

MTH307

Population Dynamics

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1 Basic Population Dynamics

Difference between Population Dynamics and other models

Types of models

Single Species Model

t be time, changing with intervals τ (e.g. 1 year).

$N(t)$ be total population of the species at time t .

$\tilde{B}(t, \tau)$ be number of births in the population between t and $t + \tau$, fertility/recruitment term.

$\tilde{D}(t, \tau)$ be number of deaths in the population between t and $t + \tau$, mortality term, $\tilde{D} \geq 0$.

$\tilde{b}(t) = \lim_{\tau \rightarrow 0} \tilde{B}(t, \tau)/\tau$ be numbers of births per unit time.

$\tilde{d}(t) = \lim_{\tau \rightarrow 0} \tilde{D}(t, \tau)/\tau$ be numbers of deaths per unit time.

Discrete Time

$$N(t + \tau) = N(t) + \tilde{B}(t, \tau) - \tilde{D}(t, \tau) \quad (1)$$

Continuous Time when $\tau \rightarrow 0$, then

$$\frac{dN}{dt} = \lim_{\tau \rightarrow 0} \frac{N(t + \tau) - N(t)}{\tau} = \tilde{b}(t) - \tilde{d}(t) \quad (2)$$

1.1 Single Species Model: Malthusian Model

Assumption 1.1 (Malthusian Assumption). *The acts of death and birth of animals (vegetation, cell) happen statistically independently of each other and independently of the current population size.*

1.1.1 Discrete Time Equation $N(t + \tau) = N(t) + \tilde{B}(t, \tau) - \tilde{D}(t, \tau)$

Total number of births and deaths happening during timestep τ proportional to current population size, we have

$$\tilde{D}(t, \tau) = D(\tau)N(t), \quad \tilde{B}(t, \tau) = B(\tau)N(t)$$

then Eq.1 becomes

$$\begin{aligned} N(t + \tau) &= (1 + B(\tau) + D(\tau))N(t) \\ N(t + \tau) &= R(\tau)N(t) \end{aligned} \tag{3}$$

where

$$R(\tau) = 1 + B(\tau) + D(\tau) = \frac{N(t + \tau)}{N(t)}$$

is called **reproduction coefficient**. $R(\tau)$ is size ratio and only $R(\tau) \geq 0$ make **biological sense**. $R(\tau) = 0$ is the trivial case: the population dies out within one time step.

Discrete Time Solution

to solve Eq.3, we set $N(\tau) = R(\tau)N(0)$, $N(0) \geq 0$ at $t = 0$. And for all $t = n\tau$, we have

$$\begin{aligned} t = 0, N(\tau) &= RN(0) \\ t = \tau, N(2\tau) &= RN(\tau) = R^2N(0) \\ t = 2\tau, N(3\tau) &= RN(2\tau) = R^3N(0) \\ &\dots \\ t = n\tau, N(n\tau) &= RN(n\tau) = R^nN(0) \end{aligned} \tag{4}$$

That is, the population grows (for $R(\tau) > 1$) or decays (for $R(\tau) < 1$) in **Geometric Progression**.

1.1.2 Continuous Time Equation $\frac{dN}{dt} = \tilde{b}(t) - \tilde{d}(t)$

Now we have the number of births and deaths happening continuously per unit time proportional to current population size.

$$\tilde{d}(t) = dN(t), \quad \tilde{b}(t) = bN(t)$$

then Eq.2 becomes

$$\begin{aligned} \frac{dN}{dt} &= \tilde{b}(t) - \tilde{d}(t) \\ \frac{dN}{dt} &= rN(t) \end{aligned} \tag{5}$$

where

$$r = b - d$$

is called ***reproduction rate***, r can be positive or negative.

Continuous Time Solution

We solve Eq.5 by **separation of variables**.

$$\begin{aligned} \frac{dN}{N(t)} &= r dt \\ \int \frac{dN}{N(t)} &= \int r dt \\ \ln N(t) &= rt + C \\ N(t) &= N(0)e^{rt} \end{aligned} \tag{6}$$

where

$$N(t=0) = N(0) \geq 0$$

This is, the population grows (for $r > 0$) or decays (for $r < 0$) ***exponentially***.

1.1.3 Link of Two Equation

Discrete \rightarrow Continuous $N(n\tau) = R^n(\tau)N(0) \rightarrow N(t) = e^{rt}N(0)$

For small τ ,

$$\begin{aligned}\lim_{\tau \rightarrow 0} \frac{R(\tau) - 1}{\tau} &= \lim_{\tau \rightarrow 0} \frac{B(\tau) - D(\tau)}{\tau} \\ R(\tau) &\approx 1 + \tau r\end{aligned}\tag{7}$$

For fixed time t ,

$$n = \frac{t}{\tau} \rightarrow \infty$$

Then we have

$$\begin{aligned}N(n\tau) &= \lim_{\tau \rightarrow 0} R(\tau)^n N(0) \\ &\approx \lim_{\tau \rightarrow 0} (1 + r\tau)^{t/\tau} N(0) \\ &= e^{rt} N(0)\end{aligned}$$

Continuous \rightarrow Discrete $N(t) = e^{rt}N(0) \rightarrow N(n\tau) = R^n(\tau)N(0)$

Let $t = n\tau$, we have

$$\begin{aligned}N(t) &= N(n\tau) \\ &= e^{(r\tau)n} N(0) \\ &\approx (1 + r\tau)^n N(0) \\ &\approx R^n(\tau) N(0)\end{aligned}$$

by Eq.7 as $\tau \rightarrow 0$ and Taylor series.

1.2 Intraspecific Competition Model

Competition: increase of population size suppresses its reproduction.

Assumption 1.2 (Intraspecific Competition Assumption). *Reproduction per capita per unit of time is **NOT** constant. Instead, reproduction is depends on the current population size.*

Discrete Time

$$N_{t+1} = R(N_t)N_t$$

Competition when

$$dR/dN < 0$$

Continuous Time

$$dN/dt = r(N)N$$

Competition when

$$dr/dN < 0$$

1.2.1 Verhulst (Logistic) Model

1.2.2 Richards Model

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Theorem 1 (Helge Tverberg 1966). *Given $(r - 1)(d + 1) + 1$ points in \mathbb{R}^d , there is a partition of them into r parts whose convex hulls intersect.*