Lecture 31: Eigenvectors of Circulant matrices in Machine Learning (Image Processing)

$$C = c_0 I + c_1 P + \dots + c_{n-1} P^{n-1}$$

eigenvalues | eigenvectors | convolution rule

$$C = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-1} \\ & \ddots & \ddots & \ddots & c_2 \\ & & \ddots & \ddots & c_1 \\ c_2 & & & \ddots & c_0 \\ c_1 & c_2 & & & \end{bmatrix} \qquad \text{Toeplitz matrix} = \begin{bmatrix} t_0 & t_1 & t_2 & \cdots \\ t_{-1} & \ddots & \ddots & \ddots \\ t_{-2} & \ddots & \ddots & \ddots \\ \vdots & & \ddots & \ddots \end{bmatrix}$$

Cyclic convolution

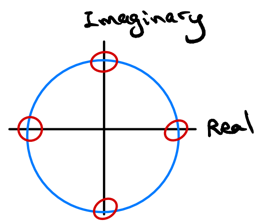
$c \circledast d$

Not cyclic

$Tv = t * v$

Eigenvalues and Eigenvectors of P

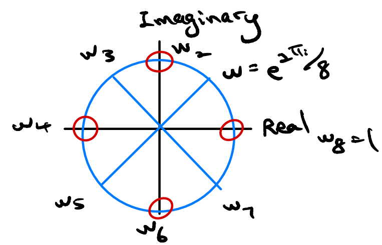
$$P = \begin{bmatrix} 0 & 1 & & 0 \\ & & \ddots & \\ 0 & & & 1 \\ 1 & & 0 & 0 \end{bmatrix} \quad \text{Cyclic shift}$$



$$\det(P - \lambda I) = \lambda^4 - 1 = 0$$

$$\lambda = 1, -1, i, -i$$

$P_{8\times 8}$



Matrices with orthogonal eigenvectors $(q_i, q_j) = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$

Those matrices (Normal matrix) $M = Q \Lambda \bar{Q}^T$ satisfy $\bar{M}^T M = M \bar{M}^T$

1) Symmetric (λ is real)
2) diagonal (eigenvectors $\to I$)
3) orthogonal ($|\lambda| = 1$)
4) Anti-symmetric $A^T = -A$ (λ is imaginary)

$\Rightarrow$ Matrix P and C are normal matrix

$C_1 C_2 = C_2 C_1$

$\Rightarrow$ Eigenvectors of P will also be eigenvectors of $C = c_0 I + c_1 P + c_2 P^2 + \dots + c_{n-1} P^{n-1}$

$$\lambda = 1 \qquad \lambda = -1 \qquad \lambda = i \qquad \lambda = -i$$

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ i \\ i^2 \\ i^3 \end{bmatrix} \quad \begin{bmatrix} 1 \\ -i \\ (-i)^2 \\ (-i)^3 \end{bmatrix}$$

Eigenvector matrix for all circulants matrix C_N including P_N

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w & w^2 & & w^7 \\ 1 & w^2 & w^4 & & w^{14} \\ 1 & w^3 & w^6 & & w^{21} \\ 1 & w^4 & w^8 & & w^{28} \\ 1 & w^5 & w^{10} & & w^{35} \\ 1 & w^6 & w^{12} & & w^{42} \\ 1 & w^7 & w^{14} & & w^{49} \end{bmatrix} = F_8$$

Fourier matrix is eigenvector matrix