ICT600 Computational Mathematics

Basic Mathematics

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1 Sets

Definition 1. A *set* is a group of objects represented as a unit. The objects in a set are called *elements* or *members* of a set.

Elements in a set can be numbers, symbols and even other sets. They are enclosed by braces. There is no notion of order in a set. There is no notion of duplication either.

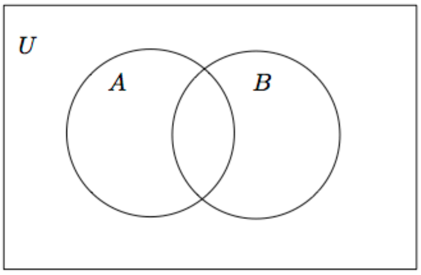
The *empty set* is the set containing no items; it is written as $\{\}$ or $\emptyset$ for short.

Notations

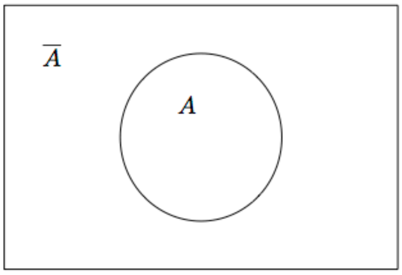
- $x \in A$ means that x is an element of a set A.
- $A \cap B$, the intersection of A and B, denotes the set containing objects that are in both A and B.
- $A \cup B$, the union of A and B, denotes the set containing objects that are in either A or B.
- $A \subseteq B$ means that every element in A is also in B. We say that A is a subset of B. If there is an element that is in B but not in A, then we can also use the notation $A \subset B$, or for emphasis $A \subsetneq B$. In this case, we call A a proper subset of B.

- $\bar{A}$ denotes the set containing objects that are not in A. For this notion to be meaningful we need a notion of the universe of objects at hand, the set U of objects, where $A \subseteq U$. Then $\bar{A}$ is the set of objects in U but not in A.
- $A - B$ is the set containing of objects contained in A that are not contained in B. Note that $A - B = A \cap \bar{B}$.
- $|A|$ denotes the number of objects in A. It is called cardinal number, or cardinal for short. If the cardinal of A is finite, then we call A a finite set. If the cardinal of A is infinite, then A is called an infinite set.
- 2^A denotes the power set of A which is a set of all subset of A.
- $\mathbb{N}$ is a set of all natural numbers i.e. $\mathbb{N} = \{1, 2, 3, ...\}$.
- $\mathbb{Z}$ is a set of all integers i.e. $\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$.
- $\mathbb{Z}_n = \{0, 1, 2, ..., n-1\}$ modulo n which is the all remainder after division by n.
- $\mathbb{R}$ is a set of all real numbers.

Venn Diagrams: These are diagrams that represent sets as regions enclosed by circular lines. They can be very useful to help use see what's going on. The following pictures are some example of Venn Diagrams.



(a)



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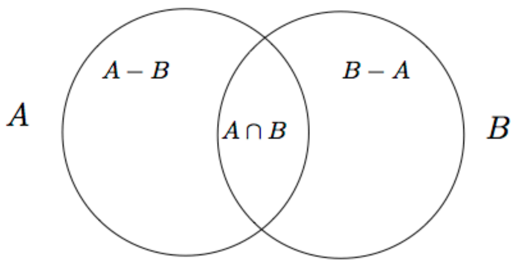


Example 1. *Let* $A = \{1, 2, 3, 4\}$ *and* $B = \{2, 4, 6\}$*. Evaluate the followings;*

1. $A \cap B$

2. $A \cup B$

3. $A - B$

4. $B - A$

5. *The cardinal of* A *and* B

6. 2^B

2 Sequences and tuples

Definition 2. A *sequence* is an ordered list of objects in which there may be repetitions.

A sequence is usually designated by writing the list within parentheses. Same as in sets, sequences can be finite or infinte as well. For example $(1, 2, 3)$ is a sequence. Also, $(2, 1, 3)$ is a sequence. Note that these two sequences contains the same objects but they are listed in the different orders. They are not equal.

Definition 3. A sequence with k elements is a k-tuple.

Definition 4. Let A and B be sets. The *Cartesian product* or *cross product* of A and B, written $A \times B$, is the set of all pairs where in the first element is a member of A and the second element is a member of B i.e. $A \times B = \{(a, b)|a \in A, b \in B\}$.

If A is a set, then $A^n = \underbrace{A \times A \times A \times A \times ... \times A}_{n \text{ times}}$.

Example 2. *Let* $A = \{1, 2, 3\}$ *and* $B = \{x, y\}$. *Calculate the followings:*

1. $A \times B$

2. $A \times B \times A$

3. B^3

3 Functions and Relations

Definition 5. Let D and R be two nonempty sets. A *function* f from D to R is a rule of correspondence that assigns to each element in set D exactly one element in R.

The set D is called the *domain* of the function f. The domain is like the set of all the inputs. And for each input, the function f gives us an output corresponding to it. The set of all these outputs is called the *range* of the function f. The letter representing elements in the domain is called the *independent variable*, and the letter representing the element in the range is called the *dependent variable*. Thus if $y = f(x)$, x is the independent variable and y is a dependent variable. A function also is called a *mapping*, and, if $f(x) = y$, we say that f maps x to y.

Definition 6. A function $f : A \to B$ is call *one to one* or *injective* if whenever $f(a) = f(a')$ for all $a, a' \in A$, then $a = a'$

Definition 7. A function $f : A \to B$ is call *onto* or *surjective* if $B = f(A)$.

A function that is both one to one and onto is called *bijective*.

Example 3. *Let* $f : \mathbb{R} \rightarrow \mathbb{R}$ *be a function defined on a set of real numbers defined by* $f(x) = x^2 - 1$.

1. *What is the domain and range of* f*?*

2. *Is* f *injective, surjective, bijective?*

When the domain of a function f is $A_1 \times A_2 \times ... \times A_k$ for some set $A_1, ..., A_k$ the input to f is a k-tuple $(a_1, a_2, ..., a_k)$ and we call the a_i the *arguments* to f. If k is 2, f is a binary function.

A *predicate* or *property* is a function whose range is $\{$TRUE, FALSE$\}$.

A property whose domain is a set of k-tuples $A \times A \times A ... \times A$ is called a *relation*. If $k = 2$, then we call that particular property a *binary relation*.

A special type of binary relation, called an *equivalence relation*, captures the notion of two objects being equal in some features.

Definition 8. A binary relation R is an equivalence relation if R satisfy three conditions;

1. R is reflexive if for every x, xRx;
2. R is symmetric if for every x and y, xRy implies yRx; and
3. R is transitive if for every x, y and z, xRy and yRz implies xRz.

The notation of '$\sim$' or '$\equiv$' sometimes are used instead of R.

An equivalence relation is a relation that partitions a set so that every element of the set is a member of one and only one cell of the partition.

Definition 9. Let n be a positive integer. For integers a and b we say that a is congruent to b modulo n, and write $a \equiv b \mod n$, provided $a - b$ is divisible by n.

Example 4. *Prove that congruent modulo n is an equivalence relation on $\mathbb{Z}$.*

Example 5. *For* $x, y \in \mathbb{R}$*, define* $x \sim y$ *to mean that* $x - y \in \mathbb{Z}$*. Prove or disprove that this relation is an equivalence relation.*

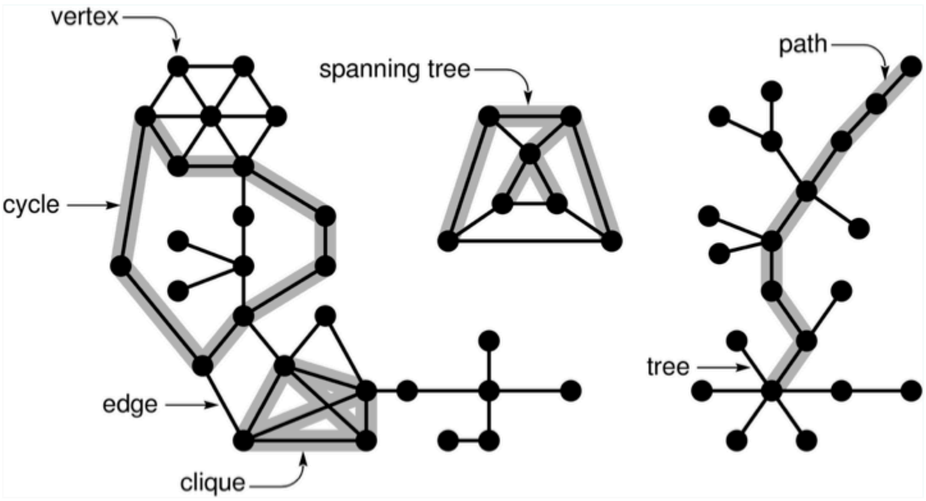
Example 6. *For $a, b \in \mathbb{R}$, define $a \sim b$ to mean that $ab = 0$. Prove or disprove that this relation is an equivalence relation.*

Example 7. *For the set of all functions $f : \mathbb{R} \to \mathbb{R}$, let $f \sim g$ mean that f is a constant multiple of g, that is, there is some constant $k \neq 0$ such that $f(x) = k \cdot g(x)$ for all $x \in \mathbb{R}$. Prove or disprove that this relation is an equivalence relation.*

4 Graphs

A graph consists of two sets, a set of vertices and a set of edges. The number of edges at a particular vertex is the degree of that vertex. No more than one edge is allowed between any two vertices. In a graph G the contain vertices i and j, the pair (i, j) represents the edges that connect i and j. Formally, a graph $G = (V, E)$ where V is a set of vertices and $E = V \times V$.

There are two types of graphs, *undirected* and *directed*. A graph G is undirected if for all $i, j \in V$, then $(i, j) = (j, i)$, otherwise G is directed.



Notations

Path A path is a sequence of vertices where successive vertices are joined by edges. A graph is *connected* if every two vertices have a path between them.

Cycle This is a path that begins and ends at the same vertex.

Simple Cycle A cycle with exactly one repeated vertex (its first and its last).

Tree A graph is a tree if it is connected and has no simple cycle. A tree may contain a specially designated vertex called the root. The vertex of degree 1 in a tree, other than the root, are called the leaves of the tree.

Spanning Tree It is a tree composed of all the vertices and some (or perhaps all) of the edges of G.

Clique A clique of G is a subset of its vertices such that every two vertices in the subset are connected by an edge.

Example 8. *Let* $G = (V, E)$ *be a graph such that* $V = \{1, 2, 3, 4\}$ *and* $E = \{(1, 2), (2, 3), (1, 3), (2, 4), (1, 4)\}$. *Draw the graph* G. *What is the degree of vertex 3. Indicate a path from vertex 3 to vertex 4 on your drawing of* G. *Draw a spanning tree of* G. *Are there any clique. If so, highlight on your drawing of a graph* G.

5 Strings and Languages

Alphabet is a set of one or more characters, e.g. $\{0\}, \{0, 1\}, \{a, b, c, ..., z\}$. The elements of alphabet are called symbols.

Strings over an alphabet is a finite sequence of symbols from that alphabet. The *length* of a string w, denoted $|w|$, is the number of symbols that w contains. The string of length zero is called the *empty string* and is written ε. The reverse of w, denoted w^R is the strong obtained by writing w in the opposite order. String z is substring of w if z appears consecutively within w. The *concatenation* of strings x and y is the string obtained by appending y to the end of x.

Language is a set of strings. e.g. $L = \{aa, ab, ba, bb\}$ is the set of all 2-character strings over the alphabet $\Sigma = \{a, b\}$.

The notation Σ^* (sometimes known as *Kleene star*) denotes the set of all strings over the alphabet Σ; e.g. $\{a, b\}^* = \{\varepsilon, a, b, aa, bb, ab, ba, aaa, ...\}$.

When S is a set of words, S^* refers the set of all finite strings formed by concatenating words from S.

Example 9. *Consider the language S^*, where $S = \{a, b\}$. How many words does this language have of length 2? of length n?*

Example 10. *Give an example of a set S such that the language S^* has more six-letter words than seven-letter words.*

Example 11. *Give an example of a set S such that the language S^* has more six-letter words than eight-letter words.*

Example 12. *Does there exist an S^* such that it has more six-letter words than twelve-letter words?*

6 Boolean Logic

Boolean logic is the algebra of variables that take on just one of two values: TRUE and FALSE or 1 and 0 respectively.

Operation

Negation or NOT $\overline{x}$ or $\neg x$ This is 1 exactly if x is 0; otherwise it is 0.

Conjunction or AND $x \wedge y$ It is 1 exactly if both x and y are 1. Otherwise, it is 0.

Disjunction or OR $x \vee y$ This is 1 if at least one of x or y is 1. Otherwise it is 0.

Exclusive or or XOR $x \oplus y$ This is 1 if either but not both of x, y are 1.

Equality $x \leftrightarrow y$ This is 1 if both of x, y have the same value.

Implication $x \rightarrow y$ This is 0 if its first operand is 1 and its second operand is 0. Otherwise it is 1.

Example 13. *Complete the following truth table for the operation* $(\neg x \vee y) \oplus y$.

x	y	$\neg x$	$\neg x \vee y$	$(\neg x \vee y) \oplus y$
1	0			
1	1			
0	0			
0	1			

References

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[2] Richard Cole *Theory of Computing. What is Feasible, Infeasible, and Impossible Computationally*, Lecture notes.

[3] Robert Sedgewick and Kevin Wayne *Algorithms and Data Structures*, Lecture notes.