Judgements: s mparen , s lparen

Reference rules:

$$\frac{}{\epsilon \text{ mparen}}\,\text{Meps} \qquad \frac{s \text{ mparen}}{(s) \text{ mparen}}\,\text{Mpar} \qquad \frac{s_1 \text{ mparen} \quad s_2 \text{ mparen}}{s_1 s_2 \text{ mparen}}\,\text{Mseq}$$

$$\frac{}{\epsilon \text{ lparen}}\,\text{Leps} \qquad \frac{s_1 \text{ lparen} \quad s_2 \text{ lparen}}{(s_1) s_2 \text{ lparen}}\,\text{Lseq}$$

Question 1.

Theorem 1.1. If s lparen, then s mparen.

Proof) By rule induction on the judgement s lparen.

Case $\dfrac{}{\epsilon \text{ lparen}}\,\text{Leps}$ where $s = \epsilon$:

ϵ mparen by Meps.

Case $\dfrac{s_1 \text{ lparen} \quad s_2 \text{ lparen}}{(s_1) s_2 \text{ lparen}}\,\text{Lseq}$ where $s = (s_1) s_2$

s_1 mparen	by inductive hypothesis on s_1 lparen
s_2 mparen	by inductive hypothesis on s_2 lparen
(s_1) mparen	by Mpar with s_1
$(s_1) s_2$ mparen	by Mseq with (s_1) and s_2

$\blacksquare$

Judgement : s tparen

Inference rule : $\dfrac{}{\epsilon \text{ tparen}}\,T_{eps}$ $\dfrac{s_1 \text{ tparen} \quad s_2 \text{ tparen}}{s_1(s_2) \text{ tparen}}\,T_{seq}$

Lemma 1.2 If s tparen and s' tparen, then ss' tparen

Question 2. Proof) We can interpret this theorem as below:

If s' tparen, then s tparen implies ss' tparen.

By rule induction on s' tparen

Case $\dfrac{}{\epsilon \text{ tparen}}\,T_{eps}$ where $s' = \epsilon$

s tparen assumption

$ss' = s\epsilon = s$

ss' tparen from s tparen

Case $\dfrac{s_1 \text{ tparen} \quad s_2 \text{ tparen}}{s_1(s_2) \text{ tparen}}$ where $s' = s_1(s_2)$

s tparen assumption

$ss' = ss_1(s_2)$

s implies ss_1 tparen by induction hypothesis on s_1 tparen

ss_1 tparen from assumption s tparen

$ss_1(s_2)$ tparen by rule T_{seq} with ss_1 tparen and s_2 tparen $\blacksquare$

Question 3.

Theorem 1.3. If s mparen, then s tparen.

Proof) By rule induction on s mparen

Case $\dfrac{}{\epsilon \text{ mparen}}$ Meps where $s = \epsilon$

ϵ tparen by Teps

Case $\dfrac{s' \text{ mparen}}{(s') \text{ mparen}}$ Mpar where $s = (s')$

s' tparen by induction hypothesis

ϵ tparen by Teps

$\epsilon(s')$ tparen from Tseq with ϵ and s' $\left(\dfrac{\epsilon \text{ tparen} \quad s' \text{ tparen}}{\epsilon \cdot (s') \text{ tparen}} \text{Tseq} \right)$

(s') tparen from $\epsilon(s') = (s')$

Case $\dfrac{s_1 \text{ mparen} \quad s_2 \text{ mparen}}{s_1 \cdot s_2 \text{ mparen}}$ Mseq where $s = s_1 s_2$

s_1 tparen by induction hypothesis on s_1 mparen

s_2 tparen by induction hypothesis on s_2 mparen

$s_1 s_2$ tparen by Lemma 1.2 $\square$