ICT 600 HW 1 Solution

Disclaimer: This is my attempt to help you learn the material. There might be typos/mistakes.

① $B - A = \{\phi, \{\{a\}\}\}$

② A is not subset of B because $a \in A$ but $a \notin B$.

③ $A \times (B - A) = \{(a, \phi), (a, \{\{a\}\}), (\{a\}, \phi), (\{a\}, \{\{a\}\})\}$

④ $f \sim g$ means $f(x) = g(x) + c, \ \forall x$

Reflexive: $f(x) = f(x) + 0$ ← $c = 0$

$\Rightarrow f \sim f$

Symmetric: $f \sim g$

$\Rightarrow f(x) = g(x) + c, \ \forall x$

$\Rightarrow f(x) - c = g(x), \ \forall x$

$\Rightarrow g(x) = f(x) + (-c), \ \forall x$

$\Rightarrow g \sim f$

Transitive $f \sim g$ and $g \sim h$

$\Rightarrow f(x) = g(x) + c$ and $g(x) = h(x) + d$

$\Rightarrow f(x) = (h(x) + d) + c = h(x) + (d + c)$

$\Rightarrow f \sim h$

Therefore $f \sim g$ is an equivalence relation.

(5) $x \sim y$ means $|x-y| \leq 1$

Let $x=1$, $y=2$, $z=3$

then $|x-y| = |1-2| = 1 \leq 1$

and $|y-z| = |2-3| = 1 \leq 1$

But $|x-z| = |1-3| = 2 > 1$

Thus transitive property fails in this relation. Therefore it is not an equivalence relation.

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x	y	$x \to \neg y$	$y \oplus \neg x$	$(x \to \neg y) \vee (y \oplus \neg x)$
1	1	0	1	1
1	0	1	0	1
0	1	1	0	1
0	0	1	1	1

⑦ Proof by contradiction

Assume that $\sqrt{3}$ is a rational number

Then there exists $a, b \in \mathbb{Z}$ such that

$\sqrt{3} = \frac{a}{b}$ and a, b have no common divisor

then $3 = \frac{a^2}{b^2} \Rightarrow 3a^2 = b^2$

$\Rightarrow 3 \mid b^2 \Rightarrow 3 \mid b$

$\Rightarrow b = 3k$ for some $k \in \mathbb{Z}$

$\Rightarrow 3a^2 = (3k)^2 \Rightarrow a^2 = 3k^2$

$\Rightarrow 3 \mid a^2 \Rightarrow 3 \mid a$ contradiction

we just show that $3 \mid a$ and $3 \mid b$

but this contradict with our assumption

that a, b have no common divisor.

Therefore $\sqrt{3}$ is an irrational number.

⑧ Direct proof

Suppose n is an integer

if n is even, then $n = 2k$, $k \in \mathbb{Z}$

$$n^2 + 2 = (2k)^2 + 2$$
$$= 4k^2 + 2 \quad \text{remainder}$$

$\Rightarrow 4 \nmid n^2+2$ if n is even

if n is odd, then $n = 2k+1$, $k \in \mathbb{Z}$

$$n^2 + 2 = (2k+1)^2 + 2$$
$$= 4k^2 + 4k + 1 + 2$$
$$= 4(k^2+k) + 3 \quad \text{remainder}$$

$\Rightarrow 4 \nmid n^2+2$ if n is odd.

Therefore if n is an integer then n^2+2 is not divisible by 4.

(9) Proof by induction

Base case $n=1$

A clique graph has 1 vertex.

Zero edges are required and

$$\frac{0(0-1)}{2} = 0$$

Inductive Step

Assume that a clique graph w/ k vertices has $\frac{1}{2}k(k-1)$ edges. This is our induction hypothesis.

We need to prove the case $k+1$ i.e when a clique graph has $k+1$ vertices.

Let us start with a k-vertices clique graph. When we add $(k+1)^{th}$ vertex, we need to connect it to k original vertices, requiring k additional edges.

Thus we will have

$$\frac{1}{2}k(k-1)+k = \frac{k^2-k+2k}{2} = \frac{k^2+k}{2}$$

$$= \frac{(k+1)(k)}{2} \text{ edges}$$

$\boxtimes$

⑩ $(\Rightarrow)$ To prove $A-(B\cup C) \subset (A-B)\cap(A-C)$

Let $x \in A-(B\cup C)$

$\Rightarrow x \in A$ but $x \notin B\cup C$

$\Rightarrow x \in A$ and $x \notin B$ and $x \notin C$

$\Rightarrow x \in (A-B)$ and $x \in (A-C)$

$\Rightarrow x \in (A-B)\cap(A-C)$

$\Rightarrow A-(B\cup C) \subset (A-B)\cap(A-C)$

$(\Leftarrow)$ To prove $(A-B)\cap(A-C) \subset A-(B\cup C)$

let $x \in (A-B)\cap(A-C)$

$\Rightarrow x \in A-B$ and $x \in A-C$

$\Rightarrow x \in A$ but $x \notin B$. Also $x \in A$ but $x \notin C$

$\Rightarrow x \in A$ but $x \notin B\cup C$

$\Rightarrow x \in A-(B\cup C)$

$\Rightarrow (A-B)\cap(A-C) \subset A-(B\cup C)$

Therefore $(A-B)\cap(A-C) = A-(B\cup C)$