Linear Algebra

주재걸
고려대학교 컴퓨터학과

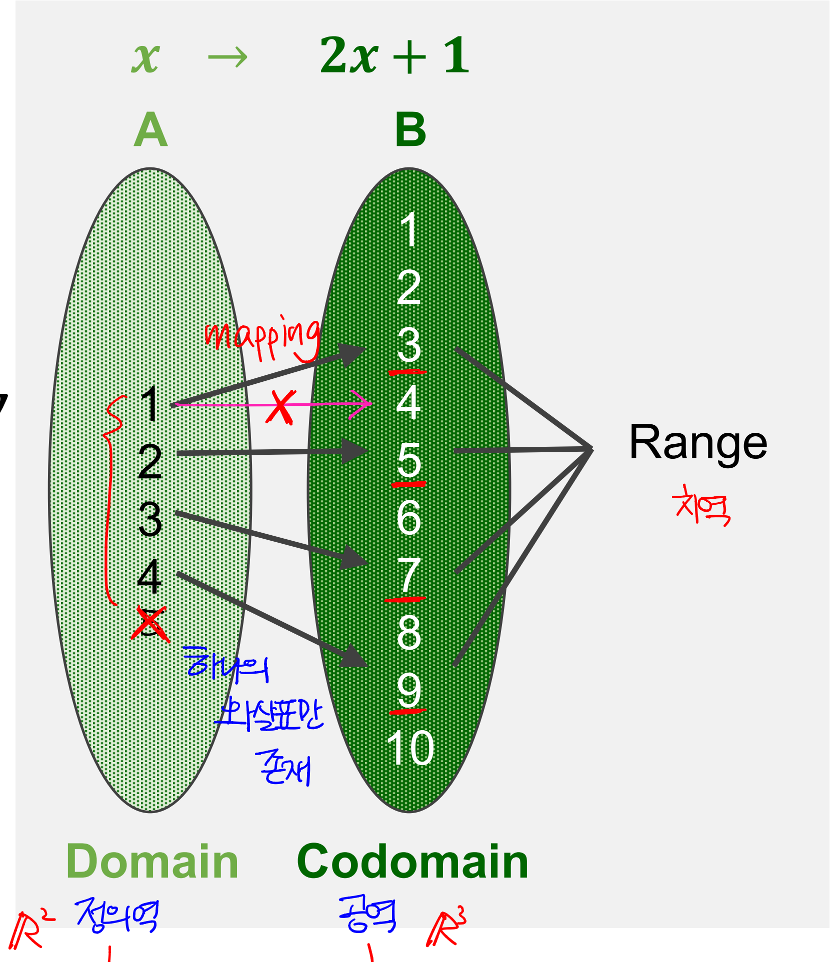
Lecture Overview

- Elements in linear algebra
- Linear system
- Linear combination, vector equation,
  Four views of matrix multiplication
- Linear independence, span, and subspace
- <u>Linear transformation</u>
- Least squares
- Eigendecomposition
- Singular value decomposition

Transformation 행렬에서의 변환

- A **transformation**, **function**, or **mapping**, T maps an input x to an output y
  - Mathematical notation: $\boxed{T: x \mapsto y}$
- **Domain**: Set of all the possible values of x
  정의역
- **Co-domain**: Set of all the possible values of y
  공역
- **Image**: a mapped output y, given x
  상 = 함수값
- **Range**: Set of all the output values mapped by each x in the domain
  치역

- **Note**: the output mapped by a particular x is uniquely determined.



기본적으로 벡터공간

Linear Transformation

선형 변환

T: $y=3x+2$ → T is linear transform?

1 → 5, 2 → 8 → $3\cdot1+4\cdot2=11$ & $3\cdot5+4\cdot8$

35 ≠ 47

→ bias가 포함되면 선형결합 아님

$T: \mathbb{R}^1 \to \mathbb{R}^1$ 선형결합이 아님 (x가 하나)

$T: \mathbb{R}^2 \to \mathbb{R}^1$ 2차원

$[3\ 2]\begin{bmatrix} x \\ 1 \end{bmatrix} = 3x+2$

선형결합이 됨

- **Definition**: A transformation (or mapping) ***T* is linear** if:

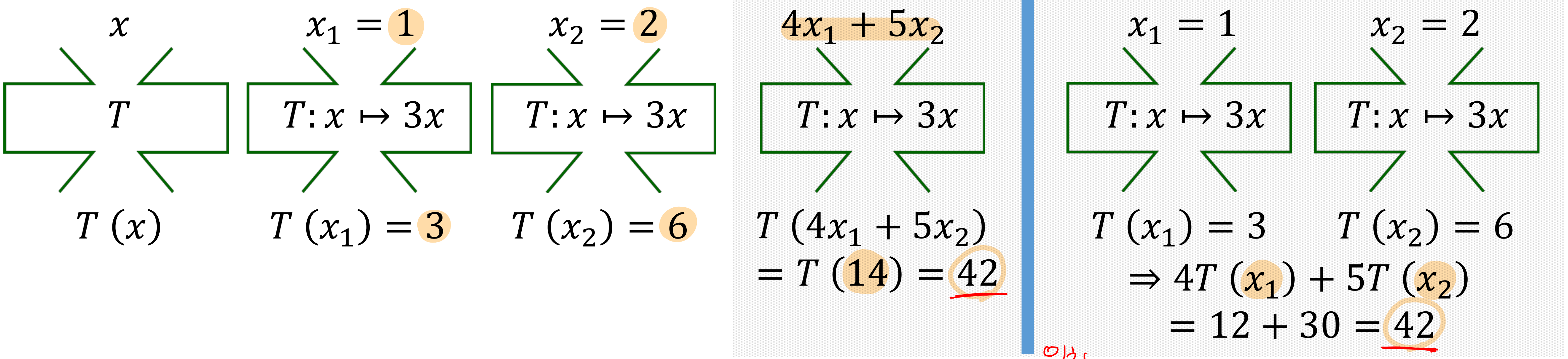
  I. $T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in the domain of T and for all scalars c and d

  일치

  정의역에서 두 개의 벡터를 뽑음

  ↳ 선형 변환

- Simple example: $T: x \mapsto y$, $T(x) = y = 3x$



일치

선형 변환

Transformations between Vectors

- $T: \mathbf{x} \in \mathbb{R}^n \mapsto \mathbf{y} \in \mathbb{R}^m$: Mapping n-dim vector to m-dim vector
- Example:

$$T: \mathbf{x} \in \mathbb{R}^3 \mapsto \mathbf{y} \in \mathbb{R}^2 \quad \mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \in \mathbb{R}^3 \quad \mapsto \quad \mathbf{y} = T(\mathbf{x}) = \begin{bmatrix} 4 \\ 5 \end{bmatrix} \in \mathbb{R}^2$$

정의역 공역

Matrix of Linear Transformation

- Example: Suppose ① T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$ such that
  ② $\boxed{T\left(\overset{\mathbb{R}^2}{\begin{bmatrix}1\\0\end{bmatrix}}\right)=\overset{\mathbb{R}^3}{\begin{bmatrix}2\\-1\\1\end{bmatrix}}}$ and $\boxed{T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=\begin{bmatrix}0\\1\\2\end{bmatrix}}$. With no additional information,
  find a formula for the image of an arbitrary $\mathbf{x}$ in $\mathbb{R}^2$

$$\mathbf{x}=\begin{bmatrix}x_1\\x_2\end{bmatrix}=x_1\boxed{\begin{bmatrix}1\\0\end{bmatrix}}+x_2\boxed{\begin{bmatrix}0\\1\end{bmatrix}}$$

standard basis (2 dim)
선형 결합으로 나타냄

$$\Rightarrow T(\mathbf{x})=\boxed{T\left(x_1\begin{bmatrix}1\\0\end{bmatrix}+x_2\begin{bmatrix}0\\1\end{bmatrix}\right)=x_1\boxed{T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)}+x_2\boxed{T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)}}$$

T is linear Trans:

$$=x_1\boxed{\begin{bmatrix}2\\-1\\1\end{bmatrix}}+x_2\boxed{\begin{bmatrix}0\\1\\2\end{bmatrix}}=\begin{bmatrix}2&0\\-1&1\\1&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

이렇게 표현가능

두 col의 linear combination

T는 이런 변환이다.

$\therefore$ T is linear transformation

⇒ 어떤 행렬과 주어진 입력 벡터와의 곱으로 $\begin{bmatrix}x_1\\x_2\end{bmatrix}$ 항상 나타낼 수 있다.

Matrix of Linear Transformation

- In general, let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation.
  Then T is always written as a matrix-vector multiplication, i.e.,

어떤 행렬 입력 벡터

$$T(\mathbf{x}) = A\mathbf{x} \text{ for all } \mathbf{x} \in \mathbb{R}^n$$

- In fact, the j-th column of $A \in \mathbb{R}^{m \times n}$ is equal to the vector $T(\mathbf{e}_j)$,
  where $\mathbf{e}_j$ is the j-th column of the identity matrix in $\mathbb{R}^{n \times n}$:

standard basis를 T에 넣은 값 ⇒ A가 됨

$$A = [T(\mathbf{e}_1) \quad \cdots \quad T(\mathbf{e}_n)]$$

- Here, the matrix A is called the **standard matrix** of the linear transformation T

Matrix of Linear Transformation

- Example: Find the standard matrix A of a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$ such that

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}1\\2\end{bmatrix}, T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}4\\3\end{bmatrix}, \text{ and } T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}5\\6\end{bmatrix}.$$

$$\mathbf{x} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = x_1\begin{bmatrix}1\\0\\0\end{bmatrix} + x_2\begin{bmatrix}0\\1\\0\end{bmatrix} + x_3\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$\Rightarrow T(\mathbf{x}) = T\left(x_1\begin{bmatrix}1\\0\\0\end{bmatrix} + x_2\begin{bmatrix}0\\1\\0\end{bmatrix} + x_3\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = x_1T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) + x_2T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) + x_3T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)$$

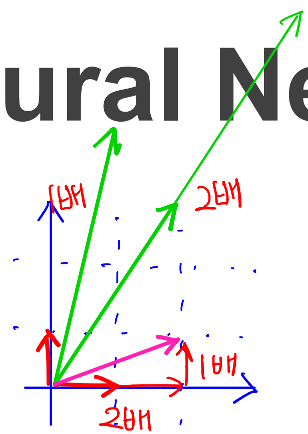
$$= x_1\begin{bmatrix}1\\2\end{bmatrix} + x_2\begin{bmatrix}4\\3\end{bmatrix} + x_2\begin{bmatrix}5\\6\end{bmatrix} = \begin{bmatrix}1 & 4 & 5\\2 & 3 & 6\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = A\mathbf{x}$$

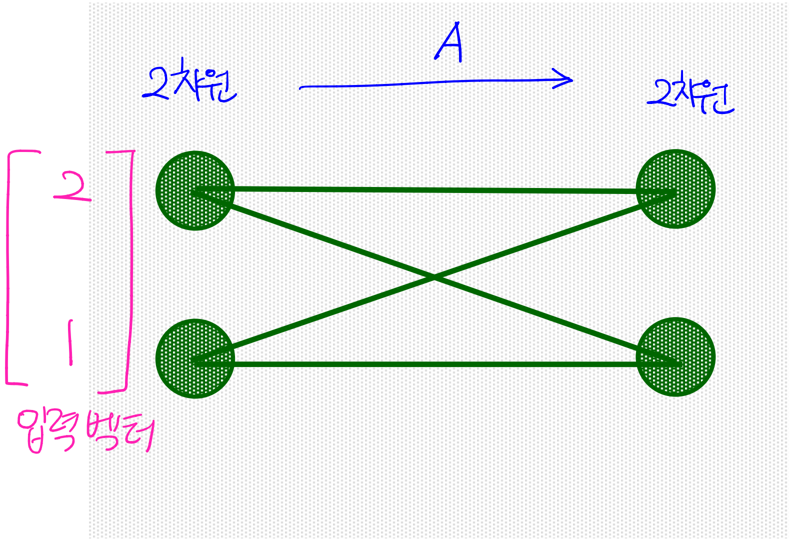
Linear Transformation in Neural Networks

- Fully-connected layers (linear layer)

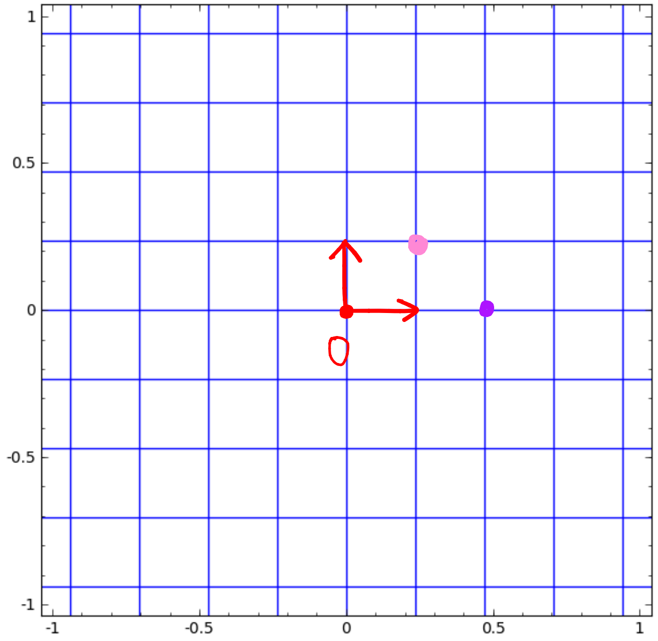
$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=\begin{bmatrix}2\\3\end{bmatrix}$

$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=\begin{bmatrix}1\\4\end{bmatrix}$

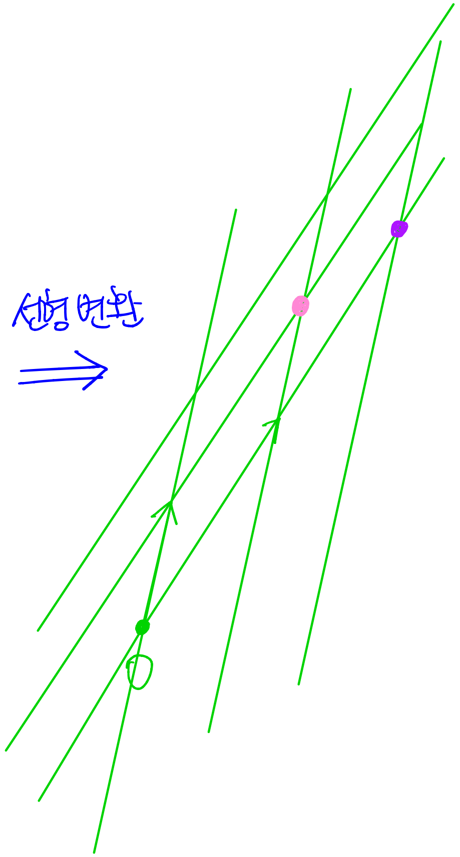




$\mathbf{x} \xrightarrow[T_1]{} \mathbf{y}$



선형변환



https://colah.github.io/posts/2014-03-NN-Manifolds-Topology/

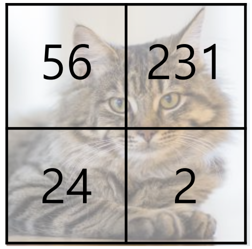
$$T\left(\begin{bmatrix}2\\1\end{bmatrix}\right)=T\left(2\begin{bmatrix}1\\0\end{bmatrix}+1\cdot\begin{bmatrix}0\\1\end{bmatrix}\right)=2\cdot T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)+1\cdot T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$$

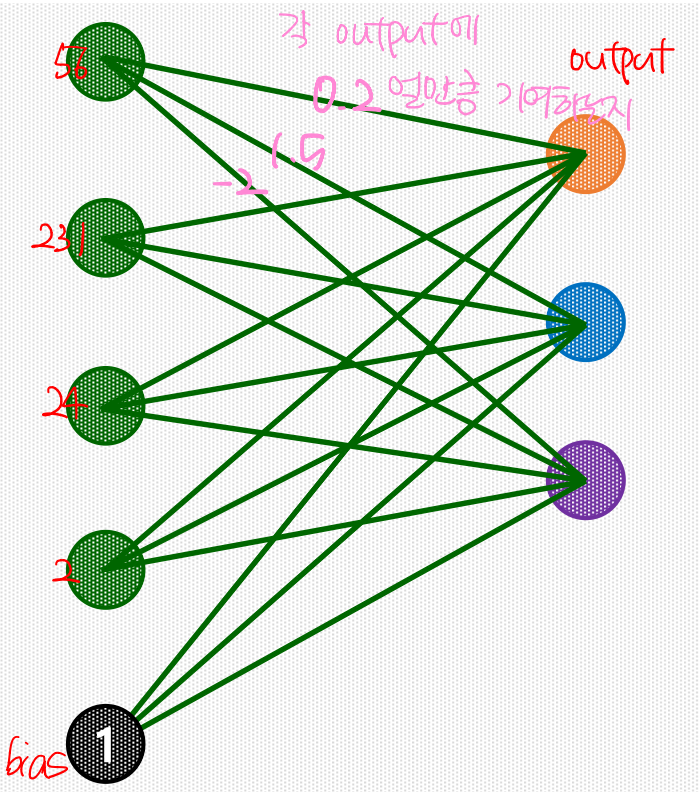
Affine Layer in Neural Networks

$y = 3x + 2$

- Fully-connected layers usually involve a bias term. That's why we call it an affine layer, but not a linear layer.
- Example: Image with 4 pixels and 3 classes (cat/dog/ship)

이미지 pixel 4개





$$\begin{bmatrix} 0.2 & -0.5 & 0.1 & 2 \\ 1.5 & 1.3 & 2.1 & 1 \\ -.2 & 0.3 & 0.7 & -1.3 \end{bmatrix} \boxed{\begin{bmatrix} 56 \\ 231 \\ 24 \\ 2 \end{bmatrix}} + \begin{bmatrix} 1.1 \\ 3.2 \\ -1.2 \end{bmatrix} = \begin{bmatrix} -96.8 \\ 439.9 \\ 71.1 \end{bmatrix}$$

선형변환

최적화 대상 (가중치)

bias

여기에 sigmoid/relu

$$= 56 \begin{bmatrix} 0.2 \\ 1.5 \\ -.2 \end{bmatrix} + 231 \begin{bmatrix} -0.5 \\ 1.3 \\ 0.3 \end{bmatrix} + 24 \begin{bmatrix} 0.1 \\ 2.1 \\ 0.7 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 1 \\ -1.3 \end{bmatrix} + 1 \begin{bmatrix} 1.1 \\ 3.2 \\ -1.2 \end{bmatrix}$$

$$= \begin{bmatrix} 0.2 & -0.5 & 0.1 & 2 & 1.1 \\ 1.5 & 1.3 & 2.1 & 1 & 3.2 \\ -.2 & 0.3 & 0.7 & -1.3 & -1.2 \end{bmatrix} \begin{bmatrix} 56 \\ 231 \\ 24 \\ 2 \\ 1 \end{bmatrix}$$

→ bias도 포함하면 선형변환으로 가능