2 Eigenvectors of PAP

Here we demonstrate the computation of the eigenvectors of PAP corresponding to the least eigenvalues with $P = I - vv^\top$ with some unit vector v.

We start with the first eigenvector of PAP. One fast method to compute the eigenvector is called Rayleigh Quotient iteration, a power iteration with adjusting shift s, which could be an estimate of the least eigenvalue of PAP as long as s is less than the smallest eigenvalue of PAP.

For simplicity, we just use the inverse power iteration, i.e., since A is positive definite, we choose $s = 0$. The algorithm consists of repeating the following updates: For $k = 1, 2, 3, \ldots$

(i) $u_{k-1} = Py_{k-1}/\|Py_{k-1}\|$.

(ii) Solve Py_k from $P(A - sI)Py_k = u_{k-1}$.

We proceed a sufficient number of iterations, so that $\|PAPu_k - s_k u_k\| < \epsilon$, where $s_k = u_k^\top A u_k$. Then the limit $w := \lim_k u_k$ is the eigenvector of PAP corresponding to the least eigenvalue. During the iterations, we can update the information of s by $s_{k-1} = u_{k-1}^\top A u_{k-1}$ when s_{k-1} is close to the least eigenvalue of PAP.

In the following, we illustrate the matrix inversion in the second step(ii), provided that $(A-sI)^{-1}u$ can be computed easily for any vector u. Note that $P(A-sI)P$ is actually singular, since it has a null vector v. Hence, instead of computing $(PAP - sI)^{-1}$, we shall compute the pseudo inverse $\{P(A - sI)^{-1}P\}^\dagger u_{k-1}$ in (7).

Proposition 2.1. *Let $P = I - vv^\top$ with some unit vector v. Suppose $A - sI$ is invertible. Let*

$$u' := (A - sI)^{-1}u_{k-1},\ v_1 := (A - sI)^{-1}v. \tag{6}$$

Let $y_k := \{P(A - sI)^{-1}P\}^\dagger u_{k-1}$. Then $v^\top v_1 \neq 0$ and

$$Py_k = \{P(A - sI)^{-1}P\}^\dagger u_{k-1} = (I - \frac{v_1 v^\top}{v^\top v_1})u'. \tag{7}$$

Proof. From the definition of y_k, we have for some $c \in \mathbb{R}$,

$$(A - sI)Py_k = u_{k-1} + cv.$$

Assume the inverse of the sparse matrix $(A - sI)$ is easy. Then (6) gives

$$Py_k = (A - sI)^{-1}\{u_{k-1} + cv\} = u' + cv_1. \tag{8}$$

Since the inner product of Py_k with v is zero, (8) gives

$$c = -\frac{v^\top (A - sI)^{-1}u_{k-1}}{v^\top (A - sI)^{-1}v} = -\frac{v^\top u'}{v^\top v_1}. \tag{9}$$

We have the expression of (7) from (8,9). □

Once the smallest eigenvector is obtained, we can proceed to compute the eigenvector of PAP corresponding to the second eigenvalue by the same approach with a minor adjustment: just replacing $P = I - vv^\top$ with $P = I - vv^\top - ww^\top$. That is, in the second step(ii), we replace (7) with

$$Py_k = (A - sI)^{-1}\{u_{k-1} + c_1 v + c_2 w\},$$

where c_1, c_2 can be solved from the small linear 2×2 system,

$$-[v, w]^\top (A - sI)^{-1}u_{k-1} = [v, w]^\top (A - sI)^{-1}[v, w][\begin{array}{c} c_1 \\ c_2 \end{array}]. \tag{10}$$