STAB52, An Introduction to Probability - Tutorial 3

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In calculus, we consider functions which map the real number line to itself (namely, $f : \mathbb{R} \to \mathbb{R}$). In probability, the analogous concept is a *random variable*; a function from the sample space to the reals ($X : \mathcal{S} \to \mathbb{R}$). When tackling real-world problems, a fundamental difficulty is selecting the appropriate random variable and sample space that best models your desired phenomena.

In this course, we will focus primarily on two types of random variables: *discrete* and *absolutely continuous*. A random variable X is said to be *discrete* if $\sum_{x\in\mathbb{R}} \mathbb{P}(X = x) = 1$. That is, the range of X is at most countable. Conversely, a random variable X is said to be *absolutely continuous* if there is a density function f such that

$$\mathbb{P}(a \leq x \leq b) = \int_a^b f(x)dx.$$

Within each dichotomy of random variables, there are various *distributions* (allocations of probability measure across various sample space). These are theoretical models which mathematicians and statisticians use to model events and systems. One should understand what each distribution aims to model (and hence when to use them) but memorizing the formulas for each distribution (e.g. the density functions) is not imperative.

Problem 1. (Evans & Rosenthal, Exercise 2.1.11.)
Suppose the sample space $\mathcal{S}$ is finite. Is it possible to define an unbounded random variables on $\mathcal{S}$? Why or why not?

Solution: Let X be any random variable on $\mathcal{S}$, $L_1 = \inf_{s\in\mathcal{S}} X(s)$ and $L_2 = \sup_{s\in\mathcal{S}} X(s)$. Then both these quantities are finite because $\mathcal{S}$ is. Then define

$$L = \max\{|L_1|, |L_2|\}$$

so that $|X(s)| \leq L$ for all $s \in \mathcal{S}$. Thus, X is a bounded.

□

Problem 2. (Evans & Rosenthal, Exercise 2.1.13.)
Suppose the sample sapce $\mathcal{S}$ is finite, of size n. How many different indicator functions can be define on $\mathcal{S}$.

Solution: This problem is equivalent to asking how many possible subsets of $\mathcal{S}$ can be constructed since for each subset A, we may define $\mathbb{I}_A$ to be a unique indicator function.

One solutions falls out immediately from combinatorics. Suppose you want to choose a subset. For each element (there are n of them), you have two choices: either you include or exclude it from your subset. Thus, by the multiplications principle, there are 2^n possible subsets.

An alternative solution is to use the Binomial Theorem:

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

First we realized that the total number of subsets is the number of subsets if size 0, 1, 2, ... n. In other wards, there are

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + ... + \binom{n}{n}$$

many possibilities. Taking $x = y = 1$ in the Binomial Theorem, shows that this sum is indeed 2^n.

$\square$

Problem 3. (Evans & Rosenthal, Exercise 2.3.26.)
Let $X \sim$ Geometric(θ_1) and $Y \sim$ Geoemetric(θ_2), with X and Y chosen independently. Compute $\mathbb{P}(X \leq Y)$.

Solution: Using the probability mass functions for a geometric distribution, we compute

$$\begin{aligned}
\mathbb{P}(X \leq Y) &= \sum_{y=0}^{\infty} \mathbb{P}(X \leq Y | Y = y) \cdot \mathbb{P}(Y = y) \\
&= \sum_{y=0}^{\infty} \mathbb{P}(X \leq y) \cdot \mathbb{P}(Y = y) \\
&= \sum_{y=0}^{\infty} \left(\sum_{x=0}^{y} \theta_1 (1-\theta_1)^x \right) \theta_2 (1-\theta_2)^y \\
&= \sum_{y=0}^{\infty} \left(1 - (1-\theta_1)^{y+1}\right) \theta_2 (1-\theta_2)^y \\
&= 1 - \theta_2 (1-\theta_1) \sum_{y=0}^{\infty} (1-\theta_1)^y (1-\theta_2)^y \\
&= 1 - \frac{\theta_2 (1-\theta_1)}{1 - (1-\theta_1)(1-\theta_2)}
\end{aligned}$$

$\square$

Problem 4. (Evans & Rosenthal, Exercise 2.4.10)
Suppose X has a density f and y has a density g. Is it possible that $f(x) > g(x)$ for all x?

Solution: Suppose $f(x) > g(x)$ for all $x \in \mathbb{R}$. Then because g is a density and by the strict monotonicity property of integrals, we have

$$1 = \int_{-\infty}^{\infty} g(x)dx < \int_{-\infty}^{\infty} f(x)dx$$

which is a contradiction to f being a density function.

$\square$

Problem 5. (Evans & Rosenthal, Exercise 2.4.13.)
Suppose $X \sim N(0, 1)$ and $Y \sim N(1, 1)$. Prove that $\mathbb{P}(X < 3) > \mathbb{P}(Y < 3)$.

Solution: Recall the density of a normal $N(\mu, \sigma^2)$ distribution is given by

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}.$$

Hence, we compute

$$\mathbb{P}(Y < 3) = \int_{\infty}^{3} \frac{1}{\sqrt{2\pi}} e^{\frac{-(y-1)^2}{2}} dy = \int_{\infty}^{2} \frac{1}{\sqrt{2\pi}} e^{\frac{-u^2}{2}} du = \mathbb{P}(X < 2)$$

where in the second identity, we made the substitution $u = y - 1$. Thus, $\mathbb{P}(Y < 3) = \mathbb{P}(X < 2) < \mathbb{P}(X < 3)$ by monotonicity of the integral.

$\square$