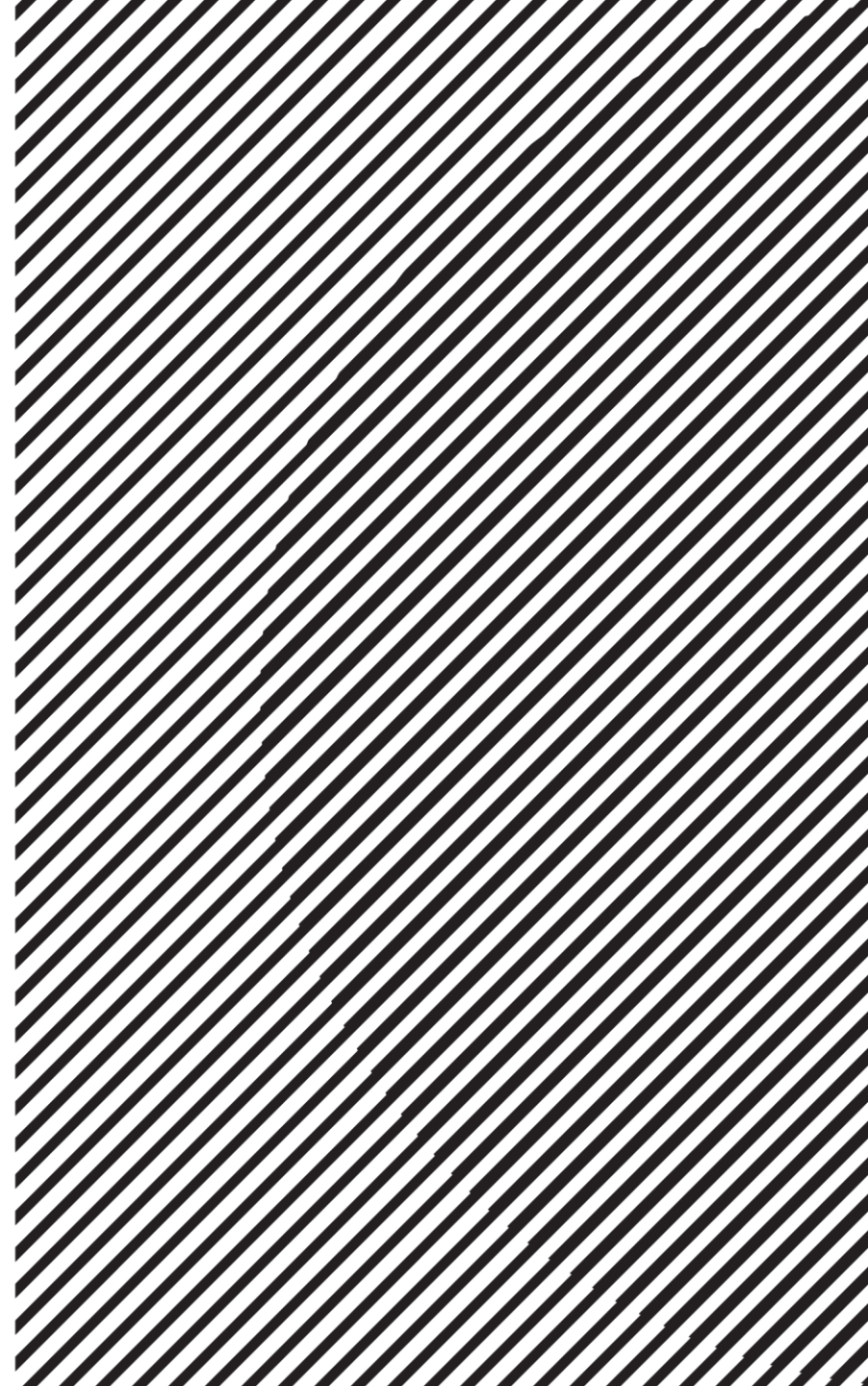


---

# Linear Algebra

주재걸  
고려대학교 컴퓨터학과





# Lecture Overview

- Elements in linear algebra
- Linear system
- Linear combination, vector equation,  
Four views of matrix multiplication
- Linear independence, span, and subspace
- Linear transformation
- Least squares
- Eigendecomposition
- Singular value decomposition

# Recall: Linear System

- Recall the matrix equation of a linear system:

Person ID	Weight	Height	Is_smoking	Life-span
1	60kg	5.5ft	Yes (=1)	66
2	65kg	5.0ft	No (=0)	74
3	55kg	6.0ft	Yes (=1)	78

$$\begin{bmatrix} 60 & 5.5 & 1 \\ 65 & 5.0 & 0 \\ 55 & 6.0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 66 \\ 74 \\ 78 \end{bmatrix}$$

$A \quad \mathbf{x} = \mathbf{b}$

- Or, a vector equation is written as

$$\begin{bmatrix} 60 \\ 65 \\ 55 \end{bmatrix} x_1 + \begin{bmatrix} 5.5 \\ 5.0 \\ 6.0 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} x_3 = \begin{bmatrix} 66 \\ 74 \\ 78 \end{bmatrix}$$

$\mathbf{a}_1 x_1 + \mathbf{a}_2 x_2 + \mathbf{a}_3 x_3 = \mathbf{b}$

# Uniqueness of Solution for $A\mathbf{x} = \mathbf{b}$

- The solution exists only when  $\mathbf{b} \in \text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ . *해가 존재.*

$$\begin{bmatrix} 60 \\ 65 \\ 55 \end{bmatrix} x_1 + \begin{bmatrix} 5.5 \\ 5.0 \\ 6.0 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} x_3 = \begin{bmatrix} 66 \\ 74 \\ 78 \end{bmatrix}$$

$$\mathbf{a}_1 x_1 + \mathbf{a}_2 x_2 + \mathbf{a}_3 x_3 = \mathbf{b}$$

- If the solution exists for  $A\mathbf{x} = \mathbf{b}$ , when is it unique? *근데 해가 하나만 존재?*
- It is unique when  $\mathbf{a}_1$ ,  $\mathbf{a}_2$ , and  $\mathbf{a}_3$  are linearly independent.
- Infinitely many solutions exist when  $\mathbf{a}_1$ ,  $\mathbf{a}_2$ , and  $\mathbf{a}_3$  are linearly dependent.

# Linear Independence

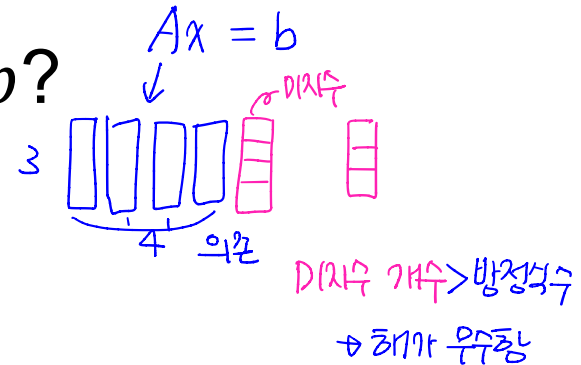
직관적인 정의

## (Practical) Definition:

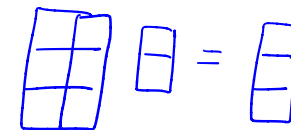
- Given a set of vectors  $\mathbf{v}_1, \dots, \mathbf{v}_p \in \mathbb{R}^n$ , check if  $\mathbf{v}_j$  can be represented as a linear combination of the previous vectors  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{j-1}\}$  for  $j = 1, \dots, p$ , e.g.,

$$\mathbf{v}_j \in \text{Span} \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{j-1}\} \text{ for some } j = 1, \dots, p?$$

첫 번째 벡터  $\rightarrow$  span 포함?  
+ 두 번째 벡터 추가  $\rightarrow$  span에 포함? span에 포함되면 좋음  
+ ...  
+ p 번째 벡터 추가  $\rightarrow$



- If at least one such  $\mathbf{v}_j$  is found, then  $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$  is linearly dependent.
- If no such  $\mathbf{v}_j$  is found, then  $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$  is linearly independent.



# Linear Independence

0 벡터는 어떤 span에 모두 들어감

homogeneous equation

## (Formal) Definition:

- Consider  $x_1 \mathbf{v}_1 + x_2 \mathbf{v}_2 \cdots + x_p \mathbf{v}_p = \mathbf{0}$ .

$b$  (0 벡터)

- Obviously, one solution is  $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ ,  $x_i$ 가 0 벡터이면 당연히 solution이 됨  $\rightarrow$  trivial sol.

which we call a trivial solution. 이 해 이외의 다른 결과가 존재한다면 linearly independent.

- $\mathbf{v}_1, \cdots, \mathbf{v}_p$  are linearly independent if this is the only solution.
- $\mathbf{v}_1, \cdots, \mathbf{v}_p$  are linearly dependent if this system also has other nontrivial solutions, e.g., at least one  $x_i$  being nonzero.

적어도 하나는 0이 아닐 때



# Two Definitions are Equivalent

- If  $\mathbf{v}_1, \dots, \mathbf{v}_p$  are linearly dependent, consider a nontrivial solution.
- In the solution, let's denote  $j$  as the last index such that  $x_j \neq 0$ .
- Then, one can write  $x_j \mathbf{v}_j = -x_1 \mathbf{v}_1 - \dots - x_{j-1} \mathbf{v}_{j-1}$ ,  
and **safely divide it by  $x_j$** , resulting in

$$\mathbf{v}_j = -\frac{x_1}{x_j} \mathbf{v}_1 - \dots - \frac{x_{j-1}}{x_j} \mathbf{v}_{j-1} \in \text{Span} \{ \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{j-1} \}$$

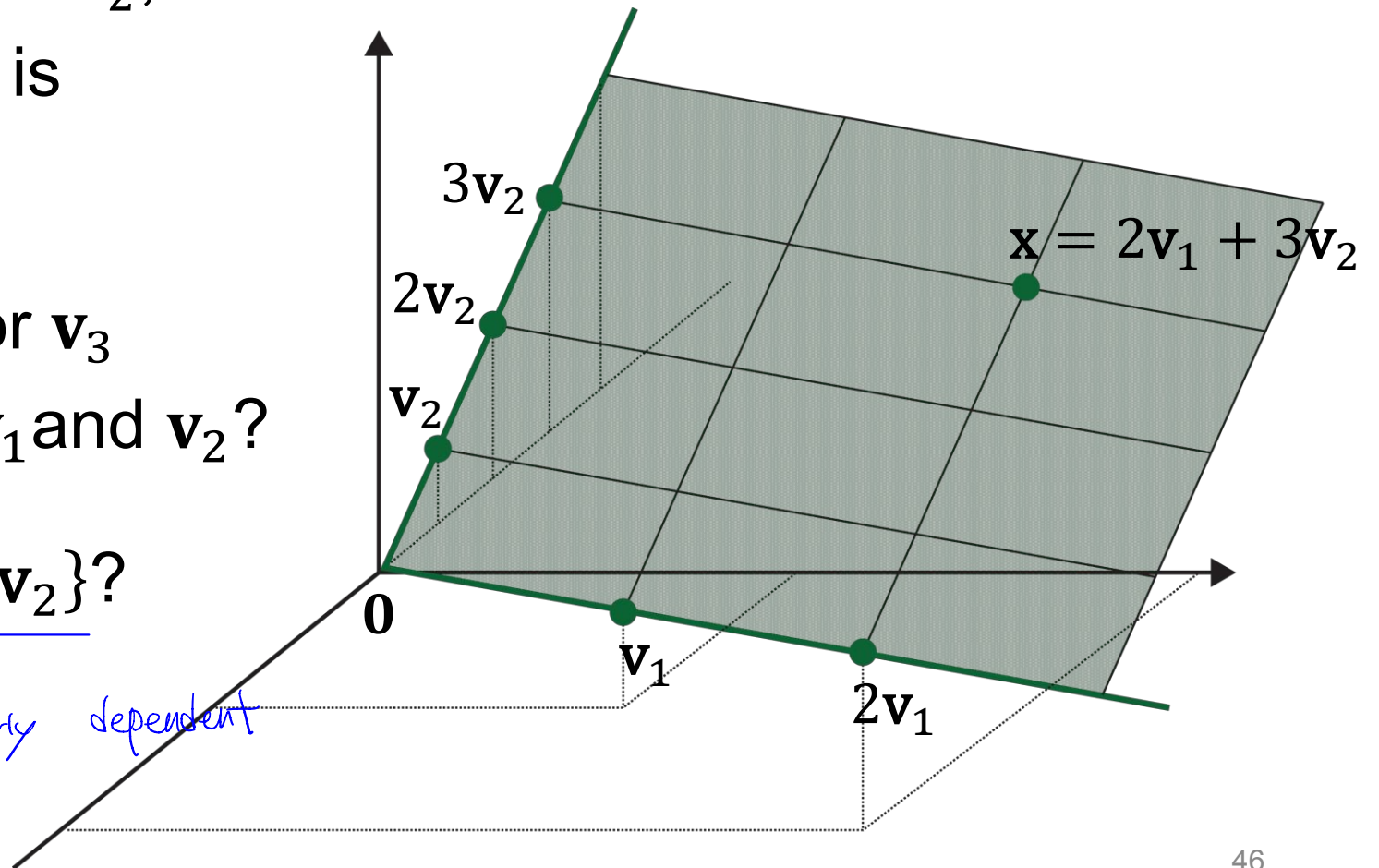
which means  $\mathbf{v}_j$  can be represented as a linear combination of the previous vectors.



# Geometric Understanding of Linear Dependence

- Given two vectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$ ,  
Suppose  $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$  is  
the plane on the right.
- When is the third vector  $\mathbf{v}_3$   
linearly dependent of  $\mathbf{v}_1$  and  $\mathbf{v}_2$ ?
- That is,  $\mathbf{v}_3 \in \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ ?

or linearly dependent





# Linear Dependence

- A linearly dependent vector does not increase Span!
- If  $\mathbf{v}_3 \in \text{Span} \{\mathbf{v}_1, \mathbf{v}_2\}$ , then
$$\text{Span} \{\mathbf{v}_1, \mathbf{v}_2\} \stackrel{\checkmark}{=} \text{Span} \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\},$$
추가해도 span 동일
- Why?
- Suppose  $\mathbf{v}_3 = d_1\mathbf{v}_1 + d_2\mathbf{v}_2$ , then the linear combination of  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  can be written as
$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = (c_1 + d_1)\mathbf{v}_1 + (c_2 + d_2)\mathbf{v}_2$$
which is also a linear combination of  $\mathbf{v}_1, \mathbf{v}_2$ .

# Linear Dependence and Linear System Solution

- Also, a linearly dependent set produces **multiple possible linear combinations** of a given vector.
- Given a vector equation  $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{b}$ , suppose the solution is  $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ , i.e.,  $\underline{3}\mathbf{v}_1 + \underline{2}\mathbf{v}_2 + \underline{1}\mathbf{v}_3 = \mathbf{b}$ .  
*homogeneous*  $2\mathbf{v}_1 + 3\mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0}$   $(\begin{bmatrix} 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & -1 \end{bmatrix}) = 0$
- Suppose also  $\mathbf{v}_3 = 2\mathbf{v}_1 + 3\mathbf{v}_2$ , a linearly dependent case.
- Then,  $3\mathbf{v}_1 + 2\mathbf{v}_2 + 1\mathbf{v}_3 = 3\mathbf{v}_1 + 2\mathbf{v}_2 + (2\mathbf{v}_1 + 3\mathbf{v}_2) = 5\mathbf{v}_1 + 5\mathbf{v}_2$ ,  
so  $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 0 \end{bmatrix}$  is another solution. Many more solutions exist.



# Linear Dependence and Linear System Solution

- Actually, many more solutions exist.
- e.g.,  $3\mathbf{v}_1 + 2\mathbf{v}_2 + 1\mathbf{v}_3 = 3\mathbf{v}_1 + 2\mathbf{v}_2 + (2\mathbf{v}_3 - 1\mathbf{v}_3)$   
 $= 3\mathbf{v}_1 + 2\mathbf{v}_2 + 2(2\mathbf{v}_1 + 3\mathbf{v}_2) - 1\mathbf{v}_3 = 7\mathbf{v}_1 + 8\mathbf{v}_2 - 1\mathbf{v}_3,$

thus  $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ -1 \end{bmatrix}$  is another solution.

# Uniqueness of Solution for $A\mathbf{x} = \mathbf{b}$

- The solution exists only when  $\mathbf{b} \in \text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ .

$$\begin{bmatrix} 60 \\ 65 \\ 55 \end{bmatrix} x_1 + \begin{bmatrix} 5.5 \\ 5.0 \\ 6.0 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} x_3 = \begin{bmatrix} 66 \\ 74 \\ 78 \end{bmatrix}$$

$$\mathbf{a}_1 x_1 + \mathbf{a}_2 x_2 + \mathbf{a}_3 x_3 = \mathbf{b}$$

- If the solution exists for  $A\mathbf{x} = \mathbf{b}$ , when is it unique?
- It is unique when  $\mathbf{a}_1$ ,  $\mathbf{a}_2$ , and  $\mathbf{a}_3$  are linearly independent.
- Infinitely many solutions exist when  $\mathbf{a}_1$ ,  $\mathbf{a}_2$ , and  $\mathbf{a}_3$  are linearly dependent.