STAB52, An Introduction to Probability - Tutorial 5

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Given a random variable X, we are often interested in the new distribution of X under some transformation. For example, if $X \sim \text{Poisson}(\lambda)$, what is the distribution of $3X + \sqrt{X}$ Perhaps a more appropriate question is whether we can always determine the distribution of $f(X)$ for a given function $f : \mathbb{R} \mapsto \mathbb{R}$? In the case when X is a discrete random variable, this problem is simple. for any function f, we know $f(X)$ will be discrete and

$$P(f(X) = k) = \sum_{\{x \in \mathbb{R}: f(x) = k\}} \mathbb{P}(X = x)$$

for each $k \in \mathbb{R}$. We often say that this is the distribution "induced by f from X" (also known as the "push-forward probability measure". This is always just a simple brute-force calculation. No machinery, is needed here!

Now, for absolutely continuous random variables, the analogous result may be less clear. However, it is indeed true that under certain conditions on f, $f(X)$ will again be absolutely continuous provided X is. The most important characterization of an absolutely continuous distribution is its probability density function (this completely described the distribution after all). To address this need, we have the following theorem:

Theorem: *(Change of Variables)*
Let X be an absolutely continuous random variable with density function $f_X(x)$. Let $Y = h(X)$, where $h : \mathbb{R} \to \mathbb{R}$ is a function that is differentiable and strictly increasing wherever $f_X(x) > 0$. Then Y is also absolutely continuous and its density is given by

$$f_Y(y) = \frac{f_X(h^{-1}(y))}{|h'(h^{-1}(y))|}.$$

Notice that this theorem tell us two results: firstly, that the distribution will again be absolutely continuous (not a trivial fact) and secondly, it gives an explicit formula for the p.d.f.

Problem 1. (Evans & Rosenthal, Exercise 2.6.5.)
Let $X \sim \text{exponential}(\lambda)$. Let $Y = X^3$. Compute the density function $f_Y(y)$ of Y.

Solution: Here, we have $f_X(x) = \lambda e^{-\lambda x} \cdot \mathbb{I}_{\{x \geq 0\}}$, $Y = h(X) = X^3$. Thus, $h'(X) = 2X$ and $X = h^{-1}(Y) = Y^{1/3}$. Notice that $h(X) = X^3$ is both differentiable and strictly increasing everywhere (and hence whenever $f_X(x) > 0$. Thus, we may invoke the change of variables formula to conclude:

$$f_Y(y) = \frac{\lambda e^{-\lambda y^{1/3}}}{|3(y^{1/3})^2|} \cdot \mathbb{I}_{\{y^{1/3} \geq 0\}} = \frac{\lambda e^{-\lambda y^{1/3}}}{3y^{2/3}} \cdot \mathbb{I}_{\{y \geq 0\}}$$

□

Problem 2. (Evans & Rosenthal, Exercise 2.6.7.)
Let $X \sim \text{uniform}([0, 3])$. Let $Y = X^2$. Compute the density function $f_Y(y)$ of Y.

Solution: Here, we have $f_X(x) = (1/3)\mathbb{I}_{\{[0,3]\}}(x)$, $Y = h(X) = X^2$. Thus, $h'(X) = 2$ and $X = h^{-1}(Y) = Y^{1/2}$. Notice that $h(X) = X^2$ is both differentiable and strictly increasing for non-negative x (and hence whenever $f_X(x) > 0$. Thus, we may invoke the change of variables formula to conclude:

$$f_Y(y) = \frac{(1/3)\mathbb{I}_{\{[0,3]\}}(\sqrt{y})}{|2|} = \frac{1}{6}\mathbb{I}_{[0,9]}(y).$$

□