ICT600 Computational Mathematics

First Order Logic

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First Order Logic (FOL) is formal system for logical reasoning. It is a language that is used to communicate with computers. It allows computers to manage complicated human reasoning processes.

First Order Logic defines the world with objects, relations and functions. It is called first-order because it allows quantifiers to range over objects (terms) but not properties, relations, or functions applied to those objects.

1 Syntax for FOL

Basic Elements of FOL

Constant Symbols A constant is an specific object such as a person name Tom, a particular apple etc.

Variable Symbols A countably infinite set of unknowns such as $x, y, a, b, \ldots$

Function Symbols A function takes n-tuples of terms (constants or variables) and return another term.

Predicate Symbols An n-ary predicate can defined as a function from tuples of n terms to True, False.

Connective Symbols $\lor$ (or), $\land$ (and), $\rightarrow$ (implies), $\leftrightarrow$ (equivalent), $\neg$ (not).
They are used to construct complex sentences.

Quantifier Symbols $\forall$ (for all), $\exists$ (there exists).
It allows statements about entire or some collections of objects rather than having to enumerate the objects by name.

Equality Symbol $=$ (equality)

2 Sentences

An **atomic** sentence is the most simplest structure in FOL. It is a predicate that is applied to a set of terms (again, terms consist of constants and variables.)

$$AtomicSetence = Predicate(term_1, term_2, ..., term_n)$$

where $term_i$ can be variable or constant.

Here are some examples:

Brother(John, Richard)

>(Length(LeftLegOf (Richard)), Length(LeftLegOf (John)))

Complicated sentences can be constructed from many atomic sentences using connectives ($\vee, \wedge, \neg, \rightarrow, \leftrightarrow$). Here are some examples

$$\text{Owns(John, Car1)} \wedge \text{Owns(John, Car2)}$$

$$\text{Sold(John, Car1, Fred)} \rightarrow \neg\text{Owns(John, Car1)}$$

3 Truth in FOL

- Sentences are **true** with respect to a *model* and an *interpretation.*
- Model contains **objects** and **relations among objects**.
- Interpretation specifies referents for

constant symbols	$\rightarrow$	objects		
predicate symbols	$\rightarrow$	relations		
function symbols	$\rightarrow$	functional relations	$\rightarrow$	objects

- An atomic sentence $predicate(term_1, \ldots, term_n)$ is <u>true</u>, if and only if the <u>objects</u> referred by $term_1, \ldots, term_n$ are in the <u>relation</u> referred by predicate.

Example 1. *Let $P(x)$ be a predicate '$x = x^2$' and the domain of x is the set of all integers. What are the truth values of the following?*

1. $P(0)$

2. $P(1)$

3. $P(2)$

Example 2. *Let $Q(x, y)$ be a predicate '$x + y = x - y$'. The domain of x, y is the set of all integers. What are the truth values of the following?*

1. $Q(0, 0)$

2. $Q(1, 1)$

3. $Q(2, 0)$

4 Qualifiers in FOL

4.1 Universal Quantification

The **universal** quantifier ($\forall$) is used to describe situation/things that are true about all objects. For example, 'Every one studying at SIIT is smart.' is represented by

$$\forall x \ \Big(\mathrm{StudyAt}(x, \mathrm{SIIT}) \rightarrow \mathrm{Smart}(x)\Big)$$

$\forall x \ P$ is true in a model m if and only if P is true with x being **each** possible object in the model. Roughly speaking, equivalent to the conjunction of instantiations of P

$$\begin{aligned} & \Big(\mathrm{StudyAt}(\mathrm{John}, \mathrm{SIIT}) \rightarrow \ \mathrm{Smart}(\mathrm{John})\Big) \\ \wedge \ & \Big(\ \mathrm{StudyAt}(\mathrm{Richard}, \mathrm{SIIT}) \rightarrow \ \mathrm{Smart}(\mathrm{Richard})\Big) \\ \wedge \ & \Big(\mathrm{StudyAt}(\mathrm{SIIT}, \mathrm{SIIT}) \rightarrow \mathrm{Smart}(\mathrm{SIIT})\Big) \\ \wedge \ & \ldots \end{aligned}$$

Typically, $\rightarrow$ is the main connective with $\forall$.

Common mistake: using $\wedge$ as the connective with $\forall$:

$$\forall x\ \Big(\mathrm{StudyAt}(x, \mathrm{SIIT}) \wedge \mathrm{Smart}(x)\Big)$$

It means 'Everyone is studying at SIIT and everyone is smart.'

4.2 Existential Quantification

The **existential** quantifier ($\exists$) is used to state properties of **some** objects without naming it. For example, 'Someone studying at SIIT is smart.'

$$\exists x\ \Big(\mathrm{StudyAt}(x, \mathrm{SIIT}) \wedge \mathrm{Smart}(x)\Big)$$

$\exists x\ P$ is true in a model m if and only if P is true with x being some possible object in the model. Roughly speaking, equivalent to the disjunction of instantiations of P

$$\begin{aligned}
& \Big(\mathrm{StudyAt}(\mathrm{John}, \mathrm{SIIT}) \wedge \mathrm{Smart}(\mathrm{John})\Big) \\
\vee\ & \Big(\mathrm{StudyAt}(\mathrm{Richard}, \mathrm{SIIT}) \wedge \mathrm{Smart}(\mathrm{Richard})\Big) \\
\vee\ & \Big(\mathrm{StudyAt}(\mathrm{SIIT}, \mathrm{SIIT}) \wedge \mathrm{Smart}(\mathrm{SIIT})\Big) \\
\vee\ & \ldots
\end{aligned}$$

Typically, $\wedge$ is the main connective with $\exists$.

Common mistake: using $\rightarrow$ as the connective with $\exists$:

$$\exists x \Big(\text{StudyAt}(x, \text{SIIT}) \rightarrow \text{Smart}(x)\Big)$$

which is equivalent to

$$\exists x\ (\neg\text{StudyAt}(x, \text{SIIT}) \vee \text{Smart}(x))$$

This means that there exists someone who is smart or does not study at SIIT. It is true if there is anyone who is not at SIIT.

4.3 Negation of quantifiers

De Morgan Rule

$$\begin{aligned}
\forall x\ (\neg P) &\equiv \neg(\exists x\ P) \\
\neg\forall x\ (\neg P) &\equiv \exists x\ (P) \\
\forall x\ (P) &\equiv \neg(\exists x\ \neg P) \\
\exists x\ (P) &\equiv \neg(\forall x\ \neg P)
\end{aligned}$$

Moreover

$$\begin{aligned}
\neg(\forall x\ P) &\equiv \exists x\ (\neg P) \\
\neg(\exists x\ P) &\equiv \forall x\ (\neg P)
\end{aligned}$$

Example 3. *Let* $Master(x)$ *be a predicate 'x is a master student' and let* $ICT600(x)$ *be a predicate 'x is enrolling in ICT600'. The domain of* x *is the set of all people.*

1. *Every master students is enrolling in ICT600.*

2. *Some master student are enrolling in ICT600.*

3. *There are some non master students who are enrolling in ICT600*

4. *There are some master students who are not enrolling in ICT600*

5 Nested Quantifiers

In working with quantifications of more than one variable, it is sometimes helpful to think of them in terms of nested loops.

For example, to see whether $\forall x \forall y P(x,y)$ is true, we loop through the values for x and for each x we loop through the values for y. If we find that $P(x,y)$ is true for all values of x and y we have determined that $\forall x \forall y P(x,y)$ is true. If we find that we ever hit a value x for which we hit a value y for which $P(x,y)$ is false, then we have shown that $\forall x \forall y P(x,y)$ is false.

Statement	When is true	When is false
$\forall x \forall y \big(P(x,y)\big)$ $\forall y \forall x \big(P(x,y)\big)$	$P(x,y)$ is true for every pair of x,y	There is a pair of x,y for which $P(x,y)$ is false.
$\exists x \exists y \big(P(x,y)\big)$ $\exists y \exists x \big(P(x,y)\big)$	There is a pair of x,y for which $P(x,y)$ is true.	$P(x,y)$ is false for every pair of x,y
$\forall x \exists y \big(P(x,y)\big)$	For every x there is a y for which $P(x,y)$ is true.	There exists an x such that $P(x,y)$ is false for every y.
$\exists x \forall y \big(P(x,y)\big)$	There is an x for which $P(x,y)$ is true for every y.	For every x there is a y for which $P(x,y)$ is false.

Here are some examples:

$$\forall y \exists x \ \Big(\text{ParentOf}(x,y) \wedge \text{Female}(x)\Big)$$

This means 'everyone has a mother'.

$$\forall x \exists t \ \Big(\text{person}(x) \rightarrow \big(\text{time}(t) \wedge \text{can-fool}(x,t)\big) \ \Big)$$

This means 'You can fool all of the people some of the time.

6 Equality

Sometimes the notion of equality is needed in an FOL statement to deal with the identity relation issue.

$$term_1 = term_2$$

is true under a given interpretation if and only if $term_1$ and $term_2$ refer to the same object.

Here are some examples:

$$\exists x \exists y \Big(\text{Owns}(\text{Mickey}, x) \wedge \text{Dog}(x) \wedge \text{Owns}(\text{Mickey}, y) \wedge \text{Dog}(y) \wedge \neg(x = y)\Big)$$

This means Mickey owns (at least) two dogs. Inequality is needed to insure that x and y are distinct.

$$\forall x \exists y \Big(\text{ married}(x, y) \wedge \forall z \big(\text{married}(x, z) \rightarrow (y = z)\big)\Big)$$

This means that everyone is married to exactly one person. Second conjunction is needed to guarantee there is only one unique spouse.

Example 4. *Let $L(x, y)$ be the predicate 'x loves y', where the domain for x and y is the set of all people in the world. Express the following in FOL.*

1. *Everybody loves Kitty.*

2. *Everybody loves somebody.*

3. *There is somebody whom everybody loves.*

4. *Everyone has someone who loves them.*

5. *Everyone loves himself or herself.*

6. *There is someone who loves no one besides himself or herself.*

7. *There is exactly one person whom everybody loves.*

8. *Kitty loves exactly two people.*

9. *At least one people don't love Kitty.*

10. *Nobody loves everybody.*

Example 5. *Let Lent(x, y) be a predecate 'x lent some money to y'. The domain of x, y is a set of all creatures in the world. Express the following FOL back in plain English.*

1. $\textit{Lent(Piglet, Pooh)}$

2. $\neg\Big(\forall x \; \textit{Lent(Pooh, x)}\Big)$

3. $\exists x \exists y \; \Big(\textit{Lent(x, Piglet)} \wedge \textit{Lent(y, Piglet)} \wedge \neg(x = y)\Big)$

4. $\forall x \forall y \; \Big(\big(\textit{Lent(x, Piglet)} \wedge \textit{Lent(y, Piglet)}\big) \rightarrow (x = y)\Big)$

5. $\exists x \; \Big(\textit{Lent(x, Piglet)} \wedge \forall y \; \big(\textit{Lent(y, Piglet)} \rightarrow (x = y)\big)\Big)$

References

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[2] Kari, Lila *Predicate Calculus (first order logic)*, Lecture notes.

[3] Mooney, Raymond J. *First-Order Logic (First-Order Predicate Calculus)*, Lecture notes.