Lecture 19: Saddle Points Continued, Maxmin Principle

Saddle Points of $\frac{x^T S x}{x^T x} = R(x)$

$$S = \begin{bmatrix} 5 & & \\ & 3 & \\ & & 1 \end{bmatrix} \quad x = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \quad R = \frac{5u^2 + 3v^2 + w^2}{u^2 + v^2 + w^2}$$

$\sigma_1 =$ max $R = 5$ at $(1, 0, 0)$

$\sigma_2 =$ min $R = 3$ at $(0, 1, 0)$

$\sigma_3 =$ saddle $R = 1$ at $(0, 0, 1)$

eigenvectors of S

Maxmin Principle

$$\lambda_2 = \max_{\text{(all 2 dim spaces } V)} \ \min_{(V)} \frac{x^T S x}{x^T x}$$

$\Rightarrow$ The maximum possible minimum is λ_2 in the 5, 3, 1 example.

e.g. $x = (u, v, 0)$, the minimum is $\lambda_2 = 3$

For every other 2D combinations, minimum is $\lambda_3 = 1$.

$\therefore$ Therefore max min is 3.

Fact:

① All partial derivatives of $R(u, v, w)$ are zeros at $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$

Gradient vector is 0.

② These 3 points are the eigenvectors of S

③ $R(x) = 5, 1, 3$ at those three points.