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# **chombo-discharge Documentation**

**Jan 19, 2023**



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**Important:** This is an beta release. Development is still in progress, and various bugs may be present. Minor changes to the user interface can still occur.

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**Important:** chombo-discharge is a modular and scalable research code for Cartesian two- and three-dimensional simulations of low-temperature plasmas in complex geometries. The code is hosted at [GitHub](#) together with the source files for this documentation.

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chombo-discharge features include:

- Fully written in C++.
- Support for complex geometries.
- Scalar advection-diffusion-reaction processes.
- Electrostatics with support for electrodes and dielectrics.
- Radiative transport as a diffusion or Monte Carlo process.
- Particle-mesh operations (like Particle-In-Cell)
- Parallel I/O with HDF5.
- Dual-grid simulations with individual load balancing of fluid and particles.
- Various multi-physics interfaces that use the above solvers.
- Various time integration schemes.

Numerical solvers are designed to run either on their own, or as a part of a larger application.

For scalability, chombo-discharge is built on top of [Chombo 3](#), and therefore additionally features

- Cut-cell representation of multi-material geometries.
- Patch based adaptive mesh refinement.
- Weak and strong scalability to thousands of computer cores.

Our goal is that users will be able to use chombo-discharge without modifying the underlying solvers. There are interfaces for describing e.g. the plasma physics, boundary conditions, mesh refinement, etc. As chombo-discharge evolves, so will these interfaces. We aim for (but cannot guarantee) backward compatibility such that existing chombo-discharge models can be run on future versions of chombo-discharge.



## INTRODUCTION

### 1.1 Using this documentation

This documentation is the user documentation for `chombo-discharge`. It includes an explanation of the data structures, algorithms, and code design. It is built as a part of the continuous integration (CI) at GitHub. The HTML, PDF, and doxygen documentation obtained from [GitHub](#) is automatically kept up-to-date with the latest version of `chombo-discharge`.

#### 1.1.1 Doxygen documentation

A separate Doxygen documentation of the `chombo-discharge` code is found at <https://chombo-discharge.github.io/chombo-discharge/doxygen/html/index.html>

### 1.2 Overview

#### 1.2.1 History

`chombo-discharge` is aimed at solving discharge problems. It was originally developed at SINTEF Energy Research between 2015-2018, and aimed at simulating discharges in high-voltage engineering. Further development was started in 2021, where much of the code was redesigned for improved modularity and performance.

#### 1.2.2 Key functionalities

`chombo-discharge` uses a Cartesian embedded boundary (EB) grid formulation and adaptive mesh refinement (AMR) formalism where the grids frequently change and are adapted to the solution as simulations progress. Key functionalities are provided in [Table 1.2.1](#).

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**Important:** `chombo-discharge` is **not** a black-box model for discharge applications. It is an evolving research code with occasionally expanded capabilities, API changes, and performance improvements. Although problems can be set up through our Python tools, users should nonetheless be willing to take time to understand how the code works. In particular, developers should invest some effort in understanding the data structures and Chombo basics (see [Chombo-3 basics](#)).

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Table 1.2.1: Key capabilities.

Capabilities	Supported?
Grids	<b>Fundamentally Cartesian.</b>
Parallelized?	<b>Yes</b> , using flat MPI.
Load balancing?	<b>Yes</b> , with support for individual particle and fluid load balancing.
Complex geometries?	<b>Yes</b> , using embedded boundaries (i.e., cut-cells).
Adaptive mesh refinement?	<b>Yes</b> , using patch-based refinement.
Subcycling in time?	<b>No</b> , only global time step methods.
Computational particles?	<b>Yes</b> .
Linear solvers?	<b>Yes</b> , using geometric multigrid in complex geometries.
Time discretizations?	<b>Mostly explicit</b> .
Parallel IO?	<b>Yes</b> , using HDF5.
Checkpoint-restart?	<b>Yes</b> , for both fluid and particles.

An early version of chombo-discharge used sub-cycling in time, but this has now been replaced with global time stepping methods. That is, all the AMR levels are advanced using the same time step. chombo-discharge has also incorporated many changes to the EB functionality supplied by Chombo. This includes much faster grid generation, support for polygon surfaces, and many performance optimizations (in particular to the EB formulation).

chombo-discharge supports both fluid and particle methods, and can use multiply parallel distributed grids (see *Realm*) for individually load balancing particle and fluid kernels. Although many abstractions are in place so that user can describe a new set of physics, or write entirely new solvers into chombo-discharge and still use the embedded boundary formalism, chombo-discharge also provides several physics modules for describing various types of problems.

## 1.3 Installation

### 1.3.1 Obtaining chombo-discharge

chombo-discharge can be freely obtained from <https://github.com/chombo-discharge/chombo-discharge>. The following packages are *required*:

- Chombo, which is supplied with chombo-discharge.
- The C++ JSON file parser <https://github.com/nlohmann/json>.
- The EBGeometry package, see <https://github.com/rmrsk/EBGeometry>.
- LAPACK and BLAS

The Chombo, nlohmann/json, and EBGeometry dependencies are automatically handled by chombo-discharge through git submodules.

**Warning:** Our version of Chombo is hosted at <https://github.com/chombo-discharge/Chombo-3.3.git>. chombo-discharge has made substantial changes to the embedded boundary generation in Chombo. It will not compile with other versions of Chombo than the one above.

Optional packages are

- A serial or parallel version of HDF5, which is used for writing plot and checkpoint files.
- An MPI installation, which is used for parallelization.
- <https://visit-dav.github.io/visit-website/> visualization, which used for visualization.

### 1.3.2 Organization

The chombo-discharge source files are organized as follows:

Table 1.3.1: Code organization.

Folder	Explanation
Source	Source files for the AMR core, solvers, and various utilities.
Physics	Various implementations that can run the chombo-discharge source code.
Geometries	Various geometries.
Submodules	Git submodule dependencies.
Exec	Various executable applications.

### 1.3.3 Cloning chombo-discharge

To obtain chombo-discharge, the simplest approach is to clone it and recursively fetch the submodules:

```
git clone --recursive git@github.com:chombo-discharge/chombo-discharge.git
```

### 1.3.4 Setting up the environment

When compiling chombo-discharge, makefiles must be able to find both chombo-discharge and Chombo. In our makefiles the paths to these are supplied through the environment variables

- DISCHARGE\_HOME, pointing to the chombo-discharge root directory.
- CHOMBO\_HOME, pointing to your Chombo library

Note that DISCHARGE\_HOME must point to the root folder in the chombo-discharge source code, while CHOMBO\_HOME must point to the lib/ folder in your Chombo root directory. When cloning with submodules, both Chombo and nlohmann/json will be placed in the Submodules folder in \$DISCHARGE\_HOME.

---

**Note:** To clone chombo-discharge directly to \$DISCHARGE\_HOME, set the environment variables and clone (using --recursive to fetch submodules):

```
export DISCHARGE_HOME=/home/foo/chombo-discharge
export CHOMBO_HOME=$DISCHARGE_HOME/Submodules/Chombo-3.3/lib

git clone --recursive git@github.com:chombo-discharge/chombo-discharge.git ${DISCHARGE_HOME}
```

---

chombo-discharge is built using a configuration file supplied to Chombo. This file must reside in \$CHOMBO\_HOME/mk. Some standard configuration files are supplied with chombo-discharge, and reside in \$DISCHARGE\_HOME/Lib/Local. These files may or may not work right off the bat.

### 1.3.5 Test build

For a quick compilation test the user can use the GNU configuration file supplied with chombo-discharge by following the steps below.

1. Copy the GNU configuration file

```
cp $DISCHARGE_HOME/Lib/Local/Make.defs.GNU $CHOMBO_HOME/mk/Make.defs.local
```

2. If you do not have the GNU compiler suite, install it by

```
sudo apt install csh gfortran g++ libblas-dev liblapack-dev
```

This will install

- LAPACK and BLAS
- GNU compilers for Fortran and C++

3. Compile chombo-discharge

```
cd $DISCHARGE_HOME  
make -s -j4
```

This will compile the chombo-discharge source code in serial and without HDF5 (using four cores for the compilation). If successful, chombo-discharge libraries will appear in \$DISCHARGE\_HOME/Lib.

### 1.3.6 Advanced configuration

chombo-discharge is compiled using GNU Make, following the Chombo. Compilers, libraries, and configuration options are defined in a file `Make.defs.local` which resides in `$CHOMBO_HOME/mk`. Users need to supply this file in order to compile chombo-discharge. Typically, these steps include

- Specifying Fortran and C++ compilers
- Specifying configurations. E.g., serial or parallel builds, and compiler flags.
- Specifying library paths (in particular for HDF5).

#### Main settings

The main variables that the user needs to set are

- `DIM = 2/3` The dimensionality (must be 2 or 3).
- `DEBUG = TRUE/FALSE` This enables or disables debugging flags and code checks/assertions.
- `OPT = FALSE/TRUE/HIGH`. Setting `OPT=TRUE/HIGH` enables optimization flags that will speed up Chombo and chombo-discharge.
- `PRECISION = DOUBLE` Currently, chombo-discharge has not been wetted with single precision. Many algorithms (like conjugate gradient) depend on the use of double precision.
- `CXX = <C++ compiler>`
- `FC = <Fortran compiler>`
- `MPI = TRUE/FALSE` This enables/disables MPI.
- `MPICXX = <MPI compiler>`
- `CXXSTD=14` Sets the C++ standard - we are currently at C++14.

- USE\_EB=TRUE Configures Chombo with embedded boundary functionality. This is a requirement.
- USE\_MF=TRUE Configures Chombo with multifluid functionality. This is a requirement.
- USE\_HDF5 = TRUE/FALSE This enables and disables HDF5 code.

## HDF5

If using HDF5, one must also set the following flags:

- HDFINCFLAGS = -I<path to hdf5-serial>/include (for serial HDF5).
- HDFLIBFLAGS = -L<path to hdf5-serial>/lib -lhdf5 -lz (for serial HDF5)
- HDFMPIINCFLAGS = -I<path to hdf5-parallel>/include (for parallel HDF5)
- HDFMPILIBFLAGS = -L<path to hdf5-parallel>/lib -lhdf5 -lz (for parallel HDF5).

## Compiler flags

Compiler flags are set through

- cxxoptflags = <C++ compiler flags
- foptflags = <Fortran compiler flags
- syslibflags = <system library flags>

Note that LAPACK and BLAS are requirements in chombo-discharge. Linking to these can often be done using

- syslibflag = -llapack -lblas (for GNU compilers)
- syslibflag = -mkl=sequential (for Intel compilers)

## Pre-defined configuration files

Some commonly used configuration files are found in \$DISCHARGE\_HOME/Lib/Local. chombo-discharge can be compiled in serial or with MPI, and with or without HDF5. The user need to configure the Chombo makefile to ensure that the chombo-discharge is properly configured. Below, we include brief instructions for compilation on a Linux workstation and for a cluster.

## GNU configuration for workstations

Here, we provide a more complete installation example using GNU compilers for a workstation. These steps are intended for users that do not have MPI or HDF5 installed. If you already have installed MPI and/or HDF5, the steps below might require modifications.

1. Ensure that \$DISCHARGE\_HOME and \$CHOMBO\_HOME point to the correct locations:

```
echo $DISCHARGE_HOME  
echo $CHOMBO_HOME
```

2. Install GNU compiler dependencies by

```
sudo apt install csh gfortran g++ libblas-dev liblapack-dev
```

This will install

- LAPACK and BLAS

- GNU compilers for Fortran and C++
3. To also install OpenMPI and HDF5:

```
sudo apt install libhdf5-dev libhdf5-openmpi-dev openmpi-bin
```

This will install

- OpenMPI
- HDF5, both serial and parallel.

Both serial and parallel HDF5 will be installed, and these are *usually* found in folders

- /usr/lib/x86\_64-linux-gnu/hdf5/serial/ for serial HDF5
- /usr/lib/x86\_64-linux-gnu/hdf5/openmpi/ for parallel HDF5 (using OpenMPI).

Before proceeding further, the user need to locate the HDF5 libraries (if building with HDF5).

4. After installing the dependencies, copy the desired configuration file to \$CHOMBO\_HOME/mk:

- **Serial build without HDF5:**

```
cp $DISCHARGE_HOME/Lib/Local/Make.defs.GNU $CHOMBO_HOME/mk/Make.defs.local
```

- **Serial build with HDF5:**

```
cp $DISCHARGE_HOME/Lib/Local/Make.defs.HDF5.GNU $CHOMBO_HOME/mk/Make.defs.local
```

- **MPI build without HDF5:**

```
cp $DISCHARGE_HOME/Lib/Local/Make.defs.MPI.GNU $CHOMBO_HOME/mk/Make.defs.local
```

- **MPI build with HDF5:**

```
cp $DISCHARGE_HOME/Lib/Local/Make.defs.MPI.HDF5.GNU $CHOMBO_HOME/mk/Make.defs.local
```

5. Compile the chombo-discharge

```
cd $DISCHARGE_HOME  
make -s -j4 discharge-lib
```

This will compile the chombo-discharge source code using the configuration settings set by the user. To compile chombo-discharge in 3D, do make -s -j4 DIM=3 discharge-lib. If successful, chombo-discharge libraries will appear in \$DISCARGE\_HOME/Lib.

### Configuration on clusters

To configure chombo-discharge for execution on a cluster, use one of the makefiles supplied in \$DISCHARGE\_HOME/Lib/Local if it exists for your computer. Alternatively, copy \$DISCHARGE\_HOME/Lib/Local/Make.defs.local template to \$CHOMBO\_HOME/mk/Make.defs.local and set the compilers, optimization flags, and paths to HDF5 library.

On clusters, MPI and HDF5 are usually already installed, but must usually be loaded (e.g. as modules) before compilation.

### 1.3.7 Running an example application

In chombo-discharge, applications are set up so that they use the chombo-discharge source code and one chombo-discharge physics module. To run one of the applications that use a particular chombo-discharge physics module, we will run a simulation of a positive streamer (in air).

The application code is located in `$DISCHARGE_HOME/Exec/Examples/CdrPlasma/DeterministicAir` and it uses the convection-diffusion-reaction plasma module (located in `$DISCHARGE_HOME/Physics/CdrPlasma`).

First, compile the application by

```
cd $DISCHARGE_HOME/Exec/Examples/CdrPlasma/DeterministicAir
make -s -j4 DIM=2 program
```

This will provide an executable named `program2d.<bunch_of_options>.ex`. If one compiles for 3D, i.e. `DIM=3`, the executable will be named `program3d.<bunch_of_options>.ex`.

To run the application do:

#### Serial build

```
./program2d.<bunch_of_options>.ex positive2d.inputs
```

#### Parallel build

```
mpirun -np 8 program2d.<bunch_of_options>.ex positive2d.inputs
```

If the user also compiled with HDF5, plot files will appear in the subfolder `plt`.

### 1.3.8 Troubleshooting

If the prerequisites are in place, compilation of chombo-discharge is usually straightforward. However, due to dependencies on Chombo and HDF5, compilation can sometimes be an issue. Our experience is that if Chombo compiles, so does chombo-discharge.

If experiencing issues, try cleaning chombo-discharge by

```
cd $DISCHARGE_HOME
make pristine
```

---

**Note:** Do not hesitate to contact us at [GitHub](#) regarding installation issues.

---

#### Recommended configurations

#### Production runs

For production runs, we generally recommend that the user compiles with `DEBUG=FALSE` and `OPT=HIGH`. These settings can be set directly in `Make.defs.local`. Alternatively, they can be included directly on the command line when compiling problems.

### Debugging

If you believe that there might be a bug in the code, one can compile with DEBUG=TRUE and OPT=TRUE. This will turn on some assertions throughout Chombo and chombo-discharge.

### Common problems

- Missing library paths:

On some installations the linker can not find the HDF5 library. To troubleshoot, make sure that the environment variable LD\_LIBRARY\_PATH can find the HDF5 libraries:

```
echo $LD_LIBRARY_PATH
```

If the path is not included, it can be defined by:

```
export LD_LIBRARY_PATH=$LD_LIBRARY_PATH:<path_to_hdf5_installation>/lib
```

## 1.4 Visualization

chombo-discharge output files are always written to HDF5. The plot files will reside in the plt subfolder where the application was run.

Currently, we have only used [VisIt](#) for visualizing the plot files. Learning how to use VisIt is not a part of this documentation; there are great tutorials on the [VisIt website](#).

## 1.5 Controlling chombo-discharge

In this chapter we give a brief overview of how to run a chombo-discharge simulation and control its behavior through input scripts or command line options.

### 1.5.1 Compiling and running

To run simulations, the user must first compile his application. Once the application has been setup, the user can compile by

```
make -s -j 32 DIM=N <application_name>
```

where  $N$  may be 2 or 3, and  $\langle\text{application\_name}\rangle$  is the name of the file that holds the `main()` function. This will compile an executable whose name depends on your application name and compiler settings. Please refer to the Chombo manual for explanation of the executable name. You may, of course, rename your application.

## Compilation options

chombo-discharge can compile with various code guards enabled, to spot bugs or potential errors in the code. To compile with these guards turned on, compile with DEBUG=TRUE, e.g. `make -s -j32 DIM=2 DEBUG=TRUE <application_name>`.

To compile for production runs, chombo-discharge should generally speaking be compiled with DEBUG=FALSE and OPT=HIGH, for example

```
make -s -j32 DIM=2 OPT=HIGH DEBUG=FALSE <application_name>
```

Recall also that default settings for the dimension (DIM), optimization level (OPT), and debug mode (DEBUG) can be set in `Make.defs.local`, see [Installation](#).

## 1.5.2 Running applications

### Serial

Next, if the application was compiled for serial execution one runs it with:

```
./<application_executable> <input_file>
```

where `<input_file>` is your input file.

### Parallel

If the executable was compiled with MPI, one executes with e.g. `mpirun` (or one of its aliases):

```
mpirun -np 32 <application_executable> <input_file>
```

On clusters, this is a little bit different and usually requires passing the above command through a batch system.

## 1.5.3 Simulation inputs

chombo-discharge simulations take their input from a single simulation input file (possibly appended with overriding options on the command line). Simulations may consist of several hundred possible switches for altering the behavior of a simulation, and all physics models in chombo-discharge are therefore equipped with Python setup tools that collect all such options into a single file when setting up a new application. Generally, these input parameters are fetched from the options file of component that is used in a simulation. Simulation options usually consist of a prefix, a suffix, and a configuration value. For example, the configuration options that adjusts the number of time steps that will be run in a simulation is

```
Driver.max_steps = 100
```

Likewise, for controlling how often plot are written:

```
Driver.plot_interval = 5
```

You may also pass input parameters through the command line. For example, running

```
mpirun -np 32 <application_executable> <input_file> Driver.max_steps=10
```

will set the `Driver.max_steps` parameter to 10. Command-line parameters override definitions in the input file. Moreover, parameters parsed through the command line become static parameters, i.e. they are not run-time configurable (see [Run-time configurations](#)). Also note that if you define a parameter multiple times in the input file, the last definition is canon.

### 1.5.4 Simulation outputs

Mesh data from chombo-discharge simulations is by default written to HDF5 files, and if HDF5 is disabled chombo-discharge will not write any plot or checkpoint files. In addition to plot files, MPI ranks can output information to separate files so that the simulation progress can be tracked.

chombo-discharge comes with controls for adjusting output. Through the *Driver* class the user may adjust the option `Driver.output_directory` to specify where output files will be placed. This directory is relative to the location where the application is run. If this directory does not exist, chombo-discharge will create it. It will also create the following subdirectories given in *Simulation output organization..*

Table 1.5.1: Simulation output organization.

Folder	Explanation
chk	Checkpoint files (these are used for restarting simulations from a specified time step).
crash	Plot files written if a simulation crashes.
geo	Plot files for geometries (if you run with <code>Driver.geometry_only = true</code> ).
mpi	Information about individual MPI ranks, such as computational loads or memory consumption per rank.
plt	All plot files.
regrid	Plot files written during regrids (if you run with <code>Driver.write_regrid_files</code> ).
restart	Plot files written during restarts (if you run with <code>Driver.write_regrid_files</code> ).

The reason for the output folder structure is that chombo-discharge can end up writing thousands of files per simulation and we feel that having a directory structure helps us navigate simulation data.

Fundamentally, there are only two types of HDF5 files written:

1. Plot files, containing plots of simulation data.
2. Checkpoint files, which are binary files used for restarting a simulation from a given time step.

The *Driver* class is responsible for writing output files at specified intervals, but the user is generally speaking responsible for specifying what goes into the plot files. Since not all variables are always of interest, solver classes have options like `plt_vars` that specify which output variables in the solver will be written to the output file. For example, one of our convection-diffusion-reaction solver classes have the following output options:

```
CdrGodunov.plt_vars = phi vel dco src ebflux # Plot variables. Options are 'phi', 'vel', 'dco', 'src', 'ebflux'
```

where `phi` is the state density, `vel` is the drift velocity, `dco` is the diffusion coefficient, `src` is the source term, and `ebflux` is the flux at embedded boundaries. If you only want to plot the density, then you should put `CdrGodunov.plt_vars = phi`. An empty entry like `CdrGodunov.plt_vars =` may lead to run-time errors, so if you do not want a class to provide plot data you may put `CdrGodunov.plt_vars = none`.

### 1.5.5 Controlling parallel processor verbosity

By default, Chombo will write a process output file *per MPI process* and this file will be named `pout.n` where `n` is the MPI rank. These files are written in the directory where you executed your application, and are *not* related to plot files or checkpoint files. However, chombo-discharge prints information to these files as simulations advance (for example by displaying information of the current time step, or convergence rates for multigrid solvers). While it is possible to monitor the evolution of chombo-discharge for each MPI rank, most of these files contain redundant information. To adjust the number of files that will be written, Chombo can read an environment variable `CH_OUTPUT_INTERVAL` that determines which MPI ranks write `pout.n` files. For example, if you only want the master MPI rank to write `pout.0`, you would do

```
export CH_OUTPUT_INTERVAL=999999999
```

---

**Important:** If you run simulations at high concurrencies, you *should* turn off the number of process output files since they impact the performance of the file system.

---

### 1.5.6 Restarting simulations

Restarting simulations is done in exactly the same way as running simulations, although the user must set the `Driver.restart` parameter. For example,

```
mpirun -np 32 <application_executable> <input_file> Driver.restart=10
```

will restart from step 10.

Specifying anything but an integer is an error. When a simulation is restarted, chombo-discharge will look for a checkpoint file with the `Driver.output_names` variable and the specified restart step. It will look for this file in the subfolder `/chk` relative to the execution directory.

If the restart file is not found, restarting will not work and chombo-discharge will abort. You must therefore ensure that your executable can locate this file. This also implies that you cannot change the `Driver.output_names` or `Driver.output_directory` variables during restarts, unless you also change the name of your checkpoint file and move it to a new directory.

---

**Note:** If you set `Driver.restart=0`, you will get a fresh simulation.

---

### 1.5.7 Run-time configurations

chombo-discharge reads input parameters before the simulation starts, but also during run-time. This is useful when your simulation waited 5 days in the queue on a cluster before starting, but you forgot to tweak one parameter and don't want to wait another 5 days.

Driver re-reads the simulation input parameters after every time step. The new options are parsed by the core classes `Driver`, `TimeStepper`, `AmrMesh`, and `CellTagger` through special routines `parseRuntimeOptions()`. Note that not all input configurations are suitable for run-time configuration. For example, increasing the size of the simulation domain does not make sense but changing the blocking factor, refinement criteria, or plot intervals do. To see which options are run-time configurable, see `Driver`, `AmrMesh`, or the `TimeStepper` and `CellTagger` that you use.

## 1.6 Examples

chombo-discharge comes with many examples for running various types of simulations. These are organized by application type (e.g. purely electrostatic, inception model, plasma simulation etc) in `$DISCHARGE_HOME/Exec/Examples`.

## 1.7 Code testing

To ensure the integrity of chombo-discharge, we include tests in

- `$DISCHARGE_HOME/Exec/Tests` for a functional test suite.
- `$DISCHARGE_HOME/Exec/Convergence` for a code verification tests.

### 1.7.1 Test suite

To make sure chombo-discharge compiles and runs as expected, we include a test suite that runs functional tests of many chombo-discharge components. The tests are defined in `$DISCHARGE_HOME/Exec/Tests`, and are organized by application type.

#### Running the test suite

To do a clean compile and run of all tests, navigate to `$DISCHARGE_HOME/Exec/Tests` and execute the following:

```
python3 tests.py --compile --clean --silent --no_compare --parallel -cores X
```

where `X` is the number of cores to use when compiling and running. If the tests should be run in serial, omit the `--parallel -cores X` flag.

---

**Tip:** Run the test suite with `DEBUG=TRUE` to catch assertion errors.

---

#### Advanced configuration

The following options are available for running the various tests:

- `--compile` Re-compile the test.
- `--clean` Do a clean recompilation of the test.
- `--silent` Turn off unnecessary output.
- `--benchmark` Generate benchmark files.
- `--no_exec` Compile, but do not run the test.
- `--no_compare` Do not compare with benchmark files.
- `--parallel` Use MPI or not
- `-cores <number>` Run with specified number of cores.
- `-suites <string>` Run a specific application test suite.
- `-tests <string>` Run a specific test.
- `-dim <number>` Run only the specified 2D or 3D tests.

For example, to only run the two-dimensional radiative transfer tests, without comparison with benchmark files, one would run

```
python3 tests.py --compile --clean --silent --no_compare --parallel -cores 4 -suites RadiativeTransfer -dim 2
```

## Using benchmark files

The test suite can generate benchmark files which can later be compared against new test suite output files. This is often a good idea if one wants to ensure that changes the chombo-discharge code does not unintentionally change simulations. In this case one can run the test suite and generate benchmark files *before* adding changes to chombo-discharge. Once the code development is completed, the benchmark files can later be bit-wise (using `h5diff`) compared against the results of a later test suite.

This consists of the following steps:

1. Before making changes to chombo-discharge, generate benchmark files with

```
python3 tests.py --compile --clean --silent --parallel -cores X --benchmark
```

2. Make the required changes to the chombo-discharge code.
3. Run the test suite again, and compare benchmark and output files as follows:

```
python3 tests.py --compile --clean --silent --parallel -cores X
```

When running the tests this way, the output files are bit-wise compared and a warning is issued if the files not exactly match.

## 1.7.2 Automated testing

On [GitHub](#), the test suite is integrated with GitHub actions and are automatically run when opening a pull request for review. In general, all tests must pass before a pull request can be merged. The test status can be observed either in the pull request, or at <https://github.com/chombo-discharge/chombo-discharge/actions>. The automated tests run chombo-discharge with DEBUG=TRUE and OPT=FALSE in order to catch assertion errors or other code breaks. They usually take 1-2 hours to complete.

The automated tests will clone, build, and run the chombo-discharge test suite for various configurations:

- Parallel and serial.
- With or without HDF5.
- In 2D and 3D.

The tests are run with the following compiler suites:

- GNU.
- Intel oneAPI.

## 1.7.3 Convergence testing

To ensure that the various components in chombo-discharge converge at desired truncation order, many modules are equipped with their own convergence tests. These are located in `$DISCHARGE_HOME/Exec/Convergence`. The tests are too extensive to include in continuous integration, and they must be run locally like a regular chombo-discharge application. Our approach for convergence testing is found in [Verification and validation](#).

## 1.8 Acknowledgements

Substantial efforts have been made in writing chombo-discharge. Publications that arise from the use of chombo-discharge must acknowledge its usage and cite the appropriate methodology papers. Currently, if you use chombo-discharge you should cite the following methodology paper:

```
@article{chombo-discharge,
    author    = {Marskar, Robert},
    doi      = {10.1016/j.jcp.2019.03.036},
    issn     = {109602716},
    journal   = {Journal of Computational Physics},
    keywords  = {Cartesian AMR,Complex geometry,HPC,Plasma,Streamer},
    title     = {{An adaptive Cartesian embedded boundary approach for fluid simulations of two- and three-dimensional low-temperature plasma filaments in complex geometries}},
    year      = {2019}
}
```

## 2.1 Overview

A fundamental design principle in `chombo-discharge` is the division between the AMR core, geometry, solvers, physics coupling, and user applications. As an example, the fundamental time integrator class `TimeStepper` in `chombo-discharge` is just an abstraction, i.e. it only presents an API which application codes must use. Because of that, `TimeStepper` can be used for solving completely unrelated problems. We have, for example, implementations of `TimeStepper` for solving radiative transfer equations, advection-diffusion problems, electrostatic problems, or for plasma problems.

The division between computational concepts (e.g., AMR functionality and solvers) exists so that users will be able to solve problems across a range of geometries, add new solvers functionality, or write entirely new applications, without requiring deep changes to `chombo-discharge`. Fig. 2.1.1 shows the basic design sketch of the `chombo-discharge` code. To the right in this figure we have the AMR core functionality, which supplies the infrastructure for running the solvers. In general, solvers may share common features (such as elliptic discretizations) or be completely disjoint. For this reason numerical solvers are asked to *register* AMR requirements. For example, elliptic solvers need functionality for interpolating ghost cells over the refinement boundary, but pure particle solvers have no need for such functionality. A consequence of this is that the numerical solvers are literally asked (during their instantiation) to register what type of AMR infrastructure they require. In return, the AMR core will allocate this infrastructure and make it available to solver, as illustrated in Fig. 2.1.1.

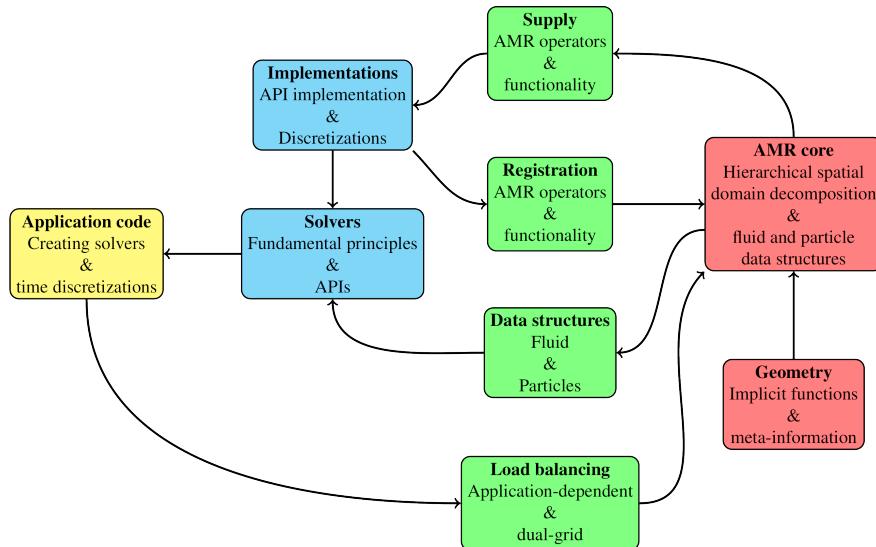


Fig. 2.1.1: Concept design sketch for `chombo-discharge`.

`chombo-discharge` also uses *loosely coupled* solvers as a foundation for the code design, where a *solver* indicates

a piece of code for solving an equation. For example, solving the Laplace equation  $\nabla^2\Phi = 0$  is encapsulated by one of the chombo-discharge solvers. Some solvers in chombo-discharge have a null-implemented API, i.e. we have enforced a strict separation of the solver interface and the solver implementation. This constraint exists because while new features may be added to a discretization, we do not want such changes to affect upstream application code. An example of this is the `FieldSolver`, which conceptualizes a numerical solver for solving for electrostatic field problems. The `FieldSolver` is an API with no fundamental discretization – it only contains high-level routines for understanding the type of solver being dealt with. Yet, it is the `FieldSolver` API which is used by application code (and not the implementation class).

All numerical solvers interact with a common AMR core that encapsulates functionality for running the solvers. All solvers are also compatible with mesh refinement and complex geometries, but they can only run through *application codes*, i.e. *physics modules*. These modules encapsulate the time advancement of either individual or coupled solvers. Solvers only interact with one another through these modules, and these modules usually advance the equations of motion using the method-of-lines.

The major design components in chombo-discharge are:

1. `Driver` for running simulations.
2. `AmrMesh` for encapsulating (almost) all AMR and EB functionality in a common core class.
3. `TimeStepper` for integrating the equations of motion.
4. `ComputationalGeometry` for representing computational geometries (such as electrodes and dielectrics).
5. `CellTagger` for flagging cells for refinement and coarsening.
6. `GeoCoarsener` for manually removing refinement flags along the EB surface.

**Caution:** In the current version of chombo-discharge, `GeoCoarsener` has become redundant (due to improvements in the algorithms). It will be removed in future versions.

## 2.2 Driver

The `Driver` class runs chombo-discharge simulations. The primary purpose of the `Driver` class is to coordinate the simulation advance (i.e. `TimeStepper` integration) together with I/O and regrid functionality. The constructor for this class is

```
Driver(const RefCountedPtr<ComputationalGeometry>& a_compgeom,
       const RefCountedPtr<TimeStepper>& a_timestepper,
       const RefCountedPtr<AmrMesh>& a_amr,
       const RefCountedPtr<CellTagger>& a_celltagger = RefCountedPtr<CellTagger>(nullptr),
       const RefCountedPtr<GeoCoarsener>& a_geocoarsen = RefCountedPtr<GeoCoarsener>(nullptr));
```

---

**Tip:** Driver C++ API.

---

## 2.2.1 Simulation setup

For setting up and running simulations, only a single routine is used:

```
void setupAndRun(const std::string a_inputFile);
```

This routine will set up and run a simulation.

### New simulations

If a simulation starts from the first time step, the `Driver` class will perform the following steps in `setupAndRun()`.

1. Ask `ComputationalGeometry` to generate the cut-cell moments.
2. Collect all the cut-cells and ask `AmrMesh` to set up an initial grid. This initial grid is always generated by flagging cells for refinement along the boundary. Various options are available for configuring this initial grid, see [Cell refinement philosophy](#). Also note that it is possible to restrict the maximum level that can be generated from the geometric tags, or remove some of the cut-cell refinement flags through the auxiliary class `GeoCoarsener`.
3. Ask the `TimeStepper` to set up relevant solvers and fill them with initial data.
4. Perform the number of initial regrids that the user asks for. After the regrid, the solvers are re-filled with initial data (rather than regridding).
5. Ask `TimeStepper` to perform a *post-initialization* routine.

### Restarting simulations

If a simulation uses restart file, i.e. one that *does not start* from the first time step, the `Driver` class will execute the following steps in `setupAndRun(...)`.

1. Ask `ComputationalGeometry` to generate the cut-cell moments.
2. Read a checkpoint file that contains the grids and all the data that have been checkpointed by the solvers. `Driver` will issue an error and abort if the checkpoint file does not exist.
3. Ask `TimeStepper` to perform a *post-checkpoint* step. This functionality has been included because not all data in every solver needs to be checkpointed. For example, an electric field solver only needs to write the electric potential to the checkpoint file because the electric field is simply obtained by taking the gradient.
4. Perform the number of initial regrids that the user asks for.

## 2.2.2 Simulation advancement

The algorithm for running a simulation is conceptually simple; the `Driver` class calls `TimeStepper::computeDt` for computing a reasonable time step for advancing the equations, and uses `Real TimeStepper::advance(Real dt)` to actually perform the advance. Regrids, plot files, and checkpoint files are written at certain step intervals (or when the `TimeStepper` demands them). In brief, the algorithm looks like this:

```
Driver::run(...){

    while(KeepRunningTheSimulation){
        if(RegridEverything){
            Driver->regrid()
        }

        tryDt    = TimeStepper->computeDt()
        actualDt = TimeStepper->advance(tryDt)

        if(WriteAPlotFile or EndOfSimulation){

    }
```

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```

        Driver->writePlotFile();
    }
    if(TimeToWriteACheckpointFile or EndOfSimulation){
        Driver->writeCheckpointFile()
    }

    KeepRunningTheSimulation = true or false
}
}

```

## 2.2.3 Regridding

Regrids are called by the `Driver` class and proceed as follows:

1. `CellTagger` generates tags for grid refinement and coarsening.
2. The `TimeStepper` class stores data that is subject to regrids so that we have access to previously defined data when we interpolate to the new grids.
3. The `AmrMesh` class generates the new grid boxes and EB information.
4. `TimeStepper` checks if the defined realms show be load balanced.
5. `AmrMesh` regrids the realms and EBAMR operators.
6. The `TimeStepper` class regrids its solvers and internal data.
7. The `TimeStepper` performs a *post-regrid* operation (e.g. filling solvers with auxiliary data).

In C++ pseudo-code, this looks something like:

```

Driver::regrid(){

    // Tag cells
    CellTagger->tagCellsForRefinement()

    // Store old data and free up some memory
    TimeStepper->storeOldGridData()

    // Generate the new grids
    AmrMesh->makeNewGrids()

    if(loadBalance) {
        TimeStepper->loadBalance();
    }

    // AmrMesh finalizes the EBAMR grids
    AmrMesh->regridOperators()

    // Regrid timestepper
    TimeStepper->regrid()

    // Do a post-regrid step
    TimeStepper->postRegrid()
}

```

---

**Note:** `Driver` class does not *require* an instance of `CellTagger` (which is responsible for flagging cells for refinement). If users decide to omit a cell tagger, regridding functionality is completely turned off and only the initially generated grids will be used throughout the simulation.

---

## 2.2.4 Class options

Various class options are available for adjusting the behavior of the `Driver` class

- `Driver.verbosity` controls output will be given to `pout.n`. We use 2 or 3 - higher values are for debugging.
- `Driver.geometry_generation` controls the grid generation method (see [Geometry generation](#)). Valid options are *chombo-discharge* or *chombo*.
- `Driver.geometry_scan_level`. Which refinement level to initiate the chombo-discharge geometry generation method. This entry indicates the number of refinements of the coarsest AMR level used in the simulation. E.g. if the `Driver.geometry_scan_level=1` and the coarsest AMR level is  $128^3$  then the signed distance pruning (see [Geometry generation](#)) begins at the AMR level  $256^3$ . Note that negative numbers are also permitted, in which case the pruning initiates at a coarsened level.
- `Driver.plot_interval`. Time steps between each plot file.
- `Driver.checkpoint_interval`. Time steps between each checkpoint file.
- `Driver.regrid_interval`. Time steps between each regrid.
- `Driver.write_regrid_files`. Write plot files during regrids. Valid options are *true* or *false*.
- `Driver.write_restart_files`. Write plot files during restarts. Valid options are *true* or *false*.
- `Driver.initial_regrids`. Number of initial regrids to perform when starting (or restarting) a simulation.
- `Driver.start_time`. Simulation start time.
- `Driver.stop_time`. Simulation stop time.
- `Driver.max_steps`. Maximum number of simulation time steps.
- `Driver.geometry_only`. If *true*, do not run the simulation and only write the geometry to file.
- `Driver.write_memory`. Write MPI memory report. Valid options are *true* or *false*.
- `Driver.write_loads`. Write computational loads. Valid options are *true* or *false*.
- `Driver.output_directory`. Output directory.
- `Driver.output_names`. Simulation file names.
- `Driver.max_plot_depth`. Maximum plot depth. Values  $< 0$  means all levels.
- `Driver.max_chk_depth`. Maximum checkpoint file depth. Values  $< 0$  means all levels.
- `Driver.num_plot_ghost`. Number of ghost cells in plot files.
- `Driver.plt_vars`. Plot variables for `Driver`. Valid options are *tags*, *mpi\_rank*, *levelset*.
- `Driver.restart`. Restart step (less or equal to 0 implies fresh simulation)
- `Driver.allow_coarsening`. Allows removal of grid levels if cell tags dont run deep enough.
- `Driver.grow_geo_tags`. How much to grow cut-cell refinement tags.
- `Driver.refine_angles`. Refine cells if the angle between normal vector in neighboring cells exceed this threshold.
- `Driver.refine_electrodes`. Refine electrode surfaces. Values  $< 0$  will refine all the way down.
- `Driver.refine_dielectrics`. Refine dielectric surfaces. Values  $< 0$  will refine all the way down.

## 2.2.5 Runtime options

Driver can parse options during run-time (i.e. between simulation steps), see [Run-time configurations](#). The following options are run-time adjustable:

- Driver.verbosity.
- Driver.plot\_interval.
- Driver.checkpoint\_interval.
- Driver.regrid\_interval.
- Driver.write\_regrid\_files.
- Driver.write\_restart\_files.
- Driver.stop\_time.
- Driver.max\_steps.
- Driver.write\_memory.
- Driver.write\_loads.
- Driver.num\_plot\_ghost.
- Driver.plt\_vars.
- Driver.allow\_coarsening.
- Driver.grow\_geo\_tags.
- Driver.refine\_angles.
- Driver.refine\_electrodes.
- Driver.refine\_dielectrics.

## 2.3 ComputationalGeometry

ComputationalGeometry is the class that implements geometries in chombo-discharge. In principle, geometries consist of electrodes and dielectrics but there are many problems where the actual nature of the EB is irrelevant (such as fluid flow).

---

**Tip:** ComputationalGeometry C++ API.

---

ComputationalGeometry is *not* an abstract class. The default implementation is an empty geometry – i.e. a geometry without any objects, but several geometries are included in \$DISCHARGE\_HOME/Geometries. Making a non-empty ComputationalGeometry class requires that you inherit from ComputationalGeometry and instantiate the following class members:

```
Real m_eps0;
Vector<Electrode> m_electrodes;
Vector<Dielectric> m_dielectrics;
```

Here, `m_eps0` is the relative gas permittivity, `m_electrodes` are the electrodes for the geometry and `m_dielectrics` are the dielectrics for the geometry. These are described in detail further down.

When geometries are created, the ComputationalGeometry class will first create the (approximations to the) signed distance functions that describe two possible material phases (gas and solid). Here, the *solid* phase is the part of the

computational domain inside the dielectrics, while the *gas phase* is the part of the computational domain that is outside both the electrodes and the dielectrics.

### 2.3.1 Electrode

The `Electrode` class is responsible for describing an electrode and potentially also its boundary condition. Internally, this class is lightweight and consists only of a tuple that holds a level-set function and an associated boolean value that tells whether or not the level-set function is at a live voltage or not. The constructor for the electrode class is:

```
Electrode(RefCountedPtr<BaseIF> a_baseIF, bool a_live, Real a_fraction = 1.0);
```

where the `a_baseIF` argument is the level-set function and the `a_live` argument is used by some solvers in order to determine if the electrode is at a live voltage or not. If `a_live` is set to false, `FieldSolver` will fetch this value and determine that the electrode is at ground. Otherwise, if `a_live` is set to true then `FieldSolver` will determine that the electrode is at live voltage, and the `a_fraction` argument is an optional argument that allows the user to set the potential to a specified fraction of the live voltage.

---

**Tip:** [Electrode C++ API](#)

---

### 2.3.2 Dielectric

The `Dielectric` class describes a dielectric. This class is lightweight and consists of a tuple that holds a level-set function and the associated permittivity. The constructors are

```
Dielectric(RefCountedPtr<BaseIF> a_baseIF, Real a_permittivity);
Dielectric(RefCountedPtr<BaseIF> a_baseIF, Real (*a_permittivity)(const RealVect a_pos));
```

where the `a_baseIF` argument is the level-set function and the second argument sets the the permittivity. The permittivity can be set to a constant (first constructor) or to a spatially varying value (second constructor).

---

**Tip:** [Dielectric C++ API](#)

---

### 2.3.3 Retrieving parts

It is possible to retrieve the implicit functions for the electrodes and dielectrics through the following member functions:

```
const Vector<Dielectric>& getDielectrics() const;
const Vector<Electrode>& getElectrodes() const;
```

### 2.3.4 Retrieving implicit functions

When generating the geometry we compute the implicit functions for each *phase*, the gas-phase and the dielectric-phase. If none of the implicit functions for the electrodes/dielectrics overlap, the resulting function will also be a signed distance function. Note that these functions are the unions/intersections of all electrodes and dielectrics.

To retrieve the implicit function corresponding to a particular phase, use

```
const RefCountedPtr<BaseIF>& getImplicitFunction(const phase::which_phase a_phase) const;
```

where `a_phase` will be `phase::gas` or `phase::solid`.

## 2.4 TimeStepper

TimeStepper represents the EB+AMR equation solving class in chombo-discharge. It owns the various numerical solvers and is responsible for setting up solvers and advancing the equations of motion. Because TimeStepper and not *Driver* owns the solvers as class members, it will also coordinate I/O (together with *Driver*).

---

**Tip:** TimeStepper C++ API

---

### 2.4.1 Basic functions

Since it is necessary to implement different solvers for different types of physics, TimeStepper is an abstract class with the following pure functions:

```
// Setup routines
virtual void setupSolvers() = 0;
virtual void allocate() = 0;
virtual void initialData() = 0;
virtual void postInitialize() = 0;
virtual void postCheckpointSetup() = 0;
virtual void registerRealms() = 0;
virtual void registerOperators() = 0;
virtual void parseRuntimeOptions();

// IO routines
virtual void writeCheckpointData(HDF5Handle& a_handle, const int a_lvl) const = 0;
virtual void readCheckpointData(HDF5Handle& a_handle, const int a_lvl) = 0;
virtual void writePlotData(EBAMRCellData& a_output, Vector<std::string>& a_plotVariableNames, int& a_icomp) const = 0;
virtual int getNumberOfPlotVariables() const = 0;
virtual Vector<long int> getCheckpointLoads(const std::string a_realm, const int a_level) const;

// Advance routines
virtual Real computeDt() = 0;
virtual Real advance(const Real a_dt) = 0;
virtual void synchronizeSolverTimes(const int a_step, const Real a_time, const Real a_dt) = 0;
virtual void printStepReport() = 0;

// Regrid routines
virtual void preRegrid(const int a_lmin, const int a_oldFinestLevel) = 0;
virtual void postRegrid() = 0;
virtual void regrid(const int a_lmin, const int a_oldFinestLevel, const int a_newFinestLevel) = 0;
virtual bool needToRegrid();
virtual bool loadBalanceThisRealm(const std::string a_realm) const;
virtual void loadBalanceBoxes(Vector<Vector<int> >& a_procs,
                           Vector<Vector<Box> >& a_boxes,
                           const std::string a_realm,
                           const Vector<DisjointBoxLayout>& a_grids,
                           const int a_lmin,
                           const int a_finestLevel);
```

These functions are all used by *Driver* class at various stages.

### 2.4.2 Setup routines

Here, we consider the various setup routines in TimeStepper. The routines are used by *Driver* in the simulation setup step.

## registerRealms

TimeStepper permits solvers to run on different realms (see *Realm*) for individual load balancing of various components. To register a Realm, users will have TimeStepper register realms in the `registerRealms()`, as follows:

```
void myTimeStepper::registerRealms(){
    m_amr->registerRealm(Realm::Primal);
    m_amr->registerRealm("particleRealm");
    m_amr->registerRealm("otherParticleRealm");
}
```

Since at least one realm is required, *Driver* will always register the realm "Primal". Fundamentally, there is no limitation to the number of realms that can be allocated.

## registerOperators

Internally, an instantiation of *Realm* contains the grids and the geometric information (e.g. EBISLayout), as well as any operators that the user has seen fit to *register*. Various operators are available for e.g. gradient stencils, conservative coarsening, ghost cell interpolation, filling a patch with interpolation data, redistribution, and so on. Since operators always incur overhead and not all applications require *all* operators, they must be *registered*. If a solver needs an operator for, say, piecewise linear ghost cell interpolation, the solver needs to *register* that operator through the AmrMesh public interface:

```
m_amr->registerOperator(s_eb_pwl_interp, m_realm, m_phase);
```

Once an operator has been registered, *Realm* will define those operators during initialization e.g. regrids. Run-time error messages are issued if an AMR operator is used, but has not been registered.

More commonly, chombo-discharge solvers will contain a routine that registers the operators that the solver needs. A valid TimeStepper implementation *must* register all required operators in the function `registerOperators()`.

Currently available operators are:

1. Gradient `s_eb_gradient`.
2. Irregular cell centroid interpolation, `s_eb_irreg_interp`.
3. Coarse grid conservative coarsening, `s_eb_coar_ave`.
4. Piecewise linear interpolation (with slope limiters), `s_eb_fill_patch`.
5. Linear ghost cell interpolation, `s_eb_fine_interp`.
6. Flux registers, `s_eb_flux_reg`.
7. Redistribution registers, `s_eb_redist`.
8. Non-conservative divergence stencils, `s_eb_noncons_div`.
9. Multigrid interpolators, `s_eb_multigrid` (used for multigrid).
10. Signed distance function defined on grid, `s_levelset`.
11. Particle-mesh support, `s_eb_particle_mesh`.

Solvers will typically allocate a subset of these operators, but for multiphysics code that use both fluid and particles, most of these will probably be in use.

### setupSolvers

`setupSolvers` is used for setting up solvers. This step is done *prior* to setting up the grids, so it is not possible to allocate mesh data inside this routine.

### allocate

`allocate` is used for allocating particle and mesh data for the solvers and `TimeStepper`. This step is done *after* the grids have been initialized by `AmrMesh` and during regrids.

### initialData

`initialData` is called by `Driver` setup routines after the `allocate` step. This routine must fill solvers with initial data.

### postInitialize

`postInitialize` is called *after* `Driver` has filled the solvers with initial data. Most data initialization steps can, however, be done in `initialData`.

### postCheckpointSetup

During simulation restarts, `Driver` will open an HDF5 file and have `TimeStepper` fill solvers with data from that file. `postCheckpointSetup` is a routine which is called immediately after the solvers have been filled with data.

## 2.4.3 I/O routines

The `TimeStepper` I/O routines serve two purposes:

1. To add solver data to HDF5 plot files.
2. To write and read data for checkpoint/restart files.

In general, plot and checkpoint data do not contain the same data.

### getNumberOfPlotVariables

`getNumberOfPlotVariables` must return the number of components that will be plotted by `TimeStepper`. Note that if `TimeStepper` will plot a single scalar, it must return a value of one. If it plots a single vector, it must return a value of `SpaceDim`.

The existence of `getNumberOfPlotVariables` is due to pre-allocation of memory that will be written to the plot file.

## writePlotData

`writePlotData` will write the plot data to the provided data holder. The signature is

```
virtual void writePlotData(EBAMRCelldata& a_output, Vector<std::string>& a_plotVariableNames, int& a_icomp) const = 0;
```

Here, `a_output` is pre-allocate block of memory that `TimeStepper` will write its components to, and `a_plotVariable` are the associated plot variable names. `a_icomp` is the starting component in `a_output` where we start writing data.

## writeCheckpointData

`writeCheckpointData` will write solver data to the provided HDF5 file handle. This data is used when restarting simulations from a checkpoint file. Note that checkpoint data is written on a level-by-level basis.

## readCheckpointData

`readCheckpointData` will read data from the provided HDF5 file handle and back into the solvers. Note that the data is read on a level-by-level basis.

### 2.4.4 Advance routines

#### computeDt

`computeDt` will compute a time step for `Driver` to use when calling the `advance` method.

#### advance

`advance` is called by `Driver` when advancing the equation of motion one time step. Note that `advancpe` takes a trial time step as input argument and returns the actual time step that was used. These do not need to be the same.

#### synchronizeSolverTimes

`synchronizeSolverTimes` is called after the `advance` method and is used to update the simulation time for all solvers.

#### printStepReport

`printStepReport` called after the `advance` method – it can be left empty but is otherwise used to print some information about the time step that was taken.

## 2.4.5 Regrid routines

For an explanation to how regridding occurs in chombo-discharge, see [Regridding](#).

### preRegrid

preRegrid should any necessary pre-regrid operation. Note that when solvers regrid their data, solution is allocated on new grids and the previously defined data is lost. For this reason most solvers have the option of putting the old grid data into temporary storage that permits us to interpolate to the new grids.

### regrid

regrid will perform the actual regrid operation.

### postRegrid

postRegrid is called after regrid can be used to perform any post-regrid operations.

## 2.4.6 Load balancing routines

During the regrid step, [Driver](#) will check if any of the realms should be load balanced. If a realm should be load balanced then [TimeStepper](#) take a [DisjointBoxLayout](#) which originally load balanced using the patch volume, and generate a new set of grids.

### loadBalanceThisRealm

Return true if a [Realm](#) should be load balanced and false otherwise.

### loadBalanceBoxes

This is called if `loadBalanceThisRealm` evaluates to true, and in this case the [TimeStepper](#) should compute a new set of rank ownership for the input grid boxes.

## 2.5 AmrMesh

[AmrMesh](#) handles (almost) all spatial operations in chombo-discharge. Internally, [AmrMesh](#) contains a bunch of operators that are useful across classes, such as ghost cell interpolation operators, coarsening operators, and stencils for interpolation and extrapolation near the embedded boundaries. [AmrMesh](#) also contains routines for generation and load-balancing of grids based and also contains simple routines for allocation and deallocation of memory.

---

**Note:** [AmrMesh](#) only handles spatial *operations*, it otherwise has limited knowledge of numerical discretizations.

---

[AmrMesh](#) is an integral part of chombo-discharge, and users will never have the need to modify it unless they are implementing something entirely new. The behavior of [AmrMesh](#) is modified through its available input parameters.

---

**Tip:** [AmrMesh C++ API](#)

---

## 2.5.1 Main functionality

There are two main functionalities in `AmrMesh`:

1. Building grid hierarchies, and providing geometric information
2. Providing AMR operators.

The grids in `AmrMesh` consist of a `DisjointBoxLayout` (see [Chombo-3 basics](#)) on each level, supported also by the EB information (`EBISLayout`). Recall that each grid patch in a `DisjointBoxLayout` is owned by a unique rank. However, since `chombo-discharge` supports multiple decompositions, we support the use of multiple `DisjointBoxLayout` describing the same grid level. Although these `DisjointBoxLayout` consist of the same boxes, the patch-to-rank mapping can be different (see [Realm](#)). To fetch a grid on a particular level, one can call `AmrMesh::getGrids(const std::string a_realm)`. E.g.

```
const std::string myRealm;
const int myLevel;

const DisjointBoxLayout& dbl = m_amr->getGrids("myRealm")[myLevel];
```

Likewise, to fetch the geometric (EB) information for a specified realm, phase, and level:

```
const std::string myRealm;
const int myLevel;
const phase::which_phase myPhase;

const EBISLayout& ebisl = m_amr->getEBISLayout(myRealm, myPhase)[myLevel];
```

In addition to the grids, the user can fetch AMR operators. These are, for example, coarsening operators, interpolation operators, ghost cell interpolators etc. To save some regrid time, we don't always build every AMR operator that we might ever need, but have solvers *register* the ones that they specifically need. See [Realm](#) for details.

The API of `AmrMesh` is quite extensive, and it has functionality both for providing grid operators as well as:

- Maintaining overview of grids and operators.
- Allocating grid and particle data.
- Interpolating to new grids.
- Coarsening data.
- Updating ghost cells.
- Interpolating data to e.g. EB centroids.

The [AmrMesh API](#) is a good place to start for figuring out the `AmrMesh` functionality.

## 2.5.2 Particle intersection

### 2.5.3 Class options

The class options below control `AmrMesh`:

- `AmrMesh.lo_corner`. Low corner of problem domain (e.g. 0 0 0)
- `AmrMesh.hi_corner`. High corner of problem domain (e.g. 1 1 1).
- `AmrMesh.verbosity`. Class verbosity. Leave to -1 unless you are debugging.
- `AmrMesh.coarsest_domain`. Number of grid cells on coarsest domain
- `AmrMesh.max_amr_depth`. Maximum number of refinement levels.

- `AmrMesh.max_sim_depth`. Maximum simulation depth. Values < 0 means that grids can be generated with depths up to `AmrMesh.max_amr_depth`.
- `AmrMesh.fill_ratio`. Fill ratio for BR grid generation
- `AmrMesh.irreg_growth`. Buffer region around irregular tagged cells.
- `AmrMesh.buffer_size`. Buffer size for BR grid generation.
- `AmrMesh.grid_algorithm`. Grid generation algorithm. Valid options are *br* or *tiled*. See [Mesh generation](#) for details.
- `AmrMesh.box_sorting`. Box sorting algorithm. Valid options are *std*, *morton*, or *shuffle*.
- `AmrMesh.blocking_factor`. Blocking factor.
- `AmrMesh.max_box_size`. Maximum box size.
- `AmrMesh.max_ebis_box`. Maximum box size during EB geometry generation.
- `AmrMesh.ref_rat`. Refinement ratios.
- `AmrMesh.num_ghost`. Number of ghost cells for mesh data.
- `AmrMesh.lsf_ghost`. Number of ghost cells when allocating level-set function on the grid.
- `AmrMesh.eb_ghost`. Number of ghost cells for EB moments.
- `AmrMesh.centroid_sten`. Which centroid interpolation stencils to use. Valid options are *pwl*, *linear*, *taylor*, *lsq*. Only *linear* is guaranteed monotone.
- `AmrMesh.eb_sten`. EB interpolation stencils.
- `AmrMesh.redist_radius`. Redistribution radius.
- `AmrMesh.ghost_interp`. Default ghost cell interpolation type. Valid options are *pwl* or *quad*.
- `AmrMesh.ebcf`. Can be set to false if refinement boundaries do not cross the EB. Valid options are *true* and *false*.

**Warning:** chombo-discharge only supports uniform resolution (i.e., cubic grid cells). I.e. the user must specify consistent domain sizes and resolutions.

## 2.5.4 Runtime options

The following options are runtime options for `AmrMesh`:

- `AmrMesh.verbosity`.
- `AmrMesh.fill_ratio`.
- `AmrMesh.irreg_growth`.
- `AmrMesh.buffer_size`.
- `AmrMesh.grid_algorithm`.
- `AmrMesh.box_sorting`.
- `AmrMesh.blocking_factor`.
- `AmrMesh.max_box_size`.

These options only affect the grid generation method and parameters, and are thus only effective after the next regrid.

## 2.6 CellTagger

The `CellTagger` class is responsible for flagging grid cells for refinement or coarsening. If the user wants to implement a new refinement or coarsening routine, he will do so by writing a new derived class from `CellTagger`. The `CellTagger` parent class is a stand-alone class - it does not have a view of `AmrMesh`, `Driver`, or `TimeStepper`. Since refinement is intended to be quite general, the user is responsible for providing `CellTagger` with the appropriate dependencies.

---

**Tip:** `CellTagger` C++ API

---

Refinement flags live in a data holder called `EBAMRTags` inside of `Driver`. This data is typedef'ed as

```
typedef Vector<RefCountedPtr<LayoutData<DenseIntVectSet>>> EBAMRTags;
```

For performance reasons, `DenseIntVectSet` only stores refinement cells on a per-patch basis. It is not possible to add a grid cell to a `DenseIntVectSet` if it falls outside the grid patch. Furthermore, the `EBAMRTags` structure is a distributed data structure which makes sure that each MPI rank is aware of the refinement flags on the corresponding grid patches. The flags themselves are owned by `Driver`, and they are copied onto the new grids in the regrid step.

### 2.6.1 User interface

To implement a new `CellTagger`, the following functions must be implemented:

```
virtual void regrid() = 0;
virtual void parseOptions() = 0;
virtual void parseRuntimeOptions();
virtual bool tagCells(EBAMRTags& a_tags) = 0;
```

Users can also parse run-time options and even have `CellTagger` write to plot files, by implementing

```
virtual int getNumberOfPlotVariables();
virtual void writePlotData(EBAMRCelldata& a_output, Vector<std::string>& a_plotvar_names, int& a_icomp);
```

This is primarily useful for debugging the tracer fields that are used for flagging cells for refinement.

#### tagCells

When the regrid routine enters, the `CellTagger` will be asked to generate the refinement flags through a function

```
bool tagCells(EBAMRTags& a_tags) = 0;
```

This routine should add cells that will be refined, and remove cells that will be coarsened.

#### regrid

`regrid` is called by `Driver` during regrids. The existence of this routine is due to the common usage pattern where `CellTagger` holds some auxiliary mesh data that is used when evaluating refinement and coarsening criteria. This routine should reallocate such temporary storage during regrids.

### parseOptions

`parseOptions` is called by *Driver* when setting the `CellTagger`. This routine should parse options into the `CellTagger` instance.

### parseRuntimeOptions

`parseRuntimeOptions` is called by *Driver* after each grid step. This routine will re-read class options into the `CellTagger` instance. See *Run-time configurations* for further details.

### getNumberOfPlotVariables

`getNumberOfPlotVariables` will return the number of plot variables that `CellTagger` will write to plot files.

### writePlotData

`writePlotData` will write the plot data to the provided data holder. The functionality is the same as for *TimeStepper*.

## 2.6.2 Restrict tagging

It is possible to prevent `CellTagger` from adding refinement flags in specified regions. The default behavior is to add a number of boxes where refinement and coarsening is allowed:

```
MyCellTagger.num_boxes = 0           # Number of allowed tag boxes (0 = tags allowed everywhere)
MyCellTagger.box1_lo   = 0.0 0.0 0.0 # Only allow tags that fall between
MyCellTagger.box1_hi   = 1.0 1.0 1.0 # these two corners
```

Here, `MyCellTagger` is a placeholder for the name of the class that is used. By adding restrictive boxes, tagging will only be allowed inside the specified box corners `box1_lo` and `box1_hi`. More boxes can be specified by following the same convention, e.g. `box2_lo` and `box2_hi` etc.

## 2.6.3 Adding a buffer

By default, each MPI rank can only tag grid cells where it owns data. This has been done for performance and communication reasons. Under the hood, the `DenseIntVectSet` is an array of boolean values on a patch which is very fast and simple to communicate with MPI. Adding a grid cell for refinement which lies outside the patch will lead to memory corruptions. It is nonetheless still possible to do this by growing the final generated tags like so:

```
MyCellTagger.buffer = 4 # Add a buffer region around the tagged cells
```

Just before passing the flags into `AmrMesh` grid generation routines, the tagged cells are put in a different data holder (`IntVectSet`) and this data holder *can* contain cells that are outside the patch boundaries.

## 2.7 GeoCoarsener

**Warning:** GeoCoarsener is a relic of an early version of chombo-discharge. It will be completely removed in a future version.

The GeoCoarsener class permits manual removement of refinement flags along a geometric surface. To remove these so-called *geometric tags*, the user specifies boxes in space where geometric tags will be removed:

```
GeoCoarsener.num_boxes = 1          # Number of coarsening boxes (0 = don't coarsen)
GeoCoarsener.box1_lo   = -1 -1 -1    # Lower-left corner
GeoCoarsener.box1_hi   = 1 1 1      # Upper-right corner
GeoCoarsener.box1_lvl  = 0          # Remove tags down to this level.
GeoCoarsener.box1_inv  = false       # Flip removal.
```

If users want more boxes, they can specify it using the same syntax, e.g.

```
GeoCoarsener.num_boxes = 2          # Number of coarsening boxes (0 = don't coarsen)
GeoCoarsener.box1_lo   = -1 -1 -1    # Lower-left corner
GeoCoarsener.box1_hi   = 1 1 1      # Upper-right corner
GeoCoarsener.box1_lvl  = 0          # Remove tags down to this level.
GeoCoarsener.box1_inv  = false       # Flip removal.

GeoCoarsener.box2_lo   = 2 2 2      # Lower-left corner
GeoCoarsener.box2_hi   = 3 3 3      # Upper-right corner
GeoCoarsener.box2_lvl  = 0          # Remove tags down to this level.
GeoCoarsener.box2_inv  = false       # Flip removal.
```



## DISCRETIZATION

### 3.1 Spatial discretization

#### 3.1.1 Cartesian AMR

chombo-discharge uses patch-based structured adaptive mesh refinement (AMR) provided by Chombo [5]. In patch-based AMR the domain is subdivided into a collection of hierarchically nested grid levels, see Fig. 3.1.1. With Cartesian AMR each patch is a Cartesian block of grid cells. A *grid level* is composed of a union of grid patches sharing the same grid resolution, with the additional requirement that the patches on a grid level are *non-overlapping*. With AMR, such levels can be hierarchically nested; finer grid levels exist on top of coarser ones. In patch-based AMR there are only a few fundamental requirements on how such grids are constructed. For example, a refined grid level must exist completely within the bounds of its parent level. In other words, grid levels  $l - 1$  and  $l + 1$  are spatially separated by a non-zero number of grid cells on level  $l$ .

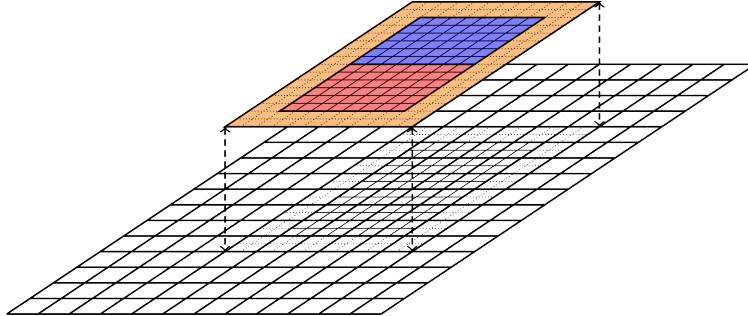


Fig. 3.1.1: Cartesian patch-based refinement showing two grid levels. The fine-grid level lives on top of the coarse level, and consists of two patches (red and blue colors) with two layers of ghost cells (dashed lines and orange shaded region).

The resolution on level  $l + 1$  is typically finer than the resolution on level  $l$  by an integer (usually power of two). However,

---

**Important:** chombo-discharge only supports refinement factors of 2 and 4.

---

### 3.1.2 Embedded boundaries

chombo-discharges uses an embedded boundary (EB) formulation for describing complex geometries. With EBs, the Cartesian grid is directly intersected by the geometry. This is fundamentally different from unstructured grid where one generates a volume mesh that conforms to the surface mesh of the input geometry. Since EBs are directly intersected by the geometry, there is no fundamental need for a surface mesh for describing the geometry. Moreover, Cartesian EBs have a data layout which remains (almost) fully structured. The connectivity of neighboring grid cells is still trivially found by fundamental strides along the data rows/columns, which allows extending the efficiency of patch-based AMR to complex geometries. Figure Fig. 3.1.2 shows an example of patch-based grid refinement for a complex surface.

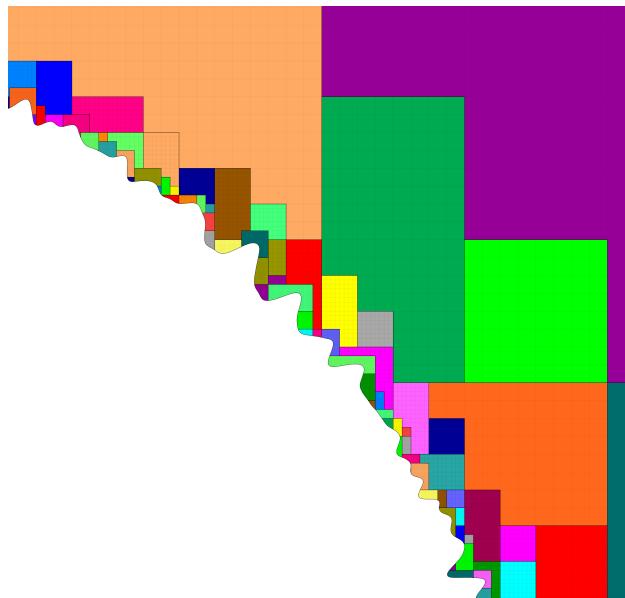


Fig. 3.1.2: Patch-based refinement (factor 4 between levels) of a complex surface. Each color shows a patch, which is a rectangular computational unit.

Since EBs are directly intersected by the geometry, pathological cases can arise where a Cartesian grid cell consists of multiple volumes. One can easily envision this case by intersecting a thin body with a Cartesian grid, as shown in Fig. 3.1.3. This figure shows a thin body which is intersected by a Cartesian grid, and this grid is then coarsened. At the coarsened level, one of the grid cells has two cell fragments on opposite sides of the body. Such multi-valued cells (a.k.a *multi-cells*) are fundamentally important for EB applications. Note that there is no fundamental difference between single-cut and multi-cut grid cells. This distinction exists primarily due to the fact that if all grid cells were single-cut cells the entire EB data structure would fit in a Cartesian grid block (say, of  $N_x \times N_y \times N_z$  grid cells). Because of multi-cells, EB data structures are not purely Cartesian. Data structures need to live on more complex graphs that describe support multi-cells and, furthermore, describe the cell connectivity. Without multi-cells it would be impossible to describe most complex geometries. It would also be extremely difficult to obtain performant geometric multigrid methods (which rely on this type of coarsening).

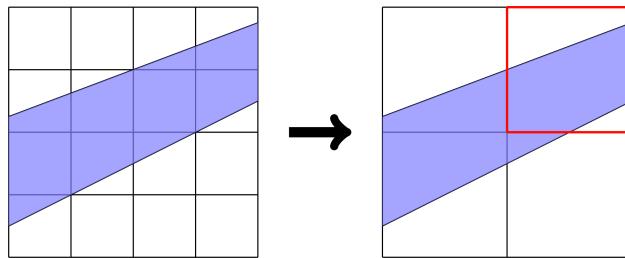


Fig. 3.1.3: Example of how multi-valued cells occur during grid coarsening. Left: Original grid. Right: Coarsened grid.

### 3.1.3 Geometry representation

chombo-discharge uses (approximations to) signed distance functions (SDFs) for describing geometries. Signed distance fields are functions  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$  that describe the distance from the object. These functions are also *implicit functions*, i.e.  $f(\mathbf{x}) = 0$  describes the surface of the object,  $f(\mathbf{x}) > 0$  describes a point inside the object and  $f(\mathbf{x}) < 0$  describes a point outside the object.

Many EB applications only use the implicit function formulation, but chombo-discharge requires (an approximation to) the signed distance field. There are two reasons for this:

1. The SDF can be used for robustly load balancing the geometry generation with orders of magnitude speedup over naive approaches.
2. The SDF is useful for resolving particle collisions with boundaries, using e.g. simple ray tracing of particle paths.

To illustrate the difference between an SDF and an implicit function, consider the implicit functions for a sphere at the origin with radius  $R$ :

$$d_1(\mathbf{x}) = R - |\mathbf{x}|, \quad (3.1.1)$$

$$d_2(\mathbf{x}) = R^2 - \mathbf{x} \cdot \mathbf{x}. \quad (3.1.2)$$

Here, only  $d_1(\mathbf{x})$  is a signed distance function.

In chombo-discharge, SDFs can be generated through analytic expressions, constructive solid geometry, or by supplying polygon tessellation. NURBS geometries are, unfortunately, not supported. Fundamentally, all geometric objects are described using BaseIF objects from Chombo, see [BaseIF](#).

#### Constructive solid geometry (CSG)

Constructive solid geometry can be used to generate complex shapes from geometric primitives. For example, to describe the union between two SDFs  $d_1(\mathbf{x})$  and  $d_2(\mathbf{x})$ :

$$d(\mathbf{x}) = \min(d_1(\mathbf{x}), d_2(\mathbf{x}))$$

Note that the resulting is an implicit function but is *not* an SDF. However, the union typically approximates the signed distance field quite well near the surface. Chombo natively supports many ways of performing CSG.

## EBGeometry

While functions like  $R - |\mathbf{x}|$  are quick to compute, a polygon surface may consist of hundreds of thousands of primitives (e.g., triangles). Generating signed distance function from polygon tessellations is quite involved as it requires computing the signed distance to the closest feature, which can be a planar polygon (e.g., a triangle), edge, or a vertex. chombo-discharge supports such functions through the [EBGeometry](#) package.

**Warning:** The signed distance function for a polygon surface is only well-defined if it is manifold-2, i.e. it is watertight and does not self-intersect. chombo-discharge should nonetheless compute the distance field as best as it can, but the final result may not make sense in an EB context.

Searching through all features (faces, edge, vertices) is unacceptably slow, and [EBGeometry](#) therefore uses a bounding volume hierarchy for accelerating these searches. The bounding volume hierarchy is top-down constructed, using a root bounding volume (typically a cube) that encloses all triangles. Using heuristics, the root bounding volume is then subdivided into two separate bounding volumes that contain roughly half of the primitives each. The process is then recursed downwards until specified recursion criteria are met. Additional details are provided in the [EBGeometry documentation](#).

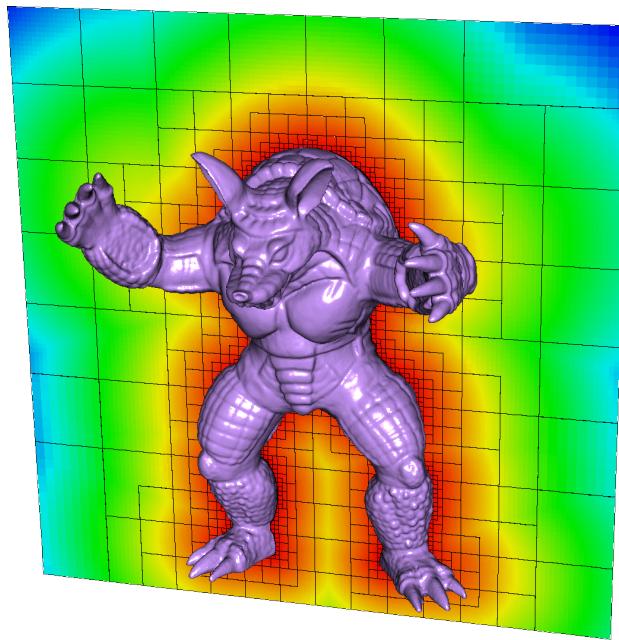


Fig. 3.1.4: Example of an SDF reconstruction and cut-cell grid from a surface tessellation in chombo-discharge.

### 3.1.4 Geometry generation

#### Chombo approach

The default geometry generation method in Chombo is to locate cut-cells on the finest AMR level first and then generate the coarser levels cells through grid coarsening. This will look through all cells on the finest level, so for a domain which is effectively  $N \times N \times N$  cells there are  $\mathcal{O}(N^3)$  implicit function queries (in 2D, the complexity is  $\mathcal{O}(N^2)$ ). Note that as  $N$  becomes large, say  $N = 10^5$ , geometric queries of this type become a bottleneck.

## chombo-discharge pruning

chombo-discharge has made modifications to the geometry generation routines in Chombo, resolving a few bugs and, most importantly, using the signed distance function for load balancing the geometry generation step. This modification to Chombo yields a reduction of the original  $\mathcal{O}(N^3)$  scaling in Chombo grid generation to an  $\mathcal{O}(N^2)$  scaling in chombo-discharge. Typically, we find that this makes geometry generation computationally trivial (in the sense that it is very fast compared to the simulation).

To understand this process, note that the SDF satisfies the Eikonal equation

$$|\nabla f| = 1, \quad (3.1.3)$$

and so it is well-behaved for all  $\mathbf{x}$ . The SDF can thus be used to prune large regions in space where cut-cells don't exist. For example, consider a Cartesian grid patch with cell size  $\Delta x$  and cell-centered grid points  $\mathbf{x}_i = (\mathbf{i} + \frac{1}{2}) \Delta x$  where  $\mathbf{i} \in \mathbb{Z}^3$  are grid cells in the patch, as shown in Fig. 3.1.5. We know that cut cells do not exist in the grid patch if  $|f(\mathbf{x}_i)| > \frac{1}{2} \Delta x$  for all  $i$  in the patch. One can use this to perform a quick scan of the SDF on a *coarse* grid level first, for example on  $l = 0$ , and recurse deeper into the grid hierarchy to locate cut-cells on the other levels. Typically, a level is decomposed into Cartesian subregions, and each subregion can be scanned independently of the other subregions (i.e. the problem is embarrassingly parallel). Subregions that can't contain cut-cells are designated as *inside* or *outside*, depending on the sign of the SDF. There is no point in recursively refining these to look for cut-cells at finer grid levels, owing to the nature of the SDF they can be safely pruned from subsequent scans at finer levels. The subregions that did contain cut-cells are refined and decomposed into sub-subregions. This procedure recurses until  $l = l_{\max}$ , at which point we have determined all sub-regions in space where cut-cells can exist (on each AMR level), and pruned the ones that don't. This process is shown in Fig. 3.1.5. Once all the grid patches that contain cut-cells have been found, these patches are distributed (i.e., load balanced) to the various MPI ranks for computing the discrete grid information.

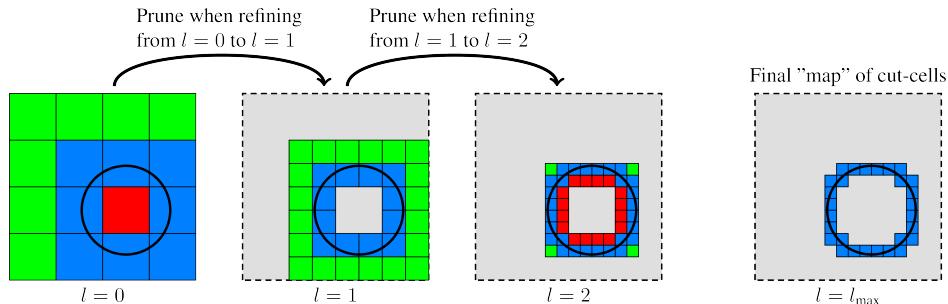


Fig. 3.1.5: Pruning cut-cells with the signed distance field. Red-colored grid patches are grid patches entirely contained inside the EB. Green-colored grid patches are entirely outside the EB, while blue-colored grid patches contain cut-cells.

The above load balancing strategy is very simple, and it reduces the original  $O(N^3)$  complexity in 3D to  $O(N^2)$  complexity (in 2D the complexity is reduced from  $O(N^2)$  to  $O(N)$ ). The strategy works for all SDFs although, strictly speaking, an SDF is not fundamentally needed. If a well-behaved Taylor series can be found for an implicit function, the bounds on the series can also be used to infer the location of the cut-cells, and the same algorithm can be used. For example, generating compound objects with CSG are typically sufficiently well behaved (provided that the components are SDFs). However, implicit functions like  $d(\mathbf{x}) = R^2 - \mathbf{x} \cdot \mathbf{x}$  must be used with caution.

When polygonal surfaces are involved the above process might lead to load imbalance if the input grids to [EBGeometry](#) do not produce well-balanced bounding volume hierarchies (which is often the case). In this case it might be beneficial to shuffle the cut-cell boxes among the ranks by specifying `ScanShop.box_sorting = shuffle`, which will normally lead to well-balanced cut-cell grid generation. Other options are `ScanShop.box_sorting = morton` and `ScanShop.box_sorting = std`. The default behavior is to use a Morton space-filling curve for organizing the cut-cell patches among the ranks.

### 3.1.5 Mesh generation

chombo-discharge supports two algorithm for AMR grid generation:

1. The classical Berger-Rigoutsos algorithm [1].
2. A *tiled* algorithm [7].

Both algorithms work by taking a set of flagged cells on each grid level and generating new boxes that cover the flags. Only *properly nested* grids are generated, in which case two grid levels  $l - 1$  and  $l + 1$  are separated by a non-zero number of grid cells on level  $l$ . This requirement is not fundamentally required for quad- and oct-tree grids, but is nevertheless usually imposed. For patch based AMR, the rationale for this requirement is that stencils on level  $l + 1$  should reach into grid cells on levels  $l$  and  $l + 1$ . For example, ghost cells on level  $l + 1$  can then be interpolated from data only on levels  $l$  and  $l + 1$ .

#### Berger-Rigoutsos algorithm

The Berger-Rigoustos grid algorithm is implemented in Chombo and is called by chombo-discharge. The classical Berger-Rigoustos algorithm is inherently serial in the sense that it collects the flagged cells onto each MPI rank and then generates the boxes, see [1] for implementation details. Typically, it is not used at large scale in 3D due to its memory consumption.

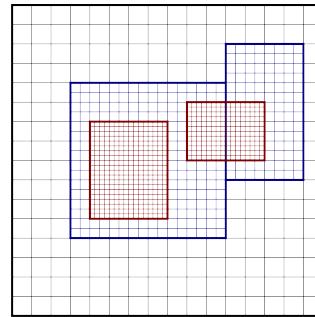


Fig. 3.1.6: Classical cartoon of patch-based refinement. Bold lines indicate entire grid blocks.

#### Tiled mesh refinement

chombo-discharge also supports a tiled algorithm where the grid boxes on each block are generated according to a predefined tiled pattern. If a tile contains a single tag, the entire tile is flagged for refinement. The tiled algorithm produces grids that are visually similar to octrees, but is slightly more general since it also supports refinement factors other than 2 and is not restricted to domain extensions that are an integer factor of 2 (e.g.  $2^{10}$  cells in each direction). Moreover, the algorithm is extremely fast and has low memory consumption even at large scales.

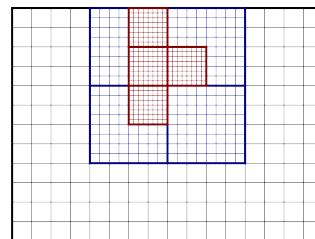


Fig. 3.1.7: Classical cartoon of tiled patch-based refinement. Bold lines indicate entire grid blocks.

### 3.1.6 Cell refinement philosophy

chombo-discharge can flag cells for refinement using various methods:

1. Refine all embedded boundaries down to a specified refinement level.
2. Refine embedded boundaries based on estimations of the surface curvature in the cut-cells.
3. Manually add refinement flags (by specifying boxes where cells will be refined).
4. Physics-based or data-based refinement where the user fetches data from solver classes (e.g., discretization errors, the electric field) and uses that for refinement.

The first two cases are covered by the `Driver` class in chombo-discharge (see [Driver](#)). In the first case the `Driver` class will simply fetch arguments from an input script which specifies the refinement depth for the embedded boundaries. In the second case, the `Driver` class will visit every cut-cell and check if the normal vectors in neighboring cut-cell deviate by more than a specified threshold angle. Given two normal vectors  $\mathbf{n}$  and  $\mathbf{n}'$ , the cell is refined if

$$\mathbf{n} \cdot \mathbf{n}' \geq \cos \theta_c,$$

where  $\theta_c$  is a threshold angle for grid refinement.

The other two cases are more complicated, and are covered by the `GeoCoarsener` and `CellTagger` classes.

## 3.2 Chombo-3 basics

To fully understand this documentation the user should be familiar with Chombo. This documentation uses class names from Chombo and the most relevant Chombo data structures are summarized here. What follows is a *very* brief introduction to these data structures, for in-depth explanations please see the [Chombo manual](#).

### 3.2.1 Real

`Real` is a `typedef`'ed structure for holding a single floating point number. Compiling with double precision will `typedef` `Real` as `double`, otherwise it is `typedef`'ed as `float`.

### 3.2.2 RealVect

`RealVect` is a spatial vector. It holds two `Real` components in 2D and three `Real` components in 3D. The `RealVect` class has floating point arithmetic, e.g. addition, subtraction, multiplication etc.

Most of chombo-discharge is written in dimension-independent code, and for cases where `RealVect` is initialized with components the constructor uses Chombo macros for expanding the correct number of arguments. For example

```
RealVect v(D_DECL(vx, vy, vz));
```

will expand to `RealVect v(vx,vy)` in 2D and `RealVect v(vx, vy, vz)` in 3D.

### 3.2.3 IntVect

`IntVect` is an integer spatial vector, and is used for indexing data structures. It works in much the same way as `RealVect`, except that the components are integers.

### 3.2.4 Box

The `Box` object describes a box in Cartesian space. The boxes are indexed by the low and high corners, both of which are an `IntVect`. The `Box` may be cell-centered or face-centered. To turn a cell-centered `Box` into a face-centered box one would do

```
Box bx(IntVect::Zero, IntVect::Unit); // Default constructor give cell centered boxes  
bx.surroundingNodes(); // Now a cell-centered box
```

This will increase the box dimensions by one in each coordinate direction.

### 3.2.5 EBCellFAB and FArrayBox

The `EBCellFAB` object is an array for holding cell-centered data in an embedded boundary context. The `EBCellFAB` has two data structures: An `FArrayBox` that holds the data on the cell centers, and a additional data structure that holds data in cells that are multiply cut. Doing arithmetic with `EBCellFAB` usually requires one to iterate over all the cell in the `FArrayBox`, and then to iterate over the *irregular cells* (i.e. cut-cells) later. A `VoFIIterator` is such as object; it can iterate over cut-cells. Usually, code for doing anything with the `EBCellFAB` looks like this:

```
// Call Fortran code  
FORT_DO_SOMETHING(....)  
  
// Iterate over cut-cells  
for (VoFIIterator vofit(...); vofit.ok(); ++vofit){  
    (...)  
}
```

---

**Important:** The `FArrayBox` stores the data in column major order.

---

### 3.2.6 Vector

`Vector<T>` is a one-dimensional array with constant-time random access and range checking. It uses `std::vector` under the hood and can access the most commonly used `std::vector` functionality through the public member functions. E.g. to obtain an element in the vector

```
Vector<T> my_vector(10, T0);  
T& element = my_vector[5];
```

Likewise, `push_back`, `resize` etc works in much the same way as for `std::vector`.

### 3.2.7 RefCountedPtr

`RefCountedPtr<T>` is a pointer class in Chombo with reference counting. That is, when objects that hold a reference to some `RefCountedPtr<T>` object goes out of scope the reference counter is decremented. If the reference counter reaches zero, the object that `RefCountedPtr<T>` points to it deallocated. Using `RefCountedPtr<T>` is much preferred over using a raw pointer `T*` to 1) avoid memory leaks and 2) compress code since no explicit deallocations need to be called.

In modern C++-speak, `RefCountedPtr<T>` can be thought of as a *very* simple version of `std::shared_ptr<T>`.

### 3.2.8 DisjointBoxLayout

The `DisjointBoxLayout` class describes a grid on an AMR level where all the boxes are *disjoint*, i.e. they don't overlap. `DisjointBoxLayout` is built upon a union of non-overlapping boxes having the same grid resolution and with unique rank-to-box ownership. The constructor is

```
Vector<Box> boxes(...); // Vector of disjoint boxes
Vector<int> ranks(...); // Ownership of each box

DisjointBoxLayout dbl(boxes, ranks);
```

In simple terms, `DisjointBoxLayout` is the decomposed grid on each level in which MPI ranks have unique ownership of specific parts of the grid.

The `DisjointBoxLayout` view is global, i.e. each MPI rank knows about all the boxes and the box ownership on the entire AMR level. However, ranks will only allocate data on the part of the grid that they own. Data iterators also exist, and the most common is to use iterators that only iterate over the part of the `DisjointBoxLayout` that the specific MPI ranks own:

```
DisjointBoxLayout dbl;
for (DataIterator dit(dbl); dit.ok(); ++dit){
    // Do something
}
```

Each MPI rank will then iterate *only* over the part of the grid where it has ownership.

Other data iterators exist that iterate over all boxes in the grid:

```
for (LayoutIterator lit = dbl.layoutIterator(); dit.ok(); ++dit){
    // Do something
}
```

This is typically used if one wants to do some global operation, e.g. count the number of cells in the grid. However, trying to use `LayoutIterator` to retrieve data that was allocated locally on a different MPI rank is an error.

### 3.2.9 LevelData

The `LevelData<T>` template structure holds data on all the grid patches of one AMR level. The data is distributed with the domain decomposition specified by `DisjointBoxLayout`, and each patch contains exactly one instance of `T`. `LevelData<T>` uses a factory pattern for creating the `T` objects, so if you have new data structures that should fit the in `LevelData<T>` structure you must also implement a factory method for `T`.

The `LevelData<T>` object provides the domain decomposition method in Chombo and chombo-discharge. Often, `T` is an `EBCellFAB`, i.e. a Cartesian grid patch that also supports EB formulations.

To iterate over `LevelData<T>` one will use the data iterator above:

```
LevelData<T> myData;
for (DataIterator dit(dbl); dit.ok(); ++dit){
    T& = myData[dit()];
}
```

`LevelData<T>` also includes the concept of ghost cells and exchange operations.

### 3.2.10 EBISLayout and EBISBox

The `EBISLayout` holds the geometric information over one `DisjointBoxLayout` level. Typically, the `EBISLayout` is used for fetching the geometric moments that are required for performing computations near cut-cells. `EBISLayout` can be thought of as an object which provides all EB-related information on a specific grid level. The EB information consists of e.g. cell flags (i.e., is the cell a cut-cell?), volume fractions, etc. This information is stored in a class `EBISBox`, which holds all the EB information for one specific grid patch. To obtain the EB-information for a specific grid patch, one will call:

```
EBISLayout ebisl;
for (DataIterator dit(dbl); dit.ok(); ++dit){
    EBISBox& ebisbox = ebisl[dit()];
}
```

where `EBISBox` contains the geometric information over only one grid patch. One can thus think of the `EBISLayout` as a `LevelData<EBISBox>` structure.

As an example, to iterate over all the cut-cells defined for a cell-centered data holder an AMR-level one would do:

```
constexpr int comp = 0;

// Assume that these exist.
LevelData<EBCellFAB> myData;
EBISLayout ebisl;

// Iterate over all the patches on a grid level.
for (DataIterator dit(dbl); dit.ok(); ++dit){
    const Box cellBox = dbl[dit()];
    EBCellFAB& patchData = myData[dit()];
    EBISBox& ebisbox = ebisl[dit()];

    // Get all the cut-cells in the grid patch
    const IntVectSet& ivs = ebisbox.getIrregIVS(cellBox);
    const EBGraph& ebggraph = ebisbox.getEBGraph();

    // Define a VoFIterator for the cut-cells and iterate over all the cut-cells.
    for (VoFIterator vofit(ivs, ebggraph); vofit.ok(); ++vofit){
        const VolIndex& vof = vofit();
        patchData(vof, comp) = ...
    }
}
```

Here, `EBGraph` is the graph that describes the connectivity of the cut cells.

### 3.2.11 BaseIF

The `BaseIF` is a Chombo class which encapsulates an implicit function (recall that all SDFs are also implicit functions, see [Geometry representation](#)). `BaseIF` is therefore used for fundamentally constructing a geometric object. Many examples of `BaseIF` are found in Chombo itself, and `chombo-discharge` includes additional ones.

To implement a new implicit function, the user must inherit from `BaseIF` and implement the pure function

```
virtual Real BaseIF::value(const RealVect& a_point) const = 0;
```

The implementation should return a positive value if the point `a_point` is inside the object and a negative value otherwise.

### 3.3 Mesh data

Mesh data structures of the type discussed in [Spatial discretization](#) are derived from a class `EBAMRData<T>` which holds a `T` in every grid patch across the AMR hierarchy. Internally, the data is stored as a `Vector<RefCountedPtr<LevelData<T>>`. Here, the `Vector` holds data on each AMR level; the data is allocated with a smart pointer called `RefCountedPtr` which points to a `LevelData` template structure, see [Chombo-3 basics](#). The first entry in the `Vector` is base AMR level and finer levels follow later in the `Vector`, see e.g. [Fig. 3.3.1](#).

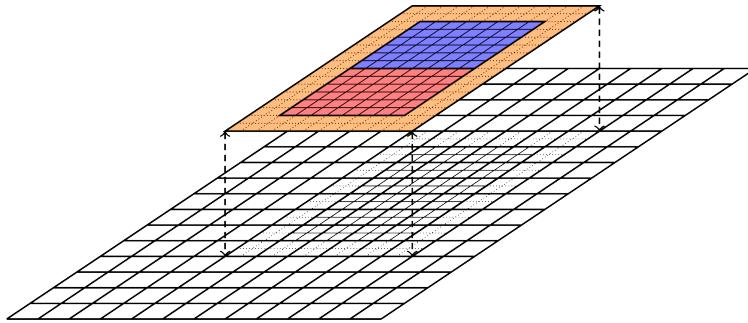


Fig. 3.3.1: Cartesian patch-based refinement showing two grid levels. This is encapsulated by `EBAMRData` where the levels are stored in a `Vector` and the grid patches in the `LevelData` object.

The reason for having class encapsulation of mesh data is due to [Realm](#), so that we can only keep track on which `Realm` the mesh data is defined. Users will interact with `EBAMRData<T>` through application code, or interacting with the core AMR functionality in [AmrMesh](#) (such as computing gradients, interpolating ghost cells etc.). `AmrMesh` (see [AmrMesh](#)) has functionality for defining most `EBAMRData<T>` types on a `Realm`, and `EBAMRData<T>` itself is typically not used anywhere else within chombo-discharge.

A number of explicit template specifications exist and are frequently used. These are outlined below:

<code>typedef EBAMRData&lt;EBCellFAB&gt;</code>	<code>EBAMRCellData;</code> // Cell-centered single-phase data
<code>typedef EBAMRData&lt;EBFluxFAB&gt;</code>	<code>EBAMRFluxData;</code> // Face-centered data in all coordinate direction
<code>typedef EBAMRData&lt;EBFaceFAB&gt;</code>	<code>EBAMRFaceData;</code> // Face-centered in a single coordinate direction
<code>typedef EBAMRData&lt;BaseIVFAB&gt; &gt;</code>	<code>EBAMRIVData;</code> // Data on irregular data centroids
<code>typedef EBAMRData&lt;DomainFluxIFFAB&gt;</code>	<code>EBAMRIFData;</code> // Data on domain phases
<code>typedef EBAMRData&lt;BaseFab&lt;bool&gt; &gt;</code>	<code>EBAMRBool;</code> // For holding bool at every cell
<code>typedef EBAMRData&lt;MFCellFAB&gt;</code>	<code>MFAMRCellData;</code> // Cell-centered multifluid data
<code>typedef EBAMRData&lt;MFFluxFAB&gt;</code>	<code>MFAMRFluxData;</code> // Face-centered multifluid data
<code>typedef EBAMRData&lt;MFBaseIVFAB&gt;</code>	<code>MFAMRIVData;</code> // Irregular face multifluid data

For example, `EBAMRCellData` is a `Vector<RefCountedPtr<LevelData<EBCellFAB>>`, describing cell-centered data across the entire AMR hierarchy. There are many more data structures in place, but the above data structures are the most commonly used ones. Here, `EBAMRFluxData` is precisely like `EBAMRCellData`, except that the data is stored on *cell faces* rather than cell centers. Likewise, `EBAMRIVData` is a `typedef`'ed data holder that holds data on each cut-cell center across the entire AMR hierarchy. In the same way, `EBAMRIFData` holds data on each face of all cut cells.

### 3.3.1 Allocating mesh data

To allocate data over a particular `Realm`, the user will interact with `AmrMesh`:

```
int nComps = 1;
EBAMRCellData myData;
m_amr->allocate(myData, "myRealm", phase::gas, nComps);
```

Here, `nComps` determine the number of cell-centered data components. Note that it *does* matter on which `Realm` and on which phase the data is defined. See [Realm](#) for details.

The user *can* specify a number of ghost cells for his/hers application code directly in the `AmrMesh::allocate` routine, like so:

```
int nComps = 1;
EBAMRCellData myData;
m_amr->allocate(myData, "myRealm", phase::gas, nComps, 5*IntVect::Unit);
```

If the user does not specify the number of ghost cells when calling `AmrMesh::allocate`, `AmrMesh` will use the default number of ghost cells specified in the input file.

### 3.3.2 Iterating over patches

To iterate over data in an AMR hierarchy, you will first iterate over levels and the patches in levels:

```
for (int lvl = 0; lvl < myData.size(); lvl++){
    LevelData<EBCellFAB>& levelData = *myData[lvl];

    const DisjointBoxLayout& levelGrids = levelData.disjointBoxLayout();

    for (DataIterator dit = levelGrids.dataIterator(); dit.ok(); ++dit){
        EBCellFAB& patchData = levelData[dit];
    }
}
```

### 3.3.3 Iterating over cells

For single-valued data, chombo-discharge uses standard loops (in column-major order) for iterating over data. For example, the standard loops for iterating over cell-centered data are

```
namespace BoxLoops {

    template <typename Functor>
    ALWAYS_INLINE void
    loop(const Box& a_computeBox, Functor&& kernel, const IntVect& a_stride = IntVect::Unit);

    template <typename Functor>
    ALWAYS_INLINE void
    loop(VoFIterator& a_iter, Functor&& a_kernel);
}
```

Here, the `Functor` argument is a C++ lambda or `std::function` which takes a grid cell as a single argument. For the first loop, we iterate over all grid cells in `a_computeBox`. In the second function we use a `VoFIterator`, which Iterating over the cells in a patch data holder (like the `EBCellFAB`) can be done with a `VoFIterator`, which can iterate through cells on an `EBCellFAB` that are not covered by the geometry For example:

```
const int component = 0;

for (int lvl = 0; lvl < myData.size(); lvl++){
    LevelData<EBCellFAB>& levelData = *myData[lvl];

    const DisjointBoxLayout& levelGrids = levelData.disjointBoxLayout();
```

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```

for (DataIterator dit = levelGrids.dataIterator(); dit.ok(); ++dit){

    EBCellFAB& patchData      = levelData[dit()];
    BaseFab<Real>& regularData = patchData.getSingleValuedFab();

    auto regularKernel = [&](const IntVect& iv) -> void {
        regularData(iv, component) = 1.0;
    };

    auto irregularKernel = [&](const VolIndex& vof) -> void {
        patchData(vof, component) = 1.0;
    };

    // Kernel regions (defined by user)
    Box computeBox;
    VoFIIterator vofit;

    BoxLoops::loop(computeBox, regularKernel);
    BoxLoops::loop(vofit, irregularKernel);
}
}

```

There are loops available for other types of data (e.g., face-centered data), see the [BoxLoop documentation](#).

### 3.3.4 Coarsening data

Conservative coarsening of data is done using the `averageDown(...)` functions in [AmrMesh](#). When using these functions, coarse-grid data is replaced by a conservative average of fine grid data throughout the entire AMR hierarchy. The signatures for various types of data are as follows:

```

// Conservatively coarsen multifluid cell-centered data
void averageDown(MFAMRCellData& a_data, const std::string a_realm) const;

// Conservatively coarsen multifluid face-centered data
void averageDown(MFAMRFluxData& a_data, const std::string a_realm) const;

// Conservatively coarsen cell-centered data
void averageDown(EBAMRCellData& a_data, const std::string a_realm, const phase::which_phase a_phase) const;

// Conservatively coarsen face-centered data
void averageDown(EBAMRFluxData& a_data, const std::string a_realm, const phase::which_phase a_phase) const;

// Conservatively coarsen EB-centered data
void averageDown(EBAMRIVData& a_data, const std::string a_realm, const phase::which_phase a_phase) const;

```

There are other types of coarsening available also. For example, the `averageFaces(...)` will use unweighted averaging, see the [AmrMesh API](#) for further details.

### 3.3.5 Filling ghost cells

Filling ghost cells is done using the `interpGhost(...)` functions in [AmrMesh](#).

```

void interpGhost(MFAMRCellData& a_data, const std::string a_realm) const;
void interpGhost(EBAMRCellData& a_data, const std::string a_realm, const phase::which_phase a_phase) const;

```

This will fill the specified number of ghost cells using data from the coarse level only, using piecewise linear interpolation.

As an alternative, one *can* interpolate a single layer of ghost cells using the multigrid interpolator (see [Ghost cell interpolation](#)). In this case only a single layer of ghost cells are filled in regular regions, but additional ghost cells (up to some specified range) are filled near the EB. This is often required when computing gradients (to avoid reaching into invalid cut-cells), see [Computing gradients](#) for details. The functions for filling ghost cells in this way are

```
void interpGhostMG(MFAMRCellData& a_data, const std::string a_realm) const;
void interpGhostMG(EBAMRCellData& a_data, const std::string a_realm, const phase::which_phase a_phase) const;
```

See the [AmrMesh API](#) for further details.

### 3.3.6 Computing gradients

In chombo-discharge gradients are computed using a standard second-order stencil based on finite differences. This is true everywhere except near the refinement boundary and EB where the coarse-side stencil will avoid using the coarsened data beneath the fine level. This is shown in Fig. 3.3.2 which shows the typical 5-point stencil in regular grid regions, and also a much larger and more complex stencil.

In Fig. 3.3.2 we have shown two regular 5-point stencils (red and green). The coarse stencil (red) reaches underneath the fine level and uses the data defined by coarsening of the fine-level data. The coarsened data in this case is just an average of the fine-level data. Likewise, the green stencil reaches over the refinement boundary and into one of the ghost cells on the coarse level.

Fig. 3.3.2 also shows a much larger stencil (blue stencil). The larger stencil is necessary because computing the  $y$  component of the gradient using a regular 5-point stencil would have the stencil reach underneath the fine level and into coarse data that is also irregular data. Since there is no unique way (that we know of) for coarsening the cut-cell fine-level data onto the coarse cut-cell without introducing spurious artifacts into the gradient, we reconstruct the gradient using a least squares procedure. In this case we fetch a sufficiently large neighborhood of cells for computing a least squares minimization of a local solution reconstruction in the neighborhood of the coarse cell. In order to avoid fetching potentially badly coarsened data, this neighborhood of cells only uses *valid* grid cells, i.e. the stencil does not reach underneath the fine level at all. Once this neighborhood of cells is obtained, we compute the gradient using the procedure in [Least squares](#).

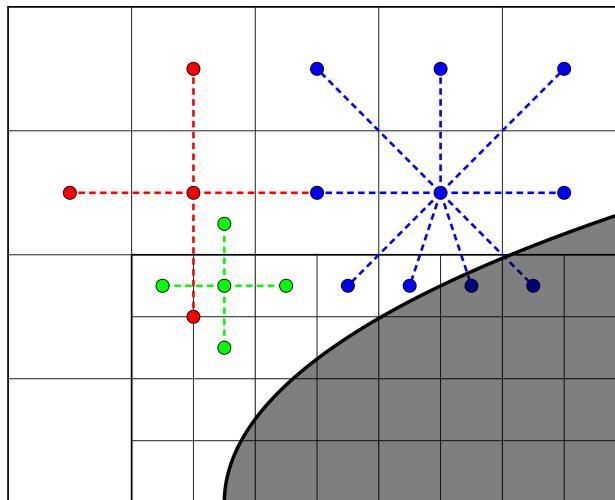


Fig. 3.3.2: Example of stencils for computing gradients near embedded boundaries. The red stencil shows a regular 5-point stencil for computing the gradient on the coarse side of the refinement boundary; it reaches into the coarsened data beneath the fine level. The green stencil shows a similar 5-point stencil on the fine side of the refinement boundary; the stencil reaches over the refinement boundary and into one ghost cell. The blue stencils shows a much more complex stencil which is computed using a least squares reconstruction procedure.

To compute gradients of a scalar, one can simply call `AmrMesh::computeGradient(...)` functions:

```
void computeGradient(EBAMRCellData& a_phi,
```

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```

const std::string a_realm,
const phase::which_phase a_phase) const;
```

```
void computeGradient(MFAMRCellData& a_gradient, const MFAMRCellData& a_phi, const std::string a_realm) const;
```

See [AmrMesh](#) or refer to the [AmrMesh API](#) for further details.

### 3.3.7 Copying data

To copy data, one may use the `EBAMRData<T>::copy(...)` function or `DataOps::copy` (see [DataOps](#)). These differ in the following way:

- `EBAMRData<T>::copy` works across realms, but will not copy ghost cells.
- `DataOps::copy` will always do a local copy, and thus the data that is copied *must* be defined on the same realm.

If you call `EBAMRData<T>::copy(...)`, the data holders will first check if they are both defined on the same realm. If they are, a purely local copy is performed, which will include ghost cells. Communication copies involving MPI are performed otherwise, in which case ghost cells are *not* copied into the new data holder.

### 3.3.8 DataOps

We have prototyped functions for many common data operations in a static class `DataOps`. For example, setting the value of various data holders can be done with

```

EBAMRFluxData cellData;
EBAMRFluxData fluxData;
EBAMRIVData irreData;

DataOps::setValue(cellData, 0.0);
DataOps::setValue(fluxData, 1.0);
DataOps::setValue(irreData, 2.0);
```

For the full API, see the [DataOps documentation](#).

## 3.4 Particles

chombo-discharge supports computational particles using native Chombo particle data. The source code for the particle functionality resides in `$DISCHARGE_HOME/Source/Particle`.

### 3.4.1 GenericParticle

`GenericParticle` is a default particle usable by the Chombo particle library. The particle type is a template

```

template <size_t M, size_t N>
class GenericParticle
{
public
    RealVect&
    position();
protected:
    RealVect m_position;
    std::array<Real, M> m_scalars;
    std::array<RealVct, N> m_vectors;
};
```

where M and N are the number of Real and RealVect variables for the particle. The `GenericParticle` also stores the position of the particle, which is available through `GenericParticle<M,N>::position`.

To fetch the Real and RealVect variables, `GenericParticle` has member functions

```
template <size_t K>
inline Real&
GenericParticle<M,N>::real();

template <size_t K>
inline RealVect&
GenericParticle<M,N>::vect();
```

If using `GenericParticle` directly, the correct C++ way of fetching one of these variables is

```
GenericParticle<2,2> p;

Real& s = p.template real<0>();
```

Note that one must include the template keyword.

---

**Tip:** The `GenericParticle` C++ API is found at <https://chombo-discharge.github.io/chombo-discharge/doxygen/html/classGenericParticle.html>.

---

### 3.4.2 Custom particles

To create a simple custom particle class with more sane signatures, one can inherit from `GenericParticle` and specify new function signatures that return the appropriate fields:

```
class KineticParticle : public GenericParticle<1,2>
{
public:
    inline
    Real& weight() {
        return this->real<0>;
    }

    inline
    RealVect& velocity() {
        return this->vect<0>;
    }

    inline
    RealVect& acceleration() {
        return this->vect<1>;
    }
};
```

There are many particles in chombo-discharge, see the `GenericParticle` C++ API for more information.

### 3.4.3 ParticleContainer

The `ParticleContainer<P>` is a template class that

1. Stores computational particles of type P over an AMR hierarchy.
2. Provides infrastructure for mapping and remapping.

`ParticleContainer<P>` uses the Chombo structure `ParticleData<P>` under the hood, and therefore has template constraints on P. The simplest way to use `ParticleContainer` for a new type of particle is to let P inherit from `GenericParticle`. However, the fundamental requirement on P is just that it must contain the appropriate Chombo linearization functions and a `const RealVect& P::position()` function.

### 3.4.4 Data structures

#### List<P> and ListBox<P>

At the lowest level the particles are always stored in a linked list `List<P>`. The class can be simply be thought of as a regular list of `P` with non-random access.

The `ListBox<P>` consists of a `List<P>` *and* a `Box`. The latter specifies the grid patch that the particles are assigned to.

To get the list of particles from a `ListBox<P>`:

```
ListBox<P> myListBox;
List<P>& myList = myListBox.listItems();
```

#### ListIterator<P>

In order to iterate over particles, use an iterator `ListIterator<P>` (which is not random access):

```
List<P> myParticles;
for (ListIterator<P> lit(myParticles); lit.ok(); ++lit){
    P& p = lit();
    // ... do something with this particle
}
```

#### ParticleData<P>

On each grid level, `ParticleContainer<P>` stores the particles in a Chombo class `ParticleData`.

```
template <class P>
ParticleData<P>
```

where `P` is the particle type. `ParticleData<P>` can be thought of as a `LevelData<ListBox<P> >`, although it actually inherits from `LayoutData<ListBox<P> >`. Each grid patch contains a `ListBox<P>` of particles.

#### AMRParticles<P>

`AMRParticles<P>` is our AMR version of `ParticleData<P>`. It is a simply a typedef of a vector of pointers to `ParticleData<P>` on each level:

```
template <class P>
using AMRParticles = Vector<RefCountedPtr<ParticleData<P> > >;
```

Again, the `Vector` indicates the AMR level and the `ParticleData<P>` is a distributed data holder that holds the particles on each AMR level.

### 3.4.5 Basic use

Here, we give some examples of basic use of `ParticleContainer`. For the full API, see the `ParticleContainer` C++ API <https://chombo-discharge.github.io/chombo-discharge/doxygen/html/classParticleContainer.html>.

#### Getting the particles

To get the particles from a `ParticleContainer<P>` one can call `AMRParticles<P>& ParticleContainer<P>::getParticles()` which will provide the particles:

```
ParticleContainer<P> myParticleContainer;
AMRParticles<P>& myParticles = myParticleContainer.getParticles();
```

Alternatively, one can fetch directly from a specified grid level as follows:

```
int lvl;
ParticleContainer<P> myParticleContainer;
ParticleData<P>& levelParticles = myParticleContainer[lvl];
```

#### Iterating over particles

To do something basic with the particle in a `ParticleContainer<P>`, one will typically iterate over the particles in all grid levels and patches.

The code bit below shows a typical example of how the particles can be moved, and then remapped onto the correct grid patches and ranks if they fall off their original one.

```
ParticleContainer<P> myParticleContainer;

// Iterate over grid levels
for (int lvl = 0; lvl <= m_amr->getFinestLevel(); lvl++){

    // Get the grid on this level.
    const DisjointBoxLayout& dbl = m_amr->getGrids(myParticleContainer.getRealm())[lvl];

    // Get the distributed particles on this level
    ParticleData<P>& levelParticles = myParticleContainer[lvl]

    // Iterate over grid patches on this level
    for (DataIterator dit(dbl); dit.ok(); ++dit){

        // Get the particles in the current patch.
        List<P>& patchParticles = levelParticles[dit].listItems();

        // Iterate over the particles in the current patch.
        for (ListIterator<P> lit(patchParticles); lit.ok(); ++lit){
            P& p = lit;

            // Move the particle
            p.position() = ...
        }
    }

    // Remap particles onto new patches and ranks (they may have moved off their original ones)
    myParticleContainer.remap();
}
```

### 3.4.6 Sorting particles

#### Sorting by cell

The particles can also be sorted by cell by calling `void ParticleContainer<P>::sortParticleByCell()`, like so:

```
ParticleContainer<P> myParticleContainer;
myParticleContainer.sortParticlesByCell();
```

Internally in `ParticleContainer<P>`, this will place the particles in another container which can be iterated over on a per-cell basis. This is different from `List<P>` and `ListBox<P>` above, which contained particles stored on a per-patch basis with no internal ordering of the particles.

The per-cell particle container is a `Vector<RefCountedPtr<LayoutData<BinFab<P>>>` type where again the `Vector` holds the particles on each AMR level and the `LayoutData<BinFab>` holds one `BinFab` on each grid patch. The `BinFab` is also a template, and it holds a `List<P>` in each grid cell. Thus, this data structure stores the particles per cell rather than per patch. Due to the horrific template depth, this container is `typedef`'ed as `AMRCellParticles<P>`.

To get cell-sorted particles one can call

```
AMRCellParticles<P>& cellSortedParticles = myParticleContainer.getCellParticles();
```

Iteration over cell-sorted particles is mostly the same as for patch-sorted particles, except that we also need to explicitly iterate over the grid cells in each grid patch:

```
const int comp = 0;

// Iterate over all AMR levels
for (int lvl = 0; lvl <= m_amr->getFinestLevel(); lvl++){

    // Get the grids on this level
    const DisjointBoxLayout& dbl = m_amr->getGrids(myParticleContainer.getRealm())[lvl];

    // Iterate over grid patches on this level
    for (DataIterator dit(dbl); dit.ok(); ++dit){

        // Get the Cartesian box for the current grid patch
        const Box cellBox = dbl[dit()];

        // Get the particles in the current grid patch.
        BinFab<P>& cellSortedBoxParticles = (*cellSortedParticles[lvl])[dit()];

        // Iterate over all cells in the current box
        for (BoxIterator bit(cellBox); bit.ok(); ++bit){
            const IntVect iv = bit();

            // Get the particles in the current grid cell.
            List<P>& cellParticles = cellSortedBoxParticles(iv, comp);

            // Do something with cellParticles
            for (ListIterator<P> lit(cellParticles); lit.ok(); ++lit){
                P& p = lit();
            }
        }
    }
}
```

## Sorting by patch

If the particles need to return to patch-sorted particles:

```
ParticleContainer<P> myParticleContainer;
myParticleContainer.sortParticlesByPatch();
```

---

**Important:** If particles are sorted by cell, calling `ParticleContainer<P>` member functions that fetch particles by patch will issue an error. This is done by design since the patch-sorted particles have been moved to a different container. Note that remapping particles also requires that the particles are patch-sorted. Calling `remap()` with cell-sorted particles will issue a run-time error.

---

## 3.4.7 Allocating particles

`AmrMesh` has a very simple function for allocating a `ParticleContainer<P>`:

```
template <typename P>
void allocate(ParticleContainer<P>& a_container, const std::string a_realm);
```

which will allocate a `ParticleContainer` on realm `a_realm`. See [AmrMesh](#) for further details.

## 3.4.8 Particle mapping

Particles that move off their original grid patch must be remapped in order to ensure that they are assigned to the correct grid. The remapping function for `ParticleContainer<P>` is

```
void
ParticleContainer<P>::remap();
```

This is simply used as follows:

```
ParticleContainer<P> myParticles;
myParticles.remap();
```

## 3.4.9 Regridding

`ParticleContainer<P>` is comparatively simple to regrid, and this is done in two steps:

1. Before creating the new grids, each MPI rank collects *all* particles on a single `List<P>` by calling

```
void ParticleContainer<P>::preRegrid(int a_base)
```

This will pull the particles off their current grids and collect them in a single list (on a per-rank basis).

2. When `ParticleContainer<P>` regrids, each rank adds his `List<P>` back into the internal particle containers.

The use case typically looks like this:

```
ParticleContainer<P> myParticleContainer;
// Each rank caches his particles
const int baseLevel = 0;
myParticleContainer.preRegrid(0);
// Driver does a regrid.
```

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```
.
.
.

// After the regrid we fetch grids from AmrMesh;
Vector<DisjointBoxLayout> grids;
Vector<ProblemDomain> domains;
Vector<Real> dx;
Vector<int> refinement_ratios;
int base;
int newFinestLevel;

myParticleContainer.regrid(grids, domains, dx, refinement_ratios, baseLevel, newFinestLevel);
```

Here, `baseLevel` is the finest level that didn't change and `newFinestLevel` is the finest AMR level after the regrid.

### 3.4.10 Masked particles

`ParticleContainer<P>` also supports the concept of *masked particles*, where one can fetch a subset of particles that live only in specified regions in space. Typically, this “specified region” is the refinement boundary, but the functionality is generic and might prove useful also in other cases.

When *masked particles* are used, the user can provide a boolean mask over the AMR hierarchy and obtain the subset of particles that live in regions where the mask evaluates to true. This functionality is for example used for some of the particle deposition methods in `chombo-discharge` where we deposit particles that live near the refinement boundary with special deposition functions.

To fill the masked particles, `ParticleContainer<P>` has members functions for copying the particles into internal data containers which the user can later fetch. The function signatures for these are

```
using AmrMask = Vector<RefCountedPtr<LevelData<BaseFab<bool>>>;
template <class P>
void copyMaskParticles(const AmrMask& a_mask) const;
template <class P>
void copyNonMaskParticles(const AmrMask& a_mask) const;
```

The argument `a_mask` holds a bool at each cell in the AMR hierarchy. Particles that live in cells where `a_mask` is true will be copied to an internal data holder in `ParticleContainer<P>` which can be retrieved through a call

```
AMRParticles<P>& maskParticles = myParticleContainer.getMaskParticles();
```

Note that `copyNonMaskParticles` is just like `copyMaskParticles` except that the bools in `a_mask` have been flipped.

In the above functions the mask particles are *copied*, and the original particles are left untouched. After the user is done with the particles, they should be deleted through the functions `void clearMaskParticles()` and `void clearNonMaskParticles`, like so:

```
AmrMask myMask;
ParticleContainer<P> myParticles;

// Copy mask particles
myParticles.copyMaskParticles(myMask);

// Do something with the mask particles
AMRParticles<P>& maskParticles = myParticleContainer.getMaskParticles();

// Release the mask particles
myParticles.clearMaskParticles();
```

## Creating particle halo masks

AmrMesh can register a *halo* mask with a specified width:

```
void registerMask(const std::string a_mask, const int a_buffer, const std::string a_realm);
```

where `a_mask` must be "s\_particle\_halo". This will register a mask which is false everywhere except in coarse-grid cells that are within a distance `a_buffer` from the refinement boundary, see Fig. 3.4.1. This functionality is useful when processing particles on the refinement boundary using special deposition functions since the halo mask allows us to straightforwardly extract those particles.

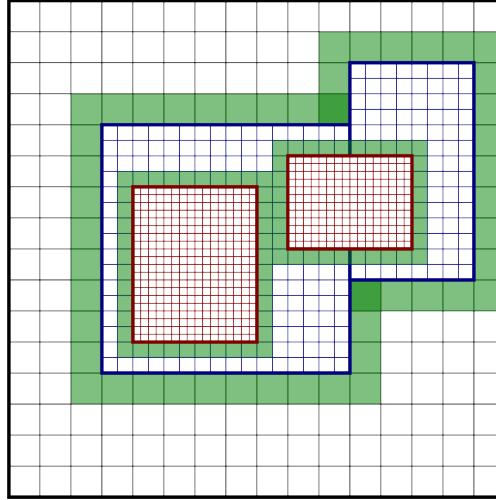


Fig. 3.4.1: Example of a particle halo mask (shaded green color) surrounding refined grid levels.

### 3.4.11 Wall interaction

`ParticleContainer<P>` is EB-agnostic and has no information about the embedded boundary. This means that particles remap just as if the EB was not there. Interaction with the EB is done via the implicit function or discrete information, as well as modifications in the interpolation and deposition steps.

#### Signed distance function

When signed distance functions are used, one can always query how far a particle is from a boundary:

```
List<P>& particles;
BaseIF distanceFunction;

for (ListIterator<P> lit(particles); lit.ok(); ++lit){
    const P& p = lit();
    const RealVect& pos = p.position();

    const Real distanceToBoundary = distanceFunction.value(pos);
}
```

If the particle is inside the EB then the signed distance function will be positive and the particle can be removed from the simulation. The distance function can also be used to detect collisions between particles and the EB. See [AmrMesh](#) for details on how to obtain the distance function.

## Domain edges

By default, the `ParticleContainer` remapping function will discard particles that fall outside of the domain. The user can also check if this happen by checking if the particle position is outside the computational domain:

```
GenericParticle<0,0> p;

const RealVect pos = p.position();
const RealVect probLo = m_amr->getProbLo();
const RealVect probHi = m_amr->getProbHi();

bool outside = false;
for (int dir = 0; dir < SpaceDim; dir++) {
    if(pos[dir] < probLo[dir] || pos[dir] > probHi[dir]) {
        outside = true;
    }
}
```

## Particle intersection

It is occasionally useful to catch particles that hit an EB or crossed a domain side. Assuming that the particle type `P` also has a member function that stores the starting position of the particle, one can compute the intersection point between the particle trajectory and the EB and domain edges/faces. Currently, `AmrMesh` supports two methods for computing this

- Using a bisection algorithm with a user-specified step.
- Using a ray-casting algorithm.

These algorithms differ in the sense that the bisection approach will check for a particle crossing between two positions  $x_0$  and  $x_1$  using a pre-defined tolerance. The ray-casting algorithm will check if the particle can move from  $x_0$  towards  $x_1$  by using a variable step along the particle trajectory. This step is selected from the signed distance from the particle position to the EB such that it uses a large step if the particle is far away from the EB. Conversely, if the particle is close to the EB a small step will be used. For the function signatures, see [Particle intersection](#). The algorithms that operate under the hood of these routines are given in `ParticleOps`, see [ParticleOps](#).

Both the bisection and ray-casting algorithm have weaknesses. The bisection algorithm algorithm requires a user-supplied step in order to operate efficiently, while the ray-casting algorithm is very slow when the particle is close to the EB and moves tangentially along it. Future versions of `chombo-discharge` will likely include more sophisticated algorithms.

---

**Tip:** `AmrMesh` also stores the implicit function on the mesh, which could also be used to resolved particle collisions with the EB/domain.

---

### 3.4.12 Particle-mesh

#### Particle deposition

To deposit particles on the mesh, the user can call the templated function `AmrMesh::depositParticles` which have a signatures

```
template <class P, const Real&(P::*particleScalarField)() const>
void depositParticles(EBAMRCellData& a_meshData,
                      const std::string& a_realm,
                      const phase::which_phase& a_phase,
                      const ParticleContainer<P>& a_particles,
                      const DepositionType a_depositionType,
                      const CoarseFineDeposition a_coarseFineDeposition,
```

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```

const bool a_forceIrregNGP);

template <class P, const RealVect&(P::*particleVectorField)() const>
void depositParticles(EBAMRCelldata&
    a_meshData,
    const std::string& a_realm,
    const phase::which_phase& a_phase,
    const ParticleContainer<P>& a_particles,
    const DepositionType a_depositionType,
    const CoarseFineDeposition a_coarseFineDeposition,
    const bool a_forceIrregNGP);

```

Here, the template parameter `P` is the particle type and the template parameter `particleScalarField` is a C++ pointer-to-member-function. This function must have the indicated signature `const Real& P::particleScalarField() const` or the signature `Real P::particleScalarField() const`. The pointer-to-member `particleScalarField` indicates the variable to be deposited on the mesh. This function pointer does not need to return a member in the particle class.

When depositing vector-quantities (such as electric currents), one must call the version which takes `RealVect P::particleVectorField() const` as a template parameter. The supplied function must return a `RealVect` and `a_meshData` must then have `SpaceDim` components.

Next, the input arguments to `depositParticles` are the output mesh data holder (must have exactly one or `SpaceDim` components), the realm and phase where the particles live, and the particles themselves (`a_particles`). Finally, the flag `a_forceIrregNGP` permits the user to enforce nearest grid-point deposition in cut-cells. This option is motivated by the fact that some applications might require hard mass conservation, and the user can ensure that mass is never deposited into covered grid cells.

The input argument `a_depositionType` indicates the deposition method, while `a_coarseFineDeposition` deposition modifications near refinement boundaries. These are discussed below.

## Base deposition

The base deposition scheme is specified by an enum `DepositionType` with valid values:

- `DepositionType::NGP` (Nearest grid-point).
- `DepositionType::CIC` (Cloud-In-Cell).
- `DepositionType::TSC` (Triangle-Shaped Cloud).
- `DepositionType::W4` (Fourth order weighted).

## Coarse-fine deposition

The input argument `a_coarseFineDeposition` determines how coarse-fine deposition is handled. Refinement boundaries introduce additional complications in the deposition scheme due to

1. Fine-grid particles whose deposition clouds hang over the refinement boundary and onto the coarse level.
2. Coarse-grid particles whose deposition clouds stick underneath the fine-level.

`chombo-discharge` currently supports three methods for handling coarse-fine deposition. In all of these methods the mass on the fine grid particles whose deposition clouds hang over the refinement boundaries is simply added to the coarse grid. For the coarse-grid particles the following processes then occur:

- `CoarseFineDeposition::Interp` This method permits the coarse-grid particles to deposit into the region underneath the fine grid. The deposited quantity is then piecewise interpolated onto the fine grid. The indicated coarse-grid deposition cloud in Fig. 3.4.2 will then add its mass into two layers of fine-grid cells.

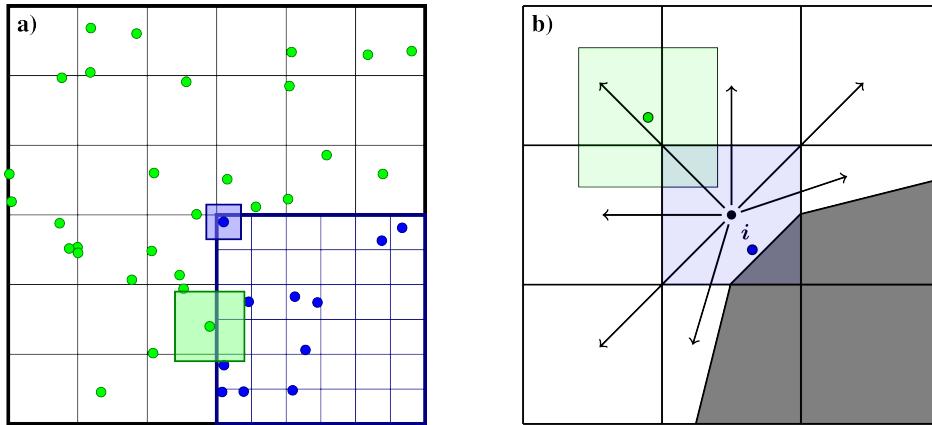


Fig. 3.4.2: Sketch of deposition schemes near refinement boundaries and cut-cells.

- `CoarseFineDeposition::Halo` This method extracts the coarse-grid particles that live on the refinement boundary. These particles are then transferred to the fine level and they are then deposit on the fine grid using the original particle width. For example, if using a CIC scheme and having a refinement factor of 2 between the coarse grid and fine grid, the particle width on the fine grid will be  $2\Delta x_{\text{fine}}$  rather than  $\Delta x_{\text{fine}}$ .

**Warning:** This functionality is currently limited to NGP and CIC schemes.

- `CoarseFineDeposition::HaloNGP` Like `CoarseFineDeposition::Halo`, this method also extracts the coarse-grid particles on the coarse side of the refinement boundary, but rather than using the original scheme these particles are deposited with an NGP scheme.

## Particle interpolation

To interpolate a field onto a particle position, the user can call the `AmrMesh` member functions

```
template <class P, Real&(P::*particleScalarField)()>
void interpolateParticles(ParticleContainer<P>& a_particles,
                         const std::string& a_realm,
                         const phase::which_phase& a_phase,
                         const EBAMRCellData& a_meshScalarField,
                         const DepositionType a_interpType,
                         const bool a_forceIrregNGP) const;

template <class P, RealVect&(P::*particleVectorField)()>
void interpolateParticles(ParticleContainer<P>& a_particles,
                         const std::string& a_realm,
                         const phase::which_phase& a_phase,
                         const EBAMRCellData& a_meshVectorField,
                         const DepositionType a_interpType,
                         const bool a_forceIrregNGP) const;
```

The function signature for particle interpolation is pretty much the same as for particle deposition, with the exception of the interpolated field. The template parameter `P` still indicates the particle type, but the user can interpolate onto either a scalar particle variable or a vector variable. For example, in order to interpolate the particle acceleration, the particle class (let's call it `MyParticleClass`) will typically have a member function `RealVect& acceleration()`, and in this case one can interpolate the acceleration by

```
RefCountedPtr<AmrMesh> amr;
amr->interpolateParticles<MyParticleClass, &MyParticleClass::acceleration>(...)
```

Note that if the user interpolates onto a scalar variable, the mesh variable must have exactly one component. Likewise, if interpolating a vector variable, the mesh variable must have exact SpaceDim components.

## Example

Assume that we have some particle class KineticParticle defined as

```
class KineticParticle : public GenericParticle<1,2>
{
public:
    inline
    Real& weight() {
        return this->real<0>();
    }

    inline
    RealVect& velocity() {
        return this->vect<0>();
    }

    inline
    RealVect& acceleration() {
        return this->vect<1>();
    }

    inline
    RealVect momentum() const {
        return this->weight() * this->velocity();
    }
};
```

To deposit the weight, velocity, and momentum on the grid we would call

```
RefCountedPtr<AmrMesh> amr;

amr->depositParticles<KineticParticle, &KineticParticle::mass>(...);
amr->depositParticles<KineticParticle, &KineticParticle::velocity>(...);
amr->depositParticles<KineticParticle, &KineticParticle::momentum>(...);
```

Likewise, to interpolate onto these fields we can call

```
RefCountedPtr<AmrMesh> amr;

amr->interpolateParticles<KineticParticle, &KineticParticle::mass>(...);
amr->interpolateParticles<KineticParticle, &KineticParticle::velocity>(...);
```

### 3.4.13 Superparticles

#### Custom approach

For a custom approach of managing superparticles, users can simply manipulate the particle lists in the grid patches or grid cells. In each case one starts with a list `List<P>` that needs to be modified.

#### kD-trees

##### Overview

chombo-discharge has functionality for spatially partitioning particles using kD-trees, which can be used as a basis for particle merging and splitting. kD-trees operate by partitioning a set of input primitives into spatially coherent subsets. At each level in the tree recursion one chooses an axis for partitioning one subset into two new subsets, and the recursion continues until the partitioning is complete. Fig. 3.4.3 shows an example where a set of initial particles are partitioned using such a tree.

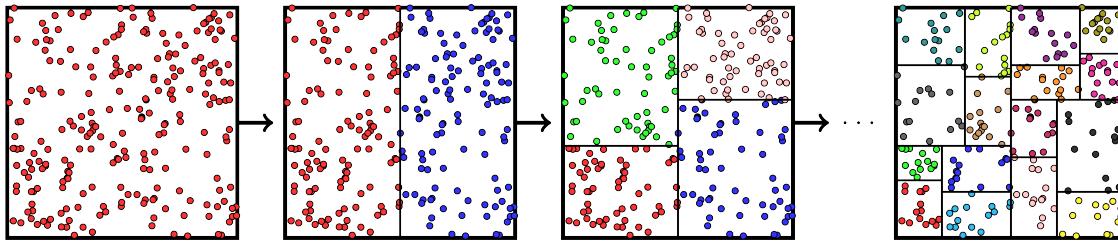


Fig. 3.4.3: Example of a kD-tree partitioning of particles in a single cell.

---

**Tip:** The source code for the kD-tree functionality is given in `$DISCHARGE_HOME/Source/Particle/CD_SuperParticles.H`.

---

#### Particle partitioners

The kD-tree partitioner requires a user-supplied criterion for particle partitioning. Only the partitioner `PartitionEqualWeight` is currently supported, and this partitioner will divide the original subset into two new subsets such that the particle weights in the two halves differ by at most one physical particle. This partitioner is implemented as

```
template <class P, Real& (P::*weight)(), const RealVect& (P::*position)() const>
typename KDNode<P>::Partitioner PartitionEqualWeight;
```

Here, `P` is the particle type, and this class *must* have function members `Real& P::weight()` and `const RealVect& P::position()` which return the particle weight and position.

**Warning:** `PartitionEqualWeight` will usually split particles to ensure that the weight in the two subsets are the same (thus creating new particles). In this case any other members in the particle type are copied over into the new particles.

The particles in each leaf of the kD-tree can then be merged into new particles. Since the weight in the nodes of the tree differ by at most one, the resulting computational particles also have weights that differ by at most one.

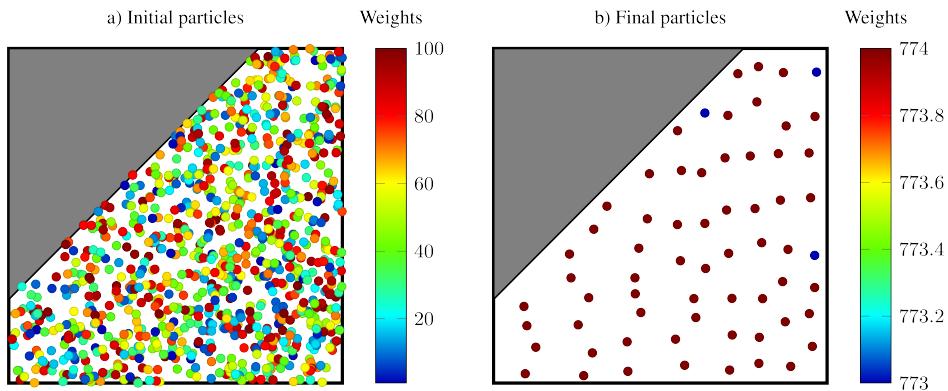


Fig. 3.4.4: kD-tree partitioning of particles into new particles whose weight differ by at most one. Left: Original particles with weights between 1 and 100. Right: Merged particles.

### 3.4.14 ParticleOps

`ParticleOps` is a static data class that provides methods for commonly used particle operations. These include

- Intersection of particles with EBs.
- Intersection of particles with domain edges/faces.
- Drawing particles from a probability distribution.

The `ParticleOps` API is found at <https://chombo-discharge.github.io/chombo-discharge/doxygen/html/classParticleOps.html>.

## 3.5 Realm

`Realm` is a class for centralizing EBAMR-related grids and operators for a specific AMR grid. For example, a `Realm` consists of a set of grids (i.e. a `Vector<DisjointBoxLayout>`) as well as *operators*, e.g. functionality for filling ghost cells or averaging down a solution from a fine level to a coarse level. One may think of a `Realm` as a fully-fledged AMR hierarchy with associated multilevel operators, i.e. how one would usually do AMR.

### 3.5.1 Dual grid

The reason why `Realm` exists at all is due to individual load balancing of algorithmic components. The terminology *dual grid* is used when more than one `Realm` is used in a simulation, and in this case the user/developer has chosen to solve the equations of motion over a different set of `DisjointBoxLayout` on each level. This approach is very useful when using computational particles since users can quickly generate separate Eulerian sets of grids for fluids and particles, and the grids can then be load balanced separately. Note that every `Realm` consists of the same boxes, i.e. the physical domain and computational grids are the same for all realms. The difference lies primarily in the assignment of MPI ranks to grids; i.e. the load-balancing and domain decomposition.

Fig. 3.5.1 shows an example of a dual-grid approach. In this figure we have a set of grid patches on a particular grid level. In the top panel the grid patches are load-balanced using the grid patch volume as a proxy for the computational load. The numbers in each grid patch indicates the MPI rank ownership of the patches. In the bottom panel we have introduced computational particles in some of the patches. For particles, the computational load is better defined by the number of computational particles assigned to the patch, and so using the number of particles as a proxy for the load yields different rank ownership over the grid patches.

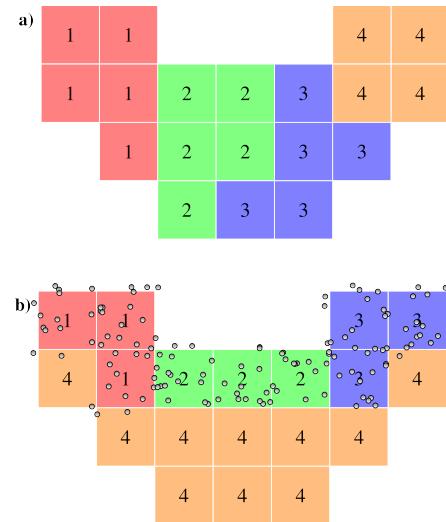


Fig. 3.5.1: Sketch of dual grid approach. Each rectangle represents a grid patch and the numbers show MPI ranks. a) Load balancing with the number of grid cells. b) Load balancing with the number of particles.

### 3.5.2 Interacting with realms

Users will not interact with `Realm` directly. Every `Realm` is owned by `AmrMesh`, and the user will only interact with realms through the public `AmrMesh` interface, for example by fetching operators for performing AMR operations. In addition, data that is defined on one realm can be copied to another; `EBAMRData<T>` takes care of this. You will simply call a copier:

```
EBAMRCellData realmOneData;
EBAMRCellData realmTwoData;

realmOneData.copy(realmTwoData);
```

The rest of the functionality uses the public interface of `AmrMesh`. For example for coarsening of multifluid data:

```
std::string multifluidRealm;
MFAMRCellData multifluidData;
AmrMesh amrMesh;

amrMesh.averageDown(multifluidData, multifluidRealm);
```

## 3.6 Regridding

Most `chombo-discharge` simulations will benefit from using AMR where grids change in time. When new grids are generated, the following situations occur:

1. Grids were removed, and data underneath the old grids must be replaced by coarsened data from the fine grids.
2. Grids were added, and data on those grids must be interpolated from the coarse data underneath them.

When grids are removed/added, computational particles must be also be associated with the new grids.

### 3.6.1 Coarsening data

When removing grids, we simply keep the old grid data. We thus assume that, prior to regridd, the user coarsened the data appropriately, see [Coarsening data](#).

### 3.6.2 Interpolation

chombo-discharge currently supports interpolation onto the new grids using:

1. Constant interpolation, where the fine-grid data is initialized using the value in the coarse grid cells.
2. With minmod slope-limiters.

The API for interpolating onto the new grids is given in [AmrMesh](#).

## 3.7 Linear solvers

### 3.7.1 Helmholtz equation

The Helmholtz equation is represented by

$$\alpha a(\mathbf{x}) \Phi + \beta \nabla \cdot [b(\mathbf{x}) \nabla \Phi] = \rho$$

where  $\alpha$  and  $\beta$  are constants and  $a(\mathbf{x})$  and  $b(\mathbf{x})$  are spatially dependent and piecewise smooth.

To solve the Helmholtz equation, it is solved in the form

$$\kappa L \Phi = \kappa \rho,$$

where  $L$  is the Helmholtz operator above. The preconditioning by the volume fraction  $\kappa$  is done in order to avoid the small-cell problem encountered in finite-volume discretizations on EB grids.

#### Discretization and fluxes

The Helmholtz equation is solved by assuming that  $\Phi$  lies on the cell-center. The  $b(\mathbf{x})$ -coefficient lies on face centers and EB faces. In the general case the cell center might lie inside the embedded boundary, and the cell-centered discretization relies on the concept of an extended state. Thus,  $\Phi$  does not satisfy a discrete maximum principle.

The finite volume update require fluxes on the face centroids rather than the centers. These are constructed by first computing the fluxes to second order on the face centers, and then interpolating them to the face centroids. For example, the flux  $F_3$  in the figure above is

$$F_3 = \beta b_{i,j+1/2} \frac{\Phi_{i,j+1} - \Phi_{i,j}}{\Delta x}.$$

#### Boundary conditions

The finite volume discretization of the Helmholtz equation require fluxes through the EB and domain faces. Below, we discuss how these are implemented.

---

**Note:** chombo-discharge supports spatially dependent boundary conditions

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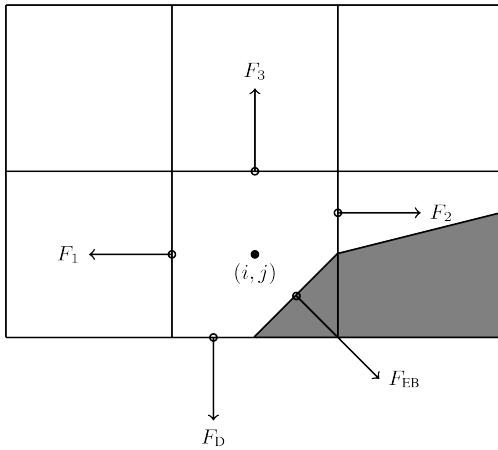


Fig. 3.7.1: Location of fluxes for finite volume discretization.

### Neumann

Neumann boundary conditions are straightforward since the flux through the EB or domain faces are specified directly. From the above figure, the fluxes  $F_{\text{EB}}$  and  $F_D$  are specified.

### Dirichlet

Dirichlet boundary conditions are more involved since only the value at the boundary is prescribed, but the finite volume discretization requires a flux. On the domain boundaries the fluxes are face-centered and we therefore use finite differencing for obtaining a second order accurate approximation to the flux at the boundary.

On the embedded boundaries the flux is more complicated to compute, and requires us to compute an approximation to the normal gradient  $\partial_n \Phi$  at the boundary. Our approach is to approximate this flux by expanding the solution as a polynomial around a specified number of grid cells. By using more grid cells than there are unknown in the Taylor series, we formulate an over-determined system of equations up to some specified order. As a first approximation we include only those cells in the quadrant or half-space defined by the normal vector, see Fig. 3.7.2. If we can not find enough equations, the strategy is to 1) drop order and 2) include all cells around the cut-cell.

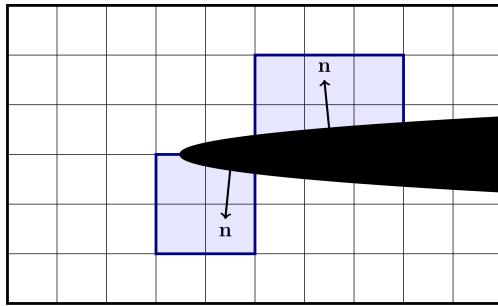


Fig. 3.7.2: Examples of neighborhoods (quadrant and half-space) used for gradient reconstruction on the EB.

Once the cells used for the gradient reconstruction have been obtained, we use weighted least squares to compute the approximation to the derivative to specified order (for details, see [Least squares](#)). The result of the least squares computation is represented as a stencil:

$$\frac{\partial \Phi}{\partial n} = w_B \Phi_B + \sum_i w_i \Phi_i,$$

where  $\Phi_B$  is the value on the boundary, the  $w$  are weights for grid points  $i$ , and the sum runs over cells in the domain.

Note that the gradient reconstruction can end up requiring more than one ghost cell layer near the embedded boundaries. For example, Fig. 3.7.3 shows a typical stencil region which is built when using second order gradient reconstruction on the EB. In this case the gradient reconstruction requires a stencil with a radius of 2, but as the cut-cell lies on the refinement boundary the stencil reaches into two layers of ghost cells. For the same reason, gradient reconstruction near the cut-cells might require interpolation of corner ghost cells on refinement boundaries.

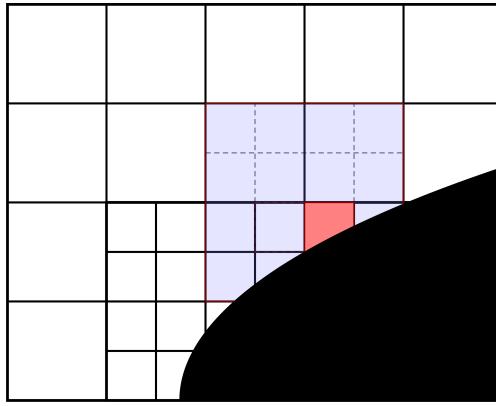


Fig. 3.7.3: Example of the region of a second order stencil for the Laplacian operator with second order gradient reconstruction on the embedded boundary.

## Robin

Robin boundary conditions are in the form

$$A\partial_n \Phi + B\Phi = C,$$

where  $A$ ,  $B$ , and  $C$  are constants. This boundary conditions is enforced through the flux

$$\partial_n \Phi = \frac{1}{A} (C - B\Phi),$$

which requires an evaluation of  $\Phi$  on the domain boundaries and the EB.

For domain boundaries we extrapolate the cell-centered solution to the domain edge, using standard first order finite differencing.

On the embedded boundary, we approximate  $\Phi(x_{EB})$  by linearly interpolating the solution with a least squares fit, using cells which can be reached with a monotone path of radius one around the EB face (see [Least squares](#) for details). The Robin boundary condition takes the form

$$\partial_n \Phi = \frac{C}{A} - \frac{B}{A} \sum_i w_i \Phi_i.$$

Currently, we include the data in the cut-cell itself in the interpolation, and thus also use unweighted least squares.

## Ghost cell interpolation

With AMR, multigrid requires ghost cells on the refinement boundary. The interior stencils for the Helmholtz operator have a radius of one and thus only require a single layer of ghost cells (and no corner ghost cells). These ghost cells are filled using a finite-difference stencil, see Fig. 3.7.4.

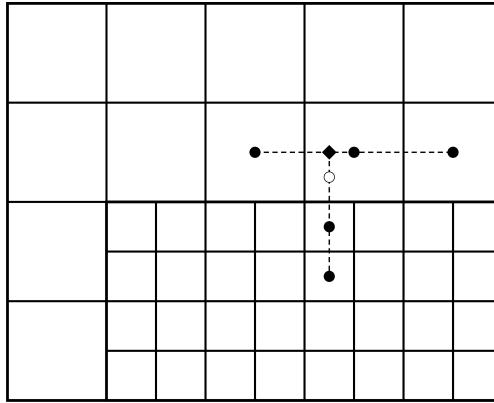


Fig. 3.7.4: Standard finite-difference stencil for ghost cell interpolation (open circle). We first interpolate the coarse-grid cells to the centerline (diamond). The coarse-grid interpolation is then used together with the fine-grid cells (filled circles) for interpolation to the ghost cell (open circle).

Embedded boundaries introduce many pathologies for multigrid:

1. Cut-cell stencils may have a large radius (see Fig. 3.7.3) and thus require more ghost cell layers.
2. The EBs cut the grid in arbitrary ways, leading to multiple pathologies regarding cell availability.

The pathologies mean that standard finite differencing fails near the EB, mandating a more general approach. Our way of handling ghost cell interpolation near EBs is to reconstruct the solution (to specified order) in the ghost cells, using the available cells around the ghost cell (see [Least squares](#) for details). As per conventional wisdom regarding multigrid interpolation, this reconstruction does *not* use coarse-level grid cells that are covered by the fine level.

Figure Fig. 3.7.5 shows a typical interpolation stencil for the stencil in Fig. 3.7.3. Here, the open circle indicates the ghost cell to be interpolated, and we interpolate the solution in this cell using neighboring grid cells (closed circles). For this particular case there are 10 nearby grid cells available, which is sufficient for second order interpolation (which require at least 6 cells in 2D).

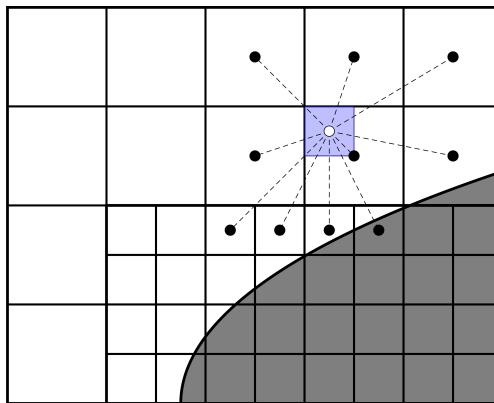


Fig. 3.7.5: Multigrid interpolation for refinement boundaries away from and close to an embedded boundary.

---

**Note:** chombo-discharge implements a fairly general ghost cell interpolation scheme near the EB. The ghost cell values can be reconstructed to specified order (and with specified least squares weights).

---

On coarse-fine interfaces the Helmholtz operators will perform a *refluxing* operations where the coarse-grid fluxes are replaced by the sum of the fine-grid fluxes. `EBHelmholtzOp` has a special flag for replacing the refluxing operation by flux coarsening, which can be specified with `EBHelmholtzOp.reflux_free = true/false`. In this case the reflux operation is turned off and we compute the fluxes on the entire fine level (not just the interface) and replace the coarse-grid fluxes by averages of the fine-grid fluxes.

### Relaxation methods

The Helmholtz equation is solved using multigrid, with various smoothers available on each grid level. The currently supported smoothers are:

1. Standard point Jacobi relaxation.
2. Red-black Gauss-Seidel relaxation in which the relaxation pattern follows that of a checkerboard.
3. Multi-colored Gauss-Seidel relaxation in which the relaxation pattern follows quadrants in 2D and octants in 3D.

Users can select between the various smoothers in solvers that use multigrid.

---

**Note:** Multi-colored Gauss-Seidel usually provide the best convergence rates. However, the multi-colored kernels are twice as expensive as red-black Gauss-Seidel relaxation in 2D, and four times as expensive in 3D.

---

### 3.7.2 Multiphase Helmholtz equation

chombo-discharge also supports a *multiphase version* where data exists on both sides of the embedded boundary. The most common case is that involving discontinuous coefficients, e.g. for

$$\nabla \cdot [b(\mathbf{x}) \nabla \Phi(\mathbf{x})] = 0.$$

where  $b(\mathbf{x})$  is only piecewise constant.

### Jump conditions

For the case of discontinuous coefficients there is a jump condition on the interface between two materials:

$$b_1 \partial_{n_1} \Phi + b_2 \partial_{n_2} \Phi = \sigma,$$

where  $b_1$  and  $b_2$  are the Helmholtz equation coefficients on each side of the interface, and  $n_1 = -n_2$  are the normal vectors pointing away from the interface in each phase.  $\sigma$  is a jump factor.

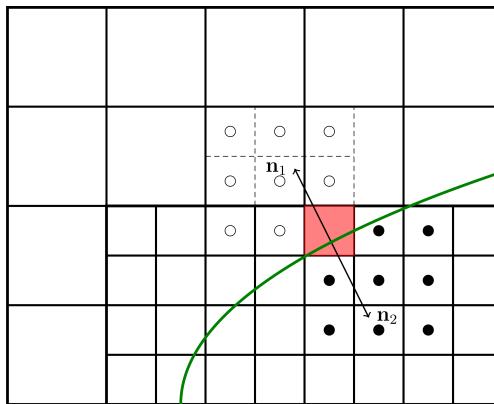


Fig. 3.7.6: Example of cells and stencils that are involved in discretizing the jump condition. Open and filled circles indicate cells in separate phases.

### Discretization

To incorporate the jump condition in the Helmholtz discretization, we use a gradient reconstruction to obtain a solution to  $\Phi$  on the boundary, and use this value to impose a Dirichlet boundary condition during multigrid relaxation. Recalling the gradient reconstruction  $\frac{\partial \Phi}{\partial n} = w_B \Phi_B + \sum_i w_i \Phi_i$ , the matching condition (see Fig. 3.7.6) can be written as

$$b_1 \left[ w_{B,1} \Phi_B + \sum_i w_{i,1} \Phi_{i,1} \right] + b_2 \left[ w_{B,2} \Phi_B + \sum_i w_{i,2} \Phi_{i,2} \right] = \sigma.$$

This equation can be solved for the boundary value  $\Phi_B$ , which can then be used to compute the finite-volume fluxes into the cut-cells.

---

**Note:** For discontinuous coefficients the gradient reconstruction on one side of the EB does not reach into the other (since the solution is not differentiable across the EB).

---

### 3.7.3 AMRMultiGrid

AMRMultiGrid is the Chombo implementation of the Martin-Cartwright multigrid algorithm. It takes an “operator factory” as an argument, and the factory can generate objects (i.e., operators) that encapsulate the discretization on each AMR level.

chombo-discharge runs its own operator, and the user can use either of:

1. EBHelmholtzOpFactory for single-phase problems.
2. MFHelmholtzOpFactory for multi-phase problems.

The source code for these are located in `$DISCHARGE_HOME/Source/Elliptic`.

## Bottom solvers

Chombo provides (at least) three bottom solvers which can be used with `AMRMultigrid`.

1. A regular smoother (e.g., point Jacobi).
2. A biconjugate gradient stabilized method (BiCGStab)
3. A generalized minimal residual method (GMRES).

The user can select between these for the various solvers that use multigrid.

## 3.8 Verification and validation

We strive to include convergence testing (verification), and in some cases comparison with various types of experimental (validation). Below, we discuss our approach to spatial and temporal convergence testing.

### 3.8.1 Spatial convergence

Assume that we have some evolution problem which provides a solution on the mesh as  $\phi_i^k(\Delta x, \Delta t)$  where  $\Delta x$  is a uniform grid resolution and  $\Delta t$  is the time step used for evolving the state from  $t = 0$  to  $t = k\Delta t$ .

To estimate the spatial order of convergence for the discretization we can use Richardson extrapolation to estimate the error, using the results from a finer grid resolution as the “exact” solution. We solve the problem on a grids with resolutions  $\Delta x_c$  and a finer resolution  $\Delta x_f$  and estimate the error in the coarse-grid solution as

$$E_i^k(\Delta x_c) = \phi_i^k(\Delta x_c, \Delta t) - \{A_{\Delta x_f \rightarrow \Delta x_c}[\phi^k(\Delta x_f, \Delta t)]\}_i.$$

where  $A_{\Delta x_f \rightarrow \Delta x_c}$  is an averaging operator which coarsens the solution from the fine grid ( $\Delta x_f$ ) to the coarse grid ( $\Delta x_c$ ). The error norms are computed from  $E_i^k(\Delta x_c; \Delta x_f)$ . Specifically:

$$\begin{aligned} L_\infty[\phi^k(\Delta x_c, \Delta t)] &= \max |E_i^k(\Delta x_c)|, \\ L_1[\phi^k(\Delta x_c, \Delta t)] &= \frac{1}{\sum_i} \sum_i |E_i^k(\Delta x_c)|, \\ L_2[\phi^k(\Delta x_c, \Delta t)] &= \sqrt{\frac{1}{\sum_i} \sum_i |E_i^k(\Delta x_c)|^2}. \end{aligned}$$

where the sums run over the grid points.

---

**Tip:** If one has an exact solution  $\phi_e(\mathbf{x}, T)$  available, one can replace  $\phi_i^k(\Delta x, \Delta t)$  by  $\phi_e(\mathbf{i}\Delta x, k\Delta t)$ .

---

### 3.8.2 Temporal convergence

For temporal convergence we compute the errors in the same way as for the spatial convergence, replacing  $\Delta t$  and  $\Delta x$  as fixed parameters. The solution error is computed as

$$E_i^k(\Delta t_c) = \phi_i^k(\Delta x, \Delta t_c) - \phi_i^{k'}(\Delta x, \Delta t_f),$$

where  $k\Delta t_c = T$  and  $k'\Delta t_f = T$ . Temporal integration errors are computed as

$$\begin{aligned} L_\infty [\phi^k (\Delta x, \Delta t_c)] &= \max |E_i^k (\Delta t_c)|, \\ L_1 [\phi^k (\Delta x, \Delta t_c)] &= \frac{1}{\sum_i} \sum_i |E_i^k (\Delta t_c)|, \\ L_2 [\phi^k (\Delta x, \Delta t_c)] &= \sqrt{\frac{1}{\sum_i} \sum_i |E_i^k (\Delta t_c)|^2}. \end{aligned}$$

---

**Tip:** If an exact solution as available, one can replace  $\phi_i^{k'} (\Delta x, \Delta t_f)$  by  $\phi_e (i\Delta x, k'\Delta t_f)$ .

---



## SOLVERS

### 4.1 Convection-Diffusion-Reaction

Here, we discuss the discretization of the equation

$$\frac{\partial \phi}{\partial t} + \nabla \cdot (\mathbf{v}\phi - D\nabla\phi) = S.$$

We assume that  $\phi$  is discretized by cell-centered averages (note that cell centers may lie inside solid boundaries), and use finite volume methods to construct fluxes in a cut-cells and regular cells. Here,  $\mathbf{v}$  indicates a drift velocity and  $D$  is the diffusion coefficient.

---

**Note:** Using cell-centered versions  $\phi$  might be problematic for some models since the state is extended outside the valid region. Models might have to recenter the state in order compute e.g. physically meaningful reaction terms in cut-cells.

---

Source code for the convection-diffusion-reaction solvers reside in `$DISCHARGE_HOME/Source/ConvectionDiffusionReaction` and the full API can be found at <https://chombo-discharge.github.io/chombo-discharge/doxygen/html/classCdrSolver.html>.

#### 4.1.1 CdrSolver

The `CdrSolver` class contains the interface for solving advection-diffusion-reaction problems. `CdrSolver` is an abstract class and does not contain any specific advective or diffusive discretization (these are added by implementation classes).

Currently, the implementation layers consist of the following:

1. `CdrMultigrid`, which inherits from `CdrSolver` and adds a second order accurate discretization for the diffusion operator.
2. `CdrCTU` and `CdrGodunov` which inherit from `CdrMultigrid` and add a second order accurate spatial discretization for the advection operator.

Currently, we mostly use the `CTU` class which contains a second order accurate discretization with slope limiters. `CdrGodunov` is a similar operator, but the advection code for this is distributed by the Chombo team. The C++ API for these classes can be obtained from <https://chombo-discharge.github.io/chombo-discharge/doxygen/html/classCdrSolver.html>.

The advance methods for `CdrSolver` are encapsulated by the following functions:

```

// For advancing the diffusion equation with an implicit Euler method.
void advanceEuler(EBAMRCellData& a_newPhi, const EBAMRCellData& a_oldPhi, const EBAMRCellData& a_source, const Real a_dt) = 0;

// For advancing the diffusion equation with a Crank-Nicholson method.
void advanceCrankNicholson(EBAMRCellData& a_newPhi, const EBAMRCellData& a_oldPhi, const EBAMRCellData& a_source, const Real a_dt) = 0;

// For computing div(v*phi - D*grad(phi)).
void computeDivJ(EBAMRCellData& a_divJ, EBAMRCellData& a_phi, const Real a_dt, const bool a_conservativeOnly, const bool a_ebFlux, const bool a_domainFlux) = 0;

// For computing div(v*phi).
void computeDivF(EBAMRCellData& a_divJ, EBAMRCellData& a_phi, const Real a_dt, const bool a_conservativeOnly, const bool a_ebFlux, const bool a_domainFlux) = 0;

// For computing div(D*grad(phi)).
void computeDivD(EBAMRCellData& a_divJ, EBAMRCellData& a_phi, const Real a_dt, const bool a_conservativeOnly, const bool a_ebFlux, const bool a_domainFlux) = 0;

```

## 4.1.2 Discretization details

### Explicit divergences and redistribution

Computing explicit divergences for equations like

$$\frac{\partial \phi}{\partial t} + \nabla \cdot \mathbf{G} = 0$$

is problematic because of the arbitrarily small volume fractions of cut cells. In general, we seek a method-of-lines update  $\phi^{k+1} = \phi^k - \Delta t [\nabla \cdot \mathbf{G}^k]$  where  $[\nabla \cdot \mathbf{G}]$  is a stable numerical approximation based on some finite volume approximation.

Pure finite volume methods use

$$\phi^{k+1} = \phi^k - \frac{\Delta t}{\kappa \Delta x^{\text{DIM}}} \int_V \nabla \cdot \mathbf{G} dV, \quad (4.1.1)$$

where  $\kappa$  is the volume fraction of a grid cell, DIM is the spatial dimension and the volume integral is written as discretized surface integral

$$\int_V \nabla \cdot \mathbf{G} dV = \sum_{f \in f(V)} (\mathbf{G}_f \cdot \mathbf{n}_f) \alpha_f \Delta x^{\text{DIM}-1}.$$

The sum runs over all cell edges (faces in 3D) of the cell where  $G_f$  is the flux on the edge centroid and  $\alpha_f$  is the edge (face) aperture.

However, taking  $[\nabla \cdot \mathbf{G}^k]$  to be this sum leads to a time step constraint proportional to  $\kappa$ , which can be arbitrarily small. This leads to an unacceptable time step constraint for Eq. 4.1.1. We use the Chombo approach and expand the range of influence of the cut cells in order to stabilize the discretization and allow the use of a normal time step constraint. First, we compute the conservative divergence

$$\kappa_i D_i^c = \sum_f G_f \alpha_f \Delta x^{\text{DIM}-1},$$

where  $G_f = \mathbf{G}_f \cdot \mathbf{n}_f$ . Next, we compute a non-conservative divergence  $D_i^{nc}$

$$D_i^{nc} = \frac{\sum_{j \in N(i)} \kappa_j D_j^c}{\sum_{j \in N(i)} \kappa_j}$$

where  $N(i)$  indicates some neighborhood of cells around cell  $i$ . Next, we compute a hybridization of the divergences,

$$D_i^H = \kappa_i D_i^c + (1 - \kappa_i) D_i^{nc},$$

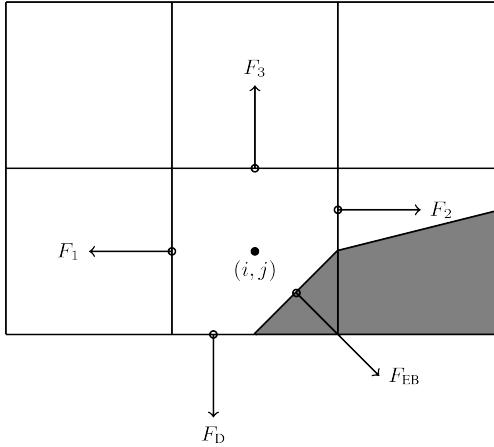


Fig. 4.1.1: Location of centroid fluxes for cut cells.

and perform an intermediate update

$$\phi_i^{k+1} = \phi_i^k - \Delta t D_i^H.$$

The hybrid divergence update fails to conserve mass by an amount  $\delta M_i = \kappa_i (1 - \kappa_i) (D_i^c - D_i^{nc})$ . In order to maintain overall conservation, the excess mass is redistributed into neighboring grid cells. Let  $\delta M_{j,i}$  be the redistributed mass from  $j$  to  $i$  where

$$\delta M_i = \sum_{j \in N(i)} \delta M_{j,i}.$$

This mass is used as a local correction in the vicinity of the cut cells, i.e.

$$\phi_i^{k+1} \rightarrow \phi_i^{k+1} + \delta M_{j \in N(i), i},$$

where  $\delta M_{j \in N(i), i}$  is the total mass redistributed to cell  $i$  from the other cells. After these steps, we define

$$[\nabla \cdot \mathbf{G}^k]_i \equiv \frac{1}{\Delta t} (\phi_i^{k+1} - \phi_i^k)$$

Numerically, the above steps for computing a conservative divergence of a one-component flux  $\mathbf{G}$  are implemented in the convection-diffusion-reaction solvers, which also respects boundary conditions (e.g. charge injection). The user will need to call the function

```
virtual void CdrSolver::computeDivG(EBAMRCellData& a_divG, EBAMRFluxData& a_G, const EBAMRIVData& a_ebG)
```

where  $a_G$  is the numerical representation of  $\mathbf{G}$  over the cut-cell AMR hierarchy and must be stored on cell-centered faces, and  $a\_ebG$  is the flux on the embedded boundary. The above steps are performed by interpolating  $a_G$  to face centroids in the cut cells for computing the conservative divergence, and the remaining steps are then performed successively. The result is put in  $a\_divG$ .

Note that when refinement boundaries intersect with embedded boundaries, the redistribution process is far more complicated since it needs to account for mass that moves over refinement boundaries. These additional complications are taken care of inside  $a\_divG$ , but are not discussed in detail here.

**Caution:** Mass redistribution has the effect of not being monotone and thus not TVD, and the discretization order is formally  $\mathcal{O}(\Delta x)$ .

### Explicit advection

Scalar advection updates follows the computation of the explicit divergence discussed in [Explicit divergences and redistribution](#). The face-centered fluxes  $\mathbf{G} = \phi\mathbf{v}$  are computed by instantiation classes for the convection-diffusion-reaction solvers. The function signature for explicit advection is

```
void computeDivF(EBAMRCellData& a_divF, const EBAMRCellData& a_state, const Real a_dt);
```

where the face-centered fluxes are computed by using the velocities and boundary conditions that reside in the solver, and result is put in `a_divF` using the procedure outlined above. The argument `a_dt` is the time step size. It is not needed in a method-of-lines context, but it is used in e.g. `CdrCTU` for computing transverse derivatives in order to expand the stability region (i.e., use larger CFL numbers).

For example, in order to perform an advective advance over a time step  $\Delta t$ , one would perform the following:

```
CdrSolver* solver;
const Real dt = 1.0;

EBAMRCellData& phi = solver->getPhi();           // Cell-centered state
EBAMRCellData& divF = solver->getScratch();       // Scratch storage in solver
solver->computeDivF(divF, phi, 0.0);             // Computes divF
DataOps::incr(phi, divF, -dt);                    // makes phi -> phi - dt*divF
```

### Explicit diffusion

Explicit diffusion is performed in much the same way as implicit advection, with the exception that the general flux  $\mathbf{G} = D\nabla\phi$  is computed by using centered differences on face centers. The function signature for explicit diffusion is

```
void computeDivD(EBAMRCellData& a_divD, const EBAMRCellData& a_state);
```

and we increment in the same way as for explicit advection:

```
CdrSolver* solver;
const Real dt = 1.0;

EBAMRCellData& phi = solver->getPhi();           // Cell-centered state
EBAMRCellData& divD = solver->getScratch();       // Scratch storage in solver
solver->computeDivD(divD, phi, 0.0);             // Computes divD
DataOps::incr(phi, divD, dt);                     // makes phi -> phi + dt*divD
```

### Explicit advection-diffusion

There is also functionality for aggregating explicit advection and diffusion advances. The reason for this is that the cut-cell overhead is only applied once on the combined flux  $\phi\mathbf{v} - D\nabla\phi$  rather than on the individual fluxes. For non-split methods this leads to some performance improvement since the interpolation of fluxes on cut-cell faces only needs to be performed once. The signature for this is precisely the same as for explicit advection only:

```
void computeDivJ(EBAMRCellData& a_divJ, const EBAMRCellData& a_state, const Real a_extrapDt)
```

where the face-centered fluxes are computed by using the velocities and boundary conditions that reside in the solver, and result is put in `a_divF`. For example, in order to perform an advective advance over a time step  $\Delta t$ , one would perform the following:

```
const Real dt = 1.0;

EBAMRCellData& phi = solver->getPhi();           // Cell-centered state
EBAMRCellData& divJ = solver->getScratch();       // Scratch storage in solver
solver->computeDivJ(divJ, phi, 0.0);             // Computes divJ
DataOps::incr(phi, divJ, -dt);                    // makes phi -> phi - dt*divJ
```

Often, time integrators have the option of using implicit or explicit diffusion. If the time-evolution is not split (i.e. not using a Strang or Godunov splitting), the integrators will often call `computeDivJ` rather separately calling `computeDivF` and `computeDivD`. If you had a split-step Godunov method, the above procedure for a forward Euler method for both parts would be:

```
CdrSolver* solver;
const Real dt = 1.0;

solver->computeDivF(divF, phi, 0.0); // Computes divF = div(n*phi)
DataOps::incr(phi, divF, -dt);      // makes phi -> phi - dt*divF

solver->computeDivD(divD, phi);    // Computes divD = div(D*nabla(phi))
DataOps::incr(phi, divD, dt);       // makes phi -> phi + dt*divD
```

However, the cut-cell redistribution dance (flux interpolation, hybrid divergence, and redistribution) would be performed twice.

### Implicit diffusion

Implicit diffusion can occasionally be necessary. The convection-diffusion-reaction solvers support two basic diffusion solves: Backward Euler and the Crank-Nicholson methods. The function signatures for these are

```
void advanceEuler(EBAMRCellData& a_newPhi, const EBAMRCellData& a_oldPhi, const EBAMRCellData& a_source, const Real a_dt) = 0;
void advanceCrankNicholson(EBAMRCellData& a_newPhi, const EBAMRCellData& a_oldPhi, const EBAMRCellData& a_source, const Real a_
→dt) = 0;
```

where `a_newPhi` is the state at the new time  $t + \Delta t$ , `a_oldPhi` is the state at time  $t$  and `a_source` is the source term which strictly speaking should be centered at time  $t + \Delta t$  for the Euler update and at time  $t + \Delta t/2$  for the Crank-Nicholson update. This may or may not be possible for your particular problem.

For example, performing a split step Godunov method for advection-diffusion is as simple as:

```
// First. Compute phi = phi - dt*div(F)
solver->computeDivF(divF, phi, 0.0);
DataOps::incr(phi, divF, -dt);

// Implicit diffusion advance over a time step dt
DataOps::copy(phiOld, phi);
solver->advanceEuler(phi, phiOld, dt);
```

#### 4.1.3 CdrMultigrid

`CdrMultigrid` adds second-order accurate implicit diffusion code to `CdrSolver`, but leaves the advection code unimplemented. The class can use either implicit or explicit diffusion using second-order cell-centered stencils. In addition, `CdrMultigrid` adds two implicit time-integrators, an implicit Euler method and a Crank-Nicholson method.

The `CdrMultigrid` layer uses the Helmholtz discretization discussed in [Helmholtz equation](#). It implements the pure functions required by `CdrSolver` but introduces a new pure function

```
virtual void advectToFaces(EBAMRFluxData& a_facePhi, const EBAMRCellData& a_phi, const Real a_dt) = 0;
```

The faces states defined by the above function are used when forming a finite-volume approximation to the divergence operators, see [Discretization details](#).

## 4.1.4 CdrCTU

CdrCTU is an implementation class that uses the corner transport upwind (CTU) discretization. The CTU discretization uses information that propagates over corners of grid cells when calculating the face states. It can combine this with use various limiters:

- No limiter (pure CTU)
- Minmod
- Superbee
- Monotonized central differences

In addition, CdrCTU can turn off the transverse terms in which case the discretization reduces to the donor cell method. Our motivation for using the CTU discretization lies in the time step selection for the CTU and donor-cell methods, see [Time step limitation](#). Typically, we want to achieve a dimensionally independent time step that is the same in 1D, 2D, and 3D, but without directional splitting.

### Face extrapolation

The finite volume discretization uses an upstream-centered Taylor expansion that extrapolates the cell-centered term to half-edges and half-steps:

$$\phi_{i+1/2,j}^{n+1/2} = \phi_{i,j,k}^n + \frac{\Delta x}{2} \frac{\partial \phi}{\partial x} + \frac{\Delta t}{2} \frac{\partial \phi}{\partial t} + \mathcal{O}(\Delta t^2) + \mathcal{O}(\Delta t \Delta x)$$

Note that the truncation order is  $\Delta t^2 + \Delta x \Delta t$  where the latter term is due to the cross-derivative  $\frac{\partial^2 \phi}{\partial t \partial x}$ . The resulting expression in 2D for a velocity field  $\mathbf{v} = (u, v)$  is

$$\phi_{i \pm 1/2,j}^{n+1/2,+} = \phi_{i,j}^n \pm \frac{1}{2} \min \left[ 1, 1 \mp \frac{\Delta t}{\Delta x} u_{i,j}^n \right] (\Delta^x \phi)_{i,j}^n - \frac{\Delta t}{2 \Delta x} v_{i,j}^n (\Delta^y \phi)_{i,j}^n,$$

Here,  $\Delta^x$  are the regular (normal) slopes whereas  $\Delta^y$  are the transverse slopes. The transverse slopes are given by

$$(\Delta^y \phi)_{i,j}^n = \begin{cases} \phi_{i,j+1}^n - \phi_{i,j}^n, & v_{i,j}^n < 0 \\ \phi_{i,j}^n - \phi_{i,j-1}^n, & v_{i,j}^n > 0 \end{cases}$$

### Slopes

For the normal slopes, the user can choose between the minmod, superbee, and monotonized central difference (MC) slopes. Let  $\Delta_l = \phi_{i,j}^n - \phi_{i-1,j}^n$  and  $\Delta_r = \phi_{i+1,j}^n - \phi_{i,j}^n$ . The slopes are given by:

$$\text{minmod: } (\Delta^x \phi)_{i,j}^n = \begin{cases} \Delta_l & |\Delta_l| < |\Delta_r| \text{ and } \Delta_l \Delta_r > 0 \\ \Delta_r & |\Delta_l| > |\Delta_r| \text{ and } \Delta_l \Delta_r > 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$\text{MC: } (\Delta^x \phi)_{i,j}^n = \text{sgn}(\Delta_l + \Delta_r) \min \left( \left| \frac{\Delta_l + \Delta_r}{2} \right|, 2|\Delta_l|, 2|\Delta_r| \right),$$

$$\text{superbee: } (\Delta^x \phi)_{i,j}^n = \begin{cases} \Delta_1 & |\Delta_1| > |\Delta_2| \text{ and } \Delta_1 \Delta_2 > 0 \\ \Delta_2 & |\Delta_1| < |\Delta_2| \text{ and } \Delta_1 \Delta_2 > 0 \\ 0 & \text{otherwise,} \end{cases}$$

where for the superbee slope we have  $\Delta_1 = \text{minmod}(\Delta_l, 2\Delta_r)$  and  $\Delta_2 = \text{minmod}(\Delta_r, 2\Delta_l)$ .

---

**Note:** When using slopes, monotonicity is not guaranteed for the CTU discretization. If slopes are turned off, however, the scheme is guaranteed to be monotone.

---

## Time step limitation

The stability region for the donor-cell and corner transport upwind methods are:

$$\text{Donor-cell : } \Delta t \leq \frac{\Delta x}{|v_x| + |v_y| + |v_z|}$$

$$\text{CTU : } \Delta t \leq \frac{\Delta x}{\max(|v_x|, |v_y|, |v_z|)}$$

Note that when the flow is diagonal to the grid, i.e.  $|v_x| = |v_y| = |v_z|$ , the CTU can use a time step that is three times larger than for the donor-cell method.

## Class options

When running the CdrCTU solver the user can adjust the advective algorithm by turning on/off slope limiters and the transverse term through the class options:

- `CdrCTU.slope_limiter`, which must be *none*, *minmod*, *mc*, or *superbee*.
- `CdrCTU.use_ctu`, which must be *true* or *false*.

If the transverse terms are turned off, CdrCTU will compute the donor-cell time step.

### 4.1.5 CdrGodunov

CdrGodunov inherits from CdrMultigrid and adds advection code for Godunov methods. This class borrows from Chombo internals (specifically, EBLevelAdvectIntegrator) and can do second-order advection with time-extrapolation.

CdrGodunov supplies (almost) the same discretization as CdrCTU with the exception that the underlying discretization can also be used for the incompressible Navier-Stokes equation. However, it only supports the monotonized central difference limiter.

**Caution:** CdrGodunov will be removed from future versions of chombo-discharge.

### 4.1.6 Using CdrSolver

#### Setting up the solver

To set up the CdrSolver, the following commands are usually sufficient:

```
// Assume m_solver and m_species are pointers to a CdrSolver and CdrSpecies
auto solver = RefCountedPtr<CdrSolver> (new MyCdrSolver());
auto species = RefCountedPtr<CdrSpecies> (new MyCdrSpecies());
```

To see an example, the advection-diffusion code in /Physics/AdvectionDiffusion/CD\_AdvectionDiffusionStepper.cpp shows how to set up a CdrSolver.

## Filling the solver

In order to obtain mesh data from the `CdrSolver`, the user should use the following public member functions:

```
EBAMRCellData& getPhi();                                // Return phi
EBAMRCellData& getSource();                            // Returns S
EBAMRCellData& getCellCenteredVelocity();             // Get cell-centered velocity
EBAMRFluxData& getFaceCenteredDiffusionCoefficient(); // Returns D
EBAMRIVData& getEbFlux();                             // Returns flux at EB
EBAMRIFData& getDomainFlux();                         // Returns flux at domain boundaries
```

To set the drift velocities, the user will fill the *cell-centered* velocities. Interpolation to face-centered transport fluxes are done by `CdrSolver` during the discretization step.

The general way of setting the velocity is to get a direct handle to the velocity data:

```
CdrSolver solver;
EBAMRCellData& veloCell = solver.getCellCenteredVelocity();
```

Then, `veloCell` can be filled with the cell-centered velocity. This would typically look something like this:

```
EBAMRCellData& veloCell = m_solver->getCellCenteredVelocity();
for (int lvl = 0; lvl <= m_amr->getFinestLevel(); lvl++){
    const DisjointBoxLayout& dbl = m_amr->getGrids()[lvl];

    for (DataIterator dit(dbl); dit.ok(); ++dit){
        EBCellFAB& patchVel = (*veloCell[lvl])[dit];
        // Do something with patchVel
    }
}
```

The same procedure goes for the source terms, diffusion coefficients, boundary conditions and so on. For example, an explicit Euler discretization for the problem  $\partial_t \phi = S$  is:

```
CdrSolver* solver;
const Real dt = 1.0;

EBAMRCellData& phi = solver->getPhi();
EBAMRCellData& src = solver->getSource();

DataOps::incr(phi, src, dt);
```

## Adjusting output

It is possible to adjust solver output when plotting data. This is done through the input file for the class that you're using. For example, for the `CdrCTU` implementation the following variables are available:

```
CdrCTU.plt_vars = phi vel src dco ebflux # Plot variables. Options are 'phi', 'vel', 'dco', 'src'
```

Here, you adjust the plotted variables by adding or omitting them from your input script. E.g. if you only want to plot the cell-centered states you would do:

```
CdrGodunov.plt_vars = phi # Plot variables. Options are 'phi', 'vel', 'dco', 'src', 'ebflux'
```

### 4.1.7 CdrSpecies

The `CdrSpecies` class is a supporting class that passes information and initial conditions into `CdrSolver` instances. `CdrSpecies` specifies whether or not the advect-diffusion solver will use only advection, diffusion, both advection and diffusion, or neither. It also specifies initial data, and provides a string identifier to the class (e.g. for identifying output in plot files). However, it does not contain any discretization.

---

**Note:** Click [here](#) for the `CdrSpecies` C++ API.

---

The below code block shows an example of how to instantiate a species. Here, diffusion code is turned off and the initial data is one everywhere.

```
class MySpecies : public CdrSpecies{
public:

    MySpecies() {
        m_mobile      = true;
        m_diffusive   = false;
        m_name        = "mySpecies";
    }

    ~MySpecies() = default;

    Real initialData(const RealVect a_pos, const Real a_time) const override {
        return 1.0;
    }
}
```

---

**Tip:** It is also possible to use computational particles as an initial condition in `CdrSpecies`. In this case you need to fill `m_initialParticles`, and these are then deposited with a nearest-grid-point scheme when instantiating the solver. See [Particles](#) for further details.

---

### 4.1.8 Example application(s)

Example applications that use the `CdrSolver` are:

- *Advection-diffusion model.*
- *CDR plasma model.*

## 4.2 Electrostatic solver

Here, we discuss the discretization of the equation

$$\nabla \cdot (\epsilon_r \nabla \Phi) = -\frac{\rho}{\epsilon_0} \quad (4.2.1)$$

where  $\Phi$  is the electric potential,  $\rho$  is the space charge density and  $\epsilon_0$  is the vacuum permittivity. The relative permittivity is  $\epsilon_r = \epsilon_r(\mathbf{x})$  and can additionally be discontinuous at gas-dielectric interfaces.

---

**Note:** All current electrostatic field solvers solve for the potential at the cell center (not the cell centroid). The code for the electrostatics solver is given in [/Source/Electrostatics](#) and [/Source/Elliptic](#).

---

## 4.2.1 FieldSolver

`FieldSolver` is an abstract class for electrostatic solves in an EB context and contains most routines required for setting up and solving electrostatic problems. `FieldSolver` can solve over three phases, gas, dielectric, and electrode, and thus it uses `MFAMRCellData` functionality where data is defined over multiple phases (see [Mesh data](#)).

Note that in order to separate the electrostatic solver interface from the implementation, `FieldSolver` is a pure class without knowledge of numerical discretizations. See the [FieldSolver C++ API](#) for additional details. Currently, our only supported implementation is `FieldSolverMultigrid` (see [FieldSolverMultigrid](#)).

On gas-dielectric interfaces we enforce an extra equation

$$\epsilon_1 \partial_{n_1} \Phi + \epsilon_2 \partial_{n_2} \Phi = \sigma / \epsilon_0 \quad (4.2.2)$$

where  $\mathbf{n}_1 = -\mathbf{n}_2$  are the normal vectors pointing away from interface, and  $\sigma$  is the surface charge density.

We point out that this equation can be enforced in various formats. The most common case is that  $\partial_n \Phi$  are free parameters and  $\sigma$  is a fixed parameter. However, we *can* also fix  $\partial_n \Phi$  on one side of the boundary and let  $\sigma$  be the free parameter. When using `FieldSolverMultigrid` (see [FieldSolverMultigrid](#)), users can choose between these two natural boundary conditions, see [EB boundary conditions](#).

## 4.2.2 Using FieldSolver

Using the `FieldSolver` is usually straightforward by first constructing the solver and then parsing the class options. Creating a solver is usually done by means of a pointer cast:

```
auto fieldSolver = RefCountedPtr<FieldSolver> (new FieldSolverMultigrid());
```

In addition, one must parse run-time options to the class, provide the `AmrMesh` and `ComputationalGeometry` instances, and set the initial conditions. This is done as follows:

```
RefCountedPtr<AmrMesh> amr;
RefCountedPtr<ComputationalGeometry> geo;

std::function<Real(const Real)> voltage;

fieldSolver->parseOptions();           // Parse class options
fieldSolver->setAmr(amr);             // Set amr - we assume that `amr` is an object
fieldSolver->setComputationalGeometry(geo); // Set the computational geometry
fieldSolver->allocateInternals();      // Allocate storage for potential etc.
fieldSolver->setVoltage(voltage);      // Set the voltage
```

The argument in the function `setVoltage(...)` is a function pointer of the type:

```
Real voltage(const Real a_time)
```

This allows setting a time-dependent voltage on electrodes and domain boundaries. As shown above, one can also use `std::function<Real(const Real)>` or lambdas to set the voltage. E.g.,

```
FieldSolver* fieldSolver;

Real myVoltage = [] (const Real a_time) -> Real {
    return 1.0*a_time;
};

fieldSolver->setVoltage(myVoltage);
```

The electrostatic solver in `chombo-discharge` has a lot of supporting functionality, but essentially relies on only one critical function: Solving for the potential. This is encapsulated by the pure member function

```
bool FieldSolver::solve(MFAMRCellData& phi, const MFAMRCellData& rho, const EBAMRIVData& sigma) = 0;
```

where `phi` is the resulting potential that was computing with the space charge density `rho` and surface charge density `sigma`.

### 4.2.3 Domain boundary conditions

Domain boundary conditions for the solver must be set by the user through an input script, whereas the boundary conditions on internal surfaces are Dirichlet by default. Note that on multifluid-boundaries the boundary condition is enforced by the conventional matching boundary condition that follows from Gauss' law.

#### General format

The most general form of setting domain boundary conditions for `FieldSolver` is to specify a boundary condition *type* (e.g., Dirichlet) together with a function specifying the value. Domain boundary condition *types* are parsed through a member function `FieldSolver::parseDomainBc`. This function will read string identifiers from the input script, and these identifiers are either in the format `<string> <float>` (simplified format) or in the format `<string>` (general format). For setting general types of Neumann or Dirichlet BCs on the domain sides, one will specify

```
FieldSolverMultigrid.bc.x.low = dirichlet_custom
FieldSolverMultigrid.bc.x.high = dirichlet_neumann
```

Unfortunately, due to the many degrees of freedom in setting domain boundary conditions, the procedure is a bit convoluted. We first explain the general procedure.

`FieldSolver` will always set individual space-time functions on each domain side, and these functions are always in the form

```
std::function<Real(const RealVect a_position, const Real a_time)> bcFunction;
```

To set a domain boundary condition function on a side, one can use the following member function:

```
void FieldSolver::setDomainSideBcFunction(const int a_dir,
                                         const Side::LoHiSide a_side,
                                         const std::function<Real(const RealVect a_position, const Real a_time)>);
```

For a general way of setting the function value on the domain side, one will use the above function together with an identifier `dirichlet_custom` or `neumann_custom` in the input script. This identifier simply tells `FieldSolver` to use that function to either specify  $\Phi$  or  $\partial_n \Phi$  on the boundary. These functions are then directly processed by the numerical discretizations.

---

**Note:** On construction, `FieldSolver` will set all the domain boundary condition functions to a constant of one (because the functions need to be populated).

---

#### Simplified format

`FieldSolver` also supports a simplified method of setting the domain boundary conditions, in which case the user will specify Neumann or Dirichlet values (rather than functions) for each domain side. These values are usually, but not necessarily, constant values.

In this case one will use an identifier `<string> <float>` in the input script, like so:

```
FieldSolverMultigrid.bc.x.low = neumann 0.0
FieldSolverMultigrid.bc.x.high = dirichlet 1.0
```

The floating point number has a slightly different interpretation for the two types of BCs. Moreover, when using the simplified format the function specified through `setDomainSideBcFunction` will be used as a multiplier rather than being parsed directly into the numerical discretization. Although this may *seem* more involved, this procedure is usually easier to use when setting constant Neumann/Dirichlet values on the domain boundaries. It also automatically provides a link between a specified voltage wave form and the boundary conditions (unlike the general format, where the user must supply that link themselves).

### Dirichlet

When using simplified parsing of Dirichlet domain BCs, `FieldSolver` will generate and parse a different function into the discretizations. This function is *not* the same function as that which is parsed through `setDomainSideBcFunction`. In C++ pseudo-code, this function is in the format

```
Real dirichletFraction;

auto f = [&func, ...](const RealVect a_pos, const Real a_time) -> Real {
    return func(a_pos, a_time) * voltage(a_time) * dirichletFraction;
};
```

where `voltage` is the voltage wave form specified through `FieldSolver::setVoltage`, and `dirichletFraction` is a placeholder for the floating point number specified in the input script, i.e. the floating point number in the input option. That is, for Dirichlet boundary conditions the solver will always multiply the provided input function by the voltage waveform. That is, the function `func(a_pos, a_time)` is the space-time function set through `setDomainSideBcFunction`. Recall that, by default, this function is set to one so that the default voltage that is parsed into the numerical discretization is simply the specified voltage multiplied by the specified fraction in the input script. For example, using

```
FieldSolverMultigrid.bc.y.low = dirichlet 0.0
FieldSolverMultigrid.bc.y.high = dirichlet 1.0
```

will set the voltage on the lower y-plane to ground and the voltage on the upper y-plane to the live voltage. Specifically, on the upper y-plane this specification will generate a potential boundary condition function of the type

```
auto func = [] (const RealVect a_pos, const Real a_time) {return 1.0};
dirichletFraction = 1.0;

auto bc = [func](const RealVect a_pos, const Real a_time) {
    return func(a_pos, a_time) * voltage(a_time) * dirichletFraction;
};
```

In order to set the voltage on the domain side to also be spatially dependent, one can either use `dirichlet_custom` as an input option, or `dirichlet <float>` and set a different multiplier on the domain edge (face). As an example, by specifying `bc.y.high = dirichlet 1.234` in the input script AND setting the multiplier on the wall as follows:

```
auto wallFunc = [] (const RealVect a_pos, const Real a_time) -> Real {
    const Real y = a_pos[1];
    return 1.0 - y;
};

fieldSolver->setDomainSideBcFunction(1, Side::Hi, wallFunc);
```

we end up with a voltage of

$$V(\mathbf{x}, t) = 1.234(1 - y)V(t)$$

on the upper y-plane.

## Neumann

When using simplified parsing of Neumann boundary conditions, the procedure is precisely like that for Dirichlet boundary conditions *except* that multiplication by the voltage wave form is not made. I.e. the boundary condition function that is passed into the numerical discretization is

```
Real neumannFraction;

auto func = [&func, ...](const RealVect a_pos, const Real a_time) -> Real {
    return func(a_pos, a_time) * neumannFraction;
};
```

Note that since `func` is initialized to one, the floating point number in the input option directly specifies the value of  $\partial_n \Phi$ .

### 4.2.4 EB boundary conditions

#### Electrodes

For the current `FieldSolver` the natural BC at the EB is Dirichlet with a specified voltage, whereas on dielectrics we enforce Eq. 4.2.2. The voltage on the electrodes are automatically retrieved from the specified voltages on the electrodes in the geometry being used (see `ComputationalGeometry`). The exception to this is that while `ComputationalGeometry` specifies that an electrode will be at some fraction of a specified voltage, `FieldSolverMultigrid` uses this fraction *and* the specified voltage wave form in `setVoltage`.

To understand how the voltage on the electrode is being set, we first remark that our implementation uses a completely general specification of the voltage on each electrode in both space and time. This voltage has the form

$$V_i = V_i(\mathbf{x}, t).$$

where  $V_i$  is the voltage on electrode  $i$ . It is possible to interact with this function directly, going through all electrodes and setting the electrode to be spatially and temporally varying. The member function that does this is

```
void FieldSolver::setElectrodeDirichletFunction(const int a_electrode,
                                                const ElectrostaticEbBc::BcFunction& a_function);
```

Here, the type `ElectrostaticEbBc::BcFunction` is just an alias:

```
using ElectrodestaticEbBc::BcFunction = std::function<Real(const RealVect a_position, const Real a_time)>;
```

The voltage on an electrode  $i$  could thus be set as

```
int electrode;

auto myElectrodeVoltage = [] (const RealVect a_position, const Real a_time) -> Real{
    return 1.0;
};

fieldSolver->setElectrodeDirichletFunction(electrode, myElectrodeVoltage);
```

where the return value can be replaced by the user' function.

In the majority of cases the voltage on electrodes is either a live voltage or ground. Thus, although the above format is a general way of setting the voltage individually on each electrode (in both space and time) `FieldSolver` supports a simpler way of generating these voltage waveforms. When `FieldSolver` is instantiated, it will interally generate these functions through simplified expression such that the user only needs to set a single wave form that applies to all electrodes. The voltages that are set on the various electrodes are thus in the form:

```

int electrode;
Real voltageFraction;
std::function<Real(const Real a_time)> voltageWaveForm;

auto defaultElectrodeVoltage = [...](const RealVect a_position, const Real a_time) -> Real{
    return voltageFraction * voltageWaveForm(a_time);
};

fieldSolver->setElectrodeDirichletFunction(electrode, defaultElectrodeVoltage);

```

Thus, the default voltage which is set on an electrode is the voltage *fraction* specified on the electrodes (in ComputationalGeometry) multiplied by a voltage wave form (specified by `FieldSolver::setVoltage`).

## Dielectrics

On dielectrics, we enforce the jump boundary condition directly.

### 4.2.5 FieldSolverMultigrid

`FieldSolverMultigrid` implements a multigrid routine for solving Eq. 4.2.1, and is currently the only implementation of `FieldSolver`.

The discretization used by `FieldSolverMultigrid` is described in [Linear solvers](#). The underlying solver type is a Helmholtz solver, but `FieldSolverMultigrid` considers only the Laplacian term. For further details on the spatial discretization, see [Linear solvers](#).

## Solver configuration

`FieldSolverMultigrid` has a number of switches for determining how it operates. Some of these switches are intended for parsing boundary conditions, whereas others are settings for operating multigrid or for I/O. The current list of configuration options are indicated below

```

# =====
# FieldSolverMultigrid class options
# =====
FieldSolverMultigrid.verbosity      = -1          # Class verbosity
FieldSolverMultigrid.jump_bc        = natural     # Jump BC type ('natural' or 'saturation_charge')
FieldSolverMultigrid.bc.x.lo        = dirichlet 0.0 # Bc type (see docs)
FieldSolverMultigrid.bc.x.hi        = dirichlet 0.0 # Bc type (see docs)
FieldSolverMultigrid.bc.y.lo        = dirichlet 0.0 # Bc type (see docs)
FieldSolverMultigrid.bc.y.hi        = dirichlet 0.0 # Bc type (see docs)
FieldSolverMultigrid.bc.z.lo        = dirichlet 0.0 # Bc type (see docs)
FieldSolverMultigrid.bc.z.hi        = dirichlet 0.0 # Bc type (see docs)
FieldSolverMultigrid.plt_vars       = phi rho E   # Plot variables. Possible vars are 'phi', 'rho', 'E', 'res', 'sigma
`-
FieldSolverMultigrid.kappa_source  = true         # Volume weighted space charge density or not (depends on algorithm)

FieldSolverMultigrid.gmg_verbosity  = -1          # GMG verbosity
FieldSolverMultigrid.gmg_pre_smooth = 12          # Number of relaxations in downswEEP
FieldSolverMultigrid.gmg_post_smooth= 12          # Number of relaxations in upswEEP
FieldSolverMultigrid.gmg_bott_smooth= 12          # Number of at bottom level (before dropping to bottom solver)
FieldSolverMultigrid.gmg_min_iter   = 5           # Minimum number of iterations
FieldSolverMultigrid.gmg_max_iter   = 32          # Maximum number of iterations
FieldSolverMultigrid.gmg_exit_tol  = 1.E-10       # Residue tolerance
FieldSolverMultigrid.gmg_exit_hang = 0.2          # Solver hang
FieldSolverMultigrid.gmg_min_cells = 16          # Bottom drop
FieldSolverMultigrid.gmg_bc_order  = 2           # Boundary condition order for multigrid
FieldSolverMultigrid.gmg_bc_weight = 2           # Boundary condition weights (for least squares)
FieldSolverMultigrid.gmg_jump_order= 2           # Boundary condition order for jump conditions
FieldSolverMultigrid.gmg_jump_weight= 2           # Boundary condition weight for jump conditions (for least squares)
FieldSolverMultigrid.gmg_bottom_solver= bicgstab  # Bottom solver type. 'simple', 'bicgstab', or 'gmres'
FieldSolverMultigrid.gmg_cycle     = vcycle       # Cycle type. Only 'vcycle' supported for now.
FieldSolverMultigrid.gmg_smoothen = red_black    # Relaxation type. 'jacobi', 'multi_color', or 'red_black'

```

Note that *all* options pertaining to IO or multigrid are run-time configurable (see [Run-time configurations](#)).

## Setting boundary conditions

The flags that are in the format `bc.coord.side` (e.g., `bc.x.low`) parse the domain boundary condition type to the solver. See [Domain boundary conditions](#) for details.

The flag `jump_bc` indicates how the dielectric jump condition is enforced. See [Saturation charge BC](#) for additional details.

---

**Note:** Currently, we only solve the dielectric jump condition on gas-dielectric interfaces and dielectric-dielectric interfaces are not supported. If you want to use numerical mock-ups of dielectric-dielectric interfaces, you can change  $\epsilon_r$  inside a dielectric, but note that the dielectric boundary condition  $\partial_{n_1}\Phi + \partial_{n_2}\Phi = \sigma/\epsilon_0$  is *not* solved in this case.

---

## Algorithmic adjustments

By default, the Helmholtz operator uses a diagonally weighting of the operator using the volume fraction as weight. This means that the quantity that is passed into `AMRMultigrid` should be weighted by the volume fraction to avoid the small-cell problem of EB grids. The flag `kappa_source` indicates whether or not we should multiply the right-hand side by the volume fraction before passing it into the solver routine. If this flag is set to `false`, it is an indication that the user has taken responsibility to perform this weighting prior to calling `FieldSolver::solve(...)`. If this flag is set to `true`, `FieldSolverMultigrid` will perform the multiplication before the multigrid solve.

## Tuning multigrid performance

Multigrid operates by coarsening the solution (and the geometry with it) on a hierarchy of grid levels, and smoothing the solution on each level. There are a number of factors that influence the multigrid performance. Often the most critical factors are the radius of the cut-cell stencils and how far multigrid is allowed to coarsen. In addition, the multigrid convergence is improved by increasing the number of smoothings per grid level (up to a certain point), as well as the type of smoother and bottom solver being used. We explain these options below:

- `FieldSolverMultigrid.gmg_verbosity`. Controls the multigrid verbosity. Setting it to a number  $> 0$  will print multigrid convergence information.
- `FieldSolverMultigrid.gmg_pre_smooth`. Controls the number of relaxations on each level during multigrid downsweeps.
- `FieldSolverMultigrid.gmg_post_smooth`. Controls the number of relaxations on each level during multigrid upsweeps.
- `FieldSolverMultigrid.gmg_bott_smooth`. Controls the number of relaxations before entering the bottom solve.
- `FieldSolverMultigrid.gmg_min_iter`. Sets the minimum number of iterations that multigrid will perform.
- `FieldSolverMultigrid.gmg_max_iter`. Sets the maximum number of iterations that multigrid will perform.
- `FieldSolverMultigrid.gmg_exit_tol`. Sets the exit tolerance for multigrid. Multigrid will exit the iterations if  $r < \lambda r_0$  where  $\lambda$  is the specified tolerance,  $r = |L\Phi - \rho|$  is the residual and  $r_0$  is the residual for  $\Phi = 0$ .
- `FieldSolverMultigrid.gmg_exit_hang`. Sets the minimum permitted reduction in the convergence rate before exiting multigrid. Letting  $r^k$  be the residual after  $k$  multigrid cycles, multigrid will abort if the residual between levels is not reduce by at least a factor of  $r^{k+1} < (1 - h)r^k$ , where  $h$  is the “hang” factor.
- `FieldSolverMultigrid.gmg_min_cells`. Sets the minimum amount of cells along any coordinate direction for coarsened levels. Note that this will control how far multigrid will coarsen. Setting a number `gmg_min_cells = 16` will terminate multigrid coarsening when the domain has 16 cells in any of the coordinate direction.

- `FieldSolverMultigrid.gmg_bc_order`. Sets the stencil order for Dirichlet boundary conditions (on electrodes). Note that this is also the stencil radius.
- `FieldSolverMultigrid.gmg_bc_weight`. Sets the least squares stencil weighting factor for least squares gradient reconstruction on EBs. See [Least squares](#) for details.
- `FieldSolverMultigrid.gmg_jump_order`. Sets the stencil order when performing least squares gradient reconstruction on dielectric interfaces. Note that this is also the stencil radius.
- `FieldSolverMultigrid.gmg_jump_weight`. Sets the least squares stencil weighting factor for least squares gradient reconstruction on dielectric interfaces. See [Least squares](#) for details.
- `FieldSolverMultigrid.gmg_bottom_solver`. Sets the bottom solver type.
- `FieldSolverMultigrid.gmg_cycle`. Sets the multigrid method. Currently, only V-cycles are supported.
- `FieldSolverMultigrid.gmg_smoothen`. Sets the multigrid smoother.

---

**Note:** When setting the bottom solver (which by default is a biconjugate gradient stabilized method) to a regular smoother, one must also specify the number of smoothings to perform. E.g., `FieldSolverMultigrid.gmg_bottom_solver = simple 64`. Setting the bottom solver to `simple` without specifying the number of smoothings that will be performed will issue a run-time error.

---

## Adjusting output

The user may plot the potential, the space charge, the electric, and the GMG residue as follows:

```
FieldSolverMultigrid.plt_vars = phi rho E res      # Plot variables. Possible vars are 'phi', 'rho', 'E', 'res'
```

## Saturation charge BC

As mentioned above, on dielectric interfaces the user can choose to specify which “form” of Eq. 4.2.2 to solve. If the user wants the natural form in which the surface charge is the free parameter, he can specify

```
FieldSolverMultigrid.which_jump = natural
```

To use the other format (in which one of the fluxes is specified), use

```
FieldSolverMultigrid.which_jump = saturation_charge
```

---

**Note:** The `saturation_charge` option will set the derivative of  $\partial_n \Phi$  to zero on the gas side. Support for setting  $\partial_n \Phi$  to a specified (e.g., non-zero) value on either side is missing, but is straightforward to implement.

---

## 4.2.6 Frequency dependent permittivity

Frequency-dependent permittivities are fundamentally supported by the chombo-discharge elliptic discretization but none of the solvers implement it. Recall that the polarization (in frequency space) is

$$\mathbf{P}(\omega) = \epsilon_0 \chi(\omega) \mathbf{E}(\omega),$$

where  $\chi(\omega)$  is the dielectric susceptibility.

There are two forms that chombo-discharge can support frequency dependent permittivities; through convolution or through auxiliary differential equations (ADEs).

### Convolution approach

In the time domain, the displacement field is,

$$\mathbf{D}(t_k) = \epsilon_0 \mathbf{E}(t_k) + \epsilon_0 \int_0^{t_k} \chi(t) \mathbf{E}(t_k - t) dt.$$

There are various forms of discretizing the integral. E.g. with the trapezoidal rule then

$$\begin{aligned} \int_0^{t_k} \chi(t) \mathbf{E}(t - t) dt &= \sum_{n=0}^{k-1} \int_{t_n}^{t_{n+1}} \chi(t) \mathbf{E}(t_k - t) dt \\ &\approx \frac{1}{2} \sum_{n=0}^{k-1} \Delta t_n [\chi(t_n) \mathbf{E}(t_k - t_n) + \chi(t_{n+1}) \mathbf{E}(t_k - t_{n+1})] \\ &= \frac{\Delta t_0}{2} \chi_0 \mathbf{E}(t_k) + \frac{1}{2} \sum_{n=1}^{k-1} \Delta t_n \chi_n \mathbf{E}(t_k - t_n) + \frac{1}{2} \sum_{n=0}^{k-1} \Delta t_n \chi_{n+1} \mathbf{E}(t_k - t_{n+1}) \end{aligned}$$

The Gauss law becomes

$$\begin{aligned} \nabla \cdot \left[ \left( 1 + \frac{\chi_0 \Delta t_0}{2} \right) \mathbf{E}(t_k) \right] &= \frac{\rho(t_k)}{\epsilon_0} \\ &- \nabla \cdot \left[ \frac{1}{2} \sum_{n=1}^{k-1} \Delta t_n \chi_n \mathbf{E}(t_k - t_n) + \frac{1}{2} \sum_{n=0}^{k-1} \Delta t_n \chi_{n+1} \mathbf{E}(t_k - t_{n+1}) \right]. \end{aligned}$$

Note that the dispersion enters as an extra term on the right-hand side, emulating a space charge. Unfortunately, inclusion of dispersion means that we must store  $\mathbf{E}(t_n)$  for all previous time steps.

### Auxiliary differential equation

With the ADE approach we seek a solution to  $\mathbf{P}(\omega) = \epsilon_0 \chi(\omega) \mathbf{E}(\omega)$  in the form

$$\sum_k a_k (i\omega)^k \mathbf{P}(\omega) = \epsilon_0 \mathbf{E}(\omega),$$

where  $\sum a_k (i\omega)^k$  is the Taylor series for  $1/\chi(\omega)$ . This can be written as a partial differential equation

$$\sum_k a_k \partial_t^k \mathbf{P}(t) = \epsilon_0 \mathbf{E}(t).$$

This equation can be discretized using finite differences, and centering the solution on  $t_k$  with backward differences yields an expression

$$\mathbf{P}^k = \epsilon_0 C_0^k \mathbf{E}^k - \sum_{m>0} C_m^k \mathbf{P}^{k-m}.$$

where  $C_k$  are stencil coefficients to be worked out for each case. The displacement field  $\mathbf{D}^k = \epsilon_0 \mathbf{E}^k + \mathbf{P}^k$  is then

$$\mathbf{D} = \epsilon_0(1 + C_0^k)\mathbf{E} - \sum_{m>0} C_m^k \mathbf{P}^{k-m}.$$

The Gauss law yields

$$\nabla \cdot [(1 + C_0^k) \mathbf{E}^k] = \frac{\rho}{\epsilon_0} - \frac{1}{\epsilon_0} \nabla \cdot \sum_{m>0} C_m^k \mathbf{P}^{k-m}.$$

Unlike the convolution approach, this only requires storing terms required for the ADE description. This depends both on the order of the ADE, as well as its discretization. Normally, the ADE is a low-order PDE and a few terms are sufficient.

## 4.2.7 Limitations

**Warning:** There is currently a bug where having a dielectric interface align *completely* with a grid face will cause the cell to be identified as an electrode EB. This bug is due to the way Chombo handles cut-cells that align completely with a grid face. In this case the cell with volume fraction  $\kappa = 1$  will be identified as an irregular cell. For the opposite phase (i.e., viewing the grids from inside the boundary) the situation is opposite and thus the two “matching cells” can appear in different grid patches. A fix for this is underway. In the meantime, a sufficient workaround is simply to displace the dielectric slightly away from the interface (any non-zero displacement will do).

## 4.2.8 Example application(s)

Example applications that use the electrostatics capabilities are:

- *Electrostatics model*.
- *CDR plasma model*.

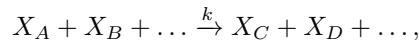
## 4.3 Kinetic Monte Carlo

### 4.3.1 Concept

Kinetic Monte Carlo solvers advance a state (or multiple states) represented e.g. as state vectors

$$\vec{X}(t) = \begin{pmatrix} X_1(t) \\ X_2(t) \\ X_3(t) \\ \vdots \end{pmatrix}$$

Each row in  $\vec{X}$  represents e.g. the population of some chemical species. Reactions between species are represented stoichiometrically as

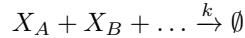


where  $k$  is the reaction rate. Each such reaction is associated with a state change in  $\vec{X}$ , e.g.

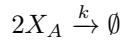
$$\vec{X} \xrightarrow{r} \vec{X} + \vec{\nu}_r,$$

where  $r$  is the reaction type and  $\nu_r$  is the state change associated with the firing of *one* reaction of type  $r$ . A set of such reactions is called the *reaction network*  $\vec{R}$ .

Propensities  $a_r(\vec{X})$  are defined such that  $a_r(\vec{X}) dt$  is the probability that exactly one reaction of type  $r$  occurs in the time interval  $[t, t + dt]$ . For unimolecular reactions of the type



with  $A \neq B \neq \dots$  the propensity function is  $kX_AX_B\dots$ . For bimolecular of the type



the propensity is  $k\frac{1}{2}X_A(X_A - 1)$  because there are  $\frac{1}{2}X_A(X_A - 1)$  unique pairs of molecules of type A. Propensities for higher-order reactions can then be expanded using the binomial theorem. For example, for a third-order reaction  $3X_A \xrightarrow{k} \emptyset$  the propensity function is  $k\frac{1}{6}X_A(X_A - 1)(X_A - 2)$ .

Various algorithms can be used for advancing the state  $\vec{X}$  for an arbitrary reaction network  $\vec{R}$ .

1. The *Stochastic simulation algorithm* (SSA). The SSA is also known as the Gillespie algorithm [6], and is an exact stochastic solution to the above problem. However, it becomes inefficient as the number of reactions per unit time grows.
2. *Tau leaping*, which is an approximation to the SSA which uses Poisson sampling of the underlying reactions.
3. Hybrid advance, see *Hybrid algorithm*. The hybrid algorithm is taken from Cao *et al.* [3], and switches between tau leaping and the SSA in their respective limits.

### 4.3.2 Stochastic simulation algorithm

For the SSA we compute the time until the next reaction by

$$T = \frac{1}{\sum_{r \in \vec{R}} a_r} \ln \left( \frac{1}{u_1} \right)$$

where  $A = \sum_{r \in \vec{R}} a_r$  and  $u_1$  is a uniformly distributed random variable between 0 and 1. The type of reaction that fires is determined from

$$r_c = \text{smallest integer satisfying } \sum_{r'=1}^{r_c} a_{r'} > u_2 A,$$

where and  $u_2$  is another uniformly distributed random variable between 0 and 1. The state is then advanced as

$$\vec{X}(t + T) = \vec{X}(t) + \vec{\nu}_{r_c}.$$

### 4.3.3 Tau leaping

With tau-leaping the state is advanced over a time  $\Delta t$  as

$$\vec{X}(t + \Delta t) = \vec{X}(t) + \sum_{\vec{R}} \vec{\nu}_r \mathcal{P} \left( a_r \left[ \vec{X}(t) \right] \Delta t \right),$$

where  $\mathcal{P}$  is a Poisson-distributed random variable. Note that tau leaping may fail to give a thermodynamically valid state, and should thus be used in combination with step rejection.

### 4.3.4 Hybrid algorithm

The hybrid algorithm is taken from Cao *et al.* [3]. Assume that we wish to integrate over some time  $\Delta t$ , which proceeds as follows:

1. Let  $\tau = 0$  be the simulated time within  $\Delta t$ .
2. Partition the reaction set  $\vec{R}$  into *critical* and *non-critical* reactions. The critical reactions are defined as the subset of  $\vec{R}$  that are within  $N_{\text{crit}}$  firings away from exhausting one of its reactants. The non-critical reactions are defined as the remaining subset.
3. Compute time steps until the firing of the next critical reaction, and a time step such that the propensities of the non-critical reactions do not change by more than some relative factor  $\epsilon$ . Let these time steps be given by  $\Delta\tau_c$  and  $\Delta\tau_{nc}$ .
4. Select a reactive substep within  $\Delta t$  from

$$\Delta\tau = \min [\Delta t - \tau, \min (\Delta\tau_c, \Delta\tau_{nc})]$$

5. Resolve reactions as follows:

- a. If  $\Delta\tau_c < \Delta\tau_{nc}$  and  $\Delta\tau_c < \Delta t - \tau$  then one critical reaction fires. Determine the reaction type using the SSA algorithm.

Next, advance the state using tau leaping for the non-critical reaction.

- b. Otherwise: No critical reactions fire. Advance the state using tau-leapnig for the non-critical reactions only. An exception is made if  $A\Delta\tau$  is smaller than some specified threshold in which case we switch to SSA advancement (which is more efficient in this limit).
6. Check if  $\vec{X}$  is a thermodynamically valid state.
  - a. If the state is valid, accept it and let  $\tau \rightarrow \tau + \Delta\tau$ .
  - b. If the state is invalid, reject the advancement. Let  $\Delta\tau_{nc} \rightarrow \Delta\tau_{nc}/2$  and return to step 4).
7. If  $\tau < \Delta t$ , return to step 2.

The Cao *et al.* [3] algorithm requires algorithmic specifications as follows:

- The factor  $\epsilon$  which determines the non-critical time step.
- The factor  $N_{\text{crit}}$  which determines which reactions are critical or not.
- Factors for determining when and how to switch to the SSA-based algorithm in step 5b.

### 4.3.5 Implementation

The Kinetic Monte Carlo solver is implemented as

```
template <typename R, typename State, typename T = long long>
class KMCsolver
{
public:
    using ReactionList = std::vector<std::shared_ptr<const R>>;
    inline KMCsolver(const ReactionList& a_reaction) noexcept;
}
```

The template parameters are:

- **R** is the type of reaction to advance with.
- **State** is the state vector that the KMC and reactions will advance.

- T is the internal floating point or integer representation.

---

**Tip:** The KMCsolver C++ API is found at <https://chombo-discharge.github.io/chombo-discharge/doxygen/html/classKMCsolver.html>.

---

## State

The State representation *must* have a member function

```
bool State::isValidState() const;
```

which determines if the state is thermodynamically valid (e.g. no negative populations). The functionality is used when using the hybrid advancement algorithm, see *Hybrid algorithm*.

## Reaction(s)

The reaction representation R *must* have the following member functions:

```
// Compute the propensity of the current reaction.
Real R::propensity(const State& s) const;

// Compute the number of reactions before exhausting one of the reactants
T R::computeCriticalNumberOfReactions(const State& s) const;

// Compute the number of reactions before exhausting one of the reactants
void R::advanceState(const State& s, const T& numReactions) const;

// Get a vector/list/deque etc. of the reactants. <some_container> can be e.g. std::vector<size_t>
<some_container> R::getReactants() const;

// Get the population corresponding to 'reactant' in the input state. If e.g. <some_container> is
// std::vector<size_t> then <some_type> will be <size_t>
T R::population(const <some_type> reactant, const State& s) const;
```

These template requirements exist so that users can define their states independent of their reactions. Likewise, reactions can be defined to operate flexibly on state, and the KMCsolver can be defined without deep restrictions on the states and reactions that are used.

## Defining states

State representations State can be defined quite simply (e.g. just a list of indices). In the absolute simplest case a state can be defined by maintaining a list of populations like below:

```
class MyState {
public:
    MyState(const size_t numSpecies) {
        m_populations.resize(numSpecies);
    }

    bool isValidState() const {
        return true;
    }

    std::vector<long long> m_populations;
};
```

More advanced examples can distinguish between different *modes* of populations, e.g. between species that can only appear on the left/right hand side of the reactions. See *Dual-state* for such an example.

### Defining reactions

See [Implementation](#) for template requirements on state-advancing reactions. Using MyState above as an example, a minimal reaction that can advance  $A \rightarrow B$  with a rate of  $k = 1$  is

```
class MyStateReaction {
public:

    // List of reactants and products
    MyStateReaction(const size_t a_A, const size_t a_B) {
        m_A = a_A;
        m_B = a_B;
    }

    // Compute propensity
    Real propensity(const State& a_state) {
        return a_state[m_A];
    }

    // Never consider these reactions to be "critical"
    long long computeCriticalNumberOfReactions(const MyState& a_state) {
        return std::numeric_limits<long long>::max();
    }

    // Get a vector/list/deque etc. of the reactant's. <some_container> can be e.g. std::vector<size_t>
    std::list<size_t> R::getReactants() const {
        return std::list<size_t>{m_A};
    }

    // Get population
    long long population(const size_t& a_reactant, const MyState& a_state) {
        return a_state.m_populations[a_reactant];
    }

    // Advance state with reaction A -> B
    void advanceState(const MyState& s, const long long& numReactions) const {
        s.populations[m_A] -= numReactions;
        s.populations[m_B] += numReactions;
    }

protected:
    size_t m_A;
    size_t m_B;
};
```

### Advancement routines

The advancement routines for the KMCsolver are

```
template <typename R, typename State, typename T = long long>
class KMCsolver
{
public:

    // Advance one step with the SSA algorithm.
    inline void
    advanceSSA(State& a_state, const Real a_dt) const;

    // Advance using tau leaping
    inline void
    advanceTau(State& a_state, const Real a_dt) const;

    // Advance using hybrid algorithm.
    inline void
    advanceHybrid(State& a_state, const Real a_dt) const;

    // Set hybrid solver parameters.
    inline void
    setSolverParameters(const T a_numCrit, const T a_numSSA, const Real a_eps, const Real a_SSALim) noexcept;
};
```

When using the hybrid algorithm, the user should set the hybrid solver parameters through `setSolverParameters`. See [Hybrid algorithm](#) for further details.

### 4.3.6 State and reaction examples

chombo-discharge maintains some states and reaction methods that can be useful when solving problems with KMCSolver.

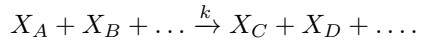
#### Single-state

The KMCSingleState class defines a single state vector  $\vec{X}$  that can appear on either side of reactions. The user defines the number of species through the constructor

```
template <typename T = long long>
class KMCSingleState {
public:
    // Define a state vector with specified number of species.
    inline KMCSingleState(const size_t a_numSpecies) noexcept;
};
```

Internally the state just uses a `std::vector<T>` for representing the populations.

KMCSingleStateReaction can be used to define reactions between species in KMCSingleState. The reaction is specified as a generic type of reaction



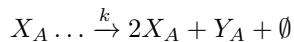
The relevant function signatures that specify the reactants, products, and the rate  $k$ , are

```
template <typename T = long long, typename State = KMCSingleState<T>>
class KMCSingleStateReaction {
public:
    // Define list of reactants/products through constructor
    inline KMCSingleStateReaction(const std::list<size_t>& a_reactants,
                                  const std::list<size_t>& a_products) noexcept;

    // For setting the reaction rate used in the propensity calculation.
    inline Real& rate() const noexcept;
};
```

#### Dual-state

KMCDualState defines two state vectors  $\vec{X}$  and  $\vec{Y}$  where  $\vec{X}$  are *reactant species* and  $\vec{Y}$  are *non-reactant species*. The intention behind this class is that reactant species are allowed on either side of the reaction, while the non-reactant species only occur on the right-hand side of the reaction. For example:



The class is implemented as

```
template <typename T = long long>
class KMCDualState {
public:
    // Define a state vector with specified number of species.
    inline KMCDualState(const size_t a_numReactiveSpecies, const size_t a_numNonReactiveSpecies) noexcept;

    // Get the reactant state (i.e., X)
    std::vector<T>& getReactiveState() noexcept;

    // Get the non-reactant state (i.e., Y)
    std::vector<T>& getNonReactiveState() noexcept;
};
```

KMCDualStateReaction can define reactions between states in the state vector  $\vec{X}$  which give products in both  $\vec{X}$  and  $\vec{Y}$  as follows.

```
template <typename T = long long, typename State = KMCDualState<T>>
class KMCDualStateReaction {
public:
    // Define list of reactants/products through constructor
    inline KMCDualStateReaction(const std::list<size_t>& a_lhsReactives,
                                const std::list<size_t>& a_rhsReactives,
                                const std::list<size_t>& a_rhsNonReactives);

    // For setting the reaction rate used in the propensity calculation.
    inline Real&
    rate() const noexcept;
};
```

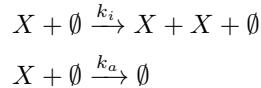
### 4.3.7 Verification

Verification tests for KMCSolver are given in

- \$DISCHARGE\_HOME/Exec/Convergence/KineticMonteCarlo/C1
- \$DISCHARGE\_HOME/Exec/Convergence/KineticMonteCarlo/C2

#### C1: Avalanche model

An electron avalanche model is given in \$DISCHARGE\_HOME/Exec/Convergence/KineticMonteCarlo/C1. The problem solves for a reaction network



In the limit  $X \gg 1$  the exact solution is

$$X(t) \approx X(0) \exp [(k_i - k_a)t].$$

Figure Fig. 4.3.1 shows the Kinetic Monte Carlo solution for  $k_i = 2k_a = 2$  and  $X(0) = 10$ .

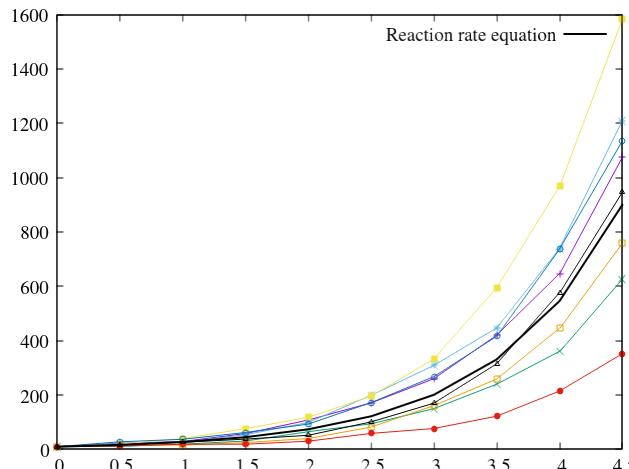
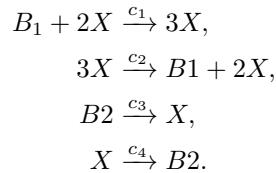


Fig. 4.3.1: Comparison of Kinetic Monte Carlo solution with reaction rate equation for an avalanche-like problem.

## C2: Schlögl model

Solution the Schlögl model are given in `$DISCHARGE_HOME/Exec/Convergence/KineticMonteCarlo/C2`. For the Schlögl model we solve for a single population  $X$  with the reactions



The states  $B_1$  and  $B_2$  are buffered states with populations that do not change during the reactions. Figure Fig. 4.3.1 shows the Kinetic Monte Carlo solutions for rates

$$\begin{aligned} c_1 &= 3 \times 10^{-7}, \\ c_2 &= 10^{-4}, \\ c_3 &= 10^{-3}, \\ c_4 &= 3.5 \end{aligned}$$

and  $B_1 = 10^5$ ,  $B_2 = 2 \times 10^5$ . The initial state is  $X(0) = 250$ .

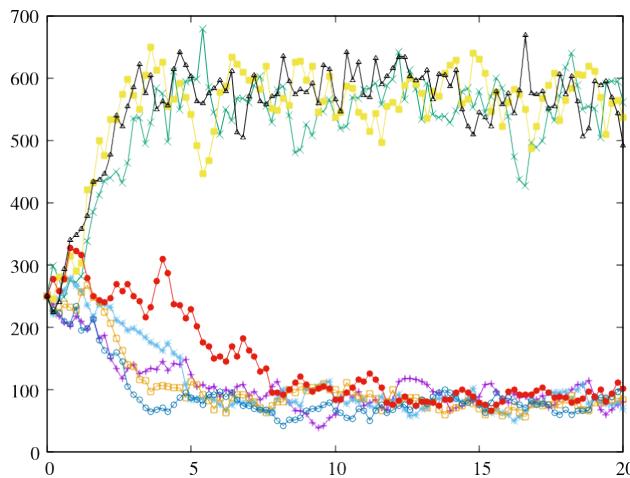


Fig. 4.3.2: Convergence to bi-stable states for the Schlögl model.

## 4.4 Mesh ODE solver

The `MeshODESolver` implements a solver for

$$\frac{\partial \vec{\phi}}{\partial t} = \vec{S},$$

where  $\vec{\phi}$  represents  $N$  unknowns on the mesh, and  $\vec{S}$  is the corresponding source term. The class is templated as

```
template <size_t N = 1>
class MeshODESolver;
```

where `N` indicates the number of variables stored on the mesh.

To instantiate the solver, use the full constructor with reference to `AmrMesh`:

```
template <size_t N>
MeshODESolver<N>::MeshODESolver(const RefCountedPtr<AmrMesh>& a_amr) noexcept;
```

If running dual grid simulations, the corresponding *Realm* can be set through the public API, see <https://chombo-discharge.github.io/chombo-discharge/doxygen/html/classMeshODESolver.html>.

---

**Note:** Source code for the MeshODESolver resides in [Source/MeshODESolver](#).

---

### 4.4.1 Setting $\vec{\phi}$

#### Mesh-based

To set the initial data in a general way, one can fetch the  $N$  mesh components from

```
template <size_t N>
EBAMRCelldata&
MeshODESolver<N>::getPhi() noexcept;
```

This will return the data holder that holds the cell centered data  $\vec{\phi}$  which the user can iterate through.

#### Analytic function

One can set  $\vec{\phi}$  by using analytic functions  $\vec{\phi}(\mathbf{x}) = \vec{f}(\mathbf{x})$  through

```
using Func1 = const std::function<Real(const RealVect& a_pos)>;
using Func2 = const std::function<std::array<Real, N>(const RealVect& a_pos)>;
template <size_t N>
void
MeshODESolver<N>::setPhi(const Func1& a_func1, const size_t a_comp) noexcept;
template <size_t N>
void
MeshODESolver<N>::setPhi(const Func2& a_func2) noexcept;
```

These differ in the sense that Func1, which is just an alias for a function  $f = f(\mathbf{x})$ , sets the value for a specified component. The other version that takes Func2 as an argument sets the corresponding values for all components.

### 4.4.2 Setting $\vec{S}$

#### General approach

For a general method of setting the source term one can fetch  $\vec{S}$  through

```
template <size_t N>
EBAMRCelldata&
MeshODESolver<N>::getRHS() noexcept;
```

This returns an l-value reference to  $\vec{S}$  which the user can iterate through and set the value in each cell.

## Spatially dependent

The source term can also be set on a component-by-component basis using

```
using Func = std::function<Real(const RealVect&)>;
template <size_t N>
void MeshODESolver<N>::setRHS(const Func& a_rhsFunction, const size_t a_comp) noexcept;
```

The above function will evaluate a function  $f(\mathbf{x})$  in each cell.

## Analytically coupled

A third option is to compute the right-hand side directly using a coupling function. `MeshODESolver` aliases a function

```
template<size_t N>
using RHSFunction = std::function<std::array<Real, N>(const std::array<Real, N>& phi, const Real& time)>;
```

which computes the source term  $\vec{S}$  as a function

$$\vec{S} = \vec{f}(\vec{\phi}, t)$$

To fill the source term using an analytic coupling function like this, one can call

```
template <size_t N>
void MeshODESolver<N>::computeRHS(const RHSFunction& a_rhsFunction) noexcept;
```

### 4.4.3 Regridding

When regridding the `MeshODESolver`, one should call

```
template <size_t N>
void MeshODESolver<N>::preRegrid(const int a_lbase, const int a_oldFinestLevel) noexcept;
```

before `AmrMesh` creates the new grids. This will store  $\vec{\phi}$  on the old mesh. After `AmrMesh` has generated the new grids,  $\vec{\phi}$  can be interpolated onto the new grids by calling

```
template <size_t N>
MeshODESolver<N>::regrid(const int a_lmin, const int a_oldFinestLevel, const int a_newFinestLevel) noexcept;
```

Users can also choose to turn on/off slope limiters when putting the solution on the new mesh, see [Regridding](#). The source term  $\vec{S}$  is also allocated on the new mesh, but is not interpolated onto the new grids.

### 4.4.4 I/O

The user can add  $\vec{\phi}$  and  $\vec{S}$  to output files by specifying these in the input script. These variables are named

```
MeshODESolver.plt_vars = phi rhs
```

Only `phi` and `rhs` are recognized as valid arguments. If choosing to omit output variables for the solver, one can put e.g. `MeshODESolver.plt_vars = -1`.

---

**Note:** `MeshODESolver` checkpoint files only contain  $\vec{\phi}$ .

---

## 4.5 Radiative transfer

Radiative transfer is supported in the diffusion (i.e. Eddington or Helmholtz) approximation and with Monte Carlo sampling of discrete photons. The solvers share a common interface (a parent class), but note that the radiative transfer equation is inherently deterministic while Monte Carlo photon transport is inherently stochastic. The diffusion approximation relies on solving an elliptic equation in the stationary case and a parabolic equation in the time-dependent case, while the Monte-Carlo approach solves for fully transient or “stationary” transport.

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**Note:** The source code for the radiative transfer solvers reside in `Source/RadiativeTransfer`

---

### 4.5.1 RtSpecies

The class `RtSpecies` is an abstract base class for parsing necessary information into radiative transfer solvers. When creating a radiative transfer solver one will need to pass in a pointer to `RtSpecies` such that the solvers can look up the required information. Currently, `RtSpecies` is a lightweight class where the user needs to implement the function

```
virtual Real RtSpecies::getAbsorptionCoefficient(const RealVect a_pos) const = 0;
```

The absorption coefficient is used in the diffusion (see *Diffusion approximation*) and Monte Carlo (see *Monte Carlo methods*) solvers.

One can also assign a name to the species through the member variable `RtSpecies::m_name`.

### 4.5.2 RtSolver

`RtSolver` is the base class for encapsulating a radiative transfer solver. The source code for the solver is located in `$DISCHARGE_HOME/Source/RadiativeTransfer` and it is a fairly lightweight abstract class. As with other solvers, `RtSolver` can use a specified *Realm*.

To use the `RtSolver` interface the user must cast from one of the inherited classes (see *Diffusion approximation* or *Monte Carlo methods*). Since most of the `RtSolver` is an interface which is implemented by other radiative transfer solvers, documentation of boundary conditions, kernels and so on are found in the implementation classes.

### 4.5.3 Diffusion approximation

#### EddingtonSP1

The first-order diffusion approximation to the radiative transfer equation is encapsulated by the `EddingtonSP1` class which implements a first order Eddington approximation of the radiative transfer equation. `EddingtonSP1` implements `RtSolver` using both stationary and transient advance methods (e.g. for stationary or time-dependent radiative transport). The source code is located in `$DISCHARGE_HOME/RadiativeTransfer`.

## Equation of motion

In the diffusion approximation, the radiative transport equation is

$$\partial_t \Psi + \kappa \Psi - \nabla \cdot \left( \frac{1}{3\kappa} \nabla \Psi \right) = \frac{\eta}{c}, \quad (4.5.1)$$

where  $\kappa$  is the absorption coefficient (i.e., inverse absorption length). Note that in the context below,  $\kappa$  is *not* the volume fraction of a grid cell. This is called the Eddington approximation, and the radiative flux is  $F = -\frac{c}{3\kappa} \nabla \Psi$ .

In the stationary case this yields a Helmholtz equation

$$\kappa \Psi - \nabla \cdot \left( \frac{1}{3\kappa} \nabla \Psi \right) = \frac{\eta}{c}, \quad (4.5.2)$$

## Implementation

EddingtonSP1 uses multigrid methods for solving Eq. 4.5.1 and Eq. 4.5.2, see [Linear solvers](#). The class implements `RtSolver::advance()`, which can switch between Eq. 4.5.1 and Eq. 4.5.2. Note that for both the stationary and time-dependent cases the absorption coefficient  $\kappa$  in Eq. 4.5.1 and Eq. 4.5.2 are filled using the `RtSpecies` implementation provided to the solver. Also note that the absorption coefficient does not need to be constant in space.

### Stationary kernel

For the stationary kernel we solve Eq. 4.5.2 directly, using a single multigrid solve. See [Linear solvers](#) for discretization details.

### Transient kernel

For solving Eq. 4.5.1, EddingtonSP1 implements both the backward Euler method and the Twizell-Gumel-Arigu (TGA) scheme. Explicit discretizations are not available. The Euler discretization is

$$(1 + \kappa \Delta t) \Psi^{k+1} - \Delta t \nabla \cdot \left( \frac{1}{3\kappa} \nabla \Psi^{k+1} \right) = \Psi^k + \frac{\Delta t \eta^{k+1}}{c},$$

Again, this is a Helmholtz equation for  $\Psi^{k+1}$  which is solved using geometric multigrid. Expressions for the TGA scheme are found in [11], but note that the TGA scheme requires a solution to two elliptic equations (thus it has approximately twice the cost).

## Boundary conditions

### Simplified domain boundary conditions

The EddingtonSP1 solver supports the following boundary conditions on domain faces and EBs. The domain boundary condition *type*, which is either Dirichlet, Neumann, or Larsen (a special type of Robin boundary condition) is always passed in through the input file. If the user passes in a value, say `neumann 0.0`, for a particular domain side/face, then the class will use a homogeneous Neumann boundary for the entire domain edge/face.

## Custom domain boundary conditions

It is possible to use more complex boundary conditions by passing in `dirichlet_custom`, `neumann_custom`, or `larsen_custom` options. In this case the EddingtonSP1 solver will use a specified function at the domain edge/face. To specify that function, EddingtonSP1 has a member function

```
void setDomainSideBcFunction(const int a_dir,
                           const Side::LoHiSide a_side,
                           const std::function<Real(const RealVect a_pos, const Real a_time)> a_function);
```

which specifies a boundary condition value for one of the edges (faces in 3D). Note that the boundary condition *type* is still Dirichlet, Neumann, or Larsen (depending on whether or not `dirichlet_custom`, `neumann_custom`, or `larsen_custom` was passed in). For example, to set the boundary condition on the left  $x$  face in the domain, one can create a `EddingtonSP1DomainBc::BcFunction` object as follows:

```
// Assume this has been instantiated.
RefCountedPtr<EddingtonSP1> eddingtonSolver;

// Make a lambda which we can bind to std::function.
auto myValue = []([const RealVect a_pos, const Real a_time] -> Real {
    return a_pos[0] * a_time;
}

// Set the domain bc function in the solver.
eddingtonSolver.setDomainSideBcFunction(0, Side::Lo, myValue);
```

---

**Note:** If the user specifies one of the custom boundary conditions but does not set the function, it will issue a run-time error.

---

## Embedded boundaries

On the EB, we currently only support constant-value boundary conditions. In the input script, the user can specify

- `dirichlet <value>` For setting a constant Dirichlet boundary condition everywhere.
- `neumann <value>` For setting a constant Neumann boundary condition everywhere.
- `larsen <value>` For setting a constant Larsen boundary condition everywhere.

## Boundary condition types

1. **Dirichlet.** For Dirichlet boundary conditions we specify the value of  $\Psi$  on the boundary. Note that this involves reconstructing the gradient  $\partial_n \Psi$  on domain faces and edges, see [Dirichlet](#).
2. **Neumann.** For Neumann boundary conditions we specify the value of  $\partial_n \Psi$  on the boundary. Note that the linear solver interface also supports setting  $B\partial_n \Psi$  on the boundary (where  $B$  is the Helmholtz equation  $B$  coefficient). However, the EddingtonSP1 solver does not use this functionality.
3. **Larsen.** The Larsen boundary condition is an absorbing boundary condition, taking the form of a Robin boundary as follows:

$$\kappa \partial_n \Psi + \frac{3\kappa^2}{2} \frac{1 - 3r_2}{1 - 2r_1} \Psi = g,$$

where  $r_1$  and  $r_2$  are reflection coefficients and  $g$  is a surface source, see [8] for details. Note that when the user specifies the boundary condition value (e.g. by setting the BC function), he is setting the surface source  $g$ . In the majority of cases, however, we will have  $r_1 = r_2 = g = 0$  and the BC becomes

$$\partial_n \Psi + \frac{3\kappa}{2} \Psi = 0.$$

## Solver configuration

The EddingtonSP1 implementation has a number of configurable options for running the solver, and these are given below:

```
# =====
# EddingtonSP1 class options
#
EddingtonSP1.stationary      = true          # Stationary solver
EddingtonSP1.reflectivity    = 0.             # Reflectivity
EddingtonSP1.use_tga          = false         # Use TGA for integration
EddingtonSP1.kappa_scale     = true          # Kappa scale source or not (depends on algorithm)
EddingtonSP1.plt_vars         = phi src       # Plot variables. Available are 'phi' and 'src'

EddingtonSP1.ebbc            = larsen 0.0    # Bc on embedded boundaries
EddingtonSP1.bc.x.lo          = larsen 0.0    # Bc on domain side. 'dirichlet', 'neuman', or 'larsen'
EddingtonSP1.bc.x.hi          = larsen 0.0    # Bc on domain side. 'dirichlet', 'neuman', or 'larsen'
EddingtonSP1.bc.y.lo          = larsen 0.0    # Bc on domain side. 'dirichlet', 'neuman', or 'larsen'
EddingtonSP1.bc.y.hi          = larsen 0.0    # Bc on domain side. 'dirichlet', 'neuman', or 'larsen'
EddingtonSP1.bc.z.lo          = larsen 0.0    # Bc on domain side. 'dirichlet', 'neuman', or 'larsen'
EddingtonSP1.bc.z.hi          = larsen 0.0    # Bc on domain side. 'dirichlet', 'neuman', or 'larsen'
EddingtonSP1.bc.z.hi          = larsen 0.0    # Boundary on domain. 'neumann' or 'larsen'

EddingtonSP1.gmg_verbosity    = -1             # GMG verbosity
EddingtonSP1.gmg_pre_smooth  = 8              # Number of relaxations in downswing
EddingtonSP1.gmg_post_smooth = 8              # Number of relaxations in upswing
EddingtonSP1.gmg_bott_smooth = 8              # NUmber of relaxations before dropping to bottom solver
EddingtonSP1.gmg_min_iter    = 5              # Minimum number of iterations
EddingtonSP1.gmg_max_iter    = 32             # Maximum number of iterations
EddingtonSP1.gmg_exit_tol   = 1.E-6          # Residue tolerance
EddingtonSP1.gmg_exit_hang  = 0.2             # Solver hang
EddingtonSP1.gmg_min_cells  = 16             # Bottom drop
EddingtonSP1.gmg_bottom_solver= bicgstab      # Bottom solver type. Valid options are 'simple <number>' and 'bicgstab'
EddingtonSP1.gmg_cycle       = vcycle          # Cycle type. Only 'vcycle' supported for now
EddingtonSP1.gmg_ebbc_weight = 2               # EBBC weight (only for Dirichlet)
EddingtonSP1.gmg_ebbc_order  = 2               # EBBC order (only for Dirichlet)
EddingtonSP1.gmg_smoothen    = red_black       # Relaxation type. 'jacobi', 'red_black', or 'multi_color'
```

## Basic options

Basic input options to EddingtonSP1 are as follows:

- `EddingtonSP1.stationary` for setting whether or not the solver is stationary.
- `EddingtonSP1.reflectivity` for controlling the reflectivity in the Larsen boundary conditions.
- `EddingtonSP1.use_tga` for switching between backward Euler and TGA time discretizations. Only relevant if `EddingtonSP1.stationary = false`.
- `EddingtonSP1.kappa_scale` Switch for multiplying the source with the volume fraction or not. Note that the multigrid Helmholtz solvers require a diagonal weighting of the operator. If `EddingtonSP1.kappa_scale = false` then the solver will assume that this weighting of the source term has already been made.
- `EddingtonSP1.plt_vars` For setting which solver plot variables are included in plot files.

## Setting boundary conditions

Boundary conditions are parsed through the flags

- `EddingtonSP1.ebbc` Which sets the boundary conditions on the EBs.
- `EddingtonSP1.bc.dim.side` Which sets the boundary conditions on the domain sides, see [Boundary conditions](#) for details.

## Tuning multigrid performance

All parameters that begin with the form `EddingtonSP1.gmg_` indicate a tuning parameter for geometric multigrid.

- `EddingtonSP1.gmg_verbose`. Controls the multigrid verbosity. Setting it to a number  $> 0$  will print multigrid convergence information.
- `EddingtonSP1.gmg_pre_smooth`. Controls the number of relaxations on each level during multigrid down-sweeps.
- `EddingtonSP1.gmg_post_smooth`. Controls the number of relaxations on each level during multigrid up-sweeps.
- `EddingtonSP1.gmg_bott_smooth`. Controls the number of relaxations before entering the bottom solve.
- `EddingtonSP1.gmg_min_iter`. Sets the minimum number of iterations that multigrid will perform.
- `EddingtonSP1.gmg_max_iter`. Sets the maximum number of iterations that multigrid will perform.
- `EddingtonSP1.gmg_exit_tol`. Sets the exit tolerance for multigrid. Multigrid will exit the iterations if  $r < \lambda r_0$  where  $\lambda$  is the specified tolerance,  $r = |L\Phi - \rho|$  is the residual and  $r_0$  is the residual for  $\Phi = 0$ .
- `EddingtonSP1.gmg_exit_hang`. Sets the minimum permitted reduction in the convergence rate before exiting multigrid. Letting  $r^k$  be the residual after  $k$  multigrid cycles, multigrid will abort if the residual between levels is not reduced by at least a factor of  $r^{k+1} < (1 - h)r^k$ , where  $h$  is the “hang” factor.
- `EddingtonSP1.gmg_min_cells`. Sets the minimum amount of cells along any coordinate direction for coarsened levels. Note that this will control how far multigrid will coarsen. Setting a number `gmg_min_cells = 16` will terminate multigrid coarsening when the domain has 16 cells in any of the coordinate direction.
- `EddingtonSP1.gmg_bottom_solver`. Sets the bottom solver type.
- `EddingtonSP1.gmg_cycle`. Sets the multigrid method. Currently, only V-cycles are supported.
- `EddingtonSP1.gmg_ebbc_order`. Sets the stencil order on EBs when using Dirichlet boundary conditions. Note that this is also the stencil radius. See [Linear solvers](#) for details.
- `EddingtonSP1.gmg_ebbc_weight`. Sets the least squares stencil weighting factor for least squares gradient reconstruction on EBs when using Dirichlet boundary conditions. See [Least squares](#) for details.
- `EddingtonSP1.gmg_smoothen`. Sets the multigrid smoother.

## Runtime parameters

The following parameters for `EddingtonSP1` are run-time configurable:

- All multigrid tuning parameters, i.e. parameters starting with `EddingtonSP1.gmg_`.
- Plot variables, i.e. `EddingtonSP1.plt_vars`.
- Kappa scaling (for algorithmic adjustments), i.e. `EddingtonSP1.kappa_scale`.

#### 4.5.4 Monte Carlo methods

All types of moment-closed radiative transfer equations contain nonphysical artifacts (which may or may not be acceptable). For example, in the diffusion approximation the radiative flux is  $F = -\frac{c}{3\kappa} \nabla \Psi$ , implying that photons can leak around boundaries. I.e. the diffusion approximation does not correctly describe shadows. It is possible to go beyond the diffusion approximation by also solving for higher-order moments like the radiative flux. While such methods can describe shadows, they do, contain other nonphysical features.

Both “stationary” and transient Monte Carlo methods are offered as an alternative to the diffusion approximation.

##### photon particle

The `Ito` particle is a computational particle class in *chombo-discharge* which can be used together with the particle tools in *Chombo*. The following data fields are implemented in the particle:

```
RealVect m_position;
RealVect m_velocity;
Real m_mass;
Real m_kappa;
```

To obtain the fields, the user will call

```
RealVect& position();
RealVect& velocity();
Real& mass();
Real& diffusion();
```

All functions also have `const` versions. Note that the field `m_mass` is the same as the *weight* of the computational particle. The following functions are used to set the various properties:

```
setPosition(const RealVect a_pos);
setVelocity(const RealVect a_vel);
setMass(const Real a_mass);
setDiffusion(const Real a_diffusion);
```

#### Interaction with boundaries

##### Stationary Monte Carlo

The stationary Monte Carlo method proceeds as follows.

1. For each cell in the mesh, draw a discrete number of photons  $\mathcal{P}(\eta \Delta V \Delta t)$  where  $\mathcal{P}$  is a Poisson distribution. The user may also choose to use pseudophotons rather than physical photons by modifying photon weights. Each photon is generated in the cell centroid  $\mathbf{x}_0$  and given a random propagation direction  $\mathbf{n}$ .
2. Draw a propagation distance  $r$  by drawing random numbers from an exponential distribution  $p(r) = \kappa \exp(-\kappa r)$ . The absorbed position of the photon is  $\mathbf{x} = \mathbf{x}_0 + r\mathbf{n}$ .
3. Check if the path from  $\mathbf{x}_0$  to  $\mathbf{x}$  intersects an internal or domain boundary. If it does, absorb the photon on the boundary. If not, move the photon to  $\mathbf{x}$  or reflect it off symmetry boundaries.
4. Rebin the absorbed photons onto the AMR grid. This involves parallel communication.
5. Compute the resulting photoionization profile. The user may choose between several different deposition schemes (like e.g. cloud-in-cell).

The Monte Carlo methods use computational particles for advancing the photons in exactly the same way a Particle-In-Cell method would use them for advancing electrons. Although a computational photon would normally live on the finest grid level that overlaps its position, this is not practical for all particle deposition kernels. For example, for cloud-in-cell deposition schemes it is useful to have the restrict the interpolation kernels to the grid level where the

particle lives. In Chombo-speak, we therefore use a buffer region that extends some cells from a refinement boundary where the photons are not allowed to live. Instead, photons in that buffer region are transferred to a coarser level, and their deposition clouds are first interpolated to the fine level before deposition on the fine level happens. Selecting a deposition scheme and adjusting the buffer region is done through an input script associated with the solver.

### Transient Monte Carlo

The transient Monte Carlo method is almost identical to the stationary method, except that it does not deposit all generated photons on the mesh but tracks them through time. The transient method is implemented as follows:

1. For each cell in the mesh, draw a discrete number of photons  $\mathcal{P}(\eta\Delta V\Delta t)$  as above, and append these to the already existing photons. Each photon is given a uniformly distributed random creation time within  $\Delta t$ .
2. Each photon is advanced over the time step  $\Delta t$  by a sequence of  $N$  substeps ( $N$  may be different for each photon).
  - a. We compute  $N$  such that we sample  $N\Delta\tau = \Delta t$  with  $c\kappa\Delta\tau < 1$ .
  - b. A photon at position  $\mathbf{x}_0$  is moved a distance  $\Delta\mathbf{x} = c\Delta\tau$ . For each step we compute the absorption probability  $p = \kappa|\Delta\mathbf{x}|$  where  $p \in [0, 1]$  is a uniform random number. If the photon is absorbed on this interval, draw a new uniform random number  $r \in [0, 1]$  and absorb the photon at the position  $\mathbf{x}_0 + r\Delta\mathbf{x}$ . If the photon is not absorbed, it is moved to position  $\mathbf{x}_0 + r\Delta\mathbf{x}$ .
3. Check if the path from  $\mathbf{x}_0$  to  $\mathbf{x}$  intersects an internal or domain boundary. If it does, absorb the photon on the boundary. If not, move the photon to  $\mathbf{x}$ .
4. Rebin the absorbed photons onto the AMR grid. This involves parallel communication.
5. Compute the resulting photoionization profile. The user may choose between several different deposition schemes (like e.g. cloud-in-cell).

#### 4.5.5 Limitations

#### 4.5.6 Example application

An example application of usage of the RtSolver is found in [Radiative transfer](#).

## 4.6 Surface ODE solver

chombo-discharge provides a simple solver for ODE equations

$$\frac{\partial \vec{\phi}}{\partial t} = \vec{F},$$

where  $\vec{\phi}$  represents  $N$  unknowns on the EB. Note that the underlying data type for  $\vec{\phi}$  and  $\vec{F}$  is `EBAMRIVData`, see [Mesh data](#). Such a solver is useful, for example, as a surface charge solver where  $\phi$  is the surface charge density and  $F$  is the charge flux onto the EB.

The surface charge solver is implemented as

```
template <int N = 1>
class SurfaceODESolver;
```

where  $N$  indicates the number of variables stored in each cut cell.

## 4.6.1 Instantiation

To instantiate the solver, use the default constructor

```
template <int N>
SurfaceODESolver<N>::SurfaceODESolver();
```

The solver also requires a reference to *AmrMesh*, and the computational geometry such that a full instantiation example is

```
SurfaceODESolver<1>* solver = new SurfaceODESolver<1>();
solver->setAmr(...);
solver->setComputationalGeometry(...);
```

## 4.6.2 Setting $\vec{\phi}$

### Mesh-based

To set  $\vec{\phi}$  on the mesh, one can fetch the underlying data by calling

```
template <int N>
EBAMRIVData&
SurfaceODESolver<N>::getPhi() noexcept;
```

This returns a reference to the underlying data which is defined on all cut-cells. The user can then iterate through this data and set the values accordingly, see *Iterating over patches*.

### Function-based

To set the data directly, *SurfaceODESolver* defines a function

```
template <int N>
void
SurfaceODESolver<N>::setPhi(std::function<std::array<Real, N>(const RealVect pos)>& a_func);
```

where the input argument represents a function  $f = \vec{f}(x)$  that returns a value for each component in  $\vec{\phi}$ .

## 4.6.3 Setting $\vec{F}$

### Mesh-based

To set  $\vec{F}$  on the mesh, one can fetch the underlying data by calling

```
template <int N>
EBAMRIVData&
SurfaceODESolver<N>::getRHS() noexcept;
```

This returns a reference to the underlying data which is defined on all cut-cells. The user can then iterate through this data and set the values accordingly, see *Iterating over patches*.

## Function-based

To set the right-hand side directly, `SurfaceODESolver` defines a function

```
template <int N>
void
SurfaceODESolver<N>::setRHS(std::function<std::array<Real, N>(>& a_func);
```

where the input argument represents a function  $\vec{f} = \vec{f}(x)$  that returns a value for each component in  $\vec{F}$ .

### 4.6.4 Resetting cells

`SurfaceODESolver` has functions for setting values in the subset of the cut-cells representing dielectrics or electrodes. The function signatures are

```
template <int N>
void
SurfaceODESolver<N>::resetElectrodeCells(const Real a_value);

template <int N>
void
SurfaceODESolver<N>::resetDielectricCells(const Real a_value);
```

Calling these functions will set the data value in electrode or dielectric cells to `a_value`. Note that one can always call `SurfaceODESolver<N>::getPhi()` to iterate over other types of cell subsets.

### 4.6.5 Regridding

When regridding the `SurfaceODESolver`, one should call

```
template <int N>
void
SurfaceODESolver<N>::preRegid(const int a_lbase, const int a_oldFinestLevel) noexcept;
```

before `AmrMesh` creates the new grids. This will store  $\vec{\phi}$  on the old mesh. After `AmrMesh` has generated the new grids,  $\vec{\phi}$  can be interpolated onto the new grids by calling

```
template <int N>
SurfaceODESolver<N>::regrid(const int a_lmin, const int a_oldFinestLevel, const int a_newFinestLevel) noexcept;
```

Note that when interpolating to the new grids one can choose to initialize data in the new cells using the value in the underlying coarse cells, i.e.

$$\vec{\phi}_{\mathbf{i}_{\text{fine}}} = \vec{\phi}_{\mathbf{i}_{\text{coarse}}}$$

Alternatively one can initialize the fine-grid data such that the area-weighted value of  $\vec{\phi}$  is conserved, i.e.

$$\sum_{\mathbf{i}_{\text{fine}}} \alpha_{\mathbf{i}_{\text{fine}}} \Delta x_{\text{fine}}^{D-1} \vec{\phi}_{\mathbf{i}_{\text{fine}}} = \alpha_{\mathbf{i}_{\text{coar}}} \Delta x_{\text{coar}}^{D-1} \vec{\phi}_{\mathbf{i}_{\text{coar}}}$$

which gives

$$\vec{\phi}_{\mathbf{i}_{\text{fine}}} = r^{D-1} \frac{\alpha_{\mathbf{i}_{\text{coar}}}}{\sum_{\mathbf{i}_{\text{fine}}} \alpha_{\mathbf{i}_{\text{fine}}}} \vec{\phi}_{\mathbf{i}_{\text{coar}}},$$

where  $\mathbf{i}_{\text{fine}}$  is set of cut-cells that occur when refining the coarse-grid cut-cell  $\mathbf{i}_{\text{coar}}$  and  $r$  is the refinement factor between the two grid levels. In this case  $\vec{\phi}$  is strictly conserved. Users can switch between these two methods by specifying the regridding type in the input script:

```
SurfaceODESolver.regridding = arithmetic
```

or

```
SurfaceODESolver.regridding = conservative
```

## 4.6.6 I/O

The user can add  $\vec{\phi}$  and  $\vec{F}$  to output files by specifying these in the input script. These variables are named

```
SurfaceODESolver.plt_vars = phi rhs
```

Only phi and rhs are recognized as valid arguments. If choosing to omit output variables for the solver, one can put e.g. SurfaceODESolver.plt\_vars = -1.

---

**Note:** SurfaceODESolver checkpoint files only contain  $\vec{\phi}$ .

---

## 4.7 Tracer particles

Tracer particles are particles that move along a velocity field

$$\frac{\partial \mathbf{X}}{\partial t} = \mathbf{V}$$

where  $\mathbf{V}$  is the particle velocity. This is interpolated from a mesh-based field as

$$\mathbf{V} = \mathbf{v}(\mathbf{X}),$$

where  $\mathbf{v}$  is a velocity field defined on the mesh. Such particles are useful, for example, for numerical integration along field lines.

chombo-discharge defines AMR-ready tracer particle solvers in \$DISCHARGE\_HOME/Source/TracerParticles.

### 4.7.1 TracerParticleSolver

The tracer particle solver is templated as

```
template <typename P>
class TracerParticleSolver;
```

where P is the particle type used for the solver. The template constraints on P are

1. It *must* contain a function RealVect& position()
2. It *must* contain a function const Real& weight() const
3. It *must* contain a function RealVect& velocity().

Users are free to provide their own particle type provided that it meets these template constraints. However, we also define a plug-and-play particle class that meets these requirements, see [TracerParticle](#).

---

**Note:** The TracerParticleSolver API is available at <https://chombo-discharge.github.io/chombo-discharge/doxygen/html/classTracerParticleSolver.html>.

---

## 4.7.2 TracerParticle

The TracerParticle type inherits from the GenericParticle particle class and is templated as

```
template <size_t M, size_t N>
class TracerParticle<M,N> : public GenericParticle<M, N>
```

and also defines two more members: A particle weight and a particle velocity. These are accesible as

```
template <size_t M, size_t N>
Real&
TracerParticle<M, N>::weight();

template <size_t M, size_t N>
RealVect&
TracerParticle<M, N>::velocity();
```

Note that, just as for GenericParticle, the template arguments M and N indicates the number of scalars and vectors allocated to the particle. See [GenericParticle](#).

---

**Note:** The TracerParticleSolver API is available at <https://chombo-discharge.github.io/chombo-discharge/doxygen/html/classTracerParticle.html>.

---

## 4.7.3 Initialization

To initialize the solver, one can use the full constructor

```
template <typename P>
TracerParticleSolver<P>::TracerParticleSolver(const RefCountedPtr<AmrMesh>& a_amr,
                                              const RefCountedPtr<ComputationalGeometry> a_compGeom);
```

## 4.7.4 Getting the particles

To obtain the solver particles simply call

```
template <typename P>
ParticleContainer<P>&
TracerParticleSolver<P>::getParticles();
```

This returns the ParticleContainer holding the particles, see [ParticleContainer](#).

## 4.7.5 Setting v

To set the velocity use

```
template <typename P>
void
TracerParticleSolver<P>::setVelocity(const EBAMRCellData& a_velocityField)
```

This will associate the input velocity a\_velocityField with v.

## 4.7.6 Interpolating velocities

To compute  $\mathbf{V} = \mathbf{v}(\mathbf{X})$  use

```
template <typename P>
void
TracerParticleSolver<P>::interpolateVelocities();
```

This will interpolate the velocities to the particle positions using the user-defined interpolation method (see *Input options*).

One can also interpolate a scalar field defined on the mesh onto the particle weight by calling

```
template <typename P>
void
TracerParticleSolver<P>::interpolateWeight(const EBAMRCellData& a_scalar) noexcept;
```

Letting  $f$  define the input field `a_scalar`, this performs the operation  $w = f(\mathbf{X})$ .

## 4.7.7 Deposit particles

To deposit the particles call

```
template <typename P>
void
TracerParticleSolver<P>::deposit(EBAMRCellData& a_phi)
```

This will deposit the particle weights onto the input data holder.

## 4.7.8 Input options

Available input options for the tracer particle solver are

```
# =====
# TracerParticleSolver class options
# =====
TracerParticleSolver.verbosity      = -1      # Solver verbosity level.
TracerParticleSolver.deposition    = cic     # Deposition method. Must be 'ngp' or 'cic'
TracerParticleSolver.interpolation = cic     # Interpolation method. Must be 'ngp' or 'cic'
TracerParticleSolver.deposition_cf = halo    # Coarse-fine deposition. Must be interp or halo
TracerParticleSolver.plot_weight   = true    # Turn on/off plotting of the particle weight.
TracerParticleSolver.plot_velocity = true    # Turn on/off plotting of the particle velocities.
TracerParticleSolver.volume_scale = false   # If true, depositions yield density * volume instead of just volume
```

The flags `deposition` and `interpolation` indicates which deposition and interpolation methods will be used. Likewise, `deposition_cf` indicates the coarse-fine deposition strategy, see *Particles*. The flags `plot_weight` and `plot_velocity` indicates whether or not to include the particle weights and velocities in plot files.

## 4.8 Îto diffusion

The Îto diffusion model advances computational particles as drifting Brownian walkers

$$\Delta \mathbf{X} = \mathbf{V} \Delta t + \sqrt{2D \Delta t} \mathbf{W}$$

where  $\mathbf{X}$  is the spatial position of a particle  $\mathbf{V}$  is the drift velocity and  $D$  is the diffusion coefficient *in the continuum limit*. The vector term  $\mathbf{W}$  is a Gaussian random field with a mean value of 0 and standard deviation of 1.

---

**Note:** The code for Îto diffusion is given in /Source/ItoSolver.

---

### 4.8.1 ItoSolver

The class `ItoSolver` encapsulates the motion of drift-diffusion particles. This class can advance a set of particles (see [ItoParticle](#)) with the following functionality:

- Move particles using a microscopic drift-diffusion model.
- Compute particle intersection with embedded boundaries and domain edges.
- Deposit particles and other particle types on the mesh.
- Interpolate velocities and diffusion coefficients to the particle positions.
- Manage superparticle splitting and merging using kD-trees.

`ItoSolver` also defines an enum `WhichContainer` for classification of `ParticleContainer` data holders for holding particles that live on:

- The embedded boundary (`WhichContainer::EB`).
- On the domain edges/faces (`WhichContainer::Domain`).
- Representing “source particles” (`WhichContainer::Source`).
- Particles that live *inside* the EB (`WhichContainer::Covered`).

The particles are available from the solver through the function

```
ParticleContainer<ItoParticle>&
ItoSolver::getParticles(const WhichContainer a_whichParticles);
```

For the `ItoSolver` the particle velocity is computed as

$$\mathbf{V} = \mu(\mathbf{X}) \mathbf{v}(\mathbf{X})$$

where  $\mu$  is a particle mobility and  $\mathbf{v}$  is a velocity field defined on the mesh. Note that both  $\mu$  and  $\mathbf{v}$  are defined on the mesh. The solver can, alternatively, also compute the velocity as

$$\mathbf{V} = (\mu\mathbf{v})(\mathbf{X}),$$

i.e. through interpolation of  $\mu\mathbf{v}$  to the particle position. Regardless of which method is chosen (see [Velocity interpolation](#)), both  $\mu$  and  $\mathbf{v}$  exist on the mesh (stored as `EBAMRCellData`).

### 4.8.2 ItoParticle

The `ItoParticle` is used as the underlying particle type for running the Ito drift-diffusion solvers. It derives from [GenericParticle](#) as follows:

```
class ItoParticle : public GenericParticle<4,2>
```

and contains the following relevant member functions

```
// Storing for particle weight
Real&
ItoParticle::weight();

// Storage for particle diffusion coefficient
Real&
ItoParticle::diffusion();

// Storage for particle mobility
Real&
ItoParticle::mobility();
```

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```
// Storage for particle energy
Real&
ItoParticle::energy();

// Storage for particle velocity
RealVect&
ItoParticle::velocity();

// Storage for previous particle position
RealVect&
ItoParticle::oldPosition();
```

All of these functions have corresponding functions with `const` qualifiers.

### 4.8.3 ItoSpecies

`ItoSpecies` is a class for parsing information into `ItoSolver`. The constructor for the `ItoSpecies` class is

```
ItoSpecies(const std::string a_name, const int a_chargeNumber, const bool a_mobile, const bool a_diffusive);
```

and this will set the name of the class, the charge, and whether or not the transport kernels use drift and/or diffusion.

---

**Note:** `ItoSpecies` API is available at <https://chombo-discharge.github.io/chombo-discharge/doxygen/html/classItoSpecies.html>.

---

Initial data for the `ItoSolver` is provided through `ItoSpecies` by providing it with a list of initial particles. `ItoSpecies` have members that provide/modify the initial particles:

```
class ItoSpecies {
public:
    List<ItoParticle>&
    getInitialParticles();

protected:
    List<ItoParticle> m_initialParticles;
};
```

When `ItoSolver` initializes the data in the solver, it transfers the particles from the species and into the solver. See [ParticleOps](#) for examples on how to draw initial particles and how to partition them when using MPI.

### 4.8.4 Transport kernel

The transport kernels for the `ItoSolver` will simply consist of particle updates of the following type:

```
Real a_dt;
List<ItoParticle> particles;

for (ListIterator<ItoParticle>& lit(particles); lit.ok(); ++lit) {
    ItoParticle& p = lit();

    p.oldPosition() = p.position();
    p.position() += p.velocity() * a_dt + sqrt(2.0*p.diffusion()*a_dt) * this->randomGaussian();
}
```

The function `randomGaussian` implements a diffusion hopping and returns a 2D/3D dimensional vector with values drawn from a normal distribution with standard width of one and mean value of zero. The implementation uses the random number generators in [Random numbers](#). The user can choose to truncate the normal distribution, see [Input options](#).

## 4.8.5 Remapping particles

To remap, call the `ItoSolver` remapping functions as

```
void  
ItoSolver::remap();
```

This will remap the particles to the correct grid patches.

To remap the other `ParticleContainer` data holders (holding e.g. particles that intersected the EB), there's an alternative function

```
void  
ItoSolver::remap(const WhichContainer a_container);
```

where `a_container` is one of the particle containers.

## 4.8.6 Deposition

To deposit the particle weights onto the grid one can use

```
void  
ItoSolver::depositParticles();
```

The particles are deposited into the data holder `m_phi`. The data can then be fetched with

```
EBAMRCellData&  
ItoSolver::getPhi();
```

To deposit a different particle data holder into `m_phi` one can use

```
void  
ItoSolver::depositParticles(const WhichContainer a_container);
```

This can be used, for example, to deposit the EB particles on the mesh. More general methods also exist, see the `ItoSolver` C++ API <https://chombo-discharge.github.io/chombo-discharge/doxygen/html/classItoSolver.html>.

One can also deposit the following quantities on the mesh:

- Conductivity, which deposits  $\mu W$ .
- Diffusivity, which deposits  $DW$ .
- Energy, which deposits  $\epsilon W$ .

Here,  $W$  is the particle weight,  $\mu$  is the particle mobility,  $D$  is the particle diffusion coefficient and  $\epsilon$  is the particle energy. These functions exist as

```
void  
ItoSolver::depositConductivity(EBAMRCellData& a_phi, ParticleContainer<ItoParticle>& a_particles) const;  
  
void  
ItoSolver::depositDiffusivity(EBAMRCellData& a_phi, ParticleContainer<ItoParticle>& a_particles) const;  
  
void  
ItoSolver::depositEnergyDensity(EBAMRCellData& a_phi, ParticleContainer<ItoParticle>& a_particles) const;
```

---

**Important:** The `ItoSolver` deposition method is specified in the input script, see [Input options](#).

---

### 4.8.7 Velocity interpolation

Computing the particle velocity is done by first computing the particle mobility and then computing the particle velocity. For interpolating the mobility to the particle position one will call

```
void
ItoSolver::interpolateMobilities();
```

which will compute  $\mu(\mathbf{X})$  using the user-specified deposition/interpolation method for computing the mobility. The solver can switch between two ways of computing the mobility. The first is to compute  $\mu(\mathbf{X})$  directly. The other is to compute the mobility as

$$\mu = \frac{(\mu |v|)(\mathbf{X})}{|v(\mathbf{X})|}.$$

When computing the particle velocity as  $\mathbf{V} = \mu(\mathbf{X}) v(\mathbf{X})$ , the latter method ensures that  $\mathbf{V} = (\mu v)(\mathbf{X})$ .

---

**Note:** The user selects between the two mobility interpolation methods in the input script. See [Input options](#).

---

After the mobility has been appropriately set, the velocity can be interpolated from

```
void
ItoSolver::interpolateVelocities();
```

The above will compute  $v(\mathbf{X})$  and set the velocity as  $\mathbf{V} = \mu(\mathbf{X}) v(\mathbf{X})$ .

---

**Important:** The ItoSolver interpolation method is specified in the input script, see [Input options](#).

---

### 4.8.8 Particle intersections

It will happen that particles occasionally hit the embedded boundary or leave through the domain sides. In this case one might want to keep the particles in separate data holders rather than discard them. ItoSolver supplies the following routine for transferring the particles to the containers that hold the EB and domain particles:

```
void
ItoSolver::intersectParticles(const EbIntersection a_ebIntersection, const bool a_deleteParticles);
```

Here, EbIntersection is a just an enum for putting logic into how the intersection is computed. Valid options are `EbIntersection::Bisection` and `EbIntersection::Raycast`. These algorithms are discussed in [Wall interaction](#). The flag `a_deleteParticles` specifies if the original particles should be deleted when populating the other particle containers.

After calling `intersectParticles`, the particles that crossed the EB or domain walls are available through the `getParticles` routine, see [ItoSolver](#).

## 4.8.9 Computing time steps

The drift time step routines are implemented such that one restricts the time step such that the fastest particle does not move more than a specified number of grid cells. This routine is implemented as

```
Real
ItoSolver::computeAdvectionDt() const;
```

which returns a CFL-like condition  $\Delta x / \max(v_x, v_y, v_z)$  on the the various AMR levels and patches.

The signatures for the diffusion time step are similar to the ones for drift:

```
Real
ItoSolver::computeDiffusiveDt() const;
```

and this returns another CFL-like condition  $\Delta x^2 / (2dD)$  for all the particles, where  $d$  is the spatial dimension. Note that there is no fundamental limitation to how far the particles can move, unless the user explicitly makes this restriction in the input options, see [Input options](#).

A combination of the advection and diffusion time step routines also exists as

```
Real
ItoSolver::computeDt() const;
```

This routine computes the time step

$$\Delta t = \frac{1}{\frac{\Delta x}{\max(v_x, v_y, v_z)} + \frac{\Delta x^2}{2dD}},$$

## 4.8.10 Superparticles

ItoSolver currently handles superparticles through a kD-tree, see [Superparticles](#). The function for splitting and merging the particles is

```
void
ItoSolver::makeSuperparticles(const WhichContainer a_container, const int a_particlesPerCell);
```

Calling this function will merge/split the particles such that their computational weights differ by at most one. ItoSolver uses the kD-node implementation from [Superparticles](#) and partitioners for splitting the particles into two subsets with equal weights.

## 4.8.11 I/O

### Plot files

ItoSolver can output the following variables to plot files:

- $\phi$ , i.e. the deposited particle weights (`ItoSolver.plt_vars = phi`)
- $v$ , the advection field (`ItoSolver.plt_vars = vel`).
- $D$ , the diffusion coefficient (`ItoSolver.plt_vars = dco`).

It can also plot the corresponding particle data holders:

- Ito particles (`ItoSolver.plt_vars = part`).
- EB particles (`ItoSolver.plt_vars = eb_part`).
- Domain particles (`ItoSolver.plt_vars = domain_part`).

- Source particles (`ItoSolver.plt_vars = source_part`).

## Checkpoint files

When writing checkpoint files, `ItoSolver` can either

- Add the particles to the HDF5 file,
- Checkpoint the corresponding fluid data.

The user specifies this through the input script variable `ItoSolver.checkpointing`, see [Input options](#). If checkpointing fluid data then a subsequent restart will generate a new set of particles.

**Warning:** If writing particle checkpoint files, simulation restarts must also *read* as if the checkpoint file contains particles.

### 4.8.12 Input options

#### I/O

Plot variables are specified using `ItoSolver.plt_vars`, see [Plot files](#)). If adding the various particle container data holders to the plot file, the deposition method for those is specified using `ItoSolver.plot_deposition`.

If using fluid checkpointing for simulation restarts, the flag `ItoSolver.ppc_restart` determines the maximum number of particles that will initialized in each grid cell during a restart.

#### Particle-mesh

To specify the mobility interpolation, use `ItoSolver.mobility_interp`. Valid options are `direct` and `velocity`, see [Velocity interpolation](#).

Deposition and coarse-fine deposition (see [Particle-mesh](#)) is controlled using the flags

- `ItoSolver.deposition` for the base deposition scheme. Valid options are `ngp`, `cic`, and `tsc`.
- `ItoSolver.deposition_cf` for the coarse-fine deposition strategy. Valid options are `interp`, `halo`, or `halo_ngp`.

To modify the deposition scheme in cut-cells, one can enforce NGP interpolation and deposition through

- `ItoSolver.irr_ngp_deposition` for enforcing NGP deposition. Valid options are `true` or `false`.
- `ItoSolver.irr_ngp_interp` for enforcing NGP interpolation. Valid options are `true` or `false`.

#### Checkpoint-restart

Available input options for the `ItoSolver` are listed below:

```
# =====
# ItoSolver class options
#
ItoSolver.verbosity      = -1          # Class verbosity
ItoSolver.plt_vars        = phi vel dco # 'phi', 'vel', 'dco', 'part', 'eb_part', 'dom_part', 'src_part', 'energy_density', 'energy'
ItoSolver.bisect_step     = 1.E-4       # Bisection step length for intersection tests
ItoSolver.normal_max      = 5.0         # Maximum value (absolute) that can be drawn from the exponential distribution.
ItoSolver.redistribute    = false        # Turn on/off redistribution.
ItoSolver.blend_conservation = false    # Turn on/off blending with nonconservative divergence
```

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ItoSolver.checkpointing	= particles	# 'particles' or 'numbers'
ItoSolver.ppc_restart	= 32	# Maximum number of computational particles to generate for restarts.
ItoSolver.irr_ngp_deposition	= false	# Force irregular deposition in cut cells or not
ItoSolver.irr_ngp_interp	= true	# Force irregular interpolation in cut cells or not
ItoSolver.mobility_interp	= direct	# How to interpolate mobility, 'direct' or 'velocity', i.e. either $\mu_p = \mu(X_p)$ ↵ or $\mu_p = (\mu * E)(X_p) / E(X_p)$
ItoSolver.plot_deposition	= cic	# Cloud-in-cell for plotting particles.
ItoSolver.deposition	= cic	# Deposition type.
ItoSolver.deposition_cf	= halo	# Coarse-fine deposition. interp, halo, or halo_ngp

#### 4.8.13 Example application

An example application of usage of the ItoSolver is found in

- \$DISCHARGE\_HOME/Physics/BrownianWalker, see *Brownian walker*.

## MULTI-PHYSICS APPLICATIONS

### 5.1 CDR plasma model

The CDR plasma model resides in `/Physics/CdrPlasma` and describes plasmas in the drift-diffusion approximation. This physics model also includes the following subfolders:

- `/Physics/CdrPlasma/PlasmaModel` which contains various implementation of some plasma models that we have used.
- `/Physics/CdrPlasma/TimeSteppers` contains various algorithms for advancing the equations of motion.
- `/Physics/CdrPlasma/CellTaggers` contains various algorithms for flagging cells for refinement and coarsening.
- `/Physics/CdrPlasma/python` contains Python source files for quickly setting up new applications.

In the CDR plasma model we are solving

$$\begin{aligned}\nabla \cdot (\epsilon_r \nabla \Phi) &= -\frac{\rho}{\epsilon_0}, \\ \frac{\partial \sigma}{\partial t} &= F_\sigma, \\ \frac{\partial n}{\partial t} + \nabla \cdot (\mathbf{v} n - D \nabla n) &= S,\end{aligned}\tag{5.1.1}$$

The above equations must be supported by additional boundary conditions on electrodes and insulating surfaces.

Radiative transport can be done either in the diffusive approximation or by means of Monte Carlo methods. Diffusive RTE methods involve solving

$$\partial_t \Psi + \kappa \Psi - \nabla \cdot \left( \frac{1}{3\kappa} \nabla \Psi \right) = \frac{\eta}{c},$$

where  $\Psi$  is the isotropic photon density,  $\kappa$  is an absorption length and  $\eta$  is an isotropic source term. I.e.,  $\eta$  is the number of photons produced per unit time and volume. The time dependent term can be turned off and the equation can be solved stationary.

The module also supports discrete photons where photon transport and absorption is done by sampling discrete photons. In general, discrete photon methods incorporate better physics (like shadows) They can easily be adapted to e.g. scattering media. They are, on the other hand, inherently stochastic which implies that some extra caution must be exercised when integrating the equations of motion.

The coupling that is (currently) available in chombo-discharge is

$$\begin{aligned}\epsilon_r &= \epsilon_r(\mathbf{x}), \\ \mathbf{v} &= \mathbf{v}(t, \mathbf{x}, \mathbf{E}, n), \\ D &= \mathbf{v}(t, \mathbf{x}, \mathbf{E}, n), \\ S &= S(t, \mathbf{x}, \mathbf{E}, \nabla \mathbf{E}, n, \nabla n, \Psi), \\ \eta &= \eta(t, \mathbf{x}, \mathbf{E}, n), \\ F &= F(t, \mathbf{x}, \mathbf{E}, n),\end{aligned}\tag{5.1.2}$$

where  $F$  is the boundary flux on insulators or electrodes (which must be separately implemented).

chombo-discharge works by embedding the equations above into an abstract C++ framework (see [CdrPlasma-Physics](#)) that the user must implement or reuse existing pieces of, and then compile into an executable.

### 5.1.1 Simulation quick start

New problems that use the CdrPlasma physics model are best set up by using the Python tools provided with the module. Navigate to `$DISCHARGE_HOME/Physics/CdrPlasma`` and set up the problem with. To see the list of available options type

```
cd $DISCHARGE_HOME/Physics/CdrPlasma
./setup.py --help
```

The following options are helpful for setting up the problem:

- `base_dir` The base directory where the application will be placed. Defaults to `$DISCHARGE_HOME/MyApplications`.
- `app_name` The application name. The application will be put in `base_dir/app_name`.
- `geometry` The geometry to be used. The geometry must be one of the ones provided in `$DISCHARGE_HOME/Geometries` (users can also provide their own models).
- `physics` The plasma physics model. This must be one of the folders/class in `$DISCHARGE_HOME/Physics/CdrPlasma/PlasmaModel` (users can also provide their own models). Defaults to `CdrPlasmaJSON` (see [JSON interface](#)).
- `time stepper` Time integrator. This must derive from `CdrPlasmaStepper` and must be one of the time steppers in `$DISCHARGE_HOME/Physics/CdrPlasma/TimeSteppers`. The default integrator is `CdrPlasmaGodunovStepper`.
- `cell_tagger` Cell tagger This must derive from `CdrPlasmaTagger` and must be one of the cell taggers in `$DISCHARGE_HOME/Physics/CdrPlasma/CellTaggers`.

For example, to set up a geometry-less that does not use AMR, do

```
cd $DISCHARGE_HOME
./setup.py -app_name=MyApplication
```

## 5.1.2 Solvers

This module uses the following solvers:

1. Advection-diffusion-reaction solver, [CdrSolver](#).
2. Electrostatics solvers, [FieldSolver](#).
3. Radiative transfer solver (either Monte-Carlo or continuum approximation), [RtSolver](#).
4. Surface charge solver, see [Surface ODE solver](#).

## 5.1.3 CdrPlasmaPhysics

[CdrPlasmaPhysics](#) is an abstract class which represents the plasma physics for the CDR plasma module, i.e. it provides the coupling functions in Eq. 5.1.2. The source code for the class resides in `/Physics/CdrPlasma/CD_CdrPlasmaPhysics.H`. Note that the entire class is an interface, whose implementations are used by the time integrators that advance the equations.

There are no default input parameters for [CdrPlasmaPhysics](#), as users must generally implement their own kinetics. The class exists solely for providing the integrators with the necessary fundamentals for filling solvers with the correct quantities at the same time, for example filling source terms and drift velocities.

A successful implementation of [CdrPlasmaPhysics](#) has the following:

1. Instantiated a list of [CdrSpecies](#). These become [Convection-Diffusion-Reaction](#) solvers and contain initial conditions and basic transport settings for the convection-diffusion-reaction solvers.
2. Instantiated a list [RtSpecies](#). These become [Radiative transfer](#) solvers and contain metadata for the radiative transport solvers.
3. Implemented the core functionality that couple the solvers together.

chombo-discharge automatically allocates the specified number of convection-diffusion-reaction and radiative transport solvers from the list of species the is intantiated. For information on how to interface into the CDR solvers, see [CdrSpecies](#). Likewise, see [RtSpecies](#) for how to interface into the RTE solvers.

Implementation of the core functionality is comparatively straightforward, but can lead to boilerplate code. For this reason we also provide an implementation layer [JSON interface](#) that provides a plug-and-play interface for specifying the plasma physics.

## API

The API for [CdrPlasmaPhysics](#) is as follows:

```

virtual Real computeAlpha(const RealVect a_E) const = 0;

virtual void advanceReactionNetwork(Vector<Real>& a_cdrSources,
                                    Vector<Real>& a_rteSources,
                                    const Vector<Real> a_cdrDensities,
                                    const Vector<RealVect> a_cdrGradients,
                                    const Vector<Real> a_rteDensities,
                                    const RealVect a_E,
                                    const RealVect a_pos,
                                    const Real a_dx,
                                    const Real a_dt,
                                    const Real a_time,
                                    const Real a_kappa) const = 0;

virtual Vector<RealVect> computeCdrDriftVelocities(const Real a_time,
                                                       const RealVect a_pos,
                                                       const RealVect a_E,
                                                       const Vector<Real> a_cdrDensities) const = 0;

```

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```

virtual Vector<Real> computeCdrDiffusionCoefficients(const Real a_time,
                                                       const RealVect a_pos,
                                                       const RealVect a_E,
                                                       const Vector<Real> a_cdrDensities) const = 0;

virtual Vector<Real> computeCdrElectrodeFluxes(const Real a_time,
                                                 const RealVect a_pos,
                                                 const RealVect a_normal,
                                                 const RealVect a_E,
                                                 const Vector<Real> a_cdrDensities,
                                                 const Vector<Real> a_cdrVelocities,
                                                 const Vector<Real> a_cdrGradients,
                                                 const Vector<Real> a_rteFluxes,
                                                 const Vector<Real> a_extrapCdrFluxes) const = 0;

virtual Vector<Real> computeCdrDielectricFluxes(const Real a_time,
                                                 const RealVect a_pos,
                                                 const RealVect a_normal,
                                                 const RealVect a_E,
                                                 const Vector<Real> a_cdrDensities,
                                                 const Vector<Real> a_cdrVelocities,
                                                 const Vector<Real> a_cdrGradients,
                                                 const Vector<Real> a_rteFluxes,
                                                 const Vector<Real> a_extrapCdrFluxes) const = 0;

virtual Vector<Real> computeCdrDomainFluxes(const Real a_time,
                                              const RealVect a_pos,
                                              const int a_dir,
                                              const Side::LoHiSide a_side,
                                              const RealVect a_E,
                                              const Vector<Real> a_cdrDensities,
                                              const Vector<Real> a_cdrVelocities,
                                              const Vector<Real> a_cdrGradients,
                                              const Vector<Real> a_rteFluxes,
                                              const Vector<Real> a_extrapCdrFluxes) const = 0;

virtual Real initialSigma(const Real a_time, const RealVect a_pos) const = 0;

```

The above code blocks do the following:

- `computeAlpha` computes the Townsend ionization coefficient. This is used by the cell tagger.
- `advanceReactionNetwork` provides the coupling  $S = S(t, \mathbf{x}, \mathbf{E}, \nabla \mathbf{E}, n, \nabla n, \Psi)$ .
- `computeCdrDriftVelocities` provides the coupling  $\mathbf{v} = \mathbf{v}(t, \mathbf{x}, \mathbf{E}, n)$ .
- `computeCdrDiffusionCoefficients` provides the coupling  $D = \mathbf{v}(t, \mathbf{x}, \mathbf{E}, n)$ .
- `computeCdrElectrodeFluxes` provides the coupling  $F = F(t, \mathbf{x}, \mathbf{E}, n)$  on electrode EBs.
- `computeCdrDielectricFluxes` provides the coupling  $F = F(t, \mathbf{x}, \mathbf{E}, n)$  on dielectric EBs.
- `computeCdrDomainFluxes` provides the coupling  $F = F(t, \mathbf{x}, \mathbf{E}, n)$  on domain sides.

For a fully documented API, see the [doxygen API](#).

Below, we include a brief overview of how `CdrPlasmaPhysics` can be directly implemented. Note that direct implements like these tend to become boilerplate, we also include an interface which implements these functions with pre-defined rules, see [JSON interface](#).

## Initializing species

In the constructor, the user should define the advected/diffused species and the radiative transfer species. These are stored in vectors `Vector<RefCountedPtr<CdrSpecies>> m_CdrSpecies` and `Vector<RefCountedPtr<RtSpecies>> m_RtSpecies`. Each species in these vectors become a convection-diffusion-reaction solver or a radiative transfer solver. See `CdrSpecies` and `RtSpecies` for details on how to implement these.

## Defining drift velocities

To set the drift velocities, implement `computeCdrDriftVelocities` – this will set the drift velocity  $\mathbf{v}$  in the CDR equations:

```
Vector<RealVect> computeCdrDriftVelocities(const Real      a_time,
                                              const RealVect   a_pos,
                                              const RealVect   a_E,
                                              const Vector<Real> a_cdrDensities) const {
    return Vector<RealVect>(m_numCdrSpecies, a_E);
}
```

This implementation is set the advection velocity equal to  $\mathbf{E}$ . For a full plasma simulation, there will also be mobilities involved, which the user is responsible for obtaining.

## Defining diffusion coefficients

To set the diffusion coefficients, implement `computeCdrDiffusionCoefficients` – this will set the diffusion coefficient  $D$  in the CDR equations:

```
Vector<Real> computeCdrDiffusionCoefficients(const Real      a_time,
                                               const RealVect   a_pos,
                                               const RealVect   a_E,
                                               const Vector<Real> a_cdrDensities) const {
    return Vector<Real>(m_numCdrSpecies, 1.0);
}
```

This sets  $D = 1$  for all species involved.

## Defining chemistry terms

To set the source terms  $S$ , implement `advanceReactionNetwork`. This routine should set the reaction terms for both the CDR equations *and* the radiative transfer equations.

---

**Note:** For the radiative transfer equations we set the isotropic source term  $\eta$  which is the number of ionizing photons produced per unit volume and time.

---

```
virtual void advanceReactionNetwork(Vector<Real>&          a_cdrSources,
                                     Vector<Real>&          a_rteSources,
                                     const Vector<Real> & a_cdrDensities,
                                     const Vector<RealVect> & a_cdrGradients,
                                     const Vector<Real> & a_rteDensities,
                                     const RealVect & a_E,
                                     const RealVect & a_pos,
                                     const Real & a_dx,
                                     const Real & a_dt,
                                     const Real & a_time,
                                     const Real & a_kappa) const {
    a_cdrSources = Vector<Real>(m_numCdrSpecies, 1.0);
    a_rteSources = Vector<Real>(m_numRteSpecies, 1.0);
}
```

The above code will set  $S = \eta = 1$  for all species.

We point out that in the plasma module the source terms are *always* used in the form

$$n^{k+1} = n^k + \Delta t S,$$

where  $S$  is the source term obtained from `advanceReactionNetwork`. This implies that it *is* possible to define fully implicit integrators directly in `advanceReactionNetwork`. For example, if the reactive problem consisted only of  $\partial_t n = -\frac{n}{\tau}$ , one could form a reactive integrator with the implicit Euler rule by first computing  $n^{k+1} = \frac{n^k}{1+\Delta t/\tau}$  and then linearizing  $S = \frac{n^{k+1}-n^k}{\Delta t}$ .

### Fluxes at electrode boundaries

To set the fluxes  $F$  on electrode EBs, implement `computeCdrElectrodeFluxes`. Note that the fluxes  $F$  are those occurring in a finite-volume context; i.e. the total injected or extracted mass.

```
Vector<Real> computeCdrElectrodeFluxes(const Real           a_time,
                                         const RealVect    a_pos,
                                         const RealVect    a_normal,
                                         const RealVect    a_E,
                                         const Vector<Real> a_cdrDensities,
                                         const Vector<Real> a_cdrVelocities,
                                         const Vector<Real> a_cdrGradients,
                                         const Vector<Real> a rteFluxes,
                                         const Vector<Real> a_extrapCdrFluxes) const {
    return Vector<Real>(m_numCdrSpecies, 0.0);
}
```

The input variable `a_extrapCdrFluxes` are cell-centered fluxes extrapolated to the EBs.

### Fluxes at dielectric boundaries

To set the fluxes  $F$  on dielectric EBs, implement `computeCdrDielectricFluxes`. Note that the fluxes  $F$  are those occurring in a finite-volume context; i.e. the total injected or extracted mass.

```
Vector<Real> computeCdrDielectricFluxes(const Real           a_time,
                                         const RealVect    a_pos,
                                         const RealVect    a_normal,
                                         const RealVect    a_E,
                                         const Vector<Real> a_cdrDensities,
                                         const Vector<Real> a_cdrVelocities,
                                         const Vector<Real> a_cdrGradients,
                                         const Vector<Real> a rteFluxes,
                                         const Vector<Real> a_extrapCdrFluxes) const {
    return Vector<Real>(m_numCdrSpecies, 0.0);
}
```

The input variable `a_extrapCdrFluxes` are cell-centered fluxes extrapolated to the EBs.

### Fluxes at domain boundaries

To set the fluxes  $F$  on dielectric EBs, implement `computeCdrDomainFluxes`. Note that the fluxes  $F$  are those occurring in a finite-volume context; i.e. the total injected or extracted mass.

```
Vector<Real> computeCdrDomainFluxes(const Real           a_time,
                                         const RealVect    a_pos,
                                         const int          a_dir,
                                         const Side::LoHiSide a_side,
                                         const RealVect    a_E,
                                         const Vector<Real> a_cdrDensities,
                                         const Vector<Real> a_cdrVelocities,
                                         const Vector<Real> a_cdrGradients,
```

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```

    const Vector<Real> a_rteFluxes,
    const Vector<Real> a_extrapCdrFluxes) const {
  return Vector<Real>(m_numCdrSpecies, 0.0);
}

```

The input variable `a_extrapCdrFluxes` are cell-centered fluxes extrapolated to the domain sides.

## Setting initial surface charge

To set the initial surface charge on dielectric boundaries, implement

```

Real initialSigma(const Real a_time, const RealVect a_pos) const{
  return 0.0;
}

```

### 5.1.4 Time discretizations

Here, we discuss two discretizations of Eq. 5.1.1. Firstly, note that there are two layers to the time integrators:

1. A pure class `CdrPlasmaStepper` which inherits from `TimeSteppers` but does not implement an `advance` method. This class simply provides the base functionality for more easily developing time integrators. `CdrPlasmaStepper` contains methods that are necessary for coupling the solvers, e.g. calling the `CdrPlasmaPhysics` methods at the correct time.
2. Implementations of `CdrPlasmaPhysics`, which implement the `advance` method and can thus be used for advancing models.

The supported time integrators are located in `$DISCHARGE_HOME/CdrPlasma/TimeSteppers`. There are two integrators that are commonly used.

- A Godunov operator splitting with either explicit or implicit diffusion. This integrator also supports semi-implicit formulations.
- A spectral deferred correction (SDC) integrator with implicit diffusion. This integrator is an implicit-explicit.

Briefly put, the Godunov operator is our most stable integrator, while the SDC integrator is our most accurate integrator.

## Godunov operator splitting

The `CdrPlasmaGodunovStepper` implements `CdrPlasmaStepper` and defines an operator splitting method between charge transport and plasma chemistry. It has a formal order of convergence of one. The source code is located in `$DISCHARGE_HOME/Physics/CdrPlasma/TimeSteppers/CdrPlasmaGodunovStepper`.

**Warning:** Splitting the terms yields *splitting errors* which can dominate for large time steps. Typically, the operator splitting discretization is not suitable for large time steps.

The basic advancement routine for `CdrPlasmaGodunovStepper` is as follows:

1. Advance the charge transport  $\phi^k \rightarrow \phi^{k+1}$  with the source terms set to zero.
2. Compute the electric field.
3. Advance the plasma chemistry over the same time step using the field computed above I.e., advance  $\partial_t \phi = S$  over a time step  $\Delta t$ .
4. Advance the radiative transport part. This can also involve discrete photons.

The transport/field steps can be done in various ways: The following transport algorithms are available:

- **Euler**, where everything is advanced with the Euler rule.
- **Semi-implicit**, where the Euler field/transport step is performed with an implicit coupling to the electric field.

In addition, diffusion can be treated

- **Explicitly**, where all diffusion advances are performed with an *explicit* rule.
- **Implicitly**, where all diffusion advances are performed with an *implicit* rule.
- **Automatically**, where diffusion advances are performed with an implicit rule only if time steps dictate it, and explicitly otherwise.

---

**Note:** When setting up a new problem with the Godunov time integrator, the default setting is to use automatic diffusion and a semi-implicit coupling. These settings tend to work for most problems.

---

## Specifying transport algorithm

To specify the transport algorithm, modify the flag `CdrPlasmaGodunovStepper.transport`, and set it to `semi_implicit` or `euler`. Everything else is an error.

Note that for the Godunov integrator, it is possible to center the advective discretization at the half time step. That is, the advancement algorithm is

$$n^{k+1} = n^k - \nabla \cdot (n^{k+1/2} \mathbf{v}) + \nabla \cdot (D \nabla \phi^k),$$

where  $n^{k+1/2}$  is obtained by also including transverse slopes (i.e., extrapolation in time). See Trebotich and Graves [10] for details. Note that the formal order of accuracy is still one, but the accuracy of the advective discretization is increased substantially.

## Specifying diffusion

To specify how diffusion is treated, modify the flag `CdrPlasmaGodunovStepper.diffusion`, and set it to `auto`, `explicit`, or `implicit`. In addition, the flag `CdrPlasmaGodunovStepper.diffusion_thresh` must be set to a number.

When diffusion is set to `auto`, the integrator switches to implicit diffusion when

$$\frac{\Delta t_A}{\Delta t_{AD}} > \epsilon,$$

where  $\Delta t_A$  is the advection-only limited time step and  $\Delta t_{AD}$  is the advection-diffusion limited time step.

---

**Note:** When there are multiple species being advected and diffused, the integrator will perform extra checks in order to maximize the time steps for the other species.

---

## Time step limitations

The basic time step limitations for the Godunov integrator are:

- Manually set maximum and minimum time steps
- Courant-Friedrichs-Lowy conditions, either on advection, diffusion, or both.
- The dielectric relaxation time.

The user is responsible for setting these when running the simulation. Note when the semi-implicit scheme is used, it is not necessary to restrict the time step by the dielectric relaxation time.

## Spectral deferred corrections

The `CdrPlasmaImExSdcStepper` uses implicit-explicit (ImEx) spectral deferred corrections (SDCs) to advance the equations. This integrator implements the advance method for `CdrPlasmStepper`, and is a high-order method with implicit diffusion.

### SDC basics

First, we provide a quick introduction to the SDC procedure. Given an ordinary differential equation (ODE) as

$$\frac{\partial u}{\partial t} = F(u, t), \quad u(t_0) = u_0,$$

the exact solution is

$$u(t) = u_0 + \int_{t_0}^t F(u, \tau) d\tau.$$

Denote an approximation to this solution by  $\tilde{u}(t)$  and the correction by  $\delta(t) = u(t) - \tilde{u}(t)$ . The measure of error in  $\tilde{u}(t)$  is then

$$R(\tilde{u}, t) = u_0 + \int_{t_0}^t F(\tilde{u}, \tau) d\tau - \tilde{u}(t).$$

Equivalently, since  $u = \tilde{u} + \delta$ , we can write

$$\tilde{u} + \delta = u_0 + \int_{t_0}^t F(\tilde{u} + \delta, \tau) d\tau.$$

This yields

$$\delta = \int_{t_0}^t [F(\tilde{u} + \delta, \tau) - F(\tilde{u}, \tau)] d\tau + R(\tilde{u}, t).$$

This is called the correction equation. The goal of SDC is to iteratively solve this equation in order to provide a high-order discretization.

The ImEx SDC method in chombo-discharge uses implicit diffusion in the SDC scheme. Coupling to the electric field is always explicit. The user is responsible for specifying the quadrature nodes, as well as setting the number of sub-intervals in the SDC integration and the number of corrections. In general, each correction raises the discretization order by one.

## Time step limitations

The ImEx SDC integrator is limited by

- The dielectric relaxation time.
- An advective CFL conditions.

In addition to this, the user can specify maximum/minimum allowed time steps.

### 5.1.5 JSON interface

Since implementations of *CdrPlasmaPhysics* are usually boilerplate, we provide a class *CdrPlasmaJSON* which can initialize and parse various types of initial conditions and reactions from a JSON input file. This class is defined in `$DISCHARGE_HOME/Physics/PlasmaModels/CdrPlasmaJSON`.

*CdrPlasmaJSON* is a full implementation of *CdrPlasmaPhysics* which supports the definition of various species (neutral, plasma species, and photons) and methods of coupling them. We expect that *CdrPlasmaJSON* provides the simplest method of setting up a new plasma model. It is also comparatively straightforward to extend the class with further required functionality.

In the JSON interface, the radiative transfer solvers always solve for the number of photons that lead to photoionization events. This means that the interpretation of  $\Psi$  is the number of photoionization events during the previous time step. This is true for both continuum and discrete radiative transfer models.

## Usage

To use this plasma model, use `-physics CdrPlasmaJSON` when setting up a new plasma problem (see [Simulation quick start](#)). When *CdrPlasmaJSON* is instantiated, the constructor will parse species, reactions, initial conditions, and boundary conditions from a JSON file that the user provides. In addition, users can parse transport data or reaction rates from tabulated ASCII files that they provide.

To specify the input plasma kinetics file, include

## Specifying input file

*CdrPlasmaJSON* will read a JSON file specified by the input variable `CdrPlasmaJSON.chemistry_file`.

## Discrete photons

There are two approaches when using discrete photons, and both rely on the user setting up the application with the Monte Carlo photon solver (rather than continuum solvers). For an introduction to the particle radiative transfer solver, see [Monte Carlo methods](#).

The user must use one of the following:

- Set the following class options:

```
CdrPlasmaJSON.discrete_photons = true  
McPhoto.photon_generation = deterministic  
McPhoto.source_type      = number
```

When specifying `CdrPlasmaJSON.discrete_photons = true`, *CdrPlasmaJSON* will do a Poisson sampling of the number of photons that are generated in each cell and put this in the radiative transfer solvers' source terms. This means that the radiative transfer solver source terms *contain*

*the physical number of photons generated in one time step.* To turn off sampling inside the radiative transfer solver, we specify `McPhoto.photon_generation = stochastic` and set `McPhoto.source_type = number` to let the solver know that the source contains the number of physical photons.

- Alternatively, set the following class options:

```
CdrPlasmaJSON.discrete_photons = false
McPhoto.photon_generation = stochastic
McPhoto.source_type      = volume_rate
```

In this case the `CdrPlasmaJSON` class will fill the solver source terms with the volumetric rate, i.e. the number of photons produced per unit volume and time. When `McPhoto` generates the photons it will compute the number of photons generated in a cell through Poisson sampling  $n = P(S_\gamma \Delta V \Delta t)$  where  $P$  indicates a Poisson sampling operator.

Fundamentally, the two approaches differ only in where the Poisson sampling is performed. With the first approach, plotting the radiative transfer solver source terms will show the number of physical photons generated. In the second approach, the source terms will show the volume photo-generation rate.

## Gas law and neutral background

### General functionality

To include the gas law and neutral species, include a JSON object `gas` with the field `law` specified. Currently, `law` can be either `ideal`, `troposphere`, or `table`.

The purpose of the gas law is to set the temperature, pressure, and neutral density of the background gas. In addition, we specify the neutral species that are used through the simulation. These species are *not* stored on the mesh; we only store function pointers to their temperature, density, and pressure.

It is also possible to include a field `plot` which will then include the temperature, pressure, and density in plot files.

### Ideal gas

To specify an ideal gas law, specify ideal gas law as follows:

```
{"gas":
{
  "law": "ideal",
  "temperature": 300,
  "pressure": 1
}}
```

In this case the gas pressure and temperatures will be as indicated, and the gas number density will be computed as

$$\rho = \frac{p'_0 N_A}{RT_0},$$

where  $p'$  is the pressure converted to Pascals.

Note that the input temperature should be specified in Kelvin, and the input pressure in atmospheres.

## Troposphere

It is also possible to specify the pressure, temperature, and density to be functions of tropospheric altitude. In this case one must specify the extra fields

- **molar mass** For specifying the molar mass (in  $\text{g} \cdot \text{mol}^{-1}$ ) of the gas.
- **gravity** Gravitational acceleration  $g$ .
- **lapse rate** Temperature lapse rate  $L$  in units of  $\text{K}/\text{m}$ .

In this case the gas temperature pressure, and number density are computed as

$$T(h) = T_0 - Lh$$

$$p(h) = p_0 \left( \left(1 - \frac{Lh}{T_0}\right)^{\frac{gM}{RL}} \right)$$

$$\rho(h) = \frac{p'(h)N_A}{RT(h)}$$

For example, specification of tropospheric conditions can be included by

```
{"gas":  
  {  
    "law": "troposphere",  
    "temperature": 300,  
    "pressure": 1,  
    "molar_mass": 28.97,  
    "gravity": 9.81,  
    "lapse_rate": 0.0065,  
    "plot": true  
  }  
}
```

## Tabulated

To specify temperature, density, and pressure as function of altitude, set **law** to **table** and include the following fields:

- **file** For specifying which file we read the data from.
- **height** For specifying the column where the height is stored (in meters).
- **temperature** For specifying the column where the temperature (in Kelvin) is stored.
- **pressure** For specifying the column where the pressure (in Pascals) is stored.
- **density** For specifying the column where the density (in  $\text{kg} \cdot \text{m}^{-3}$ ) is stored.
- **molar mass** For specifying the molar mass (in  $\text{g} \cdot \text{mol}^{-1}$ ) of the gas.
- **min height** For setting the minimum altitude in the chombo-discharge internal table.
- **max height** For setting the maximum altitude in the chombo-discharge internal table.
- **res height** For setting the height resolution in the chombo-discharge internal table.

For example, assume that our file `MyAtmosphere.dat` contains the following data:

# z [m]	rho [ $\text{kg}/\text{m}^3$ ]	T [K]	p [Pa]
0.0000000E+00	1.2900000E+00	2.7210000E+02	1.0074046E+05
1.0000000E+03	1.1500000E+00	2.6890000E+02	8.8751220E+04
2.0000000E+03	1.0320000E+00	2.6360000E+02	7.8074784E+04
3.0000000E+03	9.2860000E-01	2.5690000E+02	6.8466555E+04
4.0000000E+03	8.3540000E-01	2.4960000E+02	5.9844569E+04

If we want to truncate this data to altitude  $z$  in [1000 m, 3000 m] we specify:

```
{"gas":  
  {  
    "law": "table",  
    "file": "ENMSIS_Atmosphere.dat",  
    "molar mass": 28.97,  
    "height": 0,  
    "temperature": 2,  
    "pressure": 3,  
    "density": 1,  
    "min height": 1000,  
    "max height": 3000,  
    "res height": 10  
  }  
}
```

## Neutral species background

Neutral species are included by an array `neutral species` in the `gas` object. Each neutral species must have the fields

- `name` Species name
- `molar fraction` Molar fraction of the species.

If the molar fractions do not add up to one, they will be normalized.

**Warning:** Neutral species are *not* tracked on the mesh. They are simply stored as functions that allow us to obtain the (spatially varying) density, temperature, and pressure for each neutral species. If a neutral species needs to be tracked on the mesh (through e.g. a convection-diffusion-reaction solver) it must be defined as a plasma species. See [Plasma species](#).

For example, a standard nitrogen-oxygen atmosphere will look like:

```
{"gas":  
  {  
    "law": "ideal",  
    "temperature": 300,  
    "pressure": 1,  
    "plot": true,  
    "neutral species":  
    [  
      {  
        "name": "O2",  
        "molar_fraction": 0.2  
      },  
      {  
        "name": "N2",  
        "molar_fraction": 0.8  
      }  
    ]  
  }  
}
```

### Plasma species

The list of plasma species is included by an array `plasma species`. Each entry *must* have the entries

- `name` (string) For identifying the species name.
- `Z` (integer) Species charge number.
- `mobile` (true/false) Mobile species or not.
- `diffusive` (true/false) Diffusive species or not.

Optionally, the field `initial data`, can be included for providing initial data to the species. Details are discussed further below.

For example, a minimum version would look like

```
{"plasma species": [
  {
    "name": "N2+", "Z": 1, "mobile": false, "diffusive": false},
    {"name": "O2+", "Z": 1, "mobile": false, "diffusive": false},
    {"name": "O2-", "Z": -1, "mobile": false, "diffusive": false}
  ]
}
```

### Initial data

Initial data can be provided with

- Function based densities.
- Computational particles (deposited using a nearest-grid-point scheme).

### Density functions

To provide initial data one include `initial data` for each species. Currently, the following fields are supported:

- `uniform` For specifying a uniform background density. Simply the field `uniform` and a density (in units of  $m^{-3}$ )
- `gauss2` for specifying Gaussian seeds  $n = n_0 \exp\left(-\frac{(x-x_0)^2}{2R^2}\right)$ . `gauss2` is an array where each array entry must contain
  - `radius`, for specifying the radius  $R$ :
  - `amplitude`, for specifying the amplitude  $n_0$ .
  - `position`, for specifying the seed position  $x$ .

The position must be a 2D/3D array.

- `gauss2` for specifying Gaussian seeds  $n = n_0 \exp\left(-\frac{(x-x_0)^4}{2R^4}\right)$ . `gauss4` is an array where each array entry must contain
  - `radius`, for specifying the radius  $R$ :
  - `amplitude`, for specifying the amplitude  $n_0$ .
  - `position`, for specifying the seed position  $x$ .

The position must be a 2D/3D array.

- **height profile** For specifying a height profile along  $y$  in 2D, and  $z$  in 3D. To include it, prepare an ASCII files with at least two columns. The height (in meters) must be specified in one column and the density (in units of  $m^{-3}$ ) in another. Internally, this data is stored in a lookup table (see *Lookup tables*). Required fields are
  - **file**, for specifying the file.
  - **height**, for specifying the column that stores the height.
  - **density**, for specifying the column that stores the density.
  - **min height**, for trimming data to a minimum height.
  - **max height**, for trimming data to a maximum height.
  - **res height**, for specifying the resolution height in the chombo-discharge lookup tables.

In addition, height and density columns can be scaled in the internal tables by including

- **scale height** for scaling the height data.
- **scale density** for scaling the density data.

---

**Note:** When multiple initial data fields are specified, chombo-discharge takes the superposition of all of them.

---

## Initial particles

Initial particles can be included with the `initial_particles` field. The current implementation supports

- **uniform** For drawing initial particles randomly distributed inside a box. The user must specify the two corners `lo_corner` and `hi_corner` that indicate the spatial extents of the box, and the `number` of computational particles to draw. The weight is specified by a field `weight`. For example:

```
{"plasma_species": [
  {
    "name": "e",
    "Z": -1,
    "mobile": true,
    "diffusive": true,
    "initial_particles": {
      "uniform": {
        "lo_corner": [0,0,0],
        "hi_corner": [1,1,1],
        "number": 100,
        "weight": 1.0
      }
    }
  }
]}
```

- **sphere** For drawing initial particles randomly distributed inside a sphere. Mandatory fields are
  - `center` for specifying the sphere center.
  - `radius` for specifying the sphere radius.
  - `number` for the number of computational particles.
  - `weight` for the initial particle weight.

```
{"plasma_species": [
  {
    "name": "e",
    "Z": -1,
```

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```

    "mobile": true,
    "diffusive": true,
    "initial particles": [
        {
            "sphere": [
                {
                    "center": [0,0,0],
                    "radius": 1.0,
                    "number": 100,
                    "weight": 1.0
                }
            ]
        }
    ]
}

```

- **copy** For using an already initialized particle distribution. The only mandatory fields is **copy**, e.g.

## Complex example

For example, a species with complex initial data that combines density functions with initial particles can look like:

```

{"plasma species":
[
{
    "name": "N2+",
    "Z": 1,
    "mobile": false,
    "diffusive": false,
    "initial data": [
        "uniform": 1E10,
        "gauss2" :
        [
            {
                "radius": 100E-6,
                "amplitude": 1E18,
                "position": [0,0,0]
            },
            {
                "radius": 200E-6,
                "amplitude": 2E18,
                "position": [1,0,0]
            }
        ],
        "gauss4" :
        [
            {
                "radius": 300E-6,
                "amplitude": 3E18,
                "position": [0,1,0]
            },
            {
                "radius": 400E-6,
                "amplitude": 4E18,
                "position": [0,0,1]
            }
        ],
        "height profile": {
            "file": "MyHeightProfile.dat",
            "height": 0,
            "density": 1,
            "min height": 0,
            "max height": 100000,
            "res height": 10,
            "scale height": 100,
            "scale density": 1E6
        }
    ],
    "initial particles": {
        "sphere": [
            {
                "center": [0,0,0],
                "radius": 1.0,
                "number": 100,
                "weight": 1.0
            }
        ]
    }
}

```

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```

        }
    ]
}
```

## Mobilities

If a species is specified as mobile, the mobility is set from a field `mobility`, and the field `lookup` is used to specify the method for computing it. Currently supported are:

- Constant mobility.
- Function-based mobility, i.e.  $\mu = \mu(E, N)$ .
- Tabulated mobility, i.e.  $\mu = \mu(E, N)$ .

The cases are discussed below.

### Constant mobility

Setting `lookup` to `constant` lets the user set a constant mobility. If setting a constant mobility, the field `value` is also required. For example:

```
{"plasma species":
[
  {"name": "e", "Z": -1, "mobile": true, "diffusive": false,
   "mobility": {
     "lookup": "constant",
     "value": 0.05,
   }
]
}
```

### Function-based mobility

Setting `lookup` to `function E/N` lets the user set the mobility as a function of the reduced electric field. When setting a function-based mobility, the field `function` is also required.

Supported functions are:

- ABC, in which case the mobility is computed as

$$\mu(E) = A \frac{E^B}{N^C}.$$

The fields A, B, and C must also be specified. For example:

```
{"plasma species":
[
  {"name": "e", "Z": -1, "mobile": true, "diffusive": false,
   "mobility": {
     "lookup": "function E/N",
     "function": "ABC",
     "A": 1,
     "B": 1,
     "C": 1
   }
]
}
```

### Tabulated mobility

Specifying lookup to table E/N lets the user set the mobility from a tabulated value of the reduced electric field. BOLSIG-like files can be parsed by specifying the header which contains the tabulated data, and the columns that identify the reduced electric field and mobilities. This data is then stored in a lookup table, see [Lookup tables](#).

For example:

```
{"plasma species": [
  {"name": "e", "Z": -1, "mobile": true, "diffusive": false,
   "mobility": {
     "lookup": "table E/N",
     "file": "transport_file.txt",
     "header": "# Electron mobility (E/N, mu*N)",
     "E/N": 0,
     "mu*N": 1,
     "min E/N": 10,
     "max E/N": 1000,
     "points": 100,
     "spacing": "exponential",
     "dump": "MyMobilityTable.dat"
   }
]
}
```

In the above, the fields have the following meaning:

- **file** The file where the data is found. The data must be stored in rows and columns.
- **header**, the contents of the line preceding the table data.
- **E/N**, the column that contains  $E/N$ .
- **$\mu \cdot N$** , the column that contains  $\mu \cdot E$ .
- **min E/N**, for trimming the data range.
- **max E/N**, for trimming the data range.
- **points**, for specifying the number of points in the lookup table.
- **spacing**, for specifying how to regularize the table.
- **dump**, an optional argument (useful for debugging) which will write the table to file.

Note that the input file does *not* need regularly spaced or sorted data. For performance reasons, the tables are always resampled, see [Lookup tables](#).

## Diffusion coefficients

Setting the diffusion coefficient is done *exactly* in the same was as the mobility. If a species is diffusive, one must include the field **diffusion** as well as **lookup**. For example, the JSON input for specifying a tabulated diffusion coefficient is done by

```
{"plasma species": [
  {"name": "e", "Z": -1, "mobile": false, "true": false,
   "diffusion": {
     "lookup": "table E/N",
     "file": "transport_file.txt",
     "header": "# Electron diffusion coefficient (E/N, D*N)",
     "E/N": 0,
     "D*N": 1,
     "min E/N": 10,
     "max E/N": 1000,
     "points": 1000,
     "spacing": "exponential"
   }
]
```

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```
[  
]}
```

## Temperatures

Plasma species temperatures can be set by including a field `temperature` for the plasma species.

**Warning:** If the `temperature` field is omitted, the species temperature will be set to the gas temperature.

### Constant temperature

To set a constant temperature, include the field `temperature` and set `lookup` to `constant` and specify the temperature through the field `value` as follows:

```
{"plasma_species":  
[  
  {  
    "name": "O2",  
    "Z": 0,  
    "mobile": false,  
    "true": false,  
    "temperature": {  
      "lookup": "constant",  
      "value": 300  
    }  
  }  
]
```

### Tabulated temperature

To include a tabulated temperature  $T = T(E, N)$ , set `lookup` to `table E/N`. The temperature is then computed as

$$T = \frac{2\epsilon}{3k_B},$$

where  $\epsilon$  is the energy and  $k_B$  is the Boltzmann constant.

The following fields are required:

- `file` for specifying which file the temperature is stored.
- `header` for specifying where in the file the temperature is stored.
- `E/N` for specifying in which column we find  $E/N$ .
- `eV` for specifying in which column we find the species energy (in units of electron volts).
- `min E/N` for trimming the data range.
- `max E/N` for trimming the data range.
- `points` for setting the number of points in the lookup table.
- `spacing` for setting the grid point spacing type.
- `dump` for writing the final table to file.

For a further explanation to these fields, see [Mobilities](#).

A complete example is:

```
{"plasma species": [
  {
    "name": "e",
    "Z": -1,
    "mobile": true,
    "true": true,
    "temperature": {
      "lookup": "table E/N",
      "file": "transport_data.txt",
      "header": "# Electron mean energy (E/N, eV)",
      "E/N": 0,
      "eV": 1,
      "min E/N": 10,
      "max E/N": 1000,
      "points": 1000,
      "spacing": "exponential",
      "dump": "MyTemperatureTable.dat"
    }
  }
]}
```

### Photon species

As for the plasma species, photon species (for including radiative transfer) are included by an array `photon species`. For each species, the required fields are

- `name` For setting the species name.
- `kappa` For specifying the absorption coefficient.

Currently, `kappa` can be either

- `constant` Which lets the user set a constant absorption coefficient.
- `helmholtz` Computes the absorption coefficient as

$$\kappa = \frac{p_X \lambda}{\sqrt{3}}$$

where  $\lambda$  is a specified input parameter and  $p_X$  is the partial pressure of some species  $X$ .

- `stochastic` A which samples a random absorption coefficient as

$$\kappa = K_1 \left( \frac{K_2}{K_1} \right)^{\frac{f-f_1}{f_2-f_1}}.$$

Here,  $f_1$  and  $f_2$  are frequency ranges,  $K_1$  and  $K_2$  are absorption coefficients, and  $f$  is a stochastically sampled frequency. Note that this method is only sensible when using discrete photons.

#### Constant absorption coefficients

When specifying a constant absorption coefficient, one must include a field `value` as well. For example:

```
{"photon species": [
  {
    "name": "UVPhoton",
    "kappa": "constant",
    "value": 1E4
  }
]}
```

#### Helmholtz absorption coefficients

The interface for the Helmholtz-based absorption coefficients are inspired by Bourdon *et al.* [2] approach for computing photoionization. This method only makes sense if doing a Helmholtz-based reconstruction of the photoionization profile as a relation:

$$\left[ \nabla^2 - (p_{O_2} \lambda)^2 \right] S_\gamma = - \left( A p_{O_2}^2 \frac{p_q}{p + p_q} \xi \nu \right) S_i,$$

where

- $S_\gamma$  is the number of photoionization events per unit volume and time.
- $A$  is a model coefficient.
- $\frac{p_q}{p + p_q}$  is a quenching factor.
- $\xi$  is a photoionization efficiency.
- $\nu$  is a relative excitation efficiency.
- $S_i$  is the electron impact ionization source term.

Since the radiative transfer solver is based on the Eddington approximation, the Helmholtz reconstruction can be written as

$$\kappa \Psi - \nabla \cdot \left( \frac{1}{3\kappa} \nabla \Psi \right) = \frac{\eta}{c}$$

where the absorption coefficient is set as

$$\kappa(\mathbf{x}) = \frac{p_{O_2} \lambda}{\sqrt{3}}.$$

The photogeneration source term is still

$$\eta = \frac{p_q}{p + p_q} \xi \nu S_i,$$

but the photoionization term is

$$S_\gamma = \frac{c A p_{O_2}}{\sqrt{3} \lambda} \Psi.$$

Note that the photoionization term is, in principle, *not* an Eddington approximation. Rather, the Eddington-like equations occur here through an approximation of the exact integral solution to the radiative transfer problem. In the pure Eddington approximation, on the other hand,  $\Psi$  represents the total number of ionizing photons per unit volume, and we would have  $S_\gamma = \frac{\Psi}{\Delta t}$  where  $\Delta t$  is the time step.

When specifying the `kappa` field as `helmholtz`, the absorption coefficient is computed as

$$\kappa(\mathbf{x}) = \frac{p_X(\mathbf{x}) \lambda}{\sqrt{3}}$$

where  $p_X$  is the partial pressure of a species  $X$  and  $\lambda$  is the same input parameter as in the Helmholtz reconstruction. These are specified through fields `neutral` and `lambda` as follows:

```
{"photon species": [
  {
    "name": "UVPhoton",
    "kappa": "helmholtz",
    "lambda": 0.0415,
    "neutral": "O2"
  }
]}
```

This input will set  $\kappa(x) = \frac{p_{O_2}(x)\lambda}{\sqrt{3}}$ .

---

**Note:** The source term  $\eta$  is specified when specifying the plasma reactions, see [Plasma reactions](#).

---

### Stochastic sampling

Setting the kappa field to `stochastic A` will stochastically sample the absorption length from

$$\kappa = K_1 \left( \frac{K_2}{K_1} \right)^{\frac{f-f_1}{f_2-f_1}}.$$

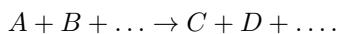
where  $K_1 = p_X \chi_{\min}$ ,  $K_2 = p_X \chi_{\max}$ , and  $f_1$  and  $f_2$  are frequency ranges. Like above,  $p_X$  is the partial pressure of some species  $X$ . Note that all input parameters are given in SI units.

Stochastic sampling of the absorption length only makes sense when using discrete photons – this particular method is inspired by the method in Chanrion and Neubert [4]. For example:

```
{"photon species": [
  {
    "name": "UVPhoton",
    "kappa": "stochastic A",
    "neutral": "O2",
    "f1": 2.925E15,
    "f2": 3.059E15,
    "chi_min": 2.625E-2,
    "chi_max": 1.5
  }
]}
```

### Plasma reactions

Plasma reactions are reactions between charged and neutral species and are written in the form



Importantly, the left hand side of the reaction can only consist of charged or neutral species. It is not permitted to put a photon species on the left hand side of these reactions; photo-ionization is handled separately by another set of reaction types (see [Photo-reactions](#)). However, photon species *can* appear on the left hand side of the equation.

When specifying reactions in this form, the reaction rate is computed as

$$R = k n_A n_B \dots$$

When computing the source term for some species  $X$ , we subtract  $R$  for each time  $X$  appears on the left hand side of the reaction and add  $R$  for each time  $X$  appears on the right-hand side of the reaction.

### Specifying reactions

Reactions of the above type are handled by a JSON array `plasma reactions`, with required fields:

- `reaction` (string) containing the reaction process.
- `lookup` (string) for determining how to compute the reaction rate.

```
{"plasma reactions": [
  {
    "reaction": "e + O2 -> e + e + O2+",
    "lookup": "constant",
    "rate": 1E-30
  }
]}
```

This adds a reaction  $e + O_2 \rightarrow e + e + O_2^+$  to the reaction set. We compute

$$R = k n_e n_{O_2^+}$$

and set

$$S_e = S_{O_2^+} = R.$$

Some caveats when setting the reaction string are:

- Whitespace are separators. For example,  $O2+e$  will be interpreted as a species with string identifier  $O2+e$ , but  $O2 + e$  will be interpreted as a reaction between  $O2$  and  $e$ .
- The reaction string *must* contain a left and right hand side separated by  $\rightarrow$ . An error will be thrown if this symbol can not be found.
- The left-hand side must consist *only* of neutral or plasma species. If the left-hand side consists of species that are not neutral or plasma species, an error will be thrown.
- The right-hand side can consist of either neutral, plasma species, or photon species. Otherwise, an error will be thrown.
- The reaction string will be checked for charge conservation.

Note that if a reaction involves a right-hand side that is not otherwise tracked, the user should omit the species from the right-hand side altogether. For example, if we have a model which tracks the species  $e$  and  $O_2^+$  but we want to include the dissociative recombination reaction  $e + O_2^+ \rightarrow O + O$ , this reaction should be added to the reaction with an empty right-hand side:

```
{"plasma reactions": [
  {
    "reaction": "e + O2 -> e + e + O2+",
    "lookup": "constant",
    "rate": 1E-30
  },
  {
    "reaction": "e + O2+ -> ",
    "lookup": "constant",
    "rate": 1E-30
  }
]}
```

## Wildcards

Reaction specifiers may include the wildcard  $@$  which is a placeholder for another species. The wildcards must be specified by including a JSON array  $@$  of the species that the wildcard is replaced by. For example:

```
{"plasma reactions": [
  {
    "reaction": "N2+ + N2 + @ -> N4+ + @",
    "@": ["N2", "O2"]
  }
]}
```

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```

    "lookup": "constant",
    "rate": 1E-30
}
]
}

```

The above code will add two reactions to the reaction set:  $N_2 + N_2 + N_2 \rightarrow N_4^+ + N_2$  and  $N_2 + N_2 + O_2 \rightarrow N_4^+ + O_2$ . It is not possible to set different reaction rates for the two reactions.

## Specifying reaction rates

### Constant reaction rates

To set a constant reaction rate for a reaction, set the field `lookup` to "constant" and specify the rate. For example:

```
{"plasma reactions":
[
{
  "reaction": "e + O2 -> e + e + O2+", 
  "lookup": "constant",
  "rate": 1E-30
}
]}
```

### Single-temperature rates

- `functionT A` To set a rate dependent on a single species temperature in the form  $k(T) = c_1 T^{c_2}$ , set `lookup` to `functionT A`. The user must specify the species from which we compute the temperature  $T$  by including a field `T`. The constants  $c_1$  and  $c_2$  must also be included.

For example, in order to add a reaction  $e + O_2 \rightarrow \emptyset$  with rate  $k = 1.138 \times 10^{-11} T_e^{-0.7}$  we can add the following:

```
{"plasma reactions":
[
{
  "reaction": "e + M+ ->",
  "lookup": "functionT A",
  "T": "e",
  "c1": 1.138
  "c2": -0.7
}
]}
```

### Two-temperature rates

- `functionT1T2 A` To set a rate dependent on two species temperature in the form  $k(T_1, T_2) = c_1 (T_1/T_2)^{c_2}$ , set `lookup` to `functionT1T2 A`. The user must specify which temperatures are involved by specifying the fields `T1`, `T2`, as well as the constants through fields `c1` and `c2`. For example, to include the reaction  $e + O_2 + O_2 \rightarrow O_2^- + O_2$  in the set, with this reaction having a rate

$$k = 2.4 \times 10^{-41} \left( \frac{T_{O_2}}{T_e} \right),$$

we add the following:

```
{"plasma reactions": [
  {
    "reaction": "e + O2 + O2 -> O2- + O2",
    "lookup": "functionT1T2 A",
    "T1": "O2",
    "T2": "e",
    "c1": 2.41E-41,
    "c2": 1
  }
]}
```

## Townsend ionization and attachment

To set standard Townsend ionization and attachment reactions, set `lookup` to `alpha*v` and `eta*v`, respectively. This will compute the rate constant  $k = \alpha |\mathbf{v}|$  where  $\mathbf{v}$  is the drift velocity of some species. To specify the species one includes the field `species`.

For example, to include the reactions  $e \rightarrow e + e + M^+$  and  $e \rightarrow M^-$  one can specify the reactions as

```
{"plasma reactions": [
  {
    "reaction": "e -> e + e + M+", "lookup": "alpha*v", "species": "e"
  },
  {
    "reaction": "e -> M-", "lookup": "eta*v", "species": "e"
  }
}]}
```

## Tabulated rates

To set a tabulated rate with  $k = k(E, N)$ , set the field `lookup` to `table E/N` and specify the file, header, and data format to be used. For example:

```
{"plasma reactions": [
  {
    "reaction": "e + O2 -> e + e + O2+", "lookup": "table E/N",
    "file": "transport_file.txt", "header": "# O2 ionization (E/N, rate/N)",
    "E/N": 0, "rate/N": 1, "min E/N": 10, "max E/N": 1000, "spacing": "exponential",
    "points": 1000, "plot": true, "dump": "O2_ionization.dat"
  }
}]}
```

The `file` field specifies which field to read the reaction rate from, while `header` indicates where in the file the reaction rate is found. The file parser will read the files below the header line until it reaches an empty line. The fields `E/N` and `rate/N` indicate the columns where the reduced electric field and reaction rates are stored.

The final fields `min E/N`, `max E/N`, and `points` are formatting fields that trim the range of the data input and organizes the data along a table with `points` entries. As with the mobilities (see [Mobilities](#)), the `spacing` argument determines

whether or not the internal interpolation table uses uniform or exponential grid point spacing. Finally, the `dump` argument will tell chombo-discharge to dump the table to file, which is useful for debugging or quality assurance of the tabulated data.

### Modifying reactions

#### Collisional quenching

To quench a reaction, include a field `quenching_pressure` and specify the *quenching pressure* (in atmospheres). When computing reaction rates, the rate for the reaction will be modified as

$$k \rightarrow k \frac{p_q}{p_q + p}$$

where  $p^q$  is the quenching pressure and  $p = p(\mathbf{x})$  is the gas pressure.

---

**Important:** The quenching pressure should be specified in Pascal.

---

For example:

```
{"plasma_reactions": [
  {
    "reaction": "e + N2 -> e + N2 + Y",
    "lookup": "table E/N",
    "file": "transport_file.txt",
    "header": "# N2 ionization (E/N, rate/N)",
    "E/N": 0,
    "rate/N": 1,
    "min E/N": 10,
    "max E/N": 1000,
    "points": 1000,
    "spacing": "exponential",
    "quenching_pressure": 4000
  }
]}
```

### Reaction efficiencies

To modify a reaction efficiency, include a field `efficiency` and specify it. This will modify the reaction rate as

$$k \rightarrow \nu k$$

where  $\nu$  is the reaction efficiency. For example:

```
{"plasma_reactions": [
  {
    "reaction": "e + N2 -> e + N2 + Y",
    "lookup": "table E/N",
    "file": "transport_file.txt",
    "header": "# N2 ionization (E/N, rate/N)",
    "E/N": 0,
    "rate/N": 1,
    "min E/N": 10,
    "max E/N": 1000,
    "points": 1000,
    "spacing": "exponential",
    "efficiency": 0.6
  }
} ]}
```

## Scaling reactions

Reactions can be scaled by including a `scale` argument to the reaction. This works exactly like the `efficiency` field outlined above.

## Energy correction

Occasionally, it can be necessary to incorporate an energy correction to models, accounting e.g. for electron energy loss near strong gradients. The JSON interface supports the correction in Soloviev and Krivtsov [9]. To use it, include an (optional) field `soloview` and specify `correction` and `species`. For example:

```
{"plasma reactions": [
  {
    "reaction": "e + N2 -> e + N2 + Y",
    "lookup": "table E/N",
    "file": "transport_file.txt",
    "header": "# N2 ionization (E/N, rate/N)",
    "E/N": 0,
    "rate/N": 1,
    "min E/N": 10,
    "max E/N": 1000,
    "points": 1000,
    "spacing": "exponential",
    "efficiency": 0.6,
    "soloview": {
      "correction": true,
      "species": "e"
    }
  }
]}
```

When this energy correction is enabled, the rate coefficient is modified as

$$k \rightarrow k \left( 1 + \frac{\mathbf{E} \cdot D_s \nabla n_s}{\mu_s n_s E^2} \right),$$

where  $s$  is the species specified in the `soloview` field,  $n_s$  is the density and  $D_s$  and  $\mu_s$  are diffusion and mobility coefficients. We point out that the correction factor is restricted such that the reaction rate is always non-negative. Note that this correction makes sense when rates are dependent only on the electric field, see Soloviev and Krivtsov [9].

---

**Note:** When using the energy correction, the species must be both mobile and diffusive.

---

## Plotting reactions

It is possible to have CdrPlasmaJSON include the reaction rates in the HDF5 output files by including a field `plot` as follows:

```
{"plasma reactions": [
  {
    "reaction": "e + O2 -> e + e + O2+",
    "plot": true,
    "lookup": "constant",
    "rate": 1E-30,
  }
]}
```

Plotting the reaction rate can be useful for debugging or analysis. Note that it is, by extension, also possible to add useful data to the I/O files from reactions that otherwise do not contribute to the discharge evolution. For example, if

we know the rate  $k$  for excitation of nitrogen to a specific excited state, but do not otherwise care about tracking the excited state, we can add the reaction as follows:

```
{"plasma reactions": [
  [
    {
      "reaction": "e + N2 -> e + N2",
      "plot": true,
      "lookup": "constant",
      "rate": 1E-30,
    }
  ]
}]
```

This reaction is a dud in terms of the discharge evolution (the left and right hand sides are the same), but it can be useful for plotting the excitation rate.

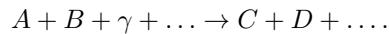
---

**Note:** This functionality should be used with care because each reaction increases the I/O load.

---

### Photo-reactions

Photo-reactions are reactions between charged/neutral and photons in the form



where species  $A, B, \dots$  are charged and neutral species and  $\gamma$  is a photon. The left hand side can contain only *one* photon species, and the right-hand side can not contain a photon species. In other words, two-photon absorption is not supported, and photons that are absorbed on the mesh cannot become new photons. This is not a fundamental limitation, but a restriction imposed by the JSON interface.

### Specifying reactions

Reactions of the above type are handled by a JSON array `photo_reactions`, with required fields:

- `reaction` (string) containing the reaction process.
- `lookup` (string) for determining how to compute the reaction rate.

For example:

```
{"photo reactions": [
  [
    {"reaction": "Y + O2 -> e + O2+"}
  ]
}]
```

The rules for specifying reaction strings are the same as for the plasma reactions, see [Plasma reactions](#). Wildcards also apply, see [Wildcards](#).

## Default behavior

Since the radiative transfer solvers solve for the number of ionizing photons, the CDR solver source terms are incremented by

$$S \rightarrow S + \frac{\Psi}{\Delta t}.$$

where  $\Psi$  is the number of ionizing photons per unit volume (i.e., the solution  $\Psi$ ).

## Helmholtz reconstruction

When performing a Helmholtz reconstruction the photoionization source term is

$$S = \frac{cA p_{O_2}}{\sqrt{3}\lambda} \Psi.$$

To modify the source term for consistency with Helmholtz reconstruction specify the field `helmholtz` with variables

- `A`. the  $A$  coefficient.
- `lambda`. the  $\lambda$  coefficient. This value will also be specified in the photon species, but it is not retrieved automatically.
- `neutral`. The neutral species for which we obtain the partial pressure.

For example:

```
{"photo_reactions": [
  {
    "reaction": "Y + O2 -> e + O2+",
    "helmholtz": {
      "A": 1.1E-4,
      "lambda": 0.0415,
      "neutral": "O2"
    }
  }
]}
```

## Scaling reactions

Photo-reactions can be scaled by including a `scale` argument. For example, to completely turn off the photoreaction above:

```
{"photo_reactions": [
  {
    "reaction": "Y + O2 -> e + O2+",
    "helmholtz": {
      "A": 1.1E-4,
      "lambda": 0.0415,
      "neutral": "O2"
    },
    "scale": 0.0
  }
]}
```

## EB boundary conditions

Boundary conditions on the embedded boundary are included by the fields

- `electrode reactions`, for specifying secondary emission on electrodes.
- `dielectric reactions`, for specifying secondary emission on dielectrics.

To include secondary emission, the user must specify a reaction string in the form  $A \rightarrow B$ , and also include an emission rate. Currently, we only support constant emission rates (i.e., secondary emission coefficients). This is likely to change in the future.

The following points furthermore apply:

- By default, standard outflow boundary conditions. When `electrode reactions` or `dielectric reactions` are specified, the user only controls the *inflow* back into the domain.
- Wildcards can appear on the left hand side of the reaction.
- If one specifies  $A + B \rightarrow C$  for a surface reaction, this is the same as specifying two reactions  $A \rightarrow C$  and  $B \rightarrow C$ . The same emission coefficient will be used for both reactions.
- Both photon species and plasma species can appear on the left hand side of the reaction.
- Photon species can not appear on the right-hand side of the reaction; we do not include surface sources for photoionization.
- To scale reactions, include a modifier `scale`.

For example, the following specification will set secondary emission efficiencies to  $10^{-3}$ :

```
{"electrode reactions":  
  [  
    { "reaction": "@ -> e",  
      "@": ["N2+", "O2+", "N4+", "O4+", "O2+N2"],  
      "lookup": "constant",  
      "value": 1E-4  
    }  
  ],  
  "dielectric reactions":  
  [  
    { "reaction": "@ -> e",  
      "@": ["N2+", "O2+", "N4+", "O4+", "O2+N2"],  
      "lookup": "constant",  
      "value": 1E-3  
    }  
  ]}
```

## Domain boundary conditions

TODO.

## 5.2 Streamer inception model

### 5.2.1 Overview

The streamer inception model computes the inception voltage and probability of streamer inception for arbitrary geometries and voltage forms.

For estimating the streamer inception, the module the electron avalanche integral

$$K(\mathbf{x}) = \int_{\mathbf{x}}^{\text{until } \alpha_{\text{eff}} \leq 0} \alpha_{\text{eff}}(E) dl$$

where  $E = |\mathbf{E}|$  and where  $\alpha_{\text{eff}}(E) = \alpha(E) - \eta(E)$  is the effective ionization coefficient. The integration runs along electric field lines.

The streamer inception model solves for  $K = K(\mathbf{x})$  for all  $\mathbf{x}$ , i.e., the inception integral is evaluated for all starting positions of the first electron. This differs from the conventional approach where the user will first extract electric field lines for post-processing.

In addition to the above, the user can specify a critical threshold value for  $K_c$  which is used for computing

- The critical volume  $V_c = \int_{K>K_c} dV$ .
- The inception voltage  $U_c$ .
- The probability of having the first electron in the critical volume,  $dP(t, t + \Delta t)$ .

Note that since the  $K(\mathbf{x}) = K(\mathbf{x}; U)$  where  $U$  is the applied voltage, these values are computed for a user-specified range of voltages. This “range of voltages” can be a series of discrete values, or a voltage curve (e.g., lightning impulse).

The streamer inception model can be run in two modes:

- *Stationary mode*.
- *Transient mode*.

These are discussed below.

## Implementation

The streamer inception model is implemented in `$DISCHARGE_HOME/Physics/StreamerInception` as

```
template <typename P, typename F, typename C>
class StreamerInceptionStepper : public TimeStepper
```

The template complexity is due to flexibility in what solver implementations we use. The following solvers are used within the model:

- The tracer particle solver (`TracerParticleSolver`) for reconstructing the inception integral.
- The `FieldSolver` for computing the electric field.
- The convection diffusion reaction module (`CdrSolver`) for modeling negative ion transport.

## Stationary mode

The stationary mode solves the  $K$  integral for a range of input voltages. The computation is done for both polarities so that the user obtains:

- $K$  values for positive and negative voltages.
- Critical volumes  $V_c$  for positive and negative voltages.

## Transient mode

In the transient mode we apply a voltage curve  $U = U(t)$  reconstruct the  $K$  value at each time step and recompute the critical volume so that we obtain

$$K = K(t)$$

$$V_c = V_c(t)$$

We also assume that ions move as drifting Brownian walkers in the electric field (see [Ito diffusion](#)). This can be written in the fluctuating hydrodynamics limit as an evolution equation for the ion distribution (not density!) as

$$\frac{\partial \langle n_- \rangle}{\partial t} = -\nabla \cdot (\mathbf{v} \langle n_- \rangle) + \nabla \cdot (D \nabla \langle n_- \rangle) + \sqrt{2D \langle n_- \rangle} \mathbf{Z},$$

where  $\mathbf{Z}$  represents uncorrelated Gaussian white noise. Note that the above equation is a mere rewrite of the Ito process for a collection of particles; it is not really useful per se since it is a tautology for the original Ito process.

However, we are interested in the average ion distribution over many experiments, so by taking the ensemble average we obtain a regular advection-diffusion equation for the evolution of the negative ion distribution (note that we redefine  $\langle n_- \rangle$  to be the ensemble average).

$$\frac{\partial \langle n_- \rangle}{\partial t} = -\nabla \cdot (\mathbf{v} \langle n_- \rangle) + \nabla \cdot (D \nabla \langle n_- \rangle).$$

This equation is sensible only when  $\langle n_- \rangle$  is interpreted as an ion density distribution (over many identical experiments).

The above quantities are then used for computing the probability of streamer inception in time  $t$  by

$$P(t) = 1 - \exp \left[ - \int_0^t \left( \int_{V_c(t')} \left\langle \frac{dn_e}{dt'} \right\rangle \left( 1 - \frac{\eta}{\alpha} \right) dV + \int_{A_c(t')} \frac{j_e}{q_e} \left( 1 - \frac{\eta}{\alpha} \right) dA \right) dt' \right]. \quad (5.2.1)$$

Here,  $\left\langle \frac{dn_e}{dt} \right\rangle$  is the electron production rate from both background ionization and electron detachment, i.e.

$$\left\langle \frac{dn_e}{dt} \right\rangle = S_{bg} + k_d \langle n_- \rangle,$$

where  $S_{bg}$  is the background ionization rate set by the user,  $k_d$  is the negative ion detachment rate, and  $\langle n_- \rangle$  is the negative ion distribution. The second integral is due to electron emission from the cathode and into the critical volume. Note that, internally, we always ensure that  $j_e dA$  evaluates to zero on anode surfaces.

We also compute the probability of a first electron appearing in the time interval  $[t, t + \Delta t]$ , given by

$$\Delta P(t, t + \Delta t) = [1 - P(t)] \left( \int_{V_c(t')} \left\langle \frac{dn_e}{dt'} \right\rangle \left( 1 - \frac{\eta}{\alpha} \right) dV + \int_{A_c(t')} \frac{j_e}{q_e} \left( 1 - \frac{\eta}{\alpha} \right) dA \right) \Delta t \quad (5.2.2)$$

When running in transient mode the user must set the voltage curve (see [Voltage curve](#)) and pay particular caution to setting the initial ion density, mobility, and detachment rates.

### 5.2.2 Input data

The input to the streamer inception model are:

1. Streamer inception threshold.
2. Townsend ionization coefficient.
3. Townsend attachment coefficients.
4. Background ionization rate (e.g., from cosmic radiation).

5. Electron detachment rate (from negative ions).
6. Negative ion mobility.
7. Negative ion diffusion coefficient.
8. Initial negative ion density.
9. Voltage curve (for transient simulations).

The input data to the streamer inception model is mostly done by passing in C++-functions to the class. These functions are mainly in the form

```
std::function<Real(const Real& E)>
```

The user can specify analytic fields or use tabulated data, and pass these in through a C++ lambda function. For defining an analytic function:

```
auto alphaCoeff = [] (const Real E) -> void {
    return 1/E;
};
```

Tabulated data (see *Lookup tables*) can also be used.

```
LookupTable1D<2> tableData;

auto alphaCoeff = [tableData] (const Real E) -> void {
    return tableData.getEntry<1>(E);
};
```

---

**Note:** The  $K$  integral is only affect by the Townsend ionization and attachment coefficients. The remaining transport data is used for calculating the inception probability (appearance of a first electron in the critical volume).

---

## Inception threshold

Use in class input value StreamerInceptionStepper.K\_inception for setting the inception threshold.

For example:

```
StreamerInceptionStepper.K_inception = 12.0
```

## Townsend ionization coefficient

To set the Townsend ionization coefficient, use the member function

```
StreamerInceptionStepper::setAlpha(const std::function<Real(const Real& E)>& a_alpha) noexcept;
```

## Townsend attachment coefficient

To set the Townsend attachment coefficient, use the member function

```
StreamerInceptionStepper::setEta(const std::function<Real(const Real& E)>& a_eta) noexcept;
```

### Negative ion mobility

To set the negative ion mobility, use the member function

```
StreamerInceptionStepper::setIonMobility(const std::function<Real(const Real& E)>& a_mobility) noexcept;
```

### Negative ion diffusion coefficient

To set the negative ion diffusion coefficient, use the member function

```
StreamerInceptionStepper::setIonDiffusion(const std::function<Real(const Real& E)>& a_diffCo) noexcept;
```

### Negative ion density

To set the negative ion density, use the member function

```
StreamerInceptionStepper::setIonDensity(const std::function<Real(const RealVect x)>& a_density) noexcept;
```

### Background ionization rate

The background ionization rate describes the appearance of a first electron from a background contribution, e.g. through cosmic radiation, decay of radioactive isotopes, etc.

To set the background ionization rate, use the member function

```
StreamerInceptionStepper::setBackgroundRate(const std::function<Real(const Real& E, const RealVect& x)>& a_backgroundRate) noexcept;
```

### Detachment rate

The detachment rate from negative describes the appearance of electrons through the equation

$$\left\langle \frac{dn_e}{dt} \right\rangle = k_d \langle n_- \rangle$$

where  $\langle n_- \rangle$  is the negative ion density in units of  $m^{-3}$  (or strictly speaking the negative ion probability density). This is used when calculating the inception probability, and the user sets the detachment rate  $k_d$  through

```
StreamerInceptionStepper::setDetachmentRate(const std::function<Real(const Real& E, const RealVect& x)>& a_backgroundRate) noexcept;
```

### Field emission

To set the field emission current, use the function

```
StreamerInceptionStepper::setFieldEmission(const std::function<Real(const Real& E, const RealVect& x)>& a_currentDensity) noexcept;
```

This will set a field-dependent emission rate from cathodes given by the input function. Note that, under the hood, the function indicates a general cathode emission current which can be the sum of several contributions (field emission, photoelectric effect etc.).

---

**Important:** The input function should provide the surface current density  $j_e$  (in units of  $C \cdot m^{-2} \cdot s^{-1}$ ).

---

## Input voltages

By default, the model will always read voltage levels from the input script. These are in the format

```
StreamerInceptionStepper.voltage_lo    = 1.0    # Low voltage multiplier
StreamerInceptionStepper.voltage_hi    = 10.0   # Highest voltage multiplier
StreamerInceptionStepper.voltage_steps = 3      # Number of voltage steps
```

## Voltage curve

To set the voltage curve, use the member function

```
StreamerInceptionStepper::setVoltageCurve(const std::function<Real(const Real& time)>& a_voltageCurve) noexcept;
```

This is relevant only when running a transient simulation.

## 5.2.3 Algorithms

The streamer inception model uses a combination of electrostatic field solves, Particle-In-Cell, and fluid advection for resolving the necessary dynamics. The various algorithms involved are discussed below.

### Field solve

Since the background field scales linearly with applied voltage, we require only a single field solve at the beginning of the simulation. This field solve is done with an applied voltage of  $U = 1 \text{ V}$  and the electric field is then simply later scaled by the actual voltage.

### Inception integral

We use a Particle-In-Cell method for computing the inception integral  $K(\mathbf{x})$  for an arbitrary electron starting position. All grid cells where  $\alpha_{\text{eff}} > 0$  are seeded with one particle on the cell centroid and the particles are then tracked through the grid. The particles move a user-specified distance along field lines  $\mathbf{E}$  and the particle weights are updated using first or second order integration. If a particle leaves through a boundary (EB or domain boundary), or enters a region  $\alpha_{\text{eff}} \leq 0$ , the integration is stopped. Once the particle integration halts, we rewind the particles back to their starting position and deposit their weight on the mesh, which provides us with  $K = K(\mathbf{x})$ .

### Euler

For the Euler rule the particle weight for a particle  $p$  the update rule is

$$\begin{aligned}\mathbf{x}_p^{k+1} &= \mathbf{x}_p^k - \hat{\mathbf{E}}(\mathbf{x}_p^k) \Delta x \\ w_p^{k+1} &= w_p^k + \alpha_{\text{eff}}(|\mathbf{E}(\mathbf{x}_p^k)|) \Delta x,\end{aligned}$$

where  $\Delta x$  is a user-specified integration length.

## Trapezoidal

With the trapezoidal rule the update is first

$$\mathbf{x}'_p = \mathbf{x}_p^k - \hat{\mathbf{E}}(\mathbf{x}_p^k) \Delta x$$

followed by

$$\begin{aligned}\mathbf{x}_p^{k+1} &= \mathbf{x}_p^k + \frac{\Delta x}{2} [\hat{\mathbf{E}}(\mathbf{x}_p^k) + \hat{\mathbf{E}}(\mathbf{x}'_p)]. \\ w_p^{k+1} &= w_p^k + \frac{\Delta x}{2} [\alpha_{\text{eff}}(|\mathbf{E}(\mathbf{x}_p^k)|) + \alpha_{\text{eff}}(|\mathbf{E}(\mathbf{x}'_p)|)]\end{aligned}$$

## Critical volume

The critical volume is computed as

$$V_c = \int_{K(\mathbf{x})>K} dV.$$

Note that the critical volume is both voltage and polarity dependent.

## Inception voltage

### Arbitrary starting electron

The inception voltage for starting a critical avalanche can be computed in the stationary solver mode. In this case we compute  $K(\mathbf{x}; U)$  for a range of voltages  $U \in U_1, U_2, \dots$

If two values of the  $K$  integral bracket  $K_c$ , i.e.

$$\begin{aligned}K_a &= K(\mathbf{x}; U_a) \leq K_c \\ K_b &= K(\mathbf{x}; U_b) \geq K_c\end{aligned}$$

then we can estimate the inception voltage for a starting electron at position  $\mathbf{x}$  through linear interpolation as

$$U_{\text{inc}}(\mathbf{x}) = U_a + \frac{K_c - K_a}{K_b - K_a} (U_b - U_a)$$

### Minimum inception voltage

The minimum inception voltage is the minimum voltage required for starting a critical avalanche for an arbitrary starting electron. From the above, this is simply

$$U_{\text{inc}}^{\min} = \min_{\forall \mathbf{x}} [U_{\text{inc}}(\mathbf{x})].$$

From the above we also determine

$$\mathbf{x}_{\text{inc}}^{\min} \leftarrow \mathbf{x} \text{ that minimizes } U_{\text{inc}}(\mathbf{x}) \quad \forall \mathbf{x},$$

which is the position of the first electron that enables a critical avalanche at the minimum inception voltage.

---

**Note:** The minimum inception voltage is the minimum voltage required for starting a critical avalanche. However, as  $U \rightarrow U_{\text{inc}}^{\min}$  we also have  $V_c \rightarrow 0$ , requires the a starting electron *precisely* in  $\mathbf{x}_{\text{inc}}^{\min}$ .

---

## Inception probability

The inception probability is given by Eq. 5.2.1 and is computed using straightforward numerical quadrature:

$$\int_{V_c} \left\langle \frac{dn_e}{dt} \right\rangle \left(1 - \frac{\eta}{\alpha}\right) dV \approx \sum_{i \in K_i > K_c} \left( \left\langle \frac{dn_e}{dt} \right\rangle \right)_i \left(1 - \frac{\eta_i}{\alpha_i}\right) \kappa_i \Delta V_i,$$

and similarly for the surface integral.

---

**Important:** The integration runs over *valid cells*, i.e. grid cells that are not covered by a finer grid.

---

## Advection algorithm

The advection algorithm for the negative ion distribution follows the time stepping algorithms described in the advection-diffusion model, see [Advection-diffusion model](#).

### 5.2.4 Simulation control

Here, we discuss simulation controls that are available for the streamer inception model. These all appear in the form `StreamerInceptionStepper.<option>`.

#### verbosity

The `verbosity` input option controls the model chattiness (to the `pout.*` files). Usually we have

```
StreamerInceptionStepper.verbosity = -1
```

#### mode

The mode flag switches between stationary and transient solves. Accepted values are `stationary` and `transient`, e.g.,

```
StreamerInceptionStepper.mode = stationary
```

---

**Important:** When running in stationary mode, set `Driver.max_steps=0`.

---

#### inception\_alg

Controls the streamer inception algorithm (for computing the  $K$  integral). This should be specified in the form

```
StreamerInceptionStepper.inception_alg = <algorithm> <mode> <value>
```

where `<algorithm>` is either `trapz` (trapezoidal rule) or `euler` and mode is either `fixed` or `dx`. These differ in the sense that the integration step in the value field either corresponds to a fixed physical size or a number relative to the grid resolution. For example, the following will set an Euler integration with a fixed step size:

```
StreamerInceptionStepper.inception_alg = euler fixed 100E-6
```

The subsequent code sets a trapezoidal rule using half the grid resolution as the particle step:

```
StreamerInceptionStepper.inception_alg = trapezoidal dx 0.5
```

### output\_file

Controls the overall report file for stationary and transient solves. The user specifies a filename for a file which will be created (in the same directory as the application is running), containing a summary of the most important simulation output variables.

**Warning:** Running a new simulation will overwrite the specified `output_file`.

For example:

```
StreamerInceptionStepper.output_file = report.txt
```

### K\_inception

Controls the critical value of the  $K$  integral. E.g.,

```
StreamerInceptionStepper.K_inception = 12
```

### plt\_vars

Controls plot variables that will be written to HDF5 outputs in the `plt` folder. Valid options are

- `K` - Inception integral
- `Uinc` - Inception voltage
- `bg_rate` - Background ionization rate
- `emission` - Field emission
- `alpha` - Effective ionization coefficient
- `eta` - Eta coefficient
- `poisson` - Poisson solver
- `tracer` - Tracer particle solver
- `ions` - Ion solver

For example:

```
StreamerInceptionStepper.plt_vars = K Uinc bg_rate emission ions
```

## For stationary mode

For the stationary mode the following input flags are required:

- `voltage_lo` Lowest simulated voltage.
- `voltage_hi` High simulated voltage.
- `voltage_steps` Extra voltage steps between `voltage_lo` and `voltage_hi`.

These voltages levels are used when running a stationary solve. For example:

```
StreamerInceptionStepper.voltage_lo    = 10E3
StreamerInceptionStepper.voltage_hi   = 30E3
StreamerInceptionStepper.voltage_steps = 5
```

## For transient mode

For the transient mode the following input options must be set:

- `ion_transport` For turning on/off ion transport.
- `transport_alg` For controlling the transport algorithm. Valid options are `euler`, `heun`, or `imex` (for semi-implicit with corner transport upwind).
- `cfl` Which controls the ion advection time step.
- `min_dt` For setting the minimum time step used.
- `max_dt` For setting the maximum time step used.

For example,

```
StreamerInceptionStepper.ion_transport = true
StreamerInceptionStepper.transport_alg = imex
StreamerInceptionStepper.cfl           = 0.8
StreamerInceptionStepper.min_dt       = 0.0
StreamerInceptionStepper.max_dt       = 1E99
```

**Warning:** The `ctu` option exists because the default advection solver for the streamer inception model is the corner transport upwind solver (see [CdrCTU](#)). Ensure that `CdrCTU.use_ctu = true` if using `StreamerInceptionStepper.transport_alg = ctu` algorithm and set `CdrCTU.use_ctu = false` otherwise.

## Caveats

The model is intended to be used with a nearest-grid-point deposition scheme (which is also volume-weighted). When running the model, ensure that the `TracerParticleSolver` flag is set as follows:

```
TracerParticleSolver.deposition = ngp
```

### 5.2.5 Adaptive mesh refinement

The streamer inception model runs its own mesh refinement routine, which refines the mesh if

$$\alpha_{\text{eff}}(|\mathbf{E}|) \Delta x > \lambda,$$

where  $\lambda$  is a user-specified refinement criterion.

This is implemented in a class

```
class StreamerInceptionTagger : public CellTagger
```

and is automatically included in simulations when setting up the application through the Python setup tools (see [Setting up a new problem](#)). The user can control refinement buffers and criterion through the following input options:

- `StreamerInceptionTagger.buffer` Adds a buffer region around tagged cells.
- `StreamerInceptionTagger.max_voltage` Maximum voltage that will be simulated.
- `StreamerInceptionTagger.ref_alpha` Sets the refinement criterion  $\lambda$  as above.

For example:

```
StreamerInceptionTagger.buffer      = 4
StreamerInceptionTagger.max_voltage = 30E3
StreamerInceptionTagger.ref_alpha   = 2.0
```

### 5.2.6 Setting up a new problem

To set up a new problem, using the Python setup tools in `$DISCHARGE_HOME/Physics/StreamerInception` is the simplest way. To see available setup options, run

```
./setup.py --help
```

For example, to set up a new problem in `$DISCHARGE_HOME/MyApplications/MyStreamerInception` for a cylinder geometry, run

```
./setup.py -base_dir=MyApplications -app_name=MyStreamerInception -geometry=Cylinder
```

This will set up a new problem in a cylinder geometry (defined in `Geometries/Cylinder`). The main file is named `program.cpp` and contains default implementations for the required input data (see [Input data](#)).

### 5.2.7 Example programs

Example programs that use the streamer inception model are given in

#### High-voltage vessel

- `$DISCHARGE_HOME/Exec/Examples/StreamerInception/Vessel`. This program is set up in 2D (stationary) and 3D (transient) for streamer inception in atmospheric air. The input data is computed using BOLSIG+.

### Electrode with surface roughness

- \$DISCHARGE\_HOME/Exec/Examples/StreamerInception/ElectrodeRoughness. This program is set up in 2D (stationary) and 3D (transient) for streamer inception on an irregular electrode surface. We use SF6 transport data as input data, computed using BOLSIG+.

## 5.3 Ito-KMC plasma model

**Warning:** This is a work-in-progress.



## SINGLE-SOLVER APPLICATIONS

### 6.1 Advection-diffusion model

The advection-diffusion model simply advects a scalar quantity with EB and AMR. The equation of motion is

$$\frac{\partial \phi}{\partial t} + \nabla \cdot (\mathbf{v}\phi - D\nabla\phi) = 0.$$

The full implementation for this model consists of the following classes:

- `AdvectionDiffusionStepper`, which implements `TimeStepper`.
- `AdvectionDiffusionSpecies`, which parses the initial conditions into the problem.
- `AdvectionDiffusionTagger`, which implements `CellTagger` and flags cells for refinement and coarsening.

The source code for this problem is found in `$DISCHARGE_HOME/Physics/AdvectionDiffusion`. See `AdvectionDiffusionStepper` for the C++ API for this time stepper.

#### 6.1.1 Solvers

This class uses the following solvers:

- `CdrSolver`.

#### 6.1.2 Time stepping

Two time stepping algorithms are supported:

1. A second-order Runge-Kutta method (Heun's method).
2. An implicit-explicit method (IMEX) which uses corner-transport upwind (CTU, see `CdrCTU`) and a Crank-Nicholson diffusion solve.

To select an integrator, set

```
AdvectionDiffusion.integrator = imex # 'imex' or 'heun'
```

The user can set the CFL number and maximum/minimum permitted time steps by

```
AdvectionDiffusion.cfl      = 0.5
AdvectionDiffusion.min_dt  = 0.0
AdvectionDiffusion.max_dt  = 1.E99
```

where `AdvectionDiffusion.cfl` is computed differently based on the chosen discretization.

## Heun's method

For Heun's method we perform the following steps: Let  $[\nabla \cdot \mathbf{J}(\phi)]$  be the finite-volume approximation to  $\nabla \cdot (\mathbf{v}\phi - D\nabla\phi)$ , where  $\mathbf{J}(\phi) = \mathbf{v}\phi - D\nabla\phi$ . The Runge-Kutta advance is then

$$\begin{aligned}\phi' &= \phi^k - \Delta t [\nabla \cdot \mathbf{J}(\phi^k)] \\ \phi^{k+1} &= \phi^k - \frac{\Delta t}{2} ([\nabla \cdot \mathbf{J}(\phi^k)] + [\nabla \cdot \mathbf{J}(\phi')])\end{aligned}$$

Note that when diffusion and advecting is coupled in this way, we do not include the transverse terms in the CTU discretization and limit the time step by

$$\Delta t \leq \frac{1}{\frac{\sum_{i=1}^d |v_i|}{\Delta x} + \frac{2dD}{\Delta x^2}},$$

where  $d$  is the dimensionality and  $D$  is the diffusion coefficient.

## IMEX

For the implicit-explicit advance, we use the CTU discretization to center the divergence at the half time step. We seek the update

$$\frac{\phi^{k+1} - \phi^k}{\Delta t} - \frac{1}{2} [\nabla (D\nabla\phi^k) + \nabla (D\nabla\phi^{k+1})] = -\nabla \cdot \mathbf{v}\phi^{k+1/2},$$

which is a Crank-Nicholson discretization of the diffusion equation with a source term centered on  $k + 1/2$ . We use the CTU discretization (see [CdrCTU](#)) for computing the edge states  $\phi^{k+1/2}$  and then complete the update by solving the corresponding Helmholtz equation

$$\phi^{k+1} - \frac{\Delta t}{2} \nabla (D\nabla\phi^{k+1}) = \phi^k - \Delta t \nabla \cdot \mathbf{v}\phi^{k+1/2} + \frac{\Delta t}{2} \nabla (D\nabla\phi^k)$$

In this case the time step limitation is

$$\Delta t \leq \frac{\Delta x}{\sum_i^d |v_i|}.$$

**Warning:** It is possible to use this module with any implementation of [CdrSolver](#), but the IMEX discretization only makes sense if the hyperbolic term can be centered on  $k + 1/2$ .

If the truncation order of  $\phi^{k+1/2}$  is  $\mathcal{O}(\Delta t^2)$ , the resulting IMEX discretization is globally second order accurate. For the [CdrCTU](#) discretization the edge states are accurate to  $\mathcal{O}(\Delta t \Delta x)$ , so the scheme is globally first order convergent (but with a small error constant).

### 6.1.3 Initial data

#### Default behavior

By default, the initial data for this problem is given by a super-Gaussian blob

$$\phi(\mathbf{x}, t = 0) = \phi_0 \exp\left(-\frac{|\mathbf{x} - \mathbf{x}_0|^4}{2R^4}\right),$$

where  $\phi_0$  is an amplitude,  $\mathbf{x}_0$  is the blob center and  $R$  is the blob radius. These are set by the input options

```
AdvectionDiffusion.blob_amplitude = 1.0
AdvectionDiffusion.blob_radius    = 0.1
AdvectionDiffusion.blob_center   = 0 0 0
```

### Custom value

For a more general way of specifying initial data, `AdvectionDiffusionStepper` has a public member function

```
void setInitialData(const std::function<Real(const RealVect& a_pos)>& a_func) noexcept;
```

## 6.1.4 Velocity field

### Default behavior

The default velocity field for this class is

$$\begin{aligned} v_x &= -r\omega \sin \theta, \\ v_y &= r\omega \cos \theta, \\ v_z &= 0, \end{aligned}$$

where  $r = \sqrt{x^2 + y^2}$ ,  $\tan \theta = \frac{x}{y}$ . I.e. the flow field is a circulation around the Cartesian grid origin.

To adjust the velocity field through  $\omega$ , set

```
AdvectionDiffusion.omega = 1.0
```

### Custom value

For a more general way of setting a user-specified velocity, `AdvectionDiffusionStepper` has a public member function

```
void setVelocity(const std::function<RealVect(const RealVect a_position)>& a_velocity) noexcept;
```

## 6.1.5 Diffusion coefficient

### Default behavior

The default diffusion coefficient for this problem is set to a constant. To adjust it,  $\omega$ , set

```
AdvectionDiffusion.diffco = 1.0
```

to a chosen value.

### Custom value

For a more general way of setting the diffusion coefficient, `AdvectionDiffusionStepper` has a public member function

```
void setDiffusionCoefficient(const std::function<Real(const RealVect a_position)>& a_diffusion) noexcept;
```

## 6.1.6 Boundary conditions

At the EB, this module uses a wall boundary condition (i.e. no flux into or out of the EB). On domain edges (faces in 3D), the user can select between wall boundary conditions or outflow boundary conditions by selecting the corresponding input option for the solver. E.g. for the `CdrCTU` discretization:

```
CdrCTU.bc.x.lo = wall # 'wall' or 'outflow'  
CdrCTU.bc.x.hi = wall # 'wall' or 'outflow'
```

The syntax for the other boundaries are completely analogous.

## 6.1.7 Cell refinement

The cell refinement is based on two criteria:

1. The amplitude of  $\phi$ .
2. The local curvature  $|\nabla\phi| \Delta x / \phi$ .

We refine if the curvature is above some threshold  $\epsilon_1$  *and* the amplitude is above some threshold  $\epsilon_2$ . These can be adjusted through

```
AdvectionDiffusion.refine_curv = 0.25  
AdvectionDiffusion.refine_magn = 1E-2
```

## 6.1.8 Setting up a new problem

To set up a new problem, using the Python setup tools in `$DISCHARGE_HOME/Physics/AdvectionDiffusion` is the simplest way. To see available setup options, run

```
./setup.py --help
```

For example, to set up a new problem in `$DISCHARGE_HOME/MyApplications/MyAdvectionDiffusionProblem` for a coaxial cable geometry, run

```
./setup.py -base_dir=MyApplications -app_name=MyAdvectionDiffusionProblem -geometry=CoaxialCable
```

This will set up a new problem in a coaxial cable geometry (defined in `Geometries/CoaxialCable`).

### 6.1.9 Example programs

Some example programs for this module are given in

- `$DISCHARGE_HOME/Exec/Examples/AdvectionDiffusion/DiagonalFlowNoEB`
- `$DISCHARGE_HOME/Exec/Examples/AdvectionDiffusion/PipeFlow`

### 6.1.10 Verification

Spatial and temporal convergence tests for this module (and thus also the underlying solver implementation) are given in

- `$DISCHARGE_HOME/Exec/Convergence/AdvectionDiffusion/C1`
- `$DISCHARGE_HOME/Exec/Convergence/AdvectionDiffusion/C2`

#### C1: Spatial convergence

A spatial convergence test is given in `$DISCHARGE_HOME/Exec/Convergence/AdvectionDiffusion/C1`. The problem solves for an advected and diffused scalar in a rotational velocity in the presence of an EB:

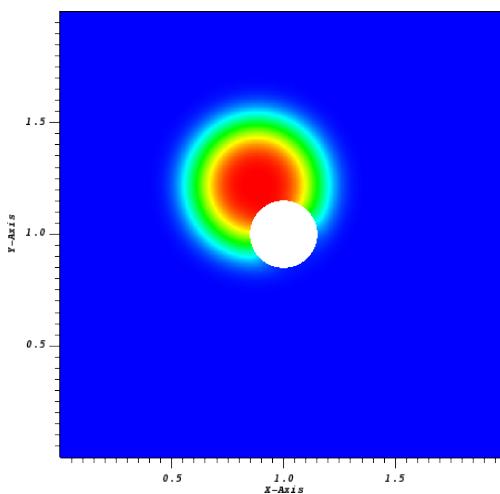


Fig. 6.1.1: Final state on a  $512^2$  uniform grid.

To compute the convergence rate we compute two solutions with grid spacings  $\Delta x$  and  $\Delta x/2$ , and estimate the solution error using the approach in [Spatial convergence](#). Figure Fig. 6.1.2 shows the computed convergence rates with various choice of slope limiters. We find 2nd order convergence in all three norms for sufficiently fine grid when using slope limiters, and first order convergence when limiters are turned off. The reduced convergence rates at coarser grids occur due to 1) insufficient resolution of the initial density profile and 2) under-resolution of the geometry.

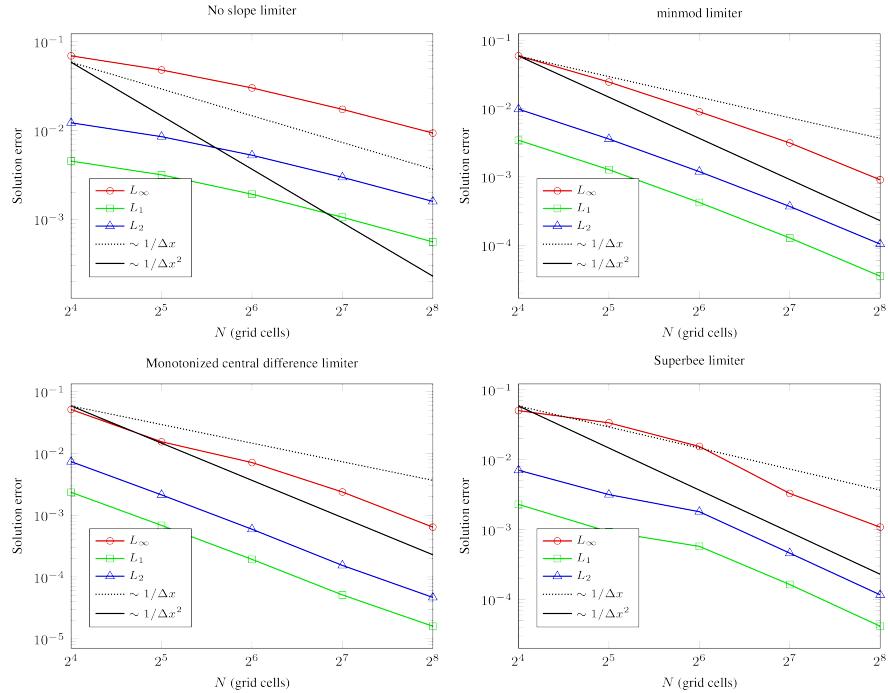


Fig. 6.1.2: Spatial convergence rates with various limiters.

## C2: Temporal convergence

A temporal convergence test is given in `$DISCHARGE_HOME/Exec/Convergence/AdvectionDiffusion/C2`. To compute the temporal convergence rate we compute two solutions using time steps  $\Delta t$  and  $\Delta t/2$ , and estimate the solution error using the approach in [Temporal convergence](#). Figure Fig. 6.1.3 shows the computed convergence rates for the second order Runge-Kutta and the IMEX discretization. As expected, we find 2nd order convergence for Heun's method and first order convergence for the IMEX discretization.

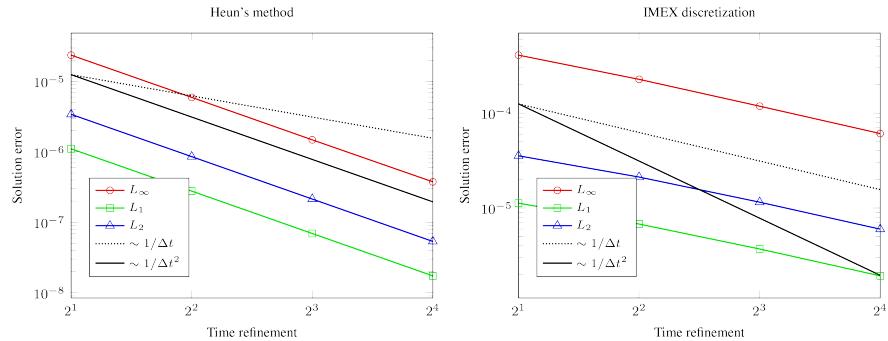


Fig. 6.1.3: Temporal convergence rates.

## 6.2 Brownian walker

## 6.3 Electrostatics model

The electrostatics model solves

$$\nabla \cdot (\epsilon_r \nabla \Phi) = -\frac{\rho}{\epsilon_0}$$

subject to the constraints and boundary conditions given in *FieldSolver*. The implementation is by a class

```
class FieldStepper : public TimeStepper
```

and is found in \$DISCHARGE\_HOME/Physics/Electrostatics. The C++ API is found [here](#).

### 6.3.1 Solvers

This module uses the following solvers:

1. *FieldSolver*.

### 6.3.2 Time stepping

**FieldStepper** is a class with only stationary solves. This is enforced by making the `TimeStepper::advance` method throw an error, while the field solve itself is done in the initialization procedure in `TimeStepper::postInitialize`.

---

**Important:** To run the solver, one must set `Driver.max_steps = 0`.

---

### 6.3.3 Setting the space charge

#### Default behavior

By default, the initial space charge for this problem is given by Gaussian

$$\rho(\mathbf{x}) = \rho_0 \exp\left(-\frac{|\mathbf{x} - \mathbf{x}_0|^2}{2R^2}\right),$$

where  $\rho_0$  is an amplitude,  $\mathbf{x}_0$  is the center and  $R$  is the Gaussian radius. These are set by the input options

```
FieldStepper.init_rho    = 1.0
FieldStepper.rho_center = 0 0 0
FieldStepper.rho_radius = 1.0
```

### Custom value

For a more general way of specifying the space charge, FieldStepper has a public member function

```
void setRho(const std::function<Real(const RealVect& a_pos)>& a_rho, const phase::which_phase a_phase) noexcept;
```

### 6.3.4 Setting the surface charge

By default, the initial surface charge is set to a constant. This constant is given by

```
FieldStepper.init_sigma = 1.0
```

### Custom value

For a more general way of specifying the surface charge, FieldStepper has a public member function

```
void setSigma(const std::function<Real(const RealVect& a_pos)>& a_sigma) noexcept;
```

### 6.3.5 Setting up a new problem

To set up a new problem, using the Python setup tools in \$DISCHARGE\_HOME/Physics/Electrostatics is the simplest way. To see available setup options, run

```
./setup.py --help
```

For example, to set up a new problem in \$DISCHARGE\_HOME/MyApplications/MyElectrostaticsProblem for a cylinder geometry, run

```
./setup.py -base_dir=MyApplications -app_name=MyElectrostaticsProblem -geometry=Cylinder
```

This will set up a new problem in a cylinder geometry (defined in Geometries/Cylinder).

### 6.3.6 Example programs

Some example programs for this module are given in

- \$DISCHARGE\_HOME/Exec/Examples/Electrostatics/Armadillo
- \$DISCHARGE\_HOME/Exec/Examples/Electrostatics/Mechshaft
- \$DISCHARGE\_HOME/Exec/Examples/Electrostatics/ProfiledSurface

### 6.3.7 Verification

Spatial convergence tests for this module (and consequently the underlying numerical discretization) are given in

- \$DISCHARGE\_HOME/Exec/Convergence/Electrostatics/C1
- \$DISCHARGE\_HOME/Exec/Convergence/Electrostatics/C2
- \$DISCHARGE\_HOME/Exec/Convergence/Electrostatics/C3

All tests operate by computing solutions on grids with resolutions  $\Delta x$  and  $\Delta x/2$  and then computing the solution error using the approach outlined in [Spatial convergence](#).

## C1: Coaxial cable

`$DISCHARGE_HOME/Exec/Convergence/Electrostatics/C1` is a spatial convergence test in a coaxial cable geometry. Figure Fig. 6.3.1 shows the field distribution. Note that there is a dielectric embedded between the two cylindrical shells.

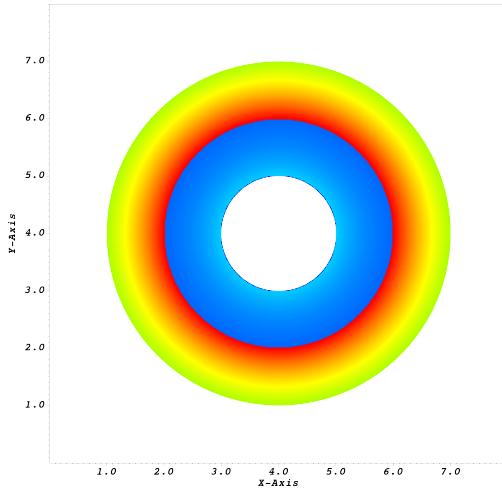


Fig. 6.3.1: Field distribution for a coaxial cable geometry on a  $512^2$  grid.

The computed convergence rates are given in Fig. 6.3.2. We find second order convergence in all three norms.

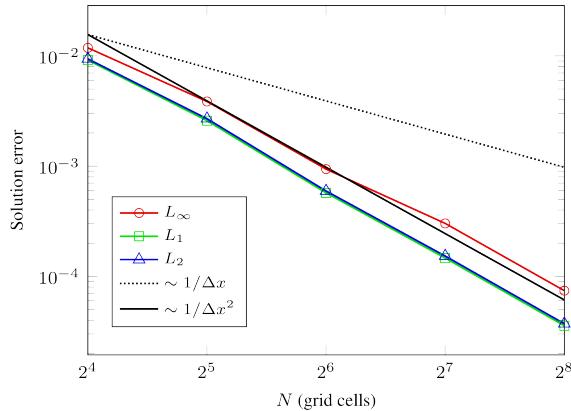


Fig. 6.3.2: Spatial convergence rates for two-dimensional coaxial cable geometry.

## C2: Profiled surface

`$DISCHARGE_HOME/Exec/Convergence/Electrostatics/C2` is a 2D spatial convergence test for an electrode and a dielectric slab with surface profiles. Figure Fig. 6.3.3 shows the field distribution.

The computed convergence rates are given in Fig. 6.3.4. We find second order convergence in all three norms.

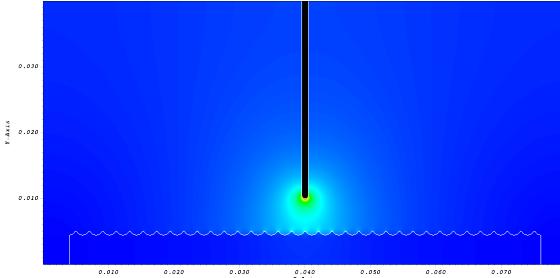


Fig. 6.3.3: Field distribution for a profiled surface geometry on a  $2048^2$  grid.

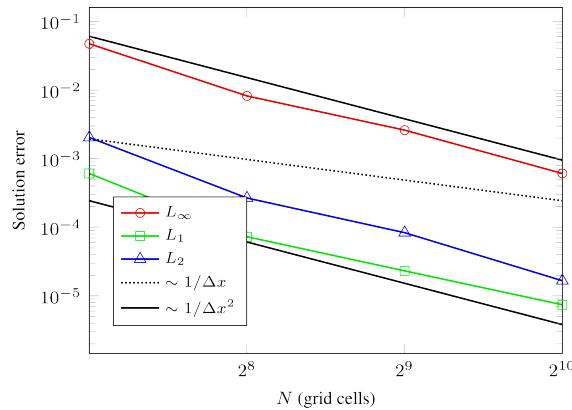


Fig. 6.3.4: Spatial convergence rates for two-dimensional dielectric surface profile.

### C3: Dielectric shaft

\$DISCHARGE\_HOME/Exec/Convergence/Electrostatics/C3 is a 3D spatial convergence test for a dielectric shaft perpendicular to the background field. Figure Fig. 6.3.5 shows the field distribution for a  $256^3$  grid.

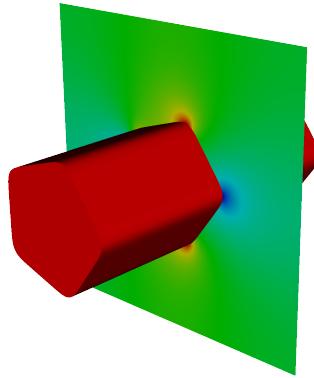


Fig. 6.3.5: Field distribution for a profiled surface geometry on a  $256^3$  grid.

The computed convergence rates are given in Fig. 6.3.6. We find second order convergence in  $L_1$  and  $L_2$  on all grids, and find second order convergence in the max-norm on sufficiently fine grids. On coarser grids, the reduced convergence rate in the max-norm is probably due to under-resolution of the geometry.

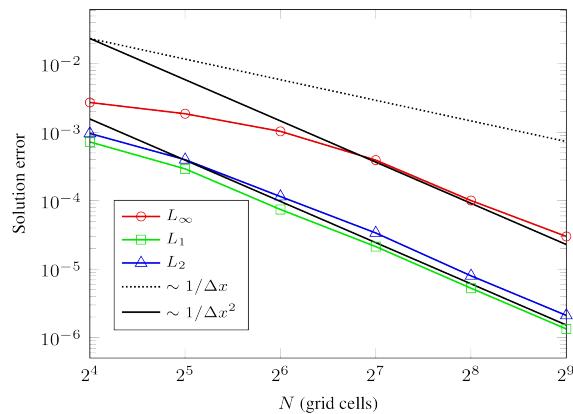


Fig. 6.3.6: Spatial convergence rates.

## 6.4 Geometry

## 6.5 Mesh ODE

## 6.6 Radiative transfer

## 6.7 Tracer particle model



## 7.1 Introduction

In this tutorial we will set up and examine the code for simulating advection-diffusion problems with AMR and regridding functionality. If you want to examine the final code, it is given in `$DISCHARGE_HOME/Physics/AdvectionDiffusion`.

## 7.2 Creating a geometry

## 7.3 Setting up a TimeStepper



## UTILITIES

### 8.1 Lookup tables

#### 8.1.1 LookupTable1D

LookupTable1D is a class for looking up and interpolation data stored in a row-column format. It is used in order to easily retrieve input data that can be stored in table formats.

---

**Important:** LookupTable1D is used for data lookup *in one independent variable*. It does not support higher-dimensional data interpolation.

---

The class is templated as

```
template <int N>
class LookupTable1D
```

where the template parameter N indicates the number of columns in the data holders. Internally, the data is stored as an `std::vector<std::array<Real, N>>` where the vector entries are rows and the `std::array<Real, N>` are data in each row. Thus, a table `LookupTable1D<2>` always has two columns.

The `LookupTable1D` is used on regularly spaced data (for performance reasons). Although the user can fill irregularly spaced data into `LookupTable1D`, the class has routines for making that data regularly spaced and sorted along of its columns. Usage of `LookupTable1D` will therefore consist of the following:

1. Add data rows into the table.
2. Swap columns if necessary.
3. Restrict data ranges if necessary.
4. Sort the table along of it's columns, smallest to largest.
5. Regularize the table with a specified number of grid points.
6. *Retrieve data.*

The steps for these processes are explained in detail below.

## 8.1.2 Inserting data

To add data to the table, one will use the member function

```
template <typename... Ts>
inline
void addEntry(const Ts&... x);
```

where the parameter pack must have N entries. For example, to add two rows of data to a table:

```
LookupTable1D<3> myTable;

myTable.add(4.0, 5.0, 6.0);
myTable.add(1.0, 2.0, 3.0);
```

This will insert two new rows at the end up the table.

---

**Important:** Input data points do not need to be uniformly spaced, or even sorted. Users will insert rows one by one; `LookupTable1D` has functions for sorting and regularizing the table.

---

## 8.1.3 Restricting ranges

To restrict the data range, call

```
void setRange(const Real a_min, const Real a_max, const int a_independentVariable)
```

where `a_min` and `a_max` are the permissible ranges for data in the input column (`a_independentVariable`). Data outside these ranges are discarded from the table.

## 8.1.4 Independent variable

To select an independent variable, the user will select one of the columns and sort the data along that column. The C++ code for this is

```
template <int N>
void LookupTable1D<N>::sort(const int a_independentVariable);
```

where the input integer indicates the column (i.e., independent variable) used for sorting. Note that the sorting is *always from smallest to largest value*. Thus, if one has two rows

```
1.0 5.0 6.0
2.0 2.0 3.0
```

and one calls `LookupTable1D<N>::sort(1)` the final table becomes

```
2.0 2.0 3.0
1.0 5.0 6.0
```

Note that the second column now becomes the independent variable.

### 8.1.5 Swapping columns

Columns can be swapped by calling `LookupTable1D<N>::swap(int, int)`, which will swap two of the columns. For example if the original data is

2.0	2.0	3.0
1.0	5.0	6.0

and one calls `swap(1,2)` the final table becomes

2.0	3.0	2.0
1.0	6.0	5.0

Note that swapping two columns destroys the sorting and one will need to set the independent variable again afterwards.

### 8.1.6 Regularize table

To regularize the table the user must first determine if the grid points should be uniformly spaced or exponentially spaced in the independent variable.

#### Setting grid point spacing

The user can set the spacing by calling

```
void setTableSpacing(const TableSpacing a_spacing);
```

where `TableSpacing::Uniform` and `TableSpacing::Exponential` are supported. For example, to use uniformly or exponentially spaced grid points:

```
LookupTable1D<2> myTable;
myTable.setTableSpacing(TableSpacing::Uniform);      // For uniformly spaced points
myTable.setTableSpacing(TableSpacing::Exponential); // For exponentially spaced points
```

#### Uniform spacing

With uniform spacing, grid points in the table are spaced as

$$x_i = x_{\min} + \frac{i}{N-1} (x_{\max} - x_{\min}), \quad i \in [0, N-1]$$

where  $x_{\min}$  and  $x_{\max}$  is the minimum and maximum data range for the independent variable (i.e., column).

#### Exponential spacing

If grid points are exponentially spaced then

$$x_i = x_{\min} \left( \frac{x_{\max}}{x_{\min}} \right)^{\frac{i-1}{N}}, \quad i \in [0, N-1].$$

### Regularizing table

```
void regularize(const int a_numRows)
```

which will make the table into a regularly spaced table with `a_numRows` rows. `LookupTable1D` will always use piecewise linear interpolation when regularizing the table. Specifying a number of rows that is smaller/larger than the original number of rows will downsample/upsample the table.

---

**Important:** When regularizing a table through `regularize`, the original data table is destroyed.

---

### 8.1.7 Retrieving data

To retrieve data from one of the columns, one can fetch either a specific value in a row, or the entire row.

```
// For fetching column K
template<int K>
Real getEntry(const Real a_x);

// For fetching the entire row
std::array<Real, N> getData(const Real a_x);
```

In the above, the template parameter `K` is the column to retrieve and `a_x` is the value of the independent variable.

---

**Important:** `LookupTable1D` will *always* use piecewise linear interpolation between two grid points.

---

For example, consider table regularized and sorted along the middle column:

```
2.0  1.0  3.0
1.0  3.0  6.0
1.0  5.0  4.0
```

To retrieve an interpolated value for  $x=2.0$  in the third column we call

```
LookupTable1D<3> myTable,
const Real val = myTable.getEntry<2>(2.0);
```

which will return a value of 4.5 (linearly interpolated).

### 8.1.8 Viewing tables

For debugging purposes, `LookupTable1D` can write the internal data to an output stream or a file through the two member functions:

```
void dumpTable(std::ostream& a_outputStream = std::cout) const;
void dumpTable(const std::string a_fileName) const;
```

For example:

```
LookupTable1D<10> myTable;

// Dump table to terminal window
myTable.dumpTable();

// Dump table to file.
myTable.dumpTable("myTable.dat");
```

## 8.1.9 Parsing data

Routines for reading column data into `LookupTable` are defined under the `DataParser` namespace. For example, for reading two columns into a lookup table:

```
// Read all rows
LookupTable1D<2>
simpleFileReadASCII(const std::string      a_fileName,
                    const int            a_xColumn    = 0,
                    const int            a_yColumn    = 1,
                    const std::vector<char> a_ignoreChars = {'#', '/'});
```

```
// Specify rows where to start and stop reading data
LookupTable1D<2>
fractionalFileReadASCII(const std::string      a_fileName,
                        const std::string    a_startRead,
                        const std::string    a_stopRead,
                        const int            a_xColumn    = 0,
                        const int            a_yColumn    = 1,
                        const std::vector<char> a_ignoreChars = {'#', '/'});
```

---

**Tip:** The `DataParser` C++ API is found at <https://chombo-discharge.github.io/chombo-discharge/doxygen/html/namespacedataparser.html>.

---

## 8.2 Random numbers

`Random` is a static class for generating pseudo-random numbers, and exist so that all random number operations can be aggregated into a single class. Internally, `Random` use a Mersenne-Twister random number generation.

To use the `Random` class, simply include `<CD_Random.H>`, e.g.

```
#include <CD_Random.H>
```

See the [Random API](#) for further details.

### 8.2.1 Drawing random numbers

The general routine for drawing a random number is

```
template<typename T>
Real get(T& a_distribution);
```

which is for example used as follows:

```
std::uniform_real_distribution<Real> dist(0.0, 100.0);
const Real randomNumber = Random::get(dist);
```

Pre-defined distributions exist for performing the following operations:

1. For drawing a real number from a uniform distribution between 0 and 1, use `Real Random::getUniformReal01()`.
2. For drawing a real number from a uniform distribution between -1 and 1, use `Real Random::getUniformReal11()`.
3. For drawing a real number from a normal distribution centered at 0 and with a variance of 1, use `Real Random::getNormal01()`.

4. For drawing an integer from a Poisson distribution with a specified mean, use `T Random::getPoisson<T>(const Real a_mean)` where `T` is an integer type.
5. For drawing a random direction in space, use `RealVect Random::getDirection()`. The implementation uses the Marsaglia algorithm for drawing coordinates uniformly distributed over the unit sphere.

## 8.2.2 Setting the seed

By default, the random number generator is seeded with the MPI rank, which we do to avoid having the MPI ranks producing the same number sequences. `Driver` (see [Driver](#)) will seed the random number generator, and user can override the seed by setting `Random.seed = <number>` in the input script. If the user sets `<number> < 0` then a random seed will be produced based on the elapsed CPU clock time. If running with MPI, this seed is obtained by only one of the MPI ranks, and this seed is then broadcast to all the other ranks. The other ranks will then increment the seed by their own MPI rank number so that each MPI rank gets a unique seed.

## 8.3 Least squares

Least squares routines are useful for reconstructing a local polynomial in the vicinity of the embedded boundary. chombo-discharge supports the expansion of such solutions in a fairly general way. These routines are often needed because the embedded boundary introduces grid pathologies which are difficult to meet with pure finite differencing, see e.g. [Ghost cell interpolation](#).

### 8.3.1 Polynomial expansion

Given some position  $\mathbf{x}$ , we expand the solution around a grid point  $\mathbf{x}_i$  to some order  $Q$ :

$$f(\mathbf{x}_i) = f(\mathbf{x}_i) + \nabla f(\mathbf{x}) \cdot (\mathbf{x}_i - \mathbf{x}) + \dots + \mathcal{O}(\Delta x^{Q+1}).$$

Using multi-index notation this is written as

$$f(\mathbf{x}_i) = \sum_{|\alpha| \leq Q} \frac{(\mathbf{x}_i - \mathbf{x})^\alpha}{\alpha!} (\partial^\alpha f)(\mathbf{x}) + \mathcal{O}(\Delta x^{Q+1}),$$

where  $\alpha$  is a multi-index. For a specified order  $Q$  there is also a specified number of unknowns. E.g. in two dimensions with  $Q = 1$  the unknowns are  $f(\mathbf{x})$ ,  $\partial_x f(\mathbf{x})$ , and  $\partial_y f(\mathbf{x})$ .

By expanding the solution around more grid points, we can formulate an over-determined system of equations  $i = 1, 2, 3, \dots, N$  that allows us to compute the coefficients (i.e., unknowns) in the Taylor expansion. By using lexicographical ordering of the multi-indices, it is straightforward to write the system out explicitly. E.g., for  $Q = 1$  in two dimensions:

$$\begin{pmatrix} 1 & (x_1 - x) & (y - y_1) \\ 1 & (x_2 - x) & (y - y_2) \\ \vdots & \ddots & \vdots \\ 1 & (x_N - x) & (y - y_N) \end{pmatrix} \begin{pmatrix} f \\ \partial_x f \\ \partial_y f \end{pmatrix}(\mathbf{x}) = \begin{pmatrix} f(\mathbf{x}_1) \\ f(\mathbf{x}_2) \\ \vdots \\ f(\mathbf{x}_N) \end{pmatrix}$$

In general, we represent this system as

$$\mathbf{A}\mathbf{u} = \mathbf{b},$$

where unknowns in  $\mathbf{u}$  are the coefficients in the Taylor series, ordered lexicographically (encoded with a Chombo `IntVect`).  $\mathbf{b}$  is a column vector of grid point values representing the local expansion around each grid point, and  $\mathbf{A}$  is the expansion matrix.

---

**Note:** chombo-discharge is not restricted to second order – it implements the above expansion to any order.

---

### 8.3.2 Neighborhood algorithm

To avoid reaching over or around embedded boundaries, the neighborhood algorithms only includes grid cells which can be reached by a *monotone* path. This path is defined by walking through neighboring grid cells without changing direction, see e.g. Fig. 8.3.1.

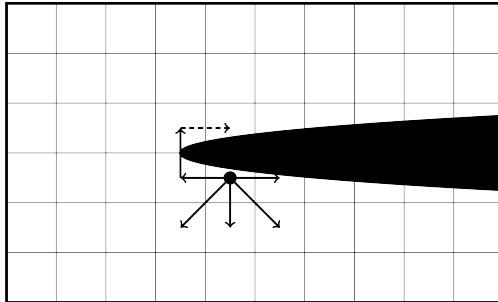


Fig. 8.3.1: Neighborhood algorithm, only reaching into grid cells that can be reached by a monotone path. The grid cell at the end of the dashed line is excluded (even though it is a neighbor to the starting grid cell) since the path circulates the embedded boundary.

### 8.3.3 Weighted equations

Weights can also be added to each equation, e.g. to ensure that close grid points are more important than remote ones:

$$\begin{pmatrix} w_1 & w_1(x_1 - x) & w_N(y - y_1) \\ w_2 & w_2(x_2 - x) & w_N(y - y_2) \\ \vdots & \ddots & \vdots \\ w_N & w_N(x_N - x) & w_N(y - y_N) \end{pmatrix} \begin{pmatrix} f \\ \partial_x f \\ \partial_y f \end{pmatrix}(\mathbf{x}) = \begin{pmatrix} w_1 f(\mathbf{x}_1) \\ w_2 f(\mathbf{x}_2) \\ \vdots \\ w_N f(\mathbf{x}_N) \end{pmatrix}$$

For weighted least squares the system is represented as

$$\mathbf{W}\mathbf{A}\mathbf{u} = \mathbf{W}\mathbf{b},$$

where  $\mathbf{W}$  are the weights. Typically, the weights are some power of the Euclidean distance

$$w_i = \frac{1}{|\mathbf{x}_i - \mathbf{x}|^p}.$$

### 8.3.4 Pseudo-inverse

An over-determined system does not have a unique solution, and so to obtain the solution to  $\mathbf{u}$  for the system  $\mathbf{W}\mathbf{A}\mathbf{u} = \mathbf{W}\mathbf{b}$  we use ordinary least squares. The solution is then

$$\mathbf{u} = [(\mathbf{W}\mathbf{A})^+ \mathbf{W}] \mathbf{b},$$

where  $(\mathbf{W}\mathbf{A})^+$  is the Moore-Penrose inverse of  $\mathbf{W}\mathbf{A}$ . The pseudo-inverse is computed using the singular value decomposition (SVD) routines in LAPACK.

Note that the column vector  $\mathbf{b}$  consist of known values (grid points), and the result  $[(\mathbf{W}\mathbf{A})^+ \mathbf{W}]$  can therefore be represented as a stencil. For example, in two dimensions with  $Q = 1$  we find

$$\begin{pmatrix} f \\ \partial_x f \\ \partial_y f \end{pmatrix}(\mathbf{x}) = \begin{pmatrix} C_{11} & C_{12} & \dots & C_{1N} \\ C_{21} & C_{22} & \ddots & C_{2N} \\ C_{31} & C_{32} & \dots & C_{3N} \end{pmatrix} \begin{pmatrix} f(\mathbf{x}_1) \\ f(\mathbf{x}_2) \\ \vdots \\ f(\mathbf{x}_N) \end{pmatrix}$$

### 8.3.5 Pruning equations

If some terms in the Taylor series are specified, one can prune equations from the systems. E.g. if  $f(\mathbf{x})$  happens to be known, the system of equations can be rewritten as

$$\begin{pmatrix} w_1(x_1 - x) & w_N(y - y_1) \\ w_2(x_2 - x) & w_N(y - y_2) \\ \vdots & \vdots \\ w_N(x_N - x) & w_N(y - y_N) \end{pmatrix} \begin{pmatrix} \partial_x f \\ \partial_y f \end{pmatrix}(\mathbf{x}) = \begin{pmatrix} w_1 f(\mathbf{x}_1) - w_1 f(\mathbf{x}) \\ w_2 f(\mathbf{x}_1) - w_2 f(\mathbf{x}) \\ \vdots \\ w_N f(\mathbf{x}_1) - w_N f(\mathbf{x}) \end{pmatrix}$$

Again, following the benefits of lexicographical ordering it is straightforward to write an arbitrary order system of equations in the form  $\mathbf{W}\mathbf{A}\mathbf{u} = \mathbf{W}\mathbf{b}$ , even with an arbitrary number of terms pruned from the Taylor series. However, note that the result of the least squares solve is now in the format

$$\begin{pmatrix} \partial_x f \\ \partial_y f \end{pmatrix}(\mathbf{x}) = \begin{pmatrix} C_{11} & C_{12} & \dots & C_{1N} \\ C_{21} & C_{22} & \dots & C_{2N} \end{pmatrix} \begin{pmatrix} f(\mathbf{x}_1) - f(\mathbf{x}) \\ f(\mathbf{x}_2) - f(\mathbf{x}) \\ \vdots \\ f(\mathbf{x}_N) - f(\mathbf{x}) \end{pmatrix}.$$

Thus, when evaluating the terms in the polynomial expansion the user must account for the modified right-hand side due to equation pruning. The modification to the right-hand side also depends on which terms are pruned from the expansion.

### 8.3.6 Source code

The source code for the least squares routines is found in `$DISCHARGE_HOME/Source/Utilities/CD_LeastSquares.*`, and the neighborhood algorithms are found in `$DISCHARGE_HOME/Source/Utilities/CD_VofUtils.*`.

## **CONTRIBUTING**

### **9.1 Contributions**

We welcome feedback, bug reports, and contributions to `chombo-discharge`. If you have feedback, questions, or general types of queries, use the issue tracker or discussion tab at <https://github.com/chombo-discharge>.

#### **9.1.1 Pull requests**

If you want to submit code to `chombo-discharge`, use the pull request system at <https://github.com/chombo-discharge>. When submitting a pull request, mark it as *draft* if you believe the pull request is not yet ready for merging. Note that when a pull request is submitted, you are asked to provide a brief summary of the changes that are made.

---

**Important:** Always squash your git commits into a single commit for the PR as a whole.

---

It will be beneficial to first compile the changes locally and run e.g. the test suite, see [Code testing](#) in debug mode. Also make sure to run e.g. `valgrind` to spot memory leaks. In serial, running valgrind can be done by calling valgrind with `--leak-check=yes`, e.g.:

```
valgrind --leak-check=full --track-origins=yes <my_executable> <my_inputs_file>
```

With MPI this looks like

```
mpirun -np 12 valgrind --leak-check=full --track-origins=yes <my_executable> <my_inputs_file>
```

#### **9.1.2 Bug reports**

`chombo-discharge` is probably not bug-free. If encountering unexpected behavior, do not hesitate to use the issue tracker at <https://github.com/chombo-discharge/chombo-discharge/issues>.

### 9.1.3 Continuous integration

chombo-discharge uses continuous integration (CI) with GitHub actions for:

- Running the test suite (see [Code testing](#)).
- Building HTML, PDF, and doxygen documentation.
- Ensuring correct code format.

When submitting pull request for review, the above tests will start. In general, all the above tests should pass before merging the pull request into the main branch. Note that all actions on GitHub run in debug mode (DEBUG=TRUE) for turning on assertions.

Upon merging with the main branch, the documentation is again rebuilt and deployed to GitHub pages (see [GitHub pages](#)). This ensures that the online documentation (HTML, PDF, and doxygen) is always up-to-date with the latest chombo-discharge release.

## 9.2 Code standard

When submitting new code to chombo-discharge, the following guidelines below show be followed.

### 9.2.1 C++ standard

We are currently at C++14.

### 9.2.2 Namespace

All code in chombo-discharge is embedded in a namespace ChomboDischarge. Embedding into a namespace is done by including header file CD\_NamespaceHeader.H that contain the necessary definitions. This is done by including after any other file includes. In addition, files must include CD\_NamespaceFooter.H at the end.

### 9.2.3 File names

Each file should contain only one class definition, and the file name must be name of the class prepended by CD\_. For example, if you are contributing a class MyClass the header files for this class must be named CD\_MyClass.H and the implementation file must be named CD\_MyClass.cpp. If your code contains templates or inlined functions, these should be defined in files appended by Implem, e.g. CD\_MyClassImplem.H.

### 9.2.4 File headers

Each file shall begin with the following note:

```
/* chombo-discharge
 * Copyright © <Copyright holder 1>
 * Copyright © <Copyright holder 2>
 * Please refer to Copyright.txt and LICENSE in the chombo-discharge root directory.
 */
```

where <Copyright holder 1>, <Copyright holder 2>, etc. are replaced by the copyright holder.

This file header shall be followed by a brief Doxygen documentation, containing at least @file, @brief, and @author. In addition, include header guards identical to the filename, replacing dots by underscores. I.e. for a file CD\_MyClass.H the header guard shall read

```
#ifndef CD_MyClass_H
#define CD_MyClass_H

#endif
```

## 9.2.5 File inclusions

File inclusions should use the follow standards for C++, Chombo, and chombo-discharge

1. C++. Use brackets, e.g. `#include <memory>`.
2. Chombo. Use brackets, e.g. `#include <LevelData.H>`.
3. chombo-discharge. Use brackets and the file name, e.g. `#include <CD_FieldSolver.H>`.

## 9.2.6 Example format

Here is a complete example of a header file in chombo-discharge:

```
/* chombo-discharge
 * Copyright © <Copyright holder 1>
 * Copyright © <Copyright holder 2>
 * Please refer to Copyright.txt and LICENSE in the chombo-discharge root directory.
 */

/*!
 @file CD_MyClass.H
 @brief This file contains ...
 @author Author name
 */

#ifndef CD_MyClass_H
#define CD_MyClass_H

// Std includes (e.g.)
#include <memory>

// Chombo includes (e.g.)
#include <LevelData.H>

// Our includes (e.g.)
#include <CD_EBAMRData.H>
#include <CD_NamespaceHeader.H>

/*!
 @brief This class does the following: ....
 */
class MyClass
{
public:

//...
};

#include <CD_NamespaceFooter.H>

#include <CD_MyClassImpl.H> // Inline and template code included at the end.

#endif
```

## 9.2.7 Code syntax

We use the following syntax:

1. Class names, structs, and namespaces should be in Pascal case where the first letter of every word is capitalized.  
E.g. a class is called `MyClass`.
2. Class functions should be in Camel case where the first letter of every word but the first is capitalized. E.g. functions should be named `MyClass::myFunction`
3. Variables should use Pascal-case, with the following requirements:
  - Arguments to functions should be prepended by `a_`. For example `MyClass::myFunction(int a_inputVariable)`.
  - Class members should always be prepended by `m_`, indicating it is a member of a class. For example `MyClass::m_functionMember`.
  - Static variables are prepended by `s_`. For example `MyClass::s_staticFunctionMember`.
  - Global variables are prepended by `//`.

## 9.2.8 Options files

Options files are named using the same convention as class files, e.g. `CD_MyClass.options`. It is the responsibility of `MyClass` to parse these variables correctly.

Everything in the options file should be lower-case, with the exception of the class name which should follow the class name syntax. If you need a separator for the variable, use an underscore `_`. For variables that should be grouped under a common block, one may use a dot `.` for grouping them. For a class `MyClass` and options file might look something like

```
MyClass.input_variable = 1.0
MyClass.bc.x.lo      = dirichlet 1.0
```

## 9.2.9 clang-format

We use `clang-format` for formatting the source code. Before opening a pull request for review, navigate to `$DISCHARGE_HOME` and format the code using

```
find Source Physics Geometries Exec \C -name "*.H" -o -name "*.cpp" \O -exec clang-format -i {} +
```

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