STAB52, An Introduction to Probability - Tutorial 8

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Consider the set of numbers $x_1, ..., x_n$. Colloquially, the average of these values is understood to be $\bar{x} = (x_1 + ... + x_n)/n$. That is, we weight each value equally (probabilistically, every value x_i is equally likely to occur i.e. a discrete uniform distribution). However, what if each x_i has probability p_i of occurring ($\sum_{i=1}^{n} p_i = 1$). Then if we average each value based on (weighting each value by) the probability of each value occurring, then the average value becomes

$$\sum_{i=1}^{n} p_i \cdot x_i.$$

This is the motivation behind the expected value of a random variable: "The average value that the random variable takes weighted by the probabilities of each value". More precisely, if X is discrete, then

$$\mathbb{E}(X) = \sum_{x \in \mathbb{R}} x \cdot \mathbb{P}(X = x) = \sum_{x \in \text{ supp}(X)} x \cdot \mathbb{P}(X = x).$$

Similarly, the expected value for an absolutely continuous random variable with density function $f_X(x)$ is given by

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x f_x(x) dx.$$

Notice that this is the direct analog of the discrete case since an integral is essentially a way of summing uncountably many values.

Problem 1. (Evans & Rosenthal, Exercise 3.1.16)
Suppose you start with one penny and repeatedly flip a fair coin. Each time you get heads, *before* the first time you get tails, your number of pennies is *doubled*. Let X be the total number of pennies you have at the end. Compute $\mathbb{E}(X)$.

Solution: Let Y be the number of heads flipped before the first tail so the $Y \sim$ geometric$(1/2)$. This implies $\mathbb{P}(Y = y) = 1/2^{y+1}, y = 0, 1, 2, ...$ It then follows that $X = 2^Y$ and so by the Theorem 3.1.1. (Evans & Rosenthal), we have

$$\mathbb{E}(X) = \mathbb{E}(2^Y) = \sum_{y} 2^y \mathbb{P}(Y = y) = \sum_{y=0}^{\infty} 2^y \cdot \frac{1}{2^{y+1}} = \sum_{y=0}^{\infty} \frac{1}{2} = \infty.$$

Notice here that we have an expected value of ∞; this not uncommon result.

□

Problem 2. (Expectation of a random variable with natural support)
Let X be a discrete random variable whose support is the natural numbers. Prove that

$$\mathbb{E}(X) = \sum_{k=1}^{\infty} \mathbb{P}(X \geq k)$$

.

Solution: Using the definition of expectation for discrete random variables, we rewrite as

$$\mathbb{E}(X) = \sum_{k=1}^{\infty} k \cdot \mathbb{P}(X = k) = \sum_{k=1}^{\infty} \mathbb{P}(X \geq k)$$

$\square$

Problem 3. (Evans & Rosenthal, Exercise 3.1.25.)
Suppose X is a discrete random variable, such that $\mathbb{E}(\min(X, M)) = \mathbb{E}(X)$. Prove that $\mathbb{P}(X > M) = 0$.

Solution: Allow me to first give you an incorrect solution. This is actually my first attempt at the problem (and in fact, a more natural approach to this problem). Using the definition of expectation and the given hypothesis, we compute

$$
\begin{aligned}
\sum_{x\in\mathbb{R}} x \cdot \mathbb{P}(X = x) = \mathbb{E}(X) \\
&= \mathbb{E}(\min(X, M)) \\
&= \sum_{y\in\mathbb{R}} y \cdot \mathbb{P}(\min(X, M) = y) \\
&= \sum_{y\leq M} y \cdot \mathbb{P}(\min(X, M) = y) + \sum_{y>M} y \cdot \mathbb{P}(\min(X, M) = y) \\
&= \sum_{y\leq M} y \cdot \mathbb{P}(X = y) + \sum_{y>M} y \cdot \mathbb{P}(\emptyset) \\
&= \sum_{y\leq M} y \cdot \mathbb{P}(X = y),
\end{aligned}
$$

which implies that $\sum_{y>M} y \cdot \mathbb{P}(X = y) = 0$. If we had the assumption that M was non-negative, we would be done. However, this is not the case and we cannot actually conclude anything.

Now, let us see the correct solution: Define $Y = X - \min(X, M)$ so then Y is a discrete random variable such that $Y \geq 0$. By the given assumption (along with linearity of expectation), we have

$$\mathbb{E}(Y) = \sum_{y\geq 0} y \cdot \mathbb{P}(Y = y) = 0.$$

Since we are summing over $\{y \geq 0\}$ and $\mathbb{P}(Y = y) \geq 0$, it must be the case that $\mathbb{P}(Y = y) = 0$ for all $y > 0$. Disjoint additivity then says $\mathbb{P}(Y > 0) = 0$ i.e.

$$0 = \mathbb{P}(X > \min(X, M)) = \mathbb{P}(X > M)$$

as required.

$\square$

Problem 4. (Evans & Rosenthal, Exercise 3.1.17.a)
Let $X \sim \text{Geometric}(\theta)$, and let $Y = \min(X, 100)$. Compute $\mathbb{E}(Y)$.

Solution: We compute

$$
\begin{aligned}
\mathbb{E}(Y) &= \sum_{y=0}^{\infty} y \cdot \mathbb{P}(Y = y) \\
&= \sum_{x=0}^{100} x \cdot \mathbb{P}(X = x) + \sum_{x=101}^{\infty} 100 \cdot \mathbb{P}(X = x) \\
&= \theta \sum_{x=0}^{100} x \cdot (1-\theta)^x + 100\theta \sum_{x=101}^{\infty} (1-\theta)^x
\end{aligned}
$$

We compute the second summation as

$$\begin{aligned}
\sum_{x=101}^{\infty} (1-\theta)^x &= \sum_{x=0}^{\infty}(1-\theta)^x - \sum_{x=0}^{100}(1-\theta)^x \\
&= \frac{1}{1-(1-\theta)} - \frac{1-(1-\theta)^{100}}{\theta} \\
&= \frac{(1-\theta)^{101}}{\theta}.
\end{aligned}$$

In the first summation, we let $S = \sum_{x=0}^{100} x \cdot (1-\theta)^x$ and hence

$$\begin{aligned}
\theta S &= S - (1-\theta)S \\
&= S - \sum_{x=0}^{100} x \cdot (1-\theta)^{x+1} \\
&= \sum_{x=0}^{100} x \cdot (1-\theta)^x - \sum_{y=1}^{101}(y-1)\cdot(1-\theta)^y \\
&= \sum_{x=1}^{100}(1-\theta)^x \\
&= \sum_{x=0}^{100}(1-\theta)^x - (1-\theta) \\
&= \frac{1-(1-\theta)^{101}}{\theta} - (1-\theta)
\end{aligned}$$

I leave the simplification of the final expression as an elementary exercise.

$\square$