STAB52, An Introduction to Probability - Tutorial 6 and 7

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Definition: Given a random variable X, it's *cumulative distribution function* is the function $F_X : \mathbb{R} \to [0,1]$ defined by

$$F_X(x) = \mathbb{P}(X \leq x).$$

One particularly special example is the normal distribution...

Problem 1. (Evans & Rosenthal, Exercise 2.7.4.a)
Given the following joint density function

$$f_{X,Y}(x,y) = \begin{cases} 2x^2y + cy^5 & : 0 \leq x, y \leq 1 \\ 0 & : \text{otherwise} \end{cases}$$

find the value of c and compute $\mathbb{P}(X \leq 0.8, Y \leq 0.6)$.

Solution: The correct value of c should be such that the joint density has volume one. i.e.

$$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)dxdy = \int_0^1\int_0^1 2x^2y + cy^5 dxdy = \int_0^1 \left(\frac{2}{3}x^3y + cy^5x \Big|_0^1 \right) dy$$

$$= \int_0^1 \frac{2}{3}y + xy^5 dy = \frac{1}{3}y^2 + \frac{c}{6}y^6 \Big|_0^1 = \frac{1}{3} + \frac{c}{6}.$$

Solving for c, we conclude $c = 4$. Using this result, we compute

$$\mathbb{P}(X \leq 0.8, Y \leq 0.6) = \int_{-\infty}^{0.8}\int_{-\infty}^{0} .62x^2y + 4y^5 dydx = \int_0^{0.8} \left(x^2y^2 + \frac{4}{6}y^6 \Big|_0^{0.6} \right) dx$$

$$= \int_0^{0.8} \frac{9}{25}x^2 + \frac{486}{15625}xdx = \frac{9}{75}x^3 + \frac{486}{15625}x \Big|_0^{0.8} \sim 0.0863$$

□

Problem 2. (Evans & Rosenthal, Exercise 2.7.5.)
Prove that $F_{X,Y}(x,y) \leq \min(F_X(x), F_Y(y))$.

Solution: Fix $x, y \in \mathbb{R}$. To clarify, we first rewrite the desired inequality as

$$\mathbb{P}(X \leq x \cap Y \leq y) \leq \min(\mathbb{P}(X \leq x), \mathbb{P}(Y \leq y)).$$

Now by monotonicity, we immediately have $\mathbb{P}(X \leq x \cap Y \leq y) \leq \mathbb{P}(X \leq x)$ and $\mathbb{P}(X \leq x \cap Y \leq y) \leq \mathbb{P}(Y \leq y)$. Since $\mathbb{P}(X \leq x \cap Y \leq y)$ serves as a lower bound for both terms, it also bounds the minimum of these two terms and the result follows.

□

Problem 3. (Evans & Rosenthal, Exercise 2.7.11.)
Let $X \sim \exp(\lambda)$ and let $Y = X^3$. Compute the joint cdf, $F_{X,Y}(x,y)$.

Solution: Using the density of the exponential distribution, we compute

$$F_{X,Y}(x,y) = \mathbb{P}(X \le x, Y \le y) = \mathbb{P}(X \le x, X^3 \le y) = \mathbb{P}(X \le x, X \le y^{1/3}) = \mathbb{P}(X \le \min(x, y^{1/3}))$$

$$= \int_0^{\min(x,y^{1/3})} \lambda e^{-\lambda x} dx = 1 - e^{-\lambda x}$$

whenever $x, y > 0$ and $F_{X,Y}(x,y) = 0$ otherwise.

$\square$

Problem 4. (Evans & Rosenthal, Exercise 2.7.12.)
Let $F_{X,Y}$ be a joint cdf. Prove that for all $y \in \mathbb{R}$, $\lim_{x\to-\infty} F_{X,Y}(x,y) = 0$.

Solution: Using Problem 2, we have

$$F_{X,Y}(x,y) \le \min(F_X(x), F_Y(y)) \le F_X(x)$$

and hence

$$\lim_{x\to-\infty} F_{X,Y}(x,y) \le \lim_{x\to-\infty} F_X(x) = 0.$$

Recalling that $F_{X,Y}(x,y)$ is indeed a probability (taking values in $[0,1]$), we have the desired result.

$\square$

Problem 5. (Evans & Rosenthal, Exercise 2.7.16.a)
Suppose the joint density $f_{X,Y}$ is given by $f_{X,Y}(x,y) = ce^{-(x+y)}$ for $0 < x < y < \infty$ and 0 otherwise. Determine the value of c so that $f_{X,Y}$ is a density.

Solution: This question is nearly identical to Problem 1. The only difference is that the integral here is harder to evaluate. Indeed, we must be careful when choosing the limits of integration (this depends on the order of variables with which you integrate). We must solve

$$1 = \int_0^\infty \int_0^y ce^{-(x+y)} dx dy$$

or equivalently

$$1 = \int_0^\infty \int_x^\infty ce^{-(x+y)} dy dx.$$

Either equation has the solution $c = 2$.

$\square$