Linear Algebra

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Lecture Overview

- Elements in linear algebra
- Linear system
- <u>Linear combination, vector equation,
Four views of matrix multiplication</u>
- Linear independence, span, and subspace
- Linear transformation
- Least squares
- Eigendecomposition
- Singular value decomposition

Linear Combinations

선형 결합

$$3\begin{array}{c} v_1 \\ \begin{bmatrix}1\\2\\3\end{bmatrix}\end{array} + (-1)\begin{array}{c} v_2 \\ \begin{bmatrix}4\\5\\6\end{bmatrix}\end{array} = \begin{bmatrix}3\\6\\9\end{bmatrix} + \begin{bmatrix}-4\\-5\\-6\end{bmatrix} \in \mathbb{R}^3$$

p개의 벡터

p=3, n=2

- Given vectors $\boxed{\mathbf{v}_1}, \boxed{\mathbf{v}_2}, \cdots, \boxed{\mathbf{v}_p}$ in $\mathbb{R}^n$ and given scalars $c_1, c_2, \cdots, c_p$,

상수배

$$\underline{c_1}\mathbf{v}_1 + \cdots + \underline{c_p}\mathbf{v}_p$$

is called a **linear combination** of $\mathbf{v}_1, \cdots, \mathbf{v}_p$ with **weights or coefficients** $c_1, \cdots, c_p$.

- The weights in a linear combination can be any real numbers, including zero.

From **Matrix** Equation to **Vector** Equation

- Recall the matrix equation of a linear system:

Person ID	Weight	Height	Is_smoking	Life-span
1	60kg	5.5ft	Yes (=1)	66
2	65kg	5.0ft	No (=0)	74
3	55kg	6.0ft	Yes (=1)	78

$$\Rightarrow \underbrace{\begin{bmatrix} 60 & 5.5 & 1 \\ 65 & 5.0 & 0 \\ 55 & 6.0 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 66 \\ 74 \\ 78 \end{bmatrix}}_{\mathbf{b}}$$

$$A\ \mathbf{x} = \mathbf{b}$$

- A matrix equation can be converted into a vector equation:

$$\Rightarrow \begin{bmatrix} 60 \\ 65 \\ 55 \end{bmatrix} x_1 + \begin{bmatrix} 5.5 \\ 5.0 \\ 6.0 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} x_3 = \begin{bmatrix} 66 \\ 74 \\ 78 \end{bmatrix}$$

$$\mathbf{a}_1 x_1 + \mathbf{a}_2 x_2 + \mathbf{a}_3 x_3 = \mathbf{b}$$

$$\begin{bmatrix} 60\,x_1 \\ 65\,x_1 \\ 55\,x_1 \end{bmatrix} + \begin{bmatrix} 5.5\,x_2 \\ 5.0\,x_2 \\ 6.0\,x_2 \end{bmatrix} + \begin{bmatrix} 1 \cdot x_3 \\ 0 \cdot x_3 \\ 1 \cdot x_3 \end{bmatrix}$$

Existence of Solution for $A\mathbf{x} = \mathbf{b}$

- Consider its vector equation:

벡터 3개의 선형결합

$$\overbrace{\begin{bmatrix}60\\65\\55\end{bmatrix}x_1 + \begin{bmatrix}5.5\\5.0\\6.0\end{bmatrix}x_2 + \cancel{\begin{bmatrix}1\\0\\1\end{bmatrix}x_3}} = \begin{bmatrix}66\\74\\78\end{bmatrix}$$

가중치들

이 값들이 span에 포함되어 있으면 solution 존재

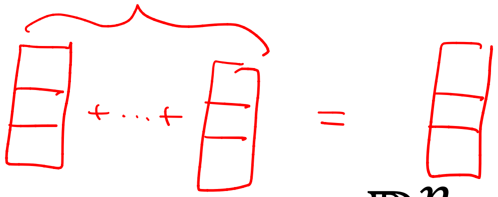
없으면 벡터 2개로 3차원 span에 들어가지 않음.

$$\mathbf{a}_1 x_1 + \mathbf{a}_2 x_2 + \mathbf{a}_3 x_3 = \mathbf{b}$$

- When does the solution exist for $A\mathbf{x} = \mathbf{b}$?

☆ 해가 존재할 것인가?

Span



가능한
결과를 모두 모은 집합: span

- **Definition**: Given a set of vectors $\mathbf{v}_1, \cdots, \mathbf{v}_p \in \mathbb{R}^n$, Span $\{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$ is defined as the set of all linear combinations of $\mathbf{v}_1, \cdots, \mathbf{v}_p$.
- That is, Span $\{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$ is the collection of all vectors that can be written in the form

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 \cdots + c_p\mathbf{v}_p$$

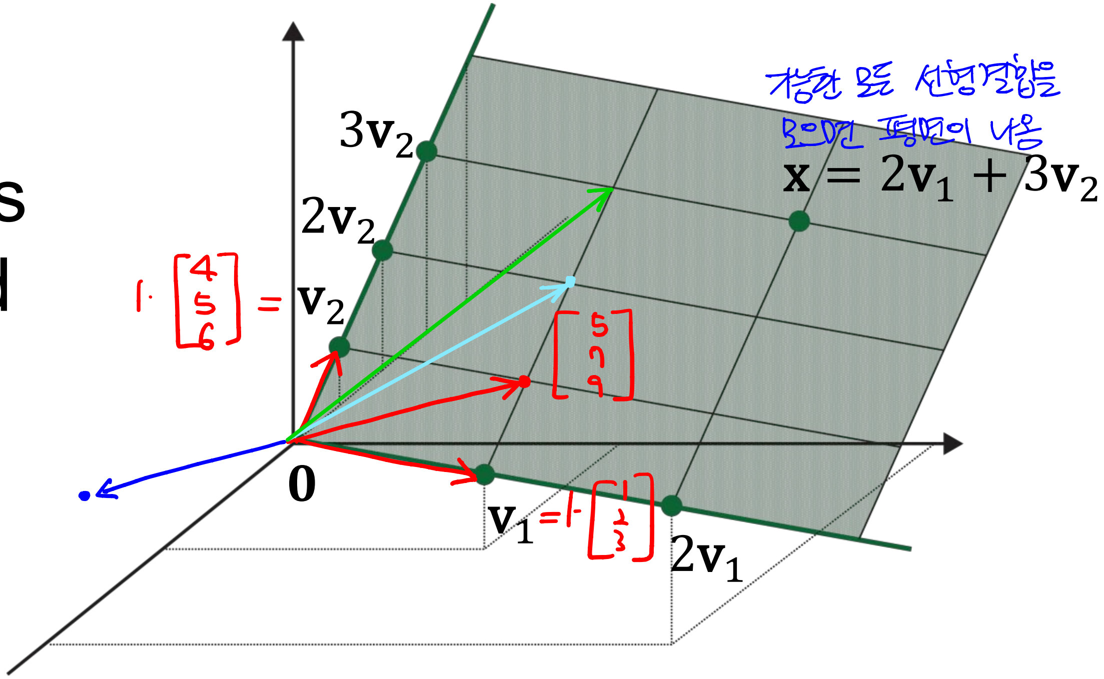
with arbitrary scalars $c_1, \cdots, c_p$.

- Span $\{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$ is also called the **subset of $\mathbb{R}^n$ spanned** (or **generated**) by $\mathbf{v}_1, \cdots, \mathbf{v}_p$.

Geometric Description of Span

- If $\mathbf{v}_1$ are $\mathbf{v}_2$ nonzero vectors in $\mathbb{R}^3$, with $\mathbf{v}_2$ not a multiple of $\mathbf{v}_1$, then Span $\{\mathbf{v}_1, \mathbf{v}_2\}$ is the plane in $\mathbb{R}^3$ that contains $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{0}$.

- In particular, Span $\{\mathbf{v}_1, \mathbf{v}_2\}$ contains the line in $\mathbb{R}^3$ through $\mathbf{v}_1$ and $\mathbf{0}$ and the line through $\mathbf{v}_2$ and $\mathbf{0}$.



Geometric Interpretation of Vector Equation

- Finding a linear combination of given vectors $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ to be equal to $\mathbf{b}$:

$$\underbrace{\begin{bmatrix} 60 \\ 65 \\ 55 \end{bmatrix}}_{a_1} x_1 + \underbrace{\begin{bmatrix} 5.5 \\ 5.0 \\ 6.0 \end{bmatrix}}_{a_2} x_2 + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}_{a_3} x_3 = \begin{bmatrix} 66 \\ 74 \\ 78 \end{bmatrix}$$

$$\mathbf{a}_1 x_1 + \mathbf{a}_2 x_2 + \mathbf{a}_3 x_3 = \mathbf{b}$$

- The solution exists only when $\boxed{\mathbf{b} \in \text{Span}\,\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}}$.

Matrix Multiplications as Linear Combinations of Vectors

- **Recall**: we defined matrix-matrix multiplications as the inner product between the row on the left and the column on the right:

내적 6번

  - e.g., $\begin{bmatrix} 1 & 6 \\ 3 & 4 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 13 & 5 \\ 11 & 1 \\ 9 & -3 \end{bmatrix}$

- Inspired by the vector equation, we can view $A\mathbf{x}$ as a linear combination of columns of the left matrix:
  - $\begin{bmatrix} 60 & 5.5 & 1 \\ 65 & 5.0 & 0 \\ 55 & 6.0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A\mathbf{x} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{a}_1 x_1 + \mathbf{a}_2 x_2 + \mathbf{a}_3 x_3$

Matrix Multiplications as Column Combinations

- Linear combinations of columns
  - Left matrix: bases, right matrix: coefficients

One column on the right

$$\underbrace{\begin{bmatrix}1 & 1 & 0\\1 & 0 & 1\\1 & -1 & 1\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}1\\2\\3\end{bmatrix}}_{x}=\underbrace{\begin{bmatrix}1\\1\\1\end{bmatrix}}_{a_1}\underset{\text{가중치}}{1}+\underbrace{\begin{bmatrix}1\\0\\-1\end{bmatrix}}_{a_2}\underset{\text{가중치}}{2}+\underbrace{\begin{bmatrix}0\\1\\1\end{bmatrix}}_{a_3}\underset{\text{가중치}}{3}$$

$$(Ax)^T = x^T A^T$$

$$\underbrace{\begin{bmatrix}1 & 2 & 3\end{bmatrix}}_{\text{가중치}}\begin{bmatrix}1 & 1 & 1\\1 & 0 & -1\\0 & 1 & 1\end{bmatrix}=\begin{bmatrix}x & y\end{bmatrix}$$

Multi-columns on the right

$$\overset{3\times 3}{\begin{bmatrix}1 & 1 & 0\\1 & 0 & 1\\1 & -1 & 1\end{bmatrix}}\overset{3\times 2}{\begin{bmatrix}1 & -1\\2 & 0\\3 & 1\end{bmatrix}}=\overset{3\times 2}{\begin{bmatrix}x_1 & y_1\\x_2 & y_2\\x_3 & y_3\end{bmatrix}}=[\mathbf{x}\ \mathbf{y}]$$

각각 따로 생각

$$\mathbf{x}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\underset{a_1}{\begin{bmatrix}1\\1\\1\end{bmatrix}}1+\underset{a_2}{\begin{bmatrix}1\\0\\-1\end{bmatrix}}2+\underset{a_3}{\begin{bmatrix}0\\1\\1\end{bmatrix}}3$$

얘는 a_1 a_2 a_3 의 span 안에 들어감

$$\mathbf{y}=\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=\begin{bmatrix}1\\1\\1\end{bmatrix}(-1)+\begin{bmatrix}1\\0\\-1\end{bmatrix}0+\begin{bmatrix}0\\1\\1\end{bmatrix}1$$

얘도 마찬가지

Matrix Multiplications as Row Combinations

행 기준

- Linear combinations of rows of the right matrix
  - Right matrix: bases, left matrix: coefficients

One row on the left

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix} = \begin{matrix} 1 \times \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \\ +2 \times \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \\ +3 \times \begin{bmatrix} 1 & -1 & 1 \end{bmatrix} \end{matrix}$$

row vector의 선형결합

Multiple rows on the left

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} = \begin{bmatrix} \mathbf{x}^{\mathrm{T}} \\ \mathbf{y}^{\mathrm{T}} \end{bmatrix}$$

$$\mathbf{x}^{\mathrm{T}} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} = 1\begin{bmatrix} 1 & 1 & 0 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 & 1 \end{bmatrix} + 3\begin{bmatrix} 1 & -1 & 1 \end{bmatrix}$$

$$\mathbf{y}^{\mathrm{T}} = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} = 1\begin{bmatrix} 1 & 1 & 0 \end{bmatrix} + 0\begin{bmatrix} 1 & 0 & 1 \end{bmatrix} + (-1)\begin{bmatrix} 1 & -1 & 1 \end{bmatrix}$$

Matrix Multiplications as Sum of (Rank-1) Outer Products

- (Rank-1) outer product 외적

$$\overset{a}{\begin{bmatrix}1\\1\\1\end{bmatrix}}\overset{b}{\begin{bmatrix}1 & 2 & 3\end{bmatrix}}=\begin{bmatrix}1 & 2 & 3\\1 & 2 & 3\\1 & 2 & 3\end{bmatrix}$$

$$[a][b]$$

$$\begin{bmatrix}a & c\end{bmatrix}\begin{bmatrix}b\\d\end{bmatrix}=ab+cd$$

- Sum of (Rank-1) outer products

pair를 뽑아 외적

$$\begin{bmatrix}1 & 1\\1 & -1\\1 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\end{bmatrix}=\begin{bmatrix}1\\1\\1\end{bmatrix}\begin{bmatrix}1 & 2 & 3\end{bmatrix}+\begin{bmatrix}1\\-1\\1\end{bmatrix}\begin{bmatrix}4 & 5 & 6\end{bmatrix}$$

$$=\begin{bmatrix}1 & 2 & 3\\1 & 2 & 3\\1 & 2 & 3\end{bmatrix}+\begin{bmatrix}4 & 5 & 6\\-4 & -5 & -6\\4 & 5 & 6\end{bmatrix}$$

Matrix Multiplications as Sum of (Rank-1) Outer Products

- Sum of (Rank-1) outer products is widely used in machine learni ng
  - Covariance matrix in multivariate Gaussian
  - Gram matrix in style transfer