MTH307

Population Dynamics

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Contents

1 Basic Population Dynamics

Difference between Population Dynamics and other models

Types of models

Single Species Model

t be time, changing with intervals τ (e.g. 1 year).

$N(t)$ be total population of the species at time t.

$\tilde{B}(t,\tau)$ be number of births in the population between t and $t+\tau$, fertility/recruitment term.

$\tilde{D}(t,\tau)$ be number of deaths in the population between t and $t+\tau$, mortality term, $\tilde{D} >= 0$.

$\tilde{b}(t) = \lim_{\tau\to 0} \tilde{B}(t,\tau)/\tau$ be numbers of births per unit time.

$\tilde{d}(t) = \lim_{\tau\to 0} \tilde{D}(t,\tau)/\tau$ be numbers of deaths per unit time.

Discrete Time

$$N(t+\tau) = N(t) + \tilde{B}(t,\tau) - \tilde{D}(t,\tau) \tag{1}$$

Continuous Time when $\tau \to 0$, then

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \lim_{\tau\to 0} \frac{N(t+\tau) - N(t)}{\tau} = \tilde{b}(t) - \tilde{d}(t) \tag{2}$$

1.1 Single Species Model: Malthusian Model

Assumption 1.1 (Malthusian Assumption). *The acts of death and birth of animals (vegetation, cell) happen statistically independently of each other and independently of the current population size.*

1.1.1 Discrete Time Equation $N(t+\tau) = N(t) + \tilde{B}(t,\tau) - \tilde{D}(t,\tau)$

Total number of births and deaths happening during timestep τ proportional to current population size, we have

$$\tilde{D}(t,\tau) = D(\tau)N(t), \quad \tilde{B}(t,\tau) = B(\tau)N(t)$$

then Eq.1 becomes

$$\begin{aligned} N(t+\tau) &= (1 + B(\tau) + D(\tau))N(t) \\ N(t+\tau) &= R(\tau)N(t) \end{aligned} \tag{3}$$

where

$$R(\tau) = 1 + B(\tau) + D(\tau) = \frac{N(t+\tau)}{N(t)}$$

is called ***reproduction coefficient***. $R(\tau)$ is size ratio and only $R(\tau) \geq 0$ make **biological sense**.$R(\tau) = 0$ is the trivial case: the population dies out within one time step.

Discrete Time Solution

to solve Eq.3, we set $N(\tau) = R(\tau)N(0), N(0) \geq 0$ at $t = 0$. And for all $t = n\tau$, we have

$$\begin{aligned} &t = 0, N(\tau) = RN(0) \\ &t = \tau, N(2\tau) = RN(\tau) = R^2N(0) \\ &t = 2\tau, N(3\tau) = RN(2\tau) = R^3N(0) \\ &\qquad\qquad\qquad \ldots \\ &t = n\tau, N(n\tau) = RN(n\tau) = R^nN(0) \end{aligned} \tag{4}$$

That is, the population grows (for $R(\tau) > 1$) or decays (for $R(\tau) < 1$) in ***Geometric Progression***.

1.1.2 Continuous Time Equation $\frac{\mathrm{d}N}{\mathrm{d}t} = \tilde{b}(t) - \tilde{d}(t)$

Now we have the number of births and deaths happening continuously per unit time proportional to current population size.

$$\tilde{d}(t) = dN(t), \quad \tilde{b}(t) = bN(t)$$

then Eq.2 becomes

$$\begin{aligned} \frac{\mathrm{d}N}{\mathrm{d}t} &= \tilde{b}(t) - \tilde{d}(t) \\ \frac{\mathrm{d}N}{\mathrm{d}t} &= rN(t) \end{aligned} \tag{5}$$

where

$$r = b - d$$

is called ***reproduction rate***, r can be positive or negative.

Continuous Time Solution

We solve Eq.5 by **separation of variables**.

$$\begin{aligned} \frac{\mathrm{d}N}{N(t)} &= r\mathrm{d}t \\ \int \frac{\mathrm{d}N}{N(t)} &= \int r dt \\ \ln N(t) &= rt + C \\ N(t) &= N(0)e^{rt} \end{aligned} \tag{6}$$

where

$$N(t = 0) = N(0) \geq 0$$

This is, the population grows (for $r > 0$) or decays (for $r < 0$) ***exponentially***.

1.1.3 Link of Two Equation

Discrete → Continuous $N(n\tau) = R^n(\tau)N(0) \to N(t) = e^{rt}N(0)$

For small τ,

$$\lim_{\tau\to 0} \frac{R(\tau)-1}{\tau} = \lim_{\tau\to 0} \frac{B(\tau)-D(\tau)}{\tau}$$
$$R(\tau) \approx 1 + \tau r \tag{7}$$

For fixed time t,

$$n = \frac{t}{\tau} \to \infty$$

Then we have

$$\begin{aligned} N(n\tau) &= \lim_{\tau\to 0} R(\tau)^n N(0) \\ &\approx \lim_{\tau\to 0}(1 + r\tau)^{t/\tau} N(0) \\ &= e^{rt}N(0) \end{aligned}$$

Continuous → Discrete $N(t) = e^{rt}N(0) \to N(n\tau) = R^n(\tau)N(0)$

Let $t = n\tau$, we have

$$\begin{aligned} N(t) &= N(n\tau) \\ &= e^{(r\tau)n}N(0) \\ &\approx (1 + r\tau)^n N(0) \\ &\approx R^n(\tau)N(0) \end{aligned}$$

by Eq.7 as $\tau \to 0$ and Taylor series.

1.2 Intraspecific Competition Model

Competition: increase of population size suppresses its reproduction.

Assumption 1.2 (Intraspecific Competition Assumption)**.** *Reproduction per capita per unit of time is* ***NOT*** *constant. Instead, reproduction is depends on the current population size.*

Discrete Time

$$N_{t+1} = R(N_t)N_t$$

Competition when

$$dR/dN < 0$$

Continuous Time

$$dN/dt = r(N)N$$

Competition when

$$dr/dN < 0$$

1.2.1 Verhulst (Logistic) Model

Equation

A simplest guess of $r(N)$ by Verhulst is

$$r = r_0(1 - N/K)$$

r_0 is called maximal per capita reproduction rate.

Then the dynamic equation is

$$\mathrm{d}N/\mathrm{d}t = r_0 N(1 - N/K) \tag{8}$$

Eq.8 is non-linear.

Solution

Solve Eq.8 with initial value $N(0) = N_0$.

When $N_0 = 0$**,** $N(t) = 0$ **for** $t \in (0, \infty)$

When $N_0 > 0, N(t) \to K$ **as** $t \to +\infty$**.**

$$\mathrm{d}N/\mathrm{d}t = rN(1 - N/K)$$
$$N(t) = \frac{K}{1 + (K/N_0 - 1)\, e^{-rt}}$$

Graph!

K is called ***carrying capacity***.

1.2.2 Richards Model

Equation

Another guess of r is

$$r = r_0 \left(1 - (N/K)^{\nu}\right)$$

$$v = \text{ constant } > 0$$

Then the dynamic equation is

$$\mathrm{d}N/\mathrm{d}t = rN(1 - (N/K)^{v}) \tag{9}$$

Solution

Solve Eq.9 using substitution (i.e. let $u = (\frac{N}{K})^{-v}$).

$$N(t) = \begin{cases} \frac{k}{\left[\left(\left(\frac{k}{N_0}\right)^{\nu} - 1\right)e^{-vrt} + 1\right]^{\frac{1}{v}}} & , N(0) \neq 0 \text{ or } k \\ 0 & , N(0) = 0 \\ k & , N(0) = k \end{cases}$$

1.2.3 General case

Any ***single-species continuous time*** equation is solved in ***quadratures***

$$\mathrm{d}N/\mathrm{d}t = r(N)N = f(N)$$
$$\int_{N_0}^{N} \frac{\mathrm{d}N}{f(N)} = \int_0^t \mathrm{d}t = t$$

with special solutions (equilibria).

1.3 Single Species continuous Model: qualitative analysis

1.4 Spruce Budworm outbreak