STAB52, An Introduction to Probability - Tutorial 1

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This tutorial focuses on one of the fundamental branches of applied probability theory: combinatorics. To deal with any of the combinatorial (counting) problems that may be encountered in the course, we require only two devices (along with some ingenuity):

The Multiplication Principle: If an event occurs in n different ways and another event occurs independently in m ways, then the two events can occur sequentially (one after the other) in m different ways.

The Binomial Coefficient: Given a set S which contains n elements, the number of different subsets of size k that can be constructed by choosing elements from S is given by the binomial coefficient

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

It is important to realize that though both principles count the number of different arrangements, the binomial coefficient does not incorporate the "order" of the elements (such rearrangements are called combination) whereas the multiplication principle does (such rearrangements are called permutations e.g. In the set $S = \{A, B, C\}$, if we count the number of subsets of size two using the binomial coefficient, there are $\binom{3}{2} = 3$ possible ways (namely, $\{A, B\}, \{A, C\}$ and $\{B, C\}$). However, according to the multiplication principle, this number is $3 \cdot 2 = 6$. The difference here, is the that the later approach cares about the "order" of the rearrangements i.e. $\{A,\, B\}$ and $\{B,\, A\}$ are two different permutations but the same combination. Let us now demonstrate the versatility of these two fundamental combinatorial principles.

Problem 1. (Evans & Rosenthal, Exercise 1.4.3.)
Suppose we flip 100 fair independent coins. What is the probability that at least three of them are heads?
Solution: Let S_{100} denote the total number of heads obtained after flipping 100 coins. Then

$$\begin{aligned}
(S_{100} \geq 3) &= 1 - \mathbb{P}(0 \leq S_{100} \leq 2) \\
&= 1 - (\mathbb{P}(S_{100} = 0) + \mathbb{P}(S_{100} = 1) + \mathbb{P}(S_{100} = 2)) \\
&= 1 - \left(\frac{1}{2^{100}} + \frac{\binom{100}{1}}{2^{100}} + \frac{\binom{100}{2}}{2^{100}} \right)
\end{aligned}$$

□

Problem 2. (Evans & Rosenthal, Exercise 1.4.4.)
Suppose we are dealt five cards from a 52-card deck. What is the probability that
(a) we get all four aces plus the king of spades?
(b) all five cards are spades?
(c) we get no pairs (i.e. all five cards are different)?
(d) we get a full house (i.e. three cards of a kind, plus a different pair)?

Solution:
(a) Assuming order does not matter (i.e. we count combinations), there is precisely one way this can occur so the probability is $1/\binom{52}{5}$.

(b) $\binom{13}{5}/\binom{52}{5}$.

(c) $\frac{52\cdot 48\cdot 44\cdot 40\cdot 36}{5!}/\binom{52}{5}$. We note here that we must divide by 5! to exclude repeating combinations (Think about why this is true by considering a simpler situation e.g. a 5 card deck).

(d) We have 13 different types of cards that can be chosen (i.e. ace, two, three, ..., king). For each type, we pick three cards of that type. Then there are twelve types of cards to choose from, of which, we pick two of that kind. Thus, the probability is $13 \cdot \binom{4}{3} \cdot 12 \cdot \binom{4}{2}/\binom{52}{5}$.

$\square$

Problem 3. (Evans & Rosenthal, Exercise 1.4.8.)
In a well-shuffled ordinary 52-card deck, what is the probability that the ace of spades and the ace of clubs are adjacent to each other?

Solution: If we imagine the ace of spades and ace of clubs being a single card that is glued together (first with the spade on top and then with the club), then it becomes clear that there are $2 \cdot 51!$ possible arrangements of the deck. So the probability is $2 \cdot 51!/52!$

$\square$

Problem 4. (Evans & Rosenthal, Exercise 1.4.12.)
Suppose we roll a fair six-sided die and flip three fair coins. What is the probability that the total number of heads is equal to the number showing on the die?

Solution: Let A be the event that the total number of heads equals the number showing on the die. Using the law of total probability (conditioned version), and conditioning on the various possible number of heads, we have

$$\begin{aligned}\mathbb{P}(A) &= \mathbb{P}(A|\text{no heads}) \cdot \mathbb{P}(\text{no heads}) + ... + \mathbb{P}(A|\text{three heads}) \cdot \mathbb{P}(\text{three heads}) \\ &= 0 \cdot \frac{1}{2^3} + \frac{1}{6}\binom{3}{1}\frac{1}{2^3} + \frac{1}{6}\binom{3}{2}\frac{1}{2^3} + \frac{1}{6}\binom{3}{3}\frac{1}{2^3}.\end{aligned}$$

$\square$