ICT600 Computational Mathematics
Homework 3

Disclaimer This is my attempt to help you learn the content of this course. There might be mistakes or typos. Do not use this material as your main study for the course.

Homework Assignment

For question (1) - (3): Two distinct dice are rolled:
Note: We can write the outcomes of this experiment as $(first, second)$, where *first* represents the outcome of the first die and *second* represents the outcome of the second die. There are 6 possible outcomes for the first die and same for the second die. So the possible out come is $6 \cdot 6 = 36$.

(1) What is the probability that at least one 4 appears?
Solution: Let

$A =$ 4 appears on the first die
$B =$ appears on the second die

Then,

$p(A) = 6/36$
$p(B) = 6/36$
$p(A \wedge B) = 1/36$

Therefore,

$$P(\textit{at least 4 appears}) = P(A \vee B) = P(A) + P(B) - P(A \wedge B)$$
$$= \frac{6}{36} + \frac{6}{36} - \frac{1}{36} = \frac{11}{36}$$

(2) Calculate the probability that the sum is 7, given that the first die is even number?

$A =$ first die is even
$B =$ sum is 7

So $A \wedge B$ are the events (2,5),(4,3), (6,1), and thus $p(A \wedge B) = 3/36$. Also, for the event A, the first die can be 2, 4, 6 and the second die can be anything. So $p(A) = (3 \cdot 6)/36$. Therefore

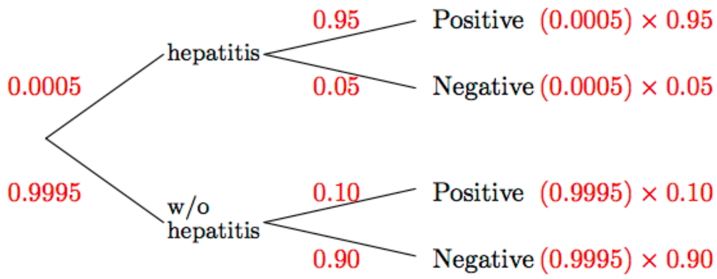
$$P(B|A) = \frac{p(A \wedge B)}{p(A)} = \frac{3/36}{(3 \cdot 6)/36} = \frac{1}{6}$$

(3) Are the events (the sum is 7) and the (first die is even) independent?
Solution: First, we need to find $P(B)$: Roll two dice, the events that sum is 7 are (1,6), (2,5), (3,4), (4,2), (5,2), (6,1). Thus $P(B) = 6/36 = 1/6$. Hence,

$$P(B|A) = \frac{1}{6} = P(B)$$

Therefore, the events (the sum is 7) and the (first die is even) are independent.

For problem (4) - (6): Suppose that the reliability of a test for hepatitis is specified as follows: Of people with hepatitis, 95% have a positive reaction and 5% have a negative reaction; of people free of hepatitis, 90% have a negative reaction and 10% have a positive reaction. From a large population of which 0.05% of the people have hepatitis, a person is selected at random and giving a test.
(4) Draw a tree diagram representing the reliability of this test for hepatitis.
Solution



(5) What is the probability that a person chosen at random will get the negative result from this test?
Solution

$$p(\text{negative}) = 0.0005 \times 0.05 + (0.9995) \times 0.90$$

(6) If the test is positive, what is the probability that the person actually has hepatitis?
Solution

$$\begin{aligned} p(\text{hepatitis}|\text{positive}) &= \frac{p(\text{hepatitis} \wedge \text{positive})}{p(\text{positive})} \\ &= \frac{(0.0005) \times 0.95}{(0.0005) \times 0.95 + (0.9995) \times 0.10} \end{aligned}$$

(7) The New York Times of January 24, 1997, discusses the recommendation of a special panel concerning mammogram for women in their 40s. About 2% of women aged 40 to 49 years old develop breast cancer in their 40s. But the mammogram used for the woman in that age group has a high rate of false positive and false negatives; the false positive rate is 0.3 and the false negative rate is 0.25. If a woman in her 40s has a positive mammogram, what is the probability that she actually has breast cancer?
Solution This problem could be solved either by Bayes' Theorem or Tree Diagram.

Bayes'Theorem We want to find $p(\text{have breast cancer } |\text{tested positive})$. So we can let

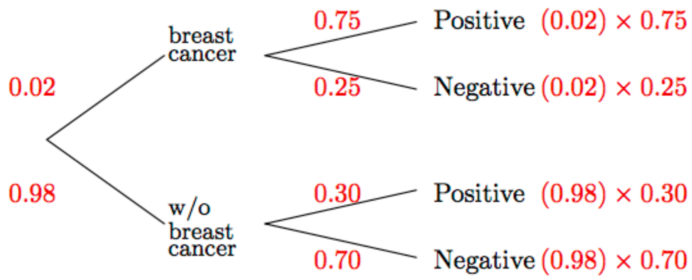
$A =$ tested positive
$B_1 =$ have breast cancer
$B_2 =$ do not have breast cancer

$p(B_1) = 0.02, p(B_2) = 0.98,$
$p(A|B_1) = 1 - (\text{false negative}) = 0.75,$
$p(A|B_2) = (\text{false positive}) = 0.3$

Thus

$$p(B_1|A) = \frac{(0.02)(0.75)}{(0.02)(0.75) + (0.98)(0.3)}$$

Tree Diagram The tree diagram of the mammograms is



Thus,

$$\begin{aligned} p(\text{have breast cancer}|\text{ positive}) &= \frac{p(\text{have breast cancer } \wedge \text{ positive})}{p(\text{positive})} \\ &= \frac{(0.02) \times 0.75}{(0.02) \times 0.75 + (0.98) \times 0.30} \end{aligned}$$

For problem (8) - (10): Metastatic cancer is a possible cause of brain tumors and is also an explanation for increased total serum calcium. In turn, either of these could explain a patient falling into a coma. Severe headache is also associated with brain tumors. For simplicity, let

$$\begin{aligned} M &= \text{Metastatic cancer} \\ B &= \text{Brain tumors} \\ S &= \text{Increased total serum calcium} \\ C &= \text{Comas} \\ H &= \text{Severe Headaches} \end{aligned}$$

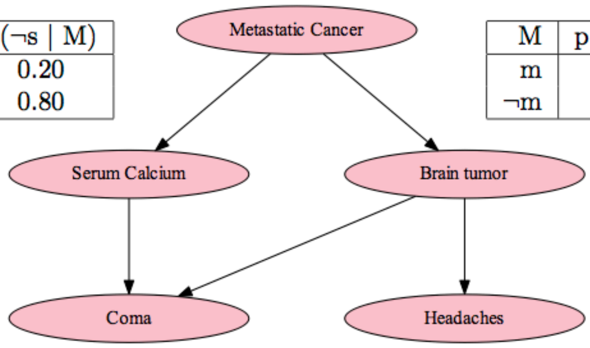
Suppose that:
$p(m) = 0.2$,
$p(s|m) = 0.8$, $p(s|\neg m) = 0.2$, $p(b|m) = 0.2$ $p(b|\neg m) = 0.05$
$p(c|s,b) = 0.8$ $p(c|s,\neg b) = 0.8$ $p(c|\neg s, b) = 0.8$ $p(c|\neg s, \neg b) = 0.05$
$p(h|b) = 0.8$ $p(h|\neg b) = 0.6$

(8) Construct a Bayesian network to represent this scenario.
Solution

p(m)	p(¬m)
0.20	0.80

M	p(s | M)	p(¬s | M)
m	0.80	0.20
¬m	0.20	0.80

M	p(b | M)	p(¬b | M)
m	0.20	0.80
¬m	0.05	0.95



S	B	p(c | S, B)	p(¬c | S, B)
s	b	0.80	0.20
s	¬b	0.80	0.20
¬s	b	0.80	0.20
¬s	¬b	0.05	0.95

B	p(h | B)	p(¬h | B)
b	0.80	0.20
¬b	0.60	0.40

(9) What is a probability that a patient is diagnosed with Metastatic cancer, has brain tumors and severe headaches but has not fallen into a coma?
Solution:

$$
\begin{aligned}
p(m, b, h, \neg c) &= \sum_S p(m, b, S, h, \neg c) \\
&= \sum_S p(m)\ p(b|m)\ p(S|m)\ p(\neg c|S, b)\ p(h|b) \\
&= p(m)\ p(b|m)\ p(s|m)\ p(\neg c|s, b)\ p(h|b) + p(m)\ p(b|m)\ p(\neg s|m)\ p(\neg c|\neg s, b)\ p(h|b) \\
&= 0.00512 + 0.00128 \\
&= 0.0064
\end{aligned}
$$

(10) Given that the patient is suffering from severe headaches but has not fallen into a coma, what is the probability that the patient has Metastatic cancer?
Solution:

$$
\begin{aligned}
p(m|h, \neg c) &= \frac{p(m, \neg c, h)}{p(h, \neg c)} \\
&= \frac{\sum_{S,B} p(m, B, S, \neg c, h)}{\sum_{S,B,M} p(M, B, S, \neg c, h)} \\
&= \frac{\sum_{S,B} p(m)\ p(B|m)\ p(S|m)\ p(\neg c|S, B)\ p(h|B)}{\sum_{S,B,M} p(M)\ p(B|M)\ p(S|M)\ p(\neg c|S, B)\ p(h|B)} \\
&= \frac{0.04}{0.4112} \\
&= 0.0973
\end{aligned}
$$