MTH 305

Risk Management

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Contents

1 Volatility

1.1 Definition

There are two almost same definitions of volatility vary by the expression of return.

Definition 1.1 (Volatility)**.** *A variable's volatility, σ, is defined as the* ***standard deviation*** *of the return provided by the variable* ***per unit of time*** *when the return is expressed using* ***continuous compounding****.*

Remark 1.1 (Unit of time)**.** *For option pricing, the unit of time is one year while for risk management, the unit of time is one day.*

Remark 1.2 (Continuously compounded)**.** *Define S_i as the value of a variable at the end of day i. The continuously compounded return per day for the variable on day i is*

$$\ln \frac{S_i}{S_{i-1}}$$

The term continuously compounded is almost the same as the proportional change during a day, that is

$$\ln \frac{S_i}{S_{i-1}} \approx \frac{S_i - S_{i-1}}{S_{i-1}}$$

Definition 1.2 (Volatility)**.** *A variable's volatility, σ, is defined as the* ***standard deviation*** *of the return provided by the variable* ***per unit of time*** *when the return is expressed using* ***proportional change****.*

Definition 1.2 is usually used in risk management.

1.1.1 Variance Rate and Days

Assumption: the returns each day are independent with the same variance, time T, then

$$\sigma_{t_0+T} = \sqrt{T}\sigma_{t_0}$$
$$\sigma^2_{t_0+T} = T\sigma^2_{t_0}$$

"uncertainty increases with the square root of time"

Volatility is much higher on business days than on non-business days. Hence, we using business days with 252 days per year.

$$\sigma_{day} = \frac{\sigma_{year}}{\sqrt{252}}$$

1.1.2 Implied Volatilities

The implied volatility of an option is the volatility that gives the market price of the option when it is substituted into the pricing model.

1.2 The Power Law

Volatility is *not* constant, hence the returns on successive days are heavy tailed compared Normal distribution. The power law is more suitable in practice.

$$f(x) = P(v > x) = Kx^{-\alpha} \tag{1}$$

where K and α are constant.Eq.1 can be convert into

$$\ln[\mathrm{P}(v > x)] = \ln K - \alpha \ln x \tag{2}$$

to fit real data.

1.3 Monitoring Daily Volatility

Volatility with Continuously Compounded

σ_n: the volatility per day of a market variable on day n, as estimated at the end of day $n-1$.

S_i: value of the market variable at the end of day i.

u_i: the continuously compounded return during day i (between the end of day $i-1$ and the end of day i).

Using $m - days$' observations to monitor σ_n, we have

$$u_i = \ln \frac{S_i}{S_{i-1}} \tag{3}$$

$$\bar{u} = \frac{1}{m} \sum_{i=1}^{m} u_{n-i} \tag{4}$$

$$\sigma_n^2 = \frac{1}{m-1} \sum_{i=1}^{m} (u_{n-i} - \bar{u})^2 \tag{5}$$

Volatility with Proportional Change

With some assumptions and tricks, we simplified Eq.5 into

$$u_i = \frac{S_i - S_{i-1}}{S_{i-1}} \tag{6}$$

$$\bar{u} = 0 \tag{7}$$

$$\sigma_n^2 = \frac{1}{m}\sum_{i=1}^{m} u_{n-i}^2 \tag{8}$$

Different between the result of Eq.5 and Eq.8 is very small.

1.3.1 Weighting Schemes

Weighting Schemes Model

Assumption 1.1 (Weighting Schemes Model)**.** *More weight given to recent data.*

Let $\sum_{i=1}^{m} \alpha_i = 1$ and when $i > j$, $\alpha_i > \alpha_j > 0$, then Eq.8 converted into

$$\sigma_n^2 = \sum_{i=1}^{m} \alpha_i u_{n-i}^2 \tag{9}$$

ARCH(m) Model

Assumption 1.2 (ARCH(m) Model)**.** *There is a long-run average variance rate and that this should be given some weight.*

Let V_L is the long-run variance rate and γ is the weight assigned to V_L and $\gamma + \sum_{i=1}^{m} \alpha_i = 1$, then Eq.8 converted into

$$\begin{aligned}\sigma_n^2 &= \gamma V_L + \sum_{i=1}^{m} \alpha_i u_{n-i}^2 \\ &= \omega + \sum_{i=1}^{m} \alpha_i u_{n-i}^2\end{aligned} \tag{10}$$

1.3.2 EWMA Model