Image Compression with Anisotropic Geodesic Triangulations

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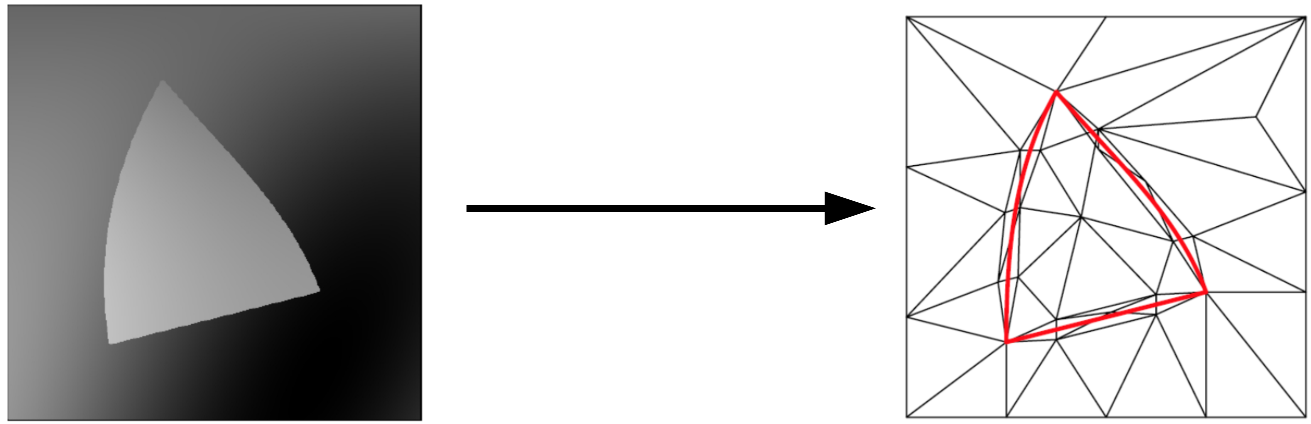
Image Approximation by Triangulations

image f



Approximation by *m* linear spline functions over a triangulation (S,T)

- vertices $S = \{p_i\}_{i=1}^{m}$
- triangles $T = \{t_i\}_{i=1}^{k}$

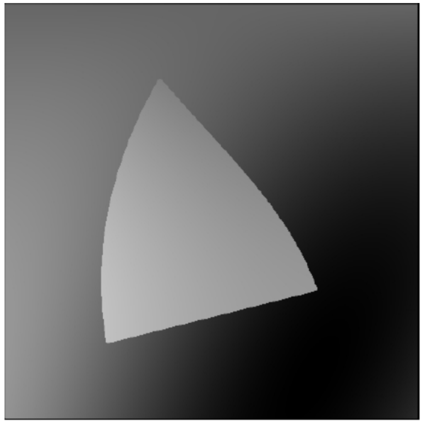
Resulted approximated image: $f_m = \sum_{i=1}^{m} \lambda_i \varphi_i$

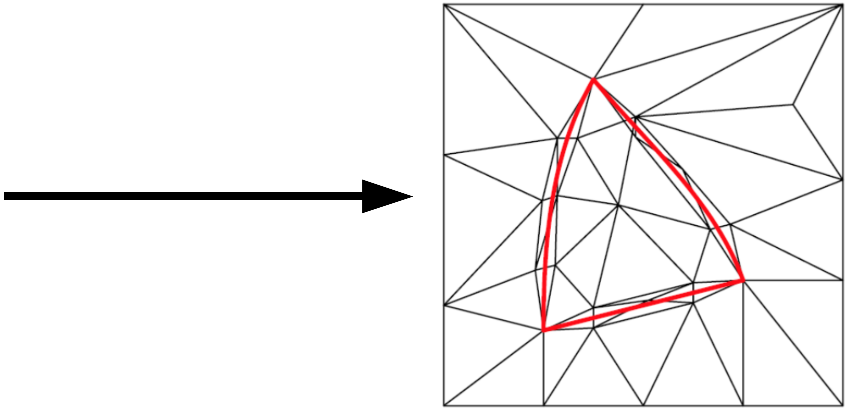
- $\{\varphi_i\}_i$ linear on each triangle
- $\varphi_i(p_j) = 1$ if $i = j$, and 0 otherwise
- coefficients λ_i computed such that f_m is the orthogonal projection of f on the space V generated by the φ_i

$$f_m = \underset{g \in V}{argmin} \|f - g\|$$

Image Approximation by Triangulations

image f





To minimize the error $\|f - f_m\|$ relative to (S,T)

- optimize position of vertices
  - ➔ more vertices near areas of strong gradient
- optimize connections between vertices
  - ➔ anistropic triangles oriented in the direction of contours

- There exists (S,T) such that $\|f - f_m\| \leqslant C_f m^{-2}$
- Optimal (S,T): NP-hard
  - ➔ provably good greedy schemes [Mirebeau-Cohen,09]

Anisotropic Metric

- Desired anisotropy of a triangulation can be represented by a tensor field

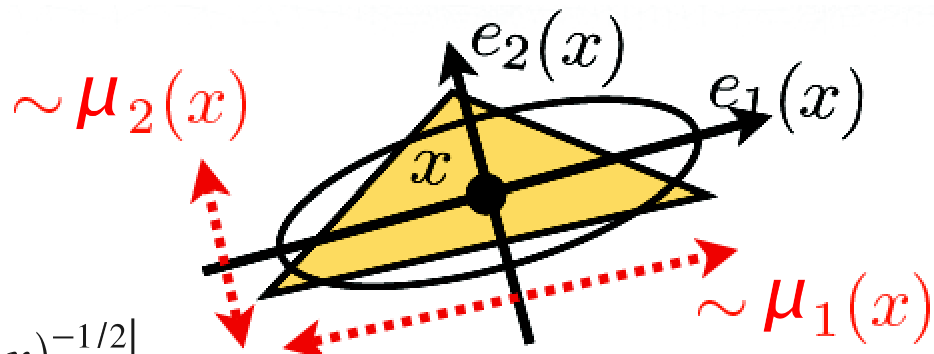
$$M : D \to \mathbb{R}^{2\times 2}$$

$$M(x) = \mu_1(x) e_1(x) e_1^T(x) + \mu_2(x) e_2(x) e_2^T(x), \quad \mu_1(x) \geqslant \mu_2(x) > 0$$

➔ characterizes the shape of triangles surrounding *x*

➔ $e_1(x)$ direction of regularity (edge, texture, …)

➔ $\mu_1(x)/\mu_2(x)$ local triangle aspect ratio



- C^2 images: $\|f - f_m\| \leqslant C_f m^{-2}$

➔ triangulation conforming to $M(x) = \left|H(x)^{-1/2}\right|$ where *H* is the Hessian of f

➔ improves the value of C_f [Cohen-Mirebeau,09]

- **Difficulties**

➔ second derivatives unstable to noise

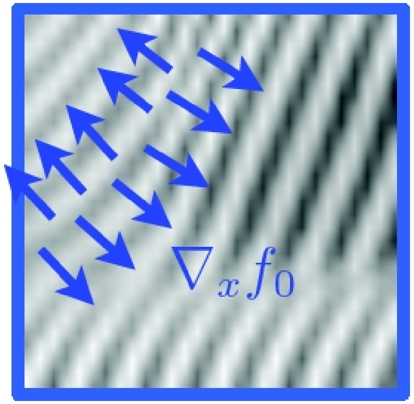
➔ does not work for cartoon images and oscillating textures

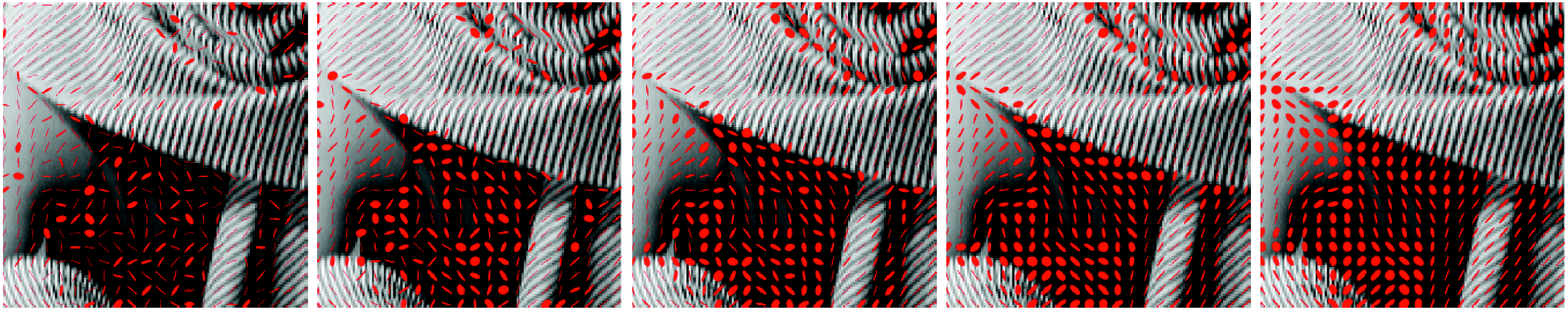
Anisotropic Metric and Structure Tensor

- To overcome this drawback
  we propose to use the structure tensor

$$M_\sigma(x) = G_\sigma(x) * (\nabla f(x) \nabla^T f(x))$$

→ robust estimation in the presence of noise





$\sigma = 1$ $\sigma = 2$ $\sigma = 4$ $\sigma = 8$ $\sigma = 16$

- **Anisotropic sizing field:** modified structure tensor

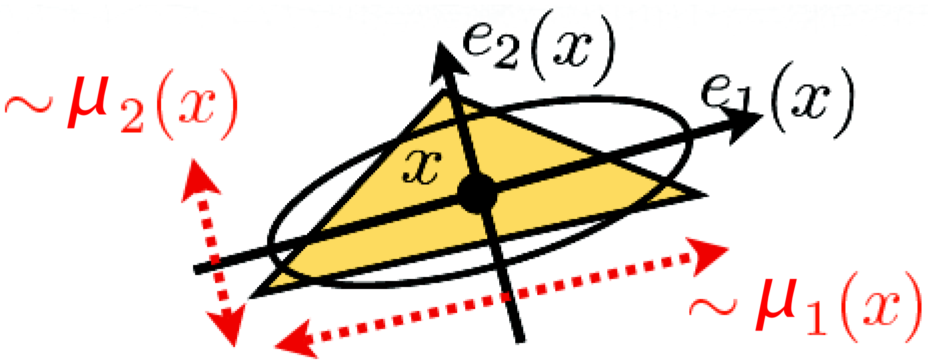
$$M(x) = (\epsilon + \mu_1)^\alpha e_1 e_1^T + (\epsilon + \mu_2)^\alpha e_2 e_2^T$$

$$\left.\begin{array}{l}\rightarrow \varepsilon \text{ isotropic adaptivity} \\ \rightarrow \alpha \text{ anisotropic adaptivity}\end{array}\right\} \text{choice of } (\varepsilon, \alpha)\text{: empirical exploration}$$

Anisotropic Metric and geodesics

- At each point x_0 of the image domain
  - ➔ local metric $M(x_0)$ defines the size of triangles around x_0
  - ➔ must be of contant size t_m for the modified metric

$$\|x_0 - x\|_{H(x_0)} = \sqrt{(x_0 - x)^T H(x_0)(x_0 - x)}$$



- Compute a point set S that achieves this condition
  - ➔ sampling the domain uniformly according to the geodesic distance

$$d_M(x, y) = \min_{\gamma} \int_0^1 \sqrt{\gamma'(t)^T M(\gamma(t)) \gamma'(t)}\, dt$$

Overview

- **Farthest Point Sampling**
- Farthest Point Meshing
- Image Approximation and Compression

Farthest Point Sampling (FPS)

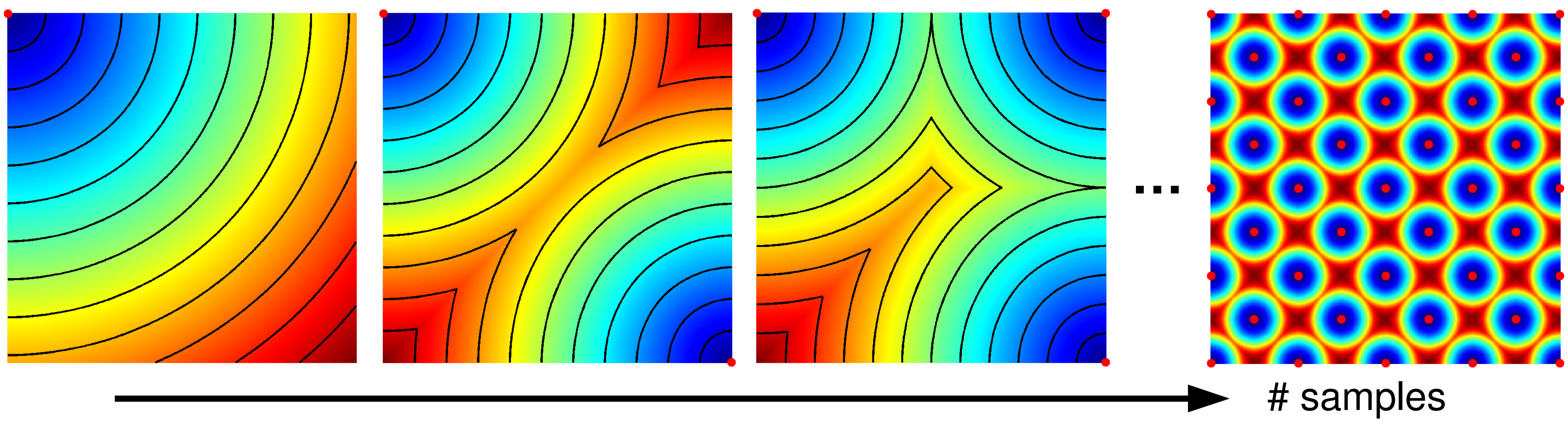
Farthest Point Sampling

Input: a point set $S \subset D$
Output: a sample *S* of *D* satisfying $U_S(p) \leqslant \varepsilon, \ \forall\, p \in D$

1) Select the farthest point of *D* from *S* : $p_{max} \leftarrow \underset{p \in D}{argmax}\ U_S(p)$

2) If $U_S(p_{max}) < \varepsilon$ then exit, else set $S \leftarrow S \cup \{p_{max}\}$ and goto 1.

- Minimal action map: $U_S(x) = \min_{y \in S} d_M(x, y)$



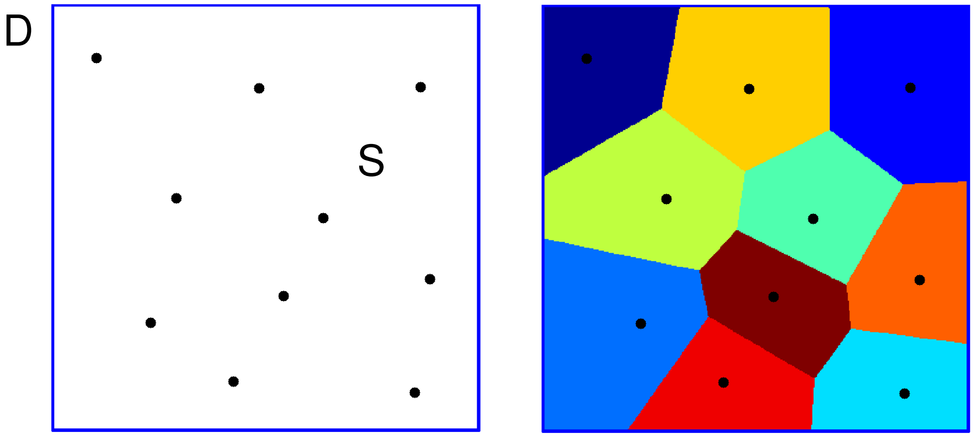
FPS and Voronoi diagram

- Voronoi diagram of *S* restricted to *D*: $V(S) = \bigcup_{p_i \in S} V(S, p_i)$

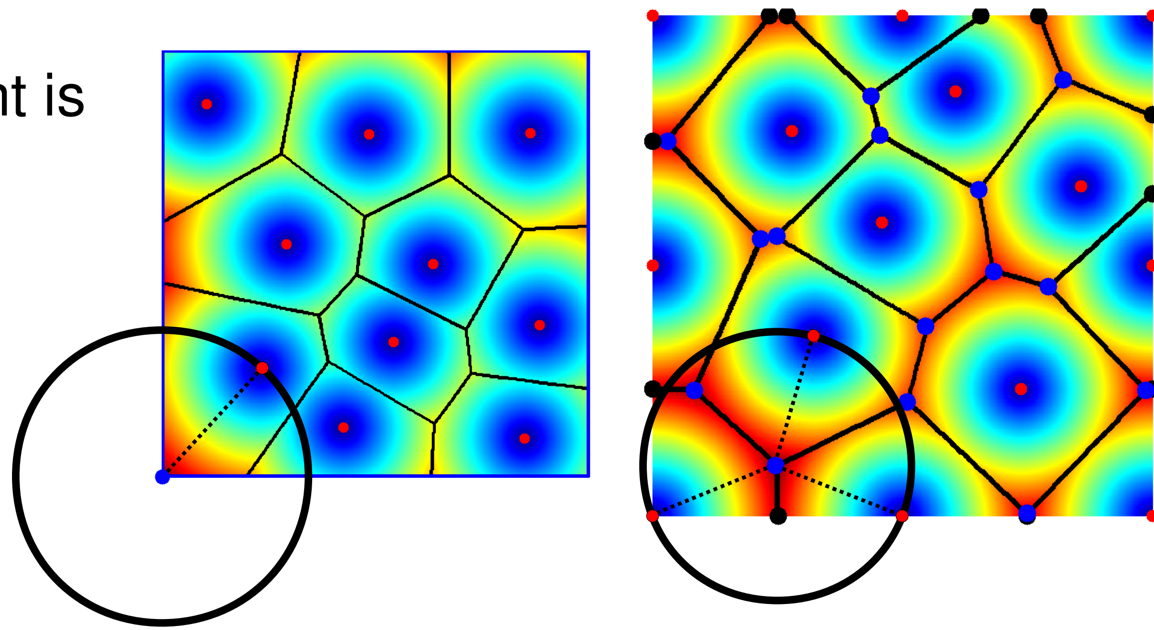
$$V(S, p_i) = \{p \in D : \ d_M(p, p_i) \leqslant d_M(p, p_j), \ \ \forall p_j \in S \setminus \{p_i\}\}$$

Euclidean case

M=Id

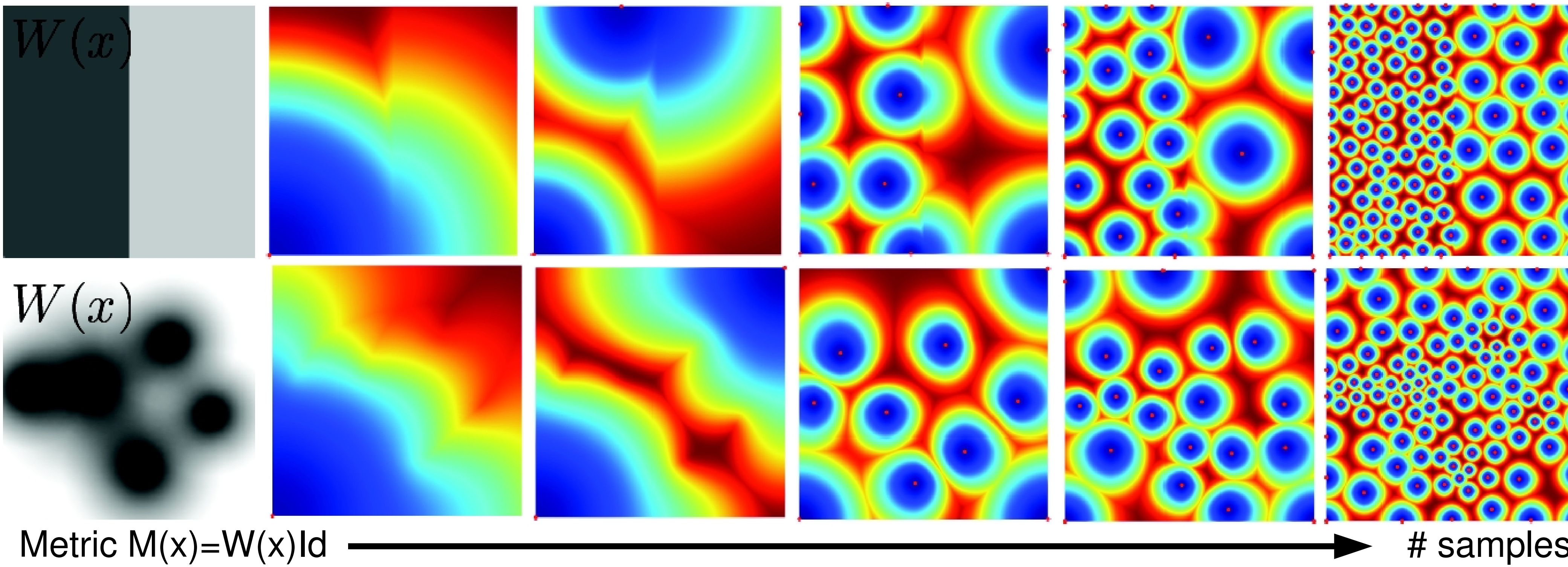


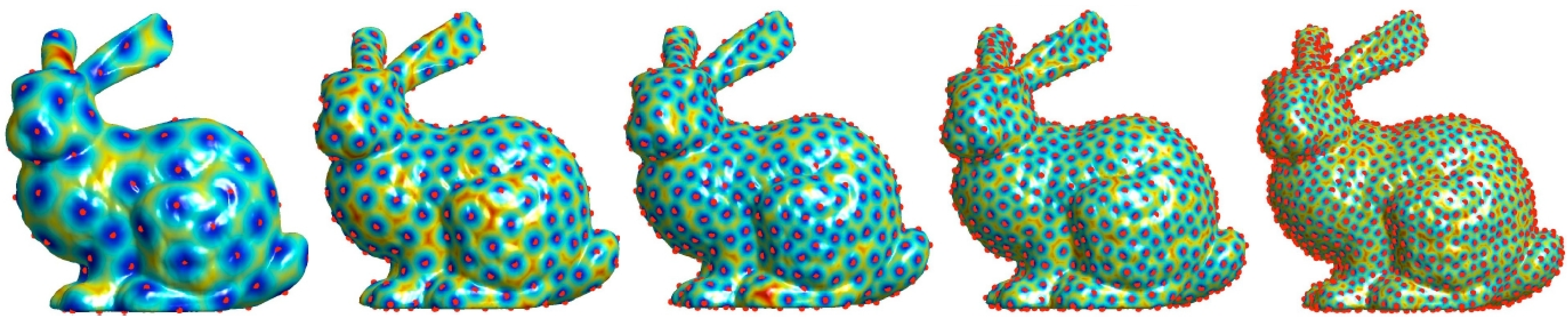
- Link to the FPS: farthest point is
  - a Voronoi vertex, or
  - a point of ∂D
  - → center of the maximal empty open disk



Geodesic FPS: adaptive sampling

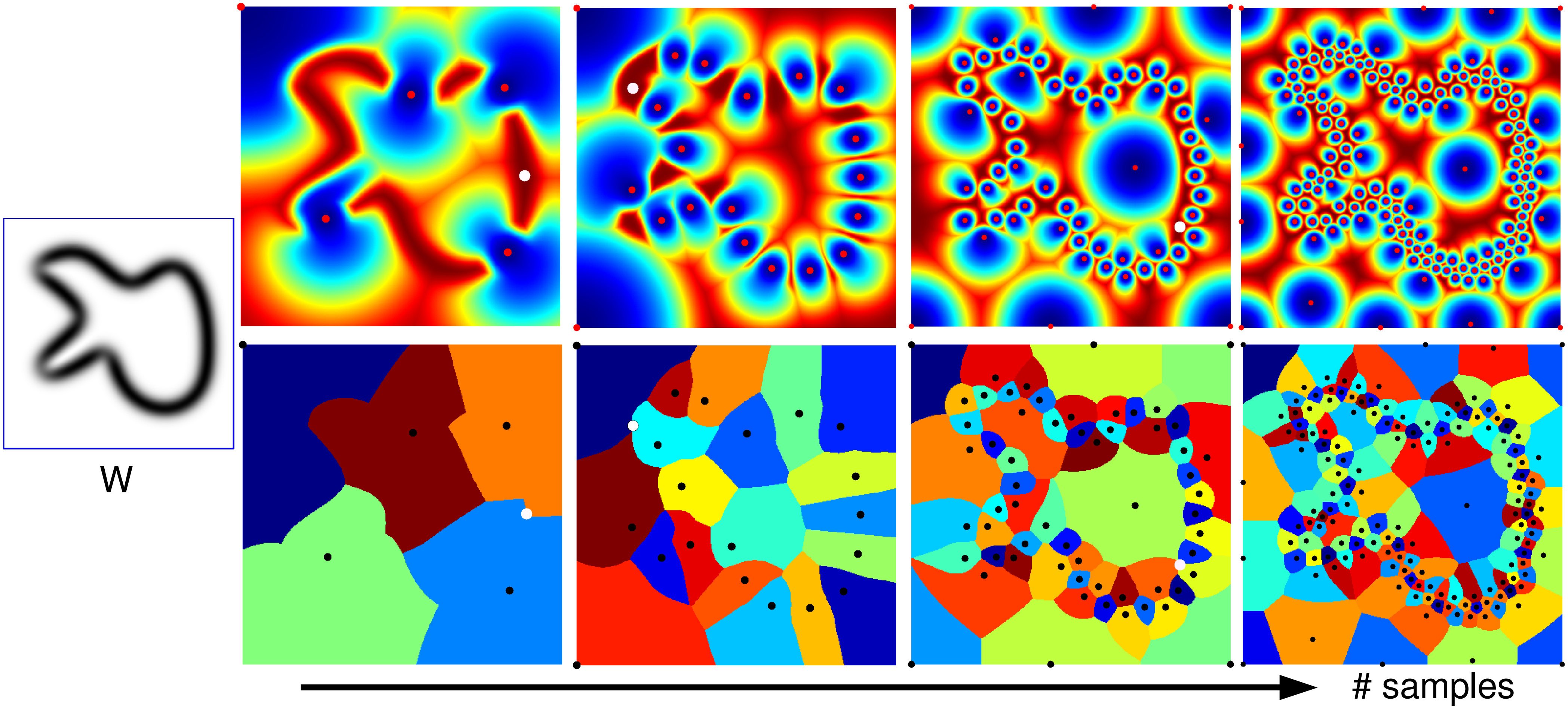
- using Euclidean Voronoi diagram [Eldar *et al.*,97]
- using Fast Marching [Peyré-Cohen,03][Moenning-Dodgson,03]





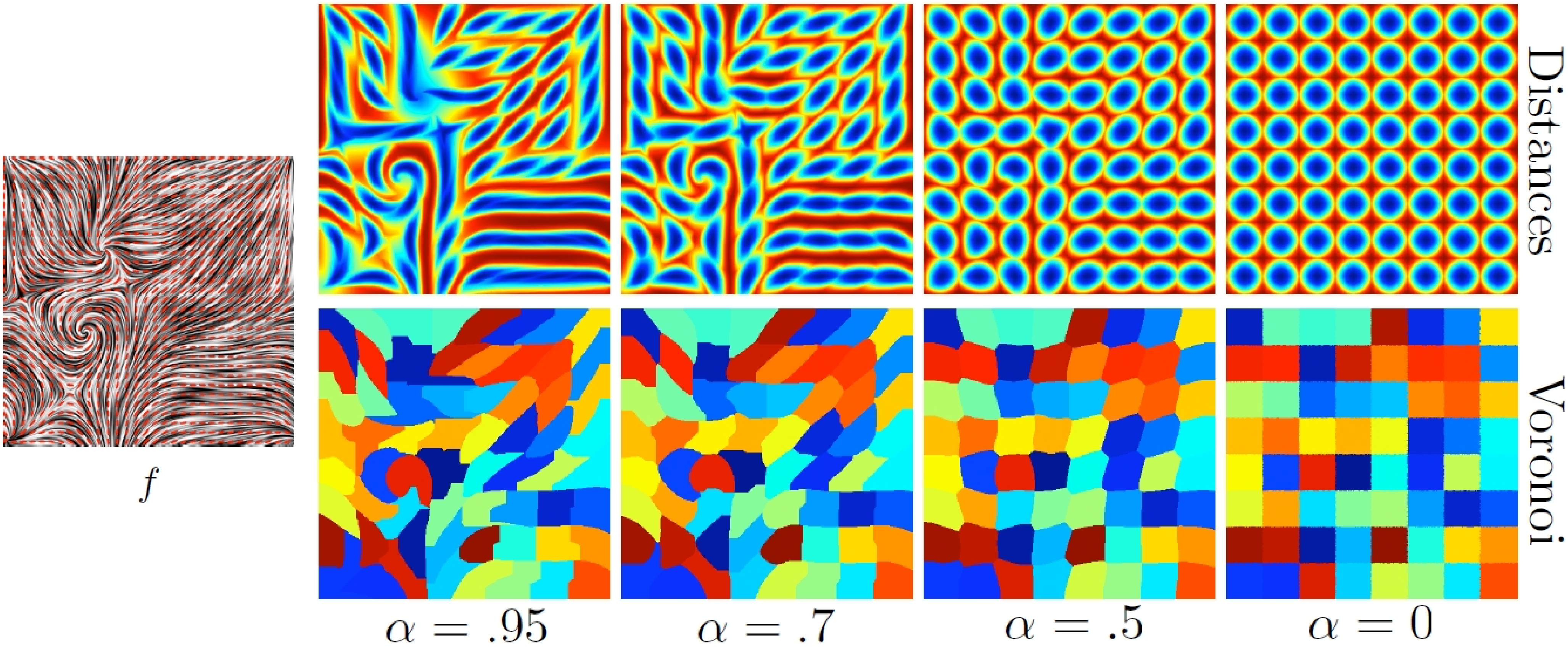
Geodesic FPS and Voronoi diagram

- farthest point is not necessarily a Voronoi vertex nor point of ∂D



- as the number of samples increases
  - farthest points are located at Voronoi vertices
  - geodesic Voronoi diagram tends to Euclidean Voronoi diagram

Anisotropic Voronoi diagram

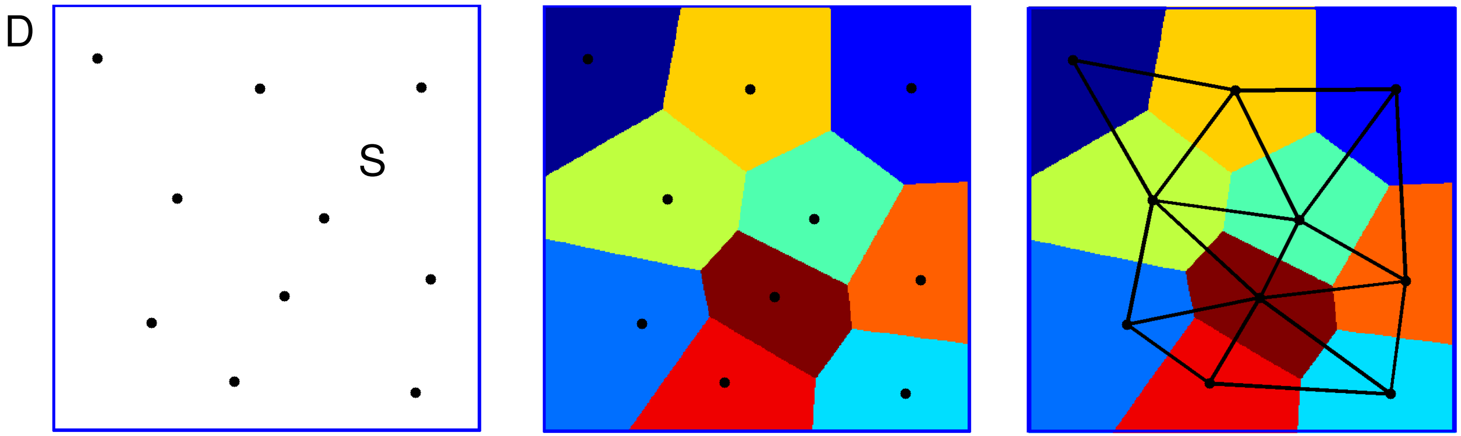


$$M(x) = \mu_1(x) e_1(x) e_1^T(x) + \mu_2(x) e_2(x) e_2^T(x), \quad \mu_1(x) \geqslant \mu_2(x) > 0$$

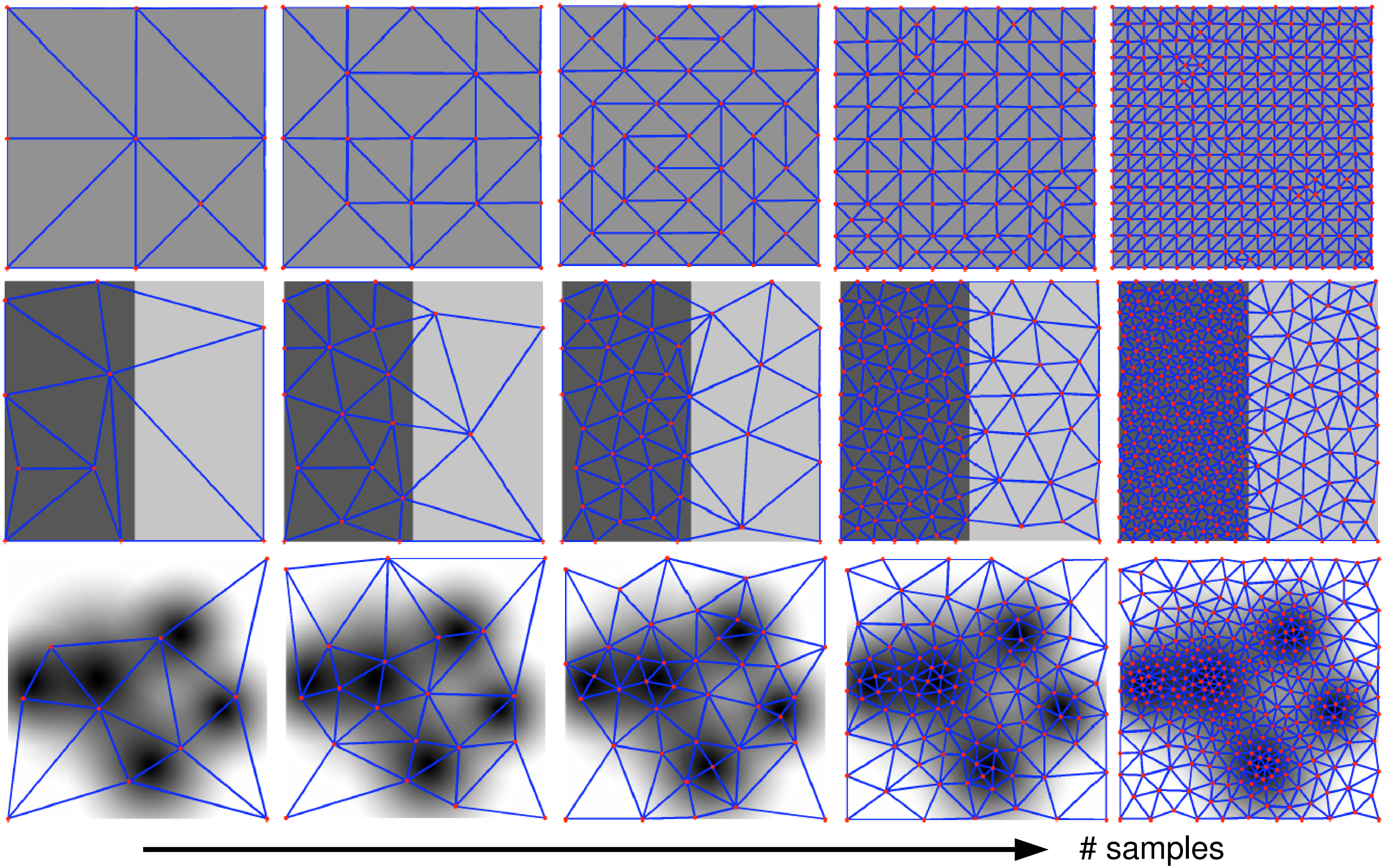
Local anisotropy of the metric: $\alpha(x) = \dfrac{\mu_1 - \mu_2}{\mu_1 + \mu_2}$

FPS and Delaunay Complex

- Delaunay complex *D*(*S*) restricted to *D*
  - edges $(p_i, p_j) \in D(S) \;\Leftrightarrow\; V(p_i, S) \cap V(p_j, S) \neq \emptyset$
  - faces $(p_i)_i \in D(S) \;\Leftrightarrow\; \bigcap_i V(p_i, S) \neq \emptyset$
  - *S* in general position ⇒ all faces are triangles

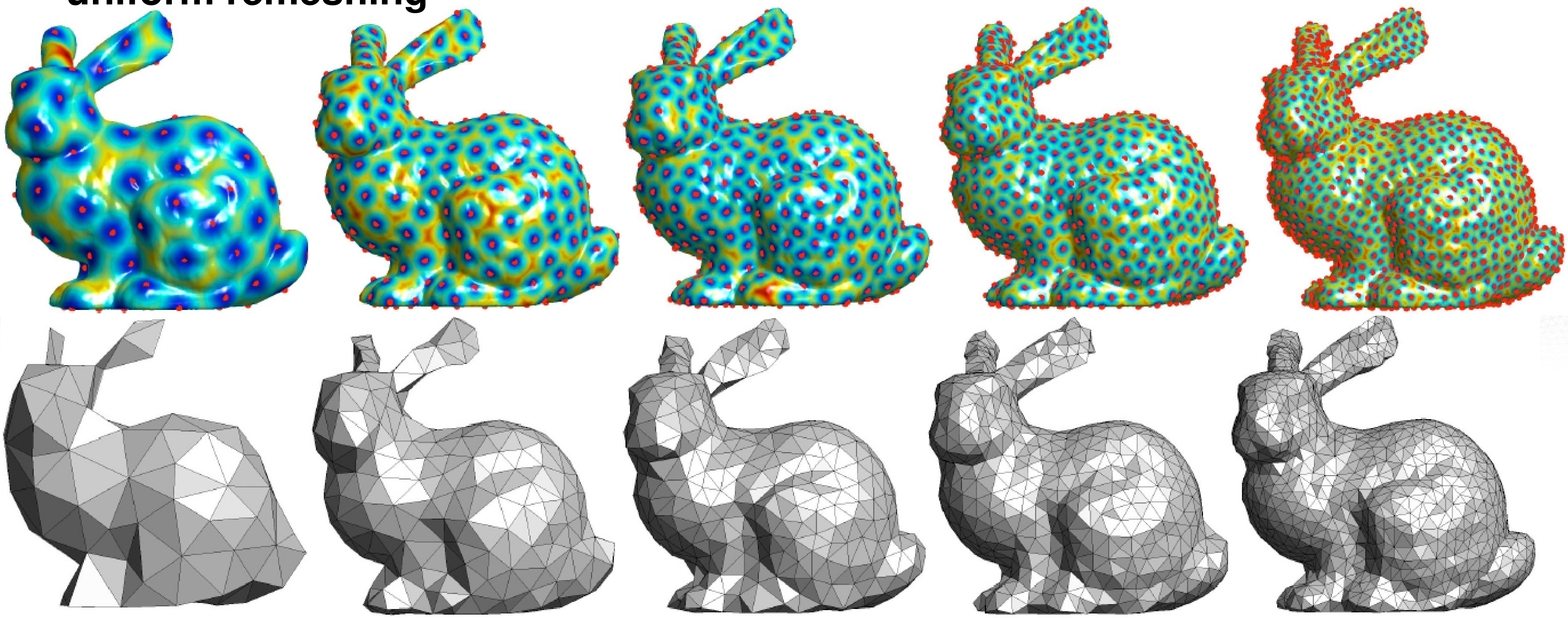


FPS and Delaunay Complex

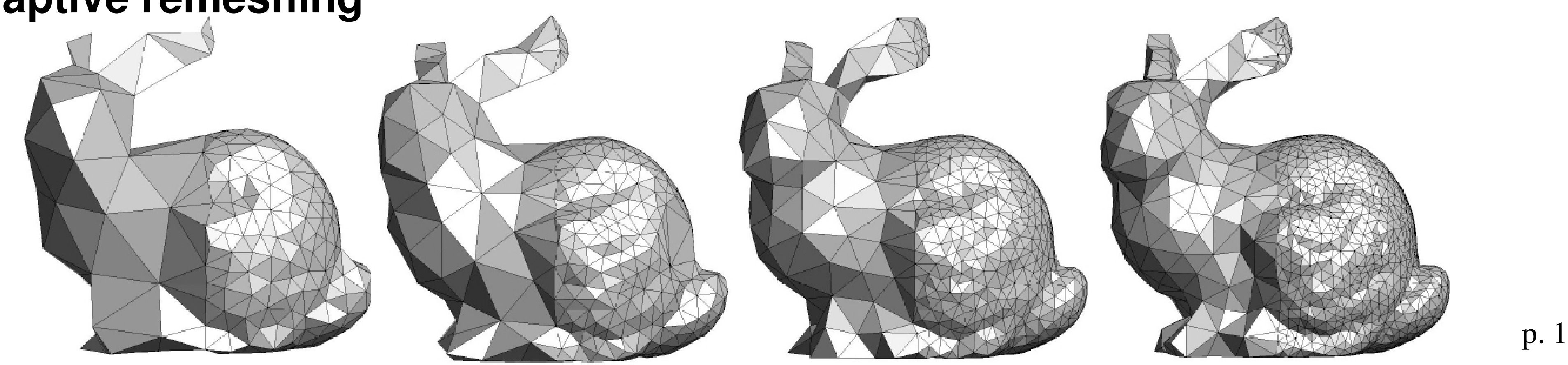


FPS and Delaunay Complex

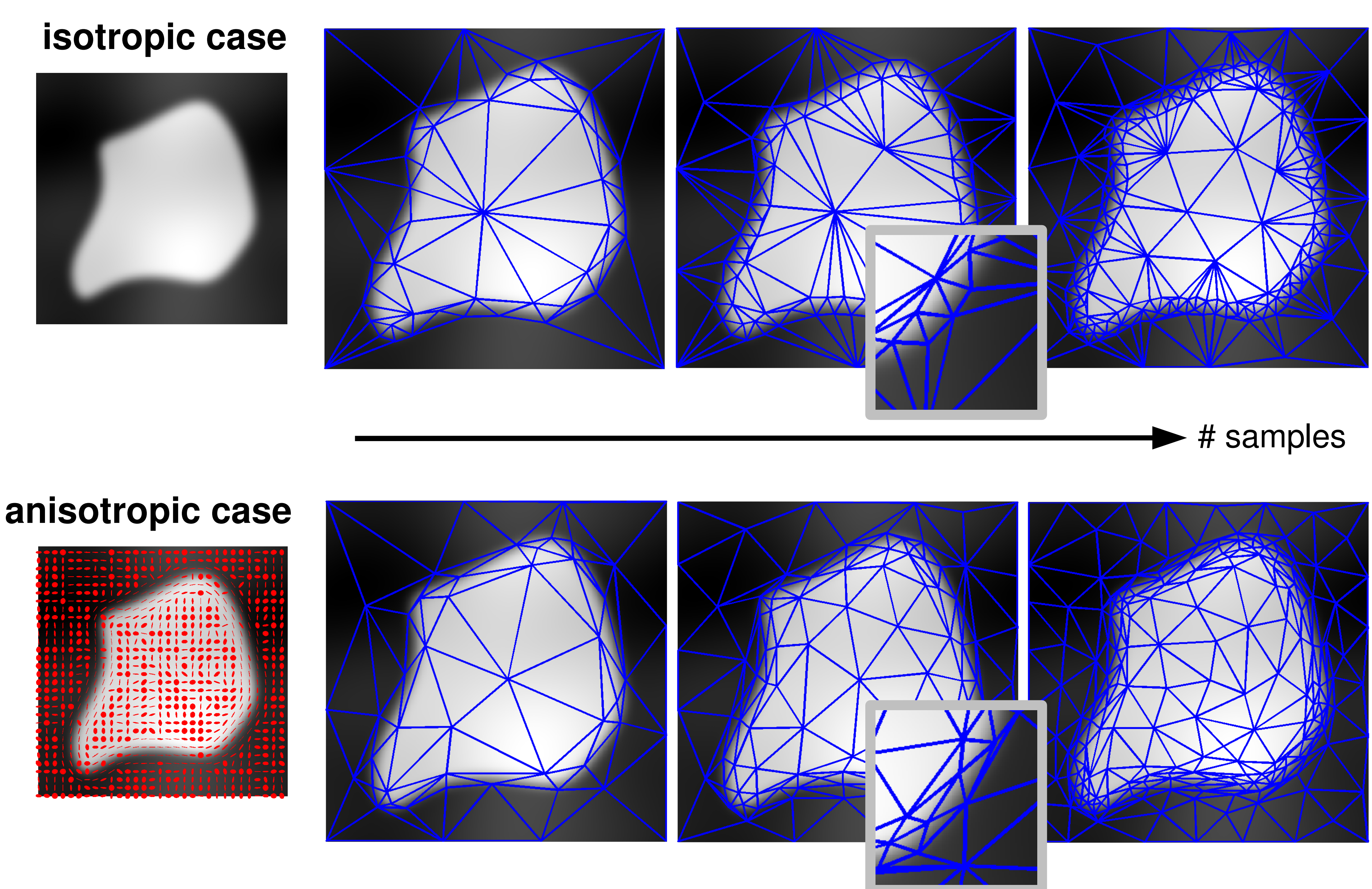
uniform remeshing



adaptive remeshing



Anisotropic FPS and Delaunay Complex



Overview

- Farthest Point Sampling
- **Farthest Point Meshing**
- Image Approximation and Compression

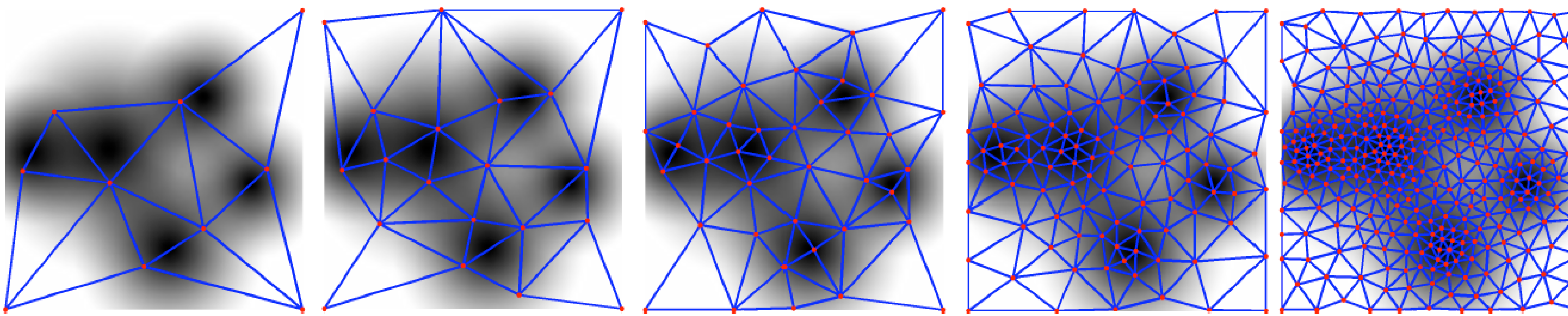
Farthest Point Sampling – 2 drawbacks

- For domains with boundaries

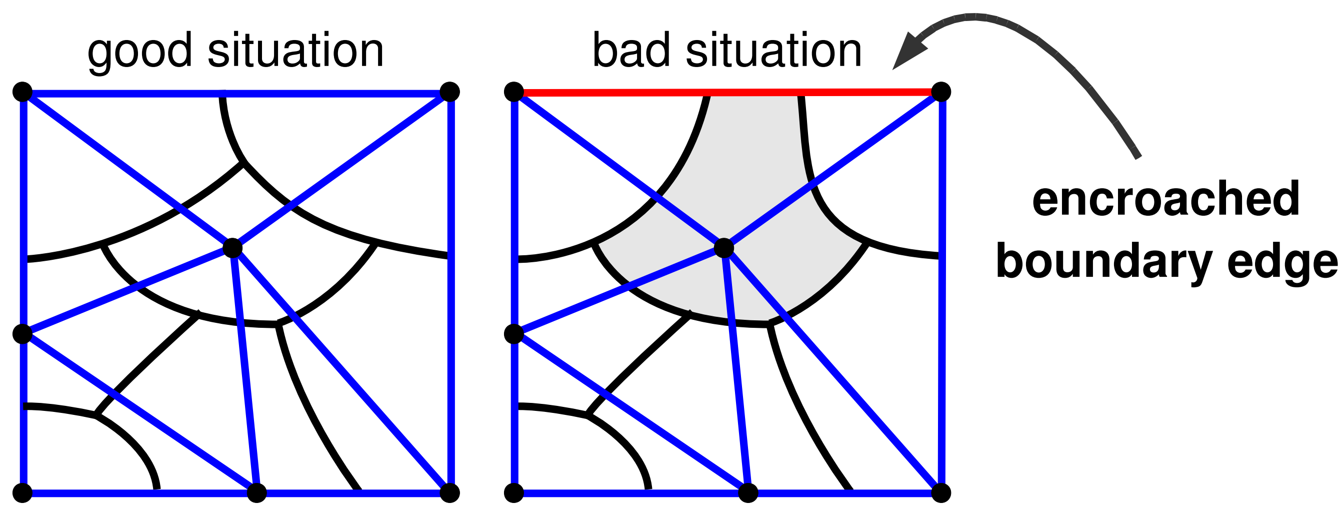
1) boundaries are not well-represented by the Delaunay complex

→ occurs both in Euclidean and (isotropic, anisotropic) geodesic cases

→ need to sample enough the boundaries



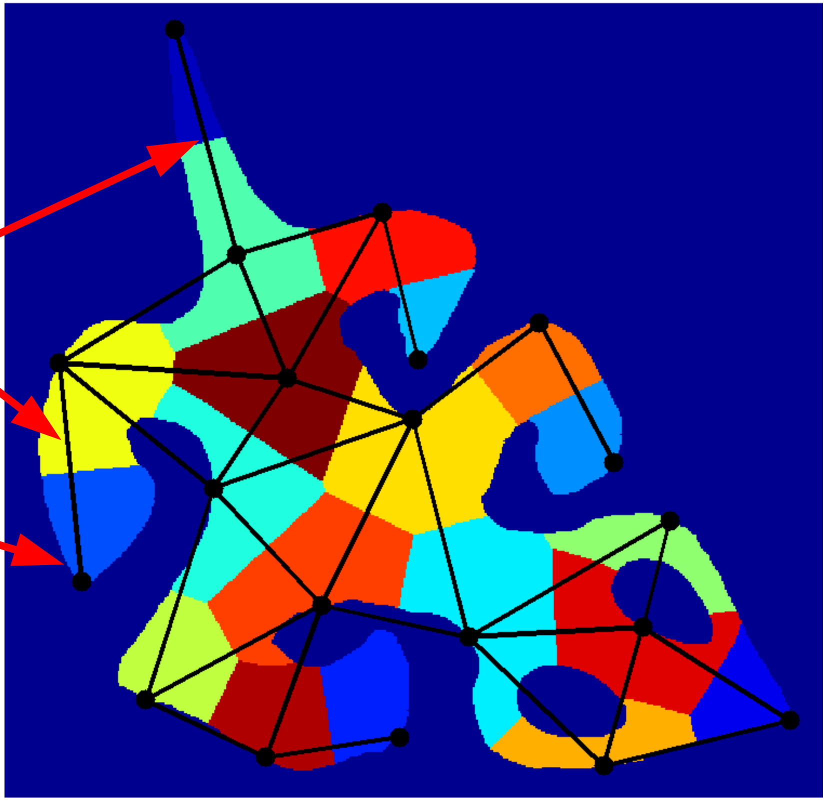
Requirement: $\partial D \subset \mathrm{D}(S)$



Farthest Point Sampling – 2 drawbacks

2) the Delaunay complex is not necessarily a triangulation

1st case
some edges are not incident to a Delaunay triangle
⇒ **isolated vertices**

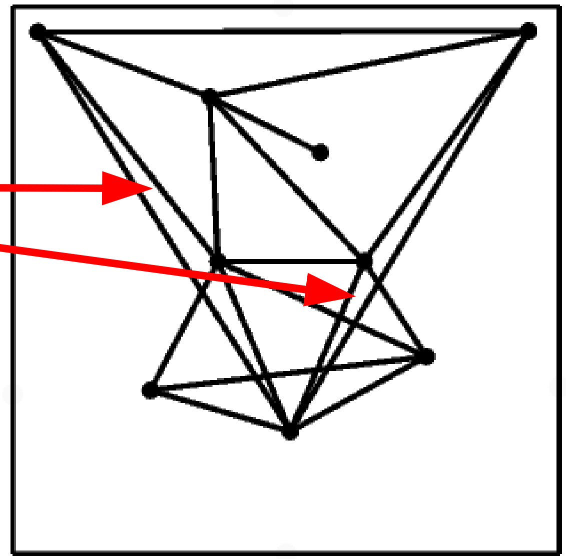


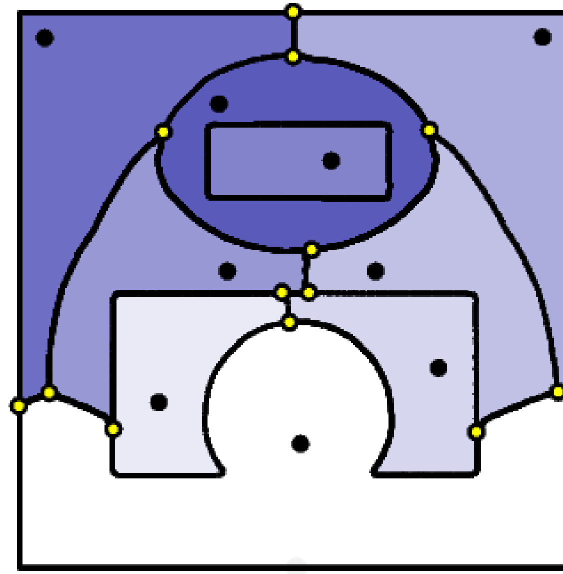
- occurs principally for
  - ➔ domains with boundaries, and/or
  - ➔ geodesic distances (isotropic and anisotropic)

Farthest Point Sampling – 2 drawbacks

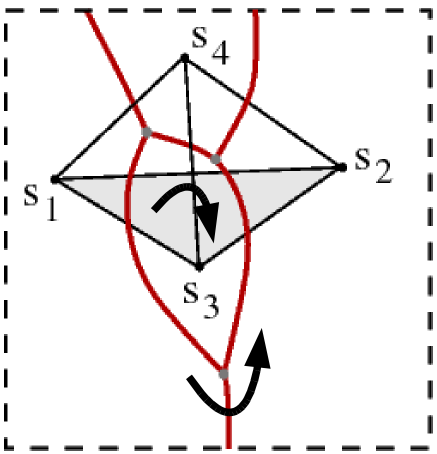
2) the Delaunay complex is not necessarily a triangulation

2nd case
some Delaunay triangles can be **inverted**





detection



- occurs for geodesic distances (isotropic and anisotropic)

→ need to sample enough the boundaries and the interior of D

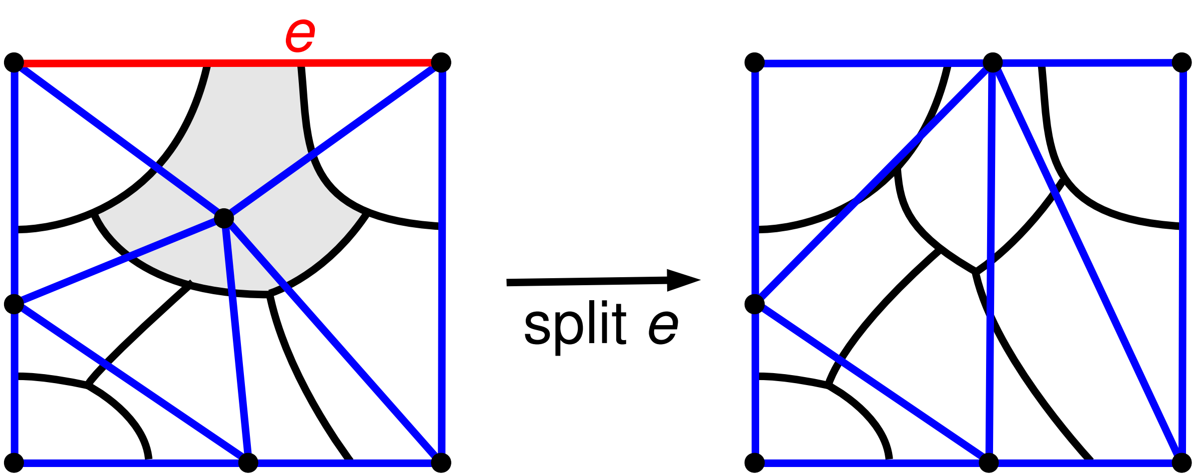
Anisotropic Farthest Point Meshing

- Anisotropic Delaunay refinement methods
  [Labelle-Shewchuk,03] [Boissonnat et al.,07]
  - discrete definition of anisotropic Voronoi diagrams based on

$$d_p(x,y) = \sqrt{(x-y)^t M_p(x-y)}, \quad d(p,q) = min(d_p(p,q), d_q(p,q))$$

  - refinement such that their dual is a triangulation

- **conditional insertion** of a point *p* of *D*:
  - if *p* encroaches a boundary edge *e*, split *e*: $S \leftarrow S \cup \underset{p \in e}{argmax}\ U_S(p)$
  - else $p_{max} \leftarrow \underset{p \in D}{argmax}\ U_S(p)$



Anisotropic Farthest Point Meshing

Farthest Point Meshing

Input: a point set $S \subset D$

Output: a triangulation of *D* satisfying $U_S(p) \leqslant \varepsilon, \; \forall\, p \in D$

1) Initialization: update S such that there is at least one point on each curve of ∂D

2) Boundary enforcement: while a boundary edge *e* is encroached by some point of *S*, split e with $S \leftarrow S \cup \underset{p \in e}{argmax}\, U_S(p)$

3) Triangulation enforcement: while there exists an isolated interior vertex, conditionally insert $p_{max} \leftarrow \underset{p \in V(p_i, S) \cap V(p_j, S)}{argmax} d_M(p, p_i)$

4) Select point: $p_{max} \leftarrow \underset{p \in D}{argmax}\, U_S(p)$

   If $U_S(p_{max}) > \varepsilon$ then conditionally insert p_{max} and goto 3

5) Triangulation enforcement: if a triangle is inverted, conditionally insert its dual Voronoi vertex and goto 3.

Overview

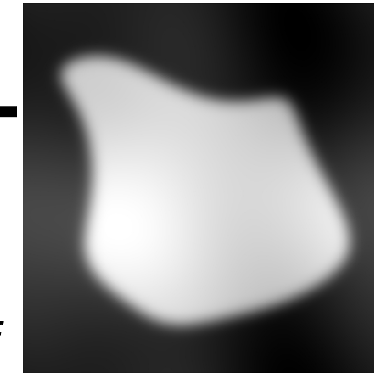
- Farthest Point Sampling
- Farthest Point Meshing
- **Image Approximation and Compression**

Image approximation

- progressive refinement of the spline approximation f_m of f

$$M(x) = (\epsilon + \mu_1)^\alpha e_1 e_1^T + (\epsilon + \mu_2)^\alpha e_2 e_2^T$$

f



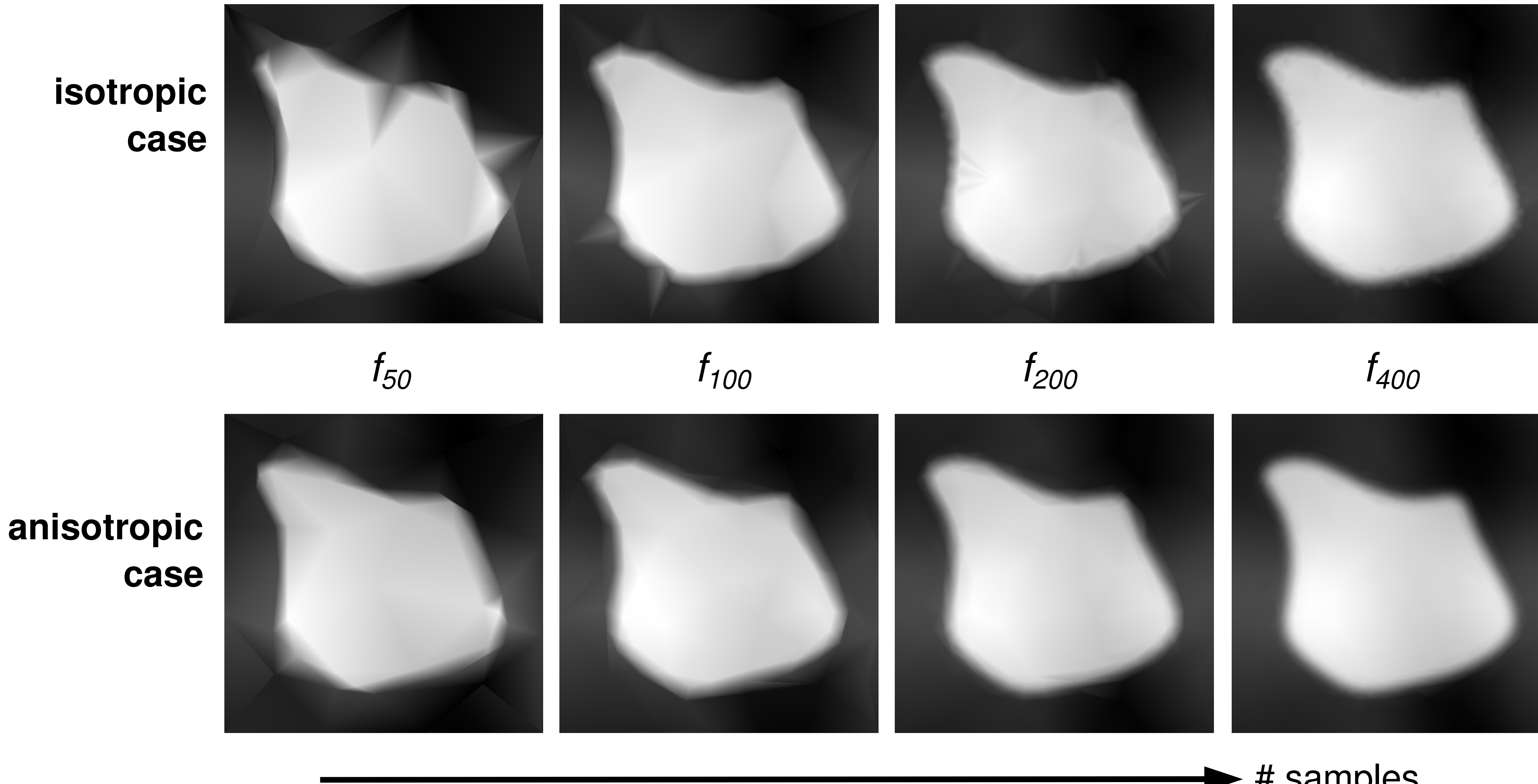


Image approximation

- progressive refinement of the spline approximation f_m of f

$$M(x) = (\epsilon + \mu_1)^{\alpha} e_1 e_1^T + (\epsilon + \mu_2)^{\alpha} e_2 e_2^T$$

- Which value of α ?
  - ➔ experimental study of the error

$$\|f - f_m\|$$

  - ➔ $\alpha \approx 0.8$

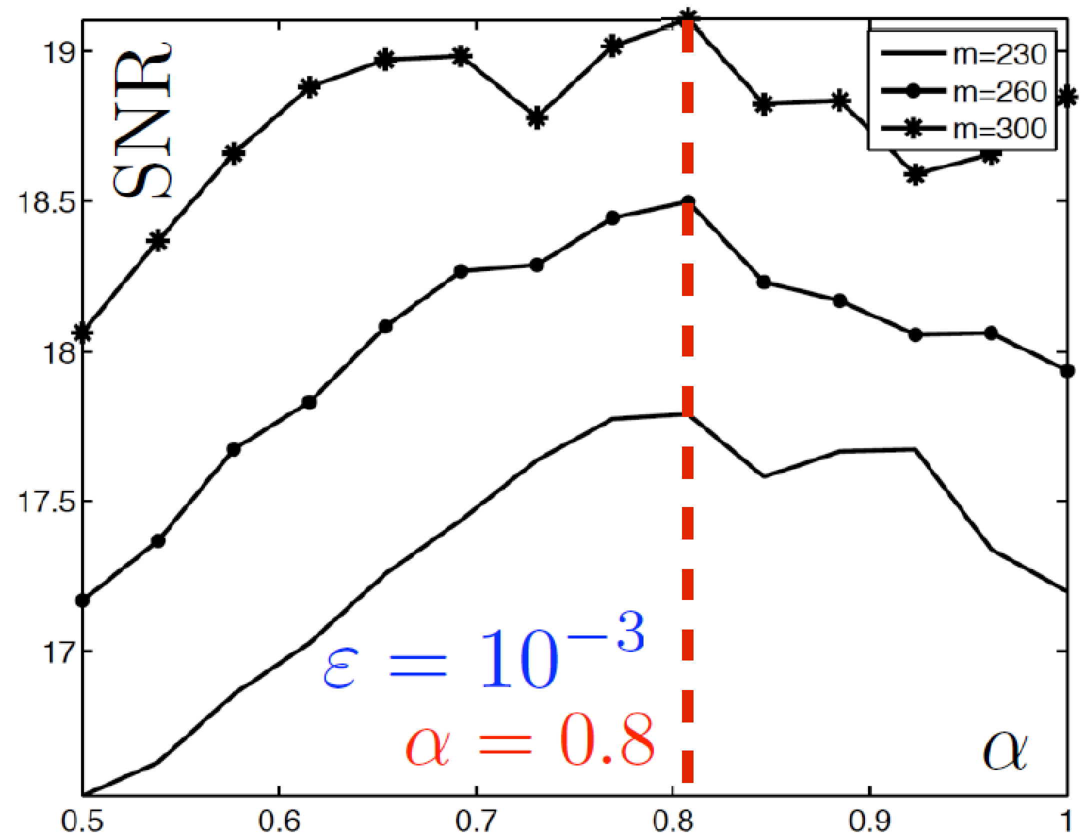


Image approximation

- progressive refinement of the spline approximation f_m of f

$$M(x) = (\epsilon + \mu_1)^\alpha e_1 e_1^T + (\epsilon + \mu_2)^\alpha e_2 e_2^T$$

- comparison between
  - anisotropic triangulation
  - Euclidean triangulation

  with same sample set for α=0.8

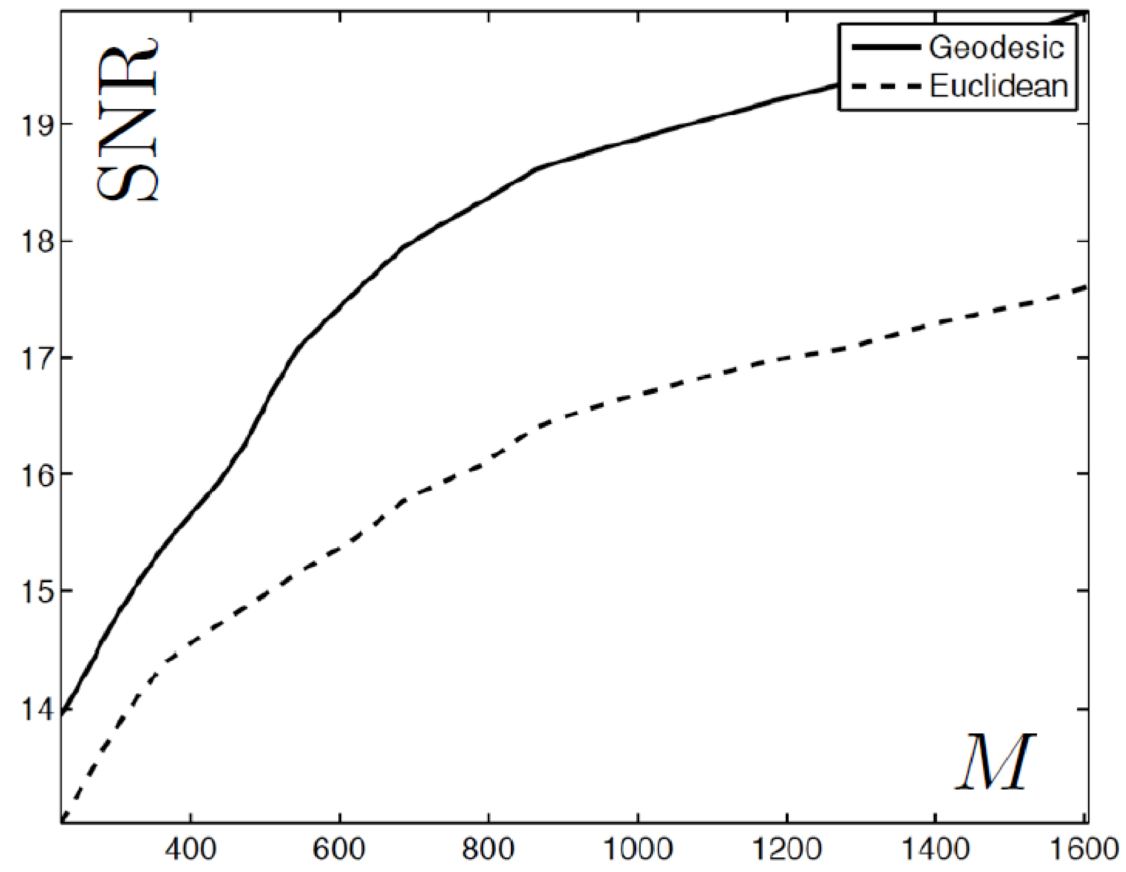
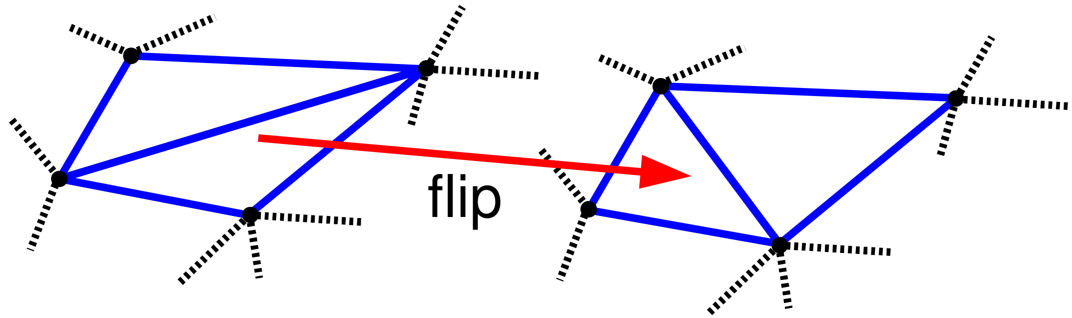


Image compression: simple encoder

Anisotropic Delaunay triangulation

- vertices $S = \{p_i\}_{i=1}^{m}$
- triangles $T = \{t_i\}_{i=1}^{k}$



1) Code the topology : sequence of flips transforming anisotropic triangulation to Euclidean Delaunay triangulation

$$T = T_0 \rightarrow T_1 \rightarrow T_2 \rightarrow \cdots \rightarrow T_q$$

$\Rightarrow$ $R_T = q \log_2(\#\text{edges})$ bits

2) Quantize + code the positions: $S = \{p_i\}_{i=1}^{m} \rightarrow \tilde{S} = \{\tilde{p}_i\}_{i=1}^{m}$

$\Rightarrow$ fixed precision (e.g. 6 bits): R_P bits

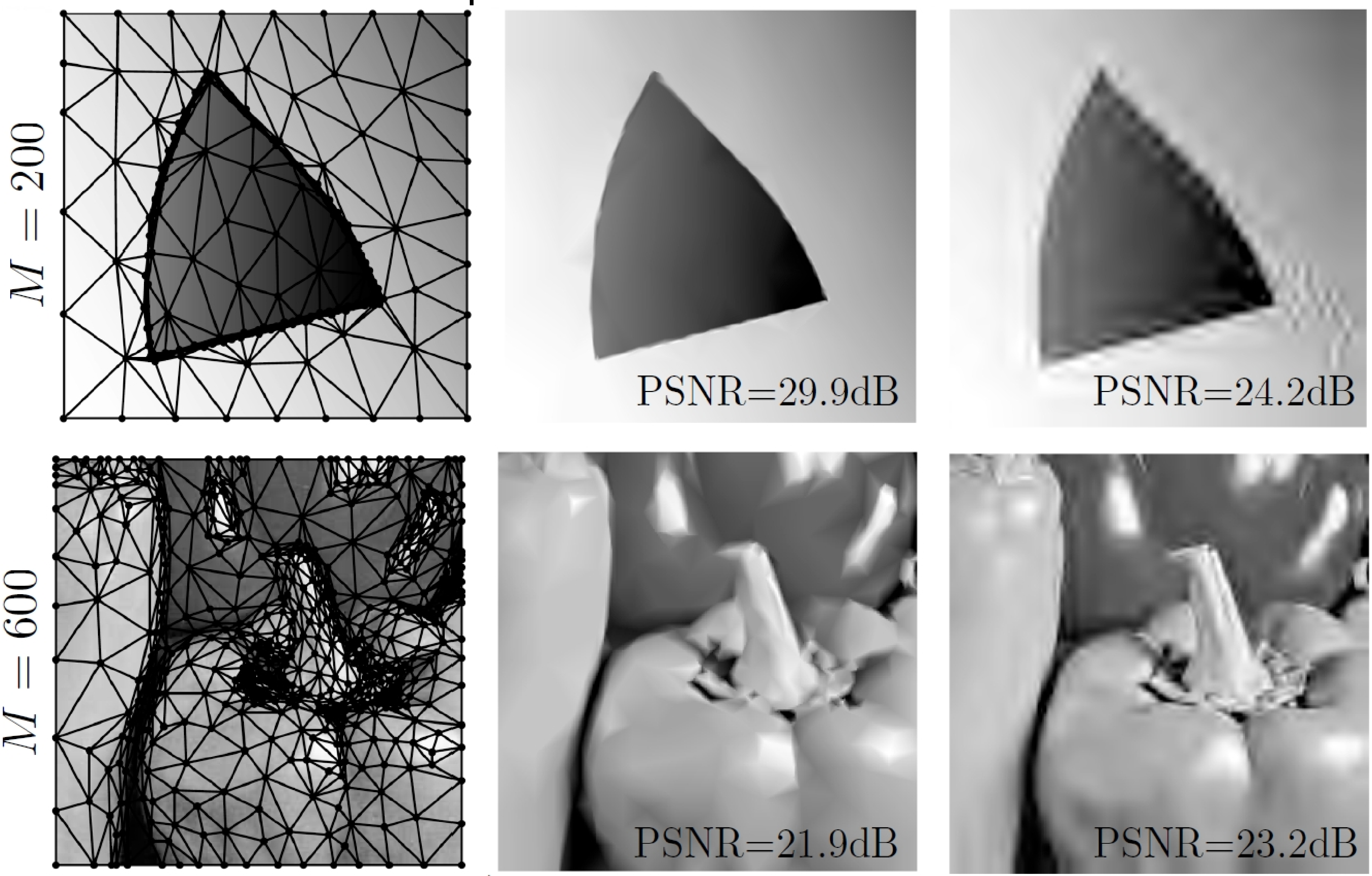
3) Quantize + code spline coefficients: $\lambda \rightarrow \tilde{\lambda}$

$\Rightarrow$ fixed precision (e.g. 6 bits): R_λ bits

Decoding with R = $R_T + R_P + R_\lambda$ bits: $f_R = \sum_{i=1}^{m} \tilde{\lambda}_i \varphi_i$

Compression results

comparisons with same bit-rate



anisotropic triangulations JPEG-2000

Conclusion

- Image compression method based on anisotropic geodesic triangulations
  - ➔ defined from a truly anisotropic geodesic Voronoi diagrams
- Anisotropy of images
  - ➔ exploited to generate triangulations through a farthest point strategy
- Extension to domain sampling/meshing/remeshing/compression

- More details in
  - ➔ Bougleux, Peyré and Cohen, *Anisotropic geodesics for perceptual grouping and domain meshing*, ECCV 2008.
  - ➔ Bougleux, Peyré and Cohen, *Image Compression with Anisotropic Triangulations*, ICCV 2009.