Lecture 18 : Counting Parameters in SVD, LU, QR, Saddle Points

Counting Parameters in the Basic Factorizations

$L =$ triangular / diagonal 1's $\quad \frac{1}{2}n(n-1)$

$U =$ triangular $\quad 1+2+\dots+n \quad \frac{1}{2}n(n+1)$

$Q =$ orthogonal $\quad (n-1)+(n-2)+\dots \quad \frac{1}{2}n(n-1)$

$\Lambda =$ diagonal $\quad n$

$X =$ eigenvectors $\quad n^2-n$

SVD

full rank $= m$

$m \leq n$

$$A = (m \times m)(m \times n)(n \times n)$$

$$\underset{m\times n}{A} \quad U \quad \Sigma \quad V^T$$

A has mn parameters

$$\frac{1}{2}m(m-1) + m + \left(mn - \frac{1}{2}m(m+1)\right)$$

SVD for any $m \times n$ matrix of rank r

$$A = \underset{m\times r}{U}\ \underset{r\times r}{\Sigma}\ \underset{r\times n}{V^T}$$

$$mr - \frac{1}{2}r(r+1) + r + rn - \frac{1}{2}r(r+1) = (m+n-r)r$$

Saddle Points

1) From constraints (Lagrangian)

$\min \frac{1}{2} x^T S x$ subject to constraint $Ax = b$

$$L(x, \lambda) = \frac{1}{2} x^T S x + \lambda^T (Ax - b)$$

$$\frac{d}{dx} \Rightarrow Sx - A^T \lambda = 0$$

$$\frac{d}{d\lambda} \Rightarrow Ax = b$$

$$\begin{bmatrix} S & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ b \end{bmatrix}$$

2) Saddles from $R(x) = \frac{x^T S x}{x^T x}$ Rayleigh Quotient

$\Rightarrow \max = \lambda_1$ at $x = q_1$ $\quad \frac{q_1^T S q_1}{q_1^T q_1} = \lambda_1$

$\Rightarrow \min = \lambda_n$ $\quad x = q_n$ $\quad R(q_n) = \lambda_n$

Fact: Derivative $= 0$ at $x =$ eigenvectors $R(x) = \lambda_i$

Also a saddle point.