COMS10013 - Analysis - WS2

Useful facts

- gradients for $f(x, y)$; $\nabla f = (f_x, f_y)$ where $f_x = \partial f/\partial x$.
- gradients for $f(x, y)$; $\nabla_{\mathbf{w}} f = \mathbf{w} \cdot \nabla f$
- the Hessian
$$H(f) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$$
- the determinant of a matrix is equal the multiple of its eigenvalues, the trace is the sum.
- the Taylor series is
$$f(a + x) = f(a) + f'(a)x + \frac{1}{2}f''(a)x^2 + \frac{1}{6}f'''(a)x^3 + \ldots$$
or
$$f(a + x) = f(a) + \sum_{n=1}^{\infty} \frac{1}{n!}x^n \left.\frac{d^n f}{dx^n}\right|_{x=a}$$
- reminder that the original 'Leibniz' approach is to expand $f(x+dx)$ and then at the end set any dxs to zero.
- $$\sin(a) - \sin(b) = 2\sin((a - b)/2)\cos((a + b)/2).$$

Questions

These are the questions you should make sure you work on in the workshop.

1. **Gradients and Hessians** Let $z(x, y) = x^2y + 3xy^2 + xy$.
   (a) Find the gradient of $z(x, y)$.
   (b) Find the derivative of $z(x, y)$ along the vector $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$.
   (c) Compute $\nabla_{\begin{pmatrix} 3 \\ 1 \end{pmatrix}} z\left(\begin{pmatrix} 2 \\ 0 \end{pmatrix}\right)$.
   (d) What is the Hessian of $z(x, y)$?

2. **Extremal points in two dimensions**; this question is pretty hard!
   (a) Find the local extrema, and determine their types, for
   $$z(x, y) = x^3 + y^3 - \frac{1}{2}(15x^2 + 9y^2) + 18x + 6y + 1.$$
   (b) Find the local extrema, and determine their types, for
   $$z(x, y) = 3xy^2 - 30y^2 + 30xy - 300y + 2x^3 - 15x^2 + 111x + 7.$$

3. **Taylor series**
   (a) Compute the Taylor series of e^x at $x = 2$.
   (b) Compute the Taylor series of $1/(1-x)^2$ at $x = 0$.
   (c) Compute the Taylor series of $1/x$ at $x = 2$.

Extra questions

These are extra questions you might attempt in the workshop or at a later time.

1. **Trigonometric functions**
   (a) Compute the derivative of the sine function the old-fashioned Newton-Leibniz way. You should get that if $y = \sin(x)$ then $\frac{dy}{dx} = \cos(x)$. The method is as follows:
   - Write out the equation for dy.
   - Use the formula
   $$\sin(a) - \sin(b) = 2\sin((a-b)/2)\cos((a+b)/2).$$
   - Use the following approximations: near 0, the sine function is roughly a straight line with slope 1 so $\sin(a) \approx a$ when a is small, and $\cos(x+a) \approx \cos x$ when a is small compared to x (for example if $a = dx$).

   (b) Now, *define*
   $$\sin(x) := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$
   and
   $$\cos(x) := \sum_{n=0}^{\infty} \frac{(-1)^n}{2n!} x^{2n}.$$
   (Recall that $0! = 1$).
   (i) Assuming that you can swap the order of $\frac{d}{dx}$ and $\sum_{n=0}^{\infty}$, prove that $\frac{d}{dx}\sin(x) = \cos(x)$ and $\frac{d}{dx}\cos(x) = -\sin(x)$.
   (ii) (***) Define i to be a number such that $i^2 = -1$, and for any integer n, that $i^{2n} = (-1)^n$ and $i^{2n+1} = (-1)^n i$. Prove that
   $$e^{ix} = \cos(x) + i\sin(x).$$
   (If you like, deduce that $e^{i\pi} = -1$).

2. **Computing with Taylor series**.
   This exercise is to approximate $\sin(\pi/4)$ without using any trigonometric functions on your calculator. Either recall from the question 3(b) or recompute, without a calculator, the Taylor series of $\sin(x)$ at $x = 0$. Compute the approximations $T_1(x)$, $T_3(x)$, $T_5(x)$, $T_7(x)$ from your series at $x = \pi/4$ to 8 decimal places (you can use a calculator). (You can check how accurate your approximations are by plugging in $\sin(\pi/4)$ to your calculator and comparing your answer.)