ICT600 Quiz:
Mathematical Proofs

Instruction: Read each question carefully. Write **all** your works in the space provided. You won't get full credits even when your answer is right without your works all written down. You may use the back of the paper to continue your work. If you do so, write "continued on next page" or so to indicate that that is not the end of your solution. **This is NOT a group work.** You have to do it yourself.

Name:... **ID:**..................................

.-**-..-**-..-**-..-**-..-**-..-**-..-**-..-**-.∼ ♡ Have Fun! ♡ ∽.-**-..-**-..-**-..-**-..-**-..-**-..-**-..-**-.

Prove the following statements. You may use any techniques we learned in class.

1. For every integer n, the integer $n^2 + n$ is even. [Hint: take cases; n odd and n even]

Proof : Direct proof

n odd : Then there exists $k \in \mathbb{Z}$ such that $n = 2k+1$

$$n^2+n = (2k+1)^2 + (2k+1) = 4k^2+4k+1+2k+1$$
$$= 4k^2+6k+2 = 2(2k^2+3k+1) \quad \text{even}$$

n even : Then there exists $k \in \mathbb{Z}$ such that $n=2k$

$$n^2+n = (2k)^2 + 2k = 4k^2+2k = 2(2k^2+k) \quad \text{even}$$

Thus n^2+n is even for all $n \in \mathbb{Z}$.

2. Let x be any integer, if $\underbrace{x^2 - 6x + 5 \text{ is even}}_{P}$, then $\underbrace{x \text{ is odd}}_{Q}$. $P \longrightarrow Q$

Proof : Contrapositive proof. We assume $\neg Q$: x is even. We need to prove $\neg P$.

Suppose x is an even integer, then $x = 2k$ for some $k \in \mathbb{Z}$

$$x^2-6x+5 = (2k)^2 - 6(2k)+5$$
$$= 4k^2-12k+5$$
$$= 4k^2-12k+4+1$$
$$= 2(2k^2+6k+2)+1 \quad \text{odd}$$

Hence $\neg P$ is true.

Therefore by contrapositive proof, if x^2+6x+5 is even then x is odd.

3. Let $A = \{x \in \mathbb{R} | -3 < x < 2\}$ and let $B = \{x \in \mathbb{R} | x^2 + x - 6 < 0\}$. Prove that $A = B$.

Proof: $(\Rightarrow)$ To prove $A \subset B$

Suppose $x \in A \Rightarrow -3 < x < 2$

$\Rightarrow (x+3) > 0$ and $(x-2) < 0 \Rightarrow (x+3)(x-2) < 0$

$\Rightarrow x^2 + x - 6 < 0 \Rightarrow x \in B \Rightarrow A \subset B$

$(\Leftarrow)$ To prove $B \subset A$.

Suppose $x \in B \Rightarrow x^2 + x - 6 < 0 \Rightarrow (x+3)(x-2) < 0$

case 1 $\quad x+3 > 0$ and $x - 2 < 0 \Rightarrow x > -3$ and $x < 2$

$\Rightarrow -3 < x < 2$

case 2 $\quad x + 3 < 0$ and $x - 2 > 0 \Rightarrow \underbrace{x < -3 \text{ and } x > 2}_{\text{impossible}}$

$\Rightarrow -3 < x < 2 \Rightarrow x \in A \Rightarrow B \subset A$

Therefore $A = B$

4. Suppose that a graph G has n edges. Then the sum of the degrees of the vertices is $2n$.

Proof: By Induction on n edges

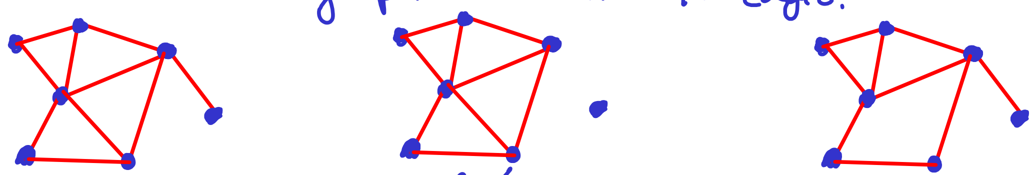
✓ Base Case $n = 1$. A graph with 1 edge has two vertices.

Degrees of each vertex is 1. Thus the sum of the vertices is $1 + 1 = 2 = 2(1)$

Inductive step: Assume that if a graph has k edges, then the sum of the degrees of the vertices is $2k$. } IH

We need to prove case $k+1$ edges.

Suppose a graph G has $k+1$ edges. Remove an edge from G. Then we have a graph G' with k edges.



Then by IH the sum of the degrees of the vertices is $2k$.
We then put the deleted edge back to G' to obtain G. This edge will connect to 2 vertices. Thus the sum of the degrees of G is $2k + 2 = 2(k+1)$.

☒