Checks on Aurora's calculations

(with additional reference material)

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This is a check of Aurora's calculations, translated into fancier (but also more powerful) technical language. References are given in case one wants to deepen one's knowledge of some topics later on.

(References of the form "[YCM III.A.1]" are to Choquet-Bruhat et al. 1996.)

On the mathematical point of view adopted

The following discussion on distributions could be mathematically approached with the hullabaloo of Lebesgue-measure theory and Lebesgue integration. Those theory and integral, however, do not extend to infinite-dimensional spaces in a straightforward way. And one eventually wants to extend this toy-discussion to physical theories which *are* based on infinite-dimensional spaces.

A more flexible approach instead is de Rham's theory of *currents* [YCM VI.B.8; de Rham 1984], essentially based on differential forms [YCM IV; Burke 1987; Bossavit 1991]. In essence this is Jaynes's "limits of finite sets" approach [Jaynes 2003 § 2.5]. The integrals then are to be understood as the more powerful Heinstock-Kurzweil a.k.a. *gauge* integrals [Fonda 2018], and singular (delta) densities in Egorov's (rather than Schwartz's) sense [Egorov 2001; Oberguggenberger 1992].

This is why a differential-geometric perspective is emphasized here. Anyway, either point of view works for the present discussion. The terminology is chosen so as not to commit to one or the other, as much as possible.

Checks

We have a map F from a 3D manifold [YCM III.A] (with boundary) Θ covered by a coordinate chart

$$\theta_1 \in [0, 2\pi[\qquad \theta_2 \in [0, \pi[\qquad r \in]2\,R, +\infty[$$

to a 2D manifold (with boundary) X covered by a coordinate chart

$$(x_1, x_2)\;, \quad |\boldsymbol{x}| \in]R, +\infty[\;.$$

The map, expressed in the two coordinate systems, is

$$F(\boldsymbol{\theta}, r) = \bigl(r\cos\theta_1 + R\cos(\theta_1+\theta_2)\;,\; r\sin\theta_1 + R\sin(\theta_1+\theta_2)\bigr) \tag{1}$$

with R a strictly positive constant.

The map F determines, as usual, an invertible set-map[1], which we also denote F, between the power sets of Θ and X. This map allows us [YCM I.D.3] to associate a distribution g on Θ to one g^* on X by

$$g^*(A) = g[F^{-1}(A)]\;, \quad A \subseteq X\;. \tag{2}$$

Our problem is to find g^*, for a specific g which will be presented later.

The map (2) requires a one-dimensional integration over the set $F^{-1}(\{\boldsymbol{x}\})$. In doing this integration we would also have to pay attention to possible "Borel-Kolmogorov paradoxes"[2]. An equivalent approach is to augment (x_1, x_2) with some function in such a way that the resulting three functions can be seen as a new coordinate system on Θ. Then we can simply express the distribution g in the new coordinate system, and obtain g^* as the marginal of g with respect to the augmented coordinate.

The Jacobian matrix (tangent map [YCM III.B.1]) of F at a generic point of Θ is

$$F'(\boldsymbol{\theta}, r) = \begin{bmatrix} -r\sin\theta_1 - R\sin(\theta_1+\theta_2) & -R\sin(\theta_1+\theta_2) & \cos\theta_1 \\ r\cos\theta_1 + R\cos(\theta_1+\theta_2) & R\cos(\theta_1+\theta_2) & \sin\theta_1 \end{bmatrix} \tag{3}$$

The minors[3] obtained from F' by eliminating the r-, θ_2-, θ_1-columns are

$$R\,r\sin\theta_2\;, \qquad -r - R\cos\theta_2\;, \qquad -R\cos\theta_2 \tag{4}$$

The r-minor is zero at $\theta_2 = 0$; this means that the map $\boldsymbol{\theta} \mapsto \boldsymbol{x}$ for constant r is invertible over the whole coordinate chart. The fact that the

[1] e.g. Simmons 1963 § 1.3. [2] see e.g. the example in Jaynes 2003 § 15.7. [3] Horn & Johnson 2013 § 0.7.

Jacobian becomes zero at the boundary value $\theta_2 = 0$ does not worry us, because the distribution we shall consider does not have any singular (delta) components (otherwise we would have to analyse the "$\infty \cdot 0$" at this boundary).

The θ_2-minor is never zero. The θ_1-minor is zero at $\theta_2 = \pi/2$; this means that the map $(\theta_2, r) \mapsto \boldsymbol{x}$ for constant θ_1 is only invertible in two subdomains. This would require the separate consideration of two integrals.

We can therefore conveniently augment $\boldsymbol{x}$ with r as an additional coordinate, as Aurora did, obtaining the coordinate chart with domain

$$(x_1, x_2, r)\,, \qquad \lvert\boldsymbol{x}\rvert > R\,,\ \max\bigl(\lvert\boldsymbol{x}\rvert - R,\ 2R\bigr) < r < \lvert\boldsymbol{x}\rvert + R \tag{5}$$

The coordinate transformation

$$\begin{gathered} C\colon (\theta_1, \theta_2, r) \mapsto (x_1, x_2, r)\,, \\ C(\boldsymbol{\theta}, r) = \bigl(r\cos\theta_1 + R\cos(\theta_1+\theta_2)\,,\ r\sin\theta_1 + R\sin(\theta_1+\theta_2)\,,\ r\bigr) \end{gathered} \tag{6}$$

has non-negative Jacobian determinant

$$\det(C')(\boldsymbol{\theta}, r) = R\,r\sin\theta_2\,. \tag{7}$$

The inverse $C^{-1}\colon (\boldsymbol{x}, r) \mapsto (\boldsymbol{\theta}, r)$ is

$$\begin{aligned} &C^{-1}(\boldsymbol{x}, r) = \\ &\Biggl(2\arctan^* \frac{2\,r\,x_2 - \sqrt{2\,r^2\,(\lvert\boldsymbol{x}\rvert^2 + R^2) - (\lvert\boldsymbol{x}\rvert^2 - R^2)^2 - r^4}}{\lvert\boldsymbol{x}\rvert^2 + r^2 + 2\,r\,x_1 - R^2}\,, \\ &\qquad\qquad - 2\arctan \frac{\sqrt{2\,r^2\,(\lvert\boldsymbol{x}\rvert^2 + R^2) - (\lvert\boldsymbol{x}\rvert^2 - R^2)^2 - r^4}}{(r - R)^2 - \lvert\boldsymbol{x}\rvert^2}\,, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad r\Biggr) \end{aligned} \tag{8}$$

where the arctan* function is defined as

$$\arctan^*(x) = \begin{cases} \arctan(x) & \text{if } \arctan(x) \ge 0 \\ \arctan(x) + \pi & \text{if } \arctan(x) < 0 \end{cases}\,.$$

The distribution considered over Θ has density

$$\begin{aligned} g(\boldsymbol{\theta}, r)\,\mathrm{d}\boldsymbol{\theta}\,\mathrm{d}r &= \frac{1}{2\pi}\frac{1}{\pi} G(r + 2\,R \mid c, s)\,\mathrm{d}\boldsymbol{\theta}\,\mathrm{d}r \\ &= \frac{s^{-c}}{2\pi^2\,R\,\Gamma(c)}\,\frac{(r-2\,R)^{c-1}}{r\sin\theta_2}\exp\frac{2\,R-r}{s}\,\mathrm{d}\boldsymbol{\theta}\,\mathrm{d}r \end{aligned} \tag{9}$$

where $G(\,\mid c, s)$ is the gamma distribution with shape parameter c and scale parameter s.

Let $g^*(\boldsymbol{x}, r)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}r$ be the expression of the distribution above in coordinates $(\boldsymbol{x}, r)$. The expressions in the two coordinate systems must satisfy the identity

$$g^*(\boldsymbol{x}, r)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}r = g(\boldsymbol{\theta}, r)\,\mathrm{d}\boldsymbol{\theta}\,\mathrm{d}r \tag{10}$$

and since $\mathrm{d}\boldsymbol{x}\,\mathrm{d}r = \det(C')(\boldsymbol{\theta}, r)\,\mathrm{d}\boldsymbol{\theta}\,\mathrm{d}r$ [YCM IV.A.1], we find

$$\begin{aligned} &g^*(\boldsymbol{x}, r)\ \det(C')(\boldsymbol{\theta}, r)\,\mathrm{d}\boldsymbol{\theta}\,\mathrm{d}r = g(\boldsymbol{\theta}, r)\,\mathrm{d}\boldsymbol{\theta}\,\mathrm{d}r \\ &\qquad\implies\qquad g^*(\boldsymbol{x}, r) = \frac{g[C^{-1}(\boldsymbol{x}, r)]}{\det(C')[C^{-1}(\boldsymbol{x}, r)]} \end{aligned} \tag{11}$$

Substituting the explicit expression (9) for g, the Jacobian determinant (7), and the expression (8) for C^{-1} we find

$$g^*(\boldsymbol{x}, r) = \frac{s^{-c}}{2\pi^2\,R\,\Gamma(c)}\,\frac{(r-2\,R)^{c-1}}{\sqrt{2\,r^2\,(|\boldsymbol{x}|^2+R^2) - (|\boldsymbol{x}|^2-R^2)^2 - r^4}}\exp\frac{2\,R-r}{s} \tag{12}$$

The marginal distribution for $\boldsymbol{x}$ is obtained by integrating the expression above in $\mathrm{d}r$, with integration limits $\max(|\boldsymbol{x}|-R, 2\,R)$ and $|\boldsymbol{x}|+R$:

$$\begin{aligned} &\int_{\max(|\boldsymbol{x}|-R, 2\,R)}^{|\boldsymbol{x}|+R} \frac{s^{-c}}{2\pi^2\,R\,\Gamma(c)} \times \\ &\qquad \frac{(r-2\,R)^{c-1}}{\sqrt{2\,r^2\,(|\boldsymbol{x}|^2+R^2) - (|\boldsymbol{x}|^2-R^2)^2 - r^4}}\exp\frac{2\,R-r}{s}\,\mathrm{d}r\ . \end{aligned} \tag{13}$$

I couldn't manage to solve this integral analytically, especially because of the peculiar boundaries. Mathematica did not find any explicit expression either.

Bibliography

("de *X*" is listed under D, "van *X*" under V, and so on, regardless of national conventions.)

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