$

$$\cdot\left|\mathcal{F}(\phi')(k_{j_1+j_2}-k_{j_1+j_2+1})\right| \tag{2346}$$

$$\cdot\prod_{d=j_1+j_2+1}^{j_1+j_2+n}\left|Q(k_d-k_{d+1})\right|\left|\mathcal{F}(\phi^{j_3}\phi')(k_{j_1+j_2+n+1})\right|. \tag{2347}$$

Moreover,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_7^3)(k_1,\beta)d\beta\right| \leq \tag{2348}$$

$$\sum_{j_1+j_2+j_3+n\geq 1}\frac{j_3C_n}{2j_1!j_2!j_3!}\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}\left|\mathcal{F}(\phi)(k_d-k_{d+1})\right| \tag{2349}$$

$$\cdot\prod_{d=j_1+j_2+1}^{j_1+j_2+n}\left|Q(k_d-k_{d+1})\right|\left|\mathcal{F}(\phi^{j_3-1}\phi'^2)(k_{j_1+j_2+n+1})\right|. \tag{2350}$$

Lastly,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_7^5)(k_1,\beta)d\beta\right| \leq \tag{2351}$$

$$\sum_{j_1+j_2+j_3+n\geq 1}\frac{nC_n}{2j_1!j_2!j_3!}\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}\left|\mathcal{F}(\phi)(k_d-k_{d+1})\right| \tag{2352}$$

$$\cdot\prod_{d=j_1+j_2+1}^{j_1+j_2+n-1}\left|Q(k_d-k_{d+1})\right|\cdot\left|\sum_{m=1}^{\infty}\frac{mi^{m-1}}{m!}\mathcal{F}(\phi^{m-1}\phi')(k_{j_1+j_2+n}-k_{j_1+j_2+n+1})\right| \tag{2353}$$

$$\cdot\left|\mathcal{F}(\phi^{j_3}\phi')(k_{j_1+j_2+n+1})\right|. \tag{2354}$$

Next,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_8^1)(k_1,\beta)d\beta\right| \leq \tag{2355}$$

$$\sum_{j_1+j_2+j_3+n\geq 1}\frac{j_1C_n}{2j_1!j_2!j_3!}\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1-1}|\mathcal{F}(\phi)(k_d-k_{d+1})|\,|\mathcal{F}(\phi')(k_{j_1}-k_{j_1+1})| \tag{2356}$$

$$\cdot\prod_{d=j_1+1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})|\prod_{d=j_1+j_2+1}^{j_1+j_2+n}|Q(k_d-k_{d+1})|\left|\mathcal{F}(\phi^{j_3}\phi')(k_{j_1+j_2+n+1})\right|. \tag{2357}$$

Moreover,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_8^2)(k_1,\beta)d\beta\right| \leq \tag{2358}$$

$$\sum_{j_1+j_2+j_3+n\geq 1}\frac{j_2C_n}{2j_1!j_2!j_3!}\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2-1}|\mathcal{F}(\phi)(k_d-k_{d+1})| \tag{2359}$$

$$\cdot|\mathcal{F}(\phi')(k_{j_1+j_2}-k_{j_1+j_2+1})| \tag{2360}$$

$$\cdot\prod_{d=j_1+j_2+1}^{j_1+j_2+n}|Q(k_d-k_{d+1})|\left|\mathcal{F}(\phi^{j_3}\phi')(k_{j_1+j_2+n+1})\right|. \tag{2361}$$

Moreover,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_8^3)(k_1,\beta)d\beta\right| \leq \tag{2362}$$

$$\sum_{j_1+j_2+j_3+n\geq 1}\frac{j_3C_n}{2j_1!j_2!j_3!}\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})| \tag{2363}$$

$$\cdot\prod_{d=j_1+j_2+1}^{j_1+j_2+n}|Q(k_d-k_{d+1})|\left|\mathcal{F}(\phi^{j_3-1}\phi'^2)(k_{j_1+j_2+n+1})\right|. \tag{2364}$$

Lastly,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_8^5)(k_1,\beta)d\beta\right| \leq \tag{2365}$$

$$\sum_{j_1+j_2+j_3+n\geq 1}\frac{nC_n}{2j_1!j_2!j_3!}\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})| \tag{2366}$$

$$\cdot\prod_{d=j_1+j_2+1}^{j_1+j_2+n-1}|Q(k_d-k_{d+1})|\cdot\left|\sum_{m=1}^{\infty}\frac{mi^{m-1}}{m!}\mathcal{F}(\phi^{m-1}\phi')(k_{j_1+j_2+n}-k_{j_1+j_2+n+1})\right| \tag{2367}$$

$$\cdot\left|\mathcal{F}(\phi^{j_3}\phi')(k_{j_1+j_2+n+1})\right|. \tag{2368}$$

Next,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}((\widetilde{B_{13}})_\alpha)(k_1,\beta)d\beta\right| \leq \sum_{j_1+j_2\geq 2}\frac{j_2}{j_1!j_2!} \tag{2369}$$

$$\cdot \sum_{k_2,\dots,k_{j_1+j_2}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2-1}|\mathcal{F}(\phi)(k_d-k_{d+1})|\,|\mathcal{F}(\phi')(k_{j_1+j_2})|\cdot\left(\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}\cdot\pi^2+3\pi\right). \tag{2370}$$

Next,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_2^{4,1})(k_1,\beta)d\beta\right| \leq \tag{2371}$$

$$\sum_{j_1+j_2+j_3+n\geq 1}\frac{D_n}{2j_1j_2!j_3!}\sum_{k_2,\dots,k_{j_1+j_2+j_3+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2+j_3}|\mathcal{F}(\phi)(k_d-k_{d+1})| \tag{2372}$$

$$\cdot\prod_{d=j_1+j_2+j_3+1}^{j_1+j_2+j_3+n}|P(k_d-k_{d+1})|\,|\mathcal{F}(\phi')(k_{j_1+j_2+j_3+n+1})|, \tag{2373}$$

where

$$D_n=\frac{\gamma}{4\pi}\left(D+(n+1)\left(\frac{\pi}{2}\right)^n\cdot\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}\cdot 2\pi\right) \tag{2374}$$

with D being an upper bound of $\left|\int_{-\pi}^{\pi}\frac{e^{i\beta x}}{\beta}d\beta\right|$, taken as a function of x. Moreover,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_2^{4,2})(k_1,\beta)d\beta\right| \leq \tag{2375}$$

$$\sum_{j_1+j_2+j_3+n\geq 1}\frac{C_n}{2j_1!j_2!j_3!}\sum_{k_2,\dots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})|\prod_{d=j_1+j_2+1}^{j_1+j_2+n}|P(k_d-k_{d+1})| \tag{2376}$$

$$\cdot\left|\mathcal{F}(\phi^{j_3}\phi')(k_{j_1+j_2+n+1})\right|. \tag{2377}$$

Moreover,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_2^{4,3})(k_1,\beta)d\beta\right| \leq \tag{2378}$$

$$\sum_{j_1+j_2+j_3+n\geq 1}\frac{D_n}{2j_1j_2!j_3!}\sum_{k_2,\dots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})|\prod_{d=j_1+j_2+1}^{j_1+j_2+n}|P(k_d-k_{d+1})| \tag{2379}$$

$$\cdot\left|\mathcal{F}(\phi^{j_3}\phi')(k_{j_1+j_2+n+1})\right|. \tag{2380}$$

Lastly,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_2^{4,4})(k_1,\beta)d\beta\right| \leq \tag{2381}$$

$$\sum_{j_1+j_2+j_3+n\geq 1}\frac{j_3C_n}{2j_1j_2!j_3!}\sum_{k_2,\dots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})|\prod_{d=j_1+j_2+1}^{j_1+j_2+n}|P(k_d-k_{d+1})| \tag{2382}$$

$$\cdot\left|\mathcal{F}(\phi^{j_3-1}\phi'^2)(k_{j_1+j_2+n+1})\right|. \tag{2383}$$

Next,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_4^{5,1})(k_1,\beta)d\beta\right| \leq \tag{2384}$$

$$\sum_{j_1+j_2+j_3+j_4+n\geq 1}(n+1)\frac{D_{n+1}}{2j_1j_2!j_3!j_4!}\sum_{k_2,\ldots,k_{j_1+j_2+j_4+n+2}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2+j_4}|\mathcal{F}(\phi)(k_d-k_{d+1})| \tag{2385}$$

$$\cdot\prod_{d=j_1+j_2+j_4+1}^{j_1+j_2+j_4+n}|Q(k_d-k_{d+1})| \tag{2386}$$

$$\cdot\left|\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+j_4+n+1}-k_{j_1+j_2+j_4+n+2})\right|\left|\mathcal{F}(\phi')(k_{j_1+j_2+j_4+n+2})\right|. \tag{2387}$$

Moreover,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_4^{5,2})(k_1,\beta)d\beta\right| \leq \tag{2388}$$

$$\sum_{j_1+j_2+j_3+j_4+n\geq 1}(n+1)\frac{C_{n+1}}{2j_1j_2!j_3!j_4!}\sum_{k_2,\ldots,k_{j_1+j_2+n+2}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})| \tag{2389}$$

$$\cdot\prod_{d=j_1+j_2+1}^{j_1+j_2+n}|Q(k_d-k_{d+1})| \tag{2390}$$

$$\cdot\left|\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+n+1}-k_{j_1+j_2+n+2})\right|\left|\mathcal{F}(\phi^{j_4}\phi')(k_{j_1+j_2+n+2})\right|. \tag{2391}$$

Moreover,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_4^{5,3})(k_1,\beta)d\beta\right| \leq \tag{2392}$$

$$\sum_{j_1+j_2+j_3+j_4+n\geq 1}(n+1)\frac{D_{n+1}}{2j_1j_2!j_3!j_4!}\sum_{k_2,\ldots,k_{j_1+j_2+n+2}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})| \tag{2393}$$

$$\cdot\prod_{d=j_1+j_2+1}^{j_1+j_2+n}|Q(k_d-k_{d+1})| \tag{2394}$$

$$\cdot\left|\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+n+1}-k_{j_1+j_2+n+2})\right|\left|\mathcal{F}(\phi^{j_4}\phi')(k_{j_1+j_2+n+2})\right|. \tag{2395}$$

Lastly,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_4^{5,4})(k_1,\beta)d\beta\right| \leq \tag{2396}$$

$$\sum_{j_1+j_2+j_3+j_4+n\geq 1}(n+1)\frac{j_4C_{n+1}}{2j_1j_2!j_3!j_4!}\sum_{k_2,\ldots,k_{j_1+j_2+n+2}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})| \tag{2397}$$

$$\cdot\prod_{d=j_1+j_2+1}^{j_1+j_2+n}|Q(k_d-k_{d+1})| \tag{2398}$$

$$\cdot\left|\mathcal{F}(\phi^{j_3})(k_{j_1+j_2+n+1}-k_{j_1+j_2+n+2})\right|\left|\mathcal{F}(\phi^{j_4-1}\phi'^2)(k_{j_1+j_2+n+2})\right|. \tag{2399}$$

Next,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_7^{4,1})(k_1,\beta)d\beta\right| \leq \tag{2400}$$

$$\sum_{j_1+j_2+j_3+n\geq 1}\frac{D_n}{2j_1j_2!j_3!}\sum_{k_2,\ldots,k_{j_1+j_2+j_3+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2+j_3}|\mathcal{F}(\phi)(k_d-k_{d+1})| \tag{2401}$$

$$\cdot\prod_{d=j_1+j_2+j_3+1}^{j_1+j_2+j_3+n}|Q(k_d-k_{d+1})| \tag{2402}$$

$$\cdot\left|\mathcal{F}(\phi')(k_{j_1+j_2+j_3+n+1})\right|. \tag{2403}$$

Moreover,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_7^{4,2})(k_1,\beta)d\beta\right| \leq \tag{2404}$$

$$\sum_{j_1+j_2+j_3+n\geq 1}\frac{C_n}{2j_1j_2!j_3!}\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})|\prod_{d=j_1+j_2+1}^{j_1+j_2+n}|Q(k_d-k_{d+1})| \tag{2405}$$

$$\cdot\left|\mathcal{F}(\phi^{j_3}\phi')(k_{j_1+j_2+n+1})\right|. \tag{2406}$$

Moreover,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_7^{4,3})(k_1,\beta)d\beta\right| \leq \tag{2407}$$

$$\sum_{j_1+j_2+j_3+n\geq 1}\frac{D_n}{2j_1j_2!j_3!}\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})|\prod_{d=j_1+j_2+1}^{j_1+j_2+n}|Q(k_d-k_{d+1})| \tag{2408}$$

$$\cdot\left|\mathcal{F}(\phi^{j_3}\phi')(k_{j_1+j_2+n+1})\right|. \tag{2409}$$

Lastly,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_7^{4,4})(k_1,\beta)d\beta\right| \leq \tag{2410}$$

$$\sum_{j_1+j_2+j_3+n\geq 1}\frac{j_3C_n}{2j_1j_2!j_3!}\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})|\prod_{d=j_1+j_2+1}^{j_1+j_2+n}|Q(k_d-k_{d+1})| \tag{2411}$$

$$\cdot\left|\mathcal{F}(\phi^{j_3-1}\phi'^2)(k_{j_1+j_2+n+1})\right|. \tag{2412}$$

Next,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_8^{4,1})(k_1,\beta)d\beta\right| \leq \tag{2413}$$

$$\sum_{j_1+j_2+j_3+n\geq 1}\frac{D_n}{2j_1j_2!j_3!}\sum_{k_2,\ldots,k_{j_1+j_2+j_3+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2+j_3}|\mathcal{F}(\phi)(k_d-k_{d+1})| \tag{2414}$$

$$\cdot\prod_{d=j_1+j_2+j_3+1}^{j_1+j_2+j_3+n}|Q(k_d-k_{d+1})|\left|\mathcal{F}(\phi')(k_{j_1+j_2+j_3+n+1})\right|. \tag{2415}$$

Moreover,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_8^{4,2})(k_1,\beta)d\beta\right| \leq \tag{2416}$$

$$\sum_{j_1+j_2+j_3+n\geq 1}\frac{C_n}{2j_1j_2!j_3!}\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})|\prod_{d=j_1+j_2+1}^{j_1+j_2+n}|Q(k_d-k_{d+1})| \tag{2417}$$

$$\cdot\left|\mathcal{F}(\phi^{j_3}\phi')(k_{j_1+j_2+n+1})\right|. \tag{2418}$$

Moreover,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_8^{4,3})(k_1,\beta)d\beta\right| \leq \tag{2419}$$

$$\sum_{j_1+j_2+j_3+n\geq 1}\frac{D_n}{2j_1j_2!j_3!}\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})|\prod_{d=j_1+j_2+1}^{j_1+j_2+n}|Q(k_d-k_{d+1})| \tag{2420}$$

$$\cdot\left|\mathcal{F}(\phi^{j_3}\phi')(k_{j_1+j_2+n+1})\right|. \tag{2421}$$

Lastly,

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(B_8^{4,4})(k_1,\beta)d\beta\right| \leq \tag{2422}$$

$$\sum_{j_1+j_2+j_3+n\geq 1}\frac{j_3C_n}{2j_1j_2!j_3!}\sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}}\prod_{d=1}^{j_1+j_2}|\mathcal{F}(\phi)(k_d-k_{d+1})|\prod_{d=j_1+j_2+1}^{j_1+j_2+n}|Q(k_d-k_{d+1})| \tag{2423}$$

$$\cdot\left|\mathcal{F}(\phi^{j_3-1}\phi'^2)(k_{j_1+j_2+n+1})\right|. \tag{2424}$$

14.2 Estimating $\|(U_{\geq 2})_\alpha\|_{\dot{\mathcal{F}}_\nu^{s,1}}$

We prove the following estimate for $\|(U_{\geq 2})_\alpha\|_{\dot{\mathcal{F}}_\nu^{s,1}}$, $s>0$.

Lemma 6. *For $s>0$,*

$$\begin{aligned}\|(U_{\geq 2})_\alpha\|_{\dot{\mathcal{F}}_\nu^{s,1}} \leq & R_1(\|\phi\|_{\mathcal{F}_\nu^{0,1}})\,\|\phi'\|_{\mathcal{F}_\nu^{0,1}}\,\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} + R_2(\|\phi\|_{\mathcal{F}_\nu^{0,1}})\,\|\phi\|_{\mathcal{F}_\nu^{0,1}}\,\|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}} && (2425)\\ &+ R_3(\|\phi\|_{\mathcal{F}_\nu^{0,1}})\,\|\phi\|_{\mathcal{F}_\nu^{0,1}}\,\|\phi'\|_{\mathcal{F}_\nu^{0,1}}\,\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} + R_4(\|\phi\|_{\mathcal{F}_\nu^{0,1}})\,\|\phi'\|^2_{\mathcal{F}_\nu^{0,1}}\,\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} && (2426)\\ &+ R_5(\|\phi\|_{\mathcal{F}_\nu^{0,1}})\,\|\phi'\|_{\mathcal{F}_\nu^{0,1}}\,\|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}}, && (2427)\end{aligned}$$

where R_1, R_2, R_3, R_4, and R_5 are monotone increasing functions of $\|\phi\|_{\mathcal{F}_\nu^{0,1}}$.

We use estimates from Section 14.1 to prove Lemma 6. We take the notational convention that if a convolution of sequences on $\mathbb{Z}$ contains a sequence of the form $|\mathcal{F}(\phi^{j_3})|$ in which $j_3=0$, then we simply ignore that sequence in the convolution. For example, in (2460), if $j_3=0$, then

$$\begin{aligned}&|\mathcal{F}(\phi)|*\cdots*|\mathcal{F}(\phi)|*|\mathcal{F}(\phi')|*|P|*\left|\mathcal{F}(\phi^{j_3})\right| && (2428)\\ =&|\mathcal{F}(\phi)|*\cdots*|\mathcal{F}(\phi)|*|\mathcal{F}(\phi')|*|P|. && (2429)\end{aligned}$$

We define

$$R(k) = \sum_{m=1}^{\infty} \frac{m(-i)^m}{m!} \mathcal{F}(\phi^{m-1}\phi')(k), \tag{2430}$$

$$S(k) = \sum_{m=1}^{\infty} \frac{m i^{m-1}}{m!} \mathcal{F}(\phi^{m-1}\phi')(k), \tag{2431}$$

and

$$D_n = \frac{\gamma}{4\pi}\left(D + (n+1)\left(\frac{\pi}{2}\right)^n \cdot \frac{1}{2}\sqrt{1+\frac{\pi^2}{4}} \cdot 2\pi\right), \tag{2432}$$

where D is an upper bound of $\left|\int_{-\pi}^{\pi} \frac{e^{i\beta x}}{\beta} d\beta\right|$, uniform in $x \in \mathbb{R}$. First,

$$\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B^1_{1,1}(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}^{s,1}_\nu} \tag{2433}$$

$$\leq \sum_{j_1+j_2+n\geq 1} \frac{j_1 C_n}{2 j_1! j_2!} \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |P| * \cdots * |P| * |\mathcal{F}(\phi')| \right\|_{l^s_\nu} \tag{2434}$$

$$\leq \sum_{j_1+j_2+n\geq 1} \frac{j_1 C_n}{2 j_1! j_2!} b(j_1+j_2+n+1, s) \tag{2435}$$

$$\cdot \Bigg(\|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu} \|\phi\|^{j_1+j_2-1}_{\mathcal{F}^{0,1}_\nu} \|\phi'\|_{\mathcal{F}^{0,1}_\nu} (e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}} - 1)^n (j_1+j_2) \tag{2436}$$

$$+ \|\phi'\|_{\dot{\mathcal{F}}^{s,1}_\nu} \|\phi\|^{j_1+j_2}_{\mathcal{F}^{0,1}_\nu} (e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}} - 1)^n \tag{2437}$$

$$+ \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|^{m-1}_{\mathcal{F}^{0,1}_\nu} \right) \|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu} \|\phi\|^{j_1+j_2}_{\mathcal{F}^{0,1}_\nu} \|\phi'\|_{\mathcal{F}^{0,1}_\nu} (e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}} - 1)^{n-1} \cdot n \Bigg). \tag{2438}$$

Moreover,

$$\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B^2_{1,1}(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}^{s,1}_\nu} \tag{2439}$$

$$\leq \sum_{j_1+j_2+n\geq 1} \frac{j_2 C_n}{2 j_1! j_2!} \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |P| * \cdots * |P| * |\mathcal{F}(\phi')| \right\|_{l^s_\nu} \tag{2440}$$

$$\leq \sum_{j_1+j_2+n\geq 1} \frac{j_2 C_n}{2 j_1! j_2!} b(j_1+j_2+n+1, s) \tag{2441}$$

$$\cdot \Bigg(\|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu} \|\phi\|^{j_1+j_2-1}_{\mathcal{F}^{0,1}_\nu} \|\phi'\|_{\mathcal{F}^{0,1}_\nu} (e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}} - 1)^n (j_1+j_2) \tag{2442}$$

$$+ \|\phi'\|_{\dot{\mathcal{F}}^{s,1}_\nu} \|\phi\|^{j_1+j_2}_{\mathcal{F}^{0,1}_\nu} (e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}} - 1)^n \tag{2443}$$

$$+ \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|^{m-1}_{\mathcal{F}^{0,1}_\nu} \right) \|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu} \|\phi\|^{j_1+j_2}_{\mathcal{F}^{0,1}_\nu} \|\phi'\|_{\mathcal{F}^{0,1}_\nu} (e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}} - 1)^{n-1} \cdot n \Bigg). \tag{2444}$$

Moreover,

$$\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B_{1,1}^3(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \tag{2445}$$

$$\leq \sum_{j_1+j_2+n\geq 1} \frac{C_n}{2j_1!j_2!} \| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |P| * \cdots * |P| * |\mathcal{F}(\phi')| \|_{l_\nu^s} \tag{2446}$$

$$\leq \sum_{j_1+j_2+n\geq 1} \frac{C_n}{2j_1!j_2!} b(j_1+j_2+n+1,s) \tag{2447}$$

$$\cdot \Bigg(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n (j_1+j_2) \tag{2448}$$

$$+ \|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n \tag{2449}$$

$$+ \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1} \right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-1} \cdot n \Bigg). \tag{2450}$$

Lastly,

$$\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B_{1,1}^4(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \tag{2451}$$

$$\leq \sum_{j_1+j_2+n\geq 1} \frac{nC_n}{2j_1!j_2!} \| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |P| * \cdots * |P| * |R| \|_{l_\nu^s} \tag{2452}$$

$$\leq \sum_{j_1+j_2+n\geq 1} \frac{nC_n}{2j_1!j_2!} b(j_1+j_2+n+1,s) \tag{2453}$$

$$\cdot \Bigg(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-1} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} (j_1+j_2+1) \tag{2454}$$

$$+ \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1} \right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-2} \tag{2455}$$

$$\cdot \|\phi'\|_{\mathcal{F}_\nu^{0,1}} e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} (n-1) \tag{2456}$$

$$+ \left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \sum_{m=2}^{\infty} \frac{b(m,s)}{(m-2)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-2} + \|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}} \sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1} \right) \tag{2457}$$

$$\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-1} \Bigg). \tag{2458}$$

Next,

$$\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B_{1,2}^1(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \tag{2459}$$

$$\leq \sum_{j_1+j_2+j_3\geq 1} \frac{j_1 C_1}{2j_1!j_2!j_3!} \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |\mathcal{F}(\phi')| * |P| * \left|\mathcal{F}(\phi^{j_3})\right| \right\|_{l_\nu^s} \tag{2460}$$

$$\leq \sum_{j_1+j_2+j_3\geq 1} \frac{j_1 C_1}{2j_1!j_2!j_3!} b(j_1+j_2+2,s) \tag{2461}$$

$$\cdot \Bigg(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-2} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1) \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3} (j_1+j_2-1) \tag{2462}$$

$$+ \|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1) \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3} \tag{2463}$$

$$+ \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1} \right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3} \tag{2464}$$

$$+ b(j_3,s) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1} \cdot j_3 \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1) \Bigg). \tag{2465}$$

Moreover,

$$\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B_{1,2}^2(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \tag{2466}$$

$$\leq \sum_{j_1+j_2+j_3\geq 1} \frac{j_2 C_1}{2j_1!j_2!j_3!} \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |\mathcal{F}(\phi')| * |P| * \left|\mathcal{F}(\phi^{j_3})\right| \right\|_{l_\nu^s} \tag{2467}$$

$$\leq \sum_{j_1+j_2+j_3\geq 1} \frac{j_2 C_1}{2j_1!j_2!j_3!} b(j_1+j_2+2,s) \tag{2468}$$

$$\cdot \Bigg(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-2} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1) \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3} (j_1+j_2-1) \tag{2469}$$

$$+ \|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1) \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3} \tag{2470}$$

$$+ \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1} \right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3} \tag{2471}$$

$$+ b(j_3,s) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1} \cdot j_3 \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1) \Bigg). \tag{2472}$$

Moreover,

$$\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B_{1,2}^3(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \tag{2473}$$

$$\leq \sum_{j_1+j_2+j_3\geq 1} \frac{j_3 C_1}{2 j_1! j_2! j_3!} \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |P| * \left|\mathcal{F}(\phi^{j_3-1}\phi')\right| \right\|_{l_\nu^s} \tag{2474}$$

$$\leq \sum_{j_1+j_2+j_3\geq 1} \frac{j_3 C_1}{2 j_1! j_2! j_3!} b(j_1+j_2+2, s) \tag{2475}$$

$$\cdot \Bigg(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1) \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} (j_1+j_2) \tag{2476}$$

$$+ \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1} \right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \tag{2477}$$

$$+ b(j_3, s)\left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-2} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} (j_3-1) + \|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1} \right) \tag{2478}$$

$$\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1) \Bigg). \tag{2479}$$

Lastly,

$$\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B_{1,2}^4(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \tag{2480}$$

$$\leq \sum_{j_1+j_2+j_3\geq 1} \frac{C_1}{2 j_1! j_2! j_3!} \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |R| * \left|\mathcal{F}(\phi^{j_3})\right| \right\|_{l_\nu^s} \tag{2481}$$

$$\leq \sum_{\substack{j_1,j_2\geq 0\\ j_3\geq 1}} \frac{C_1}{2 j_1! j_2! j_3!} b(j_1+j_2+2, s) \tag{2482}$$

$$\cdot \Bigg(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3} (j_1+j_2) \tag{2483}$$

$$+ \left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \sum_{m=2}^{\infty} \frac{b(m,s)}{(m-2)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-2} + \|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}} \sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1} \right) \tag{2484}$$

$$\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3} \tag{2485}$$

$$+ b(j_3, s) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1} j_3 \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} \Bigg). \tag{2486}$$

Next,

$$\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} ((\widetilde{B_{1,3}})_\alpha)(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \tag{2487}$$

$$\leq \frac{1}{2} C_1 \Bigg(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \sum_{m=2}^{\infty} \frac{b(m,s)(m-1)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-2} \tag{2488}$$

$$+ \|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}} \sum_{m=2}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1} \Bigg). \tag{2489}$$

Next,

$$\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B_2^1(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \tag{2490}$$

$$\leq \sum_{j_1+j_2+j_3+n\geq 1} \frac{j_1 C_n}{2 j_1! j_2! j_3!} \tag{2491}$$

$$\cdot \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |\mathcal{F}(\phi')| * |P| * \cdots * |P| * \left|\mathcal{F}(\phi^{j_3}\phi')\right| \right\|_{l_\nu^s} \tag{2492}$$

$$\leq \sum_{j_1+j_2+j_3+n\geq 1} \frac{j_1 C_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1,s) \tag{2493}$$

$$\cdot \Bigg(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-2} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} (j_1+j_2-1) \tag{2494}$$

$$+ \|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \tag{2495}$$

$$+ \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1} \right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^{n-1} \tag{2496}$$

$$\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \cdot n \tag{2497}$$

$$+ b(j_3+1,s) \left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} j_3 + \|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3} \right) \tag{2498}$$

$$\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n \Bigg). \tag{2499}$$

Moreover,

$$\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B_2^2(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \tag{2500}$$

$$\leq \sum_{j_1+j_2+j_3+n\geq 1} \frac{j_2 C_n}{2 j_1! j_2! j_3!} \tag{2501}$$

$$\cdot \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |\mathcal{F}(\phi')| * |P| * \cdots * |P| * \left|\mathcal{F}(\phi^{j_3}\phi')\right| \right\|_{l_\nu^s} \tag{2502}$$

$$\leq \sum_{j_1+j_2+j_3+n\geq 1} \frac{j_2 C_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1, s) \tag{2503}$$

$$\cdot \Bigg(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-2} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} (j_1+j_2-1) \tag{2504}$$

$$+ \|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \tag{2505}$$

$$+ \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1} \right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^{n-1} \tag{2506}$$

$$\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \cdot n \tag{2507}$$

$$+ b(j_3+1, s) \left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} j_3 + \|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3} \right) \tag{2508}$$

$$\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n \Bigg). \tag{2509}$$

Moreover,

$$\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B_2^3(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \tag{2510}$$

$$\leq \sum_{j_1+j_2+j_3+n\geq 1} \frac{j_3 C_n}{2 j_1! j_2! j_3!} \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |P| * \cdots * |P| * \left|\mathcal{F}(\phi^{j_3-1}\phi'^2)\right| \right\|_{l_\nu^s} \tag{2511}$$

$$\leq \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|^2_{\mathcal{F}_\nu^{0,1}} \tag{2512}$$

$$\cdot \sum_{j_1+j_2+j_3+n\geq 1} \frac{j_3 C_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1,s)(j_1+j_2) \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-2} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n \tag{2513}$$

$$+ \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1} \right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|^2_{\mathcal{F}_\nu^{0,1}} \tag{2514}$$

$$\cdot \sum_{j_1+j_2+j_3+n\geq 1} \frac{j_3 C_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1,s) n (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^{n-1} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-1} \tag{2515}$$

$$+ \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|^2_{\mathcal{F}_\nu^{0,1}} \sum_{j_1+j_2+j_3+n\geq 1} \frac{j_3 C_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1,s) b(j_3+1,s)(j_3-1) \tag{2516}$$

$$\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-2} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n \tag{2517}$$

$$+ \|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \sum_{j_1+j_2+j_3+n\geq 1} \frac{j_3 C_n}{j_1! j_2! j_3!} b(j_1+j_2+n+1,s) b(j_3+1,s) \tag{2518}$$

$$\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n. \tag{2519}$$

Lastly,

$$\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B_2^5(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \tag{2520}$$

$$\leq \sum_{j_1+j_2+j_3+n\geq 1} \frac{nC_n}{2j_1!j_2!j_3!} \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |P| * \cdots * |P| * |R| * \left|\mathcal{F}(\phi^{j_3}\phi')\right| \right\|_{l_\nu^s} \tag{2521}$$

$$\leq \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|_{\mathcal{F}_\nu^{0,1}}^2 e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} \sum_{j_1+j_2+j_3+n\geq 1} \frac{nC_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s)(j_1+j_2) \tag{2522}$$

$$\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^{n-1} \tag{2523}$$

$$+ \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1} \right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|_{\mathcal{F}_\nu^{0,1}}^2 e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} \tag{2524}$$

$$\cdot \sum_{j_1+j_2+j_3+n\geq 1} \frac{nC_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s)(n-1) \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^{n-2} \tag{2525}$$

$$+ \left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \sum_{m=2}^{\infty} \frac{b(m,s)}{(m-2)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-2} + \|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}} \sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1} \right) \tag{2526}$$

$$\cdot \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \sum_{j_1+j_2+j_3+n\geq 1} \frac{nC_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s) \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^{n-1} \tag{2527}$$

$$+ \|\phi'\|_{\mathcal{F}_\nu^{0,1}} e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} \sum_{j_1+j_2+j_3+n\geq 1} \frac{nC_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s) b(j_3+1,s) \tag{2528}$$

$$\cdot \left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} j_3 + \|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3} \right) \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^{n-1}. \tag{2529}$$

Next,

$$\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}B^1_{3,1}(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}^{s,1}_\nu} \tag{2530}$$

$$\leq \sum_{j_3+j_4+n\geq 2}(n+1)\frac{j_3C_{n+1}}{2j_3!j_4!}\left\||Q|*\cdots*|Q|*\left|\mathcal{F}(\phi^{j_3-1}\phi')\right|*\left|\mathcal{F}(\phi^{j_4})\right|\right\|_{l^s_\nu} \tag{2531}$$

$$\leq\left(\sum_{m=1}^{\infty}\frac{b(m,s)}{(m-1)!}\|\phi\|^{m-1}_{\mathcal{F}^{0,1}_\nu}\right)\|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu}\|\phi'\|_{\mathcal{F}^{0,1}_\nu} \tag{2532}$$

$$\cdot\sum_{j_3+j_4+n\geq 2}(n+1)\frac{j_3C_{n+1}}{2j_3!j_4!}b(n+2,s)n(e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1)^{n-1}\|\phi\|^{j_3+j_4-1}_{\mathcal{F}^{0,1}_\nu} \tag{2533}$$

$$+\sum_{j_3+j_4+n\geq 2}(n+1)\frac{j_3C_{n+1}}{2j_3!j_4!}b(n+2,s)b(j_3,s) \tag{2534}$$

$$\cdot\left(\|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu}\|\phi\|^{j_3-2}_{\mathcal{F}^{0,1}_\nu}\|\phi'\|_{\mathcal{F}^{0,1}_\nu}(j_3-1)+\|\phi'\|_{\dot{\mathcal{F}}^{s,1}_\nu}\|\phi\|^{j_3-1}_{\mathcal{F}^{0,1}_\nu}\right)(e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1)^n\|\phi\|^{j_4}_{\mathcal{F}^{0,1}_\nu} \tag{2535}$$

$$+\|\phi'\|_{\mathcal{F}^{0,1}_\nu}\sum_{j_3+j_4+n\geq 2}(n+1)\frac{j_3C_{n+1}}{2j_3!j_4!}b(n+2,s)b(j_4,s)\|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu}\|\phi\|^{j_4-1}_{\mathcal{F}^{0,1}_\nu} \tag{2536}$$

$$\cdot j_4(e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1)^n\|\phi\|^{j_3-1}_{\mathcal{F}^{0,1}_\nu}. \tag{2537}$$

Next,

$$\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}B^2_{3,1}(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}^{s,1}_\nu} \tag{2538}$$

$$\leq \sum_{j_3+j_4+n\geq 2}(n+1)\frac{j_4C_{n+1}}{2j_3!j_4!}\left\||Q|*\cdots*|Q|*\left|\mathcal{F}(\phi^{j_4-1}\phi')\right|*\left|\mathcal{F}(\phi^{j_3})\right|\right\|_{l^s_\nu} \tag{2539}$$

$$\leq\left(\sum_{m=1}^{\infty}\frac{b(m,s)}{(m-1)!}\|\phi\|^{m-1}_{\mathcal{F}^{0,1}_\nu}\right)\|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu}\|\phi'\|_{\mathcal{F}^{0,1}_\nu} \tag{2540}$$

$$\cdot\sum_{j_3+j_4+n\geq 2}(n+1)\frac{j_4C_{n+1}}{2j_3!j_4!}b(n+2,s)n(e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1)^{n-1}\|\phi\|^{j_3+j_4-1}_{\mathcal{F}^{0,1}_\nu} \tag{2541}$$

$$+\sum_{j_3+j_4+n\geq 2}(n+1)\frac{j_4C_{n+1}}{2j_3!j_4!}b(n+2,s)b(j_4,s) \tag{2542}$$

$$\cdot\left(\|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu}\|\phi\|^{j_4-2}_{\mathcal{F}^{0,1}_\nu}\|\phi'\|_{\mathcal{F}^{0,1}_\nu}(j_4-1)+\|\phi'\|_{\dot{\mathcal{F}}^{s,1}_\nu}\|\phi\|^{j_4-1}_{\mathcal{F}^{0,1}_\nu}\right)(e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1)^n\|\phi\|^{j_3}_{\mathcal{F}^{0,1}_\nu} \tag{2543}$$

$$+\|\phi'\|_{\mathcal{F}^{0,1}_\nu}\sum_{j_3+j_4+n\geq 2}(n+1)\frac{j_4C_{n+1}}{2j_3!j_4!}b(n+2,s)b(j_3,s)\|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu}\|\phi\|^{j_3-1}_{\mathcal{F}^{0,1}_\nu}j_3 \tag{2544}$$

$$\cdot(e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1)^n\|\phi\|^{j_4-1}_{\mathcal{F}^{0,1}_\nu}. \tag{2545}$$

Lastly,

$$\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B^3_{3,1}(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}^{s,1}_\nu} \tag{2546}$$

$$\leq \sum_{j_3+j_4+n\geq 2} (n+1)\frac{nC_{n+1}}{2j_3!j_4!} \left\| |Q| * \cdots * |Q| * |S| * \left|\mathcal{F}(\phi^{j_4})\right| * \left|\mathcal{F}(\phi^{j_3})\right| \right\|_{l^s_\nu} \tag{2547}$$

$$\leq \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|^{m-1}_{\mathcal{F}^{0,1}_\nu} \right) \|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu} \|\phi'\|_{\mathcal{F}^{0,1}_\nu} e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}} \tag{2548}$$

$$\cdot \sum_{j_3+j_4+n\geq 2} (n+1)\frac{nC_{n+1}}{2j_3!j_4!} b(n+2,s)(n-1)(e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1)^{n-2} \|\phi\|^{j_3+j_4}_{\mathcal{F}^{0,1}_\nu} \tag{2549}$$

$$+ \left(\|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu} \|\phi'\|_{\mathcal{F}^{0,1}_\nu} \sum_{m=2}^{\infty} \frac{b(m,s)}{(m-2)!} \|\phi\|^{m-2}_{\mathcal{F}^{0,1}_\nu} + \|\phi'\|_{\dot{\mathcal{F}}^{s,1}_\nu} \sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|^{m-1}_{\mathcal{F}^{0,1}_\nu} \right) \tag{2550}$$

$$\cdot \sum_{j_3+j_4+n\geq 2} (n+1)\frac{nC_{n+1}}{2j_3!j_4!} b(n+2,s)(e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1)^{n-1} \|\phi\|^{j_3+j_4}_{\mathcal{F}^{0,1}_\nu} \tag{2551}$$

$$+ \|\phi'\|_{\mathcal{F}^{0,1}_\nu} e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}} \sum_{j_3+j_4+n\geq 2} (n+1)\frac{nC_{n+1}}{2j_3!j_4!} b(n+2,s) b(j_4,s) \tag{2552}$$

$$\cdot \|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu} \|\phi\|^{j_4-1}_{\mathcal{F}^{0,1}_\nu} j_4 (e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1)^{n-1} \|\phi\|^{j_3}_{\mathcal{F}^{0,1}_\nu} \tag{2553}$$

$$+ \|\phi'\|_{\mathcal{F}^{0,1}_\nu} e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}} \sum_{j_3+j_4+n\geq 2} (n+1)\frac{nC_{n+1}}{2j_3!j_4!} b(n+2,s) b(j_3,s) \tag{2554}$$

$$\cdot \|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu} \|\phi\|^{j_3-1}_{\mathcal{F}^{0,1}_\nu} j_3 (e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1)^{n-1} \|\phi\|^{j_4}_{\mathcal{F}^{0,1}_\nu}. \tag{2555}$$

Next,

$$\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} B^1_{3,2}(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}^{s,1}_\nu} \tag{2556}$$

$$\leq \sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}} (n+1)\frac{j_2 C_{n+1}}{2j_1!j_2!j_3!j_4!} \tag{2557}$$

$$\cdot \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |\mathcal{F}(\phi')| * |Q| * \cdots * |Q| * \left|\mathcal{F}(\phi^{j_4})\right| * \left|\mathcal{F}(\phi^{j_3})\right| \right\|_{l^s_\nu} \tag{2558}$$

$$\leq \|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu} \|\phi'\|_{\mathcal{F}^{0,1}_\nu} \sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}} (n+1)\frac{j_2 C_{n+1}}{2j_1!j_2!j_3!j_4!} b(j_1+j_2+n+2,s)(j_1+j_2-1) \tag{2559}$$

$$\cdot \|\phi\|^{j_1+j_2+j_3+j_4-2}_{\mathcal{F}^{0,1}_\nu} (e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1)^n \tag{2560}$$

$$+ \|\phi'\|_{\dot{\mathcal{F}}^{s,1}_\nu} \sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}} (n+1)\frac{j_2 C_{n+1}}{2j_1!j_2!j_3!j_4!} b(j_1+j_2+n+2,s) \tag{2561}$$

$$\cdot \|\phi\|^{j_1+j_2+j_3+j_4-1}_{\mathcal{F}^{0,1}_\nu} (e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1)^n \tag{2562}$$

$$+ \|\phi'\|_{\mathcal{F}^{0,1}_\nu} \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|^{m-1}_{\mathcal{F}^{0,1}_\nu}\right) \|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu} \tag{2563}$$

$$\cdot \sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}} (n+1)\frac{j_2 C_{n+1}}{2j_1!j_2!j_3!j_4!} b(j_1+j_2+n+2,s)n \tag{2564}$$

$$\cdot \|\phi\|^{j_1+j_2+j_3+j_4-1}_{\mathcal{F}^{0,1}_\nu} (e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1)^{n-1} \tag{2565}$$

$$+ \|\phi'\|_{\mathcal{F}^{0,1}_\nu} \sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}} (n+1)\frac{j_2 C_{n+1}}{2j_1!j_2!j_3!j_4!} b(j_1+j_2+n+2,s) b(j_4,s) \tag{2566}$$

$$\cdot \|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu} \|\phi\|^{j_4-1}_{\mathcal{F}^{0,1}_\nu} j_4 \|\phi\|^{j_1+j_2+j_3-1}_{\mathcal{F}^{0,1}_\nu} (e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1)^n \tag{2567}$$

$$+ \|\phi'\|_{\mathcal{F}^{0,1}_\nu} \sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}} (n+1)\frac{j_2 C_{n+1}}{2j_1!j_2!j_3!j_4!} b(j_1+j_2+n+2,s) b(j_3,s) \tag{2568}$$

$$\cdot \|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu} \|\phi\|^{j_3-1}_{\mathcal{F}^{0,1}_\nu} j_3 \|\phi\|^{j_1+j_2+j_4-1}_{\mathcal{F}^{0,1}_\nu} (e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1)^n. \tag{2569}$$

Moreover,

$$\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} B_{3,2}^2(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \tag{2570}$$

$$\leq \sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}} (n+1)\frac{j_2 C_{n+1}}{2j_1!j_2!j_3!j_4!} \tag{2571}$$

$$\cdot \left\||\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |\mathcal{F}(\phi')| * |Q| * \cdots * |Q| * \left|\mathcal{F}(\phi^{j_4})\right| * \left|\mathcal{F}(\phi^{j_3})\right|\right\|_{l_\nu^s} \tag{2572}$$

$$\leq \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}} (n+1)\frac{j_2 C_{n+1}}{2j_1!j_2!j_3!j_4!} b(j_1+j_2+n+2,s)(j_1+j_2-1) \tag{2573}$$

$$\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3+j_4-2} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n \tag{2574}$$

$$+ \|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}} \sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}} (n+1)\frac{j_2 C_{n+1}}{2j_1!j_2!j_3!j_4!} b(j_1+j_2+n+2,s) \tag{2575}$$

$$\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3+j_4-1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n \tag{2576}$$

$$+ \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1}\right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \tag{2577}$$

$$\cdot \sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}} (n+1)\frac{j_2 C_{n+1}}{2j_1!j_2!j_3!j_4!} b(j_1+j_2+n+2,s)n \tag{2578}$$

$$\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3+j_4-1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-1} \tag{2579}$$

$$+ \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}} (n+1)\frac{j_2 C_{n+1}}{2j_1!j_2!j_3!j_4!} b(j_1+j_2+n+2,s) b(j_4,s) \tag{2580}$$

$$\cdot \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_4-1} j_4 \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n \tag{2581}$$

$$+ \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}} (n+1)\frac{j_2 C_{n+1}}{2j_1!j_2!j_3!j_4!} b(j_1+j_2+n+2,s) b(j_3,s) \tag{2582}$$

$$\cdot \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1} j_3 \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_4-1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n. \tag{2583}$$

Moreover,

$$\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B_{3,2}^3(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \tag{2584}$$

$$\leq \sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}} (n+1) \frac{j_3 C_{n+1}}{2 j_1! j_2! j_3! j_4!} \tag{2585}$$

$$\cdot \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |Q| * \cdots * |Q| * \left|\mathcal{F}(\phi^{j_4})\right| * \left|\mathcal{F}(\phi^{j_3-1}\phi')\right| \right\|_{l_\nu^s} \tag{2586}$$

$$\leq \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}} (n+1) \frac{j_3 C_{n+1}}{2 j_1! j_2! j_3! j_4!} b(j_1+j_2+n+2,s)(j_1+j_2) \tag{2587}$$

$$\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3+j_4-2} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n \tag{2588}$$

$$+ \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1} \right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \tag{2589}$$

$$\cdot \sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}} n(n+1) \frac{j_3 C_{n+1}}{2 j_1! j_2! j_3! j_4!} b(j_1+j_2+n+2,s) \tag{2590}$$

$$\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3+j_4-1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^{n-1} \tag{2591}$$

$$+ \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}} (n+1) \frac{j_3 C_{n+1}}{2 j_1! j_2! j_3! j_4!} b(j_1+j_2+n+2,s) b(j_4,s) \tag{2592}$$

$$\cdot \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_4-1} j_4 \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n \tag{2593}$$

$$+ \sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}} (n+1) \frac{j_3 C_{n+1}}{2 j_1! j_2! j_3! j_4!} b(j_1+j_2+n+2,s) b(j_3,s) \tag{2594}$$

$$\cdot \left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-2} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} (j_3-1) + \|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1} \right) \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_4} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n. \tag{2595}$$

Moreover,

$$\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B^4_{3,2}(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}^{s,1}_\nu} \tag{2596}$$

$$\leq \sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}} (n+1) \frac{j_4 C_{n+1}}{2 j_1! j_2! j_3! j_4!} \tag{2597}$$

$$\cdot \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |Q| * \cdots * |Q| * \left|\mathcal{F}(\phi^{j_3})\right| * \left|\mathcal{F}(\phi^{j_4-1}\phi')\right| \right\|_{l^s_\nu} \tag{2598}$$

$$\leq \|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu} \|\phi'\|_{\mathcal{F}^{0,1}_\nu} \sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}} (n+1) \frac{j_4 C_{n+1}}{2 j_1! j_2! j_3! j_4!} b(j_1+j_2+n+2,s)(j_1+j_2) \tag{2599}$$

$$\cdot \|\phi\|^{j_1+j_2+j_3+j_4-2}_{\mathcal{F}^{0,1}_\nu} (e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}} - 1)^n \tag{2600}$$

$$+ \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|^{m-1}_{\mathcal{F}^{0,1}_\nu} \right) \|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu} \|\phi'\|_{\mathcal{F}^{0,1}_\nu} \tag{2601}$$

$$\cdot \sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}} n(n+1) \frac{j_4 C_{n+1}}{2 j_1! j_2! j_3! j_4!} b(j_1+j_2+n+2,s) \tag{2602}$$

$$\cdot \|\phi\|^{j_1+j_2+j_3+j_4-1}_{\mathcal{F}^{0,1}_\nu} (e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}} - 1)^{n-1} \tag{2603}$$

$$+ \|\phi'\|_{\mathcal{F}^{0,1}_\nu} \sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}} (n+1) \frac{j_4 C_{n+1}}{2 j_1! j_2! j_3! j_4!} b(j_1+j_2+n+2,s) b(j_3,s) \tag{2604}$$

$$\cdot \|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu} \|\phi\|^{j_3-1}_{\mathcal{F}^{0,1}_\nu} j_3 \|\phi\|^{j_1+j_2+j_4-1}_{\mathcal{F}^{0,1}_\nu} (e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}} - 1)^n \tag{2605}$$

$$+ \sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}} (n+1) \frac{j_4 C_{n+1}}{2 j_1! j_2! j_3! j_4!} b(j_1+j_2+n+2,s) b(j_4,s) \tag{2606}$$

$$\cdot \left(\|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu} \|\phi\|^{j_4-2}_{\mathcal{F}^{0,1}_\nu} \|\phi'\|_{\mathcal{F}^{0,1}_\nu} (j_4-1) + \|\phi'\|_{\dot{\mathcal{F}}^{s,1}_\nu} \|\phi\|^{j_4-1}_{\mathcal{F}^{0,1}_\nu} \right) \|\phi\|^{j_1+j_2+j_3}_{\mathcal{F}^{0,1}_\nu} (e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}} - 1)^n. \tag{2607}$$

Lastly,

$$\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B_{3,2}^5(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \tag{2608}$$

$$\leq \sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}} (n+1) \frac{nC_{n+1}}{2j_1!j_2!j_3!j_4!} \tag{2609}$$

$$\cdot \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |Q| * \cdots * |Q| * |S| * \left|\mathcal{F}(\phi^{j_4})\right| * \left|\mathcal{F}(\phi^{j_3})\right| \right\|_{l_\nu^s} \tag{2610}$$

$$\leq \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} \sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}} (n+1) \frac{nC_{n+1}}{2j_1!j_2!j_3!j_4!} b(j_1+j_2+n+2,s)(j_1+j_2) \tag{2611}$$

$$\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3+j_4-1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^{n-1} \tag{2612}$$

$$+ \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1} \right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} \tag{2613}$$

$$\cdot \sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}} (n+1) \frac{nC_{n+1}}{2j_1!j_2!j_3!j_4!} b(j_1+j_2+n+2,s)(n-1) \tag{2614}$$

$$\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3+j_4} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^{n-2} \tag{2615}$$

$$+ \left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \sum_{m=2}^{\infty} \frac{b(m,s)}{(m-2)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-2} + \|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}} \sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1} \right) \tag{2616}$$

$$\cdot \sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}} (n+1) \frac{nC_{n+1}}{2j_1!j_2!j_3!j_4!} b(j_1+j_2+n+2,s) \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3+j_4} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^{n-1} \tag{2617}$$

$$+ \|\phi'\|_{\mathcal{F}_\nu^{0,1}} e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} \sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}} (n+1) \frac{nC_{n+1}}{2j_1!j_2!j_3!j_4!} b(j_1+j_2+n+2,s) b(j_4,s) \tag{2618}$$

$$\cdot \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_4-1} j_4 \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^{n-1} \tag{2619}$$

$$+ \|\phi'\|_{\mathcal{F}_\nu^{0,1}} e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} \sum_{\substack{j_1+j_2\geq 1\\ j_3+j_4+n\geq 1}} (n+1) \frac{nC_{n+1}}{2j_1!j_2!j_3!j_4!} b(j_1+j_2+n+2,s) b(j_3,s) \tag{2620}$$

$$\cdot \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1} j_3 \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_4} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^{n-1}. \tag{2621}$$

Next,

$$\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} (\widetilde{B_{3,3}})_\alpha(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \tag{2622}$$

$$\leq C_2 \left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \sum_{m=2}^{\infty} \frac{b(m,s)(m-1)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-2} + \|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}} \sum_{m=2}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1} \right). \tag{2623}$$

Next,

$$\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} B_4^1(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \tag{2624}$$

$$\leq \sum_{j_1+j_2+j_3+j_4+n\geq 1}(n+1)\frac{{}_{j_1}C_{n+1}}{2j_1!j_2!j_3!j_4!} \tag{2625}$$

$$\cdot \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |\mathcal{F}(\phi')| * |Q| * \cdots * |Q| * \left|\mathcal{F}(\phi^{j_4}\phi')\right| * \left|\mathcal{F}(\phi^{j_3})\right|\right\|_{l_\nu^s} \tag{2626}$$

$$\leq \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|^2_{\mathcal{F}_\nu^{0,1}} \sum_{j_1+j_2+j_3+j_4+n\geq 1}(n+1)\frac{{}_{j_1}C_{n+1}}{2j_1!j_2!j_3!j_4!}b(j_1+j_2+n+2,s)(j_1+j_2-1) \tag{2627}$$

$$\cdot \|\phi\|^{j_1+j_2+j_3+j_4-2}_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n \tag{2628}$$

$$+ \|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \sum_{j_1+j_2+j_3+j_4+n\geq 1}(n+1)\frac{{}_{j_1}C_{n+1}}{2j_1!j_2!j_3!j_4!}b(j_1+j_2+n+2,s) \tag{2629}$$

$$\cdot \|\phi\|^{j_1+j_2+j_3+j_4-1}_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n \tag{2630}$$

$$+ \left(\sum_{m=1}^{\infty}\frac{b(m,s)}{(m-1)!}\|\phi\|^{m-1}_{\mathcal{F}_\nu^{0,1}}\right)\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|^2_{\mathcal{F}_\nu^{0,1}} \tag{2631}$$

$$\cdot \sum_{j_1+j_2+j_3+j_4+n\geq 1}(n+1)\frac{{}_{j_1}C_{n+1}}{2j_1!j_2!j_3!j_4!}b(j_1+j_2+n+2,s)n \tag{2632}$$

$$\cdot \|\phi\|^{j_1+j_2+j_3+j_4-1}_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-1} \tag{2633}$$

$$+ \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \sum_{j_1+j_2+j_3+j_4+n\geq 1}(n+1)\frac{{}_{j_1}C_{n+1}}{2j_1!j_2!j_3!j_4!}b(j_1+j_2+n+2,s)b(j_4+1,s) \tag{2634}$$

$$\cdot \left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|^{j_4-1}_{\mathcal{F}_\nu^{0,1}} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} j_4 + \|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|^{j_4}_{\mathcal{F}_\nu^{0,1}}\right) \|\phi\|^{j_1+j_2+j_3-1}_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n \tag{2635}$$

$$+ \|\phi'\|^2_{\mathcal{F}_\nu^{0,1}} \sum_{j_1+j_2+j_3+j_4+n\geq 1}(n+1)\frac{{}_{j_1}C_{n+1}}{2j_1!j_2!j_3!j_4!}b(j_1+j_2+n+2,s)b(j_3,s) \tag{2636}$$

$$\cdot \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|^{j_3-1}_{\mathcal{F}_\nu^{0,1}} j_3 \|\phi\|^{j_1+j_2+j_4-1}_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n. \tag{2637}$$

Moreover,

$$\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B_4^2(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \tag{2638}$$

$$\leq \sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1) \frac{j_2 C_{n+1}}{2 j_1! j_2! j_3! j_4!} \tag{2639}$$

$$\cdot \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |\mathcal{F}(\phi')| * |Q| * \cdots * |Q| * \left|\mathcal{F}(\phi^{j_4}\phi')\right| * \left|\mathcal{F}(\phi^{j_3})\right| \right\|_{l_\nu^s} \tag{2640}$$

$$\leq \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|^2_{\mathcal{F}_\nu^{0,1}} \sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1) \frac{j_2 C_{n+1}}{2 j_1! j_2! j_3! j_4!} b(j_1+j_2+n+2,s)(j_1+j_2-1) \tag{2641}$$

$$\cdot \|\phi\|^{j_1+j_2+j_3+j_4-2}_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n \tag{2642}$$

$$+ \|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1) \frac{j_2 C_{n+1}}{2 j_1! j_2! j_3! j_4!} b(j_1+j_2+n+2,s) \tag{2643}$$

$$\cdot \|\phi\|^{j_1+j_2+j_3+j_4-1}_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n \tag{2644}$$

$$+ \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|^{m-1}_{\mathcal{F}_\nu^{0,1}} \right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|^2_{\mathcal{F}_\nu^{0,1}} \tag{2645}$$

$$\cdot \sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1) \frac{j_2 C_{n+1}}{2 j_1! j_2! j_3! j_4!} b(j_1+j_2+n+2,s) n \tag{2646}$$

$$\cdot \|\phi\|^{j_1+j_2+j_3+j_4-1}_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^{n-1} \tag{2647}$$

$$+ \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1) \frac{j_2 C_{n+1}}{2 j_1! j_2! j_3! j_4!} b(j_1+j_2+n+2,s) b(j_4+1,s) \tag{2648}$$

$$\cdot \left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|^{j_4-1}_{\mathcal{F}_\nu^{0,1}} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} j_4 + \|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|^{j_4}_{\mathcal{F}_\nu^{0,1}} \right) \|\phi\|^{j_1+j_2+j_3-1}_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n \tag{2649}$$

$$+ \|\phi'\|^2_{\mathcal{F}_\nu^{0,1}} \sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1) \frac{j_2 C_{n+1}}{2 j_1! j_2! j_3! j_4!} b(j_1+j_2+n+2,s) b(j_3,s) \tag{2650}$$

$$\cdot \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|^{j_3-1}_{\mathcal{F}_\nu^{0,1}} j_3 \|\phi\|^{j_1+j_2+j_4-1}_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n. \tag{2651}$$

Moreover,

$$\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B_4^3(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \tag{2652}$$

$$\leq \sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1) \frac{j_3 C_{n+1}}{2 j_1! j_2! j_3! j_4!} \tag{2653}$$

$$\cdot \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |Q| * \cdots * |Q| * \left|\mathcal{F}(\phi^{j_4}\phi')\right| * \left|\mathcal{F}(\phi^{j_3-1}\phi')\right| \right\|_{l_\nu^s} \tag{2654}$$

$$\leq \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|^2_{\mathcal{F}_\nu^{0,1}} \sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1) \frac{j_3 C_{n+1}}{2 j_1! j_2! j_3! j_4!} b(j_1+j_2+n+2,s)(j_1+j_2) \tag{2655}$$

$$\cdot \|\phi\|^{j_1+j_2+j_3+j_4-2}_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n \tag{2656}$$

$$+ \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|^{m-1}_{\mathcal{F}_\nu^{0,1}} \right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|^2_{\mathcal{F}_\nu^{0,1}} \tag{2657}$$

$$\cdot \sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1) \frac{j_3 C_{n+1}}{2 j_1! j_2! j_3! j_4!} b(j_1+j_2+n+2,s) n \tag{2658}$$

$$\cdot \|\phi\|^{j_1+j_2+j_3+j_4-1}_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^{n-1} \tag{2659}$$

$$+ \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1) \frac{j_3 C_{n+1}}{2 j_1! j_2! j_3! j_4!} b(j_1+j_2+n+2,s) b(j_4+1,s) \tag{2660}$$

$$\cdot \left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|^{j_4-1}_{\mathcal{F}_\nu^{0,1}} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} j_4 + \|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|^{j_4}_{\mathcal{F}_\nu^{0,1}} \right) \|\phi\|^{j_1+j_2+j_3-1}_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n \tag{2661}$$

$$+ \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1) \frac{j_3 C_{n+1}}{2 j_1! j_2! j_3! j_4!} b(j_1+j_2+n+2,s) b(j_3,s) \tag{2662}$$

$$\cdot \left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|^{j_3-2}_{\mathcal{F}_\nu^{0,1}} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} (j_3-1) + \|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|^{j_3-1}_{\mathcal{F}_\nu^{0,1}} \right) \|\phi\|^{j_1+j_2+j_4}_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n. \tag{2663}$$

Moreover,

$$\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B_4^4(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \tag{2664}$$

$$\leq \sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1) \frac{j_4 C_{n+1}}{2 j_1! j_2! j_3! j_4!} \tag{2665}$$

$$\cdot \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |Q| * \cdots * |Q| * \left|\mathcal{F}(\phi^{j_4-1}\phi'^2)\right| * \left|\mathcal{F}(\phi^{j_3})\right| \right\|_{l_\nu^s} \tag{2666}$$

$$\leq \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|^2_{\mathcal{F}_\nu^{0,1}} \sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1) \frac{j_4 C_{n+1}}{2 j_1! j_2! j_3! j_4!} b(j_1+j_2+n+2, s)(j_1+j_2) \tag{2667}$$

$$\cdot \|\phi\|^{j_1+j_2+j_3+j_4-2}_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n \tag{2668}$$

$$+ \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|^{m-1}_{\mathcal{F}_\nu^{0,1}} \right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|^2_{\mathcal{F}_\nu^{0,1}} \tag{2669}$$

$$\cdot \sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1) \frac{j_4 C_{n+1}}{2 j_1! j_2! j_3! j_4!} b(j_1+j_2+n+2, s) n \tag{2670}$$

$$\cdot \|\phi\|^{j_1+j_2+j_3+j_4-1}_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^{n-1} \tag{2671}$$

$$+ \sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1) \frac{j_4 C_{n+1}}{2 j_1! j_2! j_3! j_4!} b(j_1+j_2+n+2, s) b(j_4+1, s) \tag{2672}$$

$$\cdot \left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|^{j_4-2}_{\mathcal{F}_\nu^{0,1}} \|\phi'\|^2_{\mathcal{F}_\nu^{0,1}} (j_4-1) + \|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|^{j_4-1}_{\mathcal{F}_\nu^{0,1}} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \cdot 2 \right) \tag{2673}$$

$$\cdot \|\phi\|^{j_1+j_2+j_3}_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n \tag{2674}$$

$$+ \|\phi'\|^2_{\mathcal{F}_\nu^{0,1}} \sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1) \frac{j_4 C_{n+1}}{2 j_1! j_2! j_3! j_4!} b(j_1+j_2+n+2, s) b(j_3, s) \tag{2675}$$

$$\cdot \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|^{j_3-1}_{\mathcal{F}_\nu^{0,1}} j_3 \|\phi\|^{j_1+j_2+j_4-1}_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n. \tag{2676}$$

Lastly,

$$\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B_4^6(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \tag{2677}$$

$$\leq \sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1) \frac{nC_{n+1}}{2j_1!j_2!j_3!j_4!} \tag{2678}$$

$$\cdot \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |Q| * \cdots * |Q| * |S| * \left|\mathcal{F}(\phi^{j_4}\phi')\right| * \left|\mathcal{F}(\phi^{j_3})\right| \right\|_{l_\nu^s} \tag{2679}$$

$$\leq \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|_{\mathcal{F}_\nu^{0,1}}^2 e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} \tag{2680}$$

$$\cdot \sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1) \frac{nC_{n+1}}{2j_1!j_2!j_3!j_4!} b(j_1+j_2+n+2,s)(j_1+j_2) \tag{2681}$$

$$\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3+j_4-1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^{n-1} \tag{2682}$$

$$+ \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1} \right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|_{\mathcal{F}_\nu^{0,1}}^2 e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} \tag{2683}$$

$$\cdot \sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1) \frac{nC_{n+1}}{2j_1!j_2!j_3!j_4!} b(j_1+j_2+n+2,s)(n-1) \tag{2684}$$

$$\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3+j_4} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^{n-2} \tag{2685}$$

$$+ \left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \sum_{m=2}^{\infty} \frac{b(m,s)}{(m-2)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-2} + \|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}} \sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1} \right) \tag{2686}$$

$$\cdot \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1) \frac{nC_{n+1}}{2j_1!j_2!k_3!j_4!} b(j_1+j_2+n+2,s) \tag{2687}$$

$$\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3+j_4} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^{n-1} \tag{2688}$$

$$+ \|\phi'\|_{\mathcal{F}_\nu^{0,1}} e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} \tag{2689}$$

$$\cdot \sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1) \frac{nC_{n+1}}{2j_1!j_2!j_3!j_4!} b(j_1+j_2+n+2,s) b(j_4+1,s) \tag{2690}$$

$$\cdot \left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_4-1} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} j_4 + \|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_4} \right) \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^{n-1} \tag{2691}$$

$$+ \|\phi'\|_{\mathcal{F}_\nu^{0,1}}^2 e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} \sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1) \frac{nC_{n+1}}{2j_1!j_2!j_3!j_4!} b(j_1+j_2+n+2,s) b(j_3,s) \tag{2692}$$

$$\cdot \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1} j_3 \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_4} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^{n-1}. \tag{2693}$$

Next,

$$\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B_{5,1}^1(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \tag{2694}$$

$$\leq \sum_{j_3+n\geq 2} \frac{j_3 C_n}{2 j_3!} \left\| |Q| * \cdots * |Q| * \left|\mathcal{F}(\phi^{j_3-1}\phi')\right| \right\|_{l_\nu^s} \tag{2695}$$

$$\leq \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1} \right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \tag{2696}$$

$$\cdot \sum_{j_3+n\geq 2} \frac{j_3 C_n}{2 j_3!} b(n+1,s) n \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^{n-1} \tag{2697}$$

$$+ \sum_{j_3+n\geq 2} \frac{j_3 C_n}{2 j_3!} b(n+1,s) b(j_3,s) \tag{2698}$$

$$\cdot \left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-2} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} (j_3 - 1) + \|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1} \right) \tag{2699}$$

$$\cdot (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n. \tag{2700}$$

Moreover,

$$\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B_{5,1}^2(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \tag{2701}$$

$$\leq \sum_{j_3+n\geq 2} \frac{n C_n}{2 j_3!} \left\| |Q| * \cdots * |Q| * |S| * \left|\mathcal{F}(\phi^{j_3})\right| \right\|_{l_\nu^s} \tag{2702}$$

$$\leq \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1} \right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} \tag{2703}$$

$$\cdot \sum_{j_3+n\geq 2} \frac{n C_n}{2 j_3!} b(n+1,s)(n-1) \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^{n-2} \tag{2704}$$

$$+ \left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \sum_{m=2}^{\infty} \frac{b(m,s)}{(m-2)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-2} + \|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}} \sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1} \right) \tag{2705}$$

$$\cdot \sum_{j_3+n\geq 2} \frac{n C_n}{2 j_3!} b(n+1,s) \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^{n-1} \tag{2706}$$

$$+ \|\phi'\|_{\mathcal{F}_\nu^{0,1}} e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} \sum_{j_3+n\geq 2} \frac{n C_n}{2 j_3!} b(n+1,s) b(j_3,s) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1} j_3 (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^{n-1}. \tag{2707}$$

Next,

$$\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} B_{5,2}^1(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \tag{2708}$$

$$\leq \sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{j_1C_n}{2j_1!j_2!j_3!}\left\||\mathcal{F}(\phi)|*\cdots*|\mathcal{F}(\phi)|*|\mathcal{F}(\phi')|*|Q|*\cdots*|Q|*\left|\mathcal{F}(\phi^{j_3})\right|\right\|_{l_\nu^s} \tag{2709}$$

$$\leq \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi'\|_{\mathcal{F}_\nu^{0,1}} \tag{2710}$$

$$\cdot \sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{j_1C_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s)(j_1+j_2-1)\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-2}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n \tag{2711}$$

$$+\|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}} \sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{j_1C_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s)\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-1}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n \tag{2712}$$

$$+\left(\sum_{m=1}^{\infty}\frac{b(m,s)}{(m-1)!}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1}\right)\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi'\|_{\mathcal{F}_\nu^{0,1}} \tag{2713}$$

$$\cdot \sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{j_1C_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s)n\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-1}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-1} \tag{2714}$$

$$+\|\phi'\|_{\mathcal{F}_\nu^{0,1}} \sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{j_1C_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s)b(j_3,s) \tag{2715}$$

$$\cdot \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1} j_3\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n. \tag{2716}$$

Moreover,

$$\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} B_{5,2}^2(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \tag{2717}$$

$$\leq \sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{j_2 C_n}{2j_1!j_2!j_3!}\left\||\mathcal{F}(\phi)|*\cdots*|\mathcal{F}(\phi)|*|\mathcal{F}(\phi')|*|Q|*\cdots*|Q|*\left|\mathcal{F}(\phi^{j_3})\right|\right\|_{l_\nu^s} \tag{2718}$$

$$\leq \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi'\|_{\mathcal{F}_\nu^{0,1}} \tag{2719}$$

$$\cdot\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{j_2 C_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s)(j_1+j_2-1)\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-2}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n \tag{2720}$$

$$+\|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}}\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{j_2 C_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s)\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-1}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n \tag{2721}$$

$$+\left(\sum_{m=1}^{\infty}\frac{b(m,s)}{(m-1)!}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1}\right)\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi'\|_{\mathcal{F}_\nu^{0,1}} \tag{2722}$$

$$\cdot\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{j_2 C_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s)n\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-1}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-1} \tag{2723}$$

$$+\|\phi'\|_{\mathcal{F}_\nu^{0,1}}\sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{j_2 C_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s)b(j_3,s) \tag{2724}$$

$$\cdot\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1} j_3\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n. \tag{2725}$$

Moreover,

$$\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B^3_{5,2}(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}^{s,1}_\nu} \tag{2726}$$

$$\leq \sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{j_3 C_n}{2 j_1! j_2! j_3!} \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |Q| * \cdots * |Q| * \left|\mathcal{F}(\phi^{j_3-1}\phi')\right| \right\|_{l^s_\nu} \tag{2727}$$

$$\leq \|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu} \|\phi'\|_{\mathcal{F}^{0,1}_\nu} \tag{2728}$$

$$\cdot \sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{j_3 C_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1,s)(j_1+j_2) \|\phi\|^{j_1+j_2+j_3-2}_{\mathcal{F}^{0,1}_\nu} (e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1)^n \tag{2729}$$

$$+ \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|^{m-1}_{\mathcal{F}^{0,1}_\nu} \right) \|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu} \|\phi'\|_{\mathcal{F}^{0,1}_\nu} \sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{j_3 C_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1,s) n \tag{2730}$$

$$\cdot \|\phi\|^{j_1+j_2+j_3-1}_{\mathcal{F}^{0,1}_\nu} (e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1)^{n-1} \tag{2731}$$

$$+ \sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{j_3 C_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1,s) b(j_3,s) \tag{2732}$$

$$\cdot \left(\|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu} \|\phi\|^{j_3-2}_{\mathcal{F}^{0,1}_\nu} \|\phi'\|_{\mathcal{F}^{0,1}_\nu} (j_3-1) + \|\phi'\|_{\dot{\mathcal{F}}^{s,1}_\nu} \|\phi\|^{j_3-1}_{\mathcal{F}^{0,1}_\nu} \right) \|\phi\|^{j_1+j_2}_{\mathcal{F}^{0,1}_\nu} (e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1)^n. \tag{2733}$$

Lastly,

$$\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B^4_{5,2}(\alpha,\beta)d\beta \right\|_{\dot{\mathcal{F}}^{s,1}_\nu} \tag{2734}$$

$$\leq \sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{nC_n}{2j_1!j_2!j_3!} \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |Q| * \cdots * |Q| * |S| * \left|\mathcal{F}(\phi^{j_3})\right| \right\|_{l^s_\nu} \tag{2735}$$

$$\leq \|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu} \|\phi'\|_{\mathcal{F}^{0,1}_\nu} e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}} \sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{nC_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s)(j_1+j_2) \tag{2736}$$

$$\cdot \|\phi\|^{j_1+j_2+j_3-1}_{\mathcal{F}^{0,1}_\nu} (e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}} - 1)^{n-1} \tag{2737}$$

$$+ \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|^{m-1}_{\mathcal{F}^{0,1}_\nu} \right) \|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu} \|\phi'\|_{\mathcal{F}^{0,1}_\nu} e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}} \tag{2738}$$

$$\cdot \sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{nC_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s)(n-1) \|\phi\|^{j_1+j_2+j_3}_{\mathcal{F}^{0,1}_\nu} (e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}} - 1)^{n-2} \tag{2739}$$

$$+ \left(\|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu} \|\phi'\|_{\mathcal{F}^{0,1}_\nu} \sum_{m=2}^{\infty} \frac{b(m,s)}{(m-2)!} \|\phi\|^{m-2}_{\mathcal{F}^{0,1}_\nu} + \|\phi'\|_{\dot{\mathcal{F}}^{s,1}_\nu} \sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|^{m-1}_{\mathcal{F}^{0,1}_\nu} \right) \tag{2740}$$

$$\cdot \sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{nC_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s) \|\phi\|^{j_1+j_2+j_3}_{\mathcal{F}^{0,1}_\nu} (e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}} - 1)^{n-1} \tag{2741}$$

$$+ \|\phi'\|_{\mathcal{F}^{0,1}_\nu} e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}} \sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{nC_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s) b(j_3,s) \tag{2742}$$

$$\cdot \|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu} \|\phi\|^{j_3-1}_{\mathcal{F}^{0,1}_\nu} j_3 \|\phi\|^{j_1+j_2}_{\mathcal{F}^{0,1}_\nu} (e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}} - 1)^{n-1}. \tag{2743}$$

Next,

$$\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} (\widetilde{B_{5,3}})_\alpha(\alpha,\beta)d\beta \right\|_{\dot{\mathcal{F}}^{s,1}_\nu} \tag{2744}$$

$$\leq \frac{1}{2} C_1 \Bigg(\|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu} \|\phi'\|_{\mathcal{F}^{0,1}_\nu} \sum_{m=2}^{\infty} \frac{b(m,s)(m-1)}{(m-1)!} \|\phi\|^{m-2}_{\mathcal{F}^{0,1}_\nu} \tag{2745}$$

$$+ \|\phi'\|_{\dot{\mathcal{F}}^{s,1}_\nu} \sum_{m=2}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|^{m-1}_{\mathcal{F}^{0,1}_\nu} \Bigg). \tag{2746}$$

Next,

$$\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}B^1_{6,1}(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}^{s,1}_\nu} \tag{2747}$$
$$\leq \sum_{j_3+n\geq 2}\frac{j_3C_n}{2j_3!}\left\||Q|*\cdots*|Q|*\left|\mathcal{F}(\phi^{j_3-1}\phi')\right|\right\|_{l^s_\nu} \tag{2748}$$
$$\leq\left(\sum_{m=1}^{\infty}\frac{b(m,s)}{(m-1)!}\|\phi\|^{m-1}_{\mathcal{F}^{0,1}_\nu}\right)\|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu}\|\phi'\|_{\mathcal{F}^{0,1}_\nu} \tag{2749}$$
$$\cdot\sum_{j_3+n\geq 2}\frac{j_3C_n}{2j_3!}b(n+1,s)n\|\phi\|^{j_3-1}_{\mathcal{F}^{0,1}_\nu}(e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1)^{n-1} \tag{2750}$$
$$+\sum_{j_3+n\geq 2}\frac{j_3C_n}{2j_3!}b(n+1,s)b(j_3,s) \tag{2751}$$
$$\cdot\left(\|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu}\|\phi\|^{j_3-2}_{\mathcal{F}^{0,1}_\nu}\|\phi'\|_{\mathcal{F}^{0,1}_\nu}(j_3-1)+\|\phi'\|_{\dot{\mathcal{F}}^{s,1}_\nu}\|\phi\|^{j_3-1}_{\mathcal{F}^{0,1}_\nu}\right) \tag{2752}$$
$$\cdot(e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1)^n. \tag{2753}$$

Lastly,

$$\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}B^2_{6,1}(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}^{s,1}_\nu} \tag{2754}$$
$$\leq \sum_{j_3+n\geq 2}\frac{nC_n}{2j_3!}\left\||Q|*\cdots*|Q|*|S|*\left|\mathcal{F}(\phi^{j_3})\right|\right\|_{l^s_\nu} \tag{2755}$$
$$\leq\left(\sum_{m=1}^{\infty}\frac{b(m,s)}{(m-1)!}\|\phi\|^{m-1}_{\mathcal{F}^{0,1}_\nu}\right)\|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu}\|\phi'\|_{\mathcal{F}^{0,1}_\nu}e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}} \tag{2756}$$
$$\cdot\sum_{j_3+n\geq 2}\frac{nC_n}{2j_3!}b(n+1,s)(n-1)\|\phi\|^{j_3}_{\mathcal{F}^{0,1}_\nu}(e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1)^{n-2} \tag{2757}$$
$$+\left(\|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu}\|\phi'\|_{\mathcal{F}^{0,1}_\nu}\sum_{m=2}^{\infty}\frac{b(m,s)}{(m-2)!}\|\phi\|^{m-2}_{\mathcal{F}^{0,1}_\nu}+\|\phi'\|_{\dot{\mathcal{F}}^{s,1}_\nu}\sum_{m=1}^{\infty}\frac{b(m,s)}{(m-1)!}\|\phi\|^{m-1}_{\mathcal{F}^{0,1}_\nu}\right) \tag{2758}$$
$$\cdot\sum_{j_3+n\geq 2}\frac{nC_n}{2j_3!}b(n+1,s)\|\phi\|^{j_3}_{\mathcal{F}^{0,1}_\nu}(e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1)^{n-1} \tag{2759}$$
$$+\|\phi'\|_{\mathcal{F}^{0,1}_\nu}e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}\sum_{j_3+n\geq 2}\frac{nC_n}{2j_3!}b(n+1,s)b(j_3,s)\|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu}\|\phi\|^{j_3-1}_{\mathcal{F}^{0,1}_\nu}j_3(e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1)^{n-1}. \tag{2760}$$

Next,

$$\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B^1_{6,2}(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}^{s,1}_\nu} \tag{2761}$$

$$\leq \sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{j_1 C_n}{2 j_1! j_2! j_3!} \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |\mathcal{F}(\phi')| * |Q| * \cdots * |Q| * \left|\mathcal{F}(\phi^{j_3})\right| \right\|_{l^s_\nu} \tag{2762}$$

$$\leq \|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu} \|\phi'\|_{\mathcal{F}^{0,1}_\nu} \tag{2763}$$

$$\cdot \sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{j_1 C_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1,s)(j_1+j_2-1) \|\phi\|^{j_1+j_2+j_3-2}_{\mathcal{F}^{0,1}_\nu} (e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1)^n \tag{2764}$$

$$+ \|\phi'\|_{\dot{\mathcal{F}}^{s,1}_\nu} \sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{j_1 C_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1,s) \|\phi\|^{j_1+j_2+j_3-1}_{\mathcal{F}^{0,1}_\nu} (e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1)^n \tag{2765}$$

$$+ \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|^{m-1}_{\mathcal{F}^{0,1}_\nu} \right) \|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu} \|\phi'\|_{\mathcal{F}^{0,1}_\nu} \tag{2766}$$

$$\cdot \sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{j_1 C_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1,s) n \|\phi\|^{j_1+j_2+j_3-1}_{\mathcal{F}^{0,1}_\nu} (e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1)^{n-1} \tag{2767}$$

$$+ \|\phi'\|_{\mathcal{F}^{0,1}_\nu} \sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{j_1 C_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1,s) b(j_3,s) \tag{2768}$$

$$\cdot \|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu} \|\phi\|^{j_3-1}_{\mathcal{F}^{0,1}_\nu} j_3 \|\phi\|^{j_1+j_2-1}_{\mathcal{F}^{0,1}_\nu} (e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1)^n. \tag{2769}$$

Moreover,

$$\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B_{6,2}^2(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \tag{2770}$$

$$\leq \sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{j_2 C_n}{2 j_1! j_2! j_3!} \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |\mathcal{F}(\phi')| * |Q| * \cdots * |Q| * \left|\mathcal{F}(\phi^{j_3})\right| \right\|_{l_\nu^s} \tag{2771}$$

$$\leq \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \tag{2772}$$

$$\cdot \sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{j_2 C_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1,s)(j_1+j_2-1) \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-2} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n \tag{2773}$$

$$+ \|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}} \sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{j_2 C_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1,s) \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n \tag{2774}$$

$$+ \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1} \right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \tag{2775}$$

$$\cdot \sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{j_2 C_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1,s) n \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-1} \tag{2776}$$

$$+ \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{j_2 C_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1,s) b(j_3,s) \tag{2777}$$

$$\cdot \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1} j_3 \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n. \tag{2778}$$

Moreover,

$$\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} B^3_{6,2}(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}^{s,1}_\nu} \tag{2779}$$

$$\leq \sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{j_3 C_n}{2 j_1! j_2! j_3!} \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |Q| * \cdots * |Q| * \left|\mathcal{F}(\phi^{j_3-1}\phi')\right| \right\|_{l^s_\nu} \tag{2780}$$

$$\leq \|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu} \|\phi'\|_{\mathcal{F}^{0,1}_\nu} \tag{2781}$$

$$\cdot \sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{j_3 C_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1,s)(j_1+j_2) \|\phi\|^{j_1+j_2+j_3-2}_{\mathcal{F}^{0,1}_\nu} (e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1)^n \tag{2782}$$

$$+ \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|^{m-1}_{\mathcal{F}^{0,1}_\nu}\right) \|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu} \|\phi'\|_{\mathcal{F}^{0,1}_\nu} \sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{j_3 C_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1,s) n \tag{2783}$$

$$\cdot \|\phi\|^{j_1+j_2+j_3-1}_{\mathcal{F}^{0,1}_\nu} (e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1)^n \tag{2784}$$

$$+ \sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{j_3 C_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1,s) b(j_3,s) \tag{2785}$$

$$\cdot \left(\|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu} \|\phi\|^{j_3-2}_{\mathcal{F}^{0,1}_\nu} \|\phi'\|_{\mathcal{F}^{0,1}_\nu} (j_3-1) + \|\phi'\|_{\dot{\mathcal{F}}^{s,1}_\nu} \|\phi\|^{j_3-1}_{\mathcal{F}^{0,1}_\nu} \right) \|\phi\|^{j_1+j_2}_{\mathcal{F}^{0,1}_\nu} (e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1)^n. \tag{2786}$$

Lastly,

$$\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} B^4_{6,2}(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}^{s,1}_\nu} \tag{2787}$$

$$\leq \sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{nC_n}{2j_1!j_2!j_3!}\left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |Q| * \cdots * |Q| * |S| * \left|\mathcal{F}(\phi^{j_3})\right|\right\|_{l^s_\nu} \tag{2788}$$

$$\leq \|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu}\|\phi'\|_{\mathcal{F}^{0,1}_\nu} e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}} \sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{nC_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s)(j_1+j_2) \tag{2789}$$

$$\cdot \|\phi\|^{j_1+j_2+j_3-1}_{\mathcal{F}^{0,1}_\nu}(e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1)^{n-1} \tag{2790}$$

$$+\left(\sum_{m=1}^{\infty}\frac{b(m,s)}{(m-1)!}\|\phi\|^{m-1}_{\mathcal{F}^{0,1}_\nu}\right)\|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu}\|\phi'\|_{\mathcal{F}^{0,1}_\nu} e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}} \tag{2791}$$

$$\cdot \sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{nC_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s)(n-1)\|\phi\|^{j_1+j_2+j_3}_{\mathcal{F}^{0,1}_\nu}(e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1)^{n-2} \tag{2792}$$

$$+\left(\|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu}\|\phi'\|_{\mathcal{F}^{0,1}_\nu}\sum_{m=2}^{\infty}\frac{b(m,s)}{(m-2)!}\|\phi\|^{m-2}_{\mathcal{F}^{0,1}_\nu}+\|\phi'\|_{\dot{\mathcal{F}}^{s,1}_\nu}\sum_{m=1}^{\infty}\frac{b(m,s)}{(m-1)!}\|\phi\|^{m-1}_{\mathcal{F}^{0,1}_\nu}\right) \tag{2793}$$

$$\cdot \sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{nC_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s)\|\phi\|^{j_1+j_2+j_3}_{\mathcal{F}^{0,1}_\nu}(e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1)^{n-1} \tag{2794}$$

$$+\|\phi'\|_{\mathcal{F}^{0,1}_\nu} e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}} \sum_{\substack{j_1+j_2\geq 1\\ j_3+n\geq 1}} \frac{nC_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s)b(j_3,s) \tag{2795}$$

$$\cdot \|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu}\|\phi\|^{j_3-1}_{\mathcal{F}^{0,1}_\nu} j_3\|\phi\|^{j_1+j_2}_{\mathcal{F}^{0,1}_\nu}(e^{\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1)^{n-1}. \tag{2796}$$

Next,

$$\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}(\widetilde{B_{6,3}})_\alpha(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}^{s,1}_\nu} \tag{2797}$$

$$\leq \frac{1}{2}C_1\left(\|\phi\|_{\dot{\mathcal{F}}^{s,1}_\nu}\|\phi'\|_{\mathcal{F}^{0,1}_\nu}\sum_{m=2}^{\infty}\frac{b(m,s)(m-1)}{(m-1)!}\|\phi\|^{m-2}_{\mathcal{F}^{0,1}_\nu}\right. \tag{2798}$$

$$\left.+\|\phi'\|_{\dot{\mathcal{F}}^{s,1}_\nu}\sum_{m=2}^{\infty}\frac{b(m,s)}{(m-1)!}\|\phi\|^{m-1}_{\mathcal{F}^{0,1}_\nu}\right). \tag{2799}$$

Next,

$$\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} B_7^1(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \tag{2800}$$

$$\leq \sum_{j_1+j_2+j_3+n\geq 1} \frac{j_1 C_n}{2j_1!j_2!j_3!} \tag{2801}$$

$$\cdot \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |\mathcal{F}(\phi')| * |Q| * \cdots * |Q| * \left|\mathcal{F}(\phi^{j_3}\phi')\right| \right\|_{l_\nu^s} \tag{2802}$$

$$\leq \sum_{j_1+j_2+j_3+n\geq 1} \frac{j_1 C_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s) \tag{2803}$$

$$\cdot \Bigg(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-2} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} (j_1+j_2-1) \tag{2804}$$

$$+ \|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \tag{2805}$$

$$+ \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1}\right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-1} \tag{2806}$$

$$\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \cdot n \tag{2807}$$

$$+ b(j_3+1,s)\left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} j_3 + \|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3} \right) \tag{2808}$$

$$\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n \Bigg). \tag{2809}$$

Moreover,

$$\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B_7^2(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \tag{2810}$$
$$\leq \sum_{j_1+j_2+j_3+n\geq 1} \frac{j_2 C_n}{2 j_1! j_2! j_3!} \tag{2811}$$
$$\cdot \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |\mathcal{F}(\phi')| * |Q| * \cdots * |Q| * \left|\mathcal{F}(\phi^{j_3}\phi')\right| \right\|_{l_\nu^s} \tag{2812}$$
$$\leq \sum_{j_1+j_2+j_3+n\geq 1} \frac{j_2 C_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1,s) \tag{2813}$$
$$\cdot \Bigg(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-2} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} (j_1+j_2-1) \tag{2814}$$
$$+ \|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \tag{2815}$$
$$+ \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1} \right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-1} \tag{2816}$$
$$\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \cdot n \tag{2817}$$
$$+ b(j_3+1,s) \left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} j_3 + \|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3} \right) \tag{2818}$$
$$\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n \Bigg). \tag{2819}$$

Moreover,

$$\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B_7^3(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \tag{2820}$$

$$\leq \sum_{j_1+j_2+j_3+n\geq 1} \frac{j_3 C_n}{2 j_1! j_2! j_3!} \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |Q| * \cdots * |Q| * \left|\mathcal{F}(\phi^{j_3-1}\phi'^2)\right| \right\|_{l_\nu^s} \tag{2821}$$

$$\leq \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|^2_{\mathcal{F}_\nu^{0,1}} \tag{2822}$$

$$\cdot \sum_{j_1+j_2+j_3+n\geq 1} \frac{j_3 C_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1,s)(j_1+j_2) \|\phi\|^{j_1+j_2+j_3-2}_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n \tag{2823}$$

$$+ \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|^{m-1}_{\mathcal{F}_\nu^{0,1}}\right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|^2_{\mathcal{F}_\nu^{0,1}} \tag{2824}$$

$$\cdot \sum_{j_1+j_2+j_3+n\geq 1} \frac{j_3 C_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1,s) n (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-1} \|\phi\|^{j_1+j_2+j_3-1}_{\mathcal{F}_\nu^{0,1}} \tag{2825}$$

$$+ \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|^2_{\mathcal{F}_\nu^{0,1}} \sum_{j_1+j_2+j_3+n\geq 1} \frac{j_3 C_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1,s) b(j_3+1,s)(j_3-1) \tag{2826}$$

$$\cdot \|\phi\|^{j_1+j_2+j_3-2}_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n \tag{2827}$$

$$+ \|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \sum_{j_1+j_2+j_3+n\geq 1} \frac{j_3 C_n}{j_1! j_2! j_3!} b(j_1+j_2+n+1,s) b(j_3+1,s) \tag{2828}$$

$$\cdot \|\phi\|^{j_1+j_2+j_3-1}_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n. \tag{2829}$$

Lastly,

$$\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} B_7^5(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \tag{2830}$$

$$\leq \sum_{j_1+j_2+j_3+n\geq 1} \frac{nC_n}{2j_1!j_2!j_3!} \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |Q| * \cdots * |Q| * |S| * \left|\mathcal{F}(\phi^{j_3}\phi')\right| \right\|_{l_\nu^s} \tag{2831}$$

$$\leq \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|_{\mathcal{F}_\nu^{0,1}}^2 e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} \sum_{j_1+j_2+j_3+n\geq 1} \frac{nC_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s)(j_1+j_2) \tag{2832}$$

$$\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-1} \tag{2833}$$

$$+ \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1}\right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|_{\mathcal{F}_\nu^{0,1}}^2 e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} \tag{2834}$$

$$\cdot \sum_{j_1+j_2+j_3+n\geq 1} \frac{nC_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s)(n-1) \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-2} \tag{2835}$$

$$+ \left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \sum_{m=2}^{\infty} \frac{b(m,s)}{(m-2)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-2} + \|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}} \sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1} \right) \tag{2836}$$

$$\cdot \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \sum_{j_1+j_2+j_3+n\geq 1} \frac{nC_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s) \tag{2837}$$

$$\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-1} \tag{2838}$$

$$+ \|\phi'\|_{\mathcal{F}_\nu^{0,1}} e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} \sum_{j_1+j_2+j_3+n\geq 1} \frac{nC_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s) b(j_3+1,s) \tag{2839}$$

$$\cdot \left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} j_3 + \|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3} \right) \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-1}. \tag{2840}$$

Next,

$$\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B_8^1(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \tag{2841}$$

$$\leq \sum_{j_1+j_2+j_3+n\geq 1} \frac{j_1 C_n}{2 j_1! j_2! j_3!} \tag{2842}$$

$$\cdot \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |\mathcal{F}(\phi')| * |Q| * \cdots * |Q| * \left|\mathcal{F}(\phi^{j_3}\phi')\right| \right\|_{l_\nu^s} \tag{2843}$$

$$\leq \sum_{j_1+j_2+j_3+n\geq 1} \frac{j_1 C_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1,s) \tag{2844}$$

$$\cdot \Bigg(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-2} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} (j_1+j_2-1) \tag{2845}$$

$$+ \|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \tag{2846}$$

$$+ \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1} \right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^{n-1} \tag{2847}$$

$$\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \cdot n \tag{2848}$$

$$+ b(j_3+1,s) \left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} j_3 + \|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3} \right) \tag{2849}$$

$$\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n \Bigg). \tag{2850}$$

Moreover,

$$\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B_8^2(\alpha, \beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \tag{2851}$$

$$\leq \sum_{j_1+j_2+j_3+n\geq 1} \frac{j_2 C_n}{2 j_1! j_2! j_3!} \tag{2852}$$

$$\cdot \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |\mathcal{F}(\phi')| * |Q| * \cdots * |Q| * \left|\mathcal{F}(\phi^{j_3}\phi')\right| \right\|_{l_\nu^s} \tag{2853}$$

$$\leq \sum_{j_1+j_2+j_3+n\geq 1} \frac{j_2 C_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1, s) \tag{2854}$$

$$\cdot \Bigg(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-2} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} (j_1+j_2-1) \tag{2855}$$

$$+ \|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \tag{2856}$$

$$+ \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1} \right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^{n-1} \tag{2857}$$

$$\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \cdot n \tag{2858}$$

$$+ b(j_3+1, s) \left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} j_3 + \|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3} \right) \tag{2859}$$

$$\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n \Bigg). \tag{2860}$$

Moreover,

$$\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B_8^3(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \tag{2861}$$

$$\leq \sum_{j_1+j_2+j_3+n\geq 1} \frac{j_3 C_n}{2j_1!j_2!j_3!} \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |Q| * \cdots * |Q| * \left|\mathcal{F}(\phi^{j_3-1}\phi'^2)\right| \right\|_{l_\nu^s} \tag{2862}$$

$$\leq \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|_{\mathcal{F}_\nu^{0,1}}^2 \tag{2863}$$

$$\cdot \sum_{j_1+j_2+j_3+n\geq 1} \frac{j_3 C_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s)(j_1+j_2) \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-2} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n \tag{2864}$$

$$+ \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1} \right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|_{\mathcal{F}_\nu^{0,1}}^2 \tag{2865}$$

$$\cdot \sum_{j_1+j_2+j_3+n\geq 1} \frac{j_3 C_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s) n (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-1} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-1} \tag{2866}$$

$$+ \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|_{\mathcal{F}_\nu^{0,1}}^2 \sum_{j_1+j_2+j_3+n\geq 1} \frac{j_3 C_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s) b(j_3+1,s)(j_3-1) \tag{2867}$$

$$\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-2} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n \tag{2868}$$

$$+ \|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \sum_{j_1+j_2+j_3+n\geq 1} \frac{j_3 C_n}{j_1!j_2!j_3!} b(j_1+j_2+n+1,s) b(j_3+1,s) \tag{2869}$$

$$\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n. \tag{2870}$$

Lastly,

$$\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B_8^5(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \tag{2871}$$

$$\leq \sum_{j_1+j_2+j_3+n\geq 1} \frac{nC_n}{2j_1!j_2!j_3!} \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |Q| * \cdots * |Q| * |S| * \left|\mathcal{F}(\phi^{j_3}\phi')\right| \right\|_{l_\nu^s} \tag{2872}$$

$$\leq \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|_{\mathcal{F}_\nu^{0,1}}^2 e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} \sum_{j_1+j_2+j_3+n\geq 1} \frac{nC_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s)(j_1+j_2) \tag{2873}$$

$$\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^{n-1} \tag{2874}$$

$$+ \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1}\right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|_{\mathcal{F}_\nu^{0,1}}^2 e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} \tag{2875}$$

$$\cdot \sum_{j_1+j_2+j_3+n\geq 1} \frac{nC_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s)(n-1) \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^{n-2} \tag{2876}$$

$$+ \left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \sum_{m=2}^{\infty} \frac{b(m,s)}{(m-2)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-2} + \|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}} \sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1} \right) \tag{2877}$$

$$\cdot \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \sum_{j_1+j_2+j_3+n\geq 1} \frac{nC_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s) \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^{n-1} \tag{2878}$$

$$+ \|\phi'\|_{\mathcal{F}_\nu^{0,1}} e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} \sum_{j_1+j_2+j_3+n\geq 1} \frac{nC_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s) b(j_3+1,s) \tag{2879}$$

$$\cdot \left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} j_3 + \|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3} \right) \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^{n-1}. \tag{2880}$$

Next,

$$\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} (\widetilde{B_{13}})_\alpha(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \tag{2881}$$

$$\leq \left(\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}} \cdot \pi^2 + 3\pi\right) \sum_{j_1+j_2\geq 2} \frac{j_2}{j_1!j_2!} \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |\mathcal{F}(\phi')| \right\|_{l_\nu^s} \tag{2882}$$

$$\leq \left(\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}} \cdot \pi^2 + 3\pi\right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \sum_{j_1+j_2\geq 2} \frac{j_2}{j_1!j_2!} b(j_1+j_2,s)(j_1+j_2-1) \tag{2883}$$

$$\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-2} \tag{2884}$$

$$+ \left(\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}} \cdot \pi^2 + 3\pi\right) \|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}} \sum_{j_1+j_2\geq 2} \frac{j_2}{j_1!j_2!} b(j_1+j_2,s) \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2-1}. \tag{2885}$$

Next,

$$\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} B_2^{4,1}(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \tag{2886}$$

$$\leq \sum_{j_1+j_2+j_3+n\geq 1} \frac{D_n}{2j_1!j_2!j_3!} \left\||\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |P| * \cdots * |P| * |\mathcal{F}(\phi')|\right\|_{l_\nu^s} \tag{2887}$$

$$\leq \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \sum_{j_1+j_2+j_3+n\geq 1} \frac{D_n}{2j_1!j_2!j_3!} b(j_1+j_2+j_3+n+1,s)(j_1+j_2+j_3) \tag{2888}$$

$$\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n \tag{2889}$$

$$+\left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1}\right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \tag{2890}$$

$$\cdot \sum_{j_1+j_2+j_3+n\geq 1} \frac{D_n}{2j_1!j_2!j_3!} b(j_1+j_2+j_3+n+1,s) n \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-1} \tag{2891}$$

$$+ \|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}} \sum_{j_1+j_2+j_3+n\geq 1} \frac{D_n}{2j_1!j_2!j_3!} b(j_1+j_2+j_3+n+1,s) \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n. \tag{2892}$$

Moreover,

$$\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} B_2^{4,2}(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \tag{2893}$$

$$\leq \sum_{j_1+j_2+j_3+n\geq 1} \frac{C_n}{2j_1!j_2!j_3!} \left\||\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |P| * \cdots * |P| * \left|\mathcal{F}(\phi^{j_3}\phi')\right|\right\|_{l_\nu^s} \tag{2894}$$

$$\leq \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \tag{2895}$$

$$\cdot \sum_{j_1+j_2+j_3+n\geq 1} \frac{C_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s)(j_1+j_2) \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n \tag{2896}$$

$$+ \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1}\right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \tag{2897}$$

$$\cdot \sum_{j_1+j_2+j_3+n\geq 1} \frac{C_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s) n \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-1} \tag{2898}$$

$$+ \sum_{j_1+j_2+j_3+n\geq 1} \frac{C_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s) b(j_3+1,s) \tag{2899}$$

$$\cdot \left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} j_3 + \|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3}\right) \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n. \tag{2900}$$

Moreover,

$$\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} B_2^{4,3}(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \tag{2901}$$

$$\leq \sum_{j_1+j_2+j_3+n\geq 1}\frac{D_n}{2j_1!j_2!j_3!}\left\||\mathcal{F}(\phi)|*\cdots*|\mathcal{F}(\phi)|*|P|*\cdots*|P|*\left|\mathcal{F}(\phi^{j_3}\phi')\right|\right\|_{l_\nu^s} \tag{2902}$$

$$\leq \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi'\|_{\mathcal{F}_\nu^{0,1}} \tag{2903}$$

$$\cdot \sum_{j_1+j_2+j_3+n\geq 1}\frac{D_n}{2j_1!j_2!j_3!}b(j_1+j_2+n+1,s)(j_1+j_2)\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-1}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n \tag{2904}$$

$$+\left(\sum_{m=1}^{\infty}\frac{b(m,s)}{(m-1)!}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1}\right)\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi'\|_{\mathcal{F}_\nu^{0,1}} \tag{2905}$$

$$\cdot \sum_{j_1+j_2+j_3+n\geq 1}\frac{D_n}{2j_1!j_2!j_3!}b(j_1+j_2+n+1,s)n\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-1} \tag{2906}$$

$$+\sum_{j_1+j_2+j_3+n\geq 1}\frac{D_n}{2j_1!j_2!j_3!}b(j_1+j_2+n+1,s)b(j_3+1,s) \tag{2907}$$

$$\cdot\left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1}\|\phi'\|_{\mathcal{F}_\nu^{0,1}}j_3+\|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3}\right)\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n. \tag{2908}$$

Lastly,

$$\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} B_2^{4,4}(\alpha,\beta)d\beta\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \tag{2909}$$

$$\leq \sum_{j_1+j_2+j_3+n\geq 1}\frac{j_3C_n}{2j_1!j_2!j_3!}\left\||\mathcal{F}(\phi)|*\cdots*|\mathcal{F}(\phi)|*|P|*\cdots*|P|*\left|\mathcal{F}(\phi^{j_3-1}\phi'^2)\right|\right\|_{l_\nu^s} \tag{2910}$$

$$\leq \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi'\|_{\mathcal{F}_\nu^{0,1}}^2 \tag{2911}$$

$$\cdot \sum_{j_1+j_2+j_3+n\geq 1}\frac{j_3C_n}{2j_1!j_2!j_3!}b(j_1+j_2+n+1,s)(j_1+j_2)\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-2}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n \tag{2912}$$

$$+\left(\sum_{m=1}^{\infty}\frac{b(m,s)}{(m-1)!}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1}\right)\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi'\|_{\mathcal{F}_\nu^{0,1}}^2 \tag{2913}$$

$$\cdot \sum_{j_1+j_2+j_3+n\geq 1}\frac{j_3C_n}{2j_1!j_2!j_3!}b(j_1+j_2+n+1,s)n(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-1}\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-1} \tag{2914}$$

$$+\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi'\|_{\mathcal{F}_\nu^{0,1}}^2\sum_{j_1+j_2+j_3+n\geq 1}\frac{j_3C_n}{2j_1!j_2!j_3!}b(j_1+j_2+n+1,s)b(j_3+1,s)(j_3-1) \tag{2915}$$

$$\cdot\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-2}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n \tag{2916}$$

$$+\|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}}\|\phi'\|_{\mathcal{F}_\nu^{0,1}}\sum_{j_1+j_2+j_3+n\geq 1}\frac{j_3C_n}{j_1!j_2!j_3!}b(j_1+j_2+n+1,s)b(j_3+1,s) \tag{2917}$$

$$\cdot\|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-1}(e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n. \tag{2918}$$

Next,

$$\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B_4^{5,1}(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \tag{2919}$$

$$\leq \sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1) \frac{D_{n+1}}{2j_1!j_2!j_3!j_4!} \tag{2920}$$

$$\cdot \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |Q| * \cdots * |Q| * \left|\mathcal{F}(\phi^{j_3})\right| * |\mathcal{F}(\phi')| \right\|_{l_\nu^s} \tag{2921}$$

$$\leq \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1) \frac{D_{n+1}}{2j_1!j_2!j_3!j_4!} b(j_1+j_2+j_4+n+2,s)(j_1+j_2+j_4) \tag{2922}$$

$$\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3+j_4-1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n \tag{2923}$$

$$+ \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1} \right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1) \frac{D_{n+1}}{2j_1!j_2!j_3!j_4!} \tag{2924}$$

$$\cdot b(j_1+j_2+j_4+n+2,s) n \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3+j_4} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^{n-1} \tag{2925}$$

$$+ \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1) \frac{D_{n+1}}{2j_1!j_2!j_3!j_4!} b(j_1+j_2+j_4+n+2,s) b(j_3,s) \tag{2926}$$

$$\cdot \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1} j_3 \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_4} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n \tag{2927}$$

$$+ \|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}} \sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1) \frac{D_{n+1}}{2j_1!j_2!j_3!j_4!} b(j_1+j_2+j_4+n+2,s) \tag{2928}$$

$$\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3+j_4} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n. \tag{2929}$$

Moreover,

$$\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B_4^{5,2}(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \tag{2930}$$

$$\leq \sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1) \frac{C_{n+1}}{2 j_1! j_2! j_3! j_4!} \tag{2931}$$

$$\cdot \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |Q| * \cdots * |Q| * \left|\mathcal{F}(\phi^{j_3})\right| * \left|\mathcal{F}(\phi^{j_4}\phi')\right| \right\|_{l_\nu^s} \tag{2932}$$

$$\leq \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1) \frac{C_{n+1}}{2 j_1! j_2! j_3! j_4!} b(j_1+j_2+n+2,s)(j_1+j_2) \tag{2933}$$

$$\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3+j_4-1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n \tag{2934}$$

$$+ \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1} \right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1) \frac{C_{n+1}}{2 j_1! j_2! j_3! j_4!} \tag{2935}$$

$$\cdot b(j_1+j_2+n+2,s) n \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3+j_4} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^{n-1} \tag{2936}$$

$$+ \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1) \frac{C_{n+1}}{2 j_1! j_2! j_3! j_4!} b(j_1+j_2+n+2,s) b(j_3,s) \tag{2937}$$

$$\cdot \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1} j_3 \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_4} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n \tag{2938}$$

$$+ \sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1) \frac{C_{n+1}}{2 j_1! j_2! j_3! j_4!} b(j_1+j_2+n+2,s) b(j_4+1,s) \tag{2939}$$

$$\cdot \left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_4-1} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} j_4 + \|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_4} \right) \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n. \tag{2940}$$

Moreover,

$$\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B_4^{5,3}(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \tag{2941}$$

$$\leq \sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1) \frac{D_{n+1}}{2 j_1! j_2! j_3! j_4!} \tag{2942}$$

$$\cdot \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |Q| * \cdots * |Q| * \left|\mathcal{F}(\phi^{j_3})\right| * \left|\mathcal{F}(\phi^{j_4}\phi')\right| \right\|_{l_\nu^s} \tag{2943}$$

$$\leq \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1) \frac{D_{n+1}}{2 j_1! j_2! j_3! j_4!} b(j_1+j_2+n+2,s)(j_1+j_2) \tag{2944}$$

$$\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3+j_4-1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n \tag{2945}$$

$$+ \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1} \right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1) \frac{D_{n+1}}{2 j_1! j_2! j_3! j_4!} \tag{2946}$$

$$\cdot b(j_1+j_2+n+2,s) n \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3+j_4} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^{n-1} \tag{2947}$$

$$+ \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1) \frac{D_{n+1}}{2 j_1! j_2! j_3! j_4!} b(j_1+j_2+n+2,s) b(j_3,s) \tag{2948}$$

$$\cdot \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1} j_3 \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_4} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n \tag{2949}$$

$$+ \sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1) \frac{D_{n+1}}{2 j_1! j_2! j_3! j_4!} b(j_1+j_2+n+2,s) b(j_4+1,s) \tag{2950}$$

$$\cdot \left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_4-1} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} j_4 + \|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_4} \right) \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n. \tag{2951}$$

Lastly,

$$\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B_4^{5,4}(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \tag{2952}$$

$$\leq \sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1) \frac{j_4 C_{n+1}}{2 j_1! j_2! j_3! j_4!} \tag{2953}$$

$$\cdot \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |Q| * \cdots * |Q| * \left|\mathcal{F}(\phi^{j_3})\right| * \left|\mathcal{F}(\phi^{j_4-1}\phi'^2)\right| \right\|_{l_\nu^s} \tag{2954}$$

$$\leq \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|^2_{\mathcal{F}_\nu^{0,1}} \sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1) \frac{j_4 C_{n+1}}{2 j_1! j_2! j_3! j_4!} b(j_1+j_2+n+2,s)(j_1+j_2) \tag{2955}$$

$$\cdot \|\phi\|^{j_1+j_2+j_3+j_4-2}_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n \tag{2956}$$

$$+ \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|^{m-1}_{\mathcal{F}_\nu^{0,1}}\right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|^2_{\mathcal{F}_\nu^{0,1}} \sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1) \frac{j_4 C_{n+1}}{2 j_1! j_2! j_3! j_4!} \tag{2957}$$

$$\cdot b(j_1+j_2+n+2,s) n \|\phi\|^{j_1+j_2+j_3+j_4-1}_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^{n-1} \tag{2958}$$

$$+ \|\phi'\|^2_{\mathcal{F}_\nu^{0,1}} \sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1) \frac{j_4 C_{n+1}}{2 j_1! j_2! j_3! j_4!} b(j_1+j_2+n+2,s) b(j_3,s) \tag{2959}$$

$$\cdot \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|^{j_3-1}_{\mathcal{F}_\nu^{0,1}} j_3 \|\phi\|^{j_1+j_2+j_4-1}_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n \tag{2960}$$

$$+ \sum_{j_1+j_2+j_3+j_4+n\geq 1} (n+1) \frac{j_4 C_{n+1}}{2 j_1! j_2! j_3! j_4!} b(j_1+j_2+n+2,s) b(j_4+1,s) \tag{2961}$$

$$\cdot \left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|^{j_4-2}_{\mathcal{F}_\nu^{0,1}} \|\phi'\|^2_{\mathcal{F}_\nu^{0,1}} (j_4-1) + \|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|^{j_4-1}_{\mathcal{F}_\nu^{0,1}} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \cdot 2 \right) \tag{2962}$$

$$\cdot \|\phi\|^{j_1+j_2+j_3}_{\mathcal{F}_\nu^{0,1}} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n. \tag{2963}$$

Next,

$$\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B_7^{4,1}(\alpha,\beta)d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \tag{2964}$$

$$\leq \sum_{j_1+j_2+j_3+n\geq 1} \frac{D_n}{2j_1!j_2!j_3!} \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |Q| * \cdots * |Q| * |\mathcal{F}(\phi')| \right\|_{l_\nu^s} \tag{2965}$$

$$\leq \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \sum_{j_1+j_2+j_3+n\geq 1} \frac{D_n}{2j_1!j_2!j_3!} b(j_1+j_2+j_3+n+1,s)(j_1+j_2+j_3) \tag{2966}$$

$$\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n \tag{2967}$$

$$+ \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1} \right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \tag{2968}$$

$$\cdot \sum_{j_1+j_2+j_3+n\geq 1} \frac{D_n}{2j_1!j_2!j_3!} b(j_1+j_2+j_3+n+1,s) n \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-1} \tag{2969}$$

$$+ \|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}} \sum_{j_1+j_2+j_3+n\geq 1} \frac{D_n}{2j_1!j_2!j_3!} b(j_1+j_2+j_3+n+1,s) \tag{2970}$$

$$\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n. \tag{2971}$$

Moreover,

$$\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B_7^{4,2}(\alpha,\beta)d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \tag{2972}$$

$$\leq \sum_{j_1+j_2+j_3+n\geq 1} \frac{C_n}{2j_1!j_2!j_3!} \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |Q| * \cdots * |Q| * \left|\mathcal{F}(\phi^{j_3}\phi')\right| \right\|_{l_\nu^s} \tag{2973}$$

$$\leq \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \sum_{j_1+j_2+j_3+n\geq 1} \frac{C_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s)(j_1+j_2) \tag{2974}$$

$$\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n \tag{2975}$$

$$+ \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1} \right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \tag{2976}$$

$$\cdot \sum_{j_1+j_2+j_3+n\geq 1} \frac{C_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s) n \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-1} \tag{2977}$$

$$+ \sum_{j_1+j_2+j_3+n\geq 1} \frac{C_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s) b(j_3+1,s) \tag{2978}$$

$$\cdot \left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} j_3 + \|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3} \right) \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n. \tag{2979}$$

Moreover,

$$\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B_7^{4,3}(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \tag{2980}$$

$$\leq \sum_{j_1+j_2+j_3+n\geq 1} \frac{D_n}{2 j_1! j_2! j_3!} \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |Q| * \cdots * |Q| * \left|\mathcal{F}(\phi^{j_3}\phi')\right| \right\|_{l_\nu^s} \tag{2981}$$

$$\leq \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \sum_{j_1+j_2+j_3+n\geq 1} \frac{D_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1,s)(j_1+j_2) \tag{2982}$$

$$\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n \tag{2983}$$

$$+ \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1} \right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \tag{2984}$$

$$\cdot \sum_{j_1+j_2+j_3+n\geq 1} \frac{D_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1,s) n \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^{n-1} \tag{2985}$$

$$+ \sum_{j_1+j_2+j_3+n\geq 1} \frac{D_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1,s) b(j_3+1,s) \tag{2986}$$

$$\cdot \left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} j_3 + \|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3} \right) \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n. \tag{2987}$$

Lastly,

$$\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B_7^{4,4}(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \tag{2988}$$

$$\leq \sum_{j_1+j_2+j_3+n\geq 1} \frac{j_3 C_n}{2 j_1! j_2! j_3!} \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |Q| * \cdots * |Q| * \left|\mathcal{F}(\phi^{j_3-1}\phi'^2)\right| \right\|_{l_\nu^s} \tag{2989}$$

$$\leq \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|_{\mathcal{F}_\nu^{0,1}}^2 \sum_{j_1+j_2+j_3+n\geq 1} \frac{j_3 C_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1,s)(j_1+j_2) \tag{2990}$$

$$\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-2} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n \tag{2991}$$

$$+ \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1} \right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|_{\mathcal{F}_\nu^{0,1}}^2 \tag{2992}$$

$$\cdot \sum_{j_1+j_2+j_3+n\geq 1} \frac{j_3 C_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1,s) n \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^{n-1} \tag{2993}$$

$$+ \sum_{j_1+j_2+j_3+n\geq 1} \frac{j_3 C_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1,s) b(j_3+1,s) \tag{2994}$$

$$\cdot \left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-2} \|\phi'\|_{\mathcal{F}_\nu^{0,1}}^2 (j_3-1) + \|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \cdot 2 \right) \tag{2995}$$

$$\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n. \tag{2996}$$

Next,

$$\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B_8^{4,1}(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \tag{2997}$$

$$\leq \sum_{j_1+j_2+j_3+n\geq 1} \frac{D_n}{2j_1!j_2!j_3!} \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |Q| * \cdots * |Q| * |\mathcal{F}(\phi')| \right\|_{l_\nu^s} \tag{2998}$$

$$\leq \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \sum_{j_1+j_2+j_3+n\geq 1} \frac{D_n}{2j_1!j_2!j_3!} b(j_1+j_2+j_3+n+1,s)(j_1+j_2+j_3) \tag{2999}$$

$$\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n \tag{3000}$$

$$+ \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1} \right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \tag{3001}$$

$$\cdot \sum_{j_1+j_2+j_3+n\geq 1} \frac{D_n}{2j_1!j_2!j_3!} b(j_1+j_2+j_3+n+1,s) n \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-1} \tag{3002}$$

$$+ \|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}} \sum_{j_1+j_2+j_3+n\geq 1} \frac{D_n}{2j_1!j_2!j_3!} b(j_1+j_2+j_3+n+1,s) \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3} \tag{3003}$$

$$\cdot (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n. \tag{3004}$$

Moreover,

$$\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B_8^{4,2}(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \tag{3005}$$

$$\leq \sum_{j_1+j_2+j_3+n\geq 1} \frac{C_n}{2j_1!j_2!j_3!} \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |Q| * \cdots * |Q| * \left|\mathcal{F}(\phi^{j_3}\phi')\right| \right\|_{l_\nu^s} \tag{3006}$$

$$\leq \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \sum_{j_1+j_2+j_3+n\geq 1} \frac{C_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s)(j_1+j_2) \tag{3007}$$

$$\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n \tag{3008}$$

$$+ \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1} \right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \tag{3009}$$

$$\cdot \sum_{j_1+j_2+j_3+n\geq 1} \frac{C_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s) n \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^{n-1} \tag{3010}$$

$$+ \sum_{j_1+j_2+j_3+n\geq 1} \frac{C_n}{2j_1!j_2!j_3!} b(j_1+j_2+n+1,s) b(j_3+1,s) \tag{3011}$$

$$\cdot \left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} j_3 + \|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3} \right) \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}}-1)^n. \tag{3012}$$

Moreover,

$$\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B_8^{4,3}(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \tag{3013}$$

$$\leq \sum_{j_1+j_2+j_3+n\geq 1} \frac{D_n}{2 j_1! j_2! j_3!} \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |Q| * \cdots * |Q| * \left|\mathcal{F}(\phi^{j_3}\phi')\right| \right\|_{l_\nu^s} \tag{3014}$$

$$\leq \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \sum_{j_1+j_2+j_3+n\geq 1} \frac{D_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1,s)(j_1+j_2) \tag{3015}$$

$$\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n \tag{3016}$$

$$+ \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1} \right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \tag{3017}$$

$$\cdot \sum_{j_1+j_2+j_3+n\geq 1} \frac{D_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1,s) n \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^{n-1} \tag{3018}$$

$$+ \sum_{j_1+j_2+j_3+n\geq 1} \frac{D_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1,s) b(j_3+1,s) \tag{3019}$$

$$\cdot \left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} j_3 + \|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3} \right) \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n. \tag{3020}$$

Lastly,

$$\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B_8^{4,4}(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}_\nu^{s,1}} \tag{3021}$$

$$\leq \sum_{j_1+j_2+j_3+n\geq 1} \frac{j_3 C_n}{2 j_1! j_2! j_3!} \left\| |\mathcal{F}(\phi)| * \cdots * |\mathcal{F}(\phi)| * |Q| * \cdots * |Q| * \left|\mathcal{F}(\phi^{j_3-1}\phi'^2)\right| \right\|_{l_\nu^s} \tag{3022}$$

$$\leq \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|_{\mathcal{F}_\nu^{0,1}}^2 \sum_{j_1+j_2+j_3+n\geq 1} \frac{j_3 C_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1,s)(j_1+j_2) \tag{3023}$$

$$\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-2} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n \tag{3024}$$

$$+ \left(\sum_{m=1}^{\infty} \frac{b(m,s)}{(m-1)!} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{m-1} \right) \|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi'\|_{\mathcal{F}_\nu^{0,1}}^2 \tag{3025}$$

$$\cdot \sum_{j_1+j_2+j_3+n\geq 1} \frac{j_3 C_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1,s) n \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2+j_3-1} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^{n-1} \tag{3026}$$

$$+ \sum_{j_1+j_2+j_3+n\geq 1} \frac{j_3 C_n}{2 j_1! j_2! j_3!} b(j_1+j_2+n+1,s) b(j_3+1,s) \tag{3027}$$

$$\cdot \left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-2} \|\phi'\|_{\mathcal{F}_\nu^{0,1}}^2 (j_3-1) + \|\phi'\|_{\dot{\mathcal{F}}_\nu^{s,1}} \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_3-1} \|\phi'\|_{\mathcal{F}_\nu^{0,1}} \cdot 2 \right) \tag{3028}$$

$$\cdot \|\phi\|_{\mathcal{F}_\nu^{0,1}}^{j_1+j_2} (e^{\|\phi\|_{\mathcal{F}_\nu^{0,1}}} - 1)^n. \tag{3029}$$

This completes the proof of Lemma 6.

15 Proof of the Main Theorem

15.1 Proof of the Main *a priori* Estimate

To complete the estimate for the $\dot{\mathcal{F}}_\nu^{s,1}$ norm of $\widetilde{\mathcal{N}}$, we let $s = 1$. Recalling (928), we can use Lemmas 1 and 2 and the estimates of the $\mathcal{F}_\nu^{0,1}$ norm of U_1 and $U_{\geq 2}$ in Sections 12.2 and 13.2, respectively, to obtain

$$
\begin{aligned}
\left\|\widetilde{\mathcal{N}}\right\|_{\dot{\mathcal{F}}_\nu^{1,1}} &\leq \|(U_{\geq 2})_\alpha\|_{\dot{\mathcal{F}}_\nu^{1,1}} + \|T_{\geq 2}(1+\phi_\alpha)\|_{\dot{\mathcal{F}}_\nu^{1,1}} + \|T_1\phi_\alpha\|_{\dot{\mathcal{F}}_\nu^{1,1}} && (3030)\\
&\leq \|(U_{\geq 2})_\alpha\|_{\dot{\mathcal{F}}_\nu^{1,1}} + (H_3\|\phi\|_{\dot{\mathcal{F}}_\nu^{0,1}} + H_4\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}})(\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}} + 2\|\phi\|_{\dot{\mathcal{F}}_\nu^{2,1}}) && (3031)\\
&\quad + (D_1(\|\phi\|_{\mathcal{F}_\nu^{0,1}})\|\phi\|^2_{\mathcal{F}_\nu^{0,1}} + D_2(\|\phi\|_{\mathcal{F}_\nu^{0,1}})\|\phi\|_{\mathcal{F}_\nu^{0,1}}\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}})(1+2\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}}) && (3032)\\
&\quad \cdot (1+\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}} + 2\|\phi\|_{\dot{\mathcal{F}}_\nu^{2,1}}) && (3033)\\
&\quad + 2\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}}(H_3\|\phi\|_{\mathcal{F}_\nu^{0,1}} + H_4\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}}) \cdot (1+\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}} + 2\|\phi\|_{\dot{\mathcal{F}}_\nu^{2,1}}). && (3034)
\end{aligned}
$$

Using Lemma 6 and Proposition 1, we obtain

$$
\begin{aligned}
\left\|\widetilde{\mathcal{N}}\right\|_{\dot{\mathcal{F}}_\nu^{1,1}} \leq & R_1(\|\phi\|_{\mathcal{F}_\nu^{0,1}})\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}}\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}} + R_2(\|\phi\|_{\mathcal{F}_\nu^{0,1}})\|\phi\|_{\mathcal{F}_\nu^{0,1}}\|\phi\|_{\dot{\mathcal{F}}_\nu^{2,1}} && (3035)\\
&+ R_3(\|\phi\|_{\mathcal{F}_\nu^{0,1}})\|\phi\|_{\mathcal{F}_\nu^{0,1}}\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}}\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}} + R_4(\|\phi\|_{\mathcal{F}_\nu^{0,1}})\|\phi\|^2_{\dot{\mathcal{F}}_\nu^{1,1}}\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}} && (3036)\\
&+ R_5(\|\phi\|_{\mathcal{F}_\nu^{0,1}})\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}}\|\phi\|_{\dot{\mathcal{F}}_\nu^{2,1}} && (3037)\\
&+ \left(H_3\|\phi\|_{\dot{\mathcal{F}}_\nu^{0,1}} + H_4\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}}\right)\left(\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}} + 2\|\phi\|_{\dot{\mathcal{F}}_\nu^{2,1}}\right) && (3038)\\
&+ \Big(D_1(\|\phi\|_{\mathcal{F}_\nu^{0,1}})\|\phi\|^2_{\mathcal{F}_\nu^{0,1}} + D_2(\|\phi\|_{\mathcal{F}_\nu^{0,1}})\|\phi\|_{\mathcal{F}_\nu^{0,1}}\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}}\Big)\left(1+2\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}}\right) && (3039)\\
&\cdot\left(1+\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}} + 2\|\phi\|_{\dot{\mathcal{F}}_\nu^{2,1}}\right) && (3040)\\
&+ 2\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}}\left(H_3\|\phi\|_{\mathcal{F}_\nu^{0,1}} + H_4\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}}\right)\left(1+\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}} + 2\|\phi\|_{\dot{\mathcal{F}}_\nu^{2,1}}\right) && (3041)\\
\leq & \|\phi\|_{\dot{\mathcal{F}}_\nu^{2,1}}\Bigg(R_1(\|\phi\|_{\mathcal{F}_\nu^{0,1}})\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}} + R_2(\|\phi\|_{\mathcal{F}_\nu^{0,1}})\|\phi\|_{\mathcal{F}_\nu^{0,1}} && (3042)\\
&+ R_3(\|\phi\|_{\mathcal{F}_\nu^{0,1}})\|\phi\|_{\mathcal{F}_\nu^{0,1}}\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}} + R_4(\|\phi\|_{\mathcal{F}_\nu^{0,1}})\|\phi\|^2_{\dot{\mathcal{F}}_\nu^{1,1}} && (3043)\\
&+ R_5(\|\phi\|_{\mathcal{F}_\nu^{0,1}})\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}} && (3044)\\
&+ 3\left(H_3\|\phi\|_{\dot{\mathcal{F}}_\nu^{0,1}} + H_4\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}}\right) && (3045)\\
&+ 3\Big(D_1(\|\phi\|_{\mathcal{F}_\nu^{0,1}})\|\phi\|^2_{\mathcal{F}_\nu^{0,1}} + D_2(\|\phi\|_{\mathcal{F}_\nu^{0,1}})\|\phi\|_{\mathcal{F}_\nu^{0,1}}\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}}\Big)\left(1+2\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}}\right) && (3046)\\
&+ \left(D_1(\|\phi\|_{\mathcal{F}_\nu^{0,1}})\|\phi\|_{\mathcal{F}_\nu^{0,1}} + D_2(\|\phi\|_{\mathcal{F}_\nu^{0,1}})\|\phi\|_{\mathcal{F}_\nu^{0,1}}\right)\left(1+2\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}}\right) && (3047)\\
&+ 6\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}}\left(H_3\|\phi\|_{\mathcal{F}_\nu^{0,1}} + H_4\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}}\right) && (3048)\\
&+ 2\left(H_3\|\phi\|_{\mathcal{F}_\nu^{0,1}} + H_4\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}}\right)\Bigg). && (3049)
\end{aligned}
$$

Using this estimate for the $\dot{\mathcal{F}}^{1,1}_\nu$ norm of $\widetilde{\mathcal{N}}$, we obtain from (925)

$$
\begin{aligned}
&\frac{d}{dt}\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu} && (3050)\\
\leq&\Bigg(\nu'(t)-\frac{1}{2\Big(C_I(\|\phi\|_{\mathcal{F}^{0,1}})\|\phi\|_{\mathcal{F}^{0,1}}+1\Big)}\pi\frac{2}{R}\frac{\gamma}{4\pi}+\frac{\gamma}{4\pi}\frac{1}{R}A(\|\phi\|_{\mathcal{F}^{0,1}})\|\phi\|_{\mathcal{F}^{0,1}}\Bigg)\|\phi\|_{\dot{\mathcal{F}}^{2,1}_\nu} && (3051)\\
&+\frac{1}{R}\frac{1}{A_1(\|\phi\|_{\mathcal{F}^{0,1}})}\Bigg(R_1(\|\phi\|_{\mathcal{F}^{0,1}_\nu})\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu}+R_2(\|\phi\|_{\mathcal{F}^{0,1}_\nu})\|\phi\|_{\mathcal{F}^{0,1}_\nu} && (3052)\\
&+R_3(\|\phi\|_{\mathcal{F}^{0,1}_\nu})\|\phi\|_{\mathcal{F}^{0,1}_\nu}\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu}+R_4(\|\phi\|_{\mathcal{F}^{0,1}_\nu})\|\phi\|^2_{\dot{\mathcal{F}}^{1,1}_\nu} && (3053)\\
&+R_5(\|\phi\|_{\mathcal{F}^{0,1}_\nu})\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu} && (3054)\\
&+3\left(H_3\|\phi\|_{\dot{\mathcal{F}}^{0,1}_\nu}+H_4\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu}\right) && (3055)\\
&+3\left(D_1(\|\phi\|_{\mathcal{F}^{0,1}_\nu})\|\phi\|^2_{\mathcal{F}^{0,1}_\nu}+D_2(\|\phi\|_{\mathcal{F}^{0,1}_\nu})\|\phi\|_{\mathcal{F}^{0,1}_\nu}\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu}\right)\left(1+2\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu}\right) && (3056)\\
&+\left(D_1(\|\phi\|_{\mathcal{F}^{0,1}_\nu})\|\phi\|_{\mathcal{F}^{0,1}_\nu}+D_2(\|\phi\|_{\mathcal{F}^{0,1}_\nu})\|\phi\|_{\mathcal{F}^{0,1}_\nu}\right)\left(1+2\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu}\right) && (3057)\\
&+6\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu}\left(H_3\|\phi\|_{\mathcal{F}^{0,1}_\nu}+H_4\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu}\right) && (3058)\\
&+2\left(H_3\|\phi\|_{\mathcal{F}^{0,1}_\nu}+H_4\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu}\right)\Bigg)\|\phi\|_{\dot{\mathcal{F}}^{2,1}_\nu}. && (3059)
\end{aligned}
$$

Since C_I, A, A_1^{-1}, R_1, R_2, R_3, R_4, R_5, D_1, and D_2 are all monotone increasing, we can use Proposition 1 to obtain

$$
\frac{d}{dt}\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu}\leq-\Big(\Lambda(\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu})-\nu'(t)\Big)\|\phi\|_{\dot{\mathcal{F}}^{2,1}_\nu}, \tag{3060}
$$

where

$$
\begin{aligned}
\Lambda(\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu})=&\frac{1}{2\Big(C_I(\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu})\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu}+1\Big)}\pi\frac{2}{R}\frac{\gamma}{4\pi}-\frac{\gamma}{4\pi}\frac{1}{R}A(\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu})\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu} && (3061)\\
&-\frac{1}{R}\frac{1}{A_1(\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu})}\Bigg(R_1(\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu})\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu}+R_2(\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu})\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu} && (3062)\\
&+R_3(\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu})\|\phi\|^2_{\dot{\mathcal{F}}^{1,1}_\nu}+R_4(\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu})\|\phi\|^2_{\dot{\mathcal{F}}^{1,1}_\nu} && (3063)\\
&+R_5(\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu})\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu} && (3064)\\
&+3\left(H_3\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu}+H_4\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu}\right) && (3065)\\
&+3\left(D_1(\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu})\|\phi\|^2_{\dot{\mathcal{F}}^{1,1}_\nu}+D_2(\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu})\|\phi\|^2_{\dot{\mathcal{F}}^{1,1}_\nu}\right)\left(1+2\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu}\right) && (3066)\\
&+\left(D_1(\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu})\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu}+D_2(\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu})\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu}\right)\left(1+2\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu}\right) && (3067)\\
&+6\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu}\left(H_3\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu}+H_4\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu}\right) && (3068)\\
&+2\left(H_3\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu}+H_4\|\phi\|_{\dot{\mathcal{F}}^{1,1}_\nu}\right)\Bigg). && (3069)
\end{aligned}
$$

Integrating (3060) with respect to time, we obtain

$$\|\phi(t)\|_{\dot{\mathcal{F}}^{1,1}_{\nu}} + \int_0^t (\Lambda(\|\phi(\tau)\|_{\dot{\mathcal{F}}^{1,1}_{\nu}}) - \nu'(\tau)) \|\phi(\tau)\|_{\dot{\mathcal{F}}^{2,1}_{\nu}} \, d\tau \leq \|\phi_0\|_{\dot{\mathcal{F}}^{1,1}} . \tag{3070}$$

We choose the initial datum such that $\Lambda(\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}) > 0$. Then we let $\nu_0 \in (0, \Lambda(\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}))$. From (9), it follows that for all $\tau \geq 0$,

$$0 < \nu'(\tau) = \frac{\nu_0}{(1+\tau)^2} \leq \nu_0. \tag{3071}$$

Then

$$\Lambda(\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}) - \nu'(0) > 0. \tag{3072}$$

Let

$$T_1 = \sup\left\{ t_1 : \Lambda(\|\phi(\tau)\|_{\dot{\mathcal{F}}^{1,1}_{\nu}}) - \nu'(\tau) > 0 \text{ for all } \tau \in [0, t_1] \right\}. \tag{3073}$$

Since $\Lambda(\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}) - \nu'(0) > 0$ and $\Lambda(\|\phi(\cdot)\|_{\dot{\mathcal{F}}^{1,1}_{\nu}}) - \nu'(\cdot)$ is a continuous function, we have $T_1 > 0$. For any $t_1 \in [0, T_1)$,

$$\Lambda(\|\phi(\tau)\|_{\dot{\mathcal{F}}^{1,1}_{\nu}}) - \nu'(\tau) > 0 \text{ for all } \tau \in [0, t_1]. \tag{3074}$$

Then by (3070) for all $t \in [0, t_1]$,

$$\|\phi(t)\|_{\dot{\mathcal{F}}^{1,1}_{\nu}} \leq \|\phi_0\|_{\dot{\mathcal{F}}^{1,1}} . \tag{3075}$$

Fix $t_1 \in [0, T_1)$ and $t_2 \in [t_1, T_1)$. Then

$$\|\phi(t_2)\|_{\dot{\mathcal{F}}^{1,1}_{\nu}} + \int_{t_1}^{t_2} \left(\Lambda(\|\phi(\tau)\|_{\dot{\mathcal{F}}^{1,1}_{\nu}}) - \nu'(\tau) \right) \|\phi(\tau)\|_{\dot{\mathcal{F}}^{2,1}_{\nu}} \, d\tau \leq \|\phi(t_1)\|_{\dot{\mathcal{F}}^{1,1}_{\nu}} . \tag{3076}$$

Since

$$\int_{t_1}^{t_2} \left(\Lambda(\|\phi(\tau)\|_{\dot{\mathcal{F}}^{1,1}_{\nu}}) - \nu'(\tau) \right) \|\phi(\tau)\|_{\dot{\mathcal{F}}^{2,1}_{\nu}} \, d\tau > 0, \tag{3077}$$

it follows from (3639) that $\|\phi(t_2)\|_{\dot{\mathcal{F}}^{1,1}_{\nu}} \leq \|\phi(t_1)\|_{\dot{\mathcal{F}}^{1,1}_{\nu}}$. Since Λ is a monotone decreasing function of $\|\phi\|_{\dot{\mathcal{F}}^{1,1}_{\nu}}$, this means that $\Lambda(\|\phi(t_2)\|_{\dot{\mathcal{F}}^{1,1}_{\nu}}) \geq \Lambda(\|\phi(t_1)\|_{\dot{\mathcal{F}}^{1,1}_{\nu}})$, i.e., $\Lambda(\|\phi(\cdot)\|_{\dot{\mathcal{F}}^{1,1}_{\nu}})$ is a monotone increasing function on $[0, T_1)$. Suppose for contradiction that $T_1 < \infty$. We note that $\Lambda(\|\phi(T_1)\|_{\dot{\mathcal{F}}^{1,1}_{\nu}}) - \nu'(T_1) = 0$. Since $\Lambda(\|\phi(\cdot)\|_{\dot{\mathcal{F}}^{1,1}_{\nu}})$ is monotone increasing on $[0, T_1)$ and is continuous on $[0, T_1]$, it is monotone increasing on $[0, T_1]$. Then

$$\nu_0 = \nu'(0) < \Lambda(\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}) \leq \Lambda(\|\phi(T_1)\|_{\dot{\mathcal{F}}^{1,1}_{\nu}}) = \nu'(T_1) = \frac{\nu_0}{(1+T_1)^2}, \tag{3078}$$

which is a contradiction. Hence, $T_1 = \infty$. Then for all $t \in [0, \infty)$,

$$\|\phi(t)\|_{\dot{\mathcal{F}}^{1,1}_{\nu}} \leq \|\phi_0\|_{\dot{\mathcal{F}}^{1,1}} - \int_0^t \left(\Lambda(\|\phi(\tau)\|_{\dot{\mathcal{F}}^{1,1}_{\nu}}) - \nu'(\tau) \right) \|\phi(\tau)\|_{\dot{\mathcal{F}}^{2,1}_{\nu}} \, d\tau \tag{3079}$$

$$\leq \|\phi_0\|_{\dot{\mathcal{F}}^{1,1}} - \int_0^t \left(\Lambda(\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}) - \nu_0 \right) \|\phi(\tau)\|_{\dot{\mathcal{F}}^{2,1}_{\nu}} \, d\tau. \tag{3080}$$

Therefore, for all $t \in [0, \infty)$,

$$\|\phi(t)\|_{\dot{\mathcal{F}}^{1,1}_{\nu}} + \left(\Lambda(\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}) - \nu_0 \right) \int_0^t \|\phi(\tau)\|_{\dot{\mathcal{F}}^{2,1}_{\nu}} \, d\tau \leq \|\phi_0\|_{\dot{\mathcal{F}}^{1,1}} . \tag{3081}$$

15.2 Boundedness of $\mathcal{F}(\theta)(0)$

Using the *a priori* estimate for ϕ derived in Section 15.1, we now show that $\hat{\theta}(0)$ is bounded in time. To that end, we first take the zeroth Fourier mode of (46). Plugging into it

$$\mathcal{F}(U_\alpha)(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} U_\alpha(\alpha)d\alpha = \frac{1}{2\pi}(U(\pi) - U(-\pi)) = 0, \tag{3082}$$

we obtain

$$\mathcal{F}(\theta)_t(0) = \frac{2\pi}{L(t)}(\mathcal{F}(T) * \mathcal{F}(1+\theta_\alpha))(0). \tag{3083}$$

Recalling that $T = T_1 + T_{\geq 2}$, we obtain

$$\hat{\theta}(0) - \hat{\theta}_0(0) = \int_0^t \frac{2\pi}{L(\tau)}\mathcal{F}\Big(T_1(\alpha)(1+\theta_\alpha(\alpha))\Big)(0)d\tau \tag{3084}$$

$$+ \int_0^t \frac{2\pi}{L(t)}\mathcal{F}\Big(T_{\geq 2}(\alpha)(1+\theta_\alpha(\alpha))\Big)(0)d\tau. \tag{3085}$$

Using that

$$\left|\mathcal{F}\Big(T_1(\alpha)(1+\theta_\alpha(\alpha))\Big)(0)\right| \tag{3086}$$

$$= \left|\sum_{k\in\mathbb{Z}} \mathcal{F}(T_1)(k)\mathcal{F}(1+\theta_\alpha(\alpha))(-k)\right| \tag{3087}$$

$$\leq \sum_{k\in\mathbb{Z}} |\mathcal{F}(T_1)(k)|\,|\mathcal{F}(1+\theta_\alpha(\alpha))(-k)| \tag{3088}$$

$$\leq \left(\sum_{k\in\mathbb{Z}} |\mathcal{F}(T_1)(k)|\right)\left(\sum_{k\in\mathbb{Z}} |\mathcal{F}(1+\theta_\alpha(\alpha))(-k)|\right) \tag{3089}$$

$$= \left(\sum_{k\in\mathbb{Z}} |\mathcal{F}(T_1)(k)|\right)\left(\sum_{k\in\mathbb{Z}} |\mathcal{F}(1+\theta_\alpha(\alpha))(k)|\right) \tag{3090}$$

$$\leq \left(\sum_{k\in\mathbb{Z}} |\mathcal{F}(T_1)(k)|\right)\left(1 + \sum_{k\in\mathbb{Z}} |\mathcal{F}(\theta_\alpha)(k)|\right) \tag{3091}$$

$$= \|T_1\|_{\mathcal{F}^{0,1}}\left(1 + \sum_{k\neq 0} |k|\,|\mathcal{F}(\phi)(k)|\right) \tag{3092}$$

$$= \|T_1\|_{\mathcal{F}^{0,1}}(1 + \|\phi\|_{\mathcal{F}^{1,1}}), \tag{3093}$$

we obtain

$$
\begin{aligned}
&|\mathcal{F}(\theta)(0)| && (3094)\\
=&\left|\mathcal{F}(\theta_0)(0) + \int_0^t \frac{2\pi}{L(t)}\mathcal{F}\Big(T_1(\alpha)(1+\theta_\alpha(\alpha))\Big)(0)d\tau\right. && (3095)\\
&\left.+\int_0^t \frac{2\pi}{L(\tau)}\mathcal{F}\Big(T_{\geq 2}(\alpha)(1+\theta_\alpha(\alpha))\Big)(0)d\tau\right| && (3096)\\
\leq&|\mathcal{F}(\theta_0)(0)| + \int_0^t \frac{2\pi}{L(\tau)}\left|\mathcal{F}\Big(T_1(\alpha)(1+\theta_\alpha(\alpha))\Big)(0)\right|d\tau && (3097)\\
&+\int_0^t \frac{2\pi}{L(\tau)}\left|\mathcal{F}\Big(T_{\geq 2}(\alpha)(1+\theta_\alpha(\alpha))\Big)(0)\right|d\tau && (3098)\\
\leq&|\mathcal{F}(\theta_0)(0)| + \int_0^t \frac{2\pi}{L(\tau)}\|T_1\|_{\mathcal{F}^{0,1}}(1+\|\phi\|_{\dot{\mathcal{F}}^{1,1}})d\tau && (3099)\\
&+\int_0^t \frac{2\pi}{L(\tau)}\|T_{\geq 2}\|_{\mathcal{F}^{0,1}}(1+\|\phi\|_{\dot{\mathcal{F}}^{1,1}})d\tau && (3100)\\
\leq&|\mathcal{F}(\theta_0)(0)| + \int_0^t \frac{2\pi}{L(\tau)}\|T_1\|_{\mathcal{F}^{0,1}}d\tau + \int_0^t \frac{2\pi}{L(\tau)}\|T_1\|_{\mathcal{F}^{0,1}}\|\phi\|_{\dot{\mathcal{F}}^{2,1}}d\tau && (3101)\\
&+\int_0^t \frac{2\pi}{L(\tau)}\|T_{\geq 2}\|_{\mathcal{F}^{0,1}}d\tau + \int_0^t \frac{2\pi}{L(\tau)}\|T_{\geq 2}\|_{\mathcal{F}^{0,1}}\|\phi\|_{\dot{\mathcal{F}}^{2,1}}d\tau. && (3102)
\end{aligned}
$$

Recall that

$$
\frac{\sqrt{1+\frac{\pi}{2}(e^{2\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}}-1)}}{R} \geq \frac{\sqrt{1+\frac{\pi}{2}(e^{2\|\phi\|_{\mathcal{F}^{0,1}_\nu}}-1)}}{R} \geq \frac{2\pi}{L(t)}. \quad (3103)
$$

Hence,

$$
\begin{aligned}
|\mathcal{F}(\theta)(0)| \leq& |\mathcal{F}(\theta_0)(0)| + \int_0^t \frac{2\pi}{L(\tau)}\|T_1\|_{\mathcal{F}^{0,1}}d\tau + \int_0^t \frac{2\pi}{L(\tau)}\|T_1\|_{\mathcal{F}^{0,1}}\|\phi\|_{\dot{\mathcal{F}}^{2,1}}d\tau && (3104)\\
&+\int_0^t \frac{2\pi}{L(\tau)}\|T_{\geq 2}\|_{\mathcal{F}^{0,1}}d\tau + \int_0^t \frac{2\pi}{L(\tau)}\|T_{\geq 2}\|_{\mathcal{F}^{0,1}}\|\phi\|_{\dot{\mathcal{F}}^{2,1}}d\tau && (3105)\\
\leq& |\mathcal{F}(\theta_0)(0)| && (3106)\\
&+\frac{\sqrt{1+\frac{\pi}{2}(e^{2\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}}-1)}}{R}\int_0^t \|T_1\|_{\mathcal{F}^{0,1}}d\tau && (3107)\\
&+\frac{\sqrt{1+\frac{\pi}{2}(e^{2\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}}-1)}}{R}\int_0^t \|T_1\|_{\mathcal{F}^{0,1}}\|\phi\|_{\dot{\mathcal{F}}^{2,1}}d\tau && (3108)\\
&+\frac{\sqrt{1+\frac{\pi}{2}(e^{2\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}}-1)}}{R}\int_0^t \|T_{\geq 2}\|_{\mathcal{F}^{0,1}}d\tau && (3109)\\
&+\frac{\sqrt{1+\frac{\pi}{2}(e^{2\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}}-1)}}{R}\int_0^t \|T_{\geq 2}\|_{\mathcal{F}^{0,1}}\|\phi\|_{\dot{\mathcal{F}}^{2,1}}d\tau. && (3110)
\end{aligned}
$$

Using that

$$\|T_1\|_{\mathcal{F}^{0,1}} \leq 2\|U_1\|_{\dot{\mathcal{F}}_\nu^{0,1}} \leq 2\|U_1\|_{\mathcal{F}_\nu^{0,1}} \leq 2(H_3+H_4)\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}} \tag{3111}$$

and that

$$\begin{aligned}
\|T_{\geq 2}\|_{\mathcal{F}^{0,1}} \leq & \|\mathcal{M}(U_{\geq 2})\|_{\mathcal{F}^{0,1}} + \|\mathcal{M}(\phi_\alpha U_{\geq 1})\|_{\mathcal{F}^{0,1}} && (3112)\\
\leq & 2\Big(\|U_{\geq 2}\|_{\dot{\mathcal{F}}_\nu^{0,1}} + \|\phi_\alpha U_{\geq 1}\|_{\dot{\mathcal{F}}_\nu^{0,1}}\Big) && (3113)\\
\leq & 2\Big(\|U_{\geq 2}\|_{\dot{\mathcal{F}}_\nu^{0,1}} + \Big(\|\phi_\alpha\|_{\dot{\mathcal{F}}_\nu^{0,1}}\|U_{\geq 1}\|_{\mathcal{F}_\nu^{0,1}} + \|\phi_\alpha\|_{\mathcal{F}_\nu^{0,1}}\|U_{\geq 1}\|_{\dot{\mathcal{F}}_\nu^{0,1}}\Big)\Big) && (3114)\\
\leq & 2\Big(\|U_{\geq 2}\|_{\mathcal{F}_\nu^{0,1}} + 2\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}}\|U_{\geq 1}\|_{\mathcal{F}_\nu^{0,1}}\Big) && (3115)\\
\leq & 2\Big(\|U_{\geq 2}\|_{\mathcal{F}_\nu^{0,1}} + 2\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}}\|U_1\|_{\mathcal{F}_\nu^{0,1}} + 2\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}}\|U_{\geq 2}\|_{\mathcal{F}_\nu^{0,1}}\Big) && (3116)\\
\leq & 2\Big(D_1(\|\phi\|_{\mathcal{F}_\nu^{0,1}})\|\phi\|^2_{\mathcal{F}_\nu^{0,1}} + D_2(\|\phi\|_{\mathcal{F}_\nu^{0,1}})\|\phi\|_{\mathcal{F}_\nu^{0,1}}\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}} && (3117)\\
& + 2\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}}\Big(H_3\|\phi\|_{\mathcal{F}_\nu^{0,1}} + H_4\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}}\Big) && (3118)\\
& + 2\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}}\Big(D_1(\|\phi\|_{\mathcal{F}_\nu^{0,1}})\|\phi\|^2_{\mathcal{F}_\nu^{0,1}} + D_2(\|\phi\|_{\mathcal{F}_\nu^{0,1}})\|\phi\|_{\mathcal{F}_\nu^{0,1}}\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}}\Big)\Big) && (3119)\\
\leq & 2\Big(D_1(\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}})\|\phi\|^2_{\dot{\mathcal{F}}_\nu^{1,1}} + D_2(\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}})\|\phi\|^2_{\dot{\mathcal{F}}_\nu^{1,1}} && (3120)\\
& + 2\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}}\Big(H_3\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}} + H_4\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}}\Big) && (3121)\\
& + 2\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}}\Big(D_1(\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}})\|\phi\|^2_{\dot{\mathcal{F}}_\nu^{1,1}} + D_2(\|\phi\|_{\dot{\mathcal{F}}_\nu^{1,1}})\|\phi\|^2_{\dot{\mathcal{F}}_\nu^{1,1}}\Big)\Big), && (3122)
\end{aligned}$$

we obtain

$$\begin{aligned}
|\mathcal{F}(\theta)(0)| \leq & |\mathcal{F}(\theta_0)(0)| && (3123)\\
& + \frac{\sqrt{1+\frac{\pi}{2}(e^{2\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}}-1)}}{R} \int_0^t \|T_1\|_{\mathcal{F}^{0,1}}\, d\tau && (3124)\\
& + \frac{\sqrt{1+\frac{\pi}{2}(e^{2\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}}-1)}}{R} \int_0^t \|T_1\|_{\mathcal{F}^{0,1}} \|\phi\|_{\dot{\mathcal{F}}^{2,1}}\, d\tau && (3125)\\
& + \frac{\sqrt{1+\frac{\pi}{2}(e^{2\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}}-1)}}{R} \int_0^t \|T_{\geq 2}\|_{\mathcal{F}^{0,1}}\, d\tau && (3126)\\
& + \frac{\sqrt{1+\frac{\pi}{2}(e^{2\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}}-1)}}{R} \int_0^t \|T_{\geq 2}\|_{\mathcal{F}^{0,1}} \|\phi\|_{\dot{\mathcal{F}}^{2,1}}\, d\tau && (3127)\\
\leq & |\mathcal{F}(\theta_0)(0)| && (3128)\\
& + \frac{\sqrt{1+\frac{\pi}{2}(e^{2\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}}-1)}}{R} 2(H_3+H_4) \int_0^t \|\phi\|_{\dot{\mathcal{F}}^{2,1}_{\nu}}\, d\tau && (3129)\\
& + \frac{\sqrt{1+\frac{\pi}{2}(e^{2\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}}-1)}}{R} \cdot 2(H_3+H_4) \|\phi_0\|_{\dot{\mathcal{F}}^{1,1}} \int_0^t \|\phi\|_{\dot{\mathcal{F}}^{2,1}}\, d\tau && (3130)\\
& + \frac{\sqrt{1+\frac{\pi}{2}(e^{2\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}}-1)}}{R} && (3131)\\
& \cdot 2\Bigg(D_1(\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}) \|\phi_0\|_{\dot{\mathcal{F}}^{1,1}} + D_2(\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}) \|\phi_0\|_{\dot{\mathcal{F}}^{1,1}} && (3132)\\
& + 2\Big(H_3 \|\phi_0\|_{\dot{\mathcal{F}}^{1,1}} + H_4 \|\phi_0\|_{\dot{\mathcal{F}}^{1,1}} \Big) && (3133)\\
& + 2\Big(D_1(\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}) \|\phi_0\|^2_{\dot{\mathcal{F}}^{1,1}} + D_2(\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}) \|\phi_0\|^2_{\dot{\mathcal{F}}^{1,1}} \Big)\Bigg) && (3134)\\
& \cdot \int_0^t \|\phi\|_{\dot{\mathcal{F}}^{2,1}_{\nu}}\, d\tau && (3135)\\
& + \frac{\sqrt{1+\frac{\pi}{2}(e^{2\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}}-1)}}{R} && (3136)\\
& \cdot 2\Bigg(D_1(\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}) \|\phi_0\|^2_{\dot{\mathcal{F}}^{1,1}} + D_2(\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}) \|\phi_0\|^2_{\dot{\mathcal{F}}^{1,1}} && (3137)\\
& + 2\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}} \Big(H_3 \|\phi_0\|_{\dot{\mathcal{F}}^{1,1}} + H_4 \|\phi_0\|_{\dot{\mathcal{F}}^{1,1}} \Big) && (3138)\\
& + 2\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}} \Big(D_1(\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}) \|\phi_0\|^2_{\dot{\mathcal{F}}^{1,1}} + D_2(\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}) \|\phi_0\|^2_{\dot{\mathcal{F}}^{1,1}} \Big)\Bigg) && (3139)\\
& \cdot \int_0^t \|\phi\|_{\dot{\mathcal{F}}^{2,1}}\, d\tau && (3140)\\
\leq & Y(\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}), && (3141)
\end{aligned}$$

where

$$Y(\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}) \tag{3142}$$
$$= |\mathcal{F}(\theta_0)(0)| \tag{3143}$$
$$+ \frac{\sqrt{1+\frac{\pi}{2}(e^{2\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}}-1)}}{R} 2(H_3+H_4)\cdot\frac{\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}}{\Lambda(\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}})-\nu_0} \tag{3144}$$
$$+ \frac{\sqrt{1+\frac{\pi}{2}(e^{2\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}}-1)}}{R} \cdot 2(H_3+H_4)\,\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}\cdot\frac{\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}}{\Lambda(\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}})-\nu_0} \tag{3145}$$
$$+ \frac{\sqrt{1+\frac{\pi}{2}(e^{2\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}}-1)}}{R} \tag{3146}$$
$$\cdot 2\bigg(D_1(\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}})\,\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}} + D_2(\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}})\,\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}} \tag{3147}$$
$$+ 2\bigg(H_3\,\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}} + H_4\,\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}} \bigg) \tag{3148}$$
$$+ 2\bigg(D_1(\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}})\,\|\phi_0\|^2_{\dot{\mathcal{F}}^{1,1}} + D_2(\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}})\,\|\phi_0\|^2_{\dot{\mathcal{F}}^{1,1}} \bigg)\bigg) \tag{3149}$$
$$\cdot \frac{\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}}{\Lambda(\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}})-\nu_0} \tag{3150}$$
$$+ \frac{\sqrt{1+\frac{\pi}{2}(e^{2\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}}-1)}}{R} \tag{3151}$$
$$\cdot 2\bigg(D_1(\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}})\,\|\phi_0\|^2_{\dot{\mathcal{F}}^{1,1}} + D_2(\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}})\,\|\phi_0\|^2_{\dot{\mathcal{F}}^{1,1}} \tag{3152}$$
$$+ 2\,\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}} \bigg(H_3\,\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}} + H_4\,\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}} \bigg) \tag{3153}$$
$$+ 2\,\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}} \bigg(D_1(\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}})\,\|\phi_0\|^2_{\dot{\mathcal{F}}^{1,1}} + D_2(\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}})\,\|\phi_0\|^2_{\dot{\mathcal{F}}^{1,1}} \bigg)\bigg) \tag{3154}$$
$$\cdot \frac{\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}}}{\Lambda(\|\phi_0\|_{\dot{\mathcal{F}}^{1,1}})-\nu_0}. \tag{3155}$$

Hence, $\mathcal{F}(\theta)(0)$ is bounded in time.

15.3 Regularization Argument

In this Section, we present the details of the regularization argument that is necessary to construct a proof of our main theorem. First of all, based on the original equations for the dynamics of the interface, we define a collection of regularized equations for the dynamics of the interface, which is indexed by $\mathbb{N}$. The sequence of solutions to these regularized equations produces what turns out to be a solution to the original evolution equation for the interface. To obtain solutions to the regularized equations for the dynamics of the interface, we leverage Picard's theorem in the Banach space setting as stated in [3], i.e.,

Theorem 2. *Let $O \subseteq B$ be an open subset of a Banach space B with norm $\|\cdot\|_B$ and let $F : O \to B$ be a nonlinear operator satisfying the following conditions:*

1. *F maps O into B.*
2. *F is locally Lipschitz continuous, i.e., for any $X \in O$ there exists $L > 0$ and an open neighborhood $U_X \subseteq O$ of X such that*

$$\left\| F(\tilde{X}) - F(\hat{X}) \right\|_B \leq L \left\| \tilde{X} - \hat{X} \right\|_B \tag{3156}$$

   for all $\tilde{X}, \hat{X} \in U_X$.

Then for any $X_0 \in O$, there exists a time T such that the ordinary differential equation

$$\frac{dX}{dt} = F(X) \tag{3157}$$

$$X(0) = X_0 \in O \tag{3158}$$

has a unique local solution $X \in C^1((-T, T); O)$. If F does not depend explicitly on time, then solutions to the above ODE can be continued until they leave the set O.

To obtain a candidate for a solution to the original equation for the dynamics of the interface, we use the Aubin-Lions lemma as stated in [1], i.e.,

Lemma 7. *Let X_0, X, and X_1 be Banach spaces such that*

$$X_0 \subseteq X \subseteq X_1, \tag{3159}$$

with compact embedding $X_0 \hookrightarrow X$, and let $p \in (1, \infty]$. Let G be a set of functions mapping $[0, T]$ into X_1 such that G is bounded in $L^p([0, T]; X) \cap L^1_{loc}([0, T]; X_0)$ and $\partial_t G$ is bounded in $L^1_{loc}([0, T]; X_1)$. Then G is relatively compact in $L^q([0, T]; X)$, where $q \in [1, p)$.

15.3.1 Regularized Equations for Interface Dynamics

For each $N \in \mathbb{N}$, let us define regularized equations for the dynamics of the interface. We recall that, under the HLS parametrization, the dynamics of the interface are governed by

$$\theta_t(\alpha) = \frac{2\pi}{L(t)}(U_\alpha(\theta)(\alpha) + T(\theta)(\alpha)(1 + \theta_\alpha(\alpha))), \tag{3160}$$

$$L(t) = 2\pi R\left(1 + \frac{1}{2\pi}\mathrm{Im}\int_{-\pi}^{\pi}\int_0^{\alpha} e^{i(\alpha-\eta)}\sum_{n\geq 1}\frac{i^n}{n!}(\theta(\alpha) - \theta(\eta))^n d\eta d\alpha\right)^{-\frac{1}{2}}, \tag{3161}$$

as long as $\|\phi(t)\|_{\mathcal{F}^{0,1}}$ is sufficiently small for all $t \geq 0$, as indicated in the Remark for Proposition 7. As mentioned before, since the equations are written in terms of the HLS parametrization, they satisfy the identity

$$\int_{-\pi}^{\pi} e^{i(\alpha+\phi(\alpha,t))} d\alpha = 0, \tag{3162}$$

which constrains $\phi(\alpha, t)$ to have its ± 1 Fourier modes be completely determined by the rest of its nonzero Fourier modes at any given time. Therefore, we seek from the outset for a solution whose ± 1 Fourier modes remain zero in time. Throughout the rest of this Section, we exploit the fact that the analytical expressions for U and T written in terms of $\phi = \theta - \hat{\theta}(0)$ are identical to their respective analytical expressions written in terms of θ. This means that the analytical expressions for U and T in terms of θ are obtained by simply replacing ϕ with θ in the respective analytical expressions for U and T in terms of ϕ. For any fixed $N \in \mathbb{N}$, we define the regularized ordinary differential equation for the interface

$$\frac{d\theta_N}{dt} = (\mathcal{J}_N^1 \circ G_N)(\theta_N). \tag{3163}$$

Here, $\mathcal{J}_N^1$ is the high frequency cut-off operator introduced in (32) and

$$G_N(\theta_N) = R^{-1}\left(1 + \frac{1}{2\pi}\mathrm{Im}\int_{-\pi}^{\pi}\int_0^{\alpha} e^{i(\alpha-\eta)}\sum_{n\geq 1}\frac{i^n}{n!}(\theta_N(\alpha) - \theta_N(\eta))^n d\eta d\alpha\right)^{\frac{1}{2}} \tag{3164}$$

$$\cdot\left((U_\alpha)_N(\theta_N) + T_N(\theta_N)\left(1 + (\theta_N)_\alpha\right)\right), \tag{3165}$$

where

$$(U_\alpha)_N(\theta_N)(\alpha) = (\mathcal{J}_N \circ \mathrm{Re})\left(W(\theta_N)(\alpha)\right), \tag{3166}$$

$$U_N(\theta_N)(\alpha) = (\mathcal{J}_N \circ \mathrm{Re})\left(V(\theta_N)(\alpha)\right), \tag{3167}$$

$$T_N(\theta_N)(\alpha) = \mathcal{M}\left(\left(1 + (\theta_N)_\alpha(\alpha)\right)U_N(\theta_N)(\alpha)\right). \tag{3168}$$

Here, $\mathcal{J}_N$ is the high frequency cut-off operator defined in (31). Recalling (327) and (328) as well as (995) and (1191), we define

$$V(\theta_N)(\alpha) = \sum_{j=1}^{7}\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}E_j(\theta_N)(\alpha,\beta)d\beta \tag{3169}$$

$$+\sum_{j=1}^{8}\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\widetilde{B_j}(\theta_N)(\alpha,\beta)d\beta + \frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\widetilde{B_{13}}(\theta_N)(\alpha,\beta)d\beta. \tag{3170}$$

Lastly, we define

$$W(\theta_N)(\alpha) = V_\alpha(\theta_N)(\alpha) \tag{3171}$$

$$= \sum_{j=1}^{7}\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}(E_j)_\alpha(\theta_N)(\alpha,\beta)d\beta \tag{3172}$$

$$+\sum_{j=1}^{8}\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}(\widetilde{B_j})_\alpha(\theta_N)(\alpha,\beta)d\beta + \frac{\gamma}{4\pi}\int_{-\pi}^{\pi}(\widetilde{B_{13}})_\alpha(\theta_N)(\alpha,\beta)d\beta. \tag{3173}$$

In the expressions $(E_5)_\alpha(\theta_N)(\alpha,\beta)$ and $(E_6)_\alpha(\theta_N)(\alpha,\beta)$, the second derivative of θ_N shows up. We replace them with lower-order derivatives of θ_N by applying integration by parts. For example, recall that

$$(E_5)_\alpha(\theta_N)(\alpha,\beta) = \frac{-(-1+e^{i\beta})i\beta e^{i\beta}(-1+e^{i\beta})}{2(-1+e^{i\beta})^2} \tag{3174}$$

$$\cdot \int_0^1 e^{-i\beta s}(-1+s)(\theta_N)_{\alpha\alpha}(\alpha+\beta(-1+s))ds. \tag{3175}$$

Using integration by parts, we obtain

$$\int_0^1 e^{-i\beta s}(-1+s)(\theta_N)_{\alpha\alpha}(\alpha+\beta(-1+s))ds \tag{3176}$$

$$= \frac{(\theta_N)_\alpha(\alpha-\beta)}{\beta} - \int_0^1 \frac{1}{\beta}(e^{-i\beta s}(-i\beta)(-1+s)+e^{-i\beta s})(\theta_N)_\alpha(\alpha+\beta(-1+s))ds. \tag{3177}$$

15.3.2 Applying Picard's Theorem

We now specify an appropriate Banach space for Picard's theorem. For any $N \in \mathbb{N}$, let

$$H_N^m = \left\{ f \in H^m([-\pi,\pi)) : \mathrm{supp}(\hat{f}) \subseteq [-N,N],\ \hat{f}(\pm 1) = 0,\, \mathrm{Im}(f) = 0 \right\}. \tag{3178}$$

The space H_N^m contains the requirement that the ± 1 Fourier modes be zero, because we intend to find a candidate for a solution to the original equations for the dynamics of the interface with this property. The following proposition states that H_N^m is indeed a Banach space.

Proposition 12. *H_N^m is a Banach space.*

Proof. It suffices to show that H_N^m is a closed $\mathbb{R}$-subspace of $H^m([-\pi,\pi))$. It is straightforward to check that H_N^m, which is nonempty because $0 \in H_N^m$, is closed under addition and scalar multiplication. To check that the subspace is closed in $H^m([-\pi,\pi))$, consider a sequence $\{f_n\}$ in H_N^m converging to f in the H^m norm. We show that $f \in H_N^m$. Since

$$\lim_{n\to\infty} \sum_{k\in\mathbb{Z}} (1+|k|^2)^m \left|\mathcal{F}(f_n-f)(k)\right|^2 = 0, \tag{3179}$$

there exists a (non-relabeled) subsequence such that for all $k \in \mathbb{Z}$,

$$\lim_{n\to\infty} \mathcal{F}(f_n)(k) = \mathcal{F}(f)(k), \tag{3180}$$

which implies that $\operatorname{supp}(\hat{f}) \subseteq [-N, N]$ and $\hat{f}(\pm 1) = 0$. For any $\alpha \in [-\pi, \pi)$,

$$|f_n(\alpha) - f(\alpha)| = \left|\sum_{k\in\mathbb{Z}} \mathcal{F}(f_n - f)(k)e^{ik\alpha}\right| \leq \sum_{k\in\mathbb{Z}} |\mathcal{F}(f_n - f)(k)| \tag{3181}$$

$$= \sum_{|k|\leq N} |\mathcal{F}(f_n - f)(k)| \leq \sum_{|k|\leq N} (1 + |k|^2)^m |\mathcal{F}(f_n - f)(k)| \tag{3182}$$

$$\leq \left(\sum_{|k|\leq N} (1 + |k|)^m\right)^{\frac{1}{2}} \left(\sum_{|k|\leq N} (1 + |k|^2)^m |\mathcal{F}(f_n - f)(k)|^2\right)^{\frac{1}{2}} \tag{3183}$$

$$= \left(\sum_{|k|\leq N} (1 + |k|)^m\right)^{\frac{1}{2}} \|f_n - f\|_{H^m}, \tag{3184}$$

which shows that f_n converges to f pointwise. Thus,

$$\operatorname{Im}(f) = \lim_{n\to\infty} \operatorname{Im}(f_n) = 0. \tag{3185}$$

Therefore, H_N^m is a closed $\mathbb{R}$-subspace of H_N^m, as needed. ■

For any $M > 0$, let

$$O^M = \{f \in H_N^m : \|f\|_{H^m} < M\}. \tag{3186}$$

We want to apply Picard's theorem by setting $B = H_N^m$, $O = O^M$, and $F = \mathcal{J}_N^1 \circ G_N$. To check the first condition that $\mathcal{J}_N^1 \circ G_N$ maps O^M into H_N^m, let $f \in O^M$. It is immediate from the definition of the regularized equations that $\operatorname{supp}\mathcal{F}((\mathcal{J}_N^1 \circ G_N)(f)) \subseteq [-N, N]$ and $\mathcal{F}((\mathcal{J}_N^1 \circ G_N)(f))(\pm 1) = 0$. Since $G_N(f)$ is real,

$$(\mathcal{J}_N^1 \circ G_N)(f)(\alpha) = \sum_{\substack{|k|\leq N\\ |k|\neq 1}} \mathcal{F}(G_N(f))(k)e^{ik\alpha} \tag{3187}$$

$$= \mathcal{F}(G_N(f))(0) + \sum_{j=2}^{N} \sum_{|k|=j} \mathcal{F}(G_N(f))(k)e^{ik\alpha} \tag{3188}$$

$$= \mathcal{F}(G_N(f))(0) + \sum_{j=2}^{N} \left(\mathcal{F}(G_N(f))(-j)e^{-ij\alpha} + \mathcal{F}(G_N(f))(j)e^{ij\alpha}\right) \tag{3189}$$

$$= \mathcal{F}(G_N(f))(0) + \sum_{j=2}^{N} \left(\overline{\mathcal{F}(G_N(f))(j)e^{ij\alpha}} + \mathcal{F}(G_N(f))(j)e^{ij\alpha}\right), \tag{3190}$$

which is real. Hence, $\operatorname{Im}(\mathcal{J}_N^1 \circ G_N)(f) = 0$. To check that $(\mathcal{J}_N^1 \circ G_N)(f) \in H^m([-\pi, \pi))$, we note that it suffices to check the second condition that $\mathcal{J}_N^1 \circ G_N$ is locally Lipschitz continuous. In particular, we show that one can choose $U_X = O^M$ for any $X \in O^M$. Let

$\theta_N^1, \theta_N^2 \in O^M$. Then

$$G_N(\theta_N^1) - G_N(\theta_N^2) \tag{3191}$$

$$= R^{-1}\left(1 + \frac{1}{2\pi}\mathrm{Im}\int_{-\pi}^{\pi}\int_0^{\alpha} e^{i(\alpha-\eta)}\sum_{n\geq 1}\frac{i^n}{n!}(\theta_N^1(\alpha) - \theta_N^1(\eta))^n d\eta d\alpha\right)^{\frac{1}{2}} \tag{3192}$$

$$\cdot\left((U_\alpha)_N(\theta_N^1) + T_N(\theta_N^1)\left(1 + (\theta_N^1)_\alpha\right)\right) \tag{3193}$$

$$- R^{-1}\left(1 + \frac{1}{2\pi}\mathrm{Im}\int_{-\pi}^{\pi}\int_0^{\alpha} e^{i(\alpha-\eta)}\sum_{n\geq 1}\frac{i^n}{n!}(\theta_N^2(\alpha) - \theta_N^2(\eta))^n d\eta d\alpha\right)^{\frac{1}{2}} \tag{3194}$$

$$\cdot\left((U_\alpha)_N(\theta_N^2) + T_N(\theta_N^2)\left(1 + (\theta_N^2)_\alpha\right)\right) \tag{3195}$$

$$= R^{-1}\left(1 + \frac{1}{2\pi}\mathrm{Im}\int_{-\pi}^{\pi}\int_0^{\alpha} e^{i(\alpha-\eta)}\sum_{n\geq 1}\frac{i^n}{n!}(\theta_N^1(\alpha) - \theta_N^1(\eta))^n d\eta d\alpha\right)^{\frac{1}{2}} \tag{3196}$$

$$\cdot\left((U_\alpha)_N(\theta_N^1) + T_N(\theta_N^1)\left(1 + (\theta_N^1)_\alpha\right)\right) \tag{3197}$$

$$- R^{-1}\left(1 + \frac{1}{2\pi}\mathrm{Im}\int_{-\pi}^{\pi}\int_0^{\alpha} e^{i(\alpha-\eta)}\sum_{n\geq 1}\frac{i^n}{n!}(\theta_N^1(\alpha) - \theta_N^1(\eta))^n d\eta d\alpha\right)^{\frac{1}{2}} \tag{3198}$$

$$\cdot\left((U_\alpha)_N(\theta_N^2) + T_N(\theta_N^2)\left(1 + (\theta_N^2)_\alpha\right)\right) \tag{3199}$$

$$+ R^{-1}\left(1 + \frac{1}{2\pi}\mathrm{Im}\int_{-\pi}^{\pi}\int_0^{\alpha} e^{i(\alpha-\eta)}\sum_{n\geq 1}\frac{i^n}{n!}(\theta_N^1(\alpha) - \theta_N^1(\eta))^n d\eta d\alpha\right)^{\frac{1}{2}} \tag{3200}$$

$$\cdot\left((U_\alpha)_N(\theta_N^2) + T_N(\theta_N^2)\left(1 + (\theta_N^2)_\alpha\right)\right) \tag{3201}$$

$$- R^{-1}\left(1 + \frac{1}{2\pi}\mathrm{Im}\int_{-\pi}^{\pi}\int_0^{\alpha} e^{i(\alpha-\eta)}\sum_{n\geq 1}\frac{i^n}{n!}(\theta_N^2(\alpha) - \theta_N^2(\eta))^n d\eta d\alpha\right)^{\frac{1}{2}} \tag{3202}$$

$$\cdot\left((U_\alpha)_N(\theta_N^2) + T_N(\theta_N^2)\left(1 + (\theta_N^2)_\alpha\right)\right) \tag{3203}$$

$$=R^{-1}\left(1+\frac{1}{2\pi}\mathrm{Im}\int_{-\pi}^{\pi}\int_0^{\alpha}e^{i(\alpha-\eta)}\sum_{n\geq 1}\frac{i^n}{n!}(\theta_N^1(\alpha)-\theta_N^1(\eta))^n d\eta d\alpha\right)^{\frac{1}{2}} \tag{3204}$$

$$\cdot\left((U_\alpha)_N(\theta_N^1)-(U_\alpha)_N(\theta_N^2)+T_N(\theta_N^1)\left(1+(\theta_N^1)_\alpha\right)-T_N(\theta_N^2)\left(1+(\theta_N^2)_\alpha\right)\right) \tag{3205}$$

$$+\left(R^{-1}\left(1+\frac{1}{2\pi}\mathrm{Im}\int_{-\pi}^{\pi}\int_0^{\alpha}e^{i(\alpha-\eta)}\sum_{n\geq 1}\frac{i^n}{n!}(\theta_N^1(\alpha)-\theta_N^1(\eta))^n d\eta d\alpha\right)^{\frac{1}{2}}\right. \tag{3206}$$

$$\left.-R^{-1}\left(1+\frac{1}{2\pi}\mathrm{Im}\int_{-\pi}^{\pi}\int_0^{\alpha}e^{i(\alpha-\eta)}\sum_{n\geq 1}\frac{i^n}{n!}(\theta_N^2(\alpha)-\theta_N^2(\eta))^n d\eta d\alpha\right)^{\frac{1}{2}}\right) \tag{3207}$$

$$\cdot\left((U_\alpha)_N(\theta_N^2)+T_N(\theta_N^2)\left(1+(\theta_N^2)_\alpha\right)\right). \tag{3208}$$

Thus,

$$\left\|G_N(\theta_N^1)-G_N(\theta_N^2)\right\|_{H^m} \tag{3209}$$

$$\leq R^{-1}\left(1+\frac{1}{2\pi}\mathrm{Im}\int_{-\pi}^{\pi}\int_0^{\alpha}e^{i(\alpha-\eta)}\sum_{n\geq 1}\frac{i^n}{n!}(\theta_N^1(\alpha)-\theta_N^1(\eta))^n d\eta d\alpha\right)^{\frac{1}{2}} \tag{3210}$$

$$\cdot\left\|(U_\alpha)_N(\theta_N^1)-(U_\alpha)_N(\theta_N^2)+T_N(\theta_N^1)\left(1+(\theta_N^1)_\alpha\right)-T_N(\theta_N^2)\left(1+(\theta_N^2)_\alpha\right)\right\|_{H^m} \tag{3211}$$

$$+\left(R^{-1}\left(1+\frac{1}{2\pi}\mathrm{Im}\int_{-\pi}^{\pi}\int_0^{\alpha}e^{i(\alpha-\eta)}\sum_{n\geq 1}\frac{i^n}{n!}(\theta_N^1(\alpha)-\theta_N^1(\eta))^n d\eta d\alpha\right)^{\frac{1}{2}}\right. \tag{3212}$$

$$\left.-R^{-1}\left(1+\frac{1}{2\pi}\mathrm{Im}\int_{-\pi}^{\pi}\int_0^{\alpha}e^{i(\alpha-\eta)}\sum_{n\geq 1}\frac{i^n}{n!}(\theta_N^2(\alpha)-\theta_N^2(\eta))^n d\eta d\alpha\right)^{\frac{1}{2}}\right) \tag{3213}$$

$$\cdot\left\|(U_\alpha)_N(\theta_N^2)+T_N(\theta_N^2)\left(1+(\theta_N^2)_\alpha\right)\right\|_{H^m} \tag{3214}$$

$$\leq R^{-1}\left(1+\frac{1}{2\pi}\mathrm{Im}\int_{-\pi}^{\pi}\int_0^{\alpha}e^{i(\alpha-\eta)}\sum_{n\geq 1}\frac{i^n}{n!}(\theta_N^1(\alpha)-\theta_N^1(\eta))^n d\eta d\alpha\right)^{\frac{1}{2}} \tag{3215}$$

$$\cdot\left(\left\|(U_\alpha)_N(\theta_N^1)-(U_\alpha)_N(\theta_N^2)\right\|_{H^m}+\left\|T_N(\theta_N^1)-T_N(\theta_N^2)\right\|_{H^m}\right. \tag{3216}$$

$$\left.+\left\|T_N(\theta_N^1)((\theta_N^1)_\alpha-(\theta_N^2)_\alpha)\right\|_{H^m}+\left\|(T_N(\theta_N^1)-T_N(\theta_N^2))(\theta_N^2)_\alpha\right\|_{H^m}\right) \tag{3217}$$

$$+\left(R^{-1}\left(1+\frac{1}{2\pi}\mathrm{Im}\int_{-\pi}^{\pi}\int_0^{\alpha}e^{i(\alpha-\eta)}\sum_{n\geq 1}\frac{i^n}{n!}(\theta_N^1(\alpha)-\theta_N^1(\eta))^n d\eta d\alpha\right)^{\frac{1}{2}}\right. \tag{3218}$$

$$\left.-R^{-1}\left(1+\frac{1}{2\pi}\mathrm{Im}\int_{-\pi}^{\pi}\int_0^{\alpha}e^{i(\alpha-\eta)}\sum_{n\geq 1}\frac{i^n}{n!}(\theta_N^2(\alpha)-\theta_N^2(\eta))^n d\eta d\alpha\right)^{\frac{1}{2}}\right) \tag{3219}$$

$$\cdot\left(\left\|(U_\alpha)_N(\theta_N^2)\right\|_{H^m}+\left\|T_N(\theta_N^2)\right\|_{H^m}+\left\|T_N(\theta_N^2)(\theta_N^2)_\alpha\right\|_{H^m}\right). \tag{3220}$$

To check the local Lipschitz continuity of $\mathcal{J}_N^1 \circ G_N$, we need to derive an appropriate estimate for the upper bound for $\|G_N(\theta_N^1) - G_N(\theta_N^2)\|_{H^m}$, shown in (3215) through (3220). We present in detail the process of deriving such estimates for a select few terms in this upper bound, which are typical of the terms making up the upper bound. These derivations will showcase all the techniques that are necessary to derive estimates for the rest of the terms in the upper bound. First of all, we consider the term $\|(U_\alpha)_N(\theta_N^1) - (U_\alpha)_N(\theta_N^2)\|_{H^m}$, which appears in (3216). Using the definition of $(U_\alpha)_N$ in (3166), we obtain

$$\left\|(U_\alpha)_N(\theta_N^1) - (U_\alpha)_N(\theta_N^2)\right\|_{H^m} \tag{3221}$$

$$\leq \left\|\mathcal{J}_N\left(W(\theta_N^1)(\alpha) - W(\theta_N^2)(\alpha)\right)\right\|_{H^m} \tag{3222}$$

$$\leq \sum_{j=1}^{7} \left\|\mathcal{J}_N\left(\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}(E_j)_\alpha(\theta_N^1)(\alpha,\beta)d\beta - \frac{\gamma}{4\pi}\int_{-\pi}^{\pi}(E_j)_\alpha(\theta_N^2)(\alpha,\beta)d\beta\right)\right\|_{H^m} \tag{3223}$$

$$+ \sum_{j=1}^{8} \left\|\mathcal{J}_N\left(\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}(\widetilde{B_j})_\alpha(\theta_N^1)(\alpha,\beta)d\beta - \frac{\gamma}{4\pi}\int_{-\pi}^{\pi}(\widetilde{B_j})_\alpha(\theta_N^2)(\alpha,\beta)d\beta\right)\right\|_{H^m} \tag{3224}$$

$$+ \left\|\mathcal{J}_N\left(\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}(\widetilde{B_{13}})_\alpha(\theta_N^1)(\alpha,\beta)d\beta - \frac{\gamma}{4\pi}\int_{-\pi}^{\pi}(\widetilde{B_{13}})_\alpha(\theta_N^2)(\alpha,\beta)d\beta\right)\right\|_{H^m}. \tag{3225}$$

To obtain an appropriate estimate, it is necessary to build some groundwork. For any $m > 0$, we define for a sequence a defined on $k \in \mathbb{Z}$

$$\|a\|_{h^m} = \left(\sum_{k\in\mathbb{Z}}(1+|k|^2)^m\,|a(k)|^2\right)^{1/2}. \tag{3226}$$

Lemma 8. *Let $N \in \mathbb{N}$ and $m \geq 1$. If a and b are sequences on $\mathbb{Z}$, then*

$$\left\|1_{|\cdot|\leq N}(a*b)\right\|_{h^m} \tag{3227}$$

$$\leq r(m,N)\left\|1_{|\cdot|\leq N}a\right\|_{h^m}\left\|1_{|\cdot|\leq N}b\right\|_{h^m}, \tag{3228}$$

where

$$r(m,N) = 2^{2m}(1+N^2)^{m/2}\left(\sum_{|k|\leq N}(1+|k|^2)^m\right)^{1/2}. \tag{3229}$$

Proof.

$$\left\|1_{|\cdot|\leq N}(a*b)\right\|_{h^m} \tag{3230}$$

$$=\left(\sum_{|k|\leq N}(1+|k|^2)^m\,|(a*b)(k)|^2\right)^{1/2} \tag{3231}$$

$$\leq\left(\sum_{|k|\leq N}(1+|k|^2)^m\left|\sum_{j\in\mathbb{Z}}|a(k-j)|\,|b(j)|\right|^2\right)^{1/2} \tag{3232}$$

$$\leq\left(\sum_{|k|\leq N}\left|\sum_{j\in\mathbb{Z}}(1+|k|^2)^m\,|a(k-j)|\,|b(j)|\right|^2\right)^{1/2}. \tag{3233}$$

Since $m \geq 1$, for any $k, j \in \mathbb{Z}$,

$$(1+|k|^2)^m \leq \left(1+2|k-j|^2+2|j|^2\right)^m \tag{3234}$$
$$\leq \left(2(1+|k-j|^2)+2(1+|j|^2)\right)^m \tag{3235}$$
$$\leq 2^{m-1}\left(\left(2(1+|k-j|^2)\right)^m+\left(2(1+|j|^2)\right)^m\right) \tag{3236}$$
$$=2^{2m-1}\left((1+|k-j|^2)^m+(1+|j|^2)^m\right). \tag{3237}$$

Then

$$\left(\sum_{|k|\leq N}\left|\sum_{j\in\mathbb{Z}}(1+|k|^2)^m\,|a(k-j)|\,|b(j)|\right|^2\right)^{1/2} \tag{3238}$$
$$\leq\left(\sum_{|k|\leq N}\left|\sum_{j\in\mathbb{Z}}2^{2m-1}\left((1+|k-j|^2)^m+(1+|j|^2)^m\right)|a(k-j)|\,|b(j)|\right|^2\right)^{1/2} \tag{3239}$$
$$\leq\left(\sum_{|k|\leq N}\left|\sum_{j\in\mathbb{Z}}2^{2m-1}(1+|k-j|^2)^m\,|a(k-j)|\,|b(j)|\right|^2\right)^{1/2} \tag{3240}$$
$$+\left(\sum_{|k|\leq N}\left|\sum_{j\in\mathbb{Z}}2^{2m-1}(1+|j|^2)^m\,|a(k-j)|\,|b(j)|\right|^2\right)^{1/2}. \tag{3241}$$

Letting $(\mathfrak{F}a)(k)=(1+|k|^2)^m\cdot a(k)$, we obtain

$$\left(\sum_{|k|\leq N}\left|\sum_{j\in\mathbb{Z}}2^{2m-1}(1+|k-j|^2)^m\,|a(k-j)|\,|b(j)|\right|^2\right)^{1/2} \tag{3242}$$
$$+\left(\sum_{|k|\leq N}\left|\sum_{j\in\mathbb{Z}}2^{2m-1}(1+|j|^2)^m\,|a(k-j)|\,|b(j)|\right|^2\right)^{1/2} \tag{3243}$$
$$\leq 2^{2m-1}\left(\sum_{|k|\leq N}|(\mathfrak{F}\,|a|*|b|)(k)|^2\right)^{1/2}+2^{2m-1}\left(\sum_{|k|\leq N}|(\mathfrak{F}\,|b|*|a|)(k)|^2\right)^{1/2}. \tag{3244}$$

By Young's inequality,

$$2^{2m-1}\left(\sum_{|k|\le N}\left|(\mathfrak{F}\,|a|*|b|)(k)\right|^2\right)^{1/2}+2^{2m-1}\left(\sum_{|k|\le N}\left|(\mathfrak{F}\,|b|*|a|)(k)\right|^2\right)^{1/2} \tag{3245}$$

$$\le 2^{2m-1}\left(\sum_{|k|\le N}\left|(\mathfrak{F}\,|a|)(k)\right|^2\right)^{1/2}\cdot\left(\sum_{|k|\le N}|b(k)|\right) \tag{3246}$$

$$+\,2^{2m-1}\left(\sum_{|k|\le N}\left|(\mathfrak{F}\,|b|)(k)\right|^2\right)^{1/2}\cdot\left(\sum_{|k|\le N}|a(k)|\right) \tag{3247}$$

$$=2^{2m-1}\left(\sum_{|k|\le N}(1+|k|^2)^{2m}\,|a(k)|^2\right)^{1/2}\cdot\left(\sum_{|k|\le N}(1+|k|^2)^m\,|b(k)|\right) \tag{3248}$$

$$+\,2^{2m-1}\left(\sum_{|k|\le N}(1+|k|^2)^{2m}\,|b(k)|^2\right)^{1/2}\cdot\left(\sum_{|k|\le N}(1+|k|^2)^m\,|a(k)|\right) \tag{3249}$$

$$\le 2^{2m-1}(1+N^2)^{m/2}\left(\sum_{|k|\le N}(1+|k|^2)^m\,|a(k)|^2\right)^{1/2}\cdot\left(\sum_{|k|\le N}(1+|k|^2)^m\right)^{1/2} \tag{3250}$$

$$\cdot\left(\sum_{|k|\le N}(1+|k|^2)^m\,|b(k)|^2\right)^{1/2} \tag{3251}$$

$$+\,2^{2m-1}(1+N^2)^{m/2}\left(\sum_{|k|\le N}(1+|k|^2)^m\,|b(k)|^2\right)^{1/2}\cdot\left(\sum_{|k|\le N}(1+|k|^2)^m\right)^{1/2} \tag{3252}$$

$$\cdot\left(\sum_{|k|\le N}(1+|k|^2)^m\,|a(k)|^2\right)^{1/2} \tag{3253}$$

$$=2^{2m}(1+N^2)^{m/2}\left(\sum_{|k|\le N}(1+|k|^2)^m\right)^{1/2}\left\|1_{|\cdot|\le N}a\right\|_{h^m}\left\|1_{|\cdot|\le N}b\right\|_{h^m}, \tag{3254}$$

as needed. ■

Lemma 9. *Let $m \ge 0$. If f is a periodic function such that* $\operatorname{supp}\mathcal{F}(f)\subseteq[-M,M]$, *then*

$$\|f_\alpha\|_{H^m}\le \tilde{r}(M)\,\|f\|_{H^m}, \tag{3255}$$

where

$$\tilde{r}(M)=(1+M^2)^{1/2}. \tag{3256}$$

Proof.

$$\|f_\alpha\|_{H^m} \leq \left(\sum_{k\in\mathbb{Z}}(1+|k|^2)^{m+1}\left|\mathcal{F}(f)(k)\right|^2\right)^{1/2} \tag{3257}$$

$$= \left(\sum_{|k|\leq M}(1+|k|^2)^{m+1}\left|\mathcal{F}(f)(k)\right|^2\right)^{1/2} \tag{3258}$$

$$\leq (1+M^2)^{1/2}\left(\sum_{k\in\mathbb{Z}}(1+|k|^2)^{m}\left|\mathcal{F}(f)(k)\right|^2\right)^{1/2} \tag{3259}$$

$$= (1+M^2)^{1/2}\|f\|_{H^m}, \tag{3260}$$

as needed. ■

Using these lemmas, we present the derivation of an estimate for

$$\left\|\mathcal{J}_N\left(\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}B^1_{1,1}(\theta^1_N)(\alpha,\beta)d\beta - \frac{\gamma}{4\pi}\int_{-\pi}^{\pi}B^1_{1,1}(\theta^2_N)(\alpha,\beta)d\beta\right)\right\|_{H^m}, \tag{3261}$$

which is one of the terms making up the first term in the sum appearing in (3224). We recall that

$$B^1_{1,1}(\theta_N)(\alpha,\beta) = -\sum_{j_1+j_2+n\geq 1}\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}\cdot\frac{(-1)^{j_1+1}i^{j_1+j_2+1}}{2j_1!j_2!}j_1\theta_N(\alpha-\beta)^{j_1-1} \tag{3262}$$

$$\cdot(\theta_N)_\alpha(\alpha-\beta)\theta_N(\alpha)^{j_2}\cdot \tag{3263}$$

$$\int_0^1 e^{-i\beta s}\theta_N(\alpha+\beta(-1+s))(-1+s)ds \tag{3264}$$

$$\cdot\left(\sum_{m=1}^{\infty}\frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\frac{(-i\theta_N(\alpha+(s-1)\beta))^m}{m!}ds\right)^n. \tag{3265}$$

Using the telescoping sum, we obtain

$$B^1_{1,1}(\theta^1_N)(\alpha,\beta) - B^1_{1,1}(\theta^2_N)(\alpha,\beta) \tag{3266}$$

$$= -\sum_{j_1+j_2+n\geq 1}\Bigg(\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}\cdot\frac{(-1)^{j_1+1}i^{j_1+j_2+1}}{2j_1!j_2!}j_1\theta^1_N(\alpha-\beta)^{j_1-1} \tag{3267}$$

$$\cdot(\theta^1_N)_\alpha(\alpha-\beta)\theta^1_N(\alpha)^{j_2} \tag{3268}$$

$$\cdot\int_0^1 e^{-i\beta s}\theta^1_N(\alpha+\beta(-1+s))(-1+s)ds \tag{3269}$$

$$\cdot\left(\sum_{m=1}^{\infty}\frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\frac{(-i\theta^1_N(\alpha+(s-1)\beta))^m}{m!}ds\right)^n \tag{3270}$$

$$-\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}\cdot\frac{(-1)^{j_1+1}i^{j_1+j_2+1}}{2j_1!j_2!}j_1\theta^2_N(\alpha-\beta)^{j_1-1} \tag{3271}$$

$$\cdot(\theta^2_N)_\alpha(\alpha-\beta)\theta^2_N(\alpha)^{j_2} \tag{3272}$$

$$\cdot\int_0^1 e^{-i\beta s}\theta^2_N(\alpha+\beta(-1+s))(-1+s)ds \tag{3273}$$

$$\cdot\left(\sum_{m=1}^{\infty}\frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\frac{(-i\theta^2_N(\alpha+(s-1)\beta))^m}{m!}ds\right)^n\Bigg) \tag{3274}$$

$$= - \sum_{j_1+j_2+n\geq 1} \frac{-i\beta e^{2i\beta}}{1-e^{i\beta}} \cdot \frac{(-1)^{j_1+1} i^{j_1+j_2+1}}{2 j_1! j_2!} j_1 \tag{3275}$$

$$\cdot \Bigg((\theta_N^1 - \theta_N^2)(\alpha-\beta)\theta_N^1(\alpha-\beta)^{j_1-2}(\theta_N^1)_\alpha(\alpha-\beta)\theta_N^1(\alpha)^{j_2} \tag{3276}$$

$$\cdot \int_0^1 e^{-i\beta s}\theta_N^1(\alpha+\beta(-1+s))(-1+s)ds \tag{3277}$$

$$\cdot \Bigg(\sum_{m=1}^\infty \frac{i\beta}{1-e^{i\beta}} \int_0^1 e^{-i(s-1)\beta} \frac{(-i\theta_N^1(\alpha+(s-1)\beta))^m}{m!} ds\Bigg)^n \tag{3278}$$

$$+ \cdots \tag{3279}$$

$$+ \theta_N^2(\alpha-\beta)^{j_1-2}(\theta_N^1 - \theta_N^2)(\alpha-\beta)(\theta_N^1)_\alpha(\alpha-\beta)\theta_N^1(\alpha)^{j_2} \tag{3280}$$

$$\cdot \int_0^1 e^{-i\beta s}\theta_N^1(\alpha+\beta(-1+s))(-1+s)ds \tag{3281}$$

$$\cdot \Bigg(\sum_{m=1}^\infty \frac{i\beta}{1-e^{i\beta}} \int_0^1 e^{-i(s-1)\beta} \frac{(-i\theta_N^1(\alpha+(s-1)\beta))^m}{m!} ds\Bigg)^n \tag{3282}$$

$$+ \theta_N^2(\alpha-\beta)^{j_1-1}((\theta_N^1)_\alpha - (\theta_N^2)_\alpha)(\alpha-\beta)\theta_N^1(\alpha)^{j_2} \tag{3283}$$

$$\cdot \int_0^1 e^{-i\beta s}\theta_N^1(\alpha+\beta(-1+s))(-1+s)ds \tag{3284}$$

$$\cdot \Bigg(\sum_{m=1}^\infty \frac{i\beta}{1-e^{i\beta}} \int_0^1 e^{-i(s-1)\beta} \frac{(-i\theta_N^1(\alpha+(s-1)\beta))^m}{m!} ds\Bigg)^n \tag{3285}$$

$$+ \theta_N^2(\alpha-\beta)^{j_1-1}(\theta_N^2)_\alpha(\alpha-\beta)(\theta_N^1 - \theta_N^2)(\alpha)\theta_N^1(\alpha)^{j_2-1} \tag{3286}$$

$$\cdot \int_0^1 e^{-i\beta s}\theta_N^1(\alpha+\beta(-1+s))(-1+s)ds \tag{3287}$$

$$\cdot \Bigg(\sum_{m=1}^\infty \frac{i\beta}{1-e^{i\beta}} \int_0^1 e^{-i(s-1)\beta} \frac{(-i\theta_N^1(\alpha+(s-1)\beta))^m}{m!} ds\Bigg)^n \tag{3288}$$

$$+ \cdots \tag{3289}$$

$$+ \theta_N^2(\alpha-\beta)^{j_1-1}(\theta_N^2)_\alpha(\alpha-\beta)\theta_N^2(\alpha)^{j_2-1}(\theta_N^1 - \theta_N^2)(\alpha) \tag{3290}$$

$$\cdot \int_0^1 e^{-i\beta s}\theta_N^1(\alpha+\beta(-1+s))(-1+s)ds \tag{3291}$$

$$\cdot \Bigg(\sum_{m=1}^\infty \frac{i\beta}{1-e^{i\beta}} \int_0^1 e^{-i(s-1)\beta} \frac{(-i\theta_N^1(\alpha+(s-1)\beta))^m}{m!} ds\Bigg)^n \tag{3292}$$

$$+ \theta_N^2(\alpha-\beta)^{j_1-1}(\theta_N^2)_\alpha(\alpha-\beta)\theta_N^2(\alpha)^{j_2} \tag{3293}$$

$$\cdot \int_0^1 e^{-i\beta s}(\theta_N^1 - \theta_N^2)(\alpha+\beta(-1+s))(-1+s)ds \tag{3294}$$

$$\cdot \Bigg(\sum_{m=1}^\infty \frac{i\beta}{1-e^{i\beta}} \int_0^1 e^{-i(s-1)\beta} \frac{(-i\theta_N^1(\alpha+(s-1)\beta))^m}{m!} ds\Bigg)^n \tag{3295}$$

$$+\theta_N^2(\alpha-\beta)^{j_1-1}(\theta_N^2)_\alpha(\alpha-\beta)\theta_N^2(\alpha)^{j_2} \tag{3296}$$
$$\cdot\int_0^1 e^{-i\beta s}\theta_N^2(\alpha+\beta(-1+s))(-1+s)ds \tag{3297}$$
$$\cdot\left(\sum_{m=1}^\infty \frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\frac{(-i\theta_N^1(\alpha+(s-1)\beta))^m-(-i\theta_N^2(\alpha+(s-1)\beta))^m}{m!}ds\right) \tag{3298}$$
$$\cdot\left(\sum_{m=1}^\infty \frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\frac{(-i\theta_N^1(\alpha+(s-1)\beta))^m}{m!}ds\right)^{n-1} \tag{3299}$$
$$+\cdots \tag{3300}$$
$$+\theta_N^2(\alpha-\beta)^{j_1-1}(\theta_N^2)_\alpha(\alpha-\beta)\theta_N^2(\alpha)^{j_2} \tag{3301}$$
$$\cdot\int_0^1 e^{-i\beta s}\theta_N^2(\alpha+\beta(-1+s))(-1+s)ds \tag{3302}$$
$$\cdot\left(\sum_{m=1}^\infty \frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\frac{(-i\theta_N^2(\alpha+(s-1)\beta))^m}{m!}ds\right)^{n-1} \tag{3303}$$
$$\cdot\left(\sum_{m=1}^\infty \frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\frac{(-i\theta_N^1(\alpha+(s-1)\beta))^m-(-i\theta_N^2(\alpha+(s-1)\beta))^m}{m!}ds\right)\Bigg). \tag{3304}$$

Let us consider the term starting in (3276), defined as

$$S_1(\alpha,\beta)=(\theta_N^1-\theta_N^2)(\alpha-\beta)\theta_N^1(\alpha-\beta)^{j_1-2}(\theta_N^1)_\alpha(\alpha-\beta)\theta_N^1(\alpha)^{j_2} \tag{3305}$$
$$\cdot\int_0^1 e^{-i\beta s}\theta_N^1(\alpha+\beta(-1+s))(-1+s)ds \tag{3306}$$
$$\cdot\left(\sum_{m=1}^\infty \frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\frac{(-i\theta_N^1(\alpha+(s-1)\beta))^m}{m!}ds\right)^n. \tag{3307}$$

Then

$$\mathcal{F}(S_1(\cdot,\beta))(k_1) \tag{3308}$$
$$= \sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}} \mathcal{F}((\theta_N^1-\theta_N^2)(\cdot-\beta))(k_1-k_2)\prod_{d=2}^{j_1-1}\mathcal{F}(\theta_N^1(\cdot-\beta))(k_d-k_{d+1}) \tag{3309}$$
$$\cdot\,\mathcal{F}((\theta_N^1)_\alpha(\cdot-\beta))(k_{j_1}-k_{j_1+1})\prod_{d=j_1+1}^{j_1+j_2}\mathcal{F}(\theta_N^1)(k_d-k_{d+1}) \tag{3310}$$
$$\cdot\prod_{d=j_1+j_2+1}^{j_1+j_2+n}\mathcal{F}\bigg(\sum_{m=1}^{\infty}\frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\frac{(-i\theta_N^1(\cdot+(s-1)\beta))^m}{m!}ds\bigg)(k_d-k_{d+1}) \tag{3311}$$
$$\cdot\,\mathcal{F}\bigg(\int_0^1 e^{-i\beta s}\theta_N^1(\cdot+\beta(-1+s))(-1+s)ds\bigg)(k_{j_1+j_2+n+1}) \tag{3312}$$
$$= \sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}} \mathcal{F}(\theta_N^1-\theta_N^2)(k_1-k_2)\prod_{d=2}^{j_1-1}\mathcal{F}(\theta_N^1)(k_d-k_{d+1}) \tag{3313}$$
$$\cdot\,\mathcal{F}((\theta_N^1)_\alpha)(k_{j_1}-k_{j_1+1})\prod_{d=j_1+1}^{j_1+j_2}\mathcal{F}(\theta_N^1)(k_d-k_{d+1}) \tag{3314}$$
$$\cdot\prod_{d=j_1+j_2+1}^{j_1+j_2+n}\bigg(\sum_{m=1}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}((\theta_N^1)^m)(k_d-k_{d+1})\bigg)\mathcal{F}(\theta_N^1)(k_{j_1+j_2+n+1}) \tag{3315}$$
$$\cdot\, e^{-i\beta(k_1-k_2)}e^{-i\beta(k_2-k_{j_1})}e^{-i\beta(k_{j_1}-k_{j_1+1})} \tag{3316}$$
$$\cdot\prod_{d=j_1+j_2+1}^{j_1+j_2+n}\bigg(\frac{i\beta e^{i\beta}}{1-e^{i\beta}}\int_0^1 e^{-is\beta}e^{i(s-1)\beta(k_d-k_{d+1})}ds\bigg) \tag{3317}$$
$$\cdot\int_0^1 e^{-i\beta s}(-1+s)e^{i\beta(-1+s)k_{j_1+j_2+n+1}}ds. \tag{3318}$$

We use arguments as in Section 13.1 to obtain

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(S_1(\cdot,\beta))(k_1)\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}d\beta\right| \tag{3319}$$
$$\leq C_n\bigg(\left|\mathcal{F}(\theta_N^1-\theta_N^2)\right|*\left|\mathcal{F}(\theta_N^1)\right|*\cdots*\left|\mathcal{F}(\theta_N^1)\right|*\left|\mathcal{F}((\theta_N^1)_\alpha)\right| \tag{3320}$$
$$*\left|\mathcal{F}(\theta_N^1)\right|*\cdots*\left|\mathcal{F}(\theta_N^1)\right|*\left|P(\theta_N^1)\right|*\cdots*\left|P(\theta_N^1)\right|*\left|\mathcal{F}(\theta_N^1)\right|\bigg)(k_1). \tag{3321}$$

Hence,

$$\left\| \mathcal{J}_N\left(\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} S_1(\cdot,\beta)\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}d\beta\right)\right\|_{H^m} \tag{3322}$$

$$=\left(\sum_{k\in\mathbb{Z}} 1_{|k|\le N}(1+|k|^2)^m\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(S_1(\cdot,\beta))(k)\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}d\beta\right|^2\right)^{1/2} \tag{3323}$$

$$\le C_n\left\| 1_{|\cdot|\le N}\left|\mathcal{F}(\theta_N^1-\theta_N^2)\right| * \left|\mathcal{F}(\theta_N^1)\right| * \cdots * \left|\mathcal{F}(\theta_N^1)\right| * \left|\mathcal{F}((\theta_N^1)_\alpha)\right|\right. \tag{3324}$$

$$\left. * \left|\mathcal{F}(\theta_N^1)\right| * \cdots * \left|\mathcal{F}(\theta_N^1)\right| * \left|P(\theta_N^1)\right| * \cdots * \left|P(\theta_N^1)\right| * \left|\mathcal{F}(\theta_N^1)\right|\right\|_{h^m}. \tag{3325}$$

We can apply Lemma 8 to obtain

$$\left\| \mathcal{J}_N\left(\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} S_1(\cdot,\beta)\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}d\beta\right)\right\|_{H^m} \tag{3326}$$

$$\le C_n\cdot r(m,N)^{j_1+j_2+n} \tag{3327}$$

$$\cdot\left\|\theta_N^1-\theta_N^2\right\|_{H^m}\left\|\theta_N^1\right\|_{H^m}^{j_1+j_2-1}\left\|(\theta_N^1)_\alpha\right\|_{H^m}\left\|1_{|\cdot|\le N}P(\theta_N^1)\right\|_{h^m}^n. \tag{3328}$$

Using Lemma 9, we obtain

$$\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} S_1(\cdot,\beta)\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}d\beta\right\|_{H^m} \tag{3329}$$

$$\le C_n\cdot r(m,N)^{j_1+j_2+n}\cdot\tilde{r}(N) \tag{3330}$$

$$\cdot\left\|\theta_N^1-\theta_N^2\right\|_{H^m}\left\|\theta_N^1\right\|_{H^m}^{j_1+j_2}\left\|1_{|\cdot|\le N}P(\theta_N^1)\right\|_{h^m}^n. \tag{3331}$$

Now, let us consider the term starting in (3299), defined as

$$S_7(\alpha,\beta) \tag{3332}$$

$$=\theta_N^2(\alpha-\beta)^{j_1-1}(\theta_N^2)_\alpha(\alpha-\beta)\theta_N^2(\alpha)^{j_2} \tag{3333}$$

$$\cdot\int_0^1 e^{-i\beta s}\theta_N^2(\alpha+\beta(-1+s))(-1+s)ds \tag{3334}$$

$$\cdot\left(\sum_{m=1}^{\infty}\frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\frac{(-i\theta_N^1(\alpha+(s-1)\beta))^m}{m!}ds\right. \tag{3335}$$

$$\left.-\sum_{m=1}^{\infty}\frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\frac{(-i\theta_N^2(\alpha+(s-1)\beta))^m}{m!}ds\right) \tag{3336}$$

$$\cdot\left(\sum_{m=1}^{\infty}\frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\frac{(-i\theta_N^1(\alpha+(s-1)\beta))^m}{m!}ds\right)^{n-1}. \tag{3337}$$

We note that

$$\mathcal{F}\Bigg(\sum_{m=1}^{\infty}\frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\frac{(-i\theta_N^1(\alpha+(s-1)\beta))^m}{m!}ds \tag{3338}$$

$$-\sum_{m=1}^{\infty}\frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\frac{(-i\theta_N^2(\alpha+(s-1)\beta))^m}{m!}ds\Bigg)(k_1) \tag{3339}$$

$$=\mathcal{F}\Bigg(\frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}(e^{-i\theta_N^1(\alpha+(s-1)\beta)}-1)ds \tag{3340}$$

$$-\frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}(e^{-i\theta_N^2(\alpha+(s-1)\beta)}-1)ds\Bigg)(k_1) \tag{3341}$$

$$=\mathcal{F}\Bigg(\frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}(e^{-i\theta_N^1(\alpha+(s-1)\beta)}-e^{-i\theta_N^2(\alpha+(s-1)\beta)})ds\Bigg)(k_1), \tag{3342}$$

where

$$\mathcal{F}\Bigg(e^{-i\theta_N^1(\alpha+(s-1)\beta)}-e^{-i\theta_N^2(\alpha+(s-1)\beta)}\Bigg)(k_1) \tag{3343}$$

$$=\mathcal{F}\Bigg(e^{-i\theta_N^2(\alpha+(s-1)\beta)}\Bigg(e^{-i(\theta_N^1(\alpha+(s-1)\beta)-\theta_N^2(\alpha+(s-1)\beta))}-1\Bigg)\Bigg)(k_1) \tag{3344}$$

$$=\sum_{k_2\in\mathbb{Z}}\mathcal{F}\Bigg(e^{-i\theta_N^2(\alpha+(s-1)\beta)}\Bigg)(k_1-k_2)\mathcal{F}\Bigg(e^{-i(\theta_N^1(\alpha+(s-1)\beta)-\theta_N^2(\alpha+(s-1)\beta))}-1\Bigg)(k_2). \tag{3345}$$

We have

$$\mathcal{F}\Bigg(e^{-i\theta_N^2(\alpha+(s-1)\beta)}\Bigg)(k_1) \tag{3346}$$

$$=\mathcal{F}\Bigg(\sum_{m=0}^{\infty}\frac{(-i\theta_N^2(\alpha+(s-1)\beta))^m}{m!}\Bigg)(k_1) \tag{3347}$$

$$=\sum_{m=0}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}(\theta_N^2(\alpha+(s-1)\beta)^m)(k_1) \tag{3348}$$

$$=\sum_{m=0}^{\infty}\frac{(-i)^m}{m!}\sum_{k_2,\dots,k_m\in\mathbb{Z}}\prod_{d=1}^{m-1}\mathcal{F}(\theta_N^2(\alpha+(s-1)\beta))(k_d-k_{d+1})\mathcal{F}(\theta_N^2(\alpha+(s-1)\beta))(k_m) \tag{3349}$$

$$=\sum_{m=0}^{\infty}\frac{(-i)^m}{m!}\sum_{k_2,\dots,k_m\in\mathbb{Z}}e^{i(s-1)\beta(k_1-k_m)}\prod_{d=1}^{m-1}\mathcal{F}(\theta_N^2)(k_d-k_{d+1})\cdot e^{i(s-1)\beta k_m}\cdot\mathcal{F}(\theta_N^2)(k_m) \tag{3350}$$

$$=e^{i(s-1)\beta k_1}\sum_{m=0}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}((\theta_N^2)^m)(k_1) \tag{3351}$$

and

$$\mathcal{F}\left(e^{-i(\theta_N^1(\alpha+(s-1)\beta)-\theta_N^2(\alpha+(s-1)\beta))}-1\right)(k_1) \tag{3352}$$
$$=\mathcal{F}\left(\sum_{m=1}^{\infty}\frac{\left(-i\left(\theta_N^1(\alpha+(s-1)\beta)-\theta_N^2(\alpha+(s-1)\beta)\right)\right)^m}{m!}\right)(k_1) \tag{3353}$$
$$=\sum_{m=1}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}\left(\left(\theta_N^1(\alpha+(s-1)\beta)-\theta_N^2(\alpha+(s-1)\beta)\right)^m\right)(k_1) \tag{3354}$$
$$=\sum_{m=1}^{\infty}\frac{(-i)^m}{m!}\sum_{k_2,\ldots,k_m\in\mathbb{Z}}\prod_{d=1}^{m-1}\mathcal{F}\left(\theta_N^1(\alpha+(s-1)\beta)-\theta_N^2(\alpha+(s-1)\beta)\right)(k_d-k_{d+1}) \tag{3355}$$
$$\cdot\mathcal{F}\left(\theta_N^1(\alpha+(s-1)\beta)-\theta_N^2(\alpha+(s-1)\beta)\right)(k_m) \tag{3356}$$
$$=\sum_{m=1}^{\infty}\frac{(-i)^m}{m!}\sum_{k_2,\ldots,k_m\in\mathbb{Z}}\prod_{d=1}^{m-1}\left(e^{i(s-1)\beta(k_d-k_{d+1})}\mathcal{F}(\theta_N^1)(k_d-k_{d+1})\right. \tag{3357}$$
$$\left.-e^{i(s-1)\beta(k_d-k_{d+1})}\mathcal{F}(\theta_N^2)(k_d-k_{d+1})\right)\cdot e^{i(s-1)\beta k_m}\cdot(\mathcal{F}(\theta_N^1)(k_m)-\mathcal{F}(\theta_N^2)(k_m)) \tag{3358}$$
$$=\sum_{m=1}^{\infty}\frac{(-i)^m}{m!}e^{i(s-1)\beta k_1}\sum_{k_2,\ldots,k_m\in\mathbb{Z}}\prod_{d=1}^{m-1}\mathcal{F}(\theta_N^1-\theta_N^2)(k_d-k_{d+1})\cdot\mathcal{F}(\theta_N^1-\theta_N^2)(k_m) \tag{3359}$$
$$=e^{i(s-1)\beta k_1}\sum_{m=1}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}((\theta_N^1-\theta_N^2)^m)(k_1). \tag{3360}$$

Hence,

$$\mathcal{F}\left(\sum_{m=1}^{\infty}\frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\frac{(-i\theta_N^1(\alpha+(s-1)\beta))^m}{m!}ds\right. \tag{3361}$$
$$\left.-\sum_{m=1}^{\infty}\frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\frac{(-i\theta_N^2(\alpha+(s-1)\beta))^m}{m!}ds\right)(k_1) \tag{3362}$$
$$=\frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\left(\sum_{k_2\in\mathbb{Z}}e^{i(s-1)\beta(k_1-k_2)}\sum_{m=0}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}((\theta_N^2)^m)(k_1-k_2)\right. \tag{3363}$$
$$\left.\cdot e^{i(s-1)\beta k_2}\sum_{m=1}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}((\theta_N^1-\theta_N^2)^m)(k_2)\right)ds \tag{3364}$$
$$=\frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}e^{i(s-1)\beta k_1}ds \tag{3365}$$
$$\cdot\left(\left(\sum_{m=0}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}((\theta_N^2)^m)\right)*\left(\sum_{m=1}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}((\theta_N^1-\theta_N^2)^m)\right)\right)(k_1). \tag{3366}$$

Then

$$\mathcal{F}(S_7(\cdot,\beta))(k_1) \tag{3367}$$

$$= \sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}} \prod_{d=1}^{j_1-1} \mathcal{F}(\theta_N^2(\cdot-\beta))(k_d-k_{d+1})\mathcal{F}((\theta_N^2)_\alpha(\cdot-\beta))(k_{j_1}-k_{j_1+1}) \tag{3368}$$

$$\cdot \prod_{d=j_1+1}^{j_1+j_2} \mathcal{F}(\theta_N^2)(k_d-k_{d+1}) \tag{3369}$$

$$\cdot \mathcal{F}\Bigg(\sum_{m=1}^{\infty} \frac{i\beta}{1-e^{i\beta}} \int_0^1 e^{-i(s-1)\beta} \frac{(-i\theta_N^1(\alpha+(s-1)\beta))^m}{m!} ds \tag{3370}$$

$$- \sum_{m=1}^{\infty} \frac{i\beta}{1-e^{i\beta}} \int_0^1 e^{-i(s-1)\beta} \frac{(-i\theta_N^2(\alpha+(s-1)\beta))^m}{m!} ds\Bigg) \tag{3371}$$

$$(k_{j_1+j_2+1}-k_{j_1+j_2+2}) \tag{3372}$$

$$\cdot \prod_{d=j_1+j_2+2}^{j_1+j_2+n} \mathcal{F}\Bigg(\sum_{m=1}^{\infty} \frac{i\beta}{1-e^{i\beta}} \int_0^1 e^{-i(s-1)\beta} \frac{(-i\theta_N^1(\alpha+(s-1)\beta))^m}{m!} ds\Bigg)(k_d-k_{d+1}) \tag{3373}$$

$$\cdot \mathcal{F}\Bigg(\int_0^1 e^{-i\beta s}\theta_N^2(\alpha+\beta(-1+s))(-1+s)ds\Bigg)(k_{j_1+j_2+n+1}) \tag{3374}$$

$$= \sum_{k_2,\ldots,k_{j_1+j_2+n+1}\in\mathbb{Z}} \prod_{d=1}^{j_1-1} \mathcal{F}(\theta_N^2)(k_d-k_{d+1})\mathcal{F}((\theta_N^2)_\alpha)(k_{j_1}-k_{j_1+1}) \tag{3375}$$

$$\cdot \prod_{d=j_1+1}^{j_1+j_2} \mathcal{F}(\theta_N^2)(k_d-k_{d+1}) \tag{3376}$$

$$\cdot \Bigg(\Bigg(\sum_{m=0}^{\infty} \frac{(-i)^m}{m!}\mathcal{F}((\theta_N^2)^m)\Bigg) * \Bigg(\sum_{m=1}^{\infty} \frac{(-i)^m}{m!}\mathcal{F}((\theta_N^1-\theta_N^2)^m)\Bigg)\Bigg)(k_{j_1+j_2+1}-k_{j_1+j_2+2}) \tag{3377}$$

$$\cdot \prod_{d=j_1+j_2+2}^{j_1+j_2+n} \Bigg(\sum_{m=1}^{\infty} \frac{(-i)^m}{m!}\mathcal{F}((\theta_N^1)^m)(k_d-k_{d+1})\Bigg)\mathcal{F}(\theta_N^2)(k_{j_1+j_2+n+1}) \tag{3378}$$

$$\cdot e^{-i\beta(k_1-k_{j_1})}e^{-i\beta(k_{j_1}-k_{j_1+1})} \prod_{d=j_1+j_2+1}^{j_1+j_2+n} \frac{i\beta e^{i\beta}}{1-e^{i\beta}} \int_0^1 e^{-is\beta}e^{i(s-1)\beta(k_d-k_{d+1})}ds \tag{3379}$$

$$\cdot \int_0^1 e^{-i\beta s}(-1+s)e^{i\beta(-1+s)k_{j_1+j_2+n+1}}ds. \tag{3380}$$

Then

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(S_7(\cdot,\beta))(k_1)\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}d\beta\right| \tag{3381}$$
$$\leq C_n\Bigg(\left|\mathcal{F}(\theta_N^2)\right| * \cdots * \left|\mathcal{F}(\theta_N^2)\right| * \left|\mathcal{F}((\theta_N^2)_\alpha)\right| * \left|\mathcal{F}(\theta_N^2)\right| * \cdots * \left|\mathcal{F}(\theta_N^2)\right| \tag{3382}$$
$$* \left|\left(\left(\sum_{m=0}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}((\theta_N^2)^m)\right) * \left(\sum_{m=1}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}((\theta_N^1-\theta_N^2)^m)\right)\right)\right| \tag{3383}$$
$$* \left|P(\theta_N^1)\right| * \cdots * \left|P(\theta_N^1)\right| * \left|\mathcal{F}(\theta_N^2)\right|\Bigg)(k_1). \tag{3384}$$

Hence,

$$\left\|\mathcal{J}_N\left(\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}S_7(\cdot,\beta)\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}d\beta\right)\right\|_{H^{m'}} \tag{3385}$$
$$=\left(\sum_{k\in\mathbb{Z}}1_{|k|\leq N}(1+|k|^2)^{m'}\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(S_7(\cdot,\beta))(k)\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}d\beta\right|^2\right)^{1/2} \tag{3386}$$
$$\leq C_n\Bigg\|1_{|\cdot|\leq N}\left|\mathcal{F}(\theta_N^2)\right| * \cdots * \left|\mathcal{F}(\theta_N^2)\right| * \left|\mathcal{F}((\theta_N^2)_\alpha)\right| * \left|\mathcal{F}(\theta_N^2)\right| * \cdots * \left|\mathcal{F}(\theta_N^2)\right| \tag{3387}$$
$$* \left|\left(\left(\sum_{m=0}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}((\theta_N^2)^m)\right) * \left(\sum_{m=1}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}((\theta_N^1-\theta_N^2)^m)\right)\right)\right| \tag{3388}$$
$$* \left|P(\theta_N^1)\right| * \cdots * \left|P(\theta_N^1)\right| * \left|\mathcal{F}(\theta_N^2)\right|\Bigg\|_{h^{m'}} \tag{3389}$$
$$\leq C_n \cdot r(m',N)^{j_1+j_2+n}\left\|\theta_N^2\right\|_{H^{m'}}^{j_1+j_2}\left\|(\theta_N^2)_\alpha\right\|_{H^{m'}} \tag{3390}$$
$$\cdot\left\|1_{|\cdot|\leq N}\cdot\left(\left(\sum_{m=0}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}((\theta_N^2)^m)\right) * \left(\sum_{m=1}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}((\theta_N^1-\theta_N^2)^m)\right)\right)(\cdot)\right\|_{h^{m'}} \tag{3391}$$
$$\cdot\left\|1_{|\cdot|\leq N}P(\theta_N^1)\right\|_{h^{m'}}^{n-1} \tag{3392}$$
$$\leq C_n \cdot r(m',N)^{j_1+j_2+n}\cdot\tilde{r}(N)\left\|\theta_N^2\right\|_{H^{m'}}^{j_1+j_2+1} \tag{3393}$$
$$\cdot\left\|1_{|\cdot|\leq N}\cdot\left(\left(\sum_{m=0}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}((\theta_N^2)^m)\right) * \left(\sum_{m=1}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}((\theta_N^1-\theta_N^2)^m)\right)\right)(\cdot)\right\|_{h^{m'}} \tag{3394}$$
$$\cdot\left\|1_{|\cdot|\leq N}P(\theta_N^1)\right\|_{h^{m'}}^{n-1}. \tag{3395}$$

We note that

$$
\begin{aligned}
&\left\|1_{|\cdot|\le N} P(\theta_N^1)\right\|_{h^{m'}} && (3396)\\
=&\Bigg(\sum_{|k|\le N} (1+|k|^2)^{m'} \left|P(\theta_N^1)(k)\right|^2\Bigg)^{1/2} && (3397)\\
=&\Bigg(\sum_{|k|\le N} (1+|k|^2)^{m'} \left|\sum_{m=1}^{\infty} \frac{(-i)^m}{m!}\mathcal{F}((\theta_N^1)^m)(k)\right|^2\Bigg)^{1/2} && (3398)\\
\le&\Bigg(\sum_{|k|\le N} (1+|k|^2)^{m'} \sum_{m=1}^{\infty}\bigg(\frac{|\mathcal{F}((\theta_N^1)^m)(k)|}{m!}\bigg)^2\Bigg)^{1/2} && (3399)\\
=&\Bigg(\sum_{m=1}^{\infty} \frac{1}{(m!)^2}\sum_{|k|\le N} (1+|k|^2)^{m'} \left|\mathcal{F}((\theta_N^1)^m)(k)\right|^2\Bigg)^{1/2} && (3400)\\
=&\Bigg(\sum_{m=1}^{\infty} \frac{1}{(m!)^2}\left\|1_{|\cdot|\le N}(\mathcal{F}(\theta_N^1)*\cdots*\mathcal{F}(\theta_N^1))\right\|_{h^{m'}}^2\Bigg)^{1/2} && (3401)\\
\le&\Bigg(\sum_{m=1}^{\infty} \frac{1}{(m!)^2}\bigg(r(m',N)^{m-1}\left\|\theta_N^1\right\|_{H^{m'}}^m\bigg)^2\Bigg)^{1/2} && (3402)\\
\le&\Bigg(\sum_{m=1}^{\infty} \frac{1}{(m!)^2}\bigg(r(m',N)^{m-1}M^m\bigg)^2\Bigg)^{1/2} && (3403)\\
\le&\Bigg(\frac{1}{r(m',N)^2}\sum_{m=1}^{\infty} \frac{(M\cdot r(m',N))^{2m}}{m!}\Bigg)^{1/2} && (3404)\\
=&\frac{(e^{(M\cdot r(m',N))^2}-1)^{1/2}}{r(m',N)}. && (3405)
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\left\|1_{|\cdot|\le N}\cdot\Bigg(\bigg(\sum_{m=0}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}((\theta_N^2)^m)\bigg)*\bigg(\sum_{m=1}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}((\theta_N^1-\theta_N^2)^m)\bigg)\Bigg)(\cdot)\right\|_{h^{m'}} && (3406)\\
\le& r(m',N)\left\|1_{|\cdot|\le N}\bigg(\sum_{m=0}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}((\theta_N^2)^m)(\cdot)\bigg)\right\|_{h^{m'}} && (3407)\\
&\cdot\left\|1_{|\cdot|\le N}\bigg(\sum_{m=1}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}((\theta_N^1-\theta_N^2)^m)(\cdot)\bigg)\right\|_{h^{m'}}. && (3408)
\end{aligned}
$$

We note that

$$\left\| 1_{|\cdot|\leq N}\left(\sum_{m=0}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}((\theta_N^2)^m)(\cdot)\right)\right\|_{h^{m'}} \tag{3409}$$

$$=\left(\sum_{|k|\leq N}(1+|k|^2)^{m'}\left|\sum_{m=0}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}((\theta_N^2)^m)(k)\right|^2\right)^{1/2} \tag{3410}$$

$$\leq\left(\sum_{m=0}^{\infty}\frac{1}{(m!)^2}\left(r(m',N)^{m-1}\left\|\theta_N^2\right\|_{H^{m'}}^m\right)^2\right)^{1/2} \tag{3411}$$

$$\leq\frac{(e^{M^2 r(m',N)^2})^{1/2}}{r(m',N)} \tag{3412}$$

and

$$\left\| 1_{|\cdot|\leq N}\left(\sum_{m=1}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}((\theta_N^1-\theta_N^2)^m)(\cdot)\right)\right\|_{h^{m'}} \tag{3413}$$

$$=\left(\sum_{|k|\leq N}(1+|k|^2)^{m'}\left|\sum_{m=1}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}((\theta_N^1-\theta_N^2)^m)(k)\right|^2\right)^{1/2} \tag{3414}$$

$$\leq\left(\sum_{m=1}^{\infty}\frac{1}{(m!)^2}\left(r(m',N)^{m-1}\left\|\theta_N^1-\theta_N^2\right\|_{H^{m'}}^m\right)^2\right)^{1/2} \tag{3415}$$

$$=\left\|\theta_N^1-\theta_N^2\right\|_{H^{m'}}\left(\sum_{m=1}^{\infty}\left(\frac{(r(m',N)\left\|\theta_N^1-\theta_N^2\right\|_{H^{m'}})^{m-1}}{m!}\right)^2\right)^{1/2} \tag{3416}$$

$$\leq\left\|\theta_N^1-\theta_N^2\right\|_{H^{m'}}\frac{1}{r(m',N)\cdot 2M}\left(e^{r(m',N)^2(2M)^2}-1\right)^{1/2}. \tag{3417}$$

Hence,

$$\left\| 1_{|\cdot|\leq N}\cdot\left(\left(\sum_{m=0}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}((\theta_N^2)^m)\right)*\left(\sum_{m=1}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}((\theta_N^1-\theta_N^2)^m)\right)\right)(\cdot)\right\|_{h^{m'}} \tag{3418}$$

$$\leq r(m',N)\cdot\frac{(e^{M^2 r(m',N)^2})^{1/2}}{r(m',N)}\cdot\left\|\theta_N^1-\theta_N^2\right\|_{H^{m'}}\frac{1}{r(m',N)\cdot 2M}\left(e^{r(m',N)^2(2M)^2}-1\right)^{1/2}. \tag{3419}$$

Thus,

$$\left\|\mathcal{J}_N\left(\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}S_7(\cdot,\beta)\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}d\beta\right)\right\|_{H^{m'}} \tag{3420}$$

$$\leq C_n\cdot r(m',N)^{j_1+j_2+n}\cdot\tilde{r}(N)\left\|\theta_N^2\right\|_{H^{m'}}^{j_1+j_2+1}\cdot\left(\frac{(e^{(M\cdot r(m',N))^2}-1)^{1/2}}{r(m',N)}\right)^{n-1} \tag{3421}$$

$$\cdot\, r(m',N)\cdot\frac{(e^{M^2 r(m',N)^2})^{1/2}}{r(m',N)}\cdot\left\|\theta_N^1-\theta_N^2\right\|_{H^{m'}}\frac{1}{r(m',N)\cdot 2M} \tag{3422}$$

$$\cdot\left(e^{r(m',N)^2(2M)^2}-1\right)^{1/2}. \tag{3423}$$

Therefore,

$$\left\| \mathcal{J}_N\left(\frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B^1_{1,1}(\theta^1_N)(\alpha,\beta) d\beta - \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B^1_{1,1}(\theta^2_N)(\alpha,\beta) d\beta \right) \right\|_{H^{m'}} \tag{3424}$$

$$\leq \sum_{j_1+j_2+n\geq 1} \frac{j_1}{2j_1!j_2!} \left(\left\| \mathcal{J}_N\left(\frac{\gamma}{4\pi} \int_{-\pi}^{\pi} S_1(\cdot,\beta) \frac{-i\beta e^{2i\beta}}{1-e^{i\beta}} d\beta \right) \right\|_{H^{m'}} + \cdots \right. \tag{3425}$$

$$\left. + \left\| \mathcal{J}_N\left(\frac{\gamma}{4\pi} \int_{-\pi}^{\pi} S_7(\cdot,\beta) \frac{-i\beta e^{2i\beta}}{1-e^{i\beta}} d\beta \right) \right\|_{H^{m'}} + \cdots \right) \tag{3426}$$

$$\leq \sum_{j_1+j_2+n\geq 1} \frac{j_1}{2j_1!j_2!} C_n \cdot r(m',N)^{j_1+j_2+n} \cdot \tilde{r}(N) \tag{3427}$$

$$\cdot \left\| \theta^1_N - \theta^2_N \right\|_{H^{m'}} \left\| \theta^1_N \right\|^{j_1+j_2}_{H^{m'}} \left\| 1_{|\cdot|\leq N} P(\theta^1_N) \right\|^n_{h^{m'}} \tag{3428}$$

$$+ \cdots \tag{3429}$$

$$+ \sum_{j_1+j_2+n\geq 1} \frac{j_1}{2j_1!j_2!} C_n \cdot r(m',N)^{j_1+j_2+n} \cdot \tilde{r}(N) \left\| \theta^2_N \right\|^{j_1+j_2+1}_{H^{m'}} \tag{3430}$$

$$\cdot \left(\frac{(e^{(M\cdot r(m',N))^2} - 1)^{1/2}}{r(m',N)} \right)^{n-1} \tag{3431}$$

$$\cdot r(m',N) \cdot \frac{(e^{M^2 r(m',N)^2})^{1/2}}{r(m',N)} \cdot \left\| \theta^1_N - \theta^2_N \right\|_{H^{m'}} \frac{1}{r(m',N)\cdot 2M} \left(e^{r(m',N)^2(2M)^2} - 1 \right)^{1/2} \tag{3432}$$

$$+ \cdots \tag{3433}$$

$$\leq \sum_{j_1+j_2+n\geq 1} \frac{j_1}{2j_1!j_2!} C_n \cdot r(m',N)^{j_1+j_2+n} \cdot \tilde{r}(N) \tag{3434}$$

$$\cdot \left\| \theta^1_N - \theta^2_N \right\|_{H^{m'}} M^{j_1+j_2} \left(\frac{(e^{(M\cdot r(m',N))^2} - 1)^{1/2}}{r(m',N)} \right)^n \tag{3435}$$

$$+ \cdots \tag{3436}$$

$$+ \sum_{j_1+j_2+n\geq 1} \frac{j_1}{2j_1!j_2!} C_n \cdot r(m',N)^{j_1+j_2+n} \cdot \tilde{r}(N) \cdot M^{j_1+j_2+1} \tag{3437}$$

$$\cdot \left(\frac{(e^{(M\cdot r(m',N))^2} - 1)^{1/2}}{r(m',N)} \right)^{n-1} \tag{3438}$$

$$\cdot r(m',N) \cdot \frac{(e^{M^2 r(m',N)^2})^{1/2}}{r(m',N)} \cdot \left\| \theta^1_N - \theta^2_N \right\|_{H^{m'}} \frac{1}{r(m',N)\cdot 2M} \left(e^{r(m',N)^2(2M)^2} - 1 \right)^{1/2} \tag{3439}$$

$$+ \cdots . \tag{3440}$$

We choose M sufficiently small so that all the geometric series contained in the expression above converge. We can similarly derive estimates for the rest of the terms represented by the $\cdots$ above. In fact, using the techniques that have been showcased here, one can derive estimates for the rest of the terms making up the sum in (3224) and the term in (3225).

Next, we present the derivation of an estimate for

$$\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}(E_1)_\alpha(\theta_N^1)(\alpha,\beta)d\beta-\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}(E_1)_\alpha(\theta_N^2)(\alpha,\beta)d\beta\right\|_{H^m}, \tag{3441}$$

which is one of the terms making up the sum in (3223). Recalling (996), we have

$$(E_1)_\alpha(\theta_N^1)(\alpha,\beta)-(E_1)_\alpha(\theta_N^2)(\alpha,\beta) \tag{3442}$$
$$=\frac{-e^{i\beta}(-1+e^{i\beta})(i(-1+e^{i\beta})+\beta(1+e^{i\beta}))}{2(-1+e^{i\beta})^2} \tag{3443}$$
$$\cdot\int_0^1 e^{-i\beta s}((\theta_N^1)_\alpha-(\theta_N^2)_\alpha)(\alpha+\beta(-1+s))ds. \tag{3444}$$

Then

$$\mathcal{F}((E_1)_\alpha(\theta_N^1)(\cdot,\beta)-(E_1)_\alpha(\theta_N^2)(\cdot,\beta))(k) \tag{3445}$$
$$=\frac{-e^{i\beta}(-1+e^{i\beta})(i(-1+e^{i\beta})+\beta(1+e^{i\beta}))}{2(-1+e^{i\beta})^2}\cdot\int_0^1 e^{-i\beta s}e^{ik\beta(-1+s)}ds \tag{3446}$$
$$\cdot\mathcal{F}((\theta_N^1)_\alpha-(\theta_N^2)_\alpha)(k). \tag{3447}$$

Using the estimate in (1004), we obtain

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}((E_1)_\alpha(\theta_N^1)(\cdot,\beta)-(E_1)_\alpha(\theta_N^2)(\cdot,\beta))(k)d\beta\right| \tag{3448}$$
$$=\frac{\gamma}{4\pi}\left(2\pi+\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}\cdot\frac{1}{2}\pi^2\right)\left|\mathcal{F}((\theta_N^1)_\alpha-(\theta_N^2)_\alpha)(k)\right|. \tag{3449}$$

Therefore,

$$\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}(E_1)_\alpha(\theta_N^1)(\cdot,\beta)-(E_1)_\alpha(\theta_N^2)(\cdot,\beta)d\beta\right\|_{H^m} \tag{3450}$$
$$=\left(\sum_{k\in\mathbb{Z}}(1+|k|^2)^m\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}((E_1)_\alpha(\theta_N^1)(\cdot,\beta)-(E_1)_\alpha(\theta_N^2)(\cdot,\beta))(k)d\beta\right|^2\right)^{1/2} \tag{3451}$$
$$\leq\frac{\gamma}{4\pi}\left(2\pi+\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}\cdot\frac{1}{2}\pi^2\right)\left\|(\theta_N^1)_\alpha-(\theta_N^2)_\alpha\right\|_{H^m} \tag{3452}$$
$$\leq\frac{\gamma}{4\pi}\left(2\pi+\frac{1}{2}\sqrt{1+\frac{\pi^2}{4}}\cdot\frac{1}{2}\pi^2\right)\tilde{r}(N)\left\|\theta_N^1-\theta_N^2\right\|_{H^m}. \tag{3453}$$

We can similarly derive estimates for the rest of the terms in the sum in (3223). To derive estimates for the rest of the terms appearing in the upper bound shown in (3215) through (3220), the following lemmas are helpful.

Lemma 10. *If θ_N has finite support, then $T_N(\theta_N)$ has finite support.*

Proof. Using (30), we obtain that for $k \neq 0$,

$$\mathcal{F}(T_N(\theta_N))(k) = \mathcal{F}\Bigg(\mathcal{M}\Bigg((1+(\theta_N)_\alpha)U_N(\theta_N)\Bigg)\Bigg)(k) \tag{3454}$$

$$= -\frac{i}{k}\mathcal{F}\Bigg((1+(\theta_N)_\alpha)U_N(\theta_N)\Bigg)(k) \tag{3455}$$

$$= -\frac{i}{k}\Bigg(\mathcal{F}(U_N(\theta_N))(k) + \mathcal{F}((\theta_N)_\alpha U_N(\theta_N))(k)\Bigg). \tag{3456}$$

Since θ_N and $U_N(\theta_N)$ have finite support, the product $(\theta_N)_\alpha \cdot U_N(\theta_N)$ has finite support. Therefore, $T_N(\theta_N)$ has finite support as well, as needed. ∎

Lemma 11. *If f is a periodic function such that* $\operatorname{supp}\hat{f} \subseteq [-M, M]$, *then*

$$\|\mathcal{M}(f)\|_{H^m} \leq 2^M \|f\|_{H^m}. \tag{3457}$$

Proof.

$$\|\mathcal{M}(f)\|_{H^m} \tag{3458}$$

$$= \Bigg(\sum_{k\in\mathbb{Z}} (1+|k|^2)^m |\mathcal{F}(\mathcal{M}(f))(k)|^2\Bigg)^{1/2} \tag{3459}$$

$$= \Bigg(\Bigg|\sum_{j\neq 0} \frac{i}{j}\mathcal{F}(f)(j)\Bigg|^2 + \sum_{k\neq 0}(1+|k|^2)^m \Bigg|-\frac{i}{k}\mathcal{F}(f)(k)\Bigg|^2\Bigg)^{1/2} \tag{3460}$$

$$\leq \Bigg(\Bigg(\sum_{k\neq 0}\frac{1}{|k|}|\mathcal{F}(f)(k)|\Bigg)^2 + \sum_{k\neq 0}(1+|k|^2)^{m-1}\Bigg(\frac{1}{|k|^2}+1\Bigg)|\mathcal{F}(f)(k)|^2\Bigg)^{1/2} \tag{3461}$$

$$= \Bigg(\Bigg(\sum_{\substack{|k|\leq M\\ k\neq 0}}\frac{1}{|k|}|\mathcal{F}(f)(k)|\Bigg)^2 + \sum_{\substack{|k|\leq M\\ k\neq 0}}(1+|k|^2)^{m-1}\Bigg(\frac{1}{|k|^2}+1\Bigg)|\mathcal{F}(f)(k)|^2\Bigg)^{1/2} \tag{3462}$$

$$\leq \Bigg(2^{2M-1}\sum_{\substack{|k|\leq M\\ k\neq 0}}\frac{1}{|k|^2}|\mathcal{F}(f)(k)|^2 + \sum_{\substack{|k|\leq M\\ k\neq 0}}(1+|k|^2)^m|\mathcal{F}(f)(k)|^2\Bigg)^{1/2} \tag{3463}$$

$$\leq \Bigg(2^{2M}\sum_{|k|\leq M}(1+|k|^2)^m|\mathcal{F}(f)(k)|^2\Bigg)^{1/2} \tag{3464}$$

$$= 2^M \|f\|_{H^m}. \tag{3465}$$

∎

Now, we consider the term $\|T_N(\theta_N^1) - T_N(\theta_N^2)\|_{H^m}$, which appears in (3216). Using these lemmas, we observe that

$$\left\|T_N(\theta_N^1) - T_N(\theta_N^2)\right\|_{H^m} \tag{3466}$$
$$= \left\|\mathcal{M}\left((1+(\theta_N^1)_\alpha)U_N(\theta_N^1)\right) - \mathcal{M}\left((1+(\theta_N^2)_\alpha)U_N(\theta_N^2)\right)\right\|_{H^m} \tag{3467}$$
$$= \left\|\mathcal{M}\left(U_N(\theta_N^1) - U_N(\theta_N^2)\right) + \mathcal{M}\left((\theta_N^1)_\alpha U_N(\theta_N^1) - (\theta_N^2)_\alpha U_N(\theta_N^2)\right)\right\|_{H^m} \tag{3468}$$
$$\leq \left\|\mathcal{M}\left(U_N(\theta_N^1) - U_N(\theta_N^2)\right)\right\|_{H^m} \tag{3469}$$
$$+ \left\|\mathcal{M}\left((\theta_N^1)_\alpha U_N(\theta_N^1) - (\theta_N^1)_\alpha U_N(\theta_N^2) + (\theta_N^1)_\alpha U_N(\theta_N^2) - (\theta_N^2)_\alpha U_N(\theta_N^2)\right)\right\|_{H^m} \tag{3470}$$
$$\leq \left\|\mathcal{M}\left(U_N(\theta_N^1) - U_N(\theta_N^2)\right)\right\|_{H^m} + \left\|\mathcal{M}\left((\theta_N^1)_\alpha (U_N(\theta_N^1) - U_N(\theta_N^2))\right)\right\|_{H^m} \tag{3471}$$
$$+ \left\|\mathcal{M}\left(((\theta_N^1)_\alpha - (\theta_N^2)_\alpha) U_N(\theta_N^2)\right)\right\|_{H^m} \tag{3472}$$
$$\leq 2^N \left\|U_N(\theta_N^1) - U_N(\theta_N^2)\right\|_{H^m} + 2^{l(N)} \left\|(\theta_N^1)_\alpha (U_N(\theta_N^1) - U_N(\theta_N^2))\right\|_{H^m} \tag{3473}$$
$$+ 2^{l(N)} \left\|((\theta_N^1)_\alpha - (\theta_N^2)_\alpha) U_N(\theta_N^2)\right\|_{H^m} \tag{3474}$$
$$\leq 2^N \left\|U_N(\theta_N^1) - U_N(\theta_N^2)\right\|_{H^m} + 2^{l(N)} r(m,N)\tilde{r}(N) \left\|\theta_N^1\right\|_{H^m} \left\|U_N(\theta_N^1) - U_N(\theta_N^2)\right\|_{H^m} \tag{3475}$$
$$\leq 2^N \left\|U_N(\theta_N^1) - U_N(\theta_N^2)\right\|_{H^m} + 2^{l(N)} r(m,N)\tilde{r}(N) M \left\|U_N(\theta_N^1) - U_N(\theta_N^2)\right\|_{H^m} \tag{3476}$$

for some function $l(N)$ of N. Hence, it suffices to find an estimate for $\|U_N(\theta_N^1) - U_N(\theta_N^2)\|_{H^m}$. To do so, we first observe that for any periodic function f,

$$\mathcal{F}(\mathrm{Re}(f))(k) = \mathcal{F}\left(\frac{1}{2}(f+\bar{f})\right)(k) \tag{3477}$$
$$= \frac{1}{2}\left(\mathcal{F}(f)(k) + \mathcal{F}(\bar{f}(k))\right) \tag{3478}$$
$$= \frac{1}{2}\left(\mathcal{F}(f)(k) + \overline{\mathcal{F}(f)(-k)}\right). \tag{3479}$$

Then

$$\left|\mathcal{F}\left(\mathrm{Re}(V(\theta_N^1)) - \mathrm{Re}(V(\theta_N^2))\right)(k)\right| \tag{3480}$$
$$= \left|\frac{1}{2}\left(\mathcal{F}(V(\theta_N^1))(k) + \overline{\mathcal{F}(V(\theta_N^1))(-k)}\right) - \frac{1}{2}\left(\mathcal{F}(V(\theta_N^2))(k) + \overline{\mathcal{F}(V(\theta_N^2))(-k)}\right)\right| \tag{3481}$$
$$\leq \frac{1}{2}\left(\left|\mathcal{F}(V(\theta_N^1) - V(\theta_N^2))(k)\right| + \left|\mathcal{F}(V(\theta_N^1) - V(\theta_N^2))(-k)\right|\right). \tag{3482}$$

Hence,

$$\left|\mathcal{F}\left(\mathrm{Re}(V(\theta_N^1)) - \mathrm{Re}(V(\theta_N^2))\right)(k)\right|^2 \tag{3483}$$
$$= \frac{1}{2}\left(\left|\mathcal{F}(V(\theta_N^1) - V(\theta_N^2))(k)\right|^2 + \left|\mathcal{F}(V(\theta_N^1) - V(\theta_N^2))(-k)\right|^2\right). \tag{3484}$$

Thus,

$$\left\|U_N(\theta_N^1) - U_N(\theta_N^2)\right\|_{H^m} \tag{3485}$$

$$=\left(\sum_{k\in\mathbb{Z}}(1+|k|^2)^m \left|\mathcal{F}\left(U_N(\theta_N^1) - U_N(\theta_N^2)\right)(k)\right|^2\right)^{1/2} \tag{3486}$$

$$=\left(\sum_{|k|\le N}(1+|k|^2)^m \left|\mathcal{F}\left(\mathrm{Re}(V(\theta_N^1)) - \mathrm{Re}(V(\theta_N^2))\right)(k)\right|^2\right)^{1/2} \tag{3487}$$

$$\le\Big(\frac{1}{2}\sum_{|k|\le N}(1+|k|^2)^m(\left|\mathcal{F}(V(\theta_N^1) - V(\theta_N^2))(k)\right|^2 \tag{3488}$$

$$+\left|\mathcal{F}(V(\theta_N^1) - V(\theta_N^2))(-k)\right|^2)\Big)^{1/2} \tag{3489}$$

$$\le\left\|V(\theta_N^1) - V(\theta_N^2)\right\|_{H^m} \tag{3490}$$

$$\le\sum_{j=1}^{7}\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}E_j(\theta_N^1)(\alpha,\beta)d\beta - \frac{\gamma}{4\pi}\int_{-\pi}^{\pi}E_j(\theta_N^2)(\alpha,\beta)d\beta\right\|_{H^m} \tag{3491}$$

$$+\sum_{j=1}^{8}\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\widetilde{B_j}(\theta_N^1)(\alpha,\beta)d\beta - \frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\widetilde{B_j}(\theta_N^2)(\alpha,\beta)d\beta\right\|_{H^m} \tag{3492}$$

$$+\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\widetilde{B_{13}}(\theta_N^1)(\alpha,\beta)d\beta - \frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\widetilde{B_{13}}(\theta_N^2)(\alpha,\beta)d\beta\right\|_{H^m}. \tag{3493}$$

The derivation of an appropriate estimate for $\|U_N(\theta_N^1) - U_N(\theta_N^2)\|_{H^m}$ can be completed using the techniques that had been introduced earlier for $\|(U_\alpha)_N(\theta_N^1) - (U_\alpha)_N(\theta_N^2)\|_{H^m}$. Moreover, we note that using the techniques introduced thus far, appropriate estimates for the terms $\|T_N(\theta_N^1)((\theta_N^1)_\alpha - (\theta_N^2)_\alpha)\|_{H^m}$ and $\|(T_N(\theta_N^1) - T_N(\theta_N^2))(\theta_N^2)_\alpha\|_{H^m}$ in (3217) and the terms in (3220) can be derived. Lastly, we derive an appropriate estimate for

$$R^{-1}\left(1+\frac{1}{2\pi}\mathrm{Im}\int_{-\pi}^{\pi}\int_0^{\alpha}e^{i(\alpha-\eta)}\sum_{n\ge1}\frac{i^n}{n!}(\theta_N^1(\alpha) - \theta_N^1(\eta))^n d\eta d\alpha\right)^{\frac{1}{2}} \tag{3494}$$

$$-R^{-1}\left(1+\frac{1}{2\pi}\mathrm{Im}\int_{-\pi}^{\pi}\int_0^{\alpha}e^{i(\alpha-\eta)}\sum_{n\ge1}\frac{i^n}{n!}(\theta_N^2(\alpha) - \theta_N^2(\eta))^n d\eta d\alpha\right)^{\frac{1}{2}}. \tag{3495}$$

which appears in (3218) through (3219). We note that for a concave function f,

$$f(y) - f(x) \le f'(x)(y-x) \tag{3496}$$

for all $x, y \in \mathbb{R}$. If $y > x$, then

$$\frac{f(y) - f(x)}{y - x} \le f'(x). \tag{3497}$$

If f is also monotone, then

$$\left|\frac{f(y) - f(x)}{y - x}\right| \le f'(x). \tag{3498}$$

Without loss of generality, we let

$$\mathrm{Im}\int_{-\pi}^{\pi}\int_{0}^{\alpha} e^{i(\alpha-\eta)}\sum_{n\geq 1}\frac{i^n}{n!}(\theta_N^1(\alpha)-\theta_N^1(\eta))^n d\eta d\alpha \tag{3499}$$

$$>\mathrm{Im}\int_{-\pi}^{\pi}\int_{0}^{\alpha} e^{i(\alpha-\eta)}\sum_{n\geq 1}\frac{i^n}{n!}(\theta_N^2(\alpha)-\theta_N^2(\eta))^n d\eta d\alpha. \tag{3500}$$

Since the square root function is concave and monotone,

$$\left|y^{1/2}-x^{1/2}\right| \leq \frac{1}{2\sqrt{x}}\left|y-x\right| \tag{3501}$$

for $y > x$. In particular,

$$R^{-1}\left|\left(1+\frac{1}{2\pi}\mathrm{Im}\int_{-\pi}^{\pi}\int_{0}^{\alpha} e^{i(\alpha-\eta)}\sum_{n\geq 1}\frac{i^n}{n!}(\theta_N^1(\alpha)-\theta_N^1(\eta))^n d\eta d\alpha\right)^{1/2}\right. \tag{3502}$$

$$\left.-\left(1+\frac{1}{2\pi}\mathrm{Im}\int_{-\pi}^{\pi}\int_{0}^{\alpha} e^{i(\alpha-\eta)}\sum_{n\geq 1}\frac{i^n}{n!}(\theta_N^2(\alpha)-\theta_N^2(\eta))^n d\eta d\alpha\right)^{1/2}\right| \tag{3503}$$

$$\leq R^{-1}\cdot\frac{1}{2}\left(1+\frac{1}{2\pi}\mathrm{Im}\int_{-\pi}^{\pi}\int_{0}^{\alpha} e^{i(\alpha-\eta)}\sum_{n\geq 1}\frac{i^n}{n!}(\theta_N^2(\alpha)-\theta_N^2(\eta))^n d\eta d\alpha\right)^{-1/2} \tag{3504}$$

$$\cdot\left|\frac{1}{2\pi}\mathrm{Im}\int_{-\pi}^{\pi}\int_{0}^{\alpha} e^{i(\alpha-\eta)}e^{i(\theta_N^1(\alpha)-\theta_N^1(\eta))}d\eta d\alpha\right. \tag{3505}$$

$$\left.-\frac{1}{2\pi}\mathrm{Im}\int_{-\pi}^{\pi}\int_{0}^{\alpha} e^{i(\alpha-\eta)}e^{i(\theta_N^2(\alpha)-\theta_N^2(\eta))}d\eta d\alpha\right| \tag{3506}$$

$$\leq R^{-1}\cdot\frac{1}{4\pi}\left(1+\frac{1}{2\pi}\mathrm{Im}\int_{-\pi}^{\pi}\int_{0}^{\alpha} e^{i(\alpha-\eta)}\sum_{n\geq 1}\frac{i^n}{n!}(\theta_N^2(\alpha)-\theta_N^2(\eta))^n d\eta d\alpha\right)^{-1/2} \tag{3507}$$

$$\cdot\left|\mathrm{Im}\int_{-\pi}^{\pi}\int_{0}^{\alpha} e^{i(\alpha-\eta)}(e^{i(\theta_N^1(\alpha)-\theta_N^1(\eta))}-e^{i(\theta_N^2(\alpha)-\theta_N^2(\eta))})d\eta d\alpha\right|. \tag{3508}$$

We note that

$$\int_{-\pi}^{\pi}\int_{0}^{\alpha} e^{i(\alpha-\eta)}(e^{i(\theta_N^1(\alpha)-\theta_N^1(\eta))}-e^{i(\theta_N^2(\alpha)-\theta_N^2(\eta))})d\eta d\alpha \tag{3509}$$

$$=\frac{1}{i}\int_{-\pi}^{\pi}\frac{\partial}{\partial\alpha}(e^{i\alpha})\int_{0}^{\alpha} e^{-i\eta}(e^{i(\theta_N^1(\alpha)-\theta_N^1(\eta))}-e^{i(\theta_N^2(\alpha)-\theta_N^2(\eta))})d\eta d\alpha \tag{3510}$$

$$=\frac{1}{i}\Bigg(\int_{-\pi}^{\pi}\frac{\partial}{\partial\alpha}\Bigg(e^{i\alpha}\int_{0}^{\alpha} e^{-i\eta}(e^{i(\theta_N^1(\alpha)-\theta_N^1(\eta))}-e^{i(\theta_N^2(\alpha)-\theta_N^2(\eta))})d\eta\Bigg)d\alpha \tag{3511}$$

$$-\int_{-\pi}^{\pi}(e^{i(\theta_N^1(\alpha)-\theta_N^1(\alpha))}-e^{i(\theta_N^2(\alpha)-\theta_N^2(\alpha))})d\alpha\Bigg) \tag{3512}$$

$$=\frac{1}{i}\Bigg(e^{i\pi}\int_{0}^{\pi} e^{-i\eta}(e^{i(\theta_N^1(\pi)-\theta_N^1(\eta))}-e^{i(\theta_N^2(\pi)-\theta_N^2(\eta))})d\eta \tag{3513}$$

$$-e^{-i\pi}\int_{0}^{-\pi} e^{-i\eta}(e^{i(\theta_N^1(-\pi)-\theta_N^1(\eta))}-e^{i(\theta_N^2(-\pi)-\theta_N^2(\eta))})d\eta\Bigg) \tag{3514}$$

$$=i\Bigg(\int_{-\pi}^{\pi} e^{-i\eta}(e^{i(\theta_N^1(\pi)-\theta_N^1(\eta))}-e^{i(\theta_N^2(\pi)-\theta_N^2(\eta))})d\eta\Bigg) \tag{3515}$$

$$=2\pi i\cdot\mathcal{F}\Bigg(e^{i(\theta_N^1(\pi)-\theta_N^1(\eta))}-e^{i(\theta_N^2(\pi)-\theta_N^2(\eta))}\Bigg)(1) \tag{3516}$$

$$=2\pi i\mathcal{F}\Bigg(e^{i(\theta_N^1(\pi)-\theta_N^1(\eta))}-e^{i(\theta_N^1(\pi)-\theta_N^2(\eta))}+e^{i(\theta_N^1(\pi)-\theta_N^2(\eta))}-e^{i(\theta_N^2(\pi)-\theta_N^2(\eta))}\Bigg)(1) \tag{3517}$$

$$=2\pi i\mathcal{F}\Bigg(e^{i\theta_N^1(\pi)}\Bigg(e^{-i\theta_N^1(\eta)}-e^{-i\theta_N^2(\eta)}\Bigg)+e^{-i\theta_N^2(\eta)}\Bigg(e^{i\theta_N^1(\pi)}-e^{i\theta_N^2(\pi)}\Bigg)\Bigg)(1) \tag{3518}$$

$$=2\pi i\mathcal{F}\Bigg(e^{i\theta_N^1(\pi)}e^{-i\theta_N^2(\eta)}\Bigg(e^{-i(\theta_N^1(\eta)-\theta_N^2(\eta))}-1\Bigg)\Bigg)(1) \tag{3519}$$

$$+2\pi i\mathcal{F}\Bigg(e^{-i\theta_N^2(\eta)}e^{i\theta_N^2(\pi)}\Bigg(e^{i(\theta_N^1(\pi)-\theta_N^2(\pi))}-1\Bigg)\Bigg)(1), \tag{3520}$$

where

$$\Bigg|\mathcal{F}\Bigg(e^{i\theta_N^1(\pi)}e^{-i\theta_N^2(\eta)}\Bigg(e^{-i(\theta_N^1(\eta)-\theta_N^2(\eta))}-1\Bigg)\Bigg)(1)\Bigg| \tag{3521}$$

$$=\Bigg|\sum_{k_2,k_3\in\mathbb{Z}}\mathcal{F}(e^{i\theta_N^1(\pi)})(1-k_2)\mathcal{F}(e^{-i\theta_N^2(\eta)})(k_2-k_3)\mathcal{F}(e^{-i(\theta_N^1(\eta)-\theta_N^2(\eta))}-1)(k_3)\Bigg| \tag{3522}$$

$$=\Bigg|\sum_{k_3\in\mathbb{Z}}e^{i\theta_N^1(\pi)}\mathcal{F}(e^{-i\theta_N^2(\eta)})(1-k_3)\mathcal{F}(e^{-i(\theta_N^1(\eta)-\theta_N^2(\eta))}-1)(k_3)\Bigg| \tag{3523}$$

$$\le\sum_{k_3\in\mathbb{Z}}\Bigg|\mathcal{F}(e^{-i(\theta_N^1(\eta)-\theta_N^2(\eta))}-1)(k_3)\Bigg| \tag{3524}$$

and

$$\left|\mathcal{F}\left(e^{-i\theta_N^2(\eta)}e^{i\theta_N^2(\pi)}\left(e^{i(\theta_N^1(\pi)-\theta_N^2(\pi))}-1\right)\right)(1)\right| \tag{3525}$$

$$=\left|\sum_{k_2,k_3\in\mathbb{Z}}\mathcal{F}(e^{i\theta_N^2(\pi)})(1-k_2)\mathcal{F}(e^{-i\theta_N^2(\eta)})(k_2-k_3)\mathcal{F}(e^{i(\theta_N^1(\pi)-\theta_N^2(\pi))}-1)(k_3)\right| \tag{3526}$$

$$=\left|\sum_{k_3\in\mathbb{Z}}e^{i\theta_N^2(\pi)}\mathcal{F}(e^{-i\theta_N^2(\eta)})(1-k_3)\mathcal{F}(e^{i(\theta_N^1(\pi)-\theta_N^2(\pi))}-1)(k_3)\right| \tag{3527}$$

$$\leq\sum_{k_3\in\mathbb{Z}}\left|\mathcal{F}(e^{i(\theta_N^1(\pi)-\theta_N^2(\pi))}-1)(k_3)\right|. \tag{3528}$$

Since

$$\mathcal{F}(e^{-i(\theta_N^1(\eta)-\theta_N^2(\eta))}-1)(k)=\frac{1}{2\pi}\int_{-\pi}^{\pi}(e^{-i(\theta_N^1(\eta)-\theta_N^2(\eta))}-1)e^{-ik\eta}d\eta \tag{3529}$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{n=1}^{\infty}\frac{(-i)^n}{n!}(\theta_N^1(\eta)-\theta_N^2(\eta))^n e^{-ik\eta}d\eta \tag{3530}$$

$$=\sum_{n=1}^{\infty}\frac{(-i)^n}{n!}\mathcal{F}((\theta_N^1(\eta)-\theta_N^2(\eta))^n)(k), \tag{3531}$$

and

$$\mathcal{F}(e^{i(\theta_N^1(\pi)-\theta_N^2(\pi))}-1)(k)=\frac{1}{2\pi}\int_{-\pi}^{\pi}(e^{i(\theta_N^1(\pi)-\theta_N^2(\pi))}-1)e^{-ik\eta}d\eta \tag{3532}$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{n=1}^{\infty}\frac{i^n}{n!}(\theta_N^1(\pi)-\theta_N^2(\pi))^n e^{-ik\eta}d\eta \tag{3533}$$

$$=\sum_{n=1}^{\infty}\frac{i^n}{n!}\mathcal{F}((\theta_N^1(\pi)-\theta_N^2(\pi))^n)(k), \tag{3534}$$

we have

$$\left|\operatorname{Im}\int_{-\pi}^{\pi}\int_{0}^{\alpha} e^{i(\alpha-\eta)}(e^{i(\theta_N^1(\alpha)-\theta_N^1(\eta))}-e^{i(\theta_N^2(\alpha)-\theta_N^2(\eta))})d\eta d\alpha\right| \tag{3535}$$

$$\leq 2\pi\left|\mathcal{F}\left(e^{i\theta_N^1(\pi)}e^{-i\theta_N^2(\eta)}\left(e^{-i(\theta_N^1(\eta)-\theta_N^2(\eta))}-1\right)\right)(1)\right| \tag{3536}$$

$$+2\pi\left|\mathcal{F}\left(e^{-i\theta_N^2(\eta)}e^{i\theta_N^2(\pi)}\left(e^{i(\theta_N^1(\pi)-\theta_N^2(\pi))}-1\right)\right)(1)\right| \tag{3537}$$

$$\leq 2\pi\sum_{k_3\in\mathbb{Z}}\left|\mathcal{F}(e^{-i(\theta_N^1(\eta)-\theta_N^2(\eta))}-1)(k_3)\right| \tag{3538}$$

$$+2\pi\sum_{k_3\in\mathbb{Z}}\left|\mathcal{F}(e^{i(\theta_N^1(\pi)-\theta_N^2(\pi))}-1)(k_3)\right| \tag{3539}$$

$$\leq 2\pi\sum_{n=1}^{\infty}\frac{\|\theta_N^1-\theta_N^2\|_{\mathcal{F}^{0,1}}^n}{n!}+2\pi\sum_{n=1}^{\infty}\frac{\|\theta_N^1(\pi)-\theta_N^2(\pi)\|_{\mathcal{F}^{0,1}}^n}{n!} \tag{3540}$$

$$\leq 2\pi\left\|\theta_N^1-\theta_N^2\right\|_{\mathcal{F}^{0,1}}\sum_{n=1}^{\infty}\frac{(\|\theta_N^1\|_{\mathcal{F}^{0,1}}+\|\theta_N^2\|_{\mathcal{F}^{0,1}})^{n-1}}{n!} \tag{3541}$$

$$+2\pi\left\|\theta_N^1(\pi)-\theta_N^2(\pi)\right\|_{\mathcal{F}^{0,1}}\sum_{n=1}^{\infty}\frac{(\|\theta_N^1(\pi)\|_{\mathcal{F}^{0,1}}+\|\theta_N^2(\pi)\|_{\mathcal{F}^{0,1}})^{n-1}}{n!}. \tag{3542}$$

Since

$$\left\|\theta_N^2\right\|_{\mathcal{F}^{0,1}}=\sum_{k\in\mathbb{Z}}\left|\mathcal{F}(\theta_N^2)(k)\right| \tag{3543}$$

$$\leq\sum_{k\in\mathbb{Z}}1_{|k|\leq N}(1+|k|^2)^m\left|\mathcal{F}(\theta_N^2)(k)\right| \tag{3544}$$

$$\leq\left(\sum_{k\in\mathbb{Z}}\left(1_{|k|\leq N}(1+|k|^2)^{m/2}\right)^2\right)^{1/2}\left(\sum_{k\in\mathbb{Z}}\left((1+|k|^2)^{m/2}\left|\mathcal{F}(\theta_N^2)(k)\right|\right)^2\right)^{1/2} \tag{3545}$$

$$\leq\left(\sum_{|k|\leq N}(1+|k|^2)^m\right)^{1/2}\left\|\theta_N^2\right\|_{H^m} \tag{3546}$$

$$\leq\left(\sum_{|k|\leq N}(1+|k|^2)^m\right)^{1/2}M, \tag{3547}$$

and

$$\left\|\theta_N^1(\pi)-\theta_N^2(\pi)\right\|_{\mathcal{F}^{0,1}}=\sum_{k\in\mathbb{Z}}\left|\mathcal{F}(\theta_N^1(\pi)-\theta_N^2(\pi))(k)\right| \tag{3548}$$

$$=\left|\theta_N^1(\pi)-\theta_N^2(\pi)\right| \tag{3549}$$

$$=\left|\sum_{k\in\mathbb{Z}}\mathcal{F}(\theta_N^1-\theta_N^2)(k)e^{ik\pi}\right| \tag{3550}$$

$$\leq\left\|\theta_N^1-\theta_N^2\right\|_{\mathcal{F}^{0,1}}, \tag{3551}$$

we have

$$\left|\mathrm{Im}\int_{-\pi}^{\pi}\int_{0}^{\alpha} e^{i(\alpha-\eta)}(e^{i(\theta_N^1(\alpha)-\theta_N^1(\eta))}-e^{i(\theta_N^2(\alpha)-\theta_N^2(\eta))})d\eta d\alpha\right| \tag{3552}$$

$$\leq 2\pi\left\|\theta_N^1-\theta_N^2\right\|_{\mathcal{F}^{0,1}}\sum_{n=1}^{\infty}\frac{\left(2\cdot\left(\sum_{|k|\leq N}(1+|k|^2)^m\right)^{1/2}M\right)^{n-1}}{n!} \tag{3553}$$

$$+2\pi\left\|\theta_N^1-\theta_N^2\right\|_{\mathcal{F}^{0,1}}\sum_{n=1}^{\infty}\frac{\left(2\cdot\left(\sum_{|k|\leq N}(1+|k|^2)^m\right)^{1/2}M\right)^{n-1}}{n!}. \tag{3554}$$

We note that

$$\left\|\theta_N^1-\theta_N^2\right\|_{\mathcal{F}^{0,1}} = \sum_{k\in\mathbb{Z}}\left|\mathcal{F}(\theta_N^1-\theta_N^2)(k)\right| \tag{3555}$$

$$\leq \sum_{k\in\mathbb{Z}}(1+|k|^2)^m\left|\mathcal{F}(\theta_N^1-\theta_N^2)(k)\right| \tag{3556}$$

$$\leq \sum_{|k|\leq N}(1+|k|^2)^m\left|\mathcal{F}(\theta_N^1-\theta_N^2)(k)\right| \tag{3557}$$

$$\leq\left(\sum_{|k|\leq N}(1+|k|^2)^m\right)^{1/2}\left(\sum_{|k|\leq N}(1+|k|^2)^m\left|\mathcal{F}(\theta_N^1-\theta_N^2)(k)\right|^2\right)^{1/2} \tag{3558}$$

$$\leq\left(\sum_{|k|\leq N}(1+|k|^2)^m\right)^{1/2}\left\|\theta_N^1-\theta_N^2\right\|_{H^m}. \tag{3559}$$

Hence,

$$\left|\mathrm{Im}\int_{-\pi}^{\pi}\int_{0}^{\alpha} e^{i(\alpha-\eta)}(e^{i(\theta_N^1(\alpha)-\theta_N^1(\eta))}-e^{i(\theta_N^2(\alpha)-\theta_N^2(\eta))})d\eta d\alpha\right| \tag{3560}$$

$$\leq 2\pi\left(\sum_{|k|\leq N}(1+|k|^2)^m\right)^{1/2}\left\|\theta_N^1-\theta_N^2\right\|_{H^m} \tag{3561}$$

$$\cdot\sum_{n=1}^{\infty}\frac{\left(2\cdot\left(\sum_{|k|\leq N}(1+|k|^2)^m\right)^{1/2}M\right)^{n-1}}{n!} \tag{3562}$$

$$+2\pi\left(\sum_{|k|\leq N}(1+|k|^2)^m\right)^{1/2}\left\|\theta_N^1-\theta_N^2\right\|_{H^m} \tag{3563}$$

$$\cdot\sum_{n=1}^{\infty}\frac{\left(2\cdot\left(\sum_{|k|\leq N}(1+|k|^2)^m\right)^{1/2}M\right)^{n-1}}{n!}. \tag{3564}$$

As for

$$\left(1+\frac{1}{2\pi}\mathrm{Im}\int_{-\pi}^{\pi}\int_{0}^{\alpha}e^{i(\alpha-\eta)}\sum_{n\geq 1}\frac{i^n}{n!}(\theta_N^2(\alpha)-\theta_N^2(\eta))^n d\eta d\alpha\right)^{-1/2}, \tag{3565}$$

which appears in (3507), we use the estimate in (845) to obtain

$$\begin{aligned}&\left(1+\frac{1}{2\pi}\mathrm{Im}\int_{-\pi}^{\pi}\int_{0}^{\alpha}e^{i(\alpha-\eta)}\sum_{n\geq 1}\frac{i^n}{n!}(\theta_N^2(\alpha)-\theta_N^2(\eta))^n d\eta d\alpha\right)^{-1/2} &(3566)\\ \leq&\left(1-\frac{\pi}{2}\left(e^{2\|\theta_N^2\|_{\mathcal{F}^{0,1}}}-1\right)\right)^{-1/2}. &(3567)\end{aligned}$$

Using the estimate in (3543), we obtain

$$\begin{aligned}&\left(1+\frac{1}{2\pi}\mathrm{Im}\int_{-\pi}^{\pi}\int_{0}^{\alpha}e^{i(\alpha-\eta)}\sum_{n\geq 1}\frac{i^n}{n!}(\theta_N^2(\alpha)-\theta_N^2(\eta))^n d\eta d\alpha\right)^{-1/2} &(3568)\\ \leq&\left(1-\frac{\pi}{2}\left(e^{2\left(\sum_{|k|\leq N}(1+|k|^2)^m\right)^{1/2}M}-1\right)\right)^{-1/2}. &(3569)\end{aligned}$$

We choose M sufficiently small so that (3569) is well-defined. We note that an appropriate estimate for the expression in (3215) can be derived similarly. This completes the proof that the operator $\mathcal{J}_N^1\circ G_N$ is indeed locally Lipschitz continuous. Therefore, by Picard's theorem, for any $\theta_{N,0}\in O^M$, there exists a time $T_N>0$ such that the ordinary differential equation

$$\begin{aligned}\frac{d\theta_N}{dt}&=(\mathcal{J}_N^1\circ G_N)(\theta_N), &(3570)\\ \theta_N(0)&=\theta_{N,0}\in O^M &(3571)\end{aligned}$$

has a unique local solution $\theta_N\in C^1([0,T_N);O^M)$.

15.3.3 Derivation of an *a priori* Estimate

For every $n\in\mathbb{N}$, define $\phi_N(\alpha,t)=\theta_N(\alpha,t)-\hat{\theta}_N(0,t)$. We let

$$\begin{aligned}(U_\alpha)_N(\theta_N)=&(U_\alpha)_{N,0}(\theta_N)+(U_\alpha)_{N,1}(\theta_N)+(U_\alpha)_{N,2}(\theta_N), &(3572)\\ T_N(\theta_N)=&T_{N,0}(\theta_N)+T_{N,1}(\theta_N)+T_{N,2}(\theta_N), &(3573)\end{aligned}$$

where $(U_\alpha)_{N,0}(\theta_N)$, $(U_\alpha)_{N,1}(\theta_N)$, and $(U_\alpha)_{N,2}(\theta_N)$ are the parts of $(U_\alpha)_N(\theta_N)$ that are constant, linear, and superlinear in the variable θ_N; and $T_{N,0}(\theta_N)$, $T_{N,1}(\theta_N)$, and $T_{N,2}(\theta_N)$ are the parts of $T_N(\theta_N)$ that are constant, linear, and superlinear in the variable θ_N. We note that

$$\frac{d\theta_N}{dt}=\mathcal{L}_N(\theta_N)+\mathcal{N}_N(\theta_N), \tag{3574}$$

where $\mathcal{L}_N(\theta_N)$ and $\mathcal{N}_N(\theta_N)$ are the parts of the right hand side of (3570) which are linear and superlinear in the variable θ_N. In particular,

$$\mathcal{L}_N(\theta_N) = \frac{2\pi}{L(\theta_N)(t)}\Big((U_\alpha)_{N,1}(\theta_N) + T_{N,0}(\theta_N)\cdot(\theta_N)_\alpha + T_{N,1}(\theta_N)\Big) \tag{3575}$$

$$= \frac{2\pi}{L(\theta_N)(t)}\Big((U_\alpha)_{N,1}(\theta_N) + T_{N,1}(\theta_N)\Big) \tag{3576}$$

$$= \frac{2\pi}{L(\theta_N)(t)}\Big((U_\alpha)_{N,1}(\theta_N) + \mathcal{M}\Big(U_{N,1}(\theta_N)(\alpha)\Big)\Big) \tag{3577}$$

$$= \frac{2\pi}{L(\theta_N)(t)}\Big((\mathcal{J}_N\circ\mathrm{Re})\Big(\sum_{j=1}^{7}\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}(E_j)_\alpha(\theta_N)(\alpha,\beta)d\beta\Big) \tag{3578}$$

$$+ \mathcal{M}\Big((\mathcal{J}_N\circ\mathrm{Re})\Big(\sum_{j=1}^{7}\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}E_j(\theta_N)(\alpha,\beta)d\beta\Big)\Big)\Big) \tag{3579}$$

$$= \frac{2\pi}{L(\phi_N)(t)}\Big((\mathcal{J}_N\circ\mathrm{Re})\Big(\sum_{j=1}^{7}\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}(E_j)_\alpha(\phi_N)(\alpha,\beta)d\beta\Big) \tag{3580}$$

$$+ \mathcal{M}\Big((\mathcal{J}_N\circ\mathrm{Re})\Big(\sum_{j=1}^{7}\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}E_j(\phi_N)(\alpha,\beta)d\beta\Big)\Big)\Big). \tag{3581}$$

Hence, for $k \neq 0$,

$$\mathcal{F}(\mathcal{L}_N(\phi_N))(k) = \frac{2\pi}{L(\phi_N)(t)}\Big(1_{|k|\le N}\cdot\mathcal{F}\Big(\mathrm{Re}\Big(\sum_{j=1}^{7}\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}(E_j)_\alpha(\phi_N)(\alpha,\beta)d\beta\Big)\Big)(k) \tag{3582}$$

$$- 1_{|k|\le N}\cdot\frac{i}{k}\mathcal{F}\Big(\mathrm{Re}\Big(\sum_{j=1}^{7}\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}E_j(\phi_N)(\alpha,\beta)d\beta\Big)\Big)(k)\Big). \tag{3583}$$

We note that this expression differs from that of $\mathcal{F}(\mathcal{L})(k)$ in (349) only by the presence of $1_{|k|\le N}$. This means that for $1 \le |k| \le N$, the analogue of the expression in (817) holds, i.e.,

$$\mathcal{F}(\mathcal{L}_N(\phi_N))(k) = \begin{cases} 0 & |k| > N, \\ -\frac{2\pi}{L(\phi_N)(t)}\frac{\gamma}{4\pi}\mathcal{F}(\phi_N)(k)\pi\,|k| & 1 < |k| \le N, \\ 0 & |k| = 1. \end{cases} \tag{3584}$$

Defining

$$\widetilde{\mathcal{L}}_N(\theta_N) = \frac{L(\theta_N)(t)}{2\pi}\mathcal{L}_N(\theta_N), \tag{3585}$$

$$\widetilde{\mathcal{N}}_N(\theta_N) = \frac{L(\theta_N)(t)}{2\pi}\mathcal{N}_N(\theta_N), \tag{3586}$$

we note that

$$\widetilde{\mathcal{N}}_N(\phi_N) = (U_\alpha)_{N,2}(\phi_N) + T_{N,2}(\phi_N)(1+(\phi_N)_\alpha) + T_{N,1}(\phi_N)\cdot(\phi_N)_\alpha. \tag{3587}$$

The analogues of Lemmas 1 and 2 hold for $T_{N,2}(\phi_N)(1+(\phi_N)_\alpha)$ and $T_{N,1}(\phi_N)\cdot(\phi_N)_\alpha$, respectively. Hence, it suffices to derive estimates for the $\mathcal{F}_\nu^{0,1}$ and $\dot{\mathcal{F}}_\nu^{s,1}$ norms of $U_{N,1}(\phi_N)$ and $U_{N,2}(\phi_N)$, as well as the $\dot{\mathcal{F}}_\nu^{s,1}$ norm of $(U_\alpha)_{N,2}(\phi_N)$. For these norms, we can use the estimates presented in Sections 12, 13, and 14. Using (3584), we obtain for $0<|k|\le N$,

$$\frac{\partial}{\partial t}\mathcal{F}(\phi_N)(k)=\frac{2\pi}{L(t)}\frac{\gamma}{4\pi}\mathcal{F}(\phi_N)(k)(J_1(k)+J_2(k))+\frac{2\pi}{L(t)}\mathcal{F}(\widetilde{\mathcal{N}}_N(\phi_N))(k), \tag{3588}$$

where J_1 and J_2 are the same as in (859). Since ϕ_N is real-valued, for $k>0$,

$$\left|\hat{\phi_N}(-k)\right|=\left|\overline{\hat{\phi_N}(k)}\right|=\left|\hat{\phi_N}(k)\right|. \tag{3589}$$

Then for $s>0$,

$$\|\phi_N\|_{\dot{\mathcal{F}}_\nu^{s,1}}=\sum_{k\neq 0}e^{\nu(t)|k|}|k|^s\left|\hat{\phi_N}(k)\right|=2\sum_{k\ge 1}e^{\nu(t)k}k^s\left|\hat{\phi_N}(k)\right|. \tag{3590}$$

The norm $\|\phi_N\|_{\dot{\mathcal{F}}_\nu^{s,1}}$ is differentiable with respect to time with

$$\begin{aligned}\frac{d}{dt}\|\phi_N\|_{\dot{\mathcal{F}}_\nu^{s,1}}=&2\sum_{k\ge 1}e^{\nu(t)k}\nu'(t)k\cdot k^s\left|\hat{\phi_N}(k)\right|+e^{\nu(t)k}k^s\frac{\partial}{\partial t}\left|\hat{\phi_N}(k)\right| &(3591)\\ =&2\sum_{k\ge 1}e^{\nu(t)k}\nu'(t)k^{s+1}\left|\hat{\phi_N}(k)\right| &(3592)\\ &+e^{\nu(t)k}k^s\frac{1}{\left|\hat{\phi_N}(k)\right|}\frac{1}{2}\left(\hat{\phi_N}(k)\overline{\frac{\partial}{\partial t}\hat{\phi_N}(k)}+\overline{\hat{\phi_N}(k)}\frac{\partial}{\partial t}\hat{\phi_N}(k)\right) &(3593)\\ =&2\sum_{k\ge 1}e^{\nu(t)k}\nu'(t)k^{s+1}\left|\hat{\phi_N}(k)\right| &(3594)\\ &+2\sum_{k\ge 1}e^{\nu(t)k}k^s\frac{\hat{\phi_N}(k)\overline{\frac{\partial}{\partial t}\hat{\phi_N}(k)}+\overline{\hat{\phi_N}(k)}\frac{\partial}{\partial t}\hat{\phi_N}(k)}{2\left|\hat{\phi_N}(k)\right|}, &(3595)\end{aligned}$$

where $\frac{\partial}{\partial t}\hat{\phi_N}(k)$ is given in (3588). In particular, $\|\phi_N\|_{\dot{\mathcal{F}}_\nu^{s,1}}$ is continuous with respect to time. We can use the calculations presented in Section 10 to obtain

$$\begin{aligned}\frac{d}{dt}\|\phi_N\|_{\dot{\mathcal{F}}_\nu^{s,1}}\le&\nu'(t)\|\phi_N\|_{\dot{\mathcal{F}}_\nu^{s+1,1}}-\pi\frac{2}{R}\frac{\gamma}{4\pi}\sum_{k\ge 2}e^{\nu(t)k}k^{s+1}\left|\hat{\phi_N}(k)\right|+\frac{2\pi}{L(t)}\left\|\widetilde{\mathcal{N}}_N\right\|_{\dot{\mathcal{F}}_\nu^{s,1}} &(3596)\\ &+2\frac{\gamma}{4\pi}\frac{1}{R}A\|\phi_N\|_{\mathcal{F}^{0,1}}\sum_{k\ge 2}e^{\nu(t)k}k^{s+1}\left|\hat{\phi_N}(k)\right|. &(3597)\end{aligned}$$

Since $\hat{\phi_N}(1) = 0$, this is equal to

$$\frac{d}{dt}\|\phi_N\|_{\dot{\mathcal{F}}^{s,1}_\nu} \le \nu'(t)\|\phi_N\|_{\dot{\mathcal{F}}^{s+1,1}_\nu} - \pi\frac{1}{R}\frac{\gamma}{4\pi}\|\phi_N\|_{\dot{\mathcal{F}}^{s+1,1}_\nu} + \frac{2\pi}{L(t)}\left\|\widetilde{\mathcal{N}}_N\right\|_{\dot{\mathcal{F}}^{s,1}_\nu} \tag{3598}$$

$$+ \frac{\gamma}{4\pi}\frac{1}{R}A\|\phi_N\|_{\mathcal{F}^{0,1}}\|\phi_N\|_{\dot{\mathcal{F}}^{s+1,1}_\nu} \tag{3599}$$

$$\le\left(\nu'(t) - \pi\frac{1}{R}\frac{\gamma}{4\pi} + \frac{\gamma}{4\pi}\frac{1}{R}A\|\phi_N\|_{\mathcal{F}^{0,1}}\right)\|\phi_N\|_{\dot{\mathcal{F}}^{s+1,1}_\nu} \tag{3600}$$

$$+ \frac{1}{R}\frac{1}{A_1}\left\|\widetilde{\mathcal{N}}_N\right\|_{\dot{\mathcal{F}}^{s,1}_\nu}. \tag{3601}$$

Now, setting $s = 1$ and using the estimates for the $\mathcal{F}^{0,1}_\nu$ and $\dot{\mathcal{F}}^{s,1}_\nu$ norms of $U_{N,1}(\phi_N)$ and $U_{N,2}(\phi_N)$, as well as the $\dot{\mathcal{F}}^{s,1}_\nu$ norm of $(U_\alpha)_{N,2}(\phi_N)$, we obtain

$$\frac{d}{dt}\|\phi_N\|_{\dot{\mathcal{F}}^{1,1}_\nu} \le -\left(\Lambda(\|\phi_N\|_{\dot{\mathcal{F}}^{1,1}_\nu}) - \nu'(t)\right)\|\phi_N\|_{\dot{\mathcal{F}}^{2,1}_\nu}, \tag{3602}$$

where

$$\Lambda(\|\phi_N\|_{\dot{\mathcal{F}}^{1,1}_\nu}) = \pi\frac{2}{R}\frac{\gamma}{4\pi} - \frac{\gamma}{4\pi}\frac{1}{R}A(\|\phi_N\|_{\dot{\mathcal{F}}^{1,1}_\nu})\|\phi_N\|_{\dot{\mathcal{F}}^{1,1}_\nu} \tag{3603}$$

$$- \frac{1}{R}\frac{1}{A_1(\|\phi_N\|_{\dot{\mathcal{F}}^{1,1}_\nu})} \tag{3604}$$

$$\cdot\Bigg(R_1(\|\phi_N\|_{\dot{\mathcal{F}}^{1,1}_\nu})\|\phi_N\|_{\dot{\mathcal{F}}^{1,1}_\nu} + R_2(\|\phi_N\|_{\dot{\mathcal{F}}^{1,1}_\nu})\|\phi_N\|_{\dot{\mathcal{F}}^{1,1}_\nu} \tag{3605}$$

$$+ R_3(\|\phi_N\|_{\dot{\mathcal{F}}^{1,1}_\nu})\|\phi_N\|^2_{\dot{\mathcal{F}}^{1,1}_\nu} + R_4(\|\phi_N\|_{\dot{\mathcal{F}}^{1,1}_\nu})\|\phi_N\|^2_{\dot{\mathcal{F}}^{1,1}_\nu} \tag{3606}$$

$$+ R_5(\|\phi_N\|_{\dot{\mathcal{F}}^{1,1}_\nu})\|\phi_N\|_{\dot{\mathcal{F}}^{1,1}_\nu} \tag{3607}$$

$$+ 3\left(H_3\|\phi_N\|_{\dot{\mathcal{F}}^{1,1}_\nu} + H_4\|\phi_N\|_{\dot{\mathcal{F}}^{1,1}_\nu}\right) \tag{3608}$$

$$+ 3\left(D_1(\|\phi_N\|_{\dot{\mathcal{F}}^{1,1}_\nu})\|\phi_N\|^2_{\dot{\mathcal{F}}^{1,1}_\nu} + D_2(\|\phi_N\|_{\dot{\mathcal{F}}^{1,1}_\nu})\|\phi_N\|^2_{\dot{\mathcal{F}}^{1,1}_\nu}\right) \tag{3609}$$

$$\cdot\left(1 + 2\|\phi_N\|_{\dot{\mathcal{F}}^{1,1}_\nu}\right) \tag{3610}$$

$$+ \left(D_1(\|\phi_N\|_{\dot{\mathcal{F}}^{1,1}_\nu})\|\phi_N\|_{\dot{\mathcal{F}}^{1,1}_\nu} + D_2(\|\phi_N\|_{\dot{\mathcal{F}}^{1,1}_\nu})\|\phi_N\|_{\dot{\mathcal{F}}^{1,1}_\nu}\right) \tag{3611}$$

$$\cdot\left(1 + 2\|\phi_N\|_{\dot{\mathcal{F}}^{1,1}_\nu}\right) \tag{3612}$$

$$+ 6\|\phi_N\|_{\dot{\mathcal{F}}^{1,1}_\nu}\left(H_3\|\phi_N\|_{\dot{\mathcal{F}}^{1,1}_\nu} + H_4\|\phi_N\|_{\dot{\mathcal{F}}^{1,1}_\nu}\right) \tag{3613}$$

$$+ 2\left(H_3\|\phi_N\|_{\dot{\mathcal{F}}^{1,1}_\nu} + H_4\|\phi_N\|_{\dot{\mathcal{F}}^{1,1}_\nu}\right)\Bigg). \tag{3614}$$

We note that the above expression is well-defined only when $\|\phi_N\|_{\dot{\mathcal{F}}^{1,1}_\nu}$ is small enough for all the geometric series with respect to n contained in this expression to converge. To ensure that this expression is well-defined for all time, we need to choose an appropriate initial datum. We let $\theta^0 \in \dot{\mathcal{F}}^{1,1}$, $\mathrm{Im}(\theta^0) = 0$, such that $\Lambda(\|\theta^0\|_{\dot{\mathcal{F}}^{1,1}}) > 0$, where

$$\begin{aligned}
\Lambda(\left\|\theta^0\right\|_{\dot{\mathcal{F}}^{1,1}}) =& \pi\frac{2}{R}\frac{\gamma}{4\pi} - \frac{\gamma}{4\pi}\frac{1}{R}A(\left\|\theta^0\right\|_{\dot{\mathcal{F}}^{1,1}})\left\|\theta^0\right\|_{\dot{\mathcal{F}}^{1,1}} && (3615)\\
&- \frac{1}{R}\frac{1}{A_1(\|\theta^0\|_{\dot{\mathcal{F}}^{1,1}})}\Bigg(R_1(\left\|\theta^0\right\|_{\dot{\mathcal{F}}^{1,1}})\left\|\theta^0\right\|_{\dot{\mathcal{F}}^{1,1}} + R_2(\left\|\theta^0\right\|_{\dot{\mathcal{F}}^{1,1}})\left\|\theta^0\right\|_{\dot{\mathcal{F}}^{1,1}} && (3616)\\
&+ R_3(\left\|\theta^0\right\|_{\dot{\mathcal{F}}^{1,1}})\left\|\theta^0\right\|^2_{\dot{\mathcal{F}}^{1,1}} + R_4(\left\|\theta^0\right\|_{\dot{\mathcal{F}}^{1,1}})\left\|\theta^0\right\|^2_{\dot{\mathcal{F}}^{1,1}} && (3617)\\
&+ R_5(\left\|\theta^0\right\|_{\dot{\mathcal{F}}^{1,1}})\left\|\theta^0\right\|_{\dot{\mathcal{F}}^{1,1}} && (3618)\\
&+ 3\left(H_3\left\|\theta^0\right\|_{\dot{\mathcal{F}}^{1,1}} + H_4\left\|\theta^0\right\|_{\dot{\mathcal{F}}^{1,1}}\right) && (3619)\\
&+ 3\left(D_1(\left\|\theta^0\right\|_{\dot{\mathcal{F}}^{1,1}})\left\|\theta^0\right\|^2_{\dot{\mathcal{F}}^{1,1}} + D_2(\left\|\theta^0\right\|_{\dot{\mathcal{F}}^{1,1}})\left\|\theta^0\right\|^2_{\dot{\mathcal{F}}^{1,1}}\right)\left(1 + 2\left\|\theta^0\right\|_{\dot{\mathcal{F}}^{1,1}}\right) && (3620)\\
&+ \left(D_1(\left\|\theta^0\right\|_{\dot{\mathcal{F}}^{1,1}})\left\|\theta^0\right\|_{\dot{\mathcal{F}}^{1,1}} + D_2(\left\|\theta^0\right\|_{\dot{\mathcal{F}}^{1,1}})\left\|\theta^0\right\|_{\dot{\mathcal{F}}^{1,1}}\right)\left(1 + 2\left\|\theta^0\right\|_{\dot{\mathcal{F}}^{1,1}}\right) && (3621)\\
&+ 6\left\|\theta^0\right\|_{\dot{\mathcal{F}}^{1,1}}\left(H_3\left\|\theta^0\right\|_{\dot{\mathcal{F}}^{1,1}} + H_4\left\|\theta^0\right\|_{\dot{\mathcal{F}}^{1,1}}\right) && (3622)\\
&+ 2\left(H_3\left\|\theta^0\right\|_{\dot{\mathcal{F}}^{1,1}} + H_4\left\|\theta^0\right\|_{\dot{\mathcal{F}}^{1,1}}\right)\Bigg). && (3623)
\end{aligned}$$

To make $\Lambda(\|\theta^0\|_{\dot{\mathcal{F}}^{1,1}})$ well-defined, we choose $\|\theta^0\|_{\dot{\mathcal{F}}^{1,1}}$ small enough so that all of the geometric series with respect to n contained in this expression converge. We further require that

$$\left|\mathcal{F}(\theta^0)(0)\right| + Y\left(\left\|\theta^0\right\|_{\dot{\mathcal{F}}^{1,1}}\right) + 2^{1/2}\left\|\theta^0\right\|_{\dot{\mathcal{F}}^{1,1}} < M, \tag{3624}$$

where the function Y is defined in (3142). For each $N \in \mathbb{N}$, let $\theta_{N,0} = \mathcal{J}_N^1\theta^0 \in H_N^m$. Then

$$\begin{aligned}
\|\theta_{N,0}\|_{H^m} =& \left(\sum_{k\in\mathbb{Z}}(1+|k|^2)^m\left|\mathcal{F}(\theta_{N,0})(k)\right|^2\right)^{1/2} && (3625)\\
=& \left(\sum_{|k|\le N}(1+|k|^2)^m\left|\mathcal{F}(\theta_{N,0})(k)\right|^2\right)^{1/2} && (3626)\\
\le& \sum_{|k|\le N}(1+|k|^2)^{m/2}\left|\mathcal{F}(\theta_{N,0})(k)\right| && (3627)\\
\le& \left|\mathcal{F}(\theta_{N,0})(0)\right| + 2^{m/2}\sum_{1\le|k|\le N}|k|^m\left|\mathcal{F}(\theta_{N,0})(k)\right| && (3628)\\
\le& \left|\mathcal{F}(\theta^0)(0)\right| + 2^{m/2}\left\|\theta^0\right\|_{\dot{\mathcal{F}}^{m,1}} && (3629)\\
\le& \left|\mathcal{F}(\theta^0)(0)\right| + Y\left(\left\|\theta^0\right\|_{\dot{\mathcal{F}}^{m,1}}\right) + 2^{m/2}\left\|\theta^0\right\|_{\dot{\mathcal{F}}^{m,1}}. && (3630)
\end{aligned}$$

Choosing $m = 1$, we obtain

$$\|\theta_{N,0}\|_{H^1} < M, \tag{3631}$$

which ensures that the initial datum $\theta_{N,0}$ lies in O^M as prescribed by Picard's theorem. Let

$$\phi^0 = \theta^0 - \hat{\theta^0}(0), \tag{3632}$$

$$\phi_{N,0} = \theta_{N,0} - \hat{\theta_{N,0}}(0). \tag{3633}$$

We note that $\|\phi_{N,0}\|_{\dot{\mathcal{F}}^{1,1}} \leq \|\phi^0\|_{\dot{\mathcal{F}}^{1,1}}$. Since $\Lambda(\cdot)$ is monotone decreasing, for all $n \in \mathbb{N}$,

$$0 < \Lambda(\left\|\theta^0\right\|_{\dot{\mathcal{F}}^{1,1}}) = \Lambda(\left\|\phi^0\right\|_{\dot{\mathcal{F}}^{1,1}}) \leq \Lambda(\|\phi_{N,0}\|_{\dot{\mathcal{F}}^{1,1}}). \tag{3634}$$

We choose ν_0 such that $0 < \nu_0 < \Lambda(\|\phi^0\|_{\dot{\mathcal{F}}^{1,1}}) < \Lambda(\|\phi_{N,0}\|_{\dot{\mathcal{F}}^{1,1}})$. From (9), it follows that for all $\tau \geq 0$,

$$0 < \nu'(\tau) = \frac{\nu_0}{(1+\tau)^2} \leq \nu_0. \tag{3635}$$

Then

$$\Lambda(\|\phi_{N,0}\|_{\dot{\mathcal{F}}^{1,1}}) - \nu'(0) > 0. \tag{3636}$$

Let

$$T_{N,1} = \sup\left\{ t_1 \in [0, T_N] : \Lambda(\|\phi_N(\tau)\|_{\dot{\mathcal{F}}_\nu^{1,1}}) - \nu'(\tau) > 0 \text{ for all } \tau \in [0, t_1] \right\}. \tag{3637}$$

Since $\Lambda(\|\phi_{N,0}\|_{\dot{\mathcal{F}}^{1,1}}) - \nu'(0) > 0$ and $\Lambda(\|\phi_N(\cdot)\|_{\dot{\mathcal{F}}_\nu^{1,1}}) - \nu'(\cdot)$ is a continuous function of time, we have $T_{N,1} > 0$. For any $\tau \in [0, T_{N,1})$,

$$\Lambda(\|\phi_N(\tau)\|_{\dot{\mathcal{F}}_\nu^{1,1}}) - \nu'(\tau) > 0. \tag{3638}$$

If $t_1 \in [0, T_{N,1}]$ and $t_2 \in [t_1, T_{N,1}]$, then

$$\|\phi_N(t_2)\|_{\dot{\mathcal{F}}_\nu^{1,1}} + \int_{t_1}^{t_2} \left(\Lambda(\|\phi_N(\tau)\|_{\dot{\mathcal{F}}_\nu^{1,1}}) - \nu'(\tau) \right) \|\phi_N(\tau)\|_{\dot{\mathcal{F}}_\nu^{2,1}} \, d\tau \leq \|\phi_N(t_1)\|_{\dot{\mathcal{F}}_\nu^{1,1}}. \tag{3639}$$

Since

$$\int_{t_1}^{t_2} \left(\Lambda(\|\phi_N(\tau)\|_{\dot{\mathcal{F}}_\nu^{1,1}}) - \nu'(\tau) \right) \|\phi_N(\tau)\|_{\dot{\mathcal{F}}_\nu^{2,1}} \, d\tau > 0, \tag{3640}$$

it follows from (3639) that $\|\phi_N(t_2)\|_{\dot{\mathcal{F}}_\nu^{1,1}} \leq \|\phi_N(t_1)\|_{\dot{\mathcal{F}}_\nu^{1,1}}$. Since Λ is a monotone decreasing function of $\|\phi_N\|_{\dot{\mathcal{F}}_\nu^{1,1}}$, this means that $\Lambda(\|\phi_N(t_2)\|_{\dot{\mathcal{F}}_\nu^{1,1}}) \geq \Lambda(\|\phi_N(t_1)\|_{\dot{\mathcal{F}}_\nu^{1,1}})$, i.e., $\Lambda(\|\phi_N(\cdot)\|_{\dot{\mathcal{F}}_\nu^{1,1}})$ is a monotone increasing function on $[0, T_{N,1}]$. Suppose for contradiction that $T_{N,1} < T_N$. If $\Lambda(\|\phi_N(T_{N,1})\|_{\dot{\mathcal{F}}_\nu^{1,1}}) - \nu'(T_{N,1}) \leq 0$, then since $\Lambda(\|\phi_N(\cdot)\|_{\dot{\mathcal{F}}_\nu^{1,1}})$ is monotone increasing on $[0, T_{N,1}]$,

$$\nu_0 = \nu'(0) < \Lambda(\|\phi_{N,0}\|_{\dot{\mathcal{F}}^{1,1}}) \leq \Lambda(\|\phi_N(T_{N,1})\|_{\dot{\mathcal{F}}_\nu^{1,1}}) \leq \nu'(T_{N,1}) = \frac{\nu_0}{(1+T_{N,1})^2} < \nu_0, \tag{3641}$$

which is a contradiction. If on the other hand $\Lambda(\|\phi_N(T_{N,1})\|_{\dot{\mathcal{F}}_\nu^{1,1}}) - \nu'(T_{N,1}) > 0$, then the function $\Lambda(\|\phi_N(\cdot)\|_{\dot{\mathcal{F}}_\nu^{1,1}}) - \nu'(\cdot)$ is discontinuous at $T_{N,1} \in (0, T_N)$, a contradiction as well. Hence, we conclude that $T_{N,1} = T_N$. Thus, for all $t \in [0, T_N)$,

$$\begin{aligned}
\|\phi_N(t)\|_{\dot{\mathcal{F}}_\nu^{1,1}} &\leq \|\phi_{N,0}\|_{\dot{\mathcal{F}}^{1,1}} - \int_0^t \left(\Lambda(\|\phi_N(\tau)\|_{\dot{\mathcal{F}}_\nu^{1,1}}) - \nu'(\tau) \right) \|\phi_N(\tau)\|_{\dot{\mathcal{F}}_\nu^{2,1}} \, d\tau &(3642)\\
&\leq \|\phi_{N,0}\|_{\dot{\mathcal{F}}^{1,1}} - \int_0^t \left(\Lambda(\|\phi_{N,0}\|_{\dot{\mathcal{F}}^{1,1}}) - \nu_0 \right) \|\phi_N(\tau)\|_{\dot{\mathcal{F}}_\nu^{2,1}} \, d\tau &(3643)\\
&\leq \|\phi_{N,0}\|_{\dot{\mathcal{F}}^{1,1}} - \int_0^t \left(\Lambda(\left\|\theta^0\right\|_{\dot{\mathcal{F}}^{1,1}}) - \nu_0 \right) \|\phi_N(\tau)\|_{\dot{\mathcal{F}}_\nu^{2,1}} \, d\tau &(3644)\\
&\leq \left\|\theta^0\right\|_{\dot{\mathcal{F}}^{1,1}} - \int_0^t \left(\Lambda(\left\|\theta^0\right\|_{\dot{\mathcal{F}}^{1,1}}) - \nu_0 \right) \|\phi_N(\tau)\|_{\dot{\mathcal{F}}_\nu^{2,1}} \, d\tau. &(3645)
\end{aligned}$$

Therefore, for all $t \in [0, T_N)$,

$$\|\phi_N(t)\|_{\dot{\mathcal{F}}^{1,1}_\nu} + \left(\Lambda(\|\theta^0\|_{\dot{\mathcal{F}}^{1,1}}) - \nu_0\right) \int_0^t \|\phi_N(\tau)\|_{\dot{\mathcal{F}}^{2,1}_\nu} d\tau \leq \|\theta^0\|_{\dot{\mathcal{F}}^{1,1}} . \tag{3646}$$

Moreover, for all $t \in [0, T_N)$,

$$\begin{aligned}
\frac{d}{dt} \|\phi_N\|_{\dot{\mathcal{F}}^{1,1}_\nu} &\leq -\left(\Lambda(\|\phi_N(t)\|_{\dot{\mathcal{F}}^{1,1}_\nu}) - \nu'(t)\right) \|\phi_N\|_{\dot{\mathcal{F}}^{2,1}_\nu} && (3647)\\
&\leq -\left(\Lambda(\|\phi_{N,0}\|_{\dot{\mathcal{F}}^{1,1}}) - \nu_0\right) \|\phi_N\|_{\dot{\mathcal{F}}^{2,1}_\nu} && (3648)\\
&\leq -\left(\Lambda(\|\phi_{N,0}\|_{\dot{\mathcal{F}}^{1,1}}) - \nu_0\right) \|\phi_N\|_{\dot{\mathcal{F}}^{1,1}_\nu} , && (3649)
\end{aligned}$$

from which we deduce that $\|\phi_N\|_{\dot{\mathcal{F}}^{1,1}_\nu}$ decays exponentially on $[0, T_N)$.

15.3.4 A Remark on the Solution Being Global in Time

The global-in-time nature of the solution to the original equations for the dynamics of the interface is inherited from that of the solutions to the regularized equations. The latter is a consequence of the continuation property of Picard's theorem in the Banach space setting and the fact that the zeroth Fourier mode of θ_N is bounded in time. We fix $\epsilon > 0$ to be arbitrtarily small and let $0_{new} = T_N - \epsilon$ be the new initial time. Then

$$\begin{aligned}
\|\theta_{N,0_{new}}\|_{H^1} &\leq |\mathcal{F}(\theta_{N,0_{new}})(0)| + 2^{1/2} \|\theta_{N,0_{new}}\|_{\dot{\mathcal{F}}^{1,1}} && (3650)\\
&= |\mathcal{F}(\theta_N(T_N - \epsilon))(0)| + 2^{1/2} \|\theta_N(T_{N-\epsilon})\|_{\dot{\mathcal{F}}^{1,1}} && (3651)\\
&\leq |\mathcal{F}(\theta_N(T_N - \epsilon))(0)| + 2^{1/2} \|\theta_{N,0}\|_{\dot{\mathcal{F}}^{1,1}} && (3652)\\
&\leq |\mathcal{F}(\theta_N(T_N - \epsilon))(0)| + 2^{1/2} \|\theta^0\|_{\dot{\mathcal{F}}^{1,1}} && (3653)\\
&\leq Y\left(\|\theta_{N,0}\|_{\dot{\mathcal{F}}^{1,1}}\right) + 2^{1/2} \|\theta^0\|_{\dot{\mathcal{F}}^{1,1}} && (3654)\\
&\leq Y\left(\|\theta^0\|_{\dot{\mathcal{F}}^{1,1}}\right) + 2^{1/2} \|\theta^0\|_{\dot{\mathcal{F}}^{1,1}} && (3655)\\
&\leq \left|\mathcal{F}(\theta^0)(0)\right| + Y\left(\|\theta^0\|_{\dot{\mathcal{F}}^{1,1}}\right) + 2^{1/2} \|\theta^0\|_{\dot{\mathcal{F}}^{1,1}} && (3656)\\
&< M. && (3657)
\end{aligned}$$

This shows that the solution $\theta_N \in C^1([0, T_N); O^M)$ can be continued in time indefinitely due to the continuation property of Picard's theorem in the Banach space setting.

15.3.5 Applying Aubin-Lions' Lemma

To apply Aubin-Lions' lemma, we set $X_0 = \dot{\mathcal{F}}^{2,1}_\nu$, $X = \dot{\mathcal{F}}^{1,1}_\nu$, $X_1 = \dot{\mathcal{F}}^{0,1}_\nu$, $p = \infty$, and let

$$G = \{\theta_N : N \in \mathbb{N}\}. \tag{3658}$$

Let $T > 0$. To show that G is uniformly bounded in $L^\infty([0,T];\dot{\mathcal{F}}^{1,1}_\nu) \cap L^1_{loc}([0,T];\dot{\mathcal{F}}^{2,1}_\nu)$, we recall (3646), i.e., for all $t \in [0,T]$,

$$\|\theta_N(t)\|_{\dot{\mathcal{F}}^{1,1}_\nu} + \left(\Lambda(\left\|\theta^0\right\|_{\dot{\mathcal{F}}^{1,1}_\nu}) - \nu_0\right)\int_0^t \|\theta_N(\tau)\|_{\dot{\mathcal{F}}^{2,1}_\nu}\,d\tau \tag{3659}$$

$$= \|\phi_N(t)\|_{\dot{\mathcal{F}}^{1,1}_\nu} + \left(\Lambda(\left\|\theta^0\right\|_{\dot{\mathcal{F}}^{1,1}_\nu}) - \nu_0\right)\int_0^t \|\phi_N(\tau)\|_{\dot{\mathcal{F}}^{2,1}_\nu}\,d\tau \tag{3660}$$

$$\le \left\|\theta^0\right\|_{\dot{\mathcal{F}}^{1,1}}. \tag{3661}$$

To show that $\partial_t G$ is uniformly bounded in $L^1_{loc}([0,T];\dot{\mathcal{F}}^{0,1}_\nu)$, we observe that

$$\int_0^T \left\|(\mathcal{J}^1_N \circ G_N)(\theta_N)\right\|_{\dot{\mathcal{F}}^{0,1}_\nu} d\tau \tag{3662}$$

$$= \int_0^T \left\|\mathcal{J}^1_N(G_N(\theta_N))\right\|_{\dot{\mathcal{F}}^{0,1}_\nu} d\tau \tag{3663}$$

$$\le \int_0^T \frac{2\pi}{L(\tau)} \left\|\mathcal{J}^1_N\Big((U_\alpha)_N(\theta_N)\Big)\right\|_{\dot{\mathcal{F}}^{0,1}_\nu} d\tau \tag{3664}$$

$$+ \int_0^T \frac{2\pi}{L(\tau)} \left\|\mathcal{J}^1_N\Big(T_N(\theta_N)\Big)\right\|_{\dot{\mathcal{F}}^{0,1}_\nu} d\tau \tag{3665}$$

$$+ \int_0^T \frac{2\pi}{L(\tau)} \left\|\mathcal{J}^1_N\Big(T_N(\theta_N)\cdot(\theta_N)_\alpha\Big)\right\|_{\dot{\mathcal{F}}^{0,1}_\nu} d\tau \tag{3666}$$

$$\le \int_0^T \frac{2\pi}{L(\tau)} \|(U_\alpha)_N(\phi_N)\|_{\dot{\mathcal{F}}^{0,1}_\nu} d\tau \tag{3667}$$

$$+ \int_0^T \frac{2\pi}{L(\tau)} \|T_N(\phi_N)\|_{\dot{\mathcal{F}}^{0,1}_\nu} d\tau \tag{3668}$$

$$+ \int_0^T \frac{2\pi}{L(\tau)} \|T_N(\phi_N)\cdot(\phi_N)_\alpha\|_{\dot{\mathcal{F}}^{0,1}_\nu} d\tau \tag{3669}$$

$$\le \int_0^T \frac{2\pi}{L(\tau)} \|W(\phi_N)\|_{\dot{\mathcal{F}}^{0,1}_\nu} d\tau \tag{3670}$$

$$+ \int_0^T \frac{2\pi}{L(\tau)} \left\|\Big(1+(\phi_N)_\alpha\Big)U_N(\phi_N)\right\|_{\dot{\mathcal{F}}^{0,1}_\nu} d\tau \tag{3671}$$

$$+ \int_0^T \frac{2\pi}{L(\tau)} \left\|\Big(1+(\phi_N)_\alpha\Big)U_N(\phi_N)\cdot(\phi_N)_\alpha\right\|_{\dot{\mathcal{F}}^{0,1}_\nu} d\tau \tag{3672}$$

$$\leq \int_0^T \frac{2\pi}{L(\tau)} \|W(\phi_N)\|_{\dot{\mathcal{F}}_\nu^{0,1}} \, d\tau \tag{3673}$$
$$+ \int_0^T \frac{2\pi}{L(\tau)} \|V(\phi_N)\|_{\dot{\mathcal{F}}_\nu^{0,1}} \, d\tau \tag{3674}$$
$$+ \int_0^T \frac{2\pi}{L(\tau)} \|(\phi_N)_\alpha \cdot U_N(\phi_N)\|_{\dot{\mathcal{F}}_\nu^{0,1}} \, d\tau \tag{3675}$$
$$+ \int_0^T \frac{2\pi}{L(\tau)} \|U_N(\phi_N) \cdot (\phi_N)_\alpha\|_{\dot{\mathcal{F}}_\nu^{0,1}} \, d\tau \tag{3676}$$
$$+ \int_0^T \frac{2\pi}{L(\tau)} \left\|U_N(\phi_N) \cdot (\phi_N)_\alpha^2\right\|_{\dot{\mathcal{F}}_\nu^{0,1}} \, d\tau. \tag{3677}$$

Since

$$\|U_N(\phi_N) \cdot (\phi_N)_\alpha\|_{\mathcal{F}_\nu^{0,1}} \leq \|V(\phi_N)\|_{\mathcal{F}_\nu^{0,1}} \|\phi_N\|_{\dot{\mathcal{F}}_\nu^{1,1}} \tag{3678}$$
$$\left\|U_N(\phi_N) \cdot (\phi_N)_\alpha^2\right\|_{\mathcal{F}_\nu^{0,1}} \leq \|V(\phi_N)\|_{\mathcal{F}_\nu^{0,1}} \|\phi_N\|_{\dot{\mathcal{F}}_\nu^{1,1}}^2 , \tag{3679}$$

we obtain

$$\int_0^T \left\|(\mathcal{J}_N^1 \circ G_N)(\theta_N)\right\|_{\dot{\mathcal{F}}_\nu^{0,1}} \, d\tau \tag{3680}$$
$$\leq \int_0^T \frac{2\pi}{L(\tau)} \|W(\phi_N)\|_{\dot{\mathcal{F}}_\nu^{0,1}} \, d\tau \tag{3681}$$
$$+ \int_0^T \frac{2\pi}{L(\tau)} \|V(\phi_N)\|_{\dot{\mathcal{F}}_\nu^{0,1}} \, d\tau \tag{3682}$$
$$+ 2\int_0^T \frac{2\pi}{L(\tau)} \|V(\phi_N)\|_{\mathcal{F}_\nu^{0,1}} \|\phi_N\|_{\dot{\mathcal{F}}_\nu^{1,1}} \, d\tau \tag{3683}$$
$$+ \int_0^T \frac{2\pi}{L(\tau)} \|V(\phi_N)\|_{\mathcal{F}_\nu^{0,1}} \|\phi_N\|_{\dot{\mathcal{F}}_\nu^{1,1}}^2 \, d\tau. \tag{3684}$$

Using estimates from Sections 12, 13, and 14 and then (3646), we see that

$$\|\partial_t \theta_N\|_{L^1_{loc}([0,T_N];\dot{\mathcal{F}}^{0,1})} = \int_0^T \left\|(\mathcal{J}_N^1 \circ G_N)(\theta_N)\right\|_{\dot{\mathcal{F}}^{0,1}} \, d\tau \tag{3685}$$

is indeed uniformly bounded. Therefore, by Aubin-Lions' lemma, G is relatively compact in $L^2([0,T];\dot{\mathcal{F}}_\nu^{1,1})$. This means that there exists a subsequence convergent in $L^2([0,T];\dot{\mathcal{F}}_\nu^{1,1})$. For notational convenience, we will continue to use θ_N to denote the subsequence. That is, there exists $\theta \in L^2([0,T];\dot{\mathcal{F}}_\nu^{1,1})$ such that $\theta_N \to \theta$ in $L^2([0,T];\dot{\mathcal{F}}_\nu^{1,1})$ as $N \to \infty$. It is crucial to bring to our attention that even though our application of Aubin-Lions' lemma provides a candidate for a solution to the original problem, it remains silent on the dynamics of $\mathcal{F}(\theta(t))(0)$. Part of our task to show that θ is a solution is to specify its dynamics. We first articulate the sense in which θ is to become a solution.

Definition 1. *We say that $\theta \in L^\infty([0,T];\dot{\mathcal{F}}^{1,1}_\nu) \cap L^1([0,T];\dot{\mathcal{F}}^{2,1}_\nu)$ is a weak solution of (3160) through (3162) if $\mathcal{F}(\theta(t))(\pm 1) = 0$ for almost every $t \in [0,T]$ and for any $\psi \in C^\infty_0([-\pi,\pi) \times [0,T])$,*

$$\int_{-\pi}^{\pi} \theta(\alpha,T)\psi(\alpha,T)d\alpha - \int_{-\pi}^{\pi} \theta(\alpha,0)\psi(\alpha,0)d\alpha - \int_{-\pi}^{\pi}\int_0^T \theta(\alpha,t)\psi_t(\alpha,t)dtd\alpha \tag{3686}$$

$$= \int_{-\pi}^{\pi}\int_0^T R^{-1}\left(1 + \frac{1}{2\pi} Im \int_{-\pi}^{\pi}\int_0^\alpha e^{i(\alpha-\eta)} \sum_{n\geq 1} \frac{i^n}{n!}(\theta(\alpha,t)-\theta(\eta,t))^n d\eta d\alpha\right)^{1/2} \tag{3687}$$

$$\cdot \left(U_\alpha(\theta)(\alpha,t) + T(\theta)(\alpha,t)(1+\theta_\alpha(\alpha,t))\right)\psi(\alpha,t)dtd\alpha. \tag{3688}$$

To show that θ is a solution to the original problem in the sense of Definition 1, we use the following standard lemma from real analysis frequently.

Lemma 12. *For any sequence of measurable functions on a measure space, L^p convergence, $p \geq 1$, implies the existence of a subsequence convergent almost everywhere.*

Applying Lemma 12 to the fact that $\|\theta_N(\cdot) - \theta(\cdot)\|_{\dot{\mathcal{F}}^{1,1}_\nu} \to 0$ in L^2, we obtain a (non-relabeled) subsequence such that for almost every $t \in [0,T]$, $\lim_{N\to\infty} \|\theta_N(t) - \theta(t)\|_{\dot{\mathcal{F}}^{1,1}_\nu} = 0$. That is, for almost every $t \in [0,T]$, $\lim_{N\to\infty} \sum_{k\in\mathbb{Z}} |k|\,|\mathcal{F}(\theta_N(t)-\theta(t))(k)| = 0$. Applying Lemma 12 again to the fact that $a(k,t) = |k|\,|\mathcal{F}(\theta_N(t)-\theta(t))(k)| \to 0$ in l^2 for almost every $t \in [0,T]$, we obtain a (non-relabeled) subsequence such that for all $k \in \mathbb{Z}\setminus\{0\}$, $\lim_{N\to\infty} \mathcal{F}(\theta_N(t))(k) = \mathcal{F}(\theta(t))(k)$ for almost every $t \in [0,T]$. In particular, for almost every $t \in [0,T]$,

$$\mathcal{F}(\theta(t))(\pm 1) = \lim_{N\to\infty} \mathcal{F}(\theta_N(t))(k) = 0. \tag{3689}$$

Let $\phi(t) = \theta(t) - \mathcal{F}(\theta(t))(0)$. Let us specify the dynamics of $\mathcal{F}(\theta)(0)$ by requiring that

$$\frac{d}{dt}\mathcal{F}(\theta)(0) = \mathcal{J}^1\left(\frac{2\pi}{L(\phi)}\left(U_\alpha(\phi) + T(\phi)(1+\phi_\alpha)\right)\right) - \frac{d}{dt}\phi \tag{3690}$$

with the initial condition $\mathcal{F}(\theta(0))(0) = \mathcal{F}(\theta^0)(0)$. The initial condition is chosen this way because for all $N \in \mathbb{N}$, $\mathcal{F}(\theta_{N,0})(0) = \mathcal{F}(\mathcal{J}^1_N\theta^0)(0) = \mathcal{F}(\theta^0)(0)$. The dynamics equation (3690) for $\mathcal{F}(\theta)(0)$ is equivalent to

$$\frac{d}{dt}\theta = \mathcal{J}^1\left(\frac{2\pi}{L(\phi)}\left(U_\alpha(\phi) + T(\phi)(1+\phi_\alpha)\right)\right), \tag{3691}$$

which implies that θ is indeed a solution to the original problem in the sense of Definition 1.

15.3.6 Inheritance of the *a priori* Estimate

At the end of Section 15.3.3, we have obtained that for all $t \in [0,T]$,

$$\|\phi_N(t)\|_{\dot{\mathcal{F}}^{1,1}_\nu} + \left(\Lambda(\|\theta^0\|_{\dot{\mathcal{F}}^{1,1}}) - \nu_0\right)\int_0^t \|\phi_N(\tau)\|_{\dot{\mathcal{F}}^{2,1}_\nu} d\tau \leq \|\theta^0\|_{\dot{\mathcal{F}}^{1,1}}. \tag{3692}$$

By Fatou's lemma, for any $t \in [0, T]$,

$$\int_0^t \liminf_{N\to\infty} \|\phi_N(\tau)\|_{\dot{\mathcal{F}}^{2,1}_\nu} \, d\tau \leq \liminf_{N\to\infty} \int_0^t \|\phi_N(\tau)\|_{\dot{\mathcal{F}}^{2,1}_\nu} \, d\tau. \tag{3693}$$

Then, using that

$$\liminf_{N\to\infty} \|\phi_N(t)\|_{\dot{\mathcal{F}}^{1,1}_\nu} = \|\phi(t)\|_{\dot{\mathcal{F}}^{1,1}_\nu}, \tag{3694}$$
$$\liminf_{N\to\infty} \|\phi_N(t)\|_{\dot{\mathcal{F}}^{2,1}_\nu} = \|\phi(t)\|_{\dot{\mathcal{F}}^{2,1}_\nu}, \tag{3695}$$

we obtain for all $t \in [0, T]$

$$\|\phi(t)\|_{\dot{\mathcal{F}}^{1,1}_\nu} + \left(\Lambda(\left\|\theta^0\right\|_{\dot{\mathcal{F}}^{1,1}}) - \nu_0\right) \int_0^t \|\phi(\tau)\|_{\dot{\mathcal{F}}^{2,1}_\nu} \, d\tau \tag{3696}$$
$$\leq \liminf_{N\to\infty} \|\phi_N(t)\|_{\dot{\mathcal{F}}^{1,1}_\nu} + \left(\Lambda(\left\|\theta^0\right\|_{\dot{\mathcal{F}}^{1,1}}) - \nu_0\right) \liminf_{N\to\infty} \int_0^t \|\phi_N(\tau)\|_{\dot{\mathcal{F}}^{2,1}_\nu} \, d\tau \tag{3697}$$
$$\leq \liminf_{N\to\infty} \left(\|\phi_N(t)\|_{\dot{\mathcal{F}}^{1,1}_\nu} + \left(\Lambda(\left\|\theta^0\right\|_{\dot{\mathcal{F}}^{1,1}}) - \nu_0\right) \int_0^t \|\phi_N(\tau)\|_{\dot{\mathcal{F}}^{2,1}_\nu} \, d\tau \right) \tag{3698}$$
$$\leq \left\|\theta^0\right\|_{\dot{\mathcal{F}}^{1,1}}. \tag{3699}$$

In words, ϕ inherits the *a priori* estimate uniformly held for ϕ_N. As a consequence,

$$\theta \in L^\infty([0, T]; \dot{\mathcal{F}}^{1,1}_\nu) \cap L^1([0, T]; \dot{\mathcal{F}}^{2,1}_\nu) \tag{3700}$$

and $\|\phi(t)\|_{\dot{\mathcal{F}}^{1,1}_\nu}$ decays exponentially on $[0, T]$.

15.3.7 Continuity in Time

Now, we show that in fact $\theta \in C([0, T]; \dot{\mathcal{F}}^{1,1}_\nu)$. That is, letting $\mathfrak{G}(t) = \|\theta(t)\|_{\dot{\mathcal{F}}^{1,1}_{\nu(t)}}$, we show that $\mathfrak{G}$ is continuous on $[0, T]$. We assume that $T > 0$ is arbitrarily small. This assumption can be made without loss of generality, because since the solution θ exists globally in time, we can imagine that the solution θ is continued over intervals of some arbitrary fixed small length. Let $\tau \in [0, T]$ and fix $\epsilon > 0$. We prove the continuity of $\mathfrak{G}$ at $\tau \in [0, T]$ by showing left- and right-continuity at that point. First, we suppose that $\tau' > \tau$. Then

$$|\mathfrak{G}(\tau') - \mathfrak{G}(\tau)| \tag{3701}$$
$$= \left| \|\theta(\tau')\|_{\dot{\mathcal{F}}^{1,1}_{\nu(\tau')}} - \|\theta(\tau)\|_{\dot{\mathcal{F}}^{1,1}_{\nu(\tau)}} \right| \tag{3702}$$
$$\leq \left| \|\theta(\tau')\|_{\dot{\mathcal{F}}^{1,1}_{\nu(\tau')}} - \|\theta(\tau)\|_{\dot{\mathcal{F}}^{1,1}_{\nu(\tau')}} \right| + \left| \|\theta(\tau)\|_{\dot{\mathcal{F}}^{1,1}_{\nu(\tau')}} - \|\theta(\tau)\|_{\dot{\mathcal{F}}^{1,1}_{\nu(\tau)}} \right| \tag{3703}$$
$$\leq \|\theta(\tau') - \theta(\tau)\|_{\dot{\mathcal{F}}^{1,1}_{\nu(\tau')}} + \left| \|\theta(\tau)\|_{\dot{\mathcal{F}}^{1,1}_{\nu(\tau')}} - \|\theta(\tau)\|_{\dot{\mathcal{F}}^{1,1}_{\nu(\tau)}} \right|. \tag{3704}$$

First, we consider $\|\theta(\tau')-\theta(\tau)\|_{\dot{\mathcal{F}}^{1,1}_{\nu(\tau')}}$. We note that

$$\begin{aligned}
&\|\theta(\tau')-\theta(\tau)\|_{\dot{\mathcal{F}}^{1,1}_{\nu(\tau')}} && (3705)\\
&\le \left\| \int_\tau^{\tau'} \partial_t\theta(\tau'')d\tau'' \right\|_{\dot{\mathcal{F}}^{1,1}_{\nu(\tau')}} && (3706)\\
&\le \int_\tau^{\tau'} \|\partial_t\theta(\tau'')\|_{\dot{\mathcal{F}}^{1,1}_{\nu(\tau')}} d\tau'' && (3707)\\
&= \int_\tau^{\tau'} \left(\sum_{k\neq 0} e^{\nu(\tau')|k|} |k| \left|\mathcal{F}(\partial_t\theta(\tau''))(k)\right| \right) d\tau'' && (3708)\\
&= \int_\tau^{\tau'} \left(\sum_{k\neq 0} e^{\nu_0\frac{\tau'}{1+\tau'}|k|} |k| \left|\mathcal{F}(\partial_t\theta(\tau''))(k)\right| \right) d\tau'' && (3709)\\
&\le \int_\tau^{\tau'} \left(\sum_{k\neq 0} e^{\widetilde{\nu_0}\frac{\tau''}{1+\tau''}|k|} |k| \left|\mathcal{F}(\partial_t\theta(\tau''))(k)\right| \right) d\tau'' && (3710)\\
&= \int_\tau^{\tau'} \|\partial_t\theta(\tau'')\|_{\dot{\mathcal{F}}^{1,1}_{\widetilde{\nu}(\tau'')}} d\tau'', && (3711)
\end{aligned}$$

where $\nu_0 < \widetilde{\nu_0}$ such that $\nu_0\frac{\tau'}{1+\tau'} \le \widetilde{\nu_0}\frac{\tau}{1+\tau}$ for all $\tau' \in [\tau, T]$. Since T is arbitrarily small, $\widetilde{\nu_0}$ can be chosen to be arbitrarily close to ν_0. We note that $\partial_t\theta \in L^1([0,T];\dot{\mathcal{F}}^{1,1}_{\widetilde{\nu}})$, in which $\widetilde{\nu}$ indicates the use of $\widetilde{\nu_0}$, instead of ν_0. For $n\in\mathbb{N}$, define

$$a_n = \int_{[\tau,\tau+\frac{1}{n}]} \|\partial_t\theta(\tau'')\|_{\dot{\mathcal{F}}^{1,1}_{\widetilde{\nu}(\tau'')}} d\tau''. \tag{3712}$$

Since

$$\left| 1_{[\tau,\tau+\frac{1}{n}]} \|\partial_t\theta\|_{\dot{\mathcal{F}}^{1,1}_{\widetilde{\nu}(\tau'')}} \right| \le \|\partial_t\theta\|_{\dot{\mathcal{F}}^{1,1}_{\widetilde{\nu}(\tau'')}} \tag{3713}$$

and

$$\int_0^T \|\partial_t\theta\|_{\dot{\mathcal{F}}^{1,1}_{\widetilde{\nu}(\tau'')}} d\tau'' < \infty, \tag{3714}$$

by the dominated convergence theorem, we have

$$\lim_{n\to\infty} a_n = 0. \tag{3715}$$

That is, there exists $N^*\in\mathbb{N}$ such that for all $N \ge N^*$,

$$\int_{[\tau,\tau+\frac{1}{n}]} \|\partial_t\theta\|_{\dot{\mathcal{F}}^{1,1}_{\widetilde{\nu}(\tau'')}} d\tau'' < \frac{\epsilon}{2}. \tag{3716}$$

Hence, there exists $\delta>0$ such that for all $|\tau'-\tau|<\delta$,

$$\int_\tau^{\tau'} \|\partial_t\theta\|_{\dot{\mathcal{F}}^{1,1}_{\widetilde{\nu}(\tau'')}} d\tau'' < \frac{\epsilon}{2}. \tag{3717}$$

Next, we consider $\left|\|\theta(\tau)\|_{\dot{\mathcal{F}}^{1,1}_{\nu(\tau')}} - \|\theta(\tau)\|_{\dot{\mathcal{F}}^{1,1}_{\nu(\tau)}}\right|$. We note that

$$\left|\|\theta(\tau)\|_{\dot{\mathcal{F}}^{1,1}_{\nu(\tau')}} - \|\theta(\tau)\|_{\dot{\mathcal{F}}^{1,1}_{\nu(\tau)}}\right| \tag{3718}$$

$$= \left|\sum_{k\neq 0} e^{\nu(\tau')|k|} |k| \left|\mathcal{F}(\theta(\tau))(k)\right| - \sum_{k\neq 0} e^{\nu(\tau)|k|} |k| \left|\mathcal{F}(\theta(\tau))(k)\right|\right|. \tag{3719}$$

We define for $\tau' \in [0, T]$

$$\mathfrak{H}(\tau') = \sum_{k\in\mathbb{Z}} e^{\nu(\tau')|k|} |k| \left|\mathcal{F}(\theta(\tau))(k)\right|. \tag{3720}$$

Then

$$\left|\|\theta(\tau)\|_{\dot{\mathcal{F}}^{1,1}_{\nu(\tau')}} - \|\theta(\tau)\|_{\dot{\mathcal{F}}^{1,1}_{\nu(\tau)}}\right| = \left|\mathfrak{H}(\tau') - \mathfrak{H}(\tau)\right|. \tag{3721}$$

Let $\mathfrak{h}_k(\tau') = e^{\nu(\tau')|k|} |k| \left|\mathcal{F}(\theta(\tau))(k)\right|$. Since

$$\left|\mathfrak{h}_k(\tau')\right| = e^{\nu_0 \frac{\tau'}{1+\tau'}|k|} |k| \left|\mathcal{F}(\theta(\tau))(k)\right| \leq e^{\widetilde{\nu_0} \frac{\tau}{1+\tau}|k|} |k| \left|\mathcal{F}(\theta(\tau))(k)\right| \tag{3722}$$

and

$$\sum_{k\in\mathbb{Z}} e^{\widetilde{\nu_0} \frac{\tau}{1+\tau}|k|} |k| \left|\mathcal{F}(\theta(\tau))(k)\right| = \|\theta(\tau)\|_{\dot{\mathcal{F}}^{1,1}_{\widetilde{\nu}(\tau)}} < \infty, \tag{3723}$$

by the Weierstrass M-test,

$$\sum_{k\in\mathbb{Z}} e^{\nu(\tau')|k|} |k| \left|\mathcal{F}(\theta(\tau))(k)\right| \tag{3724}$$

converges absolutely and uniformly with respect to $\tau' \in [0, T]$. Since for each $k \in \mathbb{Z}$,

$$\mathfrak{h}_k(\tau') = e^{\nu(\tau')|k|} |k| \left|\mathcal{F}(\theta(\tau))(k)\right| \tag{3725}$$

is continuous, so is $\mathfrak{H}(\tau')$ on $[0, T]$. Hence, there exists $\delta > 0$ such that $|\tau' - \tau| < \delta$ implies that

$$\left|\mathfrak{H}(\tau') - \mathfrak{H}(\tau)\right| < \frac{\epsilon}{2}. \tag{3726}$$

Now, suppose that $\tau' < \tau$. Then

$$\left|\mathfrak{G}(\tau') - \mathfrak{G}(\tau)\right| \tag{3727}$$

$$\leq \|\theta(\tau) - \theta(\tau')\|_{\dot{\mathcal{F}}^{1,1}_{\nu(\tau')}} + \left|\|\theta(\tau)\|_{\dot{\mathcal{F}}^{1,1}_{\nu(\tau')}} - \|\theta(\tau)\|_{\dot{\mathcal{F}}^{1,1}_{\nu(\tau)}}\right| \tag{3728}$$

First, we consider $\|\theta(\tau)-\theta(\tau')\|_{\dot{\mathcal{F}}^{1,1}_{\nu(\tau')}}$. We observe that

$$\begin{aligned}
&\|\theta(\tau)-\theta(\tau')\|_{\dot{\mathcal{F}}^{1,1}_{\nu(\tau')}} && (3729)\\
&=\left\|\int_{\tau'}^{\tau}\partial_t\theta(\tau'')d\tau''\right\|_{\dot{\mathcal{F}}^{1,1}_{\nu(\tau')}} && (3730)\\
&\le\int_{\tau'}^{\tau}\|\partial_t\theta(\tau'')\|_{\dot{\mathcal{F}}^{1,1}_{\nu(\tau')}}\,d\tau'' && (3731)\\
&\le\int_{\tau'}^{\tau}\|\partial_t\theta(\tau'')\|_{\dot{\mathcal{F}}^{1,1}_{\nu(\tau'')}}\,d\tau''. && (3732)
\end{aligned}$$

We note that $\partial_t\theta\in L^1([0,T];\dot{\mathcal{F}}^{1,1}_\nu)$. For $n\in\mathbb{N}$, define

$$b_n=\int_{[\tau-\frac{1}{n},\tau]}\|\partial_t\theta(\tau'')\|_{\dot{\mathcal{F}}^{1,1}_{\nu(\tau'')}}\,d\tau''. \tag{3733}$$

Since

$$\left|1_{[\tau-\frac{1}{n},\tau]}\|\partial_t\theta(\tau'')\|_{\dot{\mathcal{F}}^{1,1}_{\nu(\tau'')}}\right|\le\|\partial_t\theta(\tau'')\|_{\dot{\mathcal{F}}^{1,1}_{\nu(\tau'')}} \tag{3734}$$

and

$$\int_0^T\|\partial_t\theta(\tau'')\|_{\dot{\mathcal{F}}^{1,1}_{\nu(\tau'')}}\,d\tau''<\infty, \tag{3735}$$

by the dominated convergence theorem, we have

$$\lim_{n\to\infty}b_n=0. \tag{3736}$$

That is, there exists $N^{**}\in\mathbb{N}$ such that for all $N\ge N^{**}$,

$$\int_{[\tau-\frac{1}{n},\tau]}\|\partial_t\theta(\tau'')\|_{\dot{\mathcal{F}}^{1,1}_{\nu(\tau'')}}\,d\tau''<\frac{\epsilon}{2}. \tag{3737}$$

Hence, there exists $\delta>0$ such that for all $|\tau-\tau'|<\delta$,

$$\int_{\tau'}^{\tau}\|\partial_t\theta(\tau'')\|_{\dot{\mathcal{F}}^{1,1}_{\nu(\tau'')}}\,d\tau''<\frac{\epsilon}{2}. \tag{3738}$$

The second term $\left|\|\theta(\tau)\|_{\dot{\mathcal{F}}^{1,1}_{\nu(\tau')}}-\|\theta(\tau)\|_{\dot{\mathcal{F}}^{1,1}_{\nu(\tau)}}\right|$ can be estimated as in the case where $\tau'>\tau$. Therefore, we conclude that $\theta\in C([0,T];\dot{\mathcal{F}}^{1,1}_\nu)$.

15.3.8 Instantaneous Analyticity

Now, we show that θ is instantaneously analytic. To prove this, we use the following result from standard analysis.

Lemma 13. *The function f is analytic on $\mathbb{T}$ if and only if there exist constants $K > 0$ and $a > 0$ such that*

$$|\mathcal{F}(f)(j)| \leq Ke^{-a|j|}. \tag{3739}$$

Let $t > 0$. We claim that there exist $C > 0$ and $k^* > 0$ such that for all $|k| \geq k^*$,

$$e^{\nu(t)|k|}\,|k|\,|\mathcal{F}(\phi(t))(k)| \leq C. \tag{3740}$$

Suppose the contrary for contradiction. Then, for all $k^* > 0$, there exists $|k| \geq k^*$ such that

$$e^{\nu(t)|k|}\,|k|\,|\mathcal{F}(\phi(t))(k)| > 1. \tag{3741}$$

This means that there is a sequence $\{k_j\}$ such that

$$e^{\nu(t)|k_j|}\,|k_j|\,|\mathcal{F}(\phi(t))(k_j)| > 1. \tag{3742}$$

Hence,

$$\infty > \|\phi(t)\|_{\dot{\mathcal{F}}^{1,1}_{\nu}} = \sum_{k\neq 0} e^{\nu(t)|k|}\,|k|\,|\mathcal{F}(\phi(t))(k)| \geq \sum_{j=1}^{\infty} e^{\nu(t)|k_j|}\,|k_j|\,|\mathcal{F}(\phi(t))(k_j)| = \infty, \tag{3743}$$

a contradiction, as needed. Thus, there exist $C > 0$ and $k^* > 0$ such that for all $|k| \geq k^*$,

$$e^{\nu(t)|k|}\,|\mathcal{F}(\phi(t))(k)| \leq C\,|k|^{-1} \leq \frac{C}{k^*}. \tag{3744}$$

Hence, for all $k \in \mathbb{Z}$,

$$e^{\nu(t)|k|}\,|\mathcal{F}(\theta(t))(k)| \leq \max\left\{\frac{C}{k^*}, \max_{|k|<k^*} e^{\nu(t)|k|}\,|\mathcal{F}(\theta(t))(k)|\right\}. \tag{3745}$$

By Lemma 13, we can then conclude that θ is analytic.

16 Uniqueness

Let θ_1 and θ_2 be two solutions to the original problem with the same initial datum, whose ± 1 Fourier modes remain zero in time. For $k > 0$,

$$\mathcal{F}(\theta_1 - \theta_2)(-k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} (\theta_1 - \theta_2)(\alpha)e^{ik\alpha}d\alpha \tag{3746}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{(\theta_1 - \theta_2)(\alpha)e^{-ik\alpha}}d\alpha \tag{3747}$$

$$= \overline{\mathcal{F}(\theta_1 - \theta_2)(k)}. \tag{3748}$$

Hence, we may write

$$\|\theta_1 - \theta_2\|_{\dot{\mathcal{F}}^{1,1}} = \sum_{k\neq 0} |k|\,|\mathcal{F}(\theta_1 - \theta_2)(k)| = 2\sum_{k>0} |k|\,|\mathcal{F}(\theta_1 - \theta_2)(k)|\,. \tag{3749}$$

Then

$$\frac{d}{dt}\|\theta_1-\theta_2\|_{\dot{\mathcal{F}}^{1,1}} \tag{3750}$$
$$=2\sum_{k>0}|k|\frac{d}{dt}|\mathcal{F}(\theta_1-\theta_2)(k)| \tag{3751}$$
$$=2\sum_{k>0}|k|\frac{d}{dt}\left(\mathcal{F}(\theta_1-\theta_2)(k)\cdot\overline{\mathcal{F}(\theta_1-\theta_2)(k)}\right)^{1/2} \tag{3752}$$
$$=\sum_{k>0}|k|\left(\mathcal{F}(\theta_1-\theta_2)(k)\cdot\overline{\mathcal{F}(\theta_1-\theta_2)(k)}\right)^{-1/2} \tag{3753}$$
$$\cdot\left(\frac{d}{dt}\mathcal{F}(\theta_1-\theta_2)(k)\cdot\overline{\mathcal{F}(\theta_1-\theta_2)(k)}+\mathcal{F}(\theta_1-\theta_2)(k)\cdot\frac{d}{dt}\overline{\mathcal{F}(\theta_1-\theta_2)(k)}\right) \tag{3754}$$
$$=\sum_{k>0}\frac{|k|}{|\mathcal{F}(\theta_1-\theta_2)(k)|} \tag{3755}$$
$$\cdot\left(\frac{d}{dt}\mathcal{F}(\theta_1-\theta_2)(k)\cdot\overline{\mathcal{F}(\theta_1-\theta_2)(k)}+\mathcal{F}(\theta_1-\theta_2)(k)\cdot\overline{\frac{d}{dt}\mathcal{F}(\theta_1-\theta_2)(k)}\right). \tag{3756}$$

Recalling that for a solution θ to the original problem, $\phi=\theta-\hat{\theta}(0)$ satisfies

$$\frac{d}{dt}\mathcal{F}(\phi)(k)=\frac{1}{R}\cdot\frac{\gamma}{4\pi}\mathcal{F}(\phi)(k)(J_1(k)+J_2(k))+\frac{2\pi}{L(\phi)}\mathcal{F}(\widetilde{N}(\phi))(k) \tag{3757}$$
$$+\frac{\gamma}{4\pi}\mathcal{F}(\phi)(k)(J_1(k)+J_2(k))\left(-\frac{1}{R}+\frac{2\pi}{L(\phi)}\right), \tag{3758}$$

where J_1 and J_2 are the same as in (859), we have for $k > 0$

$$\frac{d}{dt}\mathcal{F}(\theta_1 - \theta_2)(k) = \frac{d}{dt}\mathcal{F}(\phi_1 - \phi_2)(k) \tag{3759}$$

$$= \frac{1}{R} \cdot \frac{\gamma}{4\pi}\mathcal{F}(\phi_1 - \phi_2)(k)(J_1(k) + J_2(k)) + \frac{2\pi}{L(\phi_1)}\mathcal{F}(\widetilde{N}(\phi_1))(k) \tag{3760}$$

$$- \frac{2\pi}{L(\phi_2)}\mathcal{F}(\widetilde{N}(\phi_2))(k) \tag{3761}$$

$$+ \frac{\gamma}{4\pi}\mathcal{F}(\phi_1)(k)(J_1(k) + J_2(k))\left(-\frac{1}{R} + \frac{2\pi}{L(\phi_1)}\right) \tag{3762}$$

$$- \frac{\gamma}{4\pi}\mathcal{F}(\phi_2)(k)(J_1(k) + J_2(k))\left(-\frac{1}{R} + \frac{2\pi}{L(\phi_2)}\right) \tag{3763}$$

$$= \frac{1}{R} \cdot \frac{\gamma}{4\pi}\mathcal{F}(\phi_1 - \phi_2)(k)(J_1(k) + J_2(k)) \tag{3764}$$

$$+ \left(\frac{2\pi}{L(\phi_1)} - \frac{2\pi}{L(\phi_2)}\right)\mathcal{F}(\widetilde{N}(\phi_1))(k) \tag{3765}$$

$$+ \frac{2\pi}{L(\phi_2)}\mathcal{F}(\widetilde{N}(\phi_1) - \widetilde{N}(\phi_2))(k) \tag{3766}$$

$$+ \frac{\gamma}{4\pi}\mathcal{F}(\phi_1 - \phi_2)(k)(J_1(k) + J_2(k))\left(-\frac{1}{R} + \frac{2\pi}{L(\phi_1)}\right) \tag{3767}$$

$$- \frac{\gamma}{4\pi}\mathcal{F}(\phi_2)(k)(J_1(k) + J_2(k))\left(\frac{2\pi}{L(\phi_2)} - \frac{2\pi}{L(\phi_1)}\right). \tag{3768}$$

Substituting this expression into (3750), we obtain

$$
\begin{aligned}
&\frac{d}{dt}\left\|\theta_1-\theta_2\right\|_{\dot{\mathcal{F}}^{1,1}} && (3769)\\
=&\sum_{k>0}\frac{|k|}{|\mathcal{F}(\theta_1-\theta_2)(k)|}\Bigg(\frac{1}{R}\cdot\frac{\gamma}{4\pi}\left|\mathcal{F}(\phi_1-\phi_2)(k)\right|^2(J_1(k)+J_2(k)) && (3770)\\
&+\left(\frac{2\pi}{L(\phi_1)}-\frac{2\pi}{L(\phi_2)}\right)\mathcal{F}(\widetilde{N}(\phi_1))(k)\overline{\mathcal{F}(\phi_1-\phi_2)(k)} && (3771)\\
&+\frac{2\pi}{L(\phi_2)}\mathcal{F}(\widetilde{N}(\phi_1)-\widetilde{N}(\phi_2))(k)\cdot\overline{\mathcal{F}(\phi_1-\phi_2)(k)} && (3772)\\
&+\frac{\gamma}{4\pi}\left|\mathcal{F}(\phi_1-\phi_2)(k)\right|^2(J_1(k)+J_2(k))\left(-\frac{1}{R}+\frac{2\pi}{L(\phi_1)}\right) && (3773)\\
&-\frac{\gamma}{4\pi}\mathcal{F}(\phi_2)(k)(J_1(k)+J_2(k))\left(\frac{2\pi}{L(\phi_2)}-\frac{2\pi}{L(\phi_1)}\right)\overline{\mathcal{F}(\phi_1-\phi_2)(k)} && (3774)\\
&+\frac{1}{R}\cdot\frac{\gamma}{4\pi}\left|\mathcal{F}(\phi_1-\phi_2)(k)\right|^2(J_1(k)+J_2(k)) && (3775)\\
&+\left(\frac{2\pi}{L(\phi_1)}-\frac{2\pi}{L(\phi_2)}\right)\overline{\mathcal{F}(\widetilde{N}(\phi_1))(k)}\cdot\mathcal{F}(\phi_1-\phi_2)(k) && (3776)\\
&+\frac{2\pi}{L(\phi_2)}\overline{\mathcal{F}(\widetilde{N}(\phi_1)-\widetilde{N}(\phi_2))(k)}\cdot\mathcal{F}(\phi_1-\phi_2)(k) && (3777)\\
&+\frac{\gamma}{4\pi}\left|\mathcal{F}(\phi_1-\phi_2)(k)\right|^2(J_1(k)+J_2(k))\left(-\frac{1}{R}+\frac{2\pi}{L(\phi_1)}\right) && (3778)\\
&-\frac{\gamma}{4\pi}\overline{\mathcal{F}(\phi_2)(k)}(J_1(k)+J_2(k))\left(\frac{2\pi}{L(\phi_2)}-\frac{2\pi}{L(\phi_1)}\right)\mathcal{F}(\phi_1-\phi_2)(k)\Bigg) && (3779)\\
=&2\frac{1}{R}\cdot\frac{\gamma}{4\pi}\sum_{k>0}|k|\left|\mathcal{F}(\phi_1-\phi_2)(k)\right|(J_1(k)+J_1(k)) && (3780)\\
&+\left(\frac{2\pi}{L(\phi_1)}-\frac{2\pi}{L(\phi_2)}\right) && (3781)\\
&\cdot\sum_{k>0}|k|\frac{\mathcal{F}(\widetilde{N}(\phi_1))(k)\overline{\mathcal{F}(\phi_1-\phi_2)(k)}+\overline{\mathcal{F}(\widetilde{N}(\phi_1))(k)}\cdot\mathcal{F}(\phi_1-\phi_2)(k)}{|\mathcal{F}(\phi_1-\phi_2)(k)|} && (3782)\\
&+\frac{2\pi}{L(\phi_2)}\sum_{k>0}|k| && (3783)\\
&\cdot\frac{\mathcal{F}(\widetilde{N}(\phi_1)-\widetilde{N}(\phi_2))(k)\overline{\mathcal{F}(\phi_1-\phi_2)(k)}+\overline{\mathcal{F}(\widetilde{N}(\phi_1)-\widetilde{N}(\phi_2))(k)}\mathcal{F}(\phi_1-\phi_2)(k)}{|\mathcal{F}(\phi_1-\phi_2)(k)|} && (3784)\\
&+2\frac{\gamma}{4\pi}\left(-\frac{1}{R}+\frac{2\pi}{L(\phi_1)}\right)\sum_{k>0}|k|\left|\mathcal{F}(\phi_1-\phi_2)(k)\right|(J_1(k)+J_2(k)) && (3785)\\
&-\frac{\gamma}{4\pi}\left(\frac{2\pi}{L(\phi_2)}-\frac{2\pi}{L(\phi_1)}\right) && (3786)\\
&\cdot\sum_{k>0}|k|(J_1(k)+J_2(k))\frac{\mathcal{F}(\phi_2)(k)\overline{\mathcal{F}(\phi_1-\phi_2)(k)}+\overline{\mathcal{F}(\phi_2)(k)}\mathcal{F}(\phi_1-\phi_2)(k)}{|\mathcal{F}(\phi_1-\phi_2)(k)|}. && (3787)
\end{aligned}
$$

We will take a closer look at each of the five terms in (3780) through (3787) one by one. Since $\mathcal{F}(\phi_1)(\pm 1) = \mathcal{F}(\phi_2)(\pm 1) = 0$, the first time can be written as

$$2\frac{1}{R}\cdot\frac{\gamma}{4\pi}\sum_{k>0}|k|\,|\mathcal{F}(\phi_1-\phi_2)(k)|\,(J_1(k)+J_1(k)) \tag{3788}$$

$$=-\pi\cdot 2\cdot\frac{1}{R}\cdot\frac{\gamma}{4\pi}\sum_{k\geq 2}|k|^2\,|\mathcal{F}(\phi_1-\phi_2)(k)| \tag{3789}$$

$$=-\pi\cdot 2\cdot\frac{1}{R}\cdot\frac{\gamma}{4\pi}\sum_{k>0}|k|^2\,|\mathcal{F}(\phi_1-\phi_2)(k)| \tag{3790}$$

$$=-\pi\cdot\frac{1}{R}\cdot\frac{\gamma}{4\pi}\,\|\phi_1-\phi_2\|_{\dot{\mathcal{F}}^{2,1}}. \tag{3791}$$

Next, the second term can be bounded above as follows.

$$\left|\left(\frac{2\pi}{L(\phi_1)}-\frac{2\pi}{L(\phi_2)}\right)\right| \tag{3792}$$

$$\cdot\left|\sum_{k>0}|k|\,\frac{\mathcal{F}(\widetilde{N}(\phi_1))(k)\overline{\mathcal{F}(\phi_1-\phi_2)(k)}+\overline{\mathcal{F}(\widetilde{N}(\phi_1))(k)}\cdot\mathcal{F}(\phi_1-\phi_2)(k)}{|\mathcal{F}(\phi_1-\phi_2)(k)|}\right| \tag{3793}$$

$$\leq\left|\left(\frac{2\pi}{L(\phi_1)}-\frac{2\pi}{L(\phi_2)}\right)\right|\cdot\sum_{k>0}|k|\cdot 2\left|\mathcal{F}(\widetilde{N}(\phi_1))(k)\right| \tag{3794}$$

$$=\left|\left(\frac{2\pi}{L(\phi_1)}-\frac{2\pi}{L(\phi_2)}\right)\right|\cdot\left\|\widetilde{N}(\phi_1)\right\|_{\dot{\mathcal{F}}^{1,1}}. \tag{3795}$$

Similarly, the third term can be bounded above as follows.

$$\left|\frac{2\pi}{L(\phi_2)}\right|\cdot\left|\sum_{k>0}|k|\right. \tag{3796}$$

$$\left.\cdot\,\frac{\mathcal{F}(\widetilde{N}(\phi_1)-\widetilde{N}(\phi_2))(k)\overline{\mathcal{F}(\phi_1-\phi_2)(k)}+\overline{\mathcal{F}(\widetilde{N}(\phi_1)-\widetilde{N}(\phi_2))(k)}\mathcal{F}(\phi_1-\phi_2)(k)}{|\mathcal{F}(\phi_1-\phi_2)(k)|}\right| \tag{3797}$$

$$\leq\frac{2\pi}{L(\phi_2)}\cdot\sum_{k>0}|k|\cdot 2\left|\mathcal{F}(\widetilde{N}(\phi_1)-\widetilde{N}(\phi_2))(k)\right| \tag{3798}$$

$$=\frac{2\pi}{L(\phi_2)}\left\|\widetilde{N}(\phi_1)-\widetilde{N}(\phi_2)\right\|_{\dot{\mathcal{F}}^{1,1}}. \tag{3799}$$

The fourth term can be bounded above as follows.

$$\left|2\frac{\gamma}{4\pi}\left(-\frac{1}{R}+\frac{2\pi}{L(\phi_1)}\right)\sum_{k>0}|k|\left|\mathcal{F}(\phi_1-\phi_2)(k)\right|(J_1(k)+J_2(k))\right| \tag{3800}$$

$$=\left|-\pi\cdot 2\cdot\frac{\gamma}{4\pi}\left(-\frac{1}{R}+\frac{2\pi}{L(\phi_1)}\right)\sum_{k\geq 2}|k|^2\left|\mathcal{F}(\phi_1-\phi_2)(k)\right|\right| \tag{3801}$$

$$=\left|-\pi\cdot 2\cdot\frac{\gamma}{4\pi}\left(-\frac{1}{R}+\frac{2\pi}{L(\phi_1)}\right)\sum_{k>0}|k|^2\left|\mathcal{F}(\phi_1-\phi_2)(k)\right|\right| \tag{3802}$$

$$\leq\pi\cdot\frac{\gamma}{4\pi}\frac{1}{R}\cdot A(\|\phi_1\|_{\mathcal{F}^{0,1}})\|\phi_1\|_{\mathcal{F}^{0,1}}\|\phi_1-\phi_2\|_{\dot{\mathcal{F}}^{2,1}}. \tag{3803}$$

Lastly, the fifth term can be bounded above as follows.

$$\left|-\frac{\gamma}{4\pi}\left(\frac{2\pi}{L(\phi_2)}-\frac{2\pi}{L(\phi_1)}\right)\right. \tag{3804}$$

$$\left.\cdot\sum_{k>0}|k|(J_1(k)+J_2(k))\frac{\mathcal{F}(\phi_2)(k)\overline{\mathcal{F}(\phi_1-\phi_2)(k)}+\overline{\mathcal{F}(\phi_2)(k)}\mathcal{F}(\phi_1-\phi_2)(k)}{|\mathcal{F}(\phi_1-\phi_2)(k)|}\right| \tag{3805}$$

$$=\left|\pi\cdot\frac{\gamma}{4\pi}\left(\frac{2\pi}{L(\phi_2)}-\frac{2\pi}{L(\phi_1)}\right)\right. \tag{3806}$$

$$\left.\cdot\sum_{k\geq 2}|k|^2\frac{\mathcal{F}(\phi_2)(k)\overline{\mathcal{F}(\phi_1-\phi_2)(k)}+\overline{\mathcal{F}(\phi_2)(k)}\mathcal{F}(\phi_1-\phi_2)(k)}{|\mathcal{F}(\phi_1-\phi_2)(k)|}\right| \tag{3807}$$

$$\leq\pi\cdot\frac{\gamma}{4\pi}\left|\frac{2\pi}{L(\phi_2)}-\frac{2\pi}{L(\phi_1)}\right|\sum_{k\geq 2}|k|^2\cdot 2\left|\mathcal{F}(\phi_2)(k)\right| \tag{3808}$$

$$=\pi\cdot\frac{\gamma}{4\pi}\left|\frac{2\pi}{L(\phi_2)}-\frac{2\pi}{L(\phi_1)}\right|\cdot\|\phi_2\|_{\dot{\mathcal{F}}^{2,1}}. \tag{3809}$$

We note that for a solution θ to the original problem,

$$\widetilde{N}(\phi)=(U_{\geq 2})_\alpha(\phi)+T_{\geq 2}(\phi)\cdot(1+\phi_\alpha)+T_1(\phi)\cdot\phi_\alpha, \tag{3810}$$

where $\phi = \theta - \hat{\theta}(0)$. Hence,

$$\begin{aligned}
&\widetilde{N}(\phi_1) - \widetilde{N}(\phi_2) && (3811)\\
=&(U_{\geq 2})_\alpha(\phi_1) - (U_{\geq 2})_\alpha(\phi_2) + T_{\geq 2}(\phi_1)(1 + (\phi_1)_\alpha) && (3812)\\
&- T_{\geq 2}(\phi_2)(1 + (\phi_2)_\alpha) + T_1(\phi_1)(\phi_1)_\alpha - T_1(\phi_2)(\phi_2)_\alpha && (3813)\\
=&\mathrm{Re}\bigg(\sum_{j=1}^{8} \frac{\gamma}{4\pi}\int_{-\pi}^{\pi}(\widetilde{B_j})_\alpha(\phi_1)(\alpha,\beta)d\beta + \frac{\gamma}{4\pi}\int_{-\pi}^{\pi}(\widetilde{B_{13}})_\alpha(\phi_1)(\alpha,\beta)d\beta\bigg) && (3814)\\
&- \mathrm{Re}\bigg(\sum_{j=1}^{8} \frac{\gamma}{4\pi}\int_{-\pi}^{\pi}(\widetilde{B_j})_\alpha(\phi_2)(\alpha,\beta)d\beta + \frac{\gamma}{4\pi}\int_{-\pi}^{\pi}(\widetilde{B_{13}})_\alpha(\phi_2)(\alpha,\beta)d\beta\bigg) && (3815)\\
&+ T_{\geq 2}(\phi_1) - T_{\geq 2}(\phi_2) + T_{\geq 2}(\phi_1)(\phi_1)_\alpha - T_{\geq 2}(\phi_1)(\phi_2)_\alpha && (3816)\\
&+ T_{\geq 2}(\phi_1)(\phi_2)_\alpha - T_{\geq 2}(\phi_2)(\phi_2)_\alpha + T_1(\phi_1)(\phi_1)_\alpha - T_1(\phi_1)(\phi_2)_\alpha && (3817)\\
&+ T_1(\phi_1)(\phi_2)_\alpha - T_1(\phi_2)(\phi_2)_\alpha && (3818)\\
=&\sum_{j=1}^{8}\mathrm{Re}\bigg(\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}(\widetilde{B_j})_\alpha(\phi_1)(\alpha,\beta)d\beta - \frac{\gamma}{4\pi}\int_{-\pi}^{\pi}(\widetilde{B_j})_\alpha(\phi_2)(\alpha,\beta)d\beta\bigg) && (3819)\\
&+ \mathrm{Re}\bigg(\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}(\widetilde{B_{13}})_\alpha(\phi_1)(\alpha,\beta)d\beta - \frac{\gamma}{4\pi}\int_{-\pi}^{\pi}(\widetilde{B_{13}})_\alpha(\phi_2)(\alpha,\beta)d\beta\bigg) && (3820)\\
&+ T_{\geq 2}(\phi_1) - T_{\geq 2}(\phi_2) + T_{\geq 2}(\phi_1)((\phi_1)_\alpha - (\phi_2)_\alpha) + (\phi_2)_\alpha(T_{\geq 2}(\phi_1) - T_{\geq 2}(\phi_2)) && (3821)\\
&+ T_1(\phi_1)((\phi_1)_\alpha - (\phi_2)_\alpha) + (\phi_2)_\alpha(T_1(\phi_1) - T_1(\phi_2)). && (3822)
\end{aligned}$$

To derive an estimate for this expression, we present in detail the process of deriving appropriate estimates for a few select terms that make up the expression. The techniques used to estimate such terms can be applied for the rest of the terms making up the expression. First, we consider the term

$$\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} B_{1,1}^1(\phi_1)(\alpha,\beta)d\beta - \frac{\gamma}{4\pi}\int_{-\pi}^{\pi} B_{1,1}^1(\phi_2)(\alpha,\beta)d\beta, \tag{3823}$$

which makes up one of the terms in the sum in (3819) (to be precise, the $j = 1$ term in the sum). Let us derive an estimate for the integrand, i.e.,

$$B_{1,1}^1(\phi_1)(\alpha,\beta) - B_{1,1}^1(\phi_2)(\alpha,\beta). \tag{3824}$$

In Section 15.3.2, we derived an estimate for an analogous expression, which is shown in (3296). Borrowing notation used in that part of Section 15.3.2, we write

$$\begin{aligned}
&B_{1,1}^1(\phi_1)(\alpha,\beta) - B_{1,1}^1(\phi_2)(\alpha,\beta) = -\sum_{j_1+j_2+n\geq 1}\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}\cdot\frac{(-1)^{j_1+1}\cdot i^{j_1+j_2+1}}{2j_1!j_2!}\cdot j_1 && (3825)\\
&\cdot(S_1(\alpha,\beta) + \cdots + S_3(\alpha,\beta) + \cdots + S_7(\alpha,\beta) + \cdots), && (3826)
\end{aligned}$$

where

$$\begin{aligned}
&S_1(\alpha,\beta) && (3827)\\
=&(\phi_1-\phi_2)(\alpha-\beta)\cdot\phi_1(\alpha-\beta)^{j_1-2}\cdot(\phi_1)_\alpha(\alpha-\beta)\cdot\phi_1(\alpha)^{j_2} && (3828)\\
&\cdot\int_0^1 e^{-i\beta s}\phi_1(\alpha+\beta(-1+s))(-1+s)ds && (3829)\\
&\cdot\left(\sum_{m=1}^\infty\frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\frac{(-i\phi_1(\alpha+(s-1)\beta))^m}{m!}ds\right)^n, && (3830)\\
&S_3(\alpha,\beta) && (3831)\\
=&\phi_2(\alpha-\beta)^{j_1-1}\cdot((\phi_1)_\alpha-(\phi_2)_\alpha)(\alpha-\beta)\cdot\phi_1(\alpha)^{j_2} && (3832)\\
&\cdot\int_0^1 e^{-i\beta s}\phi_1(\alpha+\beta(-1+s))(-1+s)ds && (3833)\\
&\cdot\left(\sum_{m=1}^\infty\frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\frac{(-i\phi_1(\alpha+(s-1)\beta))^m}{m!}ds\right)^n, && (3834)\\
&S_7(\alpha,\beta) && (3835)\\
=&\phi_2(\alpha-\beta)^{j_1-1}(\phi_2)_\alpha(\alpha-\beta)\phi_2(\alpha)^{j_2}\cdot\int_0^1 e^{-i\beta s}\phi_2(\alpha+\beta(-1+s))(-1+s)ds && (3836)\\
&\cdot\left(\sum_{m=1}^\infty\frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\frac{(-i\phi_1(\alpha+(s-1)\beta))^m}{m!}ds\right. && (3837)\\
&\left.-\sum_{m=1}^\infty\frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\frac{(-i\phi_2(\alpha+(s-1)\beta))^m}{m!}ds\right) && (3838)\\
&\cdot\left(\sum_{m=1}^\infty\frac{i\beta}{1-e^{i\beta}}\int_0^1 e^{-i(s-1)\beta}\frac{(-i\phi_1(\alpha+(s-1)\beta))^m}{m!}ds\right)^{n-1}, && (3839)
\end{aligned}$$

and the $\cdots$ represents the other finitely many terms making up $B^1_{1,1}(\phi_1)(\alpha,\beta)-B^1_{1,1}(\phi_2)(\alpha,\beta)$. First, we study $S_1(\alpha,\beta)$ and $S_7(\alpha,\beta)$ and then turn the attention to $S_3(\alpha,\beta)$. We note that

$$\begin{aligned}
&\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi}\mathcal{F}(S_1(\cdot,\beta))(k_1)\cdot\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}d\beta\right| && (3840)\\
\le&C_n\Bigg(|\mathcal{F}(\phi_1-\phi_2)|*|\mathcal{F}(\phi_1)|*\cdots*|\mathcal{F}(\phi_1)|*|\mathcal{F}((\phi_1)_\alpha)| && (3841)\\
&*|\mathcal{F}(\phi_1)|*\cdots*|\mathcal{F}(\phi_1)|*|P(\phi_1)|*\cdots*|P(\phi_1)|*|\mathcal{F}(\phi_1)|\Bigg)(k_1). && (3842)
\end{aligned}$$

Then

$$\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} S_1(\cdot, \beta) \cdot \frac{-i\beta e^{2i\beta}}{1-e^{i\beta}} d\beta \right\|_{\dot{\mathcal{F}}^{1,1}} \tag{3843}$$

$$\leq C_n \sum_{k\neq 0} |k| \Big(|\mathcal{F}(\phi_1 - \phi_2)| * |\mathcal{F}(\phi_1)| * \cdots * |\mathcal{F}(\phi_1)| * |\mathcal{F}((\phi_1)_\alpha)| \tag{3844}$$

$$* |\mathcal{F}(\phi_1)| * \cdots * |\mathcal{F}(\phi_1)| * |P(\phi_1)| * \cdots * |P(\phi_1)| * |\mathcal{F}(\phi_1)| \Big)(k). \tag{3845}$$

Likewise,

$$\left| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} \mathcal{F}(S_7(\cdot, \beta))(k_1) \cdot \frac{-i\beta e^{2i\beta}}{1-e^{i\beta}} d\beta \right| \tag{3846}$$

$$\leq C_n \Big(|\mathcal{F}(\phi_2)| * \cdots * |\mathcal{F}(\phi_2)| * |\mathcal{F}((\phi_2)_\alpha)| \tag{3847}$$

$$* |\mathcal{F}(\phi_2)| * \cdots * |\mathcal{F}(\phi_2)| \tag{3848}$$

$$* \left| \left(\sum_{m=0}^{\infty} \frac{(-i)^m}{m!} \mathcal{F}((\phi_2)^m) \right) * \left(\sum_{m=1}^{\infty} \frac{(-i)^m}{m!} \mathcal{F}((\phi_1 - \phi_2)^m) \right) \right| \tag{3849}$$

$$* |P(\phi_1)| * \cdots * |P(\phi_1)| * |\mathcal{F}(\phi_2)| \Big)(k_1), \tag{3850}$$

from which we obtain

$$\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} S_7(\cdot, \beta) \cdot \frac{-i\beta e^{2i\beta}}{1-e^{i\beta}} d\beta \right\|_{\dot{\mathcal{F}}^{1,1}} \tag{3851}$$

$$\leq C_n \sum_{k\neq 0} |k| \Big(|\mathcal{F}(\phi_2)| * \cdots * |\mathcal{F}(\phi_2)| * |\mathcal{F}((\phi_2)_\alpha)| \tag{3852}$$

$$* |\mathcal{F}(\phi_2)| * \cdots * |\mathcal{F}(\phi_2)| \tag{3853}$$

$$* \left| \left(\sum_{m=0}^{\infty} \frac{(-i)^m}{m!} \mathcal{F}((\phi_2)^m) \right) * \left(\sum_{m=1}^{\infty} \frac{(-i)^m}{m!} \mathcal{F}((\phi_1 - \phi_2)^m) \right) \right| \tag{3854}$$

$$* |P(\phi_1)| * \cdots * |P(\phi_1)| * |\mathcal{F}(\phi_2)| \Big)(k). \tag{3855}$$

For a sequence a defined on $\mathbb{Z}$, we define for $s \geq 0$

$$\|a\|_{l^{s,1}} = \sum_{k\in\mathbb{Z}} |k|^s |a(k)|. \tag{3856}$$

Lemma 14. *For sequences $a_1, \dots, a_n$ defined on $\mathbb{Z}$,*

$$\|a_1 * \cdots * a_n\|_{l^{1,1}} \leq \sum_{j=1}^{n} \|a_j\|_{l^{1,1}} \prod_{\substack{k=1 \\ k\neq j}}^{n} \|a_k\|_{l^{0,1}}. \tag{3857}$$

Proof. This lemma can be proved by modifying the proof of Proposition 11. ■

Using Lemma 14, we obtain

$$\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} S_1(\cdot,\beta)\cdot\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}d\beta\right\|_{\dot{\mathcal{F}}^{1,1}} \tag{3858}$$

$$\leq C_n\Big(\|\phi_1-\phi_2\|_{\dot{\mathcal{F}}^{1,1}}\|\phi_1\|_{\dot{\mathcal{F}}^{0,1}}^{j_1+j_2-1}\|(\phi_1)_\alpha\|_{\dot{\mathcal{F}}^{0,1}}\|P(\phi_1)\|_{l^{0,1}}^{n} \tag{3859}$$

$$+\|\phi_1\|_{\dot{\mathcal{F}}^{1,1}}\|\phi_1-\phi_2\|_{\dot{\mathcal{F}}^{0,1}}\|\phi_1\|_{\dot{\mathcal{F}}^{0,1}}^{j_1+j_2-2}\|(\phi_1)_\alpha\|_{\dot{\mathcal{F}}^{0,1}}\|P(\phi_1)\|_{l^{0,1}}^{n}\cdot(j_1+j_2-1) \tag{3860}$$

$$+\|(\phi_1)_\alpha\|_{\dot{\mathcal{F}}^{1,1}}\|\phi_1-\phi_2\|_{\dot{\mathcal{F}}^{0,1}}\|\phi_1\|_{\dot{\mathcal{F}}^{0,1}}^{j_1+j_2-1}\|P(\phi_1)\|_{l^{0,1}}^{n} \tag{3861}$$

$$+\|P(\phi_1)\|_{l^{1,1}}\|\phi_1-\phi_2\|_{\dot{\mathcal{F}}^{0,1}}\|\phi_1\|_{\dot{\mathcal{F}}^{0,1}}^{j_1+j_2-1}\|(\phi_1)_\alpha\|_{\dot{\mathcal{F}}^{0,1}}\|P(\phi_1)\|_{l^{0,1}}^{n-1}\cdot n\Big) \tag{3862}$$

and

$$\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} S_7(\cdot,\beta)\cdot\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}d\beta\right\|_{\dot{\mathcal{F}}^{1,1}} \tag{3863}$$

$$\leq C_n\Big(\|\phi_2\|_{\dot{\mathcal{F}}^{1,1}}\|\phi_2\|_{\dot{\mathcal{F}}^{0,1}}^{j_1+j_2-1}\|(\phi_2)_\alpha\|_{\dot{\mathcal{F}}^{0,1}} \tag{3864}$$

$$\cdot\left\|\left(\sum_{m=0}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}((\phi_2)^m)\right)*\left(\sum_{m=1}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}((\phi_1-\phi_2)^m)\right)\right\|_{l^{0,1}} \tag{3865}$$

$$\cdot\|P(\phi_1)\|_{l^{0,1}}^{n-1}\cdot(j_1+j_2) \tag{3866}$$

$$+\|(\phi_2)_\alpha\|_{\dot{\mathcal{F}}^{1,1}}\|\phi_2\|_{\dot{\mathcal{F}}^{0,1}}^{j_1+j_2}\left\|\left(\sum_{m=0}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}((\phi_2)^m)\right)*\left(\sum_{m=1}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}((\phi_1\phi_2)^m)\right)\right\|_{l^{0,1}} \tag{3867}$$

$$\cdot\|P(\phi_1)\|_{l^{0,1}}^{n-1} \tag{3868}$$

$$+\left\|\left(\sum_{m=0}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}((\phi_2)^m)\right)*\left(\sum_{m=1}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}((\phi_1\phi_2)^m)\right)\right\|_{l^{1,1}} \tag{3869}$$

$$\cdot\|\phi_2\|_{\dot{\mathcal{F}}^{0,1}}^{j_1+j_2}\|(\phi_2)_\alpha\|_{\dot{\mathcal{F}}^{0,1}}\|P(\phi_1)\|_{l^{0,1}}^{n-1} \tag{3870}$$

$$+\|P(\phi_1)\|_{l^{1,1}}\|\phi_2\|_{\dot{\mathcal{F}}^{0,1}}^{j_1+j_2}\|(\phi_2)_\alpha\|_{\dot{\mathcal{F}}^{0,1}} \tag{3871}$$

$$\cdot\left\|\left(\sum_{m=0}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}((\phi_2)^m)\right)*\left(\sum_{m=1}^{\infty}\frac{(-i)^m}{m!}\mathcal{F}((\phi_1-\phi_2)^m)\right)\right\|_{l^{0,1}} \tag{3872}$$

$$\cdot\|P(\phi_1)\|_{\dot{\mathcal{F}}^{0,1}}^{n-2}\cdot(n-1)\Big) \tag{3873}$$

$$\leq C_n\Bigg(\|\phi_2\|_{\dot{\mathcal{F}}^{1,1}} \|\phi_2\|_{\dot{\mathcal{F}}^{0,1}}^{j_1+j_2-1} \|\phi_2\|_{\dot{\mathcal{F}}^{1,1}} e^{\|\phi_2\|_{\mathcal{F}^{0,1}}} \|\phi_1-\phi_2\|_{\mathcal{F}^{0,1}} \tag{3874}$$

$$\cdot \left(\sum_{m=1}^{\infty} \frac{\|\phi_1-\phi_2\|_{\mathcal{F}^{0,1}}^{m-1}}{m!}\right) \cdot (e^{\|\phi_1\|_{\mathcal{F}^{0,1}}}-1)^{n-1} \cdot (j_1+j_2) \tag{3875}$$

$$+ \|\phi_2\|_{\dot{\mathcal{F}}^{2,1}} \|\phi_2\|_{\dot{\mathcal{F}}^{0,1}}^{j_1+j_2} e^{\|\phi_2\|_{\mathcal{F}^{0,1}}} \|\phi_1-\phi_2\|_{\mathcal{F}^{0,1}} \left(\sum_{m=1}^{\infty} \frac{\|\phi_1-\phi_2\|_{\mathcal{F}^{0,1}}^{m-1}}{m!}\right)(e^{\|\phi_1\|_{\mathcal{F}^{0,1}}}-1)^{n-1} \tag{3876}$$

$$+ \Bigg(\left\|\sum_{m=0}^{\infty} \frac{(-i)^m}{m!}\mathcal{F}((\phi_2)^m)\right\|_{l^{1,1}} \cdot \left\|\sum_{m=1}^{\infty} \frac{(-i)^m}{m!}\mathcal{F}((\phi_1-\phi_2)^m)\right\|_{l^{0,1}} \tag{3877}$$

$$+ \left\|\sum_{m=1}^{\infty} \frac{(-i)^m}{m!}\mathcal{F}((\phi_1-\phi_2)^m)\right\|_{l^{1,1}} \cdot \left\|\sum_{m=0}^{\infty} \frac{(-i)^m}{m!}\mathcal{F}((\phi_2)^m)\right\|_{l^{0,1}} \Bigg) \tag{3878}$$

$$\cdot \|\phi_2\|_{\dot{\mathcal{F}}^{0,1}}^{j_1+j_2} \|\phi_2\|_{\dot{\mathcal{F}}^{1,1}} (e^{\|\phi_1\|_{\mathcal{F}^{0,1}}}-1)^{n-1} \tag{3879}$$

$$+ \|\phi_1\|_{\mathcal{F}^{1,1}} e^{\|\phi_1\|_{\mathcal{F}^{0,1}}} \|\phi_2\|_{\dot{\mathcal{F}}^{0,1}}^{j_1+j_2} \|\phi_2\|_{\dot{\mathcal{F}}^{1,1}} e^{\|\phi_2\|_{\mathcal{F}^{0,1}}} \|\phi_1-\phi_2\|_{\mathcal{F}^{0,1}} \tag{3880}$$

$$\cdot \left(\sum_{m=1}^{\infty} \frac{\|\phi_1-\phi_2\|_{\mathcal{F}^{0,1}}^{m-1}}{m!}\right) \cdot (e^{\|\phi_1\|_{\mathcal{F}^{0,1}}}-1)^{n-2} \cdot (n-1)\Bigg) \tag{3881}$$

$$\leq C_n\Bigg(\|\phi_2\|_{\dot{\mathcal{F}}^{1,1}} \|\phi_2\|_{\dot{\mathcal{F}}^{0,1}}^{j_1+j_2-1} \|\phi_2\|_{\dot{\mathcal{F}}^{1,1}} e^{\|\phi_2\|_{\mathcal{F}^{0,1}}} \|\phi_1-\phi_2\|_{\mathcal{F}^{0,1}} \left(\sum_{m=1}^{\infty} \frac{\|\phi_1-\phi_2\|_{\mathcal{F}^{0,1}}^{m-1}}{m!}\right) \tag{3882}$$

$$\cdot (e^{\|\phi_1\|_{\mathcal{F}^{0,1}}}-1)^{n-1} \cdot (j_1+j_2) \tag{3883}$$

$$+ \|\phi_2\|_{\dot{\mathcal{F}}^{2,1}} \|\phi_2\|_{\dot{\mathcal{F}}^{0,1}}^{j_1+j_2} e^{\|\phi_2\|_{\mathcal{F}^{0,1}}} \|\phi_1-\phi_2\|_{\mathcal{F}^{0,1}} \left(\sum_{m=1}^{\infty} \frac{\|\phi_1-\phi_2\|_{\mathcal{F}^{0,1}}^{m-1}}{m!}\right)(e^{\|\phi_1\|_{\mathcal{F}^{0,1}}}-1)^{n-1} \tag{3884}$$

$$+ \Bigg(\|\phi_2\|_{\mathcal{F}^{1,1}} e^{\|\phi_2\|_{\mathcal{F}^{0,1}}} \|\phi_1-\phi_2\|_{\mathcal{F}^{0,1}} \left(\sum_{m=1}^{\infty} \frac{\|\phi_1-\phi_2\|_{\mathcal{F}^{0,1}}^{m-1}}{m!}\right) \tag{3885}$$

$$+ \|\phi_1-\phi_2\|_{\mathcal{F}^{1,1}} \left(\sum_{m=1}^{\infty} \frac{\|\phi_1-\phi_2\|_{\mathcal{F}^{0,1}}^{m-1}}{(m-1)!}\right) e^{\|\phi_2\|_{\mathcal{F}^{0,1}}} \Bigg) \|\phi_2\|_{\dot{\mathcal{F}}^{0,1}}^{j_1+j_2} \|\phi_2\|_{\dot{\mathcal{F}}^{1,1}} \tag{3886}$$

$$\cdot (e^{\|\phi_1\|_{\mathcal{F}^{0,1}}}-1)^{n-1} \tag{3887}$$

$$+ \|\phi_1\|_{\mathcal{F}^{1,1}} e^{\|\phi_1\|_{\mathcal{F}^{0,1}}} \|\phi_2\|_{\dot{\mathcal{F}}^{0,1}}^{j_1+j_2} \|\phi_2\|_{\dot{\mathcal{F}}^{1,1}} e^{\|\phi_2\|_{\mathcal{F}^{0,1}}} \|\phi_1-\phi_2\|_{\mathcal{F}^{0,1}} \left(\sum_{m=1}^{\infty} \frac{\|\phi_1-\phi_2\|_{\mathcal{F}^{0,1}}^{m-1}}{m!}\right) \tag{3888}$$

$$\cdot (e^{\|\phi_1\|_{\mathcal{F}^{0,1}}}-1)^{n-2} \cdot (n-1)\Bigg). \tag{3889}$$

Now, we consider $S_3(\alpha,\beta)$. We note that

$$\left|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} \mathcal{F}(S_3(\cdot,\beta))(k_1) \cdot \frac{-i\beta e^{2i\beta}}{1-e^{i\beta}} d\beta\right| \tag{3890}$$

$$\leq C_n\Big(|\mathcal{F}(\phi_2)| * \cdots * |\mathcal{F}(\phi_2)| * |\mathcal{F}((\phi_1)_\alpha - (\phi_2)_\alpha)| * |\mathcal{F}(\phi_1)| * \cdots * |\mathcal{F}(\phi_1)| \tag{3891}$$

$$* |P(\phi_1)| * \cdots * |P(\phi_1)| * |\mathcal{F}(\phi_1)| \Big)(k_1) \tag{3892}$$

Then

$$\left\|\frac{\gamma}{4\pi}\int_{-\pi}^{\pi} S_3(\cdot,\beta)\cdot\frac{-i\beta e^{2i\beta}}{1-e^{i\beta}}d\beta\right\|_{\dot{\mathcal{F}}^{1,1}} \tag{3893}$$

$$\leq C_n\sum_{k\neq 0}|k|\Bigg(|\mathcal{F}(\phi_2)|*\cdots*|\mathcal{F}(\phi_2)|*|\mathcal{F}((\phi_1)_\alpha-(\phi_2)_\alpha)|*|\mathcal{F}(\phi_1)|*\cdots*|\mathcal{F}(\phi_1)| \tag{3894}$$

$$*|P(\phi_1)|*\cdots*|P(\phi_1)|*|\mathcal{F}(\phi_1)|\Bigg)(k) \tag{3895}$$

$$\leq C_n\Bigg(\|\phi_2\|_{\dot{\mathcal{F}}^{1,1}}\|\phi_2\|_{\dot{\mathcal{F}}^{0,1}}^{j_1-2}\|(\phi_1)_\alpha-(\phi_2)_\alpha\|_{\dot{\mathcal{F}}^{0,1}}\|\phi_1\|_{\dot{\mathcal{F}}^{0,1}}^{j_2+1}\|P(\phi_1)\|_{l^{0,1}}^{n}\cdot(j_1-1) \tag{3896}$$

$$+\|(\phi_1)_\alpha-(\phi_2)_\alpha\|_{\dot{\mathcal{F}}^{1,1}}\|\phi_2\|_{\dot{\mathcal{F}}^{0,1}}^{j_1-1}\|\phi_1\|_{\dot{\mathcal{F}}^{0,1}}^{j_2+1}\|P(\phi_1)\|_{l^{0,1}}^{n} \tag{3897}$$

$$+\|\phi_1\|_{\dot{\mathcal{F}}^{1,1}}\|\phi_2\|_{\dot{\mathcal{F}}^{0,1}}^{j_1-1}\|(\phi_1)_\alpha-(\phi_2)_\alpha\|_{\dot{\mathcal{F}}^{0,1}}\|\phi_1\|_{\dot{\mathcal{F}}^{0,1}}^{j_2}\|P(\phi_1)\|_{l^{0,1}}^{n}\cdot(j_2+1) \tag{3898}$$

$$+\|P(\phi_1)\|_{l^{1,1}}\|\phi_2\|_{\dot{\mathcal{F}}^{0,1}}^{j_1-1}\|(\phi_1)_\alpha-(\phi_2)_\alpha\|_{\dot{\mathcal{F}}^{0,1}}\|\phi_1\|_{\dot{\mathcal{F}}^{0,1}}^{j_2+1}\|P(\phi_1)\|_{l^{0,1}}^{n-1}\cdot n\Bigg) \tag{3899}$$

$$\leq C_n\Bigg(\|\phi_2\|_{\dot{\mathcal{F}}^{1,1}}\|\phi_2\|_{\dot{\mathcal{F}}^{0,1}}^{j_1-2}\|\phi_1-\phi_2\|_{\dot{\mathcal{F}}^{1,1}}\|\phi_1\|_{\dot{\mathcal{F}}^{0,1}}^{j_2+1}(e^{\|\phi_1\|_{\mathcal{F}^{0,1}}}-1)^n\cdot(j_1-1) \tag{3900}$$

$$+\|\phi_1-\phi_2\|_{\dot{\mathcal{F}}^{2,1}}\|\phi_2\|_{\dot{\mathcal{F}}^{0,1}}^{j_1-1}\|\phi_1\|_{\dot{\mathcal{F}}^{0,1}}^{j_2+1}(e^{\|\phi_1\|_{\mathcal{F}^{0,1}}}-1)^n \tag{3901}$$

$$+\|\phi_1\|_{\dot{\mathcal{F}}^{1,1}}\|\phi_2\|_{\dot{\mathcal{F}}^{0,1}}^{j_1-1}\|\phi_1-\phi_2\|_{\dot{\mathcal{F}}^{1,1}}\|\phi_1\|_{\dot{\mathcal{F}}^{0,1}}^{j_2}(e^{\|\phi_1\|_{\mathcal{F}^{0,1}}}-1)^n\cdot(j_2+1) \tag{3902}$$

$$+\|P(\phi_1)\|_{l^{1,1}}\|\phi_2\|_{\dot{\mathcal{F}}^{0,1}}^{j_1-1}\|\phi_1-\phi_2\|_{\dot{\mathcal{F}}^{1,1}}\|\phi_1\|_{\dot{\mathcal{F}}^{0,1}}^{j_2+1}(e^{\|\phi_1\|_{\mathcal{F}^{0,1}}}-1)^{n-1}\cdot n\Bigg). \tag{3903}$$

Combining these results, we obtain

$$\left\| \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B_{1,1}^1(\phi_1)(\alpha,\beta) d\beta - \frac{\gamma}{4\pi} \int_{-\pi}^{\pi} B_{1,1}^1(\phi_2)(\alpha,\beta) d\beta \right\|_{\dot{\mathcal{F}}^{1,1}} \tag{3904}$$

$$\leq \sum_{j_1+j_2+n\geq 1} \frac{j_1}{2j_1!j_2!} \Bigg(C_n \Bigg(\|\phi_1-\phi_2\|_{\dot{\mathcal{F}}^{1,1}} \|\phi_1\|_{\dot{\mathcal{F}}^{0,1}}^{j_1+j_2-1} \|\phi_1\|_{\dot{\mathcal{F}}^{1,1}} (e^{\|\phi_1\|_{\mathcal{F}^{0,1}}}-1)^n \tag{3905}$$

$$+ \|\phi_1\|_{\dot{\mathcal{F}}^{1,1}} \|\phi_1-\phi_2\|_{\dot{\mathcal{F}}^{0,1}} \|\phi_1\|_{\dot{\mathcal{F}}^{0,1}}^{j_1+j_2-2} \|\phi_1\|_{\dot{\mathcal{F}}^{1,1}} (e^{\|\phi_1\|_{\mathcal{F}^{0,1}}}-1)^n \cdot (j_1+j_2-1) \tag{3906}$$

$$+ \|\phi_1\|_{\dot{\mathcal{F}}^{2,1}} \|\phi_1-\phi_2\|_{\dot{\mathcal{F}}^{0,1}} \|\phi_1\|_{\dot{\mathcal{F}}^{0,1}}^{j_1+j_2-1} (e^{\|\phi_1\|_{\mathcal{F}^{0,1}}}-1)^n \tag{3907}$$

$$+ \|\phi_1\|_{\mathcal{F}^{1,1}} e^{\|\phi_1\|_{\mathcal{F}^{0,1}}} \|\phi_1-\phi_2\|_{\dot{\mathcal{F}}^{0,1}} \|\phi_1\|_{\dot{\mathcal{F}}^{0,1}}^{j_1+j_2-1} \|\phi_1\|_{\dot{\mathcal{F}}^{1,1}} (e^{\|\phi_1\|_{\mathcal{F}^{0,1}}}-1)^{n-1} \cdot n \Bigg) \tag{3908}$$

$$+ \cdots \tag{3909}$$

$$+ C_n \Bigg(\|\phi_2\|_{\dot{\mathcal{F}}^{1,1}} \|\phi_2\|_{\dot{\mathcal{F}}^{0,1}}^{j_1-2} \|\phi_1-\phi_2\|_{\dot{\mathcal{F}}^{1,1}} \|\phi_1\|_{\dot{\mathcal{F}}^{0,1}}^{j_2+1} (e^{\|\phi_1\|_{\mathcal{F}^{0,1}}}-1)^n \cdot (j_1-1) \tag{3910}$$

$$+ \|\phi_1-\phi_2\|_{\dot{\mathcal{F}}^{2,1}} \|\phi_2\|_{\dot{\mathcal{F}}^{0,1}}^{j_1-1} \|\phi_1\|_{\dot{\mathcal{F}}^{0,1}}^{j_2+1} (e^{\|\phi_1\|_{\mathcal{F}^{0,1}}}-1)^n \tag{3911}$$

$$+ \|\phi_1\|_{\dot{\mathcal{F}}^{1,1}} \|\phi_2\|_{\dot{\mathcal{F}}^{0,1}}^{j_1-1} \|\phi_1-\phi_2\|_{\dot{\mathcal{F}}^{1,1}} \|\phi_1\|_{\dot{\mathcal{F}}^{0,1}}^{j_2} (e^{\|\phi_1\|_{\mathcal{F}^{0,1}}}-1)^n \cdot (j_2+1) \tag{3912}$$

$$+ \|P(\phi_1)\|_{l^{1,1}} \|\phi_2\|_{\dot{\mathcal{F}}^{0,1}}^{j_1-1} \|\phi_1-\phi_2\|_{\dot{\mathcal{F}}^{1,1}} \|\phi_1\|_{\dot{\mathcal{F}}^{0,1}}^{j_2+1} (e^{\|\phi_1\|_{\mathcal{F}^{0,1}}}-1)^{n-1} \cdot n \Bigg) \tag{3913}$$

$$+ \cdots \tag{3914}$$

$$+ C_n \Bigg(\|\phi_2\|_{\dot{\mathcal{F}}^{1,1}} \|\phi_2\|_{\dot{\mathcal{F}}^{0,1}}^{j_1+j_2-1} \|\phi_2\|_{\dot{\mathcal{F}}^{1,1}} e^{\|\phi_2\|_{\mathcal{F}^{0,1}}} \|\phi_1-\phi_2\|_{\mathcal{F}^{0,1}} \left(\sum_{m=1}^{\infty} \frac{\|\phi_1-\phi_2\|_{\mathcal{F}^{0,1}}^{m-1}}{m!} \right) \tag{3915}$$

$$\cdot \left(e^{\|\phi_1\|_{\mathcal{F}^{0,1}}} - 1 \right)^{n-1} \cdot (j_1+j_2) \tag{3916}$$

$$+ \|\phi_2\|_{\dot{\mathcal{F}}^{2,1}} \|\phi_2\|_{\dot{\mathcal{F}}^{0,1}}^{j_1+j_2} e^{\|\phi_2\|_{\mathcal{F}^{0,1}}} \|\phi_1-\phi_2\|_{\mathcal{F}^{0,1}} \left(\sum_{m=1}^{\infty} \frac{\|\phi_1-\phi_2\|_{\mathcal{F}^{0,1}}^{m-1}}{m!} \right) (e^{\|\phi_1\|_{\mathcal{F}^{0,1}}}-1)^{n-1} \tag{3917}$$

$$+ \Bigg(\|\phi_2\|_{\mathcal{F}^{1,1}} e^{\|\phi_2\|_{\mathcal{F}^{0,1}}} \|\phi_1-\phi_2\|_{\mathcal{F}^{0,1}} \left(\sum_{m=1}^{\infty} \frac{\|\phi_1-\phi_2\|_{\mathcal{F}^{0,1}}^{m-1}}{m!} \right) \tag{3918}$$

$$+ \|\phi_1-\phi_2\|_{\mathcal{F}^{1,1}} \left(\sum_{m=1}^{\infty} \frac{\|\phi_1-\phi_2\|_{\mathcal{F}^{0,1}}^{m-1}}{(m-1)!} \right) e^{\|\phi_2\|_{\mathcal{F}^{0,1}}} \Bigg) \cdot \|\phi_2\|_{\dot{\mathcal{F}}^{0,1}}^{j_1+j_2} \|\phi_2\|_{\dot{\mathcal{F}}^{1,1}} (e^{\|\phi_1\|_{\mathcal{F}^{0,1}}}-1)^{n-1} \tag{3919}$$

$$+ \|\phi_1\|_{\mathcal{F}^{1,1}} e^{\|\phi_1\|_{\mathcal{F}^{0,1}}} \|\phi_2\|_{\dot{\mathcal{F}}^{0,1}}^{j_1+j_2} \|\phi_2\|_{\dot{\mathcal{F}}^{1,1}} e^{\|\phi_2\|_{\mathcal{F}^{0,1}}} \|\phi_1-\phi_2\|_{\mathcal{F}^{0,1}} \tag{3920}$$

$$\cdot \left(\sum_{m=1}^{\infty} \frac{\|\phi_1-\phi_2\|_{\mathcal{F}^{0,1}}^{m-1}}{m!} \right) (e^{\|\phi_1\|_{\mathcal{F}^{0,1}}}-1)^{n-2} \cdot (n-1) \Bigg) \tag{3921}$$

$$+ \cdots \Bigg), \tag{3922}$$

where the $\cdots$ represents the finitely many terms making up $B_{1,1}^1(\phi_1)(\alpha,\beta) - B_{1,1}^1(\phi_2)(\alpha,\beta)$ besides $S_1(\alpha,\beta)$, $S_3(\alpha,\beta)$, and $S_7(\alpha,\beta)$. If we collected all the coefficients for $\|\phi_1-\phi_2\|_{\dot{\mathcal{F}}^{2,1}}$

among the shown terms, then we obtain as its coefficient

$$\sum_{j_1+j_2+n\geq 1} \frac{j_1}{2j_1!j_2!} C_n \|\phi_2\|_{\dot{\mathcal{F}}^{0,1}}^{j_1-1} \|\phi_1\|_{\dot{\mathcal{F}}^{0,1}}^{j_2+1} (e^{\|\phi_1\|_{\mathcal{F}^{0,1}}} - 1)^n. \tag{3923}$$

We note that if we summed the coefficients for $\|\phi_1 - \phi_2\|_{\dot{\mathcal{F}}^{2,1}}$ across all the terms appearing in (3819) and (3820), then we can choose $\|\theta^0\|_{\dot{\mathcal{F}}^{1,1}}$ small enough such that the sum is smaller than $\pi \cdot \frac{1}{R} \cdot \frac{\gamma}{4\pi}$. Next, we consider the term

$$T_{\geq 2}(\phi_1) - T_{\geq 2}(\phi_2), \tag{3924}$$

where

$$T(\phi) = \mathcal{M}((1+\phi_\alpha)U(\phi)). \tag{3925}$$

Then

$$T_{\geq 2}(\phi_1) - T_{\geq 2}(\phi_2) = \mathcal{M}\bigg(U_{\geq 2}(\phi_1) - U_{\geq 2}(\phi_2)\bigg) + \mathcal{M}\bigg((\phi_1)_\alpha \cdot U_{\geq 1}(\phi_1) - (\phi_2)_\alpha \cdot U_{\geq 1}(\phi_2)\bigg) \tag{3926}$$

$$= \mathcal{M}\bigg(U_{\geq 2}(\phi_1) - U_{\geq 2}(\phi_2)\bigg) + \mathcal{M}\bigg((\phi_1)_\alpha \cdot (U_{\geq 1}(\phi_1) - U_{\geq 1}(\phi_2))\bigg) \tag{3927}$$

$$+ \mathcal{M}\bigg(U_{\geq 1}(\phi_2) \cdot ((\phi_1)_\alpha - (\phi_2)_\alpha)\bigg). \tag{3928}$$

Hence,

$$\|T_{\geq 2}(\phi_1) - T_{\geq 2}(\phi_2)\|_{\dot{\mathcal{F}}^{1,1}} \tag{3929}$$

$$\leq \|U_{\geq 2}(\phi_1) - U_{\geq 2}(\phi_2)\|_{\mathcal{F}^{0,1}} + \|(\phi_1)_\alpha \cdot (U_{\geq 1}(\phi_1) - U_{\geq 1}(\phi_2))\|_{\mathcal{F}^{0,1}} \tag{3930}$$

$$+ \|U_{\geq 1}(\phi_2) \cdot ((\phi_1)_\alpha - (\phi_2)_\alpha)\|_{\mathcal{F}^{0,1}} \tag{3931}$$

$$\leq \|U_{\geq 2}(\phi_1) - U_{\geq 2}(\phi_2)\|_{\mathcal{F}^{0,1}} \tag{3932}$$

$$+ \|\phi_1\|_{\mathcal{F}^{1,1}} \bigg(\|U_1(\phi_1) - U_1(\phi_2)\|_{\mathcal{F}^{0,1}} + \|U_{\geq 2}(\phi_1) - U_{\geq 2}(\phi_2)\|_{\mathcal{F}^{0,1}}\bigg) \tag{3933}$$

$$+ \|U_{\geq 1}(\phi_2)\|_{\mathcal{F}^{0,1}} \cdot \|\phi_1 - \phi_2\|_{\mathcal{F}^{1,1}}. \tag{3934}$$

Next, we consider the term

$$T(\phi_1) \cdot ((\phi_1)_\alpha - (\phi_2)_\alpha). \tag{3935}$$

We have

$$\|T(\phi_1) \cdot ((\phi_1)_\alpha - (\phi_2)_\alpha)\|_{\dot{\mathcal{F}}^{1,1}} \tag{3936}$$

$$\leq \|T(\phi_1)\|_{\dot{\mathcal{F}}^{1,1}} \|(\phi_1)_\alpha - (\phi_2)_\alpha\|_{\mathcal{F}^{0,1}} + \|(\phi_1)_\alpha - (\phi_2)_\alpha\|_{\dot{\mathcal{F}}^{1,1}} \|T(\phi_1)\|_{\mathcal{F}^{0,1}} \tag{3937}$$

$$\leq \|(1+(\phi_1)_\alpha) \cdot U(\phi_1)\|_{\mathcal{F}^{0,1}} \|\phi_1 - \phi_2\|_{\mathcal{F}^{1,1}} + \|\phi_1 - \phi_2\|_{\dot{\mathcal{F}}^{2,1}} \|(1+(\phi_1)_\alpha) \cdot U(\phi_1)\|_{\mathcal{F}^{0,1}} \tag{3938}$$

$$\leq \bigg(\|U(\phi_1)\|_{\mathcal{F}^{0,1}} + \|\phi_1\|_{\mathcal{F}^{1,1}} \|U(\phi_1)\|_{\mathcal{F}^{0,1}}\bigg) \|\phi_1 - \phi_2\|_{\mathcal{F}^{1,1}} \tag{3939}$$

$$+ \bigg(\|U(\phi_1)\|_{\mathcal{F}^{0,1}} + \|\phi_1\|_{\mathcal{F}^{1,1}} \cdot \|U(\phi_1)\|_{\mathcal{F}^{0,1}}\bigg) \|\phi_1 - \phi_2\|_{\dot{\mathcal{F}}^{2,1}}. \tag{3940}$$

Next, we consider the term

$$(\phi_2)_\alpha \cdot (T(\phi_1) - T(\phi_2)). \tag{3941}$$

We have

$$\begin{aligned}
&\|(\phi_2)_\alpha \cdot (T(\phi_1) - T(\phi_2))\|_{\dot{\mathcal{F}}^{1,1}} && (3942)\\
\leq &\|(\phi_2)_\alpha\|_{\dot{\mathcal{F}}^{1,1}} \|T(\phi_1) - T(\phi_2)\|_{\mathcal{F}^{0,1}} + \|T(\phi_1) - T(\phi_2)\|_{\dot{\mathcal{F}}^{1,1}} \|(\phi_2)_\alpha\|_{\mathcal{F}^{0,1}} && (3943)\\
= &\|\phi_2\|_{\dot{\mathcal{F}}^{2,1}} \|T(\phi_1) - T(\phi_2)\|_{\mathcal{F}^{0,1}} + \|T(\phi_1) - T(\phi_2)\|_{\dot{\mathcal{F}}^{1,1}} \|\phi_2\|_{\mathcal{F}^{1,1}}. && (3944)
\end{aligned}$$

We note that

$$\begin{aligned}
&T(\phi_1) - T(\phi_2) && (3945)\\
=&\mathcal{M}\Big((1 + (\phi_1)_\alpha) \cdot U(\phi_1)\Big) - \mathcal{M}\Big((1 + (\phi_2)_\alpha) \cdot U(\phi_2)\Big) && (3946)\\
=&\mathcal{M}\Big(U(\phi_1) - U(\phi_2)\Big) && (3947)\\
&+ \mathcal{M}\Big((\phi_1)_\alpha \cdot U(\phi_1) - (\phi_1)_\alpha \cdot U(\phi_2) + (\phi_1)_\alpha \cdot U(\phi_2) - (\phi_2)_\alpha \cdot U(\phi_2)\Big) && (3948)\\
=&\mathcal{M}\Big(U(\phi_1) - U(\phi_2)\Big) + \mathcal{M}\Big((\phi_1)_\alpha \cdot (U(\phi_1) - U(\phi_2))\Big) && (3949)\\
&+ \mathcal{M}\Big(U(\phi_2) \cdot ((\phi_1)_\alpha - (\phi_2)_\alpha)\Big). && (3950)
\end{aligned}$$

Since

$$\begin{aligned}
\|\mathcal{M}(f)\|_{\mathcal{F}^{0,1}} =&\sum_{k\in\mathbb{Z}} |\mathcal{F}(\mathcal{M}(f))(k)| && (3951)\\
=&|\mathcal{F}(\mathcal{M}(f))(0)| + \sum_{k\neq 0} |k|^{-1} |\mathcal{F}(f)(k)| && (3952)\\
=&2\sum_{k\neq 0} |k|^{-1} |\mathcal{F}(f)(k)| && (3953)\\
\leq&2\sum_{k\in\mathbb{Z}} |\mathcal{F}(f)(k)| && (3954)\\
=&2\|f\|_{\mathcal{F}^{0,1}}, && (3955)
\end{aligned}$$

we have

$$\begin{aligned}
&\|T(\phi_1) - T(\phi_2)\|_{\mathcal{F}^{0,1}} && (3956)\\
\leq&2\|U(\phi_1) - U(\phi_2)\|_{\mathcal{F}^{0,1}} + 2\|(\phi_1)_\alpha \cdot (U(\phi_1) - U(\phi_2))\|_{\mathcal{F}^{0,1}} && (3957)\\
&+ 2\|U(\phi_2) \cdot ((\phi_1)_\alpha - (\phi_2)_\alpha)\|_{\mathcal{F}^{0,1}} && (3958)\\
\leq&2\|U(\phi_1) - U(\phi_2)\|_{\mathcal{F}^{0,1}} + 2\|\phi_1\|_{\mathcal{F}^{1,1}} \|U(\phi_1) - U(\phi_2)\|_{\mathcal{F}^{0,1}} && (3959)\\
&+ 2\|U(\phi_2)\|_{\mathcal{F}^{0,1}} \|\phi_1 - \phi_2\|_{\mathcal{F}^{1,1}}. && (3960)
\end{aligned}$$

Moreover, since

$$\|\mathcal{M}(f)\|_{\dot{\mathcal{F}}^{1,1}} = \sum_{k\neq 0} |k| \left|\mathcal{F}(\mathcal{M}(f))(k)\right| \tag{3961}$$
$$= \sum_{k\neq 0} |k| \cdot |k|^{-1} \left|\mathcal{F}(f)(k)\right| \tag{3962}$$
$$\leq \sum_{k\in\mathbb{Z}} \left|\mathcal{F}(f)(k)\right| \tag{3963}$$
$$= \|f\|_{\mathcal{F}^{0,1}} , \tag{3964}$$

we have

$$\|T(\phi_1) - T(\phi_2)\|_{\dot{\mathcal{F}}^{1,1}} \leq \|U(\phi_1) - U(\phi_2)\|_{\mathcal{F}^{0,1}} + \|(\phi_1)_\alpha \cdot (U(\phi_1) - U(\phi_2))\|_{\mathcal{F}^{0,1}} \tag{3965}$$
$$+ \|U(\phi_2) \cdot ((\phi_1)_\alpha - (\phi_2)_\alpha)\|_{\mathcal{F}^{0,1}} \tag{3966}$$
$$\leq \|U(\phi_1) - U(\phi_2)\|_{\mathcal{F}^{0,1}} + \|\phi_1\|_{\mathcal{F}^{1,1}} \|U(\phi_1) - U(\phi_2)\|_{\mathcal{F}^{0,1}} \tag{3967}$$
$$+ \|U(\phi_2)\|_{\mathcal{F}^{0,1}} \|\phi_1 - \phi_2\|_{\mathcal{F}^{1,1}} . \tag{3968}$$

Therefore,

$$\|(\phi_2)_\alpha \cdot (T(\phi_1) - T(\phi_2))\|_{\dot{\mathcal{F}}^{1,1}} \tag{3969}$$
$$\leq \|\phi_2\|_{\dot{\mathcal{F}}^{2,1}} \Big(2\|U(\phi_1) - U(\phi_2)\|_{\mathcal{F}^{0,1}} + 2\|\phi_1\|_{\mathcal{F}^{1,1}} \|U(\phi_1) - U(\phi_2)\|_{\mathcal{F}^{0,1}} \tag{3970}$$
$$+ 2\|U(\phi_2)\|_{\mathcal{F}^{0,1}} \|\phi_1 - \phi_2\|_{\mathcal{F}^{1,1}} \Big) \tag{3971}$$
$$+ \|\phi_2\|_{\mathcal{F}^{1,1}} \Big(\|U(\phi_1) - U(\phi_2)\|_{\mathcal{F}^{0,1}} + \|\phi_1\|_{\mathcal{F}^{1,1}} \|U(\phi_1) - U(\phi_2)\|_{\mathcal{F}^{0,1}} \tag{3972}$$
$$+ \|U(\phi_2)\|_{\mathcal{F}^{0,1}} \|\phi_1 - \phi_2\|_{\mathcal{F}^{1,1}} \Big). \tag{3973}$$

Now, consider the expression

$$\left| \frac{2\pi}{L(\phi_1)} - \frac{2\pi}{L(\phi_2)} \right| . \tag{3974}$$

Without loss of generality, let

$$\mathrm{Im} \int_{-\pi}^{\pi} \int_0^{\alpha} e^{i(\alpha-\eta)} \sum_{n\geq 1} \frac{i^n}{n!} (\phi_1(\alpha) - \phi_1(\eta))^n d\eta d\alpha \tag{3975}$$
$$> \mathrm{Im} \int_{-\pi}^{\pi} \int_0^{\alpha} e^{i(\alpha-\eta)} \sum_{n\geq 1} \frac{i^n}{n!} (\phi_2(\alpha) - \phi_2(\eta))^n d\eta d\alpha. \tag{3976}$$

Then

$$\left|\frac{2\pi}{L(\phi_1)} - \frac{2\pi}{L(\phi_2)}\right| \tag{3977}$$

$$\leq R^{-1} \cdot \frac{1}{4\pi}\left(1 + \frac{1}{2\pi}\mathrm{Im}\int_{-\pi}^{\pi}\int_0^{\alpha} e^{i(\alpha-\eta)}\sum_{n\geq 1}\frac{i^n}{n!}(\phi_2(\alpha)-\phi_2(\eta))^n d\eta d\alpha\right)^{-1/2} \tag{3978}$$

$$\cdot\left|\mathrm{Im}\int_{-\pi}^{\pi}\int_0^{\alpha} e^{i(\alpha-\eta)}(e^{i(\phi_1(\alpha)-\phi_1(\eta))} - e^{i(\phi_2(\alpha)-\phi_2(\eta))})d\eta d\alpha\right| \tag{3979}$$

$$\leq R^{-1} \cdot \frac{1}{4\pi}\left(1 + \frac{1}{2\pi}\mathrm{Im}\int_{-\pi}^{\pi}\int_0^{\alpha} e^{i(\alpha-\eta)}\sum_{n\geq 1}\frac{i^n}{n!}(\phi_2(\alpha)-\phi_2(\eta))^n d\eta d\alpha\right)^{-1/2} \tag{3980}$$

$$\cdot\left(2\pi\left\|\phi_1-\phi_2\right\|_{\mathcal{F}^{0,1}}\sum_{n=1}^{\infty}\frac{(\|\phi_1\|_{\mathcal{F}^{0,1}}+\|\phi_2\|_{\mathcal{F}^{0,1}})^{n-1}}{n!}\right. \tag{3981}$$

$$\left.+\,2\pi\left\|\phi_1(\pi)-\phi_2(\pi)\right\|_{\mathcal{F}^{0,1}}\sum_{n=1}^{\infty}\frac{(\|\phi_1(\pi)\|_{\mathcal{F}^{0,1}}+\|\phi_2(\pi)\|_{\mathcal{F}^{0,1}})^{n-1}}{n!}\right) \tag{3982}$$

$$\leq R^{-1}\left(1-\frac{\pi}{2}\left(e^{2\|\phi_2\|_{\mathcal{F}^{0,1}}}-1\right)\right)^{-1/2}\left\|\phi_1-\phi_2\right\|_{\mathcal{F}^{0,1}}\sum_{n=1}^{\infty}\frac{(\|\phi_1\|_{\mathcal{F}^{0,1}}+\|\phi_2\|_{\mathcal{F}^{0,1}})^{n-1}}{n!}. \tag{3983}$$

Combining these results, we obtain

$$\frac{d}{dt}\left\|\theta_1-\theta_2\right\|_{\dot{\mathcal{F}}^{1,1}} \tag{3984}$$

$$\leq -\pi\cdot\frac{1}{R}\cdot\frac{\gamma}{4\pi}\left\|\phi_1-\phi_2\right\|_{\dot{\mathcal{F}}^{2,1}}+\left|\frac{2\pi}{L(\phi_1)}-\frac{2\pi}{L(\phi_2)}\right|\left\|\widetilde{N}(\phi_1)\right\|_{\dot{\mathcal{F}}^{1,1}} \tag{3985}$$

$$+\frac{2\pi}{L(\phi_2)}\left\|\widetilde{N}(\phi_1)-\widetilde{N}(\phi_2)\right\|_{\dot{\mathcal{F}}^{1,1}}+\pi\cdot\frac{\gamma}{4\pi}\cdot\frac{1}{R}A(\left\|\phi_1\right\|_{\mathcal{F}^{0,1}})\left\|\phi_1\right\|_{\mathcal{F}^{0,1}}\left\|\phi_1-\phi_2\right\|_{\dot{\mathcal{F}}^{2,1}} \tag{3986}$$

$$+\pi\cdot\frac{\gamma}{4\pi}\left|\frac{2\pi}{L(\phi_2)}-\frac{2\pi}{L(\phi_1)}\right|\left\|\phi_2\right\|_{\dot{\mathcal{F}}^{2,1}} \tag{3987}$$

$$\leq -\pi\cdot\frac{1}{R}\cdot\frac{\gamma}{4\pi}\left\|\phi_1-\phi_2\right\|_{\dot{\mathcal{F}}^{2,1}} \tag{3988}$$

$$+R^{-1}\left(1-\frac{\pi}{2}\left(e^{2\|\phi_2\|_{\mathcal{F}^{0,1}}}-1\right)\right)^{-1/2}\left(\sum_{n=1}^{\infty}\frac{(\|\phi_1\|_{\mathcal{F}^{0,1}}+\|\phi_2\|_{\mathcal{F}^{0,1}})^{n-1}}{n!}\right) \tag{3989}$$

$$\cdot\left\|\widetilde{N}(\phi_1)\right\|_{\dot{\mathcal{F}}^{1,1}}\left\|\phi_1-\phi_2\right\|_{\mathcal{F}^{0,1}} \tag{3990}$$

$$+\left(\frac{1+\frac{\pi}{2}\left(e^{2\|\phi_2\|_{\mathcal{F}^{0,1}}}-1\right)}{R^2}\right)^{1/2}\left\|\widetilde{N}(\phi_1)-\widetilde{N}(\phi_2)\right\|_{\dot{\mathcal{F}}^{1,1}} \tag{3991}$$

$$+\pi\cdot\frac{\gamma}{4\pi}\cdot\frac{1}{R}A(\left\|\phi_1\right\|_{\mathcal{F}^{0,1}})\left\|\phi_1\right\|_{\mathcal{F}^{0,1}}\left\|\phi_1-\phi_2\right\|_{\dot{\mathcal{F}}^{2,1}} \tag{3992}$$

$$+\pi\cdot\frac{\gamma}{4\pi}\left\|\phi_2\right\|_{\dot{\mathcal{F}}^{2,1}}R^{-1}\left(1-\frac{\pi}{2}\left(e^{2\|\phi_2\|_{\mathcal{F}^{0,1}}}-1\right)\right)^{-1/2} \tag{3993}$$

$$\cdot\left(\sum_{m=1}^{\infty}\frac{(\|\phi_1\|_{\mathcal{F}^{0,1}}+\|\phi_2\|_{\mathcal{F}^{0,1}})^{n-1}}{n!}\right)\left\|\phi_1-\phi_2\right\|_{\mathcal{F}^{0,1}}. \tag{3994}$$

Ultimately, for sufficiently small $\|\theta^0\|_{\dot{\mathcal{F}}^{1,1}}$,

$$\frac{d}{dt}\left\|\phi_1-\phi_2\right\|_{\dot{\mathcal{F}}^{1,1}}\leq\mathcal{E}\left\|\phi_1-\phi_2\right\|_{\mathcal{F}^{1,1}}, \tag{3995}$$

where $\mathcal{E}$ is a coefficient that may depend on $\|\phi_1\|_{\dot{\mathcal{F}}^{1,1}}$, $\|\phi_2\|_{\dot{\mathcal{F}}^{1,1}}$, $\|\phi_1\|_{\dot{\mathcal{F}}^{2,1}}$, and $\|\phi_2\|_{\dot{\mathcal{F}}^{2,1}}$ in such a way that it is integrable in time. By Grönwall's inequality, since the two solutions share the same initial datum, $\phi_1=\phi_2$. Since the dynamics of $\mathcal{F}(\theta_1)(0)$ and $\mathcal{F}(\theta_2)(0)$ are determined completely by ϕ_1 and ϕ_2, respectively, with the shared initial condition $\mathcal{F}(\theta^0)(0)$, we conclude that $\mathcal{F}(\theta_1)(0)=\mathcal{F}(\theta_2)(0)$ as well.

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