# Linear Algebra

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# Lecture Overview

- Elements in linear algebra
- Linear system
- Linear combination, vector equation,
  Four views of matrix multiplication
- <u>Linear independence, span, and subspace</u>
- Linear transformation
- Least squares
- Eigendecomposition
- Singular value decomposition

# Span and Subspace

Subset : 부분집합

Subset of $\mathbb{R}^3$ : $S=\left\{\begin{bmatrix}1\\2\\3\end{bmatrix},\begin{bmatrix}5\\4\\2\end{bmatrix}\right\}$

$\begin{bmatrix}5\\7\\9\end{bmatrix}$ $\begin{bmatrix}3\\3\\3\end{bmatrix}$

$3(1\cdot v_1+1\cdot v_2)$ $2((-1)v_1+1\cdot v_2)$

들어있으면 닫혀있다

$3v_1+3v_2-2v_1+2v_2$
$=v_1+5v_2$

$0\,\boxed{\phantom{x}}+0\,\boxed{\phantom{x}}=\boxed{\phantom{x}}$

선형결합

- **Definition**: A **subspace** $H$ is defined as a subset of $\mathbb{R}^n$ closed under linear combination:
  - 닫혀있다 선형 결합에
  - For any two vectors, $\mathbf{u}_1,\mathbf{u}_2 \in H$, and any two scalars $c$ and $d$, $c\mathbf{u}_1+d\mathbf{u}_2 \in H$.

곱셈의 닫힘

$2\in S$

$S=\{2,4,8,16,32\cdots\}$

$\{2^n : n=1\cdots\infty\}$

- Span $\{\mathbf{v}_1,\cdots,\mathbf{v}_p\}$ is always a subspace. Why?
  - $\mathbf{u}_1 = a_1\mathbf{v}_1+\cdots+a_p\mathbf{v}_p$, $\mathbf{u}_2 = b_1\mathbf{v}_1+\cdots+b_p\mathbf{v}_p$
  - $c\mathbf{u}_1+d\mathbf{u}_2 = c\left(a_1\mathbf{v}_1+\cdots+a_p\mathbf{v}_p\right)+d\left(b_1\mathbf{v}_1+\cdots+b_p\mathbf{v}_p\right)$
    $= (ca_1+db_1)\mathbf{v}_1+\cdots+\left(ca_p+db_p\right)\mathbf{v}_p$
- In fact, a subspace is always represented as Span $\{\mathbf{v}_1,\cdots,\mathbf{v}_p\}$.

# Basis of a Subspace

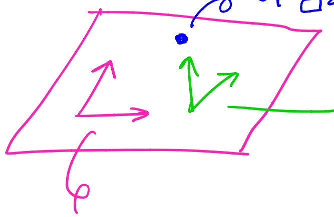
평면이 먼저 주어짐. 이 subspace를 fully span하는 벡터들 : 기저벡터

기저벡터

- **Definition**: A **basis** of a subspace $H$ is a set of vectors that satisfies both of the following:
  1) - Fully spans the given subspace $H$
  2) - Linearly independent (i.e., no redundancy) 중복이 있으면 안됨!! linearly independent
- In the previous example, where $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, Span $\{\mathbf{v}_1, \mathbf{v}_2\}$ forms a plane, but $\mathbf{v}_3 = 2\mathbf{v}_1 + 3\mathbf{v}_2 \in \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$, $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis of $H$, but not $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ nor $\{\mathbf{v}_1\}$ is a basis.

# Non-Uniqueness of Basis

이 점을 표현하는데 각각의 기저벡터의 가중치가 다르다



이것도 span

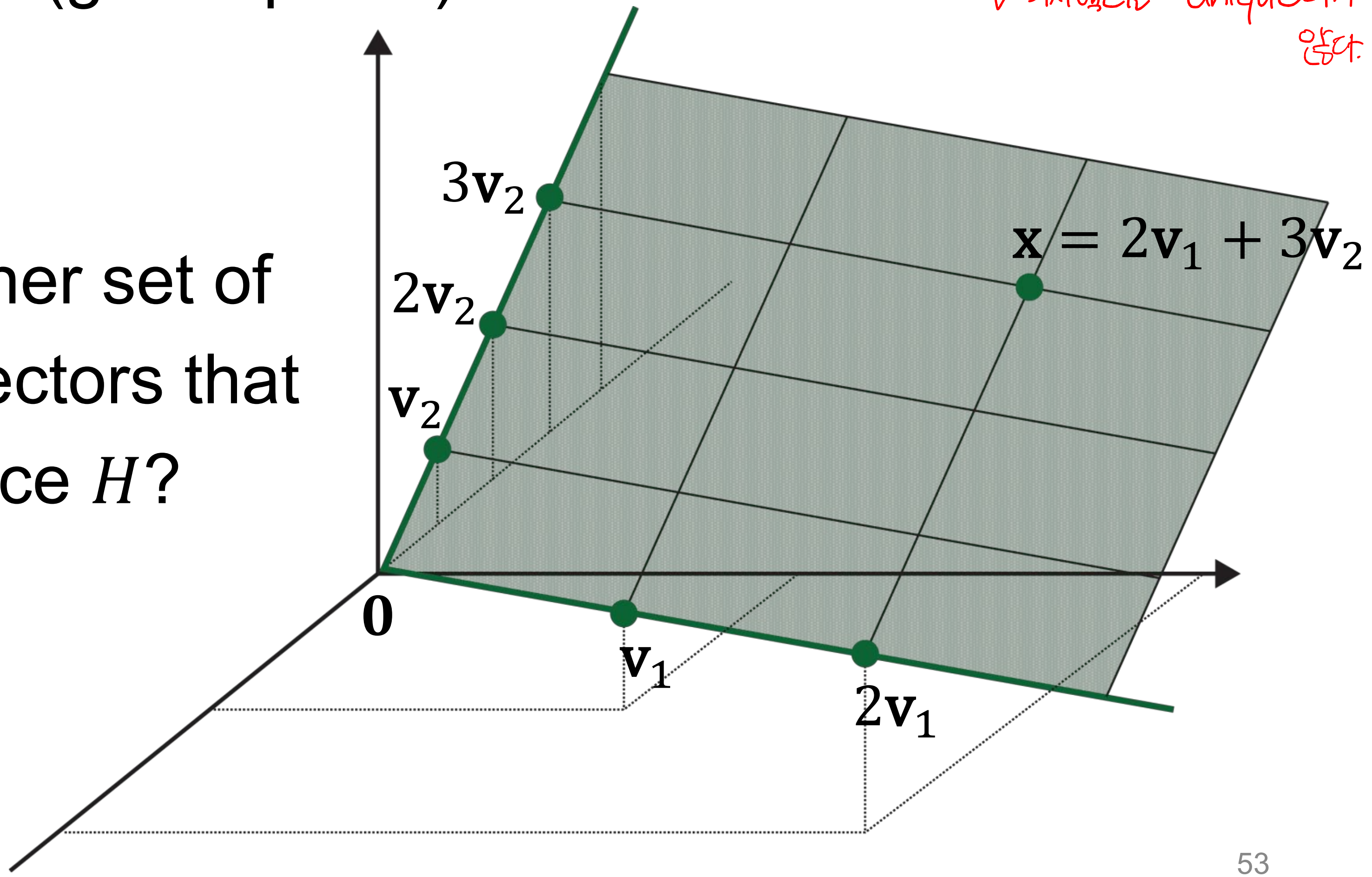
⇓

change of basis

이것도 subspace를 span

→ 기저벡터는 unique하지 않다.

- Consider a subspace $H$ (green plane).
- Is a basis unique?
- That is, is there any other set of linearly independent vectors that span the same subspace $H$?



# Dimension of Subspace

subspace의 기저벡터의 개수.

Standard basis vector

$$\underset{e_1}{\begin{bmatrix}1\\0\\0\end{bmatrix}} \underset{e_2}{\begin{bmatrix}0\\1\\0\end{bmatrix}} \underset{e_3}{\begin{bmatrix}0\\0\\1\end{bmatrix}}$$

- What is then unique, given a particular subspace $H$?
- Even though different bases exist for $H$, the number of vectors in any basis for $H$ will be unique.
- We call this number as the dimension of $H$, denoted as dim $H$.
- In the previous example, the dimension of the plane is 2, meaning any basis for this subspace contains exactly two vectors.

# Column Space of Matrix

- **Definition**: The **column space** of a matrix $A$ is the subspace spanned by the columns of $A$. We call the column space of $A$ as Col $A$.

subspace 조건 만족

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \Longrightarrow \quad \text{Col } A = \text{Span}\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\}$$

- What is dim Col $A$?

# Matrix with Linearly Dependent Columns

$\oplus$

- Given $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, note that $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$,

  i.e., the third column is a linear combination of the first two.

$$\text{Col } A = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\} \Rightarrow \text{Col } A = \text{Span}\left\{ \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}} \right\}$$

rank = 2 ↳ 없어두 됨

rank A = 2

- What is dim Col $A$?

# Rank of Matrix

- **Definition**: The **rank** of a matrix $A$, denoted by rank $A$, is the dimension of the column space of $A$:
  - rank $A$ = dim Col $A$

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# Summary So Far

- Scalars, vectors, matrices, and their operations such as addition, scalar multiple, matrix multiplication, transpose
- Linear system: solving using inverse matrix
- Matrix equation and vector equation
- Linear combination and Span
  - When does the solution of a linear system exist?
- Four views of matrix multiplication: inner product, column combination, row combination, sum of rank-1 outer products
- Linear independence
  - If the solution of a linear system exists, when is it unique or many?
- Subspace
  - Subset of vectors in $\mathbb{R}^n$ closed under linear combination
  - Basis and dimension
  - Column space and rank of a matrix