Master's Thesis

Portfolio optimization with KOSPI200 and House price index

Minky Cho

Department of Mathematical Sciences

Graduate School of UNIST

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Department of Mathematical Sciences

Graduate School of UNIST

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Minky Cho

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Approved by

Advisor
Jin Hyuk Choi

ULSAN NATIONAL INSTITUTE OF SCIENCE AND TECHNOLOGY

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This certifies that the thesis/dissertation of Minky Cho is approved.

12. 24. 2019

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Advisor: Jin Hyuk Choi

signature

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Chang Hyeong Lee: Thesis Committee Member #1

signature

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Pilwon Kim: Thesis Committee Member #2

signature

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typed name: Thesis Committee Member #3

signature

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typed name: Thesis Committee Member #4;

three signatures total in case of masters

Abstract

In this paper, we will investigate optimal investment and consumption strategies in the market with CRRA utility function. In this market, the investor chooses a portfolio that consists of one bond (with zero interest rates), one illiquid risky asset (having transaction costs), and one liquid risky asset (having no transaction costs). Using shadow price which is the virtual price between the bid and ask price, we can derive the optimal investment and consumption strategies.

From these results we obtained, we will apply to the market having stock and residential real estate. For the analyze, we chose KOSPI200 (the index consists of 200 big companies of the Stock Market Division in Korea) for the stock and house price index (the price of the residential real estate in Korea) for the real estate. With KOSPI200 and house price index data, we obtained the optimal investment and consumption strategies and also checked the effect of the transaction costs on two risky assets and consumption.

Contents

I Introduction

In one of his seminal papers, Merton solved an optimal investment and consumption problem in the market where the investor allocate his wealth between a liquid asset(stock) and a risk-free asset to maximize his utility function. This problem is called Merton's portfolio problem. The objective of this problem is maximizing the following expectation.

$$\mathbb{E}\left[\int_0^T e^{-\delta t}u(c_t) + \epsilon^\gamma e^{-\delta T}u(W_T)\right] \quad \text{where} \quad T \in [0,\infty]$$

Here, δ is discount rate, c_t is consumption rate at time t, W_t is his wealth at time t, ϵ is desired level of bequest. And utility function u is of the constant relative risk aversion (CRRA) form:

$$u(c) = \frac{c^p}{p}, \quad c \geq 0 \;\; \text{and} \;\; U(0) = \begin{cases} 0, & \text{p} > 0, \\ -\infty & \text{p} \leq 0. \end{cases}$$

In this frictionless market where there are one risk-free asset and one risky asset with CRRA utility function, Merton showed that the optimal allocation strategy is investing a constant proportion of wealth in the risky asset. The line whose slope is a proportion derived from Merton is called Merton's line. When the value process is finite, Merton's portfolio problem with an infinite horizon is reduced to maximizing the following expectation.

$$\mathbb{E}\int_0^\infty e^{-\delta t}u(c_t)dt$$

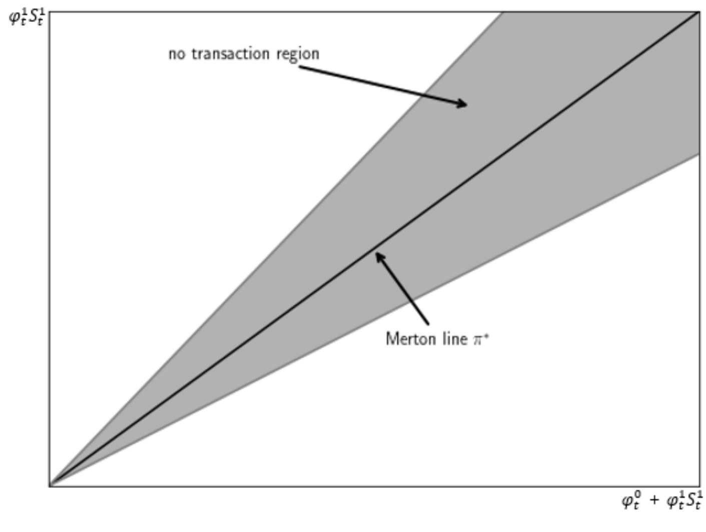


Figure 1: Optimal strategy in frictionless market and in market with transaction costs

Figure 1 shows the optimal strategy in the frictionless market and in the market with transaction costs. The x-axis denotes the sum of the risky and risk-free asset value and the y-axis denotes risky asset. So, the slope means the proportion of risky asset. In the market with transaction costs, the optimal strategy is defined by two straight lines. Merton's line lies between these two straight lines. Inside the two straight lines, the optimal strategy for the investor is not trading. Outside of the no transaction region, the optimal strategy for the investor is adjusting his portfolio to the nearest line so that the proportion could be in the no transaction region.

Many kinds of researches have been done by many researchers to remove the frictionless market condition. That is, many researchers have tried to consider transaction costs. Also, there were some trials considering not one risky asset but two risky assets. So in this paper, we consider the model with one risky asset with transaction costs and one risky asset having no transaction costs.

[1], [2] solved this problem with one bond and two risky assets. One is the liquid asset that has no transaction costs and the other is an illiquid asset that has proportional transaction costs. To solve this problem, [1] used the shadow price which is the virtual price between the bid and ask prices of the original market. This shadow price approach makes the original optimization problem solved in the frictionless market. So using the shadow price approach, he derived the optimal investment and consumption strategies by solving the optimization problem in the frictionless market.

[1] and [2] heuristically showed that the HJB equation can be transformed into a free boundary problem with a first order differential equation. The main difference between [1] and [2] is that [2] just analyzed the HJB equation but [1] took the dual approach. But since they are actually solving the same problem, we only focus on the [1].

In this paper, using the optimal investment and consumption strategies obtained in [1], we applied the results to the real data. We considered two risky assets, stock and residential real estate assets which represents liquid and illiquid asset respectively. the stock also has the transaction cost but many papers consider it as a liquid asset because the transaction cost of it is small. To analyze the market with stock and real estate, we used two monthly data associated with them. One is the KOSPI200 (index consists of 200 big companies of the Stock Market Division in Korea) and the other is the house price index(the price of the residential real estate in Korea) from January 2004 to August 2019.

To apply the result, we have to estimate the parameters $\mu_1, \mu_2, \sigma_1, \sigma_2, \rho, \delta, p, \lambda$. Using the KOSPI200 and house price index and other related data, we can estimate $\mu_1, \mu_2, \sigma_1, \sigma_2, \rho, \lambda$. The way how we estimated will be discussed in section 5. But parameters δ, λ is hard to estimate using these data. So we followed [3], [4] to estimate δ, λ respectively.

For estimating the parameter δ, we followed the method used in [3]. In [3], they estimated the discount factor β as $\beta = 1/(1+\tilde{r})$ where $\tilde{r}$ is real interest rate which is the sum of nominal interest rate and expected inflation rate. β and δ have the following relation:

$$e^{-\delta} = \beta = 1/(1+\tilde{r})$$

So using this, we can estimate the δ.

For estimating the parameter p, we just used the value estimated in [4]. In [4], they estimated the risk aversion coefficients of real estate asset owners based on the theory of consumption based capital asset pricing model(CCAPM). For the CRRA utility function, estimated relative risk aversion coefficients were 3.85 in 2012. In other words, since $(1-p) = 3.85$, we used the value $p = -2.85$.

The remainder of the paper is organized as follows. In Section 2, we will discuss the model and formulate the problem. And In section 3, we will derive the optimal investment and consumption strategies in the frictionless market with two risky assets and briefly review the paper [1]. In the next section, Asymptotic ally optimal trading and consumption strategies obtained from [1] will be introduced without the proof. And finally, the optimal strategies using KOSPI200 and house price index data will be shown.

II Model

2.1 The market

We will consider the market with one bond having zero interest rates and two risky assets. A frequently used model for modeling risky asset prices is the geometric Brownian motion. If $S^{(i)}$ denotes the price process of each risky asset, then $S^{(i)}$ follows a geometric Brownian motion if it satisfies the following stochastic differential equation.

$$dS^{(i)} = S^{(i)}\left(\mu_i dt + \sigma_i dB_t^{(i)}\right), \quad S_0^{(i)} > 0, \quad i = 1, 2 \tag{1}$$

Here, μ_i and σ_i are positive constants and $B^{(1)}$ and $B^{(2)}$ are standard Brownian motion with correlation $\rho \in (-1, 1)$. The information structure is given by the augmented filtration generated by $B^{(1)}$ and $B^{(2)}$. We assume that $S^{(1)}$ is a risky asset having the proportional transacion costs (i.e. illiquid asset) and $S^{(2)}$ is a risky asset having no transacion costs (i.e. liquid asset). In other words, $S^{(1)}$ has proportional transaction costs whenever the investor trades $S^{(1)}$. Specifically, for the illiquid asset price process $S_t^{(1)}$, there exists $\bar{\lambda} > 0$, $\underline{\lambda} \in (0, 1)$ such that buying and selling price can be represented as $\bar{S_t}^{(1)} = (1 + \bar{\lambda})S_t^{(1)}$ and $\underline{S_t}^{(1)} = (1 - \underline{\lambda})S_t^{(1)}$.

Let η_0, η_1, η_2 be the investor's initial share of bond, illiquid asset and liquid asset. And similarly, let the triple $(\varphi_t^{(0)}, \varphi_t^{(1)}, \varphi_t^{(2)})$ the number of shares in the bond, illiquid asset and liquid asset at time t. Also, let c_t be the consumption rate at time t. To include the possible case of the initial jump, we set $(\varphi_{0-}^{(0)}, \varphi_{0-}^{(1)}, \varphi_{0-}^{(2)}) = (\eta_0, \eta_1, \eta_2)$ and right-continuous after that.

First, we define the two conditions, self-financing and admissibility.

Definition 2.1.1. *A strategy $(\varphi^{(0)}, \varphi^{(1)}, \varphi^{(2)}, c)$ is said to be self-financing if*

$$\varphi_t^{(0)} + \varphi_t^{(2)} S_t^{(2)} = \eta_0 + \eta_2 S_0^{(2)} + \int_0^t \varphi_u^{(2)} dS_u^{(2)} - \int_0^t \bar{S}_u^{(1)} d(\varphi_u^{(1)})^{\uparrow} + \int_0^t \underline{S}_u^{(1)} d(\varphi_u^{(1)})^{\downarrow} - \int_0^t c_u du \tag{2}$$

where $(\varphi_u^{(1)})^{\uparrow}$ and $(\varphi_u^{(1)})^{\downarrow}$ are the cumulative numbers of illiquid asset bought and sold up to time t.

In other words, No funds are added or subtracted to the investor.

Definition 2.1.2. *A self-financing strategy $(\varphi^{(0)}, \varphi^{(1)}, \varphi^{(2)}, c)$ is called admissible if*

$$\varphi_t^{(0)} + \underline{S}_t^{(1)}(\varphi_t^1)^+ - \bar{S}_t^{(1)}(\varphi_t^{(1)})^- + S_t^{(2)}\varphi_t^{(2)} \geq 0, \quad t \geq 0 \tag{3}$$

We assume that

$$\eta_0 + \underline{S}_t^{(1)}(\eta_1)^+ - \bar{S}_t^{(1)}(\eta_1)^- + S_t^{(2)}\eta_2 \geq 0.$$

for the initial admissiability.

Assume that $\varphi^{(1)}$ is of finite variation a.s. The set of admissible strategies is denoted by $\mathcal{A}$ and the set of all c such that $(\varphi^{(0)}, \varphi^{(1)}, \varphi^{(2)}, c) \in \mathcal{A}$ for some $(\varphi^{(0)}, \varphi^{(1)}, \varphi^{(2)})$ is denoted by $\mathcal{C}$.

2.2 Utility function

For $p \in (-\infty, 1)$, we consider the constant relative risk aversion (CRRA) type utility function $U : [0, \infty] \to [-\infty, \infty)$

$$U(c) = \frac{c^p}{p}, \quad c \geq 0 \;\text{ and }\; U(0) = \begin{cases} 0, & \text{p} > 0, \\ -\infty & \text{p} \leq 0. \end{cases}$$

Our goal is to analysis the optimal consumption and investment problem

$$\sup_{c \in \mathcal{C}} \mathbb{E}\left[\int_0^\infty e^{-\delta t} U(c_t) dt\right], \tag{4}$$

where the constant δ denotes discount factor. Because people prefer the present than the future, the discounting term is multiplied to the utility function.

To make the optimization problem (4) well-posed, we assume the two conditions below.

$$\begin{aligned} &(1) \;\; \delta > \frac{q\mu_2}{2\sigma_2^2} \text{ and } \delta > \frac{q(2\mu_1(1+q) - \sigma_1^2)}{2(1+q)^2} \\ &(2) \;\; \mu_1 \neq \frac{\rho\mu_2\sigma_1}{\sigma_2} \text{ and } \mu_2 \neq \frac{\rho\sigma_1\sigma_2}{1+q}. \end{aligned}$$

where $q := p/(1-p)$. These two conditions are covered in [5] and [6].

2.3 Shadow price approach

In this subsection, we introduce the shadow price which is used in [1]. Simply speaking, the shadow price is the virtual price which is between the bid price and ask price satisfying its maximal expected utility is equal to the maximal utility of the original problem. The shadow price approach makes the transaction cost problem to be solved by constructing a frictionless market model. To define the shadow price process, we will first define the set of the consistent price process.

Definition 2.3.1. *The set of consistent price processes $\mathcal{S}$ is defined as*

$$\mathcal{S} = \left\{ \tilde{S} : \tilde{S} \textit{ is an Ito-process, and } \underline{S}_t^{(1)} \leq \tilde{S}_t \leq \bar{S}_t^{(1)} \textit{for all } t \geq 0, a.s. \right\} \tag{5}$$

Definition 2.3.2. *The set of financeable consumptions $\mathcal{C}(\tilde{S})$ is defined as a set of nonnegative, locally integrable progressively measurable processes $c \in \mathcal{C}(\tilde{S})$ if there exist progressively measurable processes $(\varphi^{(0)}, \varphi^{(1)}, \varphi^{(2)})$ that satisfy the following two conditions:*
(i) Total wealth is nonnegative.

$$W_t := \varphi_t^{(0)} + \tilde{S}_t \varphi_t^{(0)} + S_t^{(2)} \varphi_t^{(2)} \geq 0, \quad t \geq 0 \tag{6}$$

(ii) The consumption stream is financeable.

$$W_t = W_{0-} + \int_0^t \varphi_u^{(1)} d\tilde{S}_t + \int_0^t \varphi_u^{(2)} dS_u^{(2)} - \int_0^t c_u du, \quad t \geq 0 \tag{7}$$

The connection between the frictionless market problem and transaction cost problem will be described in the following two propositions.

Proposition 2.3.1. *Since $\mathcal{C}(\tilde{S})$ is a set of financeable consumptions in the frictionless market, the following inequality always holds.*

$$\sup_{c \in \mathcal{C}(\tilde{S})} \mathbb{E}\left[\int_0^\infty e^{-\delta t} U(c_t) dt \right] \geq \sup_{c \in \mathcal{C}} \mathbb{E}\left[\int_0^\infty e^{-\delta t} U(c_t) dt \right] \tag{8}$$

Proof. To prove the **Proposition 2.3.1**, we prove $\mathcal{C} \in \mathcal{C}(\tilde{S})$. For any c $\in \mathcal{C}$, there exist $(\varphi^{(0)}, \varphi^{(1)}, \varphi^{(2)})$ which satisfies (2). So,

$$\begin{aligned}\varphi_t^{(0)} + \varphi_t^{(2)} S_t^{(2)} =& \eta_0 + \eta_2 S_0^{(2)} + \int_0^t \varphi_u^{(2)} dS_u^{(2)} - \int_0^t \bar{S}_u^{(1)} d(\varphi_u^{(1)})^{\uparrow} + \int_0^t \underline{S}_u^{(1)} d(\varphi_u^{(1)})^{\downarrow} - \int_0^t c_u du \\ \leq& \eta_0 + \eta_2 S_0^{(2)} + \int_0^t \varphi_u^{(2)} dS_u^{(2)} - \int_0^t \tilde{S}_u d(\varphi_u^{(1)}) - \int_0^t c_u du\end{aligned}$$

Here, inequality holds because $\tilde{S} \in \mathcal{S}$. Then, using intgration-by-parts, we can get

$$\varphi_t^{(0)} + \varphi_t^{(1)} S_t^{(1)} + \varphi_t^{(2)} S_t^{(2)} \leq \eta_0 + \eta_1 \tilde{S}_0 + \eta_2 S_0^{(2)} + \int_0^t \varphi_u^{(1)} dS_u^{(1)} + \int_0^t \varphi_u^{(2)} dS_u^{(2)} - \int_0^t c_u du$$

Define $\tilde{\varphi}^{(0)}$ as

$$\tilde{\varphi}^{(0)} := \eta_0 + \eta_1 \tilde{S}_0 + \eta_2 S_0^{(2)} + \int_0^t \varphi_u^{(1)} dS_u^{(1)} + \int_0^t \varphi_u^{(2)} dS_u^{(2)} - \int_0^t c_u du - \varphi_t^{(1)} S_t^{(1)} - \varphi_t^{(2)} S_t^{(2)}$$

Then, $\tilde{\varphi}^{(0)} \geq \varphi^{(0)}$ and (7) is satisfied with $(\tilde{\varphi}^{(0)}, \varphi^{(1)}, \varphi^{(2)}, c)$. Also,

$$0 \leq \varphi_t^{(0)} + \underline{S}_t^{(1)} \left(\varphi_t^{(1)}\right)^{+} - \bar{S}_t^{(1)} \left(\varphi_t^{(1)}\right)^{-} + S_t^{(2)} \varphi_t^{(2)} \leq \tilde{\varphi}_t^{(0)} + \varphi_t^{(1)} \tilde{S}_t + \varphi_t^{(2)} S_t^{(2)}$$

which means that (6) is satisfied. Therefore, $c \in \mathcal{C}(\tilde{S})$ and complete the proof.

$\square$

Proposition 2.3.2. *Given $\tilde{S} \in \mathcal{S}$, let $\hat{c} \in \mathcal{C}(\tilde{S})$ solve the frictionless optimization problem, that is*

$$\mathbb{E}\left[\int_0^\infty e^{-\delta t} U(\hat{c}_t) dt\right] = \sup_{c \in \mathcal{C}(\tilde{S})} \mathbb{E}\left[\int_0^\infty e^{-\delta t} U(c_t) dt\right] \tag{9}$$

with $(\hat{\varphi}^{(0)}, \hat{\varphi}^{(1)}, \hat{\varphi}^{(2)})$ that satisfies the (7). Assume that

(i) *$\hat{\varphi}^{(1)}$ is right-continuous process of finite variation.*

(ii) *$(\hat{\varphi}^{(0)}, \hat{\varphi}^{(1)}, \hat{\varphi}^{(2)})$ satisfies (3).*

(iii) *$d(\hat{\varphi}^{(1)})^{\uparrow} = 1_{\{\tilde{S}_t = \bar{S}_t\}} d(\hat{\varphi}^{(1)})^{\uparrow}$*

(iv) *$\hat{c}, \hat{\varphi}^{(0)}, \hat{\varphi}^{(1)}, \hat{\varphi}^{(2)}$ are continuous processes except for a possible initial jump at $t = 0$-.*

Then $\hat{c} \in \mathcal{C}$ and $\hat{c}$ solves the optimization problem (4). i.e.

$$\mathbb{E}\left[\int_0^\infty e^{-\delta t} U(\hat{c}_t) dt\right] = \sup_{c \in \mathcal{C}} \mathbb{E}\left[\int_0^\infty e^{-\delta t} U(c_t) dt\right] \tag{10}$$

Proof. Let $(\hat{c}, \hat{\varphi}^{(0)}, \hat{\varphi}^{(1)}, \hat{\varphi}^{(2)})$ satisfy the assumption (i) to (iv) in the proposition. Then, since it satisfies (7), we have

$$\hat{\varphi}_t^{(0)} + \hat{\varphi}_t^{(1)} S_t^{(1)} + \hat{\varphi}_t^{(2)} S_t^{(2)} = \eta_0 + \eta_1 \tilde{S}_0 + \eta_2 \tilde{S}_0^{(2)} + \int_0^t \hat{\varphi}_u^{(1)} d\tilde{S}_u + \int_0^t \hat{\varphi}_u^{(2)} d\tilde{S}_u^{(2)} - \int_0^t \hat{c}_u du$$

And using integration-by-part, we can get

$$\begin{aligned}
\hat{\varphi}_t^{(0)} + \hat{\varphi}_t^{(2)} S_t^{(2)} &= -\hat{\varphi}_t^{(1)} S_t^{(1)} + \eta_0 + \eta_1 \tilde{S}_0 + \eta_2 \tilde{S}_0^{(2)} + \int_0^t \hat{\varphi}_u^{(1)} d\tilde{S}_u + \int_0^t \hat{\varphi}_u^{(2)} d\tilde{S}_u^{(2)} - \int_0^t \hat{c}_u du \\
&= \eta_0 + \eta_2 \tilde{S}_0^{(2)} - \int_0^t \tilde{S}_u d\hat{\varphi}_u^{(1)} + \int_0^t \hat{\varphi}_u^{(2)} d\tilde{S}_u^{(2)} - \int_0^t \hat{c}_u du \\
&= \eta_0 + \eta_2 \tilde{S}_0^{(2)} - \int_0^t \bar{S}_u d(\hat{\varphi}^{(1)})_u^{\uparrow} + \int_0^t \underline{S}_u d(\hat{\varphi}^{(1)})_u^{\downarrow} + \int_0^t \hat{\varphi}_u^{(2)} d\tilde{S}_u^{(2)} - \int_0^t \hat{c}_u du
\end{aligned}$$

which means it satisfies (2) and $\hat{c} \in \mathcal{C}$. And (9) and (10) imply (11). □

Definition 2.3.3. *Let $\tilde{S} \in \mathcal{S}$. $\tilde{S}$ is called a shadow price process if*

$$\sup_{c \in \mathcal{C}} \mathbb{E}\Big[\int_0^\infty e^{-\delta t} U(c_t) dt\Big] = \sup_{c \in \mathcal{C}(\tilde{S})} \mathbb{E}\Big[\int_0^\infty e^{-\delta t} U(c_t) dt\Big] < \infty. \tag{11}$$

Remark 2.3.1. *From **Proposition 2.3.1.** and **Proposition 2.3.2.** we can represent maximization problem (4) to the following minimization problem.*

$$\sup_{c \in \mathcal{C}} \mathbb{E}\Big[\int_0^\infty e^{-\delta t} U(c_t) dt\Big] = \inf_{\tilde{S} \in \mathcal{S}} \Big(\sup_{c \in \mathcal{C}(\tilde{S})} \mathbb{E}\Big[\int_0^\infty e^{-\delta t} U(c_t) dt\Big]\Big) \tag{12}$$

III Review of Choi [1]

In this section, we review the work of Choi. But before we review the paper, we analyze the frictionless market with two risky assets.

3.1 Frictionless market with two risky assets

Suppose that there are two risky assets with no transaction costs. We define π_1, π_2 by

$$\pi_1 = \frac{\varphi^{(1)}S^{(1)}}{\varphi^{(0)} + \varphi^{(1)}S^{(1)} + \varphi^{(2)}S^{(2)}}, \quad \pi_2 = \frac{\varphi^{(2)}S^{(2)}}{\varphi^{(0)} + \varphi^{(1)}S^{(1)} + \varphi^{(2)}S^{(2)}}$$

which are the proportion of the each risky asset over risk-free and risky assets.

Theorem 3.1.1. *Suppose that the following condition holds.*

$$\delta > \frac{q}{2(1-\rho^2)}\left(\left(\frac{\mu_1}{\sigma_1}\right)^2 + \left(\frac{\mu_2}{\sigma_2}\right)^2 - 2\rho\frac{\mu_1\mu_2}{\sigma_1\sigma_2}\right)$$

Then, the optimal policy is

$$c_t^* = Cw_t, \quad \pi_{1t}^* = \frac{(1+q)(\mu_1 - \frac{\rho\sigma_1}{\sigma_2}\mu_2)}{(1-\rho^2)\sigma_1^2}, \quad \pi_{2t}^* = \frac{(1+q)(\mu_2 - \frac{\rho\sigma_2}{\sigma_1}\mu_1)}{(1-\rho^2)\sigma_2^2}$$

where C is defined as

$$C = (1+q)(\delta - \frac{q}{2(1-\rho^2)}\left(\left(\frac{\mu_1}{\sigma_1}\right)^2 + \left(\frac{\mu_2}{\sigma_2}\right)^2 - 2\rho\frac{\mu_1\mu_2}{\sigma_1\sigma_2}\right)).$$

To prove Theorem 3.1.1, we first huristicaly derive the optimal investment and consumption solution. And then, we rigorously verify that these solutions are really the solutions of the objective function. This theorem and proof are given in [6] for the one risky asset case. We follow the way in [6] to prove for the two risky assets case.

3.1.1. Heuristic derivation

Using Ito's lemma, we can get

$$\begin{aligned} dW_t &= d\varphi_{0t} + S_{1t}d\varphi_{1t} + \varphi_{1t}dS_{1t} + S_{2t}d\varphi_{2t} + \varphi_{2t}dS_{2t} \\ &= -c_t dt + \varphi_{1t}S_{1t}(\mu_1 dt + \sigma_1 dB_{1t}) + \varphi_{2t}S_{2t}(\mu_2 dt + \sigma_2 dB_{2t}) \\ &= (\mu_1\pi_{1t}W_t + \mu_2\pi_{2t}W_t - c_t)dt + \sigma_1\pi_{1t}W_t dB_t^{(1)} + \sigma_2\pi_{2t}W_t dB_t^{(2)} \end{aligned}$$

The self-financing condition is used for the second equality.

Define the function $v(x)$ by

$$v(x) = \sup_{(\pi_1,\pi_2,c)\in\mathcal{A}(\pi_1,\pi_2)} \mathbb{E}[\int_0^\infty e^{-\delta t}U(c_t)dt]$$

where $x = \eta_0 + \eta_1 S_0^{(1)} + \eta_2 S_0^{(2)} = W_0$ is the initial wealth. Assume that the process W_t is finite. Then, we can use the dynamic programming principle (See Corollary 4.2 in [7]). The following lemma is suitable for the model.

Lemma 3.1.1. *(Dynamic programming principle) Let $(\pi, c) \in \mathcal{A}(\pi)$ be given. Then it satisfies*

$$v(W_0) = \sup_{(\pi_1,\pi_2,c)\in\mathcal{A}(\pi_1,\pi_2)} \mathbb{E}[\int_0^\infty e^{-\delta t}U(c_t)dt + e^{-\delta t}v(W_t)]$$

Here, π defined by W_0.

Assume v is C^2 function. Then, by Ito's lemma and Lemma 3.1.1,

$$\begin{aligned}
0 &= \sup_{(\pi_1,\pi_2,c)\in\mathcal{A}(\pi_1,\pi_2)} \mathbb{E}[\int_0^\infty e^{-\delta t}U(c_t)dt + e^{-\delta t}v(W_t) - v(W_0)] \\
&= \sup_{(\pi_1,\pi_2,c)\in\mathcal{A}(\pi_1,\pi_2)} \mathbb{E}[\int_0^\infty e^{-\delta t}\Big(U(c) - \delta v(W_s) + (\mu_1\pi_{1s}W_s + \mu_2\pi_{2s}W_s - c_s)v'(W_s) \\
&\quad + \frac{1}{2}(\sigma_1^2\pi_{1s}^2W_s^2 + \sigma_2^2\pi_{2s}^2W_s^2 + 2\sigma_1\sigma_2\pi_{1s}\pi_{2s}W_s^2\rho)\Big)ds + \int_0^t e^{-\delta s}\sigma_1\pi_{1s}W_s v'(W_s)dB_s^{(1)} \\
&\quad + \int_0^t e^{-\delta s}\sigma_2\pi_{2s}W_s v'(W_s)dB_s^{(2)}
\end{aligned}$$

Suppose that $v'(x) = \frac{dv}{dx}$ is bounded. Then, The last two terms are zero and Hamilton-Jacobi-Bellman(HJB) equation is made as

$$max_{(\pi_1,\pi_2,c)\in\mathcal{A}(\pi_1,\pi_2)}\left(\frac{1}{p}c^p + (\mu_1\pi_1 x + \mu_2\pi_2 x - c)v' + \frac{1}{2}(\sigma_1^2\pi_1^2x^2 + \sigma_2^2\pi_2^2x^2 + \sigma_1\sigma_2\pi_1\pi_2x^2\rho)v'' - \delta v\right) = 0 \tag{13}$$

From the FOC, the maximam is obtained at $(\tilde{c}, \tilde{\pi}_1, \tilde{\pi}_2)$ where

$$\tilde{c} = (v')^{\frac{1}{p-1}}, \quad \tilde{\pi}_1 = \frac{v'}{(1-\rho^2)\sigma_1 x v''}(-\frac{\mu_1}{\sigma_1} + \frac{\mu_2\rho}{\sigma_2}), \quad \tilde{\pi}_2 = \frac{v'}{(1-\rho^2)\sigma_2 x v''}(-\frac{\mu_2}{\sigma_2} + \frac{\mu_1\rho}{\sigma_1})$$

Then (13) changes to

$$-\delta v + \frac{1-p}{p}(v')^{\frac{p}{1-p}} - \frac{1}{2(1-\rho^2)}\left(\left(\frac{\mu_1}{\sigma_1}\right)^2 + \left(\frac{\mu_2}{\sigma_2}\right)^2 - 2\rho\frac{\mu_1\mu_2}{\sigma_1\sigma_2}\right)\frac{(v')^2}{v''} = 0 \tag{14}$$

Define $\tilde{v}(x)$ as

$$\tilde{v}(x) = \frac{1}{p}C^{p-1}x^p$$

Then, $\tilde{v}(x)$ satisfies (14) and maximizing π_1, π_2, c under $\tilde{v}(x)$ are equal to $\tilde{\pi}_1, \tilde{\pi}_2, \tilde{c}$. These $\tilde{\pi}_1, \tilde{\pi}_2, \tilde{c}$ are the candidate solution of the optimization problem.

3.1.2. Verification of the optimal strategy

In this subsection, we will see that the candidate solution we derived in section 3.1.1 is an optimal solution. Let $(\pi_1, \pi_2, c) \in \mathcal{A}(\pi_1, \pi_2)$ be an arbitrary policy. Then, W_t is given by

$$\begin{aligned} W_t =& e^{\int_0^t (\mu_1\pi_{1s}+\mu_2\pi_{2s}-\frac{1}{2}\sigma_1^2\pi_{1s}^2-\frac{1}{2}\sigma_2^2\pi_{2s}^2-\sigma_1\sigma_2\pi_{1s}\pi_{2s}\rho)ds+\int_0^t \sigma_1\pi_{1s}dB_s^{(1)}+\int_0^t \sigma_1\pi_{2s}dB_s^{(2)}} \\ & (W_0 - \int_0^t c_s e^{-\int_0^t (\mu_1\pi_{1s}+\mu_2\pi_{2s}-\frac{1}{2}\sigma_1^2\pi_{1s}^2-\frac{1}{2}\sigma_2^2\pi_{2s}^2-\sigma_1\sigma_2\pi_{1s}\pi_{2s}\rho)ds-\int_0^t \sigma_1\pi_{1s}dB_s^{(1)}-\int_0^t \sigma_1\pi_{2s}dB_s^{(2)}}) \end{aligned}$$

From Hölder inequality and boundness of π_{1t}, π_{2t}, W_t has finite moments of all orders. Define M_t as

$$M_t = \int_0^t e^{-\delta s} U(c_s)ds + e^{-\delta t}\tilde{v}(W_t)$$

From Ito's lemma,

$$\begin{aligned} M_t - M_0 =& \int_0^t e^{-\delta s}\Big((\mu_1\pi_{1s}W_s + \mu_2\pi_{2s} - c_s)\tilde{v}'(W_s) + \frac{1}{2}(\sigma_1^2\pi_1^2x^2 + \sigma_2^2\pi_2^2x^2 + \sigma_1\sigma_2\pi_1\pi_2x^2\rho)\tilde{v}''(W_s)\Big) \\ & + \sigma_1 C^{p-1}\int_0^t e^{-\delta s}\pi_{1s}W_s^p dB_s^{(1)} + \sigma_2 C^{p-1}\int_0^t e^{-\delta s}\pi_{2s}W_s^p dB_s^{(2)} \end{aligned}$$

Since π_{1s}, π_{2s} and W_t^p is bounded (W_t^p is bounded from Jensen's inequality), expectation of last two terms are zero. Also, Since the integrand of the first term is nonpositive, M_t is supermartingale, and is a martingale when $(c, \pi_1, \pi_2) = (\tilde{c}, \tilde{\pi_1}, \tilde{\pi_2})$, so that

$$\tilde{v}(W_0) = M_0 \geq \mathbb{E}[M_t] = \mathbb{E}[\int_0^t e^{-\delta s}U(c_s)]ds + e^{-\delta t}\mathbb{E}[\tilde{v}(W_t)] \tag{15}$$

From Ito's lemma and wealth equation, we can see that

$$\mathbb{E}[e^{-\delta t}\tilde{v}(W_t)] = \frac{C^{p-1}W_0^p}{p}\mathbb{E}[G_t e^{\int_0^t a(s)ds}],$$

where $G_t = e^{\frac{1}{2}\int_0^t p^2\sigma_1^2\pi_{1s}^2+p^2\sigma_2^2\pi_{2s}^2 ds+\int_0^t p\sigma_1\pi_{1s}dB_s^{(1)}+\int_0^t p\sigma_2\pi_{2s}dB_s^{(2)}}$ and $a(s) = p(\mu_1\pi_{1s} + \mu_2\pi_{2s} - \frac{c_s}{W_s} - \frac{1}{2}(1-p)(\pi_{1s}^2\sigma_1^2 + \pi_{2s}^2\sigma_2^2)) - \delta$. Note that from the Novikov's condition, G_t is a martingale since π_1, π_2 is bounded. Also, when (c,π_1, π_2) = $(\tilde{c}, \tilde{\pi_1}, \tilde{\pi_2})$, $a(s) = -C$. Therefore, the last term of (15), $e^{-\delta t}\mathbb{E}[\tilde{v}(W_t)] \to 0$ as $t \to \infty$

To complete the proof, we divide the cases $0 < p < 1$ and $p < 0$. For $0 < p < 1$, $a(s)$ has upper bound $(p-1)C$ which is less than 0. So, the last term of (15), $e^{-\delta t}\mathbb{E}[\tilde{v}(W_t)] \to 0$ as $t \to \infty$. So, $\tilde{v} = v$ and $(\tilde{c}, \tilde{\pi_1}, \tilde{\pi_2})$ is an optimal.

Let's consider the case $p < 0$. For $\epsilon > 0$, consider

$$\tilde{v}_\epsilon(x) = \frac{1}{p}C^{p-1}(x+\epsilon)^p.$$

Then, it satisfies

$$-\delta\tilde{v}_\epsilon + \frac{1-p}{p}(\tilde{v}_\epsilon')^{\frac{p}{1-p}} - \frac{1}{2(1-\rho^2)}\left(\left(\frac{\mu_1}{\sigma_1}\right)^2 + \left(\frac{\mu_2}{\sigma_2}\right)^2 - 2\rho\frac{\mu_1\mu_2}{\sigma_1\sigma_2}\right)\frac{(\tilde{v}_\epsilon')^2}{\tilde{v}_\epsilon''} = 0 \tag{16}$$

Using same procedures for $\tilde{v}$, we can get $\tilde{v}_\epsilon(W_0) \geq \mathbb{E}\int_0^t e^{-\delta t}U(c_s)ds$ for all $(\pi_1, \pi_2, c) \in \mathcal{A}(\pi_1, \pi_2)$. Since $\tilde{v}_\epsilon(W_0) \to \tilde{v}(W_0)$ as $\epsilon \to 0$, $\tilde{v} = v$ and $(\tilde{c}, \tilde{\pi_1}, \tilde{\pi_2})$ is an optimal.

3.2 Result of Choi [1]

3.2.1 Heuristic derivation of the free boundary ODE

We will heuristically derive a free boundary ODE from the HJB equation for the optimization problem (4). For $\tilde{S} \in \mathcal{S}$, we can write $\tilde{S}_t = S_t^{(1)} e^{Y_t}$ for an Ito-Process Y. Then $Y_t \in [\underline{y}, \bar{y}]$, where $\underline{y} := ln(1-\underline{\lambda})$ and $\bar{y} := ln(1+\bar{\lambda})$. Assume that the dynamics of Y is given by

$$dY_t = m_t dt + s_{1t} dB_t^{(1)} + s_{2t} dB_t^{(2)} \tag{17}$$

for some processes m, s_1, s_2. Then, the state price density process H in the market with $\tilde{S}$ and $S^{(2)}$ satisfies the stochastic differential equation

$$dH_t = -H_t\bigg(\theta_1(m_t, s_{1t}, s_{2t})dB_t(1) + \theta_2(m_t, s_{1t}, s_{2t})dB_t^{(2)}\bigg), \quad H_0 = 1 \tag{18}$$

where θ_1 and θ_2 are

$$\begin{aligned}
\theta_1(m, s_1, s_2) &:= \frac{\rho(\sigma_2 s_2 - \mu_2)}{(1-\rho^2)\sigma_2} - \frac{\mu_2 s_2 - (m + \mu_1 + s_1\sigma_1 + \frac{1}{2}(s_1^2 + s_2^2))\sigma_2}{(1-\rho^2)\sigma_2(s_1+\sigma_1)} \\
\theta_2(m, s_1, s_2) &:= \frac{\mu_2}{\sigma_2} - \rho\theta_1(m, s_1, s_2)
\end{aligned}$$

Since the frictionless market model with stock prices $\tilde{S}$ and $S^{(2)}$ is complete, the standard duality theory can be applied(c.f. [8] Karatzas)

$$\begin{aligned}
&\sup_{c \in \mathcal{C}(\tilde{S})} \mathbb{E}\bigg[\int_0^\infty e^{-\delta t} U(c_t) dt\bigg] \\
&= \inf_{z>0}\Bigg(\sup_c\Bigg(\mathbb{E}\bigg[\int_0^\infty e^{-\delta t U(c_t) dt}\bigg] + z\Bigg((\eta_0 + \tilde{S}_0\eta_1 + S_0^{(2)}\eta_2) - \mathbb{E}\bigg[\int_0^\infty c_t H_t dt\bigg]\Bigg)\Bigg)\Bigg) \\
&= \frac{(\eta_0 + \tilde{S}_0\eta_1 + S_0^{(2)}\eta_2)^p}{p}\Bigg(\mathbb{E}\bigg[\int_0^\infty e^{-(1+q)\delta t} H_t^{-q} dt\bigg]\Bigg)^{1-p}
\end{aligned}$$

Then, We can write (12) as

$$\inf_{\tilde{S}\in\mathcal{S}}\Bigg(\sup_{c\in\mathcal{C}(\tilde{S})} \mathbb{E}\bigg[\int_0^\infty e^{-\delta t} U(c_t) dt\bigg]\Bigg) = \inf_{Y_0}\Bigg\{\frac{(\eta_0 + \tilde{S}_0\eta_1 + S_0^{(2)}\eta_2)^p}{p}|w(Y_0)|^{1-p}\Bigg\}$$

where

$$w(y) := \inf_{m,s_1,s_2}\Bigg\{sgn(p)\mathbb{E}\bigg[\int_0^\infty e^{-(1+q)\delta t} H_t^{-q} dt\bigg| Y_0 = y\bigg]\Bigg\} \tag{19}$$

The HJB equation for (19) is

$$\inf_{m,s_1,s_2}\{-\alpha(m,s_1,s_2)w(y) + (m + \beta(m,s_1,s_2))w'(y) + \gamma(s_1,s_2)w''(y) + sgn(p)\} = 0 \tag{20}$$

where

$$\alpha(m, s_1, s_2) := (1+q)\delta - \frac{q(1+q)}{2}(\theta_1^2 + \theta_2^2 + 2\rho\theta_1\theta_2),$$
$$\beta(m, s_1, s_2) := q((s_1 + \rho s_2)\theta_1 + (\rho s_1 + s_2)\theta_2),$$
$$\gamma(s_1, s_2) := \frac{1}{2}(s_1^2 + s_2^2 + 2\rho s_1 s_2).$$

To incorporate the requirement $Y_t \in [\underline{y}, \bar{y}]$, we set $s_{1t} = s_{2t} = 0$ whenever Y_t reaches the boundary $\underline{y}$ or $\bar{y}$. From (20), it is expected that the boundary condition

$$w''(\underline{y}) = w''(\bar{y}) = \infty. \tag{21}$$

To handle this boundary condition, we reduce the order by changing variable. Let $x = -w'(y)$ and $g : [\underline{x}, \bar{x}] \mapsto \mathbb{R}$ as $g(x) = w(y)$, where $\underline{x} = -w'(\bar{y})$ and $\bar{x} = -w'(\underline{y})$.. Then, we can write (20) as

$$\inf_{m,s_1,s_2} \left\{ -\alpha(m, s_1, s_2)g(x) - (m + \beta(m, s_1, s_2))x + \gamma(s_1, s_2)\frac{x}{g'(x)} + sgn(p) \right\} = 0 \quad x \in [\underline{x}, \bar{x}]. \tag{22}$$

Then, (21) gives $g'(\underline{x}) = g'(\bar{x}) = 0$ and $\int_{\underline{x}}^{\bar{x}} \frac{g'(x)}{x} dx = \bar{y} - \bar{y}$.. So, (22) produces a boundary condition and an integral constraint :

$$g'(\underline{x}) = g'(\bar{x}) = 0, \qquad \int_{\underline{x}}^{\bar{x}} \frac{g'(x)}{x} dx = \bar{y} - \bar{y}. \tag{23}$$

Note that changing variable from w' to g gives order reduction.

3.2.2 Main result

In section 3.1.1, we derived the free boundary problem heuristically. In this subsection, we will see the main result of [1] without proof. Proposition 3.2.1 shows the existence for the solution of the free boundary problem in the previous subsection. And Theorem 3.1.1 provide the explicit characterization of well-posedness of the problem.

Proposition 3.2.1. *Assume that the model parameters satisfy one of the following conditions:*

(i) $p \leq 0$,

(ii) $0 < p < 1$ *and* $\delta > \frac{q}{2(1-\rho^2)}((\frac{\mu_1}{\sigma_1})^2 + (\frac{\mu_2}{\sigma_2})^2 - 2\rho\frac{\mu_1\mu_2}{\sigma_1\sigma_2})$, *and*

(iii) $0 < p < 1$, $\delta \leq \frac{q}{2(1-\rho^2)}((\frac{\mu_1}{\sigma_1})^2 + (\frac{\mu_2}{\sigma_2})^2 - 2\rho\frac{\mu_1\mu_2}{\sigma_1\sigma_2})$ *and* $c^* < ln(\frac{1+\bar{\lambda}}{1-\underline{\lambda}})$,

where c^* *is a constant defined in [1] Defnition 6.9.*

Then, there exist constants $\underline{x}$, $\bar{x}$ and a function $g \in C^2[\underline{x}, \bar{x}]$ that satisfy following conditions:

(1) *If $\mu_1 > \frac{\rho\mu_2\sigma_1}{\sigma_2}$, then $0 < \underline{x} < \bar{x}$. If $\mu_1 < \frac{\rho\mu_2\sigma_1}{\sigma_2}$, then $\underline{x} < \bar{x} < 0$.*

(2) *For $x \in [\underline{x}, \bar{x}]$, g satisfies the differential equation*

$$\inf_{m,s_1,s_2} \left\{ -\alpha(m,s_1,s_2)g(x) - (m + \beta(m,s_1,s_2))x + \gamma(s_1,s_2)\frac{x}{g'(x)} + sgn(p) \right\} = 0$$

where α, β, γ are given in section 3.2.1

(3) *The following boundary/integral conditions are satisfied.*

$$g'(\underline{x}) = g'(\bar{x}) = 0 \quad and \quad \int_{\underline{x}}^{\bar{x}} \frac{g'(x)}{x} dx = ln\left(\frac{1+\bar{\lambda}}{1-\underline{\lambda}}\right)$$

(4) *The functions*

$$qg(x), \quad qg(x)(g'(x)+1) - (1+q)xg'(x), \quad q(g(x) - xg'(x)), \quad and \quad g'(x)+1$$

are strictly positive on $[\underline{x}, \bar{x}]$.

(5) $\frac{g'(x)}{x} > 0 \quad for \quad x \in (\underline{x}, \bar{x})$

Theorem 3.2.1. *The following statements are equivalent.*

(1) *The optimization problem is well-posed, that is,*

$$\sup_{c \in \mathcal{C}} \mathbb{E}\left[\int_0^\infty e^{-\delta t} U(c_t) dt\right] < \infty$$

(2) *There exists a shadow price process.*

(3) *The model parameters satisfy one of the following three conditions:*

(i) $p \leq 0$,

(ii) $0 < p < 1$ *and* $\delta > \frac{q}{2(1-\rho^2)}((\frac{\mu_1}{\sigma_1})^2 + (\frac{\mu_2}{\sigma_2})^2 - 2\rho\frac{\mu_1\mu_2}{\sigma_1\sigma_2})$, *and*

(iii) $0 < p < 1$, $\delta \leq \frac{q}{2(1-\rho^2)}((\frac{\mu_1}{\sigma_1})^2 + (\frac{\mu_2}{\sigma_2})^2 - 2\rho\frac{\mu_1\mu_2}{\sigma_1\sigma_2})$ *and $c^* < ln(\frac{1+\bar{\lambda}}{1-\underline{\lambda}})$, where c^* is a constant defined in [1] Defnition 6.9.*

IV Asymptotic ally optimal trading and consumption strategies

In this section, we investigate the optimal trading of the liquid, illiquid risky asset and consumption. For convenience, we assume that $\bar{\lambda} = 0$ and $\underline{\lambda} = \lambda$. Also assume that

$$p > 0, \quad \mu_1 > \frac{\rho\mu_2\sigma_1}{\sigma_2}, \quad \delta > \frac{q}{2(1-\rho^2)}\left(\left(\frac{\mu_1}{\sigma_1}\right)^2 + \left(\frac{\mu_2}{\sigma_2}\right)^2 - 2\rho\frac{\mu_1\mu_2}{\sigma_1\sigma_2}\right). \tag{24}$$

This means that the proportion invested in the illiquid risky asset should be positive.

First, we consider the case when there is no transaction costs. Recall that from the Section 3.1.1, the optimal proportion of liquid, illiquid risky asset and consumption c^M, π_1^M, π_2^M in the frictionless market are given as

$$\begin{aligned}
c^M &:= (1+q)\left(\delta - \frac{q}{2(1-\rho^2)}\left(\left(\frac{\mu_1}{\sigma_1}\right)^2 + \left(\frac{\mu_2}{\sigma_2}\right)^2 - 2\rho\frac{\mu_1\mu_2}{\sigma_1\sigma_2}\right)\right) \\
\pi_1^M &:= \frac{(1+q)(\mu_1 - \frac{\rho\sigma_1}{\sigma_2}\mu_2)}{(1-\rho^2)\sigma_1^2} \\
\pi_2^M &:= \frac{(1+q)(\mu_2 - \frac{\rho\sigma_2}{\sigma_1}\mu_1)}{(1-\rho^2)\sigma_2^2}
\end{aligned}$$

And $\underline{x}, \bar{x}, g(\underline{x}), g(\underline{x})$ in Proposition 3.2.1 have the following expansions for small transaction cost λ:

$$\begin{aligned}
\underline{x} &= -\frac{q\pi_1^M}{c^M} + \frac{q\zeta}{c^M}\lambda^{1/3} + O\left(\lambda^{2/3}\right) \\
\bar{x} &= -\frac{q\pi_1^M}{c^M} - \frac{q\zeta}{c^M}\lambda^{1/3} + O\left(\lambda^{2/3}\right) \\
g(\underline{x}) &= -\frac{1}{c^M} + \frac{q(1-\rho^2)\sigma_1^2\zeta^2}{2(1+q)(c^M)^2}\lambda^{2/3} + O\left(\lambda^{2/3}\right) \\
g(\underline{x}) &= -\frac{1}{c^M} - \frac{q(1-\rho^2)\sigma_1^2\zeta^2}{2(1+q)(c^M)^2}\lambda^{2/3} + O\left(\lambda^{2/3}\right)
\end{aligned}$$

where

$$\zeta := \left(\frac{3(1+q)(\pi_1^M)^2(1-\pi_1^M)^2}{4} + \frac{3(1+q)(\mu_2(1+q) - \rho\sigma_1\sigma_2)^2(\pi_1^M)^2}{4(1-\rho^2)\sigma_1^2\sigma_2^2}\right)^{\frac{1}{3}}$$

4.1 Optimal strategy

In this subsection, we will see the optimal consumption and trading strategies of liquid and illiquid asset. The proofs are in the section 5 in [1].

4.1.1. Optimal trading of the illiquid asset

Corollary 4.1.1. *Under the assumption (24), minimally trading the proportion of illiquid asset within the interval* $[\underline{\pi}_1, \bar{\pi}_1]$ *is optimal. In othere words,*

$$\underline{\pi}_1 \leq \frac{\hat{\varphi}_t^{(1)} S_t^{(1)}}{\hat{\varphi}_t^{(0)} + \hat{\varphi}_t^{(1)} S_t^{(1)} + \hat{\varphi}_t^{(2)} S_t^{(2)}} \leq \bar{\pi}_1$$

Corollary 4.1.1 says that the optimal trading of the illiquid asset is maintaining the proportion of investment in the illiquid asset within some interval $[\underline{\pi}_1, \bar{\pi}_1]$. That is, $[\underline{\pi}_1, \bar{\pi}_1]$ is the no transaction region. If the proportion reaches $\underline{\pi}_1$ or $\bar{\pi}_1$, investor minimally buys or sells the illiquid asset to make the fraction inside the no transaction region. And $\underline{\pi}_1, \bar{\pi}_1$ has the following expansions for small transaction cost λ:

$$\underline{\pi}_1 = \pi_1^M - \zeta \lambda^{\frac{1}{3}} + O\Big(\lambda^{2/3}\Big)$$
$$\bar{\pi}_1 = \pi_1^M + \zeta \lambda^{\frac{1}{3}} + O\Big(\lambda^{2/3}\Big)$$

4.1.2 Optimal trading of the liquid asset

Corollary 4.1.2. *Under the assumption (24), optimal proportion of investment in the liquid risky asset has the following expansions for small transaction cost* λ*:*

$$\begin{aligned} &\pi_2^M - \frac{\rho\sigma_1\zeta}{\sigma_2}\lambda^{1/3} + O\Big(\lambda^{2/3}\Big) \quad \textit{when selling the illiquid asset,} \\ &\pi_2^M + \frac{\rho\sigma_1\zeta}{\sigma_2}\lambda^{1/3} + O\Big(\lambda^{2/3}\Big) \quad \textit{when buying the illiquid asset.} \end{aligned}$$

Recall that according to Corollary 4.1.1, when the fraction of the illiquid asset reaches $\underline{\pi}_1$ or $\bar{\pi}_1$, the optimal strategy for the investor is minimally buying or selling the illiquid asset. Corollary 4.1.2 says that when the fraction of the illiquid asset reaches $\underline{\pi}_1$ or $\bar{\pi}_1$ and buys or sells the illiquid asset, the optimal proportion of the liquid asset has the above expansion for small transaction cost λ.

4.1.3 Optimal consumption rate

Corollary 4.1.3. *Under the assumption (24), optimal consumption rate proportion has the following expansions for small transaction cost λ:*

$$\frac{\hat{c}_t}{\hat{\varphi}_t^{(0)} + \hat{\varphi}_1^{(1)} S_t^{(1)} + \hat{\varphi}_t^{(2)} S_t^{(2)}} = c^M - \frac{q(1-\rho^2)\sigma_1^2\zeta^2}{2(1+q)}\lambda^{\frac{2}{3}} + O(\lambda), \quad a.s.$$

Recall that C^M is the optimal consumption rate in the frictionless market. From Corollary 4.1.3, we can see that the optimal consumption rate in the market having transaction costs always bigger than the one without transaction costs. One possible guess of this effect is that the existence of the transaction costs makes investment less attractive, and causes an increase in the consumption rate.

V Parameter Estimation

In this section, we will apply the real data to the model we discussed. In the previous model, we assumed zero interest rates but since interest rates are not zero in the real world, we will consider the interest rates in this section.

5.1 Data description

To apply the model, we have used data related to the liquid and illiquid assets. One is KOSPI200 data (index consists of 200 big companies of the Stock Market Division) and the other is house price index (the price of the residential real estate in Korea) data which are obtained from Korean Statistical Information Service (KOSIS) and Korea Appraisal Board respectively. The data concerns the monthly time period between January 2004 to August 2019. Figure 2 shows the graph of the KOSPI200 and house price index from January 2004 to August 2019.

The parameters for the liquid and illiquid asset are calculated as follows. First, the illiquid asset parameters μ_1 and σ_1 is determined as

$$\mu_1 = \frac{1}{N}\sum_{i=1}^{N}\Big((return_{S^{(1)}})_i - r_i\Big) \tag{25}$$

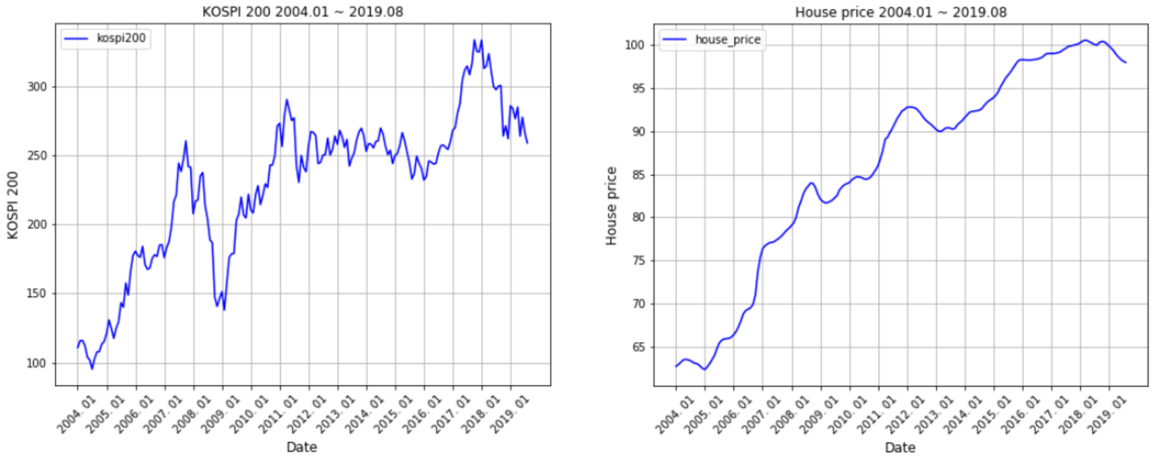
$$\sigma_1 = \sqrt{\frac{\sum_{i=1}^{N}\Big(((return_{S^{(1)}})_i - r_i - \mu_1)^2\Big)}{N}} \tag{26}$$

Similary, the liquid asset parameter μ_2 and σ_2 is determined as

$$\mu_2 = \frac{1}{N}\sum_{i=1}^{N}\Big((return_{S^{(2)}})_i - r_i + d_i\Big) \tag{27}$$

$$\sigma_2 = \sqrt{\frac{\sum_{i=1}^{N}\Big(((return_{S^{(2)}})_i - r_i + d_i - \mu_2)^2\Big)}{N}} \tag{28}$$

where r_i and d_i denotes interest rate and dividend respectively. Here, return of $S^{(j)}, j = 1, 2$ is calculated as $(return_{S^{(j)}})_i = \frac{S_i^{(j)} - S_{i-1}^{(j)}}{S_{i-1}^{(j)}}$



(a) KOSPI200

(b) House price index

Figure 2: KOSPI 200 and House price index from 2004. 01 to 2019. 08

Other parameters such as δ, p, λ, ρ will be determined as follows.

To decide the discount factor δ, we will calculate it as suggested in [3]. In [3], discount facter β is represented as $\beta = 1/(1+\tilde{r})$ where $\tilde{r}$ is real interest rate. That is, $\tilde{r}_i = r_i + \pi_i^e$ where $\tilde{r}_i$ is real interest rate, r_i is nominal interest rate (just interest rate we mostly use), and π^e is the expected inflation rate. And by the adaptive inflation expectation, expected inflation rate is based on lagged inflation rate. The relation between δ in this paper and β is

$$e^{-\delta} = \beta = 1/(1+\tilde{r}) \tag{29}$$

So, we calculate discount factor δ as $\delta = -log(\beta) = -log(1/(1+\tilde{r}))$.

For the utility function parameter p, we will just use the value in [4]. According to [4], relative risk aversion coefficient of real estate asset owners in 2012 is 3.85. It means that $(1-p) = 3.85$ i.e. $p = -2.85$ in our utility function.

Transaction cost λ mostly consists of the acquisition tax, local education tax, agricultural special tax, real estate agent's commission and legal fees. So we will only consider these. The rates of components are listed below.

$$\text{(acquisition tax)} + \text{(local education tax)} + \text{(agricultural special tax)} \Rightarrow 1.1 \sim 3.5\%$$

$$\text{real estate agent's commission} \Rightarrow 0.4 \sim 0.9\%$$

$$\text{legal fees} \Rightarrow 0.2 \sim 0.3\%$$

Considering these, we will use $\lambda = 3\%$

The correlation ρ is the correlation between KOSPI200 return and house price return. Figure 3 represents the graph of kospi200 return and house price return.

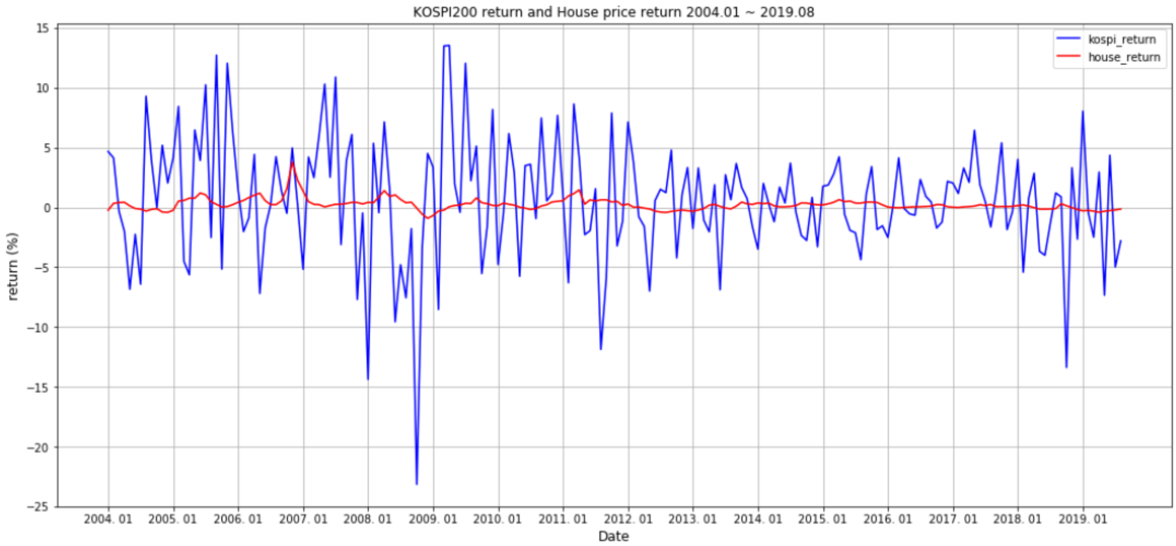


Figure 3: KOSPI200 return and House price return

5.2 Results

In this subsection, using the KOSPI200 and House price index data, we will apply to the results in Section 4 and see the effect of transaction costs. The procedure will be the following. First, the estimated parameters using the formula described in Section 5.1 will be shown. And then, we check the assumptions used in Section 4. Lastly, we will apply to the results in Section 4.

As described in Section 5.1, we can get the parameter estimates as follows.

$$\mu_1 = 0.0001, \quad \sigma_1 = 0.0047 \quad \text{illiquid asset}$$
$$\mu_2 = 0.0052, \quad \sigma_2 = 0.051 \quad \text{liquid asset}$$
$$p = -2.85, \quad \rho = 0.008, \quad \delta = 0.0004, \quad \lambda = 0.03$$

Here, $\mu_1, \mu_2, \sigma_1, \sigma_2$ was calculatied by (25) to (28).

So, what we want to do is applying these estimated parameters to the results in Section 4. But before we apply the estimated parameters to the results in Section 4, we should check the assumptions in Section 4.

The assumptions in Section 4 were

$$p > 0, \quad \mu_1 > \frac{\rho\mu_2\sigma_1}{\sigma_2}, \quad \delta > \frac{q}{2(1-\rho^2)}\left(\left(\frac{\mu_1}{\sigma_1}\right)^2 + \left(\frac{\mu_2}{\sigma_2}\right)^2 - 2\rho\frac{\mu_1\mu_2}{\sigma_1\sigma_2}\right).$$

From our estimated parameters, we can check that

$$p < 0, \quad \mu_1 > \frac{\rho\mu_2\sigma_1}{\sigma_2}, \quad \delta > \frac{q}{2(1-\rho^2)}\left(\left(\frac{\mu_1}{\sigma_1}\right)^2 + \left(\frac{\mu_2}{\sigma_2}\right)^2 - 2\rho\frac{\mu_1\mu_2}{\sigma_1\sigma_2}\right).$$

The only difference with the assumptions is $p < 0$. This induces some changes in some parts of the results. For the case of transaction cost $\lambda = 0$, there are no changes. So, we can calculate C^M, π_1^M, π_2^M as

$$\begin{aligned}
c^M &:= (1+q)\left(\delta - \frac{q}{2(1-\rho^2)}\left(\left(\frac{\mu_1}{\sigma_1}\right)^2 + \left(\frac{\mu_2}{\sigma_2}\right)^2 - 2\rho\frac{\mu_1\mu_2}{\sigma_1\sigma_2}\right)\right) = 0.00114\\
\pi_1^M &:= \frac{(1+q)(\mu_1 - \frac{\rho\sigma_1}{\sigma_2}\mu_2)}{(1-\rho^2)\sigma_1^2} = 1.13\\
\pi_2^M &:= \frac{(1+q)(\mu_2 - \frac{\rho\sigma_2}{\sigma_1}\mu_1)}{(1-\rho^2)\sigma_2^2} = 0.52
\end{aligned}$$

And $\underline{x}, \bar{x}, g(\underline{x}), g(\bar{x})$ changes to

$$\begin{aligned}
\underline{x} &= -\frac{q\pi_1^M}{c^M} + \frac{q\zeta}{c^M}\lambda^{1/3} + O\left(\lambda^{2/3}\right) = 331.5\\
\bar{x} &= -\frac{q\pi_1^M}{c^M} - \frac{q\zeta}{c^M}\lambda^{1/3} + O\left(\lambda^{2/3}\right) = 1132.4\\
g(\underline{x}) &= -\frac{1}{c^M} + \frac{q(1-\rho^2)\sigma_1^2\zeta^2}{2(1+q)(c^M)^2}\lambda^{2/3} + O\left(\lambda^{2/3}\right) = -883.6\\
g(\underline{x}) &= -\frac{1}{c^M} - \frac{q(1-\rho^2)\sigma_1^2\zeta^2}{2(1+q)(c^M)^2}\lambda^{2/3} + O\left(\lambda^{2/3}\right) = -865.2
\end{aligned}$$

where

$$\zeta := \left(\frac{3(1+q)(\pi_1^M)^2(1-\pi_1^M)^2}{4} + \frac{3(1+q)(\mu_2(1+q) - \rho\sigma_1\sigma_2)^2(\pi_1^M)^2}{4(1-\rho^2)\sigma_1^2\sigma_2^2}\right)^{\frac{1}{3}} = 1.99$$

Here, the only change from section 4 is just the sign of $\underline{x}, \bar{x}, g(\underline{x}), g(\bar{x})$ and ζ doesn't change.

5.2.1 Parameter estimation in the optimal trading of the illiquid asset.

According to optimal strategy of illiquid asset in Section 4, trading the proportion of illiquid asset within the interval $[\underline{\pi}_1, \bar{\pi}_1]$ is optimal. In this case, $p < 0$ doesn't generate any change. So, $\underline{\pi}_1, \bar{\pi}_1$ can be calculated as

$$\underline{\pi}_1 = \pi_1^M - \zeta\lambda^{\frac{1}{3}} + O\left(\lambda^{2/3}\right) = 0.51$$
$$\bar{\pi}_1 = \pi_1^M + \zeta\lambda^{\frac{1}{3}} + O\left(\lambda^{2/3}\right) = 1.75$$

where transaction cost $\lambda = 0.03$. In other words, the optimal trading of the illiquid asset is maintaining the proportion of investment in the illiquid asset as

$$\underline{\pi}_1 = 0.51 \leq \frac{\hat{\varphi}_t^{(1)} S_t^{(1)}}{\hat{\varphi}_t^{(0)} + \hat{\varphi}_t^{(1)} S_t^{(1)} + \hat{\varphi}_t^{(2)} S_t^{(2)}} \leq 1.75 = \bar{\pi}_1$$

Figure 4 shows the optimal trading of the illiquid asset with different transaction costs and Table 1 represents the values of $\underline{\pi}_1, \bar{\pi}_1$ in terms of percentage. For example, suppose that the transaction cost is 1%. Then, it is optimal not to trade illiquid asset when the proportion of investment in the illiquid asset is within 70% to 156%. From Figure 4 and Table 1, We can see that transaction cost does have an effect on the interval $[\underline{\pi}_1, \bar{\pi}_1]$. It is because the coefficient of $\lambda^{\frac{1}{3}}$, which represents the sensitivity of the increase in the investment proportion of the illiquid risky asset due to transaction costs, is about 1.99 which is large compared to the liquid asset.

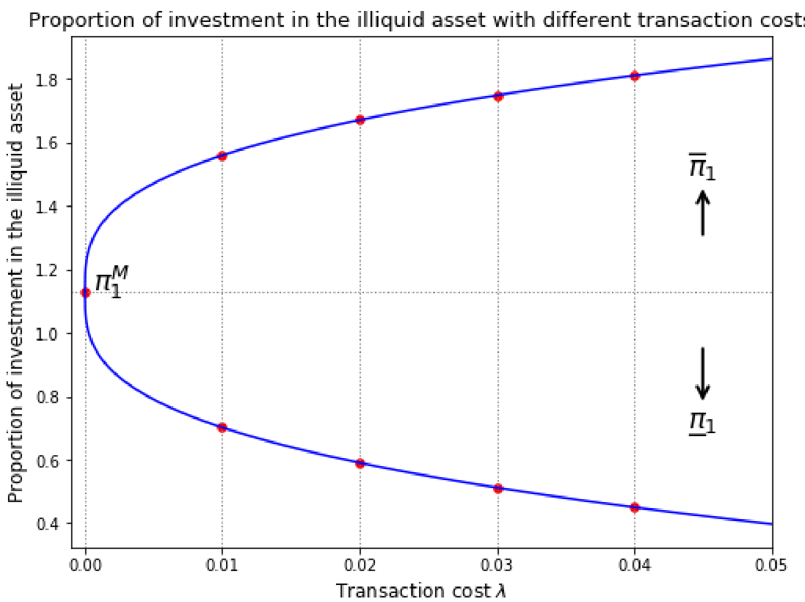


Figure 4: Proportion of investment in the illiquid asset

$\lambda(\%)$	0	1	2	3	4
$\bar{\pi}_1(\%)$	113.082	155.975	167.124	174.945	181.172
$\underline{\pi}_1(\%)$	113.082	70.189	59.040	51.219	44.993

Table 1: Values of $\underline{\pi}_1, \bar{\pi}_1$ on certain points

5.2.2 Parameter estimation in the optimal trading of the liquid asset.

In Section 4, we have seen the optimal proportion of the liquid asset when investor sells or buys the illiquid asset. Similar to the case of illiquid asset, $p < 0$ doesn't generate any change in the case of liquid asset. So, $\underline{\pi}_2, \bar{\pi}_2$ can be calculated as

$$\underline{\pi}_2 := \pi_2^M - \frac{\rho\sigma_1\zeta}{\sigma_2}\lambda^{1/3} + O\left(\lambda^{2/3}\right) = 0.518 \quad \text{when selling the illiquid asset}$$
$$\bar{\pi}_2 := \pi_2^M + \frac{\rho\sigma_1\zeta}{\sigma_2}\lambda^{1/3} + O\left(\lambda^{2/3}\right) = 0.519 \quad \text{when buying the illiquid asset}$$

where transaction cost $\lambda = 0.03$. In other words, when the invested proportion of the illiquid asset reaches $\bar{\pi}_1 = 1.75$, investor sells illiquid asset and the proportion of the liquid asset is $\underline{\pi}_2 = 0.518$ at that point. Similarly, when the invested proportion of the illiquid asset reaches $\underline{\pi}_1 = 0.51$, investor sells illiquid asset and the proportion of the liquid asset is $\bar{\pi}_2 = 0.519$ at that point.

Similar to illiquid risky asset case, Figure 5 shows the optimal trading of the liquid asset with different transaction costs and Table 2 represents the values of $\underline{\pi}_2, \bar{\pi}_2$ in terms of percentage. For example, suppose that transaction cost is 1%. Then, when an investor sells (respectively buys) illiquid asset, the optimal proportion of the liquid asset is 51.876%(respectively 51.813%). From Figure 5 and Table 2, We can see that transaction cost have less effect on the interval $[\underline{\pi}_2, \bar{\pi}_2]$. It is because the coefficient of $\lambda^{\frac{1}{3}}$, which represents the sensitivity of the increase in the investment proportion of the liquid risky asset due to transaction costs, is 0.00147 which is smaller than the case of the illiquid risky asset.

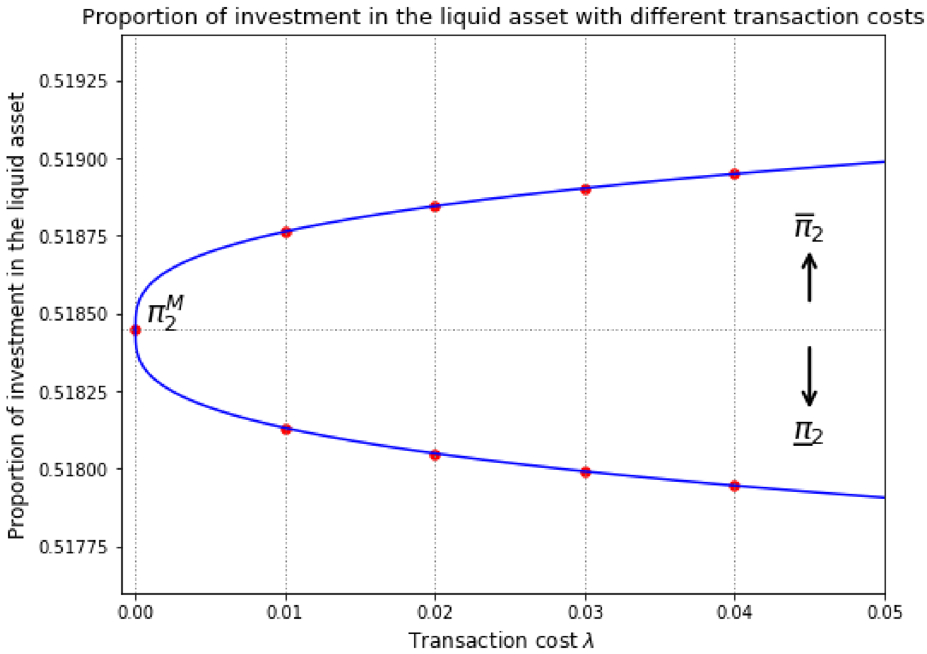


Figure 5: Proportion of investment in the liquid asset

$\lambda(\%)$	0	1	2	3	4
$\bar{\pi}_2(\%)$	51.844	51.876	51.885	51.890	51.895
$\underline{\pi}_2(\%)$	51.844	51.813	51.805	51.799	51.795

Table 2: Values of $\underline{\pi}_2, \bar{\pi}_2$ on certain points

5.2.3 Parameter estimation in the optimal consumption rate

In Section 4, we got the asymptotic expansion of the optimal consumption rate proportion for small transaction cost λ. In this case, $p < 0$ changes the result of the before. The only change is the sign in the coefficient of $\lambda^{\frac{2}{3}}$. So, the optimal consumption rate proportion is

$$\frac{\hat{c}_t}{\hat{\varphi}_t^{(0)} + \hat{\varphi}_1^{(1)} S_t^{(1)} + \hat{\varphi}_t^{(2)} S_t^{(2)}} = c^M - \frac{q(1-\rho^2)\sigma_1^2\zeta^2}{2(1+q)}\lambda^{\frac{2}{3}} + O(\lambda) = 0.00116, \quad a.s.$$

where transaction costs $\lambda = 0.03$

Figure 6 shows the optimal consumption rate with different transaction costs and Table 3 represents the values of it in terms of percentage. For example, when transaction cost is 1%, the optimal consumption rate is 0.11494%. Similar to the case of a liquid asset, the optimal consumption rate isn't much affected by the transaction cost λ. It is because the coefficient of $\lambda^{\frac{2}{3}}$, the sensitivity of the increase in the consumption rate due to transaction costs is 0.000125 which is small compared to the illiquid asset case. Also, we can see that when transaction cost changes 0% to 4%, optimal consumption rate changes to 0.1144% to 0.1158%. That is, compared to the case having no transaction costs, the investor increases his consumption rate by about 1.2%.

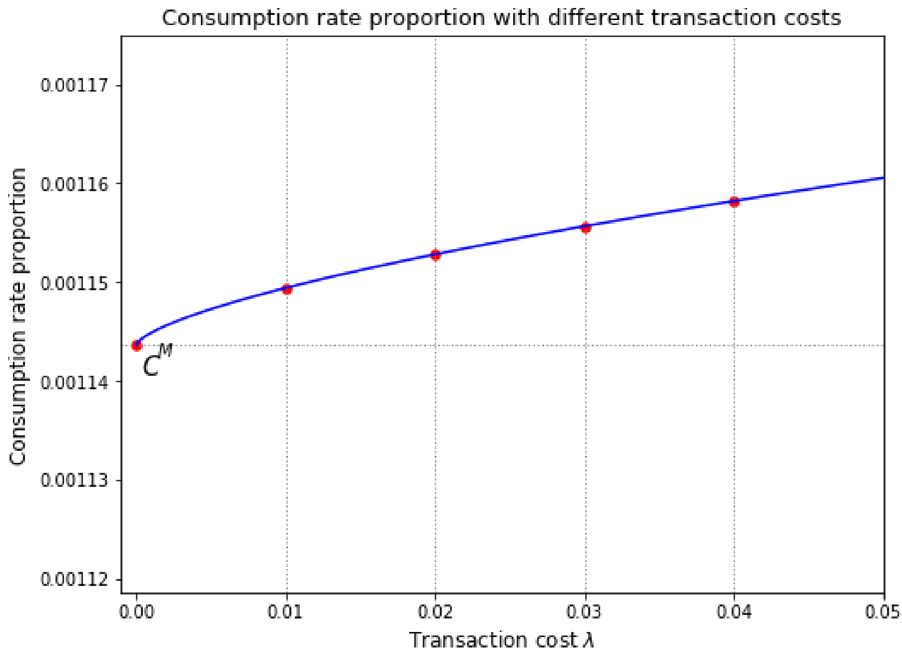


Figure 6: Proportion of investment in the illiquid asset

$\lambda(\%)$	0	1	2	3	4
c(%)	0.11436	0.11494	0.11528	0.11556	0.11582

Table 3: Values of optimal consumption rate on certain points

We can have the following interpretation. First, Note that when the transaction cost $\lambda = 0$, the optimal consumption rate is C^M. So, since the coefficient of $\lambda^{\frac{2}{3}}$, $-\frac{q(1-\rho^2)\sigma_1^2\zeta^2}{2(1+q)} = 0.000125$ which is positive, we can see that the optimal consumption rate proportion is always greater than C^M in the market having transaction costs. One of the possible explanation is that the existence of the transaction costs makes the investment unappealing. So the investor increases his consumption rate.

Also, recall that $-\frac{q(1-\rho^2)\sigma_1^2\zeta^2}{2(1+q)}$, the coefficient of $\lambda^{\frac{2}{3}}$ represents the sensitivity of the increase in the consumption rate which is generated by the transaction costs. From [1], the original market model is equal to the market having only illiquid asset when $\mu_2 = \rho = 0$. Therefore, to compare with the effect of the transaction costs on consumption where the market model has the illiquid asset only, we consider $\left(-\frac{q(1-\rho^2)\sigma_1^2\zeta^2}{2(1+q)}\Big|_{\mu_2=\rho=0} \right)$, that is, the sensitivity of the increase in the consumption rate which is generated by the transaction costs when there is only an illiquid asset.

From the data, we can get $\left(-\frac{q(1-\rho^2)\sigma_1^2\zeta^2}{2(1+q)}\Big|_{\mu_2=\rho=0} \right) = 0.00000129$. So, we can get the following inequality.

$$-\frac{q(1-\rho^2)\sigma_1^2\zeta^2}{2(1+q)} \geq \left(-\frac{q(1-\rho^2)\sigma_1^2\zeta^2}{2(1+q)}\Big|_{\mu_2=\rho=0} \right), \quad \text{if} \quad \rho = 0$$

It means that the coefficient of $\lambda^{\frac{2}{3}}$ in the market having liquid and illiquid assets is larger than the market having illiquid assets only. In other words, The effect of the transaction costs on consumption is more noticeable when there are liquid and illiquid assets. Since $\rho = 0$, the optimal investment proportion of the illiquid risky asset is π_1^M in both markets. The existence of the liquid asset makes investor also take an exposure to the risk of the liquid risky asset, That is $\pi_2^M \neq 0$. And this induces the increase in the volatility of the total wealth process. Therefore, the market having liquid asset trading opportunity has more frequent trading of the illiquid asset (from the optimal trading of the illiquid asset) and the model has more trading costs due to rebalancing. So, in the market with a liquid asset, the effect of the transaction costs on the consumption is stronger.

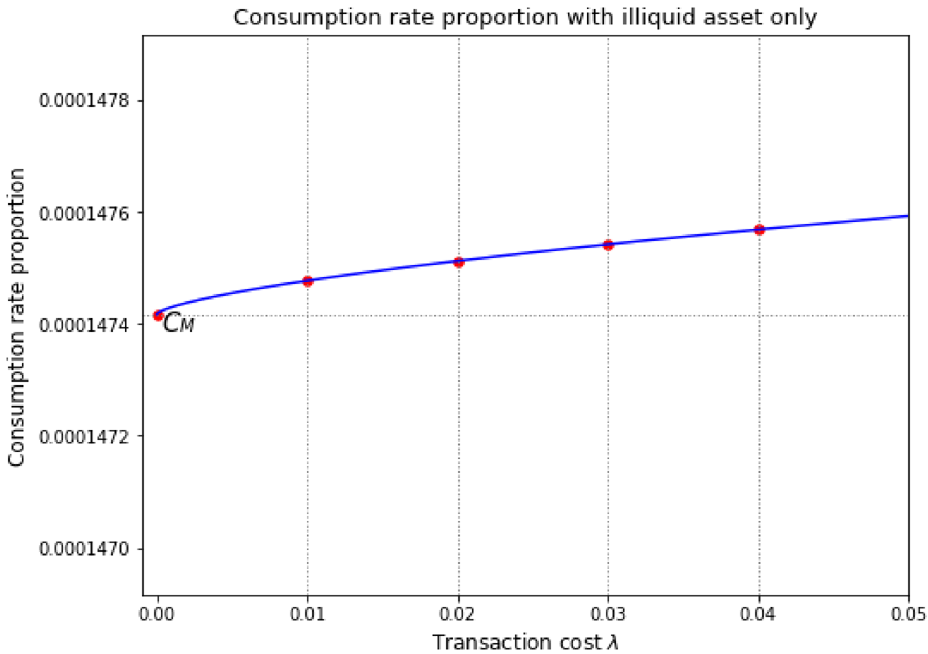


Figure 7: consumption rate with illiquid asset only

$\lambda(\%)$	0	1	2	3	4
c(%)	0.014742	0.014748	0.014751	0.014754	0.014757

Table 4: Values of optimal consumption rate on certain points with illiquid asset only

Figure 7 and Table 4 shows the effect of the transaction cost to the consumption rate with the illiquid asset only. For example, suppose that the transaction cost is 1%. Then, the optimal consumption rate is 0.014748% when there is only an illiquid asset. And when transaction cost changes 0% to 4%, the optimal consumption rate changes to 0.01474% to 0.01476%. That is, compared to the case having no transaction costs, the investor increases his consumption rate by about 0.14%. Also, we can see that the incremental amount of the optimal consumption rate is much smaller than the case with a liquid asset. In other words, compared with the market having a liquid asset, the optimal consumption rate is less affected by the transaction cost λ.

VI Conclusion

In this paper, we have considered the model with two risky assets, which are liquid and illiquid assets. From [1], we could get the asymptotic ally optimal trading and consumption strategies as in Section 4. The optimal investment of the illiquid asset is trading the proportion of illiquid asset within some interval(no transaction region). And using these strategies, we applied to the real data. For the liquid asset, we considered monthly KOSPI200 data which is the index consists of 200 big companies of the Stock Market Division in Korea. For the illiquid asset, we used monthly house price index data which represents the price of the residential real estate in Korea. Using these two data, we obtained the optimal investment and consumption rate and checked how they change as transaction costs become different. Also, we calculated the optimal consumption rate in the market having illiquid asset only to compare it with the optimal consumption rate in the original model.

The optimal strategies we obtained are the following. First, the optimal investment of illiquid asset is investing 113% of the total wealth to the illiquid asset and investor doesn't trade when it is within 51% to 175% of the total wealth(no transaction region). And for the liquid asset, the optimal investment of the liquid asset is investing 51.844% of the total wealth to the liquid asset. And when an investor sells(respectively buys) the illiquid asset, the optimal proportion of the liquid asset is 51.8%(respectively, 51.9%). Lastly, the optimal strategy of the consumption rate is consuming 0.12% of the total wealth in a month.

For example, suppose that we have the total wealth 1,000M KRW. Then, we have the following optimal strategies. First, we lend money from the bank about 650M KRW. And then, we invest around 1,130M KRW to the residential real estate and don't sell or buy when it is from 510M KRW to 1,750M KRW. Also, we invest 518.4M won to the stock and when selling(respectively buying) the illiquid asset, the amount of the investment in the stock is 518M KRW(respectively, 519M KRW) And optimal consumption rate is consuming 120M KRW in a month.

As transaction costs change, we could see that the effect of the transaction costs is large on the illiquid asset. But illiquid assets and consumption are less affected by the transaction costs. Compared with the model having an illiquid asset only, the effect of the transaction costs on the consumption is more noticeable. It is because the existence of the liquid asset causes an increase in the volatility of the total wealth process, and therefore more frequent trading of the illiquid asset.

We have seen that the existence of the liquid asset does have an effect compared to the market having illiquid asset only. So we can conclude that It is important to consider liquid asset when analyzing the market having an illiquid asset.

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