ICT600 COMPUTATIONAL MATHEMATICS

Eigenvalues and Eigenvectors

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Disclaimer: this partial note is my attempt to help you learn the material in this course. Things in here might not be in the real exam and vice versa. Don't use this as your main study. There might be some typos and/or mistakes.

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1 Computing Eigenvalues

We will use a determinant to find eigenvalues of a square matrix A; the computational technique is practical only for relatively small matrices.
Notice that;

$$\begin{aligned} A\mathbf{v} &= \lambda\mathbf{v} \\ A\mathbf{v} - \lambda\mathbf{v} &= \mathbf{0} \\ A\mathbf{v} - \lambda I\mathbf{v} &= \mathbf{0} \\ (A - \lambda I)\mathbf{v} &= \mathbf{0} \end{aligned}$$

Where I is the $n \times n$ identity matrix and so $\mathbf{v}$ must be a solution of the homogeneous linear system

$$(A - \lambda I)\mathbf{v} = \mathbf{0}$$

An eigenvalue of A is thus a scalar λ for which system has a nontrivial solution $\mathbf{v}$. (Recall that an eigenvector is nonzero by definition.) We know that the system has a nontrivial precisely when the determinant of the coefficient matrix is zero– that is, if and only if

$$\det(A - \lambda I) = 0$$

If we expand the determinant above, we obtain a polynomial expression $p(\lambda)$ of degree n with coefficients involving the a_{ij}. That is

$$\det(A - \lambda I) = p(\lambda)$$

The polynomial $p(\lambda)$ is the characteristic polynomial of matrix A. The eigenvalues of A are precisely the solution of the characteristic equation $p(\lambda) = 0$.

Example 1. *Find the eigenvalues of the following matrices.*

1.

$$A = \begin{bmatrix} 3 & 2 \\ 2 & 0 \end{bmatrix}.$$

2.

$$A = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 3 & 1 \end{bmatrix}.$$

3.

$$A = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}.$$

4.

$$A = \begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix}.$$

2 Computation of Eigenvectors

Before computing Eigenvectors, we need to know couple concepts first.

2.1 Elementary row operations

Elementary row operations are

1. Interchange two rows of the matrix.
2. Multiply (or divide) all the numbers in a row by the same nonzero constant.
3. Add a constant times a row to another row and substitute the sum for one of the rows.

2.2 Reduced row echelon forms

Definition 1. A matrix is in reduced row echelon form if it satisfies the following properties.

1. Looking from the left to right along each row, either the first nonzero number is 1 (called the leading 1) or the row (to the left of the vertical line) consists only of zeros.
2. The row of zeros all line below the rows with leading 1s.
3. Every leading 1 is the only nonzero number in its column.
4. Every leading 1 is to the right of the leading 1s in the row above it.

Example 2. *Determine whether any of the matrices is in the reduced row echelon form.*

1. $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$
2. $\begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & -2 \end{bmatrix}$
3. $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \end{bmatrix}$
4. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 3 & 0 \end{bmatrix}$
5. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

2.3 Computing Eigenvector

Having found the eigenvalue, we substitute it in homogeneous system and solved to find the nontrivial solutions of the system. We are doing so by applying elementary row operations until the matrix is in the reduced row echelon form. We will obtained an infinite number of nontrivial solutions, each of which is an eigenvector corresponding to the eigenvalue λ. Here are the detailed steps:

1. Set $[A - \lambda I|0]$
2. Apply elementary row operations until the left hand-side is in the reduced row echelon form.
3. Solve for the leading variables in terms of any remaining free variables.

Example 3. *Find the eigenvectors corresponding to each eigenvalues of the following matrices.*

1.

$$A = \begin{bmatrix} 3 & 2 \\ 2 & 0 \end{bmatrix}.$$

2.

$$A = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 3 & 1 \end{bmatrix}.$$

3.

$$A = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}.$$

4.

$$A = \begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix}.$$

3 Properties of Eigenvalues and Eigenvectors

Let A be an $n \times n$ matrix.

1. If λ is an eigenvalue of A with $\mathbf{v}$ as a corresponding eigenvector, then λ^k is an eigenvalue of A^k, with $\mathbf{v}$ as a corresponding eigenvector, for any positive integer k.

2. If λ is an eigenvalue of an invertible matrix A with $\mathbf{v}$ as a corresponding eigenvector, the $\lambda \neq 0$ and $1/\lambda$ is an eigenvalue of A^{-1} with $\mathbf{v}$ as a corresponding eigenvector.

3. If λ is an eigenvalue of A, then the set E_λ consisting of the zero vector together with all eigenvectors of A for this eigenvalue λ is a subspace of $n-$space, the **eigenspace of λ.**

Example 4. *Let* $\{2; [1, 0, 1]\}$, *{-1; [0,2,-1], [-1,1,-3]} are eigenvalues and eigenvectors of a matrix A. Which of the following statements are true? Explain your reasons.*

1. Eigenvalues and Eigenvectors of A^3 are $\{8; [1, 0, 1]\}$, *{-1; [0,8,-1], [-1,1,-27]}*

2. Eigenvalues and Eigenvectors of A^5 are $\{32; [1, 0, 1]\}$, *{-1; [0,2,-1], [-1,1,-3]}*

3. Eigenvalues and Eigenvectors of A^{-1} are $\{1/2; [1, 0, 1]\}$, *{-1; [0,1/2,-1], [-1,1,-1/3]}*

4 Diagonalization

Definition 2. An $n \times n$ matrix A is diagonalizable if there exists an invertible matrix P such that $P^{-1}AP = D$, a diagonal matrix. The matrix D is said to diagonalize the matrix A.

Note that

1. An $n \times n$ matrix A is diagonalizable if and only if the n-vector space (vector space with dimension n) has a basis consisting of eigenvectors of A.

2. If an $n \times n$ matrix A has n eigenvectors and eigenvalues and all n eigenvectors are independent, then D is an invertible matrix and

$$A^k = PD^kP^{-1}$$

3. If $\mathbf{v}_1, ..., \mathbf{v}_n$ are eigenvectors of A corresponding to distinct eigenvalues $\lambda_1, ..., \lambda_n$ respectively, the set $\{\mathbf{v}_1, ..., \mathbf{v}_n\}$ is linearly independent and A is diagonalizable.

4. It is not always essential that a matrix have distinct eigenvalues in order to be diagonalizable. As long as the $n \times n$ matrix A has n independent eigenvectors to form the column vectors of an invertible D.

Example 5. *If possible, diagonalize the following matrices*

1.

$$A = \begin{bmatrix} 3 & 2 \\ 2 & 0 \end{bmatrix}.$$

2.

$$A = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 3 & 1 \end{bmatrix}.$$

3.

$$A = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}.$$

4.

$$A = \begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix}.$$