Endogenous Risk-Exposure and Systemic Instability *

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Abstract

Most research on systemic stability assumes an economy in which banks are subject to exogenous shocks, but in practice, banks choose their exposure to risk. This paper studies the determinants of this endogenous risk exposure when banks are connected in a financial network. I show that there exists a network risk-taking externality: connected banks' choices of risk exposure are strategically complementary. Banks in financial networks, particularly densely connected ones, are endogenously exposed to greater risks and correlated risks. The model generalizes the canonical asset-substitution problem to connected economies, showing that the "effective" level (rather than the face level) of debt affects a bank's risk-taking incentives.

Keywords: systemic risks, financial networks, moral hazard, asset substitution

JEL Classification: G21, G28, L14

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Introduction

Since the 2008 financial crisis, the relationship between financial networks and systemic stability has been an important subject of research (Glasserman and Young, 2016). Most of the existing literature assumes exogenous shocks and studies how these idiosyncratic shocks propagate throughout a financial network.[1] However, banks' exposure to which particular shock is an endogenous choice variable. For example, a bank chooses between safe and subprime borrowers or determines its exposure to asset-backed securities.[2] This paper extends the theory of interbank networks and systemic stability by incorporating endogenous risk exposure. The introduction of a risk exposure choice changes the received intuition about financial stability in an important way and yields novel policy implications.

Pioneering work by Allen and Gale (2000) and Freixas et al. (2000) has shown that connected networks are more resilient to the contagion of exogenous shocks than unconnected networks due to a co-insurance mechanism, concluding that a highly connected banking sector promotes financial stability. In contrast, I show that although shocks are co-insured better in densely connected networks, banks in such networks initially choose greater risk exposure. Furthermore, they adopt correlated risks. In other words, in densely connected networks, bank-specific endogenous losses are more likely, and they tend to occur simultaneously. As a result, the banking sector as a whole becomes more fragile.

The basic intuition for this result relies on a network risk-taking externality. Banks in networks, if solvent, need to reimburse failed banks through interbank payments, which I dub as cross-subsidies. Cross-subsidies reduce the upside payoffs for banks (the payoffs when their own assets succeed). Conversely, banks' downside payoffs are protected by limited liabilities. This asymmetric distortion disincentivizes banks from being prudent, because it lessens their interest in the probability of success when trading off risk and return. Moreover, the risk-taking distortion is greater when the bank anticipates a higher likelihood of having to cross-subsidize other banks, which is when its counterparties choose greater risks. As a result, banks' choices of risk exposure are strategically complementary.

Banks that are in more extensively connected networks are more affected by such risk-taking externality. In particular, I show that banks in networks with a greater level of connections, in a maximally connected complete structure, or in networks with more counterparties will choose greater risk exposure in equilibrium. My model contributes to the debate on the relationship between a financial network's connectedness and systemic stability.[3] The "connected-stability"

[1] For example, Allen and Gale (2000), Freixas et al. (2000), and Gai et al. (2011) study exogenous liquidity shocks. Shin (2009), Elliott et al. (2014), and Acemoglu et al. (2015) examine exogenous economic shocks.

[2] Mian and Sufi (2009) empirically documented the unprecedented growth of subprime credit immediately prior to the 2008 financial crisis. They also found a concurrent, rapid increase in the securitization of subprime mortgages.

[3] For the "connected-stability" view, Allen and Gale (2000) show that a complete network is more robust to the loss contagion due to a co-insurance mechanism. For the "connected-fragility" view, Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015) argue that the "complete-stability" relationship does not apply to larger shocks due to a propagation mechanism. Elliott, Golub, and Jackson (2014) find similar non-monotonic relationships for equity networks.

view by Allen and Gale (2000) argues that a financial network's connectedness provides banks with co-insurance, resulting in greater stability. My result stands in contrast to their view, arguing that the losses that are better co-insured, as in Allen and Gale (2000)'s complete network, are more likely to evolve endogenously in the first place. Moreover, I show that banks' choices of risk exposure are not monotonically increasing in the network's degree of connectedness. On the one hand, greater connectedness increases a bank's exposure to more risk-taking externalities. On the other hand, the bank becomes less sensitive to a particular bank's failure. This non-monotonicity result is related to the observation of Elliott, Golub, and Jackson (2014), who use random networks to show that the ex-post contagion is not monotonic to a financial system's connectedness.

In addition to distorted risk-taking incentives, connected banks have incentives to expose themselves to a systemic risk endogenously. Intuitively, correlated portfolios reduce the possibility of a successful bank having to cross-subsidize other banks. In other words, choosing correlated risks reduces a bank's network distortion and thus increases its expected profit. As a result, a financial crisis (or the simultaneous failure of several banks) can evolve endogenously in connected financial systems. This result is consistent with the empirical observation by the Financial Crisis Inquiry Commission (2011) on the 2008 financial crisis when the financial system was highly connected. The commission concluded that "some financial institutions failed because of a common shock: they made similar failed bets on housing."

The theory begins with a stylized model in which banks are connected in a regular network, where each bank's interbank liabilities equal its interbank claims. In the initial date, each bank simultaneously chooses a risky project. I show that there exists a network-induced risk-taking distortion that determines each connected bank's "effective" debt – a sufficient statistic for the bank's choice of risk exposure. Moreover, I show that the strategic complementarity result is robust to a variety of model extensions that, for example, relax the binary return assumption and relax the zero downside return assumption. Finally, I show that the model can be extended to nonregular networks, in which a failed bank can earn a positive profit from its interbank claims.

This paper extends the canonical asset-substitution problem to a more complex setting, in which banks or firms are connected in a financial network. The seminal paper by Jensen and Meckling (1976) showed that the level of debt affects a firm's risk-taking behavior. Using the technique of networks, I extend this asset-substitution result by demonstrating that it is the "effective" level of a firm's debt that affects its risk-taking incentive. Specifically, I show that the "effective" debt not only consists of the face value of a firm's total debt – it is also affected by a network distortion, which is determined by the topology of the financial system, the debt and equity of the firm's counterparties, and the equilibrium risk exposure of the counterparties. My model builds on a payment equilibrium model developed by Eisenberg and Noe (2001). The innovation is that it allows banks to choose their risk exposure endogenously after anticipating the payment equilibrium and their counterparties' risk exposure. One important contribution of this model is to show that the standard intuition regarding the stabilizing effect of financial networks reverses with endogenous risk-taking. The theory also yields several novel perspectives on policy debates:

- **Capital Regulation**. I show that an individual bank's equity has a network-wide effect in reducing the entire financial system's risk exposure. A bank's equity will reduce not only the same bank's risk exposure but also the risk-taking incentives of other banks in the financial system. Intuitively, the equity first absorbs part of the loss that would otherwise propagate to other banks. Every bank in the financial system will now anticipate fewer cross-subsidies. As a result, it will optimally choose to be exposed to fewer risks.

- **Bailouts**. The conventional wisdom is that a government bailout, or simply the anticipation of one, is harmful to financial stability, because a bailout reduces banks' "skin in the game," thus encouraging excessive risk-taking. In contrast, my model suggests that a government bailout, if it is limited to interbank exposures, can actually reduce endogenous systemic risk. In the presence of the bailout, each bank anticipates fewer cross-subsidies to its counterparties. As a result, each bank is incentivized to take fewer risks.

- **Tailored Prudential Policies**. The model suggests that policymakers should consider each bank's systemic footprint instead of merely its balance sheet when deciding prudential policies. For example, I show that a policy is more effective in reducing an entire financial system's risk-taking incentives if the government targets the policy to banks that lie at the center of the financial system, are more upstream in the payment chain, or form larger clusters.[4]

The paper makes several contributions to the topic of systemic stability. In contrast to previous papers that study ex-post contagion, this paper provides a tractable model to analyze connected banks' ex-ante choices of risk exposure. My result reverses the previous intuition about the stabilizing effect of a highly connected financial system. The model also helps explain the observation that connected banks tend to make similar bets, as seen especially during the 2008 global financial crisis. The model generalizes the canonical asset-substitution problem to connected economies, arguing that it is the "effective" level (rather than the face level) of debt that affects a firm's risk-taking incentive. The theory also yields several novel perspectives on policy debates. It implores regulators to consider each bank's systemic footprint when designing prudential policies.

Related Literature This paper is related to a growing literature on the relationship between the interconnectedness of financial institutions and systemic stability. Most research focuses on

[4]This observation comports with a recently proposed rule by the Federal Reserve Board to tailor each individual bank's leverage ratio requirement based on "measures of systemic risk" rather than "a fixed leverage standard." The proposed rule will "replace the current 2 percent leverage buffer that applies uniformly to all GSIBs with a leverage buffer tailored to each GSIB." See https://www.federalreserve.gov/newsevents/pressreleases/bcreg20180411a.htm.

the question of whether more connections amplify or dampen systemic shocks. Glasserman and Young (2016) and Jackson and Pernoud (2021c) provide a survey of this literature, and here I will summarize a few related to the present paper. One branch of literature conforms to a "connected-stability" view: a connected network provides better liquidity insurance against some exogenous shocks to one individual bank. The view is supported by Allen and Gale (2000), Freixas, Parigi, and Rochet (2000), Leitner (2005). Allen and Gale (2000) argue that the initial loss will be widely divided in a complete network. Therefore banks will be less likely to default in such a network. In Freixas et al. (2000), depositors face uncertainties about where they will consume. They show that interbank connections can enhance resiliency. Leitner (2005) argues that the interbank connection is optimal ex-ante due to the possibility of private-sector bailouts.

On the other hand, the "connected-fragility" view is supported by Gai, Haldane, and Kapadia (2011), Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015), and Donaldson and Piacentino (2018). Using numerical simulations, Gai et al. (2011) demonstrate that a more complex and concentrated financial network may amplify the fragility. Acemoglu et al. (2015) use Eisenberg and Noe (2001)'s model to study the shock propagation. They conclude that a highly connected complete network becomes least stable under a large exogenous shock. Donaldson and Piacentino (2018) study the liquidity co-insurance benefits of long-term interbank debts. None of the above papers, nevertheless, studies how those initial shocks evolved in the first place.

There is sparse research on banks' portfolio choices when they are connected in financial networks. Brusco and Castiglionesi (2007) study banks' contracting behaviors in financial networks. They utilize the model of Diamond and Dybvig (1983) to study bankers' private benefit of gambling and their contracting behaviors with depositors. Contemporaneous papers such as Elliott, Georg, and Hazell (2021) and Jackson and Pernoud (2021b) study banks' choices of correlation with each other in financial systems. Elliott et al. (2021) show that German commercial banks are more likely to form connections with the ones with similar exposure to the real economy. Jackson and Pernoud (2021b) argue that banks do not internalize the inefficiency of their counterparties' bankruptcy cost. Galeotti and Ghiglino (2021) and Vohra, Xing, and Zhu (2021) study firms' incentives under common ownership structure with equity interdependencies.

1 Model

The economy consists of $N \in \mathbb{N}^+$ risk-neutral banks that are interconnected through cross-holdings of unsecured debt contracts $\bar{d}_{ij} > 0$, where $\bar{d}_{ij}$ is the face value of the interbank debt that bank j owes to bank i. Assume that all interbank liabilities have equal seniority. Denote $\bar{d}_j \equiv \sum_i \bar{d}_{ij}$ as bank j's total interbank liabilities. I restrict most of the analysis to regular network structures in which the total interbank liabilities and claims are equal for all banks to abstract away the effect of network asymmetry. I will show later that the model can be extended to nonregular networks. Define $\theta_{ij} \equiv \bar{d}_{ij}/\bar{d}_j$ as bank i's share in j's total interbank liabilities. Under the regularity assumption, $\sum_j \theta_{ij} = \sum_i \theta_{ij} = 1$. Denote $\mathbf{\Theta} \equiv [\theta_{ij}]$ as an $N \times N$ matrix, which determines the network connectedness and will be further discussed in section 3. A topology $\mathbf{\Theta}$ is path-connected if every two nodes in the network can be connected by some path. It is symmetric if each row of $\mathbf{\Theta}$ has the same set of elements. In summary, a regular network can be characterized by $(\bar{d}, \mathbf{\Theta})$.

Besides the interbank liabilities, each bank also owes a deposit $v_i = v$ that needs to be paid in full before the payment of its interbank debt. In the initial date, each bank i simultaneously chooses a risky project Z_i among a set of available projects $[\underline{Z}, \overline{Z}]$. This project Z_i will produce a random return $\tilde{e}_i(Z_i)$ with the following payoff distribution.

$$\tilde{e}_i = \begin{cases} Z_i & \text{w.p} \quad P(Z_i) \\ 0 & \text{w.p} \quad 1 - P(Z_i) \end{cases} \tag{1}$$

where $P(Z) \in (0,1)$ is some deterministic function that denotes the probability of project Z's success. In the benchmark model, let each bank's project be independent. This assumption will later be relaxed in an extension. It's worth noting that $P(Z_i)$ denotes the probability of bank i's primitive asset being successful rather than bank being "solvent" (able to fully pay back its deposits) or the bank being "out of default" (able to fully pay back its total debt).[5] To capture the main intuition, let's first consider that a bank will not default whenever its project succeeds, or in other words, there are no systemic defaults. This can be modeled by the assumption $\underline{Z} \geqslant v + \bar{d}$. In section 4.1, I will extend the model to consider the possibility of systemic defaults.

Before we proceed, let us first map elements of the model to the real world. The interbank claims and liabilities considered here are not overnight repurchase agreements but instead are long-term loans. As Donaldson and Piacentino (2018) argue, banks tend to maintain off-setting long-term debts without netting them out. For example, the authors show that Barclays' gross interbank position is almost ten times as large as its net position. Craig and Ma (2018) show that around 60% of all German interbank loans have maturities over a year. On the asset side, the risky projects in the model correspond to financial investments such as bond portfolios as in Mink et al. (2020).

ASSUMPTION 1. *$P(Z)$ is decreasing in Z, and $P(Z) \cdot Z$ is concave in Z.*

Suppose this assumption holds throughout the rest of the paper. The first part of the assumption captures the fact that high-return projects come with high risks. Each bank faces a trade-off between project payoff and project safety. A large Z represents a project with a large return along with a high risk. Therefore, we can interpret Z_i as bank i's choice of its risk exposure. The efficient risk exposure for each individual bank is when $\mathbb{E}[\tilde{e}]$ is maximized: $\hat{Z} = \text{argmax}_Z P(Z)Z$. The second part of the assumption is to ensure a unique risk exposure. A sufficient condition is to let $P()$ be concave: the project risk increases at a growing rate in the project return.

[5] To avoid confusion, throughout the rest of the paper, I will use the word "successful" to denote that the bank's primitive asset pays off, the word "solvent" to denote that the bank can fully pay back its deposits, and the phrase "out of default" to denote that the bank can fully pay back its total debt (deposits + interbank debt). The terminology difference between "default" and "insolvency" has been used by Allen and Gale (2000) where the difference in meaning results from whether interbank debt and claims are considered.

After every bank has chosen its risk exposure $\boldsymbol{Z} = (Z_1, ..., Z_N)$, the state of nature $\boldsymbol{\omega} = (\omega_1, ..., \omega_N)$ will be independently drawn from the distribution according to equation 1. For each bank, ω_i can take one of the two values: success ($\omega_i = s$) or fail ($\omega_i = f$). As a result, $\boldsymbol{\omega} \in \boldsymbol{\Omega} = \{s, f\}^N$. After realization of the state of nature, interbank debts' reimbursement will be determined from a payment equilibrium. A bank's total payments depend on what it possesses, which depends on the interbank payments from other banks. As a result, the payment equilibrium is solved by a fixed point system. This notion of the payment equilibrium is introduced by Eisenberg and Noe (2001). The current paper differs from theirs in that the my model's payment vector is parametrized by a vector of risk exposure $\boldsymbol{Z}$ and a vector of states $\boldsymbol{\omega}$. Definition 1 formally defines the payment equilibrium.

DEFINITION 1. *Given a risk vector* $\boldsymbol{Z}$*, the payment equilibrium in state* $\boldsymbol{\omega}$ *is a vector of functions* $\boldsymbol{d}^*(\boldsymbol{\omega}; \boldsymbol{Z}) = [d_1^*(\boldsymbol{\omega}; \boldsymbol{Z}), ..., d_N^*(\boldsymbol{\omega}; \boldsymbol{Z})]$ *that solves*

$$d_i^*(\boldsymbol{\omega}; \boldsymbol{Z}) = \left\{ \min \Big[\sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}; \boldsymbol{Z}) + e_i(\omega_i, Z_i) - v, \bar{d} \Big] \right\}^+ \quad \forall i \in \mathcal{N} \quad \forall \boldsymbol{\omega} \in \boldsymbol{\Omega} \tag{2}$$

$d_i^*(\boldsymbol{\omega}; \boldsymbol{Z})$ denotes bank i's total payment of its interbank liabilities in state $\boldsymbol{\omega}$ after banks choosing risk exposure $\boldsymbol{Z}$. On the right-hand side, $\sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}; \boldsymbol{Z}) + e_i(\boldsymbol{Z}, \boldsymbol{\omega})$ is bank i's available resources for payments to its total liabilities (deposits and interbank debts). The function $\min\left[., \bar{d}\right]$ captures banks' limited liabilities, so they pay either what they owe or what they have, whichever is smaller. $\{.\}^+ \equiv \max\{., 0\}$ represents the fact that banks' interbank payments are non-negative. It binds when the bank is not solvent – a bank only starts to pay its interbank liabilities after it fully fulfills its deposits.

We observe that the payment $d_i^*(\boldsymbol{\omega}; \boldsymbol{Z})$ is a function of $\boldsymbol{\omega}$. For each state of nature $\boldsymbol{\omega}$, we will have a separate fixed-point system. Therefore, given a risk vector $\boldsymbol{Z}$, we need to solve 2^N fixed-point systems, one for each state of nature. Before we proceed, one immediate task is to show that the above payment equilibrium exists and is unique.

LEMMA 1. [Eisenberg-Noe] *For any risk vector* $\boldsymbol{Z}$*, the payment equilibrium exists and is generic unique.*

The proof is a simple utilization of the Brouwer fixed-point theorem and is identical to Eisenberg and Noe (2001). Part of the proof is subsumed in the proof of proposition 2. Hence, it is omitted here to conserve space. Eisenberg and Noe (2001) show that for each $\tilde{\boldsymbol{e}}$, the fixed point exists and is generic unique. It is identical to say that for every combination of $(\boldsymbol{\omega}, \boldsymbol{Z})$, the fixed point exists and is generic unique. Hence lemma 1 naturally follows.

After the realization of $\boldsymbol{\omega}$ and the interbank payments $\boldsymbol{d}^*(\boldsymbol{\omega}; \boldsymbol{Z})$, each bank's profit at the final date becomes

$$\Pi_i(\boldsymbol{\omega}; \boldsymbol{Z}) = \Big(\sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}) + e_i(\boldsymbol{Z}, \boldsymbol{\omega}) - v_i - d_i^*(\boldsymbol{\omega}; \boldsymbol{Z}) \Big)^+ \tag{3}$$

The profit $\Pi_i(\boldsymbol{\omega}; \boldsymbol{Z})$ depends on the risk exposure of all other banks. With equation 3, we can derive each bank's expected profit as

$$\mathbb{E}\Big[\Pi_i(\boldsymbol{\omega}; \boldsymbol{Z})\Big] = \sum_{\boldsymbol{\omega} \in \boldsymbol{\Omega}} \Pi_i(\boldsymbol{\omega}; \boldsymbol{Z}) \cdot \Pr(\boldsymbol{\omega}) = \sum_{\boldsymbol{\omega} \in \boldsymbol{\Omega}} \Big[\Pi_i(\boldsymbol{\omega}; \boldsymbol{Z}) \cdot \prod_j \Pr(\omega_j)\Big]$$

The last equality is due to the assumption that each bank's project outcome is independent. In equilibrium, each bank choose its risk exposure Z_i to maximize its expected payoff $\mathbb{E}_{\boldsymbol{\omega}}[\Pi_i(\boldsymbol{\omega}; \boldsymbol{Z})]$. As a result, the Nash Equilibrium for banks' risk exposure can be expressed as the solution of the following fixed-point system:

$$Z_i^* = \underset{Z_i}{\operatorname{argmax}} \sum_{\boldsymbol{\omega} \in \boldsymbol{\Omega}} \Big[\Pi_i(\boldsymbol{\omega}; Z_i, \boldsymbol{Z}^*_{-\boldsymbol{i}}) \cdot \prod_j \Pr(\omega_j)\Big] \quad \forall i \in \mathcal{N} \tag{4}$$

We observe that $\boldsymbol{Z}_{-\boldsymbol{i}}$ affects bank i's expected profit in two ways: first through the distribution of the state of nature, $Pr(\omega_j = s) = P(Z_j)$, and second through the payment equilibrium $\boldsymbol{d}^*(\boldsymbol{\omega}, \boldsymbol{Z})$.

2 Risk-Taking Equilibrium and Network Distortion

It's immediate that we can define a risk-taking equilibrium as every bank chooses its risk exposure simultaneously, anticipating other banks' optimal risk exposure and the resulting payment equilibrium.

DEFINITION 2. *The risk-taking equilibrium in a financial network is a pair $(\boldsymbol{d}^*(\boldsymbol{\omega}; \boldsymbol{Z}), \boldsymbol{Z}^*)$ consisting of a vector of payment functions $\boldsymbol{d}^*(\boldsymbol{\omega}; \boldsymbol{Z})$ and a vector of risk exposure $\boldsymbol{Z}^*$ such that:*

1. *The vector of functions $\boldsymbol{d}^*(\boldsymbol{\omega}; \boldsymbol{Z})$ is a payment equilibrium for any $\boldsymbol{Z}$.*
2. *For each $i \in \mathcal{N}$, Z_i^* is optimal and solves equation 4, given $\boldsymbol{d}^*(\boldsymbol{\omega}; \boldsymbol{Z})$ and $\boldsymbol{Z}^*_{-i}$.*

We first observe that the above risk-taking equilibrium is the solution of two intertwined systems of equations (equation 2 and 4): when choosing the risk vector Z_i, each bank anticipates the payment equilibrium. When determining the interbank debt payment $\boldsymbol{d}^*(\boldsymbol{\omega}; \boldsymbol{Z})$, banks' chosen risk vector is a parameter.

At first glance, the fixed point solutions to the two intertwined systems look complicated to derive. Thanks to the following lemma, the existence and analytical solutions of the risk-taking equilibrium can be obtained.

LEMMA 2. *The payment equilibrium $\boldsymbol{d}^*(\boldsymbol{\omega}; \boldsymbol{Z})$ is constant in the risk exposure vector $\boldsymbol{Z}$.*

Proof. In the Appendix

The idea is that when a bank's project succeeds, its total interbank payment is the face value $\bar{d}$, independent of any bank's chosen risk exposure. On the other hand, when a bank's project fails, its contribution to the payment system is 0, also independent of any bank's chosen risk exposure. Therefore, the payment equilibrium is independent of the risk exposure vector $\boldsymbol{Z}$. As a result, we can rewrite $\boldsymbol{d}^*(\boldsymbol{\omega}) = \boldsymbol{d}^*(\boldsymbol{\omega}; \boldsymbol{Z})$. We are now able to disentangle the two intertwined fixed-point systems. We first solve the fixed-point vectors for the payment equilibrium (equation 2) and then use them to derive the fixed-point solution to the Nash equilibrium (equation 4).

We observe that banks will earn positive profits only if their projects succeed. For a failed bank, its available resources will be at most $\max_\omega \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}) = \bar{d}$, which is when its interbank claims are paid in full. This implies that a bank with a failed project will surely default on its interbank debt because $\sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}) - v < \bar{d}$.[6] From here, we can rewrite bank i's expected profit as:

$$\mathbb{E}\Big[\Pi_i(\boldsymbol{\omega}; \boldsymbol{Z})\Big] = P(Z_i) \cdot \sum_{\boldsymbol{\omega_{-i}}} \Big[Z_i - v - \Big(\bar{d} - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega^{i=s}})\Big)\Big] \cdot \Pr(\boldsymbol{\omega_{-i}})$$

where $\boldsymbol{\omega_{-i}} \in \{s, f\}^{N-1}$ denotes the states of nature of all banks except bank i. With a slight abuse of notation, let $\boldsymbol{\omega^{i=s}} \equiv (\omega_1 ..., \omega_{i-1}, s, \omega_{i+1}, ...\omega_N)$ denote a vector that appends bank i's success to other banks' states of nature $\boldsymbol{\omega_{-i}}$. From the independence assumption, we have $\Pr(\boldsymbol{\omega_{-i}} | \omega_i = s) = \Pr(\boldsymbol{\omega_{-i}})$. Now define $\mathcal{D}_i(\boldsymbol{Z_{-i}})$ as

$$\mathcal{D}_i(\boldsymbol{Z_{-i}}) \equiv \sum_{\boldsymbol{\omega_{-i}}} \Big(\bar{d} - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega^{i=s}})\Big) \cdot \Pr(\boldsymbol{\omega_{-i}}) \tag{5}$$

Note that $\mathcal{D}_i(\boldsymbol{Z_{-i}}) \in (0, \bar{d})$ and is a function of the network structure $(\bar{d}, \boldsymbol{\Theta})$ and the counterparty risk exposure $\boldsymbol{Z_{-i}}$. We can rewrite bank i's expected profit as

$$\mathbb{E}\Big[\Pi_i(\boldsymbol{\omega}; \boldsymbol{Z})\Big] = P(Z_i) \cdot \big(Z_i - v\big) - P(Z_i) \cdot \mathcal{D}_i(\boldsymbol{Z_{-i}}) \tag{6}$$

The first part $P(Z_i) \cdot \big(Z_i - v\big)$ is identical to the expected payoff of a stand-alone bank. From equation 5, we can interpret $\mathcal{D}_i(\boldsymbol{Z_{-i}})$ as bank i's expected net interbank payment (or "cross-subsidy" to other banks) when its project succeeds. Equation 6 implies that bank i's expected payoff is affected by counterparty risk $\boldsymbol{Z_{-i}}$ through the cross-subsidy $\mathcal{D}_i(\boldsymbol{Z_{-i}})$. We are interested in knowing how counterparty risk $\boldsymbol{Z_{-i}}$ affects bank i's risk-taking incentive. Proposition 1 establishes the relationship.

PROPOSITION 1. *The choices of risk exposure $\boldsymbol{Z}$ are strategically complementary among all banks in the same financial network.*

Proof. In the Appendix

The proposition states that a bank's optimal risk exposure is increasing in the risk exposure of any other banks in the network. To see the intuition, suppose a counterparty bank, say bank

[6] This fact does not necessarily hold when banks are connected in nonregular networks. In those cases, a failed bank can possess enough resources to pay off its total debt and earn a positive profit. Section 4.2 provides a more detailed discussion.

m, increases its risk exposure. As a result, bank m's project becomes more likely to fail. When it does fail, bank i's cross-subsidies to other banks will increase. This will reduce bank i's upside payoff (the payoff when its project succeeds). As a result, bank i will be less interested in the probability of success when trading off risk and return. Thus, it will optimally choose a greater risk exposure in response to bank m's increased risk exposure.

Proposition 1 suggests that $\mathcal{D}_i(\boldsymbol{Z}_{-\boldsymbol{i}})$ can also be interpreted as a network risk-taking distortion. The canonical asset-substitution problem argues that the amount of debt affects a bank's risk-taking incentive (Jensen and Meckling, 1976). My model suggests that, in a connected financial system, it is the "effective" level of debt $v + \mathcal{D}_i(\boldsymbol{Z}_{-\boldsymbol{i}})$ that matters to a bank's risk-taking incentive. The size of this effective debt is between the bank's net debt v and gross debt $v + \bar{d}$. As we will see in Sections 3 and 5, the information about a bank's effective debt is beyond its own balance sheet. It can be affected by the topology of the financial system, the debt and equity of the firm's counterparties, and government policies.

The proposition assigns a new meaning to the phrase "too connected to fail" in the sense that a bank does not only affect other connected banks through an ex-post loss contagion, as in Elliott et al. (2014) and Acemoglu et al. (2015). The interbank connections also create an ex-ante moral hazard problem due to the network risk-taking distortion. With the supermodular property for banks' choices of risk exposure at hand, we can now establish the existence of the risk-taking equilibrium.

PROPOSITION 2. *In any network structure $(\bar{d}, \boldsymbol{\Theta})$, the risk-taking equilibrium exists.*

Proof. In the Appendix

The proof is a simple application of the Tarski (1955) fixed point theorem to a supermodular game. In general, the equilibrium is not unique. Following the standard practice in this literature, let's focus on the Pareto-dominant equilibrium when $\boldsymbol{Z}$ is the smallest among the set of fixed points (Elliott et al., 2014; Capponi et al., 2020).[7] After establishing the existence of the risk-taking equilibrium, we can now compare connected banks' choices of risk exposure with that of a stand-alone bank. The following proposition shows that interconnectedness indeed encourages banks to expose themselves to greater risks.

COROLLARY 1. *A bank in any network structure $(\bar{d}, \boldsymbol{\Theta})$ will choose a greater exposure to risks than a stand-alone bank.*

Proof. In the Appendix.

In a financial network, banks with successful projects cross-subsidize failed banks to pay deposits. Those cross-subsidies are reflected in each bank's expected payoff as a network risk-taking

[7]Focusing on the least exposure equilibrium is to abstract away from a self-fulfilling failure. See Elliott et al. (2014) for a more detailed discussion. The authors consider the "best-case" equilibrium, in which as few organizations as possible fail. A financial network's multiple equilibria is also interesting but yet out of the scope of the present paper. For more details, see Jackson and Pernoud (2021a).

distortion $\mathcal{D}_i(\mathbf{Z}_{-i})$. As Proposition 1 shows, each bank in the financial network, anticipating the distortion, optimally increases its exposure to risks. In equilibrium, banks do not internalize the effect of their risk exposure on other banks' payoffs. In other words, there exists a risk-taking externality, and connected banks endogenously expose themselves to greater risks than stand-alone banks.

3 Network Structures

3.1 Size of Interbank Liabilities

So far, we have seen that connected banks endogenously expose themselves to greater risks due to cross-subsidies. We also know that this distortion determines the effective debt that affects banks' risk-taking incentives. Let's now examine the extent of this network risk-taking distortion for different parameters of the network structure.

LEMMA 3. *The network risk-taking distortion $\mathcal{D}_i(\mathbf{Z}_{-i})$ is increasing and concave in the size of interbank liabilities $\bar{d}$.*

Proof. In the Appendix.

To understand the intuition behind lemma 3, it is helpful first to notice that there are three types of bank outcomes at the final date. The first type contains banks with successful projects. Denote them by $\mathcal{S}_{\boldsymbol{\omega}} \equiv \{i : \omega_i = s\}$. The second type contains banks that failed their projects but are still "solvent" (can fully fulfill their deposits). Denote them by $\mathcal{F}^+_{\boldsymbol{\omega}} \equiv \{i : \omega_i = f, \sum_j \theta_{ij} d^*_j(\boldsymbol{\omega}) \geqslant v\}$. Since those banks can fulfill their deposits, they will contribute back to the interbank payment system. Finally, the third type contains banks that failed their projects and cannot fully fulfill their deposits. Denote them by $\mathcal{F}^-_{\boldsymbol{\omega}} \equiv \{i : \omega_i = f, \sum_j \theta_{ij} d^*_j(\boldsymbol{\omega}) < v\}$ and call them "insolvent" failed banks. The depositors of those banks will suffer losses.

In a network with greater interbank claims and liabilities, successful banks anticipate larger cross-subsidies to failed banks. This is because those cross-subsidies result from the difference between what successful banks pay, which is $\bar{d}$, and what they receive, which is $\sum_j \theta_{ij} d^*_j(\boldsymbol{\omega})$. Because this difference results from the interbank connection, banks' cross-subsidies (or their network risk-taking distortions) are increasing in the extent of the connection $\bar{d}$. Note that larger cross-subsidies can increase the likelihood of a failed bank being solvent ($\mathcal{F}^- \to \mathcal{F}^+$). Solvent banks contribute back to the payment system, lowering the marginal effect of $\bar{d}$ on successful banks' cross-subsidies. As a result, a bank's network risk-taking distortion is increasing (due to larger cross-subsidies) at a decreasing rate (due to more failed banks becoming solvent) in the size of interbank liabilities. We can then apply lemma 3 to obtain the following equilibrium result on banks' choices of risk exposure.

PROPOSITION 3. *Each bank's choice of risk exposure Z^*_i is increasing in the size of interbank liabilities $\bar{d}$.*

Proof. In the Appendix.

From Lemma 3, we know that each bank has a larger risk-taking distortion if $\bar{d}$ is larger. The increased distortion directly raises each bank's risk-taking incentive. Furthermore, due to strategic complementarity, the bank's counterparties will also be incentivized to take greater risks, feeding back to the risk-taking distortion. As a result, Proposition 3 shows that a larger size of interbank liabilities induces every bank, in equilibrium, to expose to greater risk.

3.2 Symmetric Networks

Let's now turn our attention to two particular symmetric network structures: a complete network and a ring network. The ex-post loss contagion in those two types of networks has been widely analyzed, for example, by Allen and Gale (2000) and Acemoglu et al. (2015). In this section, I will study banks' ex-ante risk-taking incentives. In a ring network, every bank is connected only to its direct neighbor. In a complete network, every bank is connected to every other bank.

DEFINITION 3. *In a financial network with N banks, a ring network and a complete network are defined as*

$$\mathbf{\Theta}^{\boldsymbol{R}} = \begin{bmatrix} \mathbf{0}'_{N-1} & 1 \\ \boldsymbol{I}_{N-1} & \mathbf{0}_{N-1} \end{bmatrix} \quad \textit{and} \quad \mathbf{\Theta}^{\boldsymbol{C}} = \frac{1}{N-1}(\mathbb{1}_{N,N} - \boldsymbol{I}_N)$$

where $\mathbf{0}_{N-1}$ is a vector of $N-1$ zeros, $\mathbb{1}_{N,N}$ is a matrix of ones with a dimension (N, N), and $\boldsymbol{I}$ is an identity matrix. Figure 1 illustrates a complete and a ring network with 5 banks.

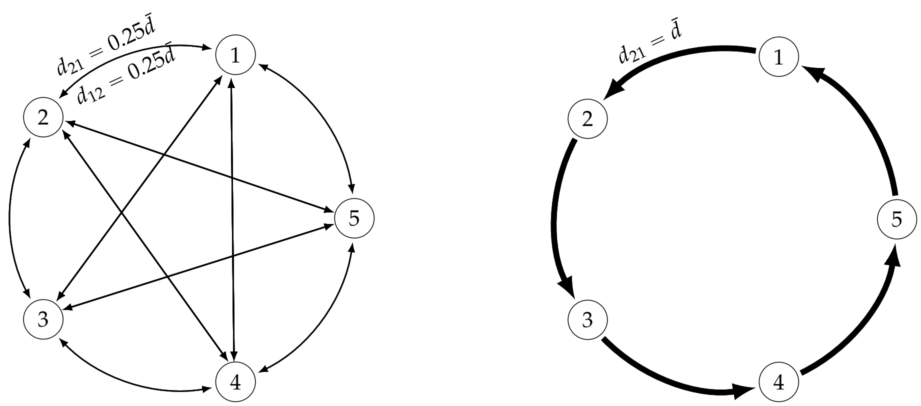


Figure 1 This figure displays a complete network (left) and a ring network (right). In both networks, each bank's total interbank liabilities equal $\bar{d}$.

We observe that each bank's total and net debt in those two networks are identical. This suggests that the canonical asset-substitution theory – which shows that leverage affects risk – is not well-positioned to study banks' risk-taking incentives in connected financial systems. With the help of my model, the following proposition compares banks' equilibrium risk exposure in a complete network and a ring network.

PROPOSITION 4. *Each bank's choice of risk exposure Z_i^* is larger in a complete network than in a ring network.*

Proof. In the Appendix.

The above proposition states that banks in a complete network choose greater risk exposure than banks in a ring network. The result stands in sharp contrast to the view of Allen and Gale (2000). They argue that a complete network is better at the co-insurance of ex-post losses and hence is more resilient. On the contrary, Proposition 4 shows that such co-insurance actually increases banks' ex-ante risk-taking incentives. This is because the co-insurance is a result of successful banks' cross-subsidies to failed banks. As argued earlier, those cross-subsidies distort banks' upside payoffs, incentivizing them to take more risks. As a result, in equilibrium, every bank in a complete network chooses to expose itself to greater risk.

The same intuition can be applied to networks with greater numbers of banks. Because the dimension of $\mathbf{\Theta}_N$ varies with N, the matrix $\mathbf{\Theta}_{N+1}$ will not be well-defined from an arbitrary network structure. Let's hence analyze the upper bound of a connected bank's risk exposure, which, according to part (b) of the following proposition, is independent of the network topology.

PROPOSITION 5.

(a) The network distortion $\mathcal{D}_i(\mathbf{Z}_{-i})$ is bounded from above for all $\bar{d}$.

(b) In any symmetric financial network, the upper bound of each bank's risk exposure is increasing in the number of banks in the network and is independent of the network topology.

Proof. In the Appendix.

Part (a) of the proposition indicates that there exists an upper bound for the network risk-taking distortion regardless of how large the size of the interbank connection is. Intuitively, the distortion results from a successful bank's cross-subsidies to compensate failed banks' depositors. This implies that a bank's network risk-taking distortion cannot exceed the amount when its cross-subsidies encompass every failed bank's deposits in every state of nature. Part (b) shows that this upper bound, which is not affected by the network topology, is increasing in the number of banks in the financial network. Moreover, the upper bound is tight only if the financial network is path-connected; otherwise, bank i does not need to cross-subsidize every other bank's depositors.

Figure 2 exhibits the results of numeral analyses that summarize our findings thus far for the relationship between the network topology and banks' risk-taking distortion. The first figure plots the network risk-taking distortion as a function of the size of interbank liabilities for a complete and a ring network. It shows that the distortion is increasing and concave in the size of interbank liabilities, confirming proposition 3. It also shows that the distortion is larger in a complete network than a ring network, confirming proposition 4. The numerical analysis in the

(a) Complete v.s. Ring **(b)** Fewer Banks **(c)** Central Clearing

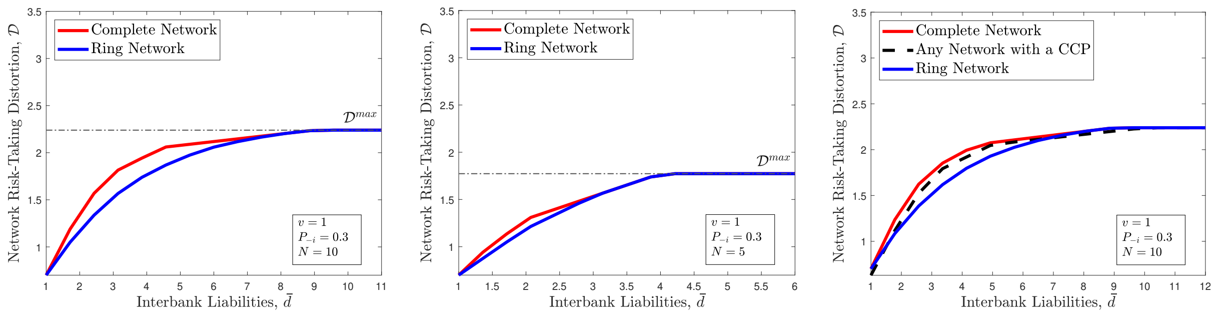
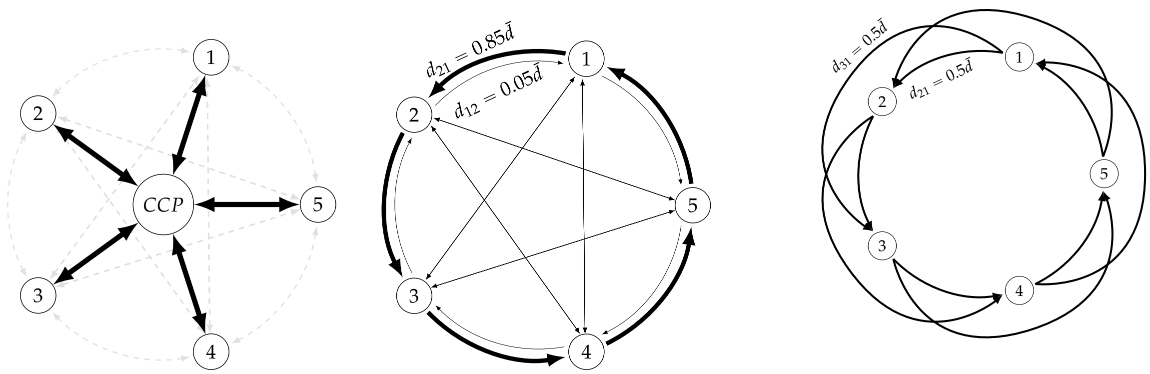


Figure 2 This figure plots $\mathcal{D}_i(\mathbf{Z}_{-i})$ as a function of $\bar{d}$ for a complete network and a ring network. Figure (a) establishes the benchmark: $N = 10$, $v = 1$, and $P(Z_{-i}) = 0.3$. Figure (b) plots the result when N is reduced to 5. Figure (3) plots the result when there is a central clearing counterparty.

second figure reduces the number of banks in the network. We see that the distortion is reduced for both the ring and the complete network.

3.3 Other Regular Networks

The machinery of networks allows us to analyze banks' risk-taking incentives in the current complex financial systems, for example, the ones with central clearing counterparties (CCP). A CCP acts as the buyer of interbank debts from all sellers and the seller to all buyers.[8] In other words, any financial structure with a CCP is effectively a core-periphery network where the core – the clearing party – has no assets and no deposits. For example, Figure 3.(a) displays the effective structure of a complete network when payments are cleared through a CCP.



(a) Central Clearing **(b)** $\lambda = 0.2$ network **(c)** generalized ring network

Figure 3 Figure (a) displays the effective network structure of a complete network when there is a central clearing counterparty. Dashed lines represent the original network, and solid lines represent the effective network. Figure (b) displays a network structure with $\mathbf{\Theta} = 0.8 \cdot \mathbf{\Theta}^{\boldsymbol{R}} + 0.2 \cdot \mathbf{\Theta}^{\boldsymbol{C}}$. Figure (c) displays a generalized ring network where each bank is connected to its neighbor and its neighbor's neighbor.

[8] According to LCH (originally London Clearing House), the world's leading multinational clearinghouse, a CCP nets down payment obligations across all the cleared contracts to one payment obligation to the CCP per member. For more details, see https://www.newyorkfed.org/medialibrary/media/banking/international/11-LCH-Credit-Risk-2015-Lee.pdf

PROPOSITION 6. *In any network structure ($\bar{d}, \mathbf{\Theta}$) with a central clearing counterparty, the risk-taking equilibrium is equivalent to that of a complete network with ($\frac{N-1}{N}\bar{d}, \mathbf{\Theta^C}$).*

Proof. In the Appendix.

Proposition 6 applies to any regular financial network regardless of whether $\mathbf{\Theta}$ is symmetric or path-connected. The proposition indicates that central clearing has two opposite effects on its member banks' ex-ante risk-taking incentives. On the one hand, a CCP increases banks' risk-taking incentives by raising the network's completeness. Intuitively, central clearing collects all interbank payments into a pool and redistributes them back to member banks. Because failed banks can only contribute to the pool after their depositors are paid in full, central clearing effectively makes a successful bank cross-subsidize *every* failed bank. As a result, each bank in any network structure with a CCP becomes *equally* exposed to the risk-taking externalities of every other bank in the network, which is identical to what happens in a complete network.

On the other hand, central clearing can reduce banks' risk-taking incentives because it nets out some ex-post interbank payments – the netting effectively reduces the size of the connection from $\bar{d}$ to $\frac{N-1}{N}\bar{d}$. This result is the same as the netting benefit studied by Duffie and Zhu (2011).[9] Because ex-post payments are reduced from central clearing, a CCP can mitigate banks' risk-taking incentives by decreasing the cross-subsidies that a successful bank needs to pay.

As a result of the two forces, whether central clearing reduces banks' risk-taking incentives depends on the banking system's original network structure. For numerical analysis, Figure 2.(c) displays the effects of central clearing on the network risk-taking distortion in a complete network and a ring network with different sizes of interbank liabilities. In a complete network, a CCP decreases the network risk-taking distortion for all $\bar{d}$. This is because banks in a complete network benefit from the CCP's netting efficiency. In a ring network, on the other hand, a CCP can increase the network risk-taking distortion. This is because the CCP "forces" each member bank to be exposed to every other bank's risk-taking externalities while the netting benefit is small. As a result, in a loosely connected financial system such as a ring network, the "CCP-riskier" effect dominates and central clearing creates systemic instability, in contrast to conventional wisdom.

Notwithstanding greater risk exposure, as Section 4.4 will show, banks still have incentives to join a CCP if they are sufficiently concerned about their charter values. Moreover, it is worth mentioning that, in reality, a CCP often requires collateral accounts that limit banks' risk-taking. As we will see in Section 5.1, a margin requirement, which is similar to a capital requirement, indeed has a network effect in reducing banks' risk-taking incentives.

We have studied the risk-taking externalities in two extreme networks: a fully connected complete network and a loosely connected ring network. Proposition 4 shows that banks in a complete network choose greater risks than banks in a ring network. One may think that banks

[9] Duffie and Zhu (2011) study a CCP's netting efficiency when assuming bank defaults are uncorrelated events. The netting efficiency in my model is identical to theirs except that I also consider the joint determination of defaults using the technique of Eisenberg and Noe (2001).

in an intermediately connected network will choose a risk exposure somewhere between that of a complete and a ring network. The basis for this conjecture relies on the fact that the payment equilibrium of an intermediately connected network is between that of a complete and a ring network, as shown by Eisenberg and Noe (2001).[10] However, as we will see, this is not true for the network risk-taking externalities.

There are two ways to define an intermediately connected network. A λ network is the convex combination of a ring and a complete network: $\boldsymbol{\Theta}^{\lambda} = (1-\lambda)\boldsymbol{\Theta}^{\boldsymbol{R}} + \lambda\boldsymbol{\Theta}^{\boldsymbol{C}}$. According to Elliott et al. (2014), λ can be interpreted as a financial network's degree of diversification. From this definition, $\lambda = 0$ is a ring network and $\lambda = 1$ is a complete network. Another way to define an intermediately connected structure is by a generalized ring network: each bank connects to r adjacent neighbors. From this definition, $r = 1$ is a ring network and $r = N-1$ is a complete network. Figure 3.(b) and (c) display a $\lambda = 0.2$ network and a $r = 2$ generalized ring network. To illustrate the relationship between a network's connectedness and the risk-taking distortion it induces, Figure 4 displays the size of the distortion as a function of different parameter values of λ, r, and $\bar{d}$ for an 8-bank network. The Online Appendix provides a numerical example.

(a) λ networks **(b)** generalized ring networks

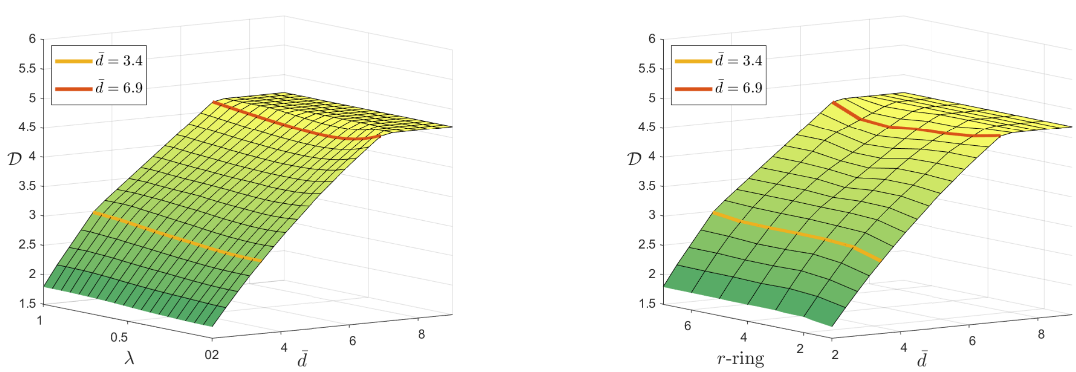


Figure 4 The figure on the left plots each bank's network risk-taking distortion $\mathcal{D}_i(\boldsymbol{Z}_{-\boldsymbol{i}})$ as a function of $\bar{d}$ and λ where $\boldsymbol{\Theta} = (1-\lambda)\boldsymbol{\Theta}^{\boldsymbol{R}} + \lambda\boldsymbol{\Theta}^{\boldsymbol{C}}$. The figure on the right plots $\mathcal{D}_i(\boldsymbol{Z}_{-\boldsymbol{i}})$ as a function of $\bar{d}$ and r for generalized ring networks. An r-generalized ring network is a network where each bank connects to r adjacent neighbors. In both figures, the network has eight banks and each bank's deposit equals 1.

We notice that when $\bar{d}$ is small (e.g., $\bar{d} = 3.4$), the network distortion is increasing in both λ and r. This is because greater connectedness exposes banks to their counterparties' risk-taking externalities to a greater extent. This effect is consistent with the findings of Eisenberg and Noe (2001), who show that the payment equilibrium is increasing in λ. On the other hand, when $\bar{d}$ is large (e.g., $\bar{d} = 6.9$), the network distortion becomes not monotone in either λ or r. In this case, banks in a λ-network or a generalized ring network may have a lower network distortion than banks in a ring network. This is because as λ or r increases, banks become less

[10] Suppose an intermediately connected network has a $\boldsymbol{\Theta}$ that is the λ-convex combination of a ring and complete network. Then the RHS of equation 2 is monotone in λ. As a result, the fixed point solution is monotone in λ. For more details, see Eisenberg and Noe (2001) Lemma 6.

sensitive to particular other banks' externalities. The non-monotonicity result is consistent with the observation of Elliott et al. (2014), who show that the ex-post contagion is most likely to occur when the integration (similar to $\bar{d}$) and the diversification (similar to λ) are in the middle range. Finally, when $\bar{d}$ is large enough such that all failed banks' depositors can be repaid (e.g. $\bar{d} \geqslant 7$), the degree of connectedness λ or r is irrelevant (Proposition 5).

4 Model Robustness and Extension

4.1 Distribution of the Primitive Asset and Systemic Defaults

In the benchmark model, there are only two states of nature for a bank's primitive asset with the upside payoff above each bank's total liabilities. This setup disentangles the Nash equilibrium and the payment equilibrium. It provides us with crucial tractability to derive intuitions about banks' risk-taking externalities in financial networks. However, it also implies that whether a bank defaults depends entirely on whether its primitive asset succeeds and that there is no cascading default. In the section, I will extend the cash flow distribution to allow all of the following three possibilities: (i) a bank has a successful project and never defaults; (ii) a bank has a successful project, but still, a cascading default is possible; (iii) a bank has a failed project and defaults.[11] Specifically, bank i's cash flow distribution is

$$\tilde{e}_i = \begin{cases} \alpha & \text{where } \alpha \geqslant v + \bar{d} & \text{Successful project, no contagion} \\ \beta & \text{where } v \leqslant \beta < v + \bar{d} & \text{Successful project, contagion possible} \\ \gamma & \text{where } \gamma < v & \text{Failed project, default} \end{cases} \tag{7}$$

Importantly, the new cash flow distribution introduces the possibility of $\tilde{e}_i \in (v, v + \bar{d})$. This implies that whether a bank defaults now depends on the project outcomes of other banks. A "systemic default" is the event when a bank survives if it is stand-alone but defaults if it is connected. A project return $\tilde{e} \in (v, v + \bar{d})$ allows for such possibility: $\tilde{e} > v$ indicates that the bank can pay off its debt if it is stand-alone, and $\tilde{e} < v + \bar{d}$ means that the bank may default if it is connected.

To model banks' risk-taking incentives, let them choose their projects' risk and return – each bank chooses the project return in its good state: $\alpha = Z_i$. Because a high-return project comes with high risk, the probabilities that the bank has a successful project, $P_\alpha(Z_i)$ and $P_\beta(Z_i)$, are decreasing in Z_i. As a result, the probability that the bank has a failed project, $P_\gamma(Z_i) \equiv 1 - P_\alpha(Z_i) - P_\beta(Z_i)$, is increasing in Z_i. This implies that a bank is more likely to fail its project if it chooses a greater risk. With the new cash flow distribution, each bank i's expected profit

[11] In nonregular networks, there is a fourth possibility: a bank has a failed project but earns a positive profit. Section 4.2 will discuss it in greater details. The term "successful" indicates that the bank does not default in the absence of contagion (i.e., if it is stand-alone).

becomes:

$$\mathbb{E}\Big[\Pi_i(\boldsymbol{\omega};\mathbf{Z})\Big] = P_\alpha(Z_i)\sum_{\boldsymbol{\omega}_{-i}}\Big[\underbrace{Z_i - v - \Big(\bar{d} - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}^{i=\alpha})\Big)}_{>0}\Big]\cdot \Pr(\boldsymbol{\omega}_{-i})$$

$$+ P_\beta(Z_i)\sum_{\boldsymbol{\omega}_{-i}}\Big[\underbrace{\beta - v - \Big(d_i^*(\boldsymbol{\omega}^{i=\beta}) - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}^{i=\beta})\Big)}_{>0 \text{ or } =0}\Big]^+\cdot \Pr(\boldsymbol{\omega}_{-i})$$

$$+ P_\gamma(Z_i)\sum_{\boldsymbol{\omega}_{-i}}\Big[\underbrace{\gamma - v - \Big(d_i^*(\boldsymbol{\omega}^{i=\gamma}) - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}^{i=\gamma})\Big)}_{=0}\Big]^+\cdot \Pr(\boldsymbol{\omega}_{-i})$$

The payment equilibrium, $d_i^*(\boldsymbol{\omega}) = \big\{\min\big[\sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}) + e_i(\omega_i, Z_i) - v, \bar{d}\big]\big\}^+$, is identical to the one in the benchmark model. Bank i's expected profit when it has a failed project, $\big[\gamma - v - \big(d_i^*(\boldsymbol{\omega}^{i=\gamma}) - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}^{i=\gamma})\big)\big]^+$, is zero for all $\boldsymbol{\omega}_{-i}$. Its expected profit in the contagion state, $\big[\beta - v - \big(d_i^*(\boldsymbol{\omega}^{i=\beta}) - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}^{i=\beta})\big)\big]^+$, can be either positive or zero depending on $\boldsymbol{\omega}_{-i}$. Its expected profit in the non-contagion state, $\big[\alpha - v - \big(\bar{d} - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}^{i=\alpha})\big)\big]$, is always positive.

We observe that counterparty risk Z_m affects bank i's expected payoff in two ways. In the non-contagion state (i.e., when $\tilde{e}_i = \alpha$), an increase in counterparty risk decreases bank i's expected profit, because bank i needs to pay more cross-subsidies to other banks. Moreover, in the state where bank i has a successful project but a contagion is possible (i.e., when $\tilde{e}_i = \beta$), higher counterparty risk increases the chance that bank i defaults due to contagion. This effect does not exist in the benchmark model in which there are no cascading defaults. The next proposition shows that allowing for the possibilities of cascading defaults does not alter the strategic complementarity result.

PROPOSITION 7. *With the cash flow distribution in equation 7, banks' choices of risk exposure* $\mathbf{Z}$ *are strategically complementary.*

Proof. In the Appendix.

In the non-contagion state, higher counterparty risk means that each bank becomes more likely to cross-subsidize other banks. Proposition 1 shows that those cross-subsidies distort a bank's upside payoff, incentivizing it to take greater risks. In the contagion state, higher counterparty risk means that the bank is more likely to default itself due to contagion, further reducing its upside payoff. The two distortions both induce the bank to be less interested in the probability of success when trading off risk and return. As a result, a bank will choose to be exposed to greater risks in response to higher counterparty risk.

4.2 Nonregular Networks

So far, the analysis has been focused on regular networks in which banks' interbank liabilities and interbank claims are equal. The regularity assumption implies that a bank with a failed project will always default – it will not be able to fulfill its interbank liabilities even if its interbank claims have been paid in full. In this section, I will show that the model can be extended to nonregular networks as well.

Suppose banks are connected in a nonregular network in which $\sum_j \theta_{ij}\bar{d}_j \neq \bar{d}_i$. The nonregularity condition raises the possibility of $\sum_j \theta_{ij} d_j^*$ being greater than $\bar{d}_i + v$. This implies that a bank with a failed project will not default if a sufficient number of its counterparties succeed. In other words, a failed bank can be "bailed out" by other successful banks in the financial network and earns a positive profit. With this extension, each bank's expected profit becomes

$$\begin{aligned}\mathbb{E}\Big[\Pi_i(\boldsymbol{\omega};\boldsymbol{Z})\Big] &= P(Z_i)\sum_{\boldsymbol{\omega}_{-i}}\Big[Z_i - v - \Big(\bar{d}_i - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}^{\boldsymbol{i}=\boldsymbol{s}})\Big)\Big]\cdot \Pr(\boldsymbol{\omega}_{-\boldsymbol{i}}) \\ &\quad + \Big(1-P(Z_i)\Big)\sum_{\boldsymbol{\omega}_{-i}}\Big[0 - v - \Big(d_i^*(\boldsymbol{\omega}^{\boldsymbol{i}=\boldsymbol{f}}) - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}^{\boldsymbol{i}=\boldsymbol{f}})\Big)\Big]^+\cdot \Pr(\boldsymbol{\omega}_{-\boldsymbol{i}})\end{aligned} \tag{8}$$

The payment equilibrium, $d_i^*(\boldsymbol{\omega}) = \Big\{\min\Big[\sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}) + e_i(\omega_i, Z_i) - v, \bar{d}\Big]\Big\}^+$, is identical to the one in the benchmark model. In contrast to the benchmark model, counterparty risk Z_m now enters bank i's downside payoff – the payoff when its project fails. Interestingly, the effect of counterparty risk on a bank's downside payoff can reduce its risk-taking incentive. Intuitively, an increase in counterparty risk increases a bank's cross-subsidies to other banks when it fails but does not default, just as when the bank succeeds. Furthermore, an increase in counterparty risk also reduces the possibility of the bank being "bailed out". Both effects imply that the bank will have a smaller downside return if there is larger counterparty risk. A smaller downside return discourages the bank from taking risks.

On the other hand, Proposition 1 has shown that an increase in counterparty risk also reduces a bank's upside return by enlarging its cross-subsides. Summing up, the effects of a bank's counterparty risk on its risk-taking incentive are opposite in the lens of its upside payoff (a larger cross-subsidy encourages risk-taking) and downside payoff (a smaller downside return discourages risk-taking). The next proposition shows that, even with the possibility of positive downside payoff, each bank's optimal choice of risk exposure is still an increasing function of its counterparty risk.

PROPOSITION 8. *In any financial network, regular or nonregular, banks' choices of risk exposure* $\boldsymbol{Z}$ *are strategically complementary.*

Proof. In the Appendix.

If a bank does not default when it fails, the payment equilibrium will be identical regardless of whether the bank's project succeeds or not. This is because the bank will pay its interbank debt in full if it does not default, just as when its project succeeds. Because the payment equilibrium is the same, the effects of counterparty risk on the bank's upside and downside payoffs will be the same. Intuitively, in a world where a bank earns a positive profit regardless of whether its project succeeds or not, its interbank claims become an independent asset that does not affect its risk-taking incentive. In other words, the effects of counterparty risk on a bank's upside and downside payoff exactly cancel out if the bank does not default when it fails. In those cases, counterparty risk becomes what Galeotti, Golub, and Goyal (2020) called pure externalities – that is, spillovers that do not affect best responses.

On the contrary, if a bank does default when it fails, its counterparty risk will not affect its downside payoff, which is zero. The counterparty risk, however, reduces its upside payoff as shown in Proposition 1. Of course, there are possibilities that a bank defaults when it fails – for example, when all of its counterparties fail. As a result, an increase in counterparty risk increases the bank's optimal choice of risk exposure.

4.3 Correlated Risk Exposure

In previous sections, we assumed that banks' project outcomes are independent. In this section, I model each bank's decision whether to be exposed to correlated risks and explain why a systemic crisis can endogenously evolve due to interbank connectedness.

Suppose each bank, besides choosing the marginal distribution $P(Z_i)$ of its project, also chooses the conditional distribution $\boldsymbol{\phi}_i = [\phi_{i1}, ..., \phi_{iN}]$ of its project on the project outcomes of other banks in the network. Define each element of the matrix $\mathbf{\Phi} \equiv [\phi_{ij}]$ as

$$\phi_{ij} \equiv Pr(\omega_i = s | \omega_j = s)$$

where $0 \leqslant \phi_{ij} \leqslant 1$. From this definition, the pairwise correlation between bank i and j's projects is

$$\rho_{ij} \equiv \frac{\phi_{ij} P(Z_j) - P(Z_i) P(Z_j)}{P(Z_i)^{1/2} P(Z_j)^{1/2} [1 - P(Z_i)]^{1/2} [1 - P(Z_j)]^{1/2}}$$

In contrast to the benchmark model, each banks' project outcomes are no longer independent. Bank i's expected profit becomes

$$\mathbb{E}\Big[\Pi_i(\boldsymbol{\omega}; \mathbf{Z}, \mathbf{\Phi})\Big] = P(Z_i)(Z_i - v) - \sum_{\boldsymbol{\omega}_{-i}} \Big(\bar{d} - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}^{\boldsymbol{i}=\boldsymbol{s}})\Big) \cdot \Pr(\boldsymbol{\omega}_{-\boldsymbol{i}}) \cdot \Pr(\omega_i = s | \boldsymbol{\omega}_{-\boldsymbol{i}})$$

The above equation uses the property $\Pr(\boldsymbol{\omega}^{\boldsymbol{i}=\boldsymbol{s}}) = \Pr(\boldsymbol{\omega}_{-\boldsymbol{i}}) \cdot \Pr(\omega_i = s | \boldsymbol{\omega}_{-\boldsymbol{i}})$. The dependence vector $\boldsymbol{\phi}_i$ enters the last term $\Pr(\omega_i = s | \boldsymbol{\omega}_{-\boldsymbol{i}})$.

DEFINITION 4. *The correlated risk-taking equilibrium in a financial network is a triplet* $\big(\boldsymbol{d}^*(\boldsymbol{\omega}; \mathbf{Z}), \mathbf{Z}^*, \mathbf{\Phi}^*\big)$ *consisting of a vector of payment functions* $\boldsymbol{d}^*(\boldsymbol{\omega}; \mathbf{Z})$*, a vector of risk exposure* $\mathbf{Z}^*$*, and a matrix of conditional distribution* $\mathbf{\Phi}^* \equiv [\phi_{ij}^*]$ *such that:*

1. *The vector of functions $\boldsymbol{d}^*(\boldsymbol{\omega};\boldsymbol{Z})$ is a payment equilibrium for any $\boldsymbol{Z}$.*

$$d_i^*(\boldsymbol{\omega};\boldsymbol{Z}) = \left\{ \min\Big[\sum_j \theta_{ij} d_j^*(\boldsymbol{\omega};\boldsymbol{Z}) + e_i(\omega_i, Z_i) - v, \bar{d},\Big] \right\}^+ \quad \forall i \in \mathcal{N} \quad \forall \boldsymbol{\omega} \in \boldsymbol{\Omega}$$

2. *For each bank $i \in \mathcal{N}$, $(Z_i^*, \boldsymbol{\phi_i^*})$ is optimal and solves the following equation, given $\boldsymbol{d}^*(\boldsymbol{\omega};\boldsymbol{Z})$, $\boldsymbol{Z}_{-i}^*$ and $\boldsymbol{\Phi}_{-i}^*$*

$$(Z_i^*, \boldsymbol{\phi_i^*}) = \underset{\substack{\underline{Z} \leqslant Z_i \leqslant \overline{Z} \\ \boldsymbol{0} \leqslant \boldsymbol{\phi}_i \leqslant \mathbb{1}}}{\operatorname{argmax}} \mathbb{E}\Big[\Pi_i(\boldsymbol{\omega}; \boldsymbol{Z_{-i}^*}, \boldsymbol{\phi_{-i}^*})\Big] \quad \forall i \in \mathcal{N}$$

3. *The pairwise correlations are compatible among all banks. i.e. $\boldsymbol{P} = [\rho_{ij}]$ is symmetric and positive semi-definite.*

Part 2 of the above definition implies that banks are unrestricted in choosing their conditional dependence with their counterparties. Any bank can choose an arbitrarily correlated project with any other bank: a notion similar to Denti (2018). However, part 3 of the above definition states that the conditional dependence has to be mutually and jointly compatible in equilibrium.[12] Part 3 also implies $\phi_{ij}^*/\phi_{ji}^* = P(Z_i^*)/P(Z_j^*)$ for all i, j. It means that there is a dependence between $\boldsymbol{\Phi}^*$ and $\boldsymbol{Z}^*$: in equilibrium, the marginal and conditional distribution should also be compatible.

PROPOSITION 9. *In any network structure $(\bar{d}, \boldsymbol{\Theta})$, the correlated risk-taking equilibrium exists and every bank's risk exposure is perfectly correlated: $\phi_{ij}^* = 1$ for all $i, j \in \mathcal{N}$.*

Proof. In the Appendix.

The proposition shows that connected banks choose to expose themselves to one single systemic risk. In anticipation of cross-subsidies to failed banks, each bank optimally correlates its project outcomes with other connected banks. By doing so, it will face no distortion in its upside payoff and hence will enjoy a higher expected profit. The correlation, however, is harmful to the economy as a whole. Since every bank chooses to expose itself to one single systemic risk, there is no co-insurance within the economy.

Proposition 9 predicts that a systemic crisis can *endogenously* evolve in a connected banking system. It comports with the empirical findings of International Monetary Fund (2009) and Bisias et al. (2012) that there existed a large distress dependence among major banks before the 2008 financial crisis when the banking system became unprecedentedly connected. The result is related to Acharya (2009), who argues that banks choose correlated investments due to a pecuniary externality, and recent papers by Elliott et al. (2021) and Jackson and Pernoud (2021b). Using data on German banks, Elliott et al. (2021) illustrate banks' incentive to form partners with

[12] For example, $\rho_{ij} = 1$ and $\rho_{ji} = 0$ is not compatible because $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ is not symmetric. For another example, $\rho_{ij} = 1$, $\rho_{jk} = 1$, $\rho_{ik} = 0$ is not compatible because $\left(\begin{smallmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{smallmatrix}\right)$ is not positive semi-definite.

similar portfolios. Jackson and Pernoud (2021b) show that banks have incentives to minimize the set of states where they pay debts and have have their values diluted.

4.4 Banks' Incentives to Form Links

A natural question to ask is why banks have incentives to join a network in the first place. The literature has proposed several reasons. Acemoglu et al. (2014) show that banks form interbank contracts because they have heterogeneous investment opportunities. Donaldson and Piacentino (2018) show that interbank debts embed the option to dilute with new debt to a third party. In this section, I will show that banks want to maintain interbank claims and liabilities without netting them out for co-insurance purposes.

A successful bank pays cross-subsidies to its counterparties while it does not benefit from those cross-subsidies when it fails. In this section, I will illustrate that banks, if possessing valuable expected present value of their future profits (charter values), have incentives to form interbank connections, notwithstanding the risk-taking distortion. The introduction of banks' charter values is relevant to their risk-taking incentives. For example, Hellmann, Murdock, and Stiglitz (2000) show that reducing banks' charter values can create instability. It also describes reality well: in financial systems with deposit insurance, regulators usually seize insolvent banks and put them into receivership.[13] As a result, banks do not want to risk defaulting on their deposits to protect their continuation values.

To model this, let $c_i \in \mathbb{R}^+$ denote bank i's charter value. The bank can preserve this charter value if and only if its depositors are paid in full, either through its own project or other banks' cross-subsidies. From here, we can rewrite bank i's expected payoff as

$$\begin{aligned}\mathbb{E}\Big[\Pi_i(\boldsymbol{\omega};\mathbf{Z})\Big] &= P(Z_i)\Big[Z_i - v - \mathcal{D}_i(\mathbf{Z}_{-i})\Big] + \underbrace{c_i - \Big[1-P(Z_i)\Big]\Pr\Big(i \in \mathcal{F}_{\boldsymbol{\omega}}^- | \omega_i = f\Big)\cdot c_i}_{\text{expected charter value}} \qquad (9)\\ &= P(Z_i)\Big[Z_i - v - \mathcal{D}_i(\mathbf{Z}_{-i}) + \Pr\Big(i \in \mathcal{F}_{\boldsymbol{\omega}}^- | \omega_i = f\Big)\cdot c_i\Big] + c_i - \Pr\Big(i \in \mathcal{F}_{\boldsymbol{\omega}}^- | \omega_i = f\Big)\cdot c_i\end{aligned}$$

where $\mathcal{F}_{\boldsymbol{\omega}}^- \equiv \{i : \omega_i = f, \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}) < v\}$ denotes the set of insolvent banks – the ones that cannot adequately reimburse their depositors. Thus $\Pr(i \in \mathcal{F}_{\boldsymbol{\omega}}^- | \omega_i = f)$ can be interpreted as the probability that bank i is insolvent given that its project fails. For example, if the financial network is "maximally" connected as in Proposition 5, $\Pr(i \in \mathcal{F}_{\boldsymbol{\omega}}^- | \omega_i = f) = \prod_{j\neq i} 1 - P(Z_j)$. It means that bank i will become insolvent and lose its charter value only when all of its counterparties fail in addition to its own failure. In contrast, if bank i is stand-alone, it will lose its charter value simply when its own project fails. This implies that a stand-alone bank has an expected payoff of $\mathbb{E}\Big[\Pi_i^{SL}(\boldsymbol{\omega};\mathbf{Z})\Big] = P(Z_i)(Z_i - v) + P(Z_i)\cdot c_i$. Comparing this with equation 9, we observe that being in a financial network can increase the probability of a bank being solvent, hence protecting its charter value.

[13] During the global financial crisis of 2008, FDIC seized over 500 banks. For example, Washington Mutual, the sixth-largest bank in the United States at the time, ceased to exist after FDIC placed it into receivership.

Because $-\mathcal{D}_i(\boldsymbol{Z}_{-\boldsymbol{i}}) + \Pr(i \in \mathcal{F}_{\omega}^{-}|\omega_i = f) \cdot c_i < c_i$, we can verify that Corollary 1 still holds: connected banks choose greater risk exposure than stand-alone banks. Intuitively, there are two forces that make a connected bank choose greater risks: (i) a network risk-taking distortion as in the benchmark model, and (ii) a downside protection from the financial network's co-insurance. The second force is new here due to the introduction of banks' charter values. Both forces induce banks to become less interested in increasing the probability of success, hence creating systemic instability.

Let's now examine whether banks have incentives to form interbank links in the face of the network risk-taking distortion.

PROPOSITION 10. *There exists $\bar{c} \in \mathbb{R}^+$, such that if $\min\{c_i\} > \bar{c}$, banks have incentives to form links.*

Proof. In the Appendix.

To decide whether to form links, banks face three considerations : (i) protection of their charter values, (ii) cross-subsidy $\mathcal{D}_i(\boldsymbol{Z}_{-\boldsymbol{i}})$, and (iii) distorted investment Z^*. On the one hand, being in a network can protect banks from losing their valuable charter values, as it provides co-insurance to their depositors. On the other hand, due to this co-insurance, banks expect to cross-subsidize other banks, decreasing their upside payoffs. This also distorts investment and results in systemic instability. Proposition 10 states that banks will join an interbank network if they care sufficiently about their charter values.

The proposition is silent on the optimal topology that banks want to connect. A natural direction for further research is to fully endogenize the network formation while taking into account the risk-taking externalities.

5 Policy Implications

5.1 Capital Requirement

So far, we have been studying banks' risk-taking equilibrium in financial networks where banks do not hold any equity. Since the 1980s, regulators began using capital adequacy requirements to ensure that banks do not take excessive risks (Gorton, 2012). With more "skin in the game", banks are less willing to gamble with their equity (Jensen and Meckling, 1976). In this section, I will study the network effect of each individual bank's equity on the financial system's risk exposure.

Suppose that each bank i is required to hold equity of size $\epsilon_i < v$ in substitute of deposits.[14] Because equity is junior to both the deposits and the interbank liabilities, a bank's equity holders will be the first ones to incur losses when its total cash flow is smaller than its total liabilities. The payment equilibrium becomes

[14]Alternatively, we can think that the government imposes a reserve requirement or a CCP imposes a margin requirement.

$$d_i^*(\boldsymbol{\omega};\boldsymbol{\epsilon}) = \left\{ \min\Big[\sum_j \theta_{ij} d_j^*(\boldsymbol{\omega};\boldsymbol{\epsilon}) + e_i(\mathbf{Z},\boldsymbol{\omega}) - v + \epsilon_i, \bar{d} \Big] \right\}^+ \quad \forall i \tag{10}$$

The payment equilibrium is now a function of the vector of equity $\boldsymbol{\epsilon} = [\epsilon_1, \epsilon_2, ...\epsilon_N]$. The expected profit of bank i is

$$\mathbb{E}\Big[\Pi_i(\boldsymbol{\omega};\mathbf{Z},\boldsymbol{\epsilon})\Big] = P(Z_i)(Z_i - v + \epsilon_i) - P(Z_i)\mathcal{D}_i(\mathbf{Z}_{-i};\boldsymbol{\epsilon}) \tag{11}$$

where the network risk-taking distortion $\mathcal{D}_i(\mathbf{Z}_{-i};\boldsymbol{\epsilon})$ is also a function the vector of equity.

$$\mathcal{D}_i(\mathbf{Z}_{-i};\boldsymbol{\epsilon}) \equiv \sum_{\boldsymbol{\omega}_{-i}} \Big(\bar{d} - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}^{i=s};\boldsymbol{\epsilon}) \Big) \cdot \Pr(\boldsymbol{\omega}_{-i})$$

We notice that the vector of equity $[\epsilon_1, \epsilon_2, ...\epsilon_N]$ affects bank i's expected payoff in two ways. First, the bank's own equity ϵ_i affects its expected payoff. This is because the value of equity will be lost if the bank's project fails. This observation leads to the canonical argument that equity reduces a bank's risk-taking behavior. More interestingly, a counterparty bank's equity ϵ_j also affects bank i's expected payoff and hence its risk-taking choice. The reason, as the next proposition will show, is that a counterparty's equity can affect bank i's network risk-taking distortion.

PROPOSITION 11. *In a path-connected network structure ($\bar{d}, \boldsymbol{\Theta}$),*

(a) Bank i's network risk-taking distortion $\mathcal{D}_i(\mathbf{Z}_{-i};\boldsymbol{\epsilon})$ is a decreasing and concave function of the size of a counterparty bank's equity ϵ_j for all $j \in \mathcal{N}/\{i\}$.

(b) Bank i's equilibrium risk exposure Z_i^* is decreasing in the size of bank j's equity ϵ_j for all $j \in \mathcal{N}$.

Proof. In the Appendix.

Part (a) shows that any counterparty bank's equity ϵ_j can reduce bank i's network risk-taking distortion. Intuitively, when bank j fails, its equity holders will be the first ones to incur the loss before its depositors or interbank counterparties. The loss that may be otherwise propagated to other banks will now be absorbed by bank j's equity holders to the limit ϵ_j. In other words, ϵ_j decreases the cross-subsidies that each bank i needs to pay. As a result, bank i's network risk-taking distortion is reduced. Moreover, with greater ϵ_j, failed banks are more likely to become solvent, contributing back to the payment system. This further reduces a successful bank's cross-subsidies. As a result, bank i's network risk-taking distortion is decreasing at a growing rate in the size of bank j's equity, illustrated in the right figure below.

For banks' equilibrium choices of risk exposure, the financial system's overall equity $\boldsymbol{\epsilon}$ has two effects. First, each bank's own equity ϵ_i has a direct impact on the bank's risk-taking incentive: its effective debt is decreasing in its own equity, illustrated in the left figure above. More interestingly, part (a) of the proposition says that a bank's equity has a network effect in reducing

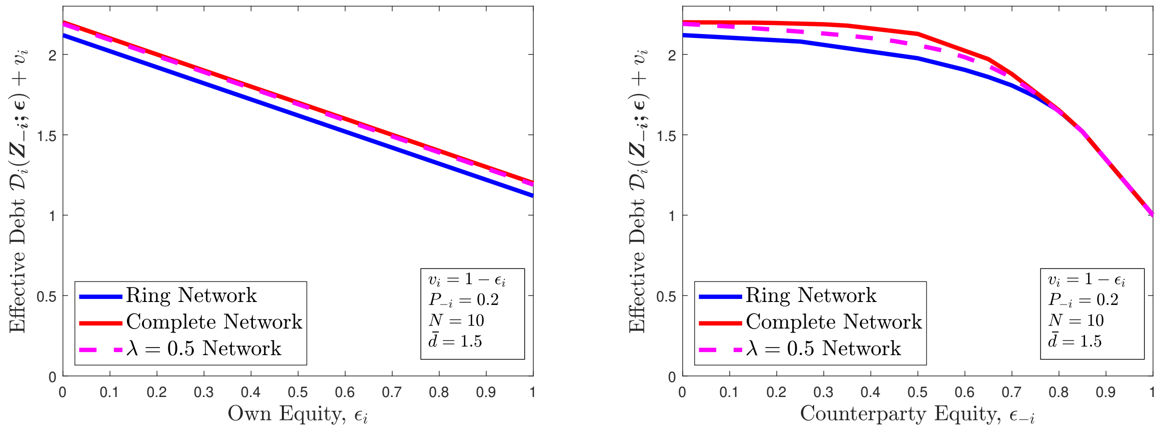


Figure 5 This figure plots a bank's effective debt as a function of its own equity ϵ_i (left figure) and its counterparties' equity ϵ_{-i} (right figure). Both figures include a complete network, a ring network, and a $\lambda = 0.5$ network.

the risk-taking incentive of every bank in the financial system. This is because each bank's effective debt level is decreasing in its counterparty's equity ϵ_j, even though the face value of its debt is not affected by ϵ_j. As a result, part (b) of the proposition shows that each bank i's equilibrium risk exposure is decreasing in the equity of any of its counterparty banks (in addition to its own equity).

The result is related to Erol and Ordoñez (2017), who also study the network response of a capital regulation. They show that a capital requirement can discontinuously discourage interbank connections, hence reducing the ex-post co-insurance benefits. Our results imply that a tighter regulation can, on the one hand, decreases the interbank network's risk-taking externalities; on the other hand, it can also break the interbank connections if beyond a tipping point.

5.2 Bailout

The 2008 bailout of Bear Stearns and the subsequent Troubled Asset Relief Program have sparked debates among policymakers and academics. Government bailouts have been widely criticized for incentivizing harmful ex-ante behaviors (Gale and Vives, 2002; Farhi and Tirole, 2012; Erol, 2019). In this section, I will study the effect of a government bailout when banks are in a connected financial system.

Following Erol (2019), I assume that government bailouts occur in crisis when a large number of banks fail. Formally, define a government bailout (τ, η) as a transfer τ from the government to each failed bank if the number of failed banks exceeds η. With the government bailout in place, the payment equilibrium becomes

$$d_i^*(\boldsymbol{\omega}; \mathbf{Z}, \tau) = \left\{ \min \left[\sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}; \mathbf{Z}) + e_i(\omega_i, Z_i) + \tau_i(\boldsymbol{\omega}) - v, \bar{d}, \right] \right\}^+ \quad \forall i \in \mathcal{N} \quad \forall \boldsymbol{\omega} \in \boldsymbol{\Omega} \qquad (12)$$

The transfer $\tau_i(\boldsymbol{\omega})$ is state-contingent and is defined as

$$\tau_i(\boldsymbol{\omega}) \equiv \tau \cdot \mathbb{1}(\omega_i = f) \cdot \mathbb{1}(\#\text{ failed banks} \geqslant \eta)$$

The bank's payoff in state $\boldsymbol{\omega}$ is $\left(\sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}; \mathbf{Z}) + e_i(\omega_i, Z_i) + \tau_i(\boldsymbol{\omega}) - v - d_i^*(\boldsymbol{\omega}; \mathbf{Z})\right)^+$. The rest of the definition for the network risk-taking equilibrium remains unchanged from Equation 4. The following proposition shows that a government bailout, when limited to banks' interbank exposure, can contribute to systemic stability by reducing the network risk-taking distortion.

PROPOSITION 12. *In any financial network $(\bar{d}, \boldsymbol{\Theta})$, each bank's equilibrium risk exposure is reduced if there exists a government bailout with $\tau \leqslant v$.*

Proof. In the Appendix.

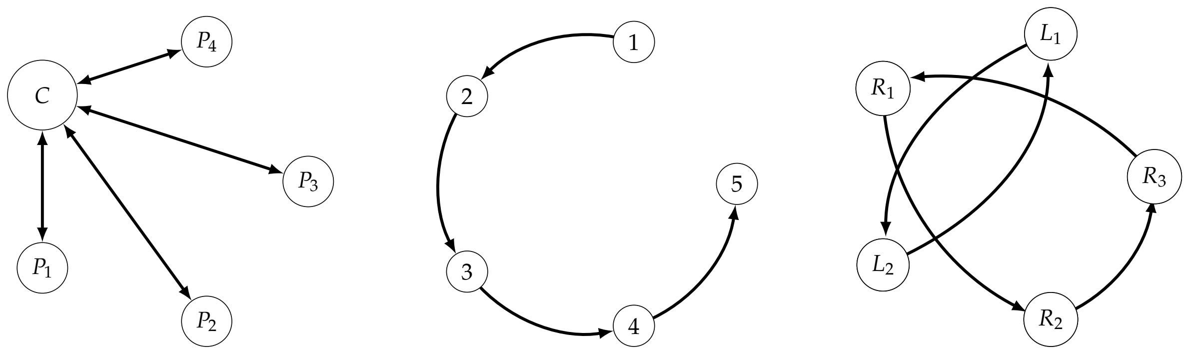
In contrast to the conventional wisdom, Proposition 12 argues that a government bailout, if limited to banks' interbank exposures, can decrease the financial system's risk-taking incentives. Intuitively, during a crisis, a government bailout absorbs each failed bank's loss before it is spread to successful banks. The network risk-taking distortion resulting from successful banks' cross-subsidies is hence reduced. This will decrease each bank's effective debt and encourage banks to reduce their choices of risk exposure.

The condition $\tau \leqslant v$ guarantees that the bailout is limited to each bank's interbank exposure. This is because, at most, a loss of v can be propagated to its counterparties. This condition ensures that a failed bank's shareholders will always earn a zero payoff in any state of nature $\boldsymbol{\omega}_{-i}$. On the contrary, if $\tau > v$, then the bailout will be extended to the shareholders of failed banks: they will earn positive profits in some state of nature. Such bailout reduces bank shareholders' "skin-in-the-game" and can create the conventional moral hazard problem, as shown by Gale and Vives (2002) and Farhi and Tirole (2012).

Proposition 12 stands in contrast to Erol (2019), who shows that a government bailout can create systemic instability by encouraging excessive interbank connections. In his model, if there exists a government bailout, banks will be less concerned with contagion during the network formation stage. As a result, they will form overly connected relationships. In contrast, my result shows that because banks are less concerned with contagion or cross-subsidies, they will be subject to smaller network risk-taking externalities and hence make safer investments.

5.3 Asymmetric Effect of Regulations

We have studied prudential policies on networks where $\boldsymbol{\Theta}$ is symmetric and $\bar{d}_i$ is identical across all banks. The basic model provides us with insights on the network effect of different policies. Nevertheless, the symmetry assumption leaves out important questions such as the optimal target of a prudential policy. In this section, I use three stylized network structures (a core-periphery network, a chain network, and a path-disconnected network) to show that when designing policies to curb the financial system's risk-taking incentives, regulators need to take into consideration each individual bank's systemic footprint. I will use government guarantees as illustrating examples, but the intuition applies to other prudential policies as well.



(a) Different centrality **(b)** Different upstreamness **(c)** Identical balance sheets

Figure 6 Figure (a) displays a core-peripheral network. Each peripheral has a debt and a claim of size $\bar{d}$ with the core. Figure (b) displays a chain network that is similar to a ring network except that bank 1 and bank 5 are disconnected. Figure (c) displays a network where each bank has an identical balance sheet. The network is path-disconnected and banks belong to different clusters.

Figure 6.(a) displays a core-periphery network with $N = 4$ peripheral banks. Each peripheral bank P_i has a debt and a claim $\bar{d}$ with the core. The extension in Section 4.1 can be applied to a core-periphery network, so let's assume that each bank can fulfill its total liabilities if its project succeeds. In the initial date, the core bank chooses its risk exposure Z_c, and each peripheral bank i chooses Z_i. For expository simplicity, let's assume that $\bar{d}$ is smaller than v_p, which is the amount of each peripheral bank's deposits. The core bank's expected profit is

$$\mathbb{E}\Big[\Pi_c(\boldsymbol{\omega};\boldsymbol{Z})\Big] = P(Z_c)\sum_{m=1}^{N}(Z_c - v_c - m\cdot\bar{d})\cdot\underbrace{\binom{N}{m}\Big(1-P(Z_i)\Big)^m\Big(P(Z_i)\Big)^{N-m}}_{=\ \Pr\ (m \text{ peripheral banks fail})}$$

And each peripheral bank's expected profit is

$$\begin{aligned}\mathbb{E}\Big[\Pi_i(\boldsymbol{\omega};\boldsymbol{Z})\Big] = P(Z_i)\sum_{m=1}^{N-1}\Bigg\{&\Big(1-P(Z_c)\Big)\Big[Z_i - v_p + \Big(\frac{(N-m)\cdot\bar{d}-v_c}{N}\Big)^+ - \bar{d}\Big]\\ &+ P(Z_c)\Big(Z_i - v_p\Big)\Bigg\}\cdot\underbrace{\binom{N-1}{m}\Big(1-P(Z_{-i})\Big)^m\Big(P(Z_{-i})\Big)^{N-1-m}}_{=\ \Pr\ (m \text{ other peripheral banks fail})}\end{aligned}$$

The core and each peripheral bank's network risk-taking distortion is

$$\mathcal{D}_c(\boldsymbol{Z_i}) = \sum_{m=1}^{N} m\cdot\bar{d}\cdot\binom{N}{m}\Big(1-P(Z_i)\Big)^m\Big(P(Z_i)\Big)^{N-m}$$

$$\mathcal{D}_i(Z_c,\boldsymbol{Z_{-i}}) = \Big(1-P(Z_c)\Big)\sum_{m=1}^{N-1}\Big[\bar{d} - \Big(\frac{(N-m)\cdot\bar{d}-v_c}{N}\Big)^+\Big]\cdot\binom{N-1}{m}\Big(1-P(Z_{-i})\Big)^m\Big(P(Z_{-i})\Big)^{N-1-m}$$

The game is supermodular and banks' choices of risk-exposure are strategically complementary. We notice that the core bank's risk-exposure Z_c directly affects each periphery bank i's risk-taking distortion $\mathcal{D}_i(Z_c, \boldsymbol{Z_{-i}})$. On the contrary, a peripheral bank's risk-exposure Z_{-i} has an effect on bank i's risk-taking distortion only if the core bank fails. This observation suggests that a prudential policy can be more effective in reducing the financial system's risk exposure if it is imposed on the core bank than a particular peripheral bank.[15]

To illustrate this point, consider that the government can guarantee either the core's or a particular peripheral bank's liabilities. If it guarantees the core bank, then each peripheral bank's network risk-taking distortion becomes zero. As a result, each peripheral bank will choose risk exposure at the stand-alone level. On the contrary, if the government guarantees a peripheral bank, then other peripheral banks still face a positive risk-taking distortion. As a result, they will choose risk exposure that is greater than the stand-alone level.

The same logic can be applied to a chain network. Figure 6.(b) provides an example of a chain network with five banks. To equalize each bank's claims and liabilities, assume that an outside firm owes bank 1 a debt $\bar{d}$ and that bank 5 owes an outside firm a debt $\bar{d}$.[16] Each bank's expected profit is $P(Z_i)(Z_i - v) - P(Z_i) \cdot \big(1 - P(Z_{i-1})\big) \cdot \bar{d}$. From here, we can see that guaranteeing bank 1's liabilities reduces the risk-taking incentives every other bank $j \in \{2, 3, 4, 5\}$ in the financial system. This is because the guarantee decreases each bank j's risk-taking distortion by affecting its counterparty risk Z_{j-1}. On the contrary, guaranteeing bank 5's liabilities will not affect any other bank's risk-taking incentives. This observation is related to Liu (2019), who studies input-output production networks and shows that it's optimal for the government to subsidize upstream industries because they pose the greatest distortionary effects.

One may be concerned that the above observations are driven by different balance sheets of the core bank or the upstream bank. In other words, bailing out the core or the upstream bank may inherently require more resources from the government than bailing out a peripheral or a downstream bank. To further illustrate that the target of prudential policies matters to systemic risk, let's consider the network structure in Figure 6.(c) in which each bank has the same amount of deposits, the same amount of interbank liabilities, and the same set of risk choices. The only difference is that banks are connected in different clusters of the network (denoted L and R). When unregulated, each bank's risk exposure in equilibrium is identical regardless of whether it is in the L or R sub-network. Specifically, the equilibrium risk exposure is determined by the following fixed point

$$\tilde{Z} = \underset{Z}{\operatorname{argmax}}\, P(Z)(Z - v) - P(Z) \cdot \big[1 - P(\tilde{Z})\big] \cdot \bar{d}$$

Now suppose that the government can guarantee either bank L_1 or bank R_1. If it chooses to

[15] Identifying the key player for policy intervention is the notion studied by Ballester et al. (2006) and Galeotti et al. (2020). Both papers assume that agents' utilities are linear-quadratic and separable. They show that the best policy intervention under those utility classes is propositional to the network's Bonacich centrality or eigenvector centrality.

[16] For notational simplicity, denote $P(Z_0)$ as the probability that bank 1's claims are paid. The probabilities of bank 2 to bank 5's claims being paid are determined in the equilibrium. Again let's assume that $\bar{d}$ is smaller than v.

guarantee bank L_1, then bank L_2 will choose the stand-alone level of risk-exposure, and banks $R_1/R_2/R_3$ will choose $\tilde{Z}$, which is larger than the stand-alone level. On the other hand, if the government chooses to guarantee bank R_1 instead, then bank R_2 will choose the stand-alone level of risk-exposure, banks L_1/L_2 will choose $\tilde{Z}$, and bank R_3 will choose the risk-exposure that is between $\tilde{Z}$ and the stand-alone level. From here, we can see that the policy to guarantee bank R_1 results in less endogenous risk exposure in the financial system than the policy to guarantee bank L_1.

6 Concluding Remarks

This paper studies banks' incentives to choose their risk exposure in financial networks, in which banks are connected through cross-holdings of unsecured debts. In contrast to previous literature that studied the co-insurance benefit of a financial network, I show that banks choose to expose themselves to greater risks if they are highly connected. In addition, they choose correlated risks, aggravating the systemic fragility.

The theory brings about several testable empirical predictions. For example, the strategic complementarity result suggests that an individual bank's risk-taking incentive correlates positively to the risks of the entire financial system. This is precisely what Mink, Ramcharan, and van Lelyveld (2020) have found. Using granular bond portfolios of banks in the European Union, the authors show that regulatory solvency shocks (proxied by the banking system's tier 1 capital ratio) induce banks to shift into riskier assets (higher-yielding sovereign debt) and correlated assets (domestic bonds). Another recent empirical study by Ellul and Kim (2021) provides empirical evidence for the existence of endogenous risk-taking behavior of banks. They use regulatory data on counterparty exposure in the OTC derivative market to show that banks, when densely connected, choose riskier non-bank counterparties for their most material exposures.

By studying financial networks, this paper provides new insights into several government policies when a financial system is interconnected. For example, the paper suggests that a government bailout, if limited to banks' interbank exposures, can reduce a financial system's risk-taking incentives. The model also implores regulators to consider each bank's systemic footprint when designing prudential policies. Nevertheless, the paper does not aim to calibrate actual government policies or provide a holistic cost-benefit study of each particular policy. A natural step for future research is to empirically examine how each government policy affects risk-shifting behaviors in connected financial systems.

References

Acemoglu, Daron, Asuman Ozdaglar, and Alireza Tahbaz-Salehi, 2015, Systemic risk and stability in financial networks, *American Economic Review* 105, 564–608.

Acemoglu, Daron, Asuman E. Ozdaglar, and Alireza Tahbaz-Salehi, 2014, Systemic risk in endogenous financial networks, *MIT and Northwestern University Working Paper* .

Acharya, Viral V, 2009, A theory of systemic risk and design of prudential bank regulation, *Journal of Financial Stability* 5, 224–255.

Allen, Franklin, and Douglas Gale, 2000, Financial contagion, *Journal of Political Economy* 108, 1–33.

Ballester, Coralio, Antoni Calvó-Armengol, and Yves Zenou, 2006, Who's who in networks. wanted: The key player, *Econometrica* 74, 1403–1417.

Bisias, Dimitrios, Mark Flood, Andrew W Lo, and Stavros Valavanis, 2012, A survey of systemic risk analytics, *Annu. Rev. Financ. Econ.* 4, 255–296.

Brusco, Sandro, and Fabio Castiglionesi, 2007, Liquidity coinsurance, moral hazard, and financial contagion, *Journal of Finance* 62, 2275–2302.

Capponi, Agostino, Felix C Corell, and Joseph E Stiglitz, 2020, Optimal bailouts and the doom loop with a financial network, Technical report, National Bureau of Economic Research.

Craig, Ben R, and Yiming Ma, 2018, Intermediation in the interbank lending market, *Columbia Business School Research Paper* .

Denti, Tommaso, 2018, Unrestricted information acquisition, *Cornell Working Paper* .

Diamond, Douglas W., and Philip H. Dybvig, 1983, Bank runs, deposit insurance, and liquidity, *Journal of Political Economy* 91, 401–419.

Donaldson, Jason Roderick, and Giorgia Piacentino, 2018, Netting, *Washington University in St. Louis and Columbia University Working Paper* .

Duffie, Darrell, and Haoxiang Zhu, 2011, Does a central clearing counterparty reduce counterparty risk?, *The Review of Asset Pricing Studies* 1, 74–95.

Eisenberg, Larry, and Thomas H. Noe, 2001, Systemic risk in financial systems, *Management Science* 47, 236–249.

Elliott, Matthew, Co-Pierre Georg, and Jonathon Hazell, 2021, Systemic risk shifting in financial networks, *Journal of Economic Theory* 191, 105157.

Elliott, Matthew, Benjamin Golub, and Matthew O. Jackson, 2014, Financial networks and contagion, *American Economic Review* 104, 3115–53.

Ellul, Andrew, and Dasol Kim, 2021, Counterparty choice, bank interconnectedness, and systemic risk, *OFR Working Paper* .

Erol, Selman, 2019, Network hazard and bailouts, *Carnegie Mellon University Working Paper* .

Erol, Selman, and Guillermo Ordoñez, 2017, Network reactions to banking regulations, *Journal of Monetary Economics* 89, 51–67.

Farhi, Emmanuel, and Jean Tirole, 2012, Collective moral hazard, maturity mismatch, and systemic bailouts, *American Economic Review* 102, 60–93.

Financial Crisis Inquiry Commission, 2011, *The Financial Crisis Inquiry Report* (U.S. Government Printing Office).

Freixas, Xavier, Bruno M. Parigi, and Jean-Charles Rochet, 2000, Systemic risk, interbank relations, and liquidity provision by the central bank, *Journal of Money, Credit and Banking* 32, 611–638.

Gai, Prasanna, Andrew Haldane, and Sujit Kapadia, 2011, Complexity, concentration and contagion, *Journal of Monetary Economics* 58, 453–470.

Gale, Douglas, and Xavier Vives, 2002, Dollarization, bailouts, and the stability of the banking system, *The Quarterly Journal of Economics* 117, 467–502.

Galeotti, Andrea, and Christian Ghiglino, 2021, Cross-ownership and portfolio choice, *Journal of Economic Theory* 192, 105194.

Galeotti, Andrea, Benjamin Golub, and Sanjeev Goyal, 2020, Targeting interventions in networks, *Econometrica* 88, 2445–2471.

Glasserman, Paul, and H. Peyton Young, 2016, Contagion in financial networks, *Journal of Economic Literature* 54, 779–831.

Gorton, Gary B., 2012, *Misunderstanding Financial Crises: Why We Don't See Them Coming* (Oxford University Press).

Hellmann, Thomas F, Kevin C Murdock, and Joseph E Stiglitz, 2000, Liberalization, moral hazard in banking, and prudential regulation: Are capital requirements enough?, *American economic review* 90, 147–165.

International Monetary Fund, 2009, Assessing the systemic implications of financial linkages, *IMF Global Financial Stability Report, Vol. 2* .

Jackson, Matthew O, and Agathe Pernoud, 2021a, Credit freezes, equilibrium multiplicity, and optimal bailouts in financial networks, *Working Paper* .

Jackson, Matthew O, and Agathe Pernoud, 2021b, Investment incentives and regulation in financial networks, *Stanford University Working Paper* .

Jackson, Matthew O., and Agathe Pernoud, 2021c, Systemic risk in financial networks: A survey, *Annual Review of Economics* Vol. 13.

Jensen, Michael C, and William H Meckling, 1976, Theory of the firm: Managerial behavior, agency costs and ownership structure, *Journal of Financial Economics* 3, 305–360.

Leitner, Yaron, 2005, Financial networks: Contagion, commitment, and private sector bailouts, *Journal of Finance* 60, 2925–2953.

Liu, Ernest, 2019, Industrial policies in production networks, *The Quarterly Journal of Economics* 134, 1883–1948.

Mian, Atif, and Amir Sufi, 2009, The consequences of mortgage credit expansion: Evidence from the us mortgage default crisis, *The Quarterly Journal of Economics* 124, 1449–1496.

Milgrom, Paul, and John Roberts, 1990, Rationalizability, learning, and equilibrium in games with strategic complementarities, *Econometrica: Journal of the Econometric Society* 1255–1277.

Mink, Mark, Rodney Ramcharan, and Iman van Lelyveld, 2020, How banks respond to distress: Shifting risks in europe's banking union, *De Nederlandsche Bank Working Paper* .

Shin, Hyun Song, 2009, Securitisation and financial stability, *The Economic Journal* 119, 309–332.

Vohra, Rakesh, Yiqing Xing, and Wu Zhu, 2021, The network effects of agency conflicts, *Working Paper* .

PROOFS

PROOF OF LEMMA 2: From the assumption $\underline{Z} \geqslant v + \bar{d}$, a successful bank's interbank payment is $d_i = \bar{d}$, independent of its choice of risk exposure Z_i. A failed bank's cash flow that will contribute to the interbank payment system is $e_i = 0$, also independent of its choice of risk exposure. Reordering equation 2 gives us,

$$\begin{aligned} d_i^*(\boldsymbol{\omega}; \boldsymbol{Z}) &= \bar{d} && \forall \omega_i = s \\ d_i^*(\boldsymbol{\omega}; \boldsymbol{Z}) &= \left\{ \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}; \boldsymbol{Z}) - v \right\}^+ && \forall \omega_i = f \end{aligned} \tag{13}$$

We can see that the vector of risk exposure $\boldsymbol{Z}$ does not enter the system of equations. As a result, the fixed point $(d_1^*(\boldsymbol{\omega}), ...d_N^*(\boldsymbol{\omega}))$ is constant in $\boldsymbol{Z}$. □

Before proving proposition 1, it is useful to have the following auxiliary lemma.

AUXILIARY LEMMA: *the payment vector $\boldsymbol{d}^*$ is weakly increasing in any bank's cash flow $\tilde{e}_j$. In particular, $\boldsymbol{d}^*(\boldsymbol{\omega})$ is higher when any bank's project succeeds ($\omega_j = s$) compared with when it fails ($\omega_j = f$).*[17]

Proof. The above lemma is identical to Eisenberg and Noe (2000) Lemma 5. The payment equilibrium (equation 2) is a fixed point solution of a function $\boldsymbol{d}^* = \Phi(\boldsymbol{d}^*; \tilde{e}_j)$. Since both min and max operator preserve monotonicity, Φ is increasing in $\tilde{e}_j$. By monotone selection theorem (Milgrom and Roberts (1990), Theorem 1), the fixed point $\boldsymbol{d}^*$ is increasing in $\tilde{e}_j$.

□

PROOF OF PROPOSITION 1: Taking the first- and second-order conditions of the equation 6, we have

$$\begin{aligned} F(Z_i; \boldsymbol{Z_{-i}}) &= P'(Z_i)(Z_i - v) + P(Z_i) - P(Z_i)'\mathcal{D}_i(\boldsymbol{Z_{-i}}) = 0 \\ S(Z_i; \boldsymbol{Z_{-i}}) &= P''(Z_i)(Z_i - v) + 2P'(Z_i) - P(Z_i)''\mathcal{D}_i(\boldsymbol{Z_{-i}}) < 0 \end{aligned}$$

From assumption 1, we obtain $S(Z_i; \boldsymbol{Z_{-i}}) < 0$. From the total derivative of the FOC, we have

$$\frac{\mathrm{d}\widehat{Z}_i}{\mathrm{d}\mathcal{D}_i(\boldsymbol{Z_{-i}})} = -\frac{\partial F(\widehat{Z}_i; \boldsymbol{Z_{-i}})/\partial \mathcal{D}_i(\boldsymbol{Z_{-i}})}{\partial F(\widehat{Z}_i; \boldsymbol{Z_{-i}})/\partial Z_i} = \frac{P'(\widehat{Z}_i)}{S(\widehat{Z}_i; \boldsymbol{Z_{-i}})} > 0 \tag{14}$$

The above inequality implies that whatever increases $\mathcal{D}_i(\boldsymbol{Z_{-i}})$ will increase bank i's optimal $\widehat{Z}_i$. Intuitively, the distortion $\mathcal{D}_i(\boldsymbol{Z_{-i}})$ decreases bank i's upside payoff (the payoff when its project succeeds). As a result, it will make bank i care less about the probability of success when trading off risk and return.

To see the effect from bank m's risk exposure Z_m on bank i's risk-taking distortion $\mathcal{D}_i(\boldsymbol{Z_{-i}})$, let's vary it from Z_m to Z_m' with $Z_m' > Z_m$. Let $\boldsymbol{Z'_{-i}}$ denote the new risk-exposure vector that differs from $\boldsymbol{Z}_{-i}$ only

[17] Throughout this paper, whenever the ordering of a vector is mentioned, it refers to a pointwise ordering. i.e $\boldsymbol{x} \geqslant \boldsymbol{y} \Leftrightarrow x_i \geqslant y_i$ for all i. For the following text, all orderings are weakly.

in Z_m. We have

$$\begin{aligned}
&\mathcal{D}_i(\mathbf{Z'_{-i}}) - \mathcal{D}_i(\mathbf{Z_{-i}}) \\
&\quad = \sum_{\omega_{-i-m}} \Pr(\boldsymbol{\omega_{-i-m}}) \Big[P(Z'_m) \Big(\bar{d} - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega^{m=s}}) \Big) + \Big(1 - P(Z'_m)\Big) \Big(\bar{d} - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega^{m=f}}) \Big) \Big] \\
&\qquad - \sum_{\omega_{-i-m}} \Pr(\boldsymbol{\omega_{-i-m}}) \Big[P(Z_m) \Big(\bar{d} - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega^{m=s}}) \Big) + \Big(1 - P(Z_m)\Big) \Big(\bar{d} - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega^{m=f}}) \Big) \Big] \\
&\quad = \sum_{\omega_{-i-m}} \Pr(\boldsymbol{\omega_{-i-m}}) \Big[\Big(P(Z'_m) - P(Z_m) \Big) \Big(\sum_j \theta_{ij} d_j^*(\boldsymbol{\omega^{m=f}}) - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega^{m=s}}) \Big) \Big] \geqslant 0
\end{aligned} \tag{15}$$

With slight abuse of notation, let $\boldsymbol{\omega_{-i-m}}$ denote a vector of $\boldsymbol{\omega}$ without the element i and m, let $\boldsymbol{\omega^{m=s}}$ denote a vector that appends $\boldsymbol{\omega_{-i-m}}$ with $\omega_m = s$ and $\omega_i = s$, and let $\boldsymbol{\omega^{m=f}}$ denote a vector that appends $\boldsymbol{\omega_{-i-m}}$ with $\omega_m = f$ and $\omega_i = s$. The last inequality results from the auxiliary lemma. The inequality states that bank i's risk-taking distortion is increasing in bank m's risk-exposure. Intuitively, if bank m chooses greater risk exposure, then its project becomes more likely to fail. When bank m's project fails, bank i's cross-subsidies to other banks become higher. Combining Equation 14 and Equation 15, we have

$$\frac{\mathrm{d}\widehat{Z}_i}{\mathrm{d}Z_{-i}} = \frac{\mathrm{d}\widehat{Z}_i}{\mathrm{d}\mathcal{D}_i(\mathbf{Z_{-i}})} \frac{\mathrm{d}\mathcal{D}_i(\mathbf{Z_{-i}})}{\mathrm{d}Z_{-i}} > 0 \quad \forall i \quad \text{and} \quad -i$$

As a result, bank i will choose greater risk exposure in response to an increase in its counterparty risk. □

PROOF OF PROPOSITION 2: The payment equilibrium in any state of nature is the fixed-point solution to a system of equations (equation 13). Denote the fixed point as $\boldsymbol{d^*} = \Phi(\boldsymbol{d^*})$, where Φ a continuous mapping with a convex and compact domain $[0, \bar{d}]^N$. By the Brouwer fixed point theorem, the payment equilibrium $\boldsymbol{d}^*(\boldsymbol{\omega}; \boldsymbol{Z})$ exists for all $\boldsymbol{\omega}$ and $\boldsymbol{Z}$ (Eisenberg and Noe, 2001). This establishes the existence of the payment equilibrium for all $\boldsymbol{\omega}$ and $\boldsymbol{Z}$. From proposition 1, $\mathrm{d}\widehat{Z}_i / Z_{-i} \geqslant 0$ for all i and $-i$. This implies that the Nash equilibrium is a supermodular game. The domain for the risk-exposure vector $[\underline{Z}, \overline{Z}]^N$ is a complete lattice. By Tarski's theorem, the fixed-point solution to the first order conditions $F(Z_i^*; \boldsymbol{Z^*_{-i}}) = 0$ exists. The equilibrium risk exposure $\boldsymbol{Z}^* = (Z_1^*, ..., Z_N^*)$ is this fixed point. □

PROOF OF CORROLARY 1: Denote Z^N and Z^S as the equilibrium risk exposure of a bank in a financial network and a stand-alone bank respectively. Formally, they are the solutions to their respective first order conditions, i.e.,

$$\begin{aligned}
&P'(Z^N)(Z^N - v) + P(Z^N) - P(Z^N)'\mathcal{D}_i(\boldsymbol{Z^N}) = 0 \\
&P'(Z^S)(Z^S - v) + P(Z^S) = 0
\end{aligned}$$

By equation 14, $\mathrm{d}Z^N / \mathrm{d}\mathcal{D}_i(\boldsymbol{Z^N}) > 0$. We also know that the distortion $\mathcal{D}_i(\boldsymbol{Z^N})$ is positive because $P(Z_j) < 1$ for all Z_j. Therefore, $Z^N > Z^S$. □

PROOF OF LEMMA 3: For any state of nature $\boldsymbol{\omega}$, suppose that there exists two vectors, $\boldsymbol{a}(\boldsymbol{\omega})$ and $\boldsymbol{b}(\boldsymbol{\omega})$, such that $d_i^*(\boldsymbol{\omega}) = \{a_i(\boldsymbol{\omega})\bar{d} - b_i(\boldsymbol{\omega})v\}^+$. By definition, they should satisfy equation 13. After plugging $\boldsymbol{a}(\boldsymbol{\omega})$

and $\boldsymbol{b}(\boldsymbol{\omega})$ into equation 13, we have $(a_i, b_i) = (1, 0)\ \forall \omega_i = s$, and

$$
\begin{aligned}
d_i^*(\boldsymbol{\omega}) &= \left\{ \sum_{\omega_j = s} \theta_{ij}\bar{d} + \sum_{j \in \mathcal{F}_{\boldsymbol{\omega}}^+} \theta_{ij}\Big(a_j(\boldsymbol{\omega})\bar{d} - b_j(\boldsymbol{\omega})v\Big) - v \right\}^+ \\
&= \left\{ \Big(\sum_{j \in \mathcal{F}_{\boldsymbol{\omega}}^+} \theta_{ij} a_j(\boldsymbol{\omega}) + \sum_{\omega_j = s} \theta_{ij} \Big)\bar{d} - \Big(\sum_{j \in \mathcal{F}_{\boldsymbol{\omega}}^+} \theta_{ij} b_j(\boldsymbol{\omega}) + 1 \Big) v \right\}^+ \qquad \forall \omega_i = f
\end{aligned}
$$

where $\mathcal{F}_{\boldsymbol{\omega}}^+ \equiv \{i : \omega_i = f, a_i\bar{d} - b_i v \geqslant 0\}$. The set contains failed banks that are solvent (i.e., being able to fulfill their deposits). Similarly, denote $\mathcal{F}_{\boldsymbol{\omega}}^- \equiv \{i : \omega_i = f, a_i\bar{d} - b_i v < 0\}$ as the set of insolvent failed banks, and $\mathcal{S}_{\boldsymbol{\omega}} \equiv \{i : \omega_i = s)$ as the set of successful banks. By the conjecture, we need $\forall \omega_i \in f$,

$$
a_i(\boldsymbol{\omega}) = \sum_{j \in \mathcal{F}_{\boldsymbol{\omega}}^+} \theta_{ij} a_j(\boldsymbol{\omega}) + \sum_{\omega_j = s} \theta_{ij} \tag{16}
$$

$$
b_i(\boldsymbol{\omega}) = \sum_{j \in \mathcal{F}_{\boldsymbol{\omega}}^+} \theta_{ij} b_j(\boldsymbol{\omega}) + 1 \tag{17}
$$

Since the RHS of above equations are increasing in $\boldsymbol{a}(\boldsymbol{\omega})$ and $\boldsymbol{b}(\boldsymbol{\omega})$ respectively, the fixed points exist by the Tarski's theorem. The conjecture is hence verified. Let's rewrite the above equations in a matrix form for banks in $\mathcal{F}_{\boldsymbol{\omega}}^+$.

$$
\boldsymbol{a}_+(\boldsymbol{\omega}) = \boldsymbol{\Theta}_{++}\boldsymbol{a}_+(\boldsymbol{\omega}) + \boldsymbol{\Theta}_{+\boldsymbol{s}}\mathbb{1}_s \tag{18}
$$

$$
\boldsymbol{b}_+(\boldsymbol{\omega}) = \boldsymbol{\Theta}_{++}\boldsymbol{b}_+(\boldsymbol{\omega}) + \mathbb{1}_+ \tag{19}
$$

where $\boldsymbol{a}_+(\boldsymbol{\omega})$ and $\boldsymbol{b}_+(\boldsymbol{\omega})$ are truncated vectors of $\boldsymbol{a}(\boldsymbol{\omega})$ and $\boldsymbol{b}(\boldsymbol{\omega})$ with only the rows that belong to $\mathcal{F}_{\boldsymbol{\omega}}^+$. Similarly, $\boldsymbol{\Theta}_{++}$ is a truncated matrix of $\boldsymbol{\Theta}$ with only the rows and columns that belong to $\mathcal{F}_{\boldsymbol{\omega}}^+$, and $\boldsymbol{\Theta}_{+\boldsymbol{s}}$ is the truncated matrix of $\boldsymbol{\Theta}$ where each row belongs to $\mathcal{F}_{\boldsymbol{\omega}}^+$ and each column belongs to $\mathcal{S}$. $\mathbb{1}_+$ and $\mathbb{1}_s$ are column vectors of ones with appropriate dimension. Note that $\boldsymbol{\Theta}_{++}$, $\boldsymbol{\Theta}_{+\boldsymbol{s}}$, $\mathbb{1}_+$, and $\mathbb{1}_s$ are all state-contingent. To conserve space, I suppress their underscript $\boldsymbol{\omega}$.

By the Markovian property of $\boldsymbol{\Theta}$ (row-sum equals to one), we have $\boldsymbol{\Theta}_{++}\mathbb{1}_+ + \boldsymbol{\Theta}_{+-}\mathbb{1}_- + \boldsymbol{\Theta}_{+\boldsymbol{s}}\mathbb{1}_s = \mathbb{1}_+$. As a result, Equation 18 implies that

$$
\boldsymbol{a}_+(\boldsymbol{\omega}) = (\boldsymbol{I}_+ - \boldsymbol{\Theta}_{++})^{-1}\boldsymbol{\Theta}_{+\boldsymbol{s}}\mathbb{1}_s < \mathbb{1}_+ \tag{20}
$$

Plugging $(\boldsymbol{a}_+, \boldsymbol{b}_+)$ into the network risk-taking distortion, we can rewrite $\mathcal{D}_i(\boldsymbol{Z}_{-\boldsymbol{i}})$ in a matrix form as

$$
\begin{aligned}
\mathcal{D}_i(\boldsymbol{Z}_{-\boldsymbol{i}}) &= \sum_{\boldsymbol{\omega}_{-\boldsymbol{i}}} \Pr(\boldsymbol{\omega}_{-\boldsymbol{i}}) \Big[\bar{d} - \Big(\Theta_{is}\mathbb{1}_s\bar{d} + \Theta_{i+}(\boldsymbol{a}_+\bar{d} - \boldsymbol{b}_+ v) + \Theta_{i-} \cdot 0 \Big) \Big] \\
&= \sum_{\boldsymbol{\omega}_{-\boldsymbol{i}}} \Pr(\boldsymbol{\omega}_{-\boldsymbol{i}}) \Big[\boldsymbol{\Theta}_{i+}\Big((\mathbb{1}_+ - \boldsymbol{a}_+)\bar{d} + \boldsymbol{b}_+ v \Big) + \boldsymbol{\Theta}_{i-}\mathbb{1}_-\bar{d} + \boldsymbol{\Theta}_{is}\mathbb{1}_s \cdot 0 \Big]
\end{aligned} \tag{21}
$$

Each part of Equation 21 has a clear interpretation: $\boldsymbol{\Theta}_{i+}\big((\mathbb{1}_+ - \boldsymbol{a}_+)\bar{d} + \boldsymbol{b}_+ v\big)$ is bank i's subsidies to solvent failed banks, $\boldsymbol{\Theta}_{i-}\mathbb{1}_-\bar{d}$ is bank i's subsidies to insolvent failed banks, and $\boldsymbol{\Theta}_{is}\mathbb{1}_s \cdot 0$ is bank i's subsidies to successful banks, which is zero.

To prove Lemma 3, let's compare three financial networks with same topology $\boldsymbol{\Theta}$ but different interbank liabilities $\bar{d}_1$, $\bar{d}_2$, and $\bar{d}_3$, with $\bar{d}_3 - \bar{d}_2 = \bar{d}_2 - \bar{d}_1 = \xi$. To prove the monotonicity and concavity, we need to show $\mathcal{D}_i^3(\mathbf{Z}_{-i}) \geqslant \mathcal{D}_i^2(\mathbf{Z}_{-i}) \geqslant \mathcal{D}_i^1(\mathbf{Z}_{-i})$ and $\mathcal{D}_i^2(\mathbf{Z}_{-i}) - \mathcal{D}_i^1(\mathbf{Z}_{-i}) \geqslant \mathcal{D}_i^3(\mathbf{Z}_{-i}) - \mathcal{D}_i^2(\mathbf{Z}_{-i})$ with inequality happens somewhere.

Note that the set $\mathcal{F}_\omega^+ \equiv \{i : \omega_i = f, a_i\bar{d} - b_i v \geqslant 0\}$ depends on $\bar{d}$. We hence denote $\mathcal{F}_1^+(\boldsymbol{\omega})$, $\mathcal{F}_2^+(\boldsymbol{\omega})$, and $\mathcal{F}_3^+(\boldsymbol{\omega})$ the set of solvent failed bank in state $\boldsymbol{\omega}$ for network $(\bar{d}_1, \boldsymbol{\Theta})$, $(\bar{d}_2, \boldsymbol{\Theta})$, and $(\bar{d}_3, \boldsymbol{\Theta})$ respectively. From monotone selection theorem (see auxiliary lemma), we have $d_i^{3*}(\boldsymbol{\omega}) \geqslant d_i^{2*}(\boldsymbol{\omega}) \geqslant d_i^{1*}(\boldsymbol{\omega}), \forall i \in \mathcal{N}$ and $\boldsymbol{\omega} \in \Omega$. That implies $\mathcal{F}_1^+(\boldsymbol{\omega}) \subseteq \mathcal{F}_2^+(\boldsymbol{\omega}) \subseteq \mathcal{F}_3^+(\boldsymbol{\omega})$ for all $\boldsymbol{\omega} \in \Omega$. Intuitively, a larger $\bar{d}$ can make more failed banks solvent.

Let's consider the following four cases: (1) $\mathcal{F}_1^+(\boldsymbol{\omega}) = \mathcal{F}_2^+(\boldsymbol{\omega}) = \mathcal{F}_3^+(\boldsymbol{\omega})$ for all $\boldsymbol{\omega}$. (2) $\mathcal{F}_1^+(\boldsymbol{\omega}) \subset \mathcal{F}_2^+(\boldsymbol{\omega}) = \mathcal{F}_3^+(\boldsymbol{\omega})$ for some $\boldsymbol{\omega}$. (3) $\mathcal{F}_1^+(\boldsymbol{\omega}) = \mathcal{F}_2^+(\boldsymbol{\omega}) \subset \mathcal{F}_3^+(\boldsymbol{\omega})$ for some $\boldsymbol{\omega}$. (4) $\mathcal{F}_1^+(\boldsymbol{\omega}) \subset \mathcal{F}_2^+(\boldsymbol{\omega}) \subset \mathcal{F}_3^+(\boldsymbol{\omega})$ for some $\boldsymbol{\omega}$.

Case I: $\mathcal{F}_1^+(\boldsymbol{\omega}) = \mathcal{F}_2^+(\boldsymbol{\omega}) = \mathcal{F}_3^+(\boldsymbol{\omega})$ for all $\boldsymbol{\omega}$

From equation 18 and 19, it's easy to see that $\boldsymbol{a}_+^1 = \boldsymbol{a}_+^2 = \boldsymbol{a}_+^3$ and $\boldsymbol{b}_+^1 = \boldsymbol{b}_+^2 = \boldsymbol{b}_+^3$. We also have $\boldsymbol{\Theta}_{i+}$, $\mathbb{1}_+$, $\boldsymbol{\Theta}_{i-}$, and $\mathbb{1}_-$ unchanged for the three networks. Therefore,

$$\mathcal{D}_i^3(\mathbf{Z}_{-i}) - \mathcal{D}_i^2(\mathbf{Z}_{-i}) = \sum_{\boldsymbol{\omega}_{-i}} \Pr(\boldsymbol{\omega}_{-i}) \Big[\boldsymbol{\Theta}_{i+}(\mathbb{1}_+ - \boldsymbol{a}_+)(\bar{d}_3 - \bar{d}_2) + \boldsymbol{\Theta}_{i-}\mathbb{1}_-(\bar{d}_3 - \bar{d}_2) \Big] > 0$$
$$\mathcal{D}_i^2(\mathbf{Z}_{-i}) - \mathcal{D}_i^1(\mathbf{Z}_{-i}) = \sum_{\boldsymbol{\omega}_{-i}} \Pr(\boldsymbol{\omega}_{-i}) \Big[\boldsymbol{\Theta}_{i+}(\mathbb{1}_+ - \boldsymbol{a}_+)(\bar{d}_2 - \bar{d}_1) + \boldsymbol{\Theta}_{i-}\mathbb{1}_-(\bar{d}_2 - \bar{d}_1) \Big] > 0$$

The last inequality is due to equation 20. With $\bar{d}_3 - \bar{d}_2 = \bar{d}_2 - \bar{d}_1 = \xi$, we have $\mathcal{D}_i^3(\mathbf{Z}_{-i}) - \mathcal{D}_i^2(\mathbf{Z}_{-i}) = \mathcal{D}_i^2(\mathbf{Z}_{-i}) - \mathcal{D}_i^1(\mathbf{Z}_{-i}) > 0$. Intuitively, this case means that the network risk-taking is linearly increasing in $\bar{d}$, if the change of $\bar{d}$ does not make additional insolvent banks "solvenet".

Case II: $\mathcal{F}_1^+(\boldsymbol{\omega}) \subset \mathcal{F}_2^+(\boldsymbol{\omega}) = \mathcal{F}_3^+(\boldsymbol{\omega})$ for some $\boldsymbol{\omega}$.

Let's first compare the distortion when the interbank liabilities are $\bar{d}_2$ and $\bar{d}_1$. Because $\mathcal{F}_1^+(\boldsymbol{\omega}) \subset \mathcal{F}_2^+(\boldsymbol{\omega})$, some otherwise insolvent failed banks for $(\bar{d}_1, \boldsymbol{\Theta})$ become solvent for $(\bar{d}_2, \boldsymbol{\Theta})$ in some state of nature $\boldsymbol{\omega}$. Denote those banks $t_1, t_2, ..., t_T$, where $T \geqslant 1$. From the continuity of the payment equilibrium in terms of $\bar{d}$ (equation 2), there exists $\bar{d}_1 < \tilde{d}_1 < \tilde{d}_2 ... < ... < \tilde{d}_S < \bar{d}_2$ (where $1 \leqslant S \leqslant T$) such that when the interbank liabilities $\bar{d} = \tilde{d}_s$, some bank t_t is exactly solvent, or $\tilde{a}_t(\boldsymbol{\omega})\tilde{d}_s - \tilde{b}_t(\boldsymbol{\omega})v = 0$. In other words, this margin bank t is solvent when $\bar{d} \in [\tilde{d}_s, \tilde{d}_{s+1})$ and insolvent when $\bar{d} \in (\tilde{d}_{s-1}, \tilde{d}_s]$ respectively. Denote $\widetilde{\mathcal{D}}_i^s(\mathbf{Z}_{-i})$ as the network risk-taking distortion at those cut-offs $\tilde{d}_s$. We have

$$\begin{aligned}
\mathcal{D}_i^2(\mathbf{Z}_{-i}) - \widetilde{\mathcal{D}}_i^S(\mathbf{Z}_{-i}) &= \sum_{\boldsymbol{\omega}_{-i}} \Pr(\boldsymbol{\omega}_{-i}) \Big[\boldsymbol{\Theta}_{i+}^2(\mathbb{1}_+ - \boldsymbol{a}_+^2)(\bar{d}_2 - \tilde{d}_S) + \boldsymbol{\Theta}_{i-}^2\mathbb{1}_-(\bar{d}_2 - \tilde{d}_S) \Big] > 0 \\
\widetilde{\mathcal{D}}_i^{s+1}(\mathbf{Z}_{-i}) - \widetilde{\mathcal{D}}_i^s(\mathbf{Z}_{-i}) &= \sum_{\boldsymbol{\omega}_{-i}} \Pr(\boldsymbol{\omega}_{-i}) \Big[\widetilde{\boldsymbol{\Theta}}_{i+}^s(\mathbb{1}_+ - \tilde{\boldsymbol{a}}_+^s)(\tilde{d}_{s+1} - \tilde{d}_s) + \widetilde{\boldsymbol{\Theta}}_{i-}^s\mathbb{1}_-(\tilde{d}_{s+1} - \tilde{d}_s) \Big] > 0 \\
\widetilde{\mathcal{D}}_i^1(\mathbf{Z}_{-i}) - \mathcal{D}_i^1(\mathbf{Z}_{-i}) &= \sum_{\boldsymbol{\omega}_{-i}} \Pr(\boldsymbol{\omega}_{-i}) \Big[\boldsymbol{\Theta}_{i+}^1(\mathbb{1}_+ - \boldsymbol{a}_+^1)(\tilde{d}_1 - \bar{d}_1) + \boldsymbol{\Theta}_{i-}^1\mathbb{1}_-(\tilde{d}_1 - \bar{d}_1) \Big] > 0
\end{aligned} \tag{22}$$

Each column of $\boldsymbol{\Theta}_{i+}^1$ corresponds to a solvent failed bank if $\bar{d} \in [\bar{d}_1, \tilde{d}_1]$. Each column of $\widetilde{\boldsymbol{\Theta}}_{i+}^s$ corresponds to a solvent failed bank if $\bar{d} \in [\tilde{d}_s, \tilde{d}_{s+1}]$. Each column of $\boldsymbol{\Theta}_{i+}^2$ corresponds to a solvent failed bank if $\bar{d} \in [\tilde{d}_S, \bar{d}_2]$. The same notation applies to $\tilde{\boldsymbol{a}}_+$ and $\widetilde{\boldsymbol{\Theta}}_{i-}$ as well. They are state-contingent, and to conserve space we suppress the state.

The above inequalities showed that $\mathcal{D}_i^2(\mathbf{Z}_{-\boldsymbol{i}}) > \widetilde{\mathcal{D}}_i^S(\mathbf{Z}_{-\boldsymbol{i}}) > ... > \widetilde{\mathcal{D}}_i^1(\mathbf{Z}_{-\boldsymbol{i}}) > \mathcal{D}_i^1(\mathbf{Z}_{-\boldsymbol{i}})$ and hence the monotonicity result follows. To prove the concavity, we observe that $\widetilde{\boldsymbol{\Theta}}_{\boldsymbol{i}+}^{\boldsymbol{s}}\mathbb{1}_+ + \widetilde{\boldsymbol{\Theta}}_{\boldsymbol{i}-}^{s}\mathbb{1}_- = \boldsymbol{\Theta}_{\boldsymbol{if}}\mathbb{1}_f$ for all s and $\boldsymbol{\omega}$. By definition, $\widetilde{\boldsymbol{\Theta}}_{\boldsymbol{i}+}^{\boldsymbol{s}}$ and $\tilde{\boldsymbol{a}}_+^{\boldsymbol{s}}$ are sub-matrix of $\widetilde{\boldsymbol{\Theta}}_{i+}^{\boldsymbol{s+1}}$ and $\tilde{\boldsymbol{a}}_+^{\boldsymbol{s+1}}$ respectively. Hence we have

$$\widetilde{\boldsymbol{\Theta}}_{\boldsymbol{i}+}^{\boldsymbol{s}}(\mathbb{1}_+ - \tilde{\boldsymbol{a}}_+^{\boldsymbol{s}}) + \widetilde{\boldsymbol{\Theta}}_{\boldsymbol{i}-}^{\boldsymbol{s}}\mathbb{1}_- > \widetilde{\boldsymbol{\Theta}}_{\boldsymbol{i}+}^{\boldsymbol{s+1}}(\mathbb{1}_+ - \tilde{\boldsymbol{a}}_+^{\boldsymbol{s+1}}) + \widetilde{\boldsymbol{\Theta}}_{\boldsymbol{i}-}^{\boldsymbol{s+1}}\mathbb{1}_- \qquad \forall s = 1, ..., S-1$$

After summing every difference in equation 22 and replacing all of RHS with the first line, i.e., the smallest, we have

$$\mathcal{D}_i^2(\mathbf{Z}_{-\boldsymbol{i}}) - \mathcal{D}_i^1(\mathbf{Z}_{-\boldsymbol{i}}) > \sum_{\boldsymbol{\omega}_{-i}} \Pr(\boldsymbol{\omega}_{-\boldsymbol{i}})\Big[\boldsymbol{\Theta}_{\boldsymbol{i}+}^{\boldsymbol{2}}(\mathbb{1}_+ - \boldsymbol{a}_+^{\boldsymbol{2}})(\bar{d}_2 - \bar{d}_1) + \boldsymbol{\Theta}_{\boldsymbol{i}-}^{\boldsymbol{2}}\mathbb{1}_-(\bar{d}_2 - \bar{d}_1)\Big]$$

Since $\mathcal{F}_2^+(\boldsymbol{\omega}) = \mathcal{F}_3^+(\boldsymbol{\omega})$, we have the following identity as in case I,

$$\mathcal{D}_i^3(\mathbf{Z}_{-\boldsymbol{i}}) - \mathcal{D}_i^2(\mathbf{Z}_{-\boldsymbol{i}}) = \sum_{\boldsymbol{\omega}_{-i}} \Pr(\boldsymbol{\omega}_{-\boldsymbol{i}})\Big[\boldsymbol{\Theta}_{\boldsymbol{i}+}^{\boldsymbol{2}}(\mathbb{1}_+ - \boldsymbol{a}_+^{\boldsymbol{2}})(\bar{d}_3 - \bar{d}_2) + \boldsymbol{\Theta}_{\boldsymbol{i}-}^{\boldsymbol{2}}\mathbb{1}_-(\bar{d}_3 - \bar{d}_2)\Big]$$

Hence $\mathcal{D}_i^3(\mathbf{Z}_{-\boldsymbol{i}}) - \mathcal{D}_i^2(\mathbf{Z}_{-\boldsymbol{i}}) < \mathcal{D}_i^2(\mathbf{Z}_{-\boldsymbol{i}}) - \mathcal{D}_i^1(\mathbf{Z}_{-\boldsymbol{i}})$ and the concavity follows.

Case III: $\mathcal{F}_1^+(\boldsymbol{\omega}) = \mathcal{F}_2^+(\boldsymbol{\omega}) \subset \mathcal{F}_3^+(\boldsymbol{\omega})$ for some $\boldsymbol{\omega}$.

The proof is identical to case II with a slight twist. Instead of replacing all RHS of equation 22 with the first line, we replace it with the last line. Hence, we obtain,

$$\mathcal{D}_i^3(\mathbf{Z}_{-\boldsymbol{i}}) - \mathcal{D}_i^2(\mathbf{Z}_{-\boldsymbol{i}}) < \sum_{\boldsymbol{\omega}_{-i}} \Pr(\boldsymbol{\omega}_{-\boldsymbol{i}})\Big[\boldsymbol{\Theta}_{\boldsymbol{i}+}^{\boldsymbol{2}}(\mathbb{1}_+ - \boldsymbol{a}_+^{\boldsymbol{2}})(\bar{d}_3 - \bar{d}_2) + \boldsymbol{\Theta}_{\boldsymbol{i}-}^{\boldsymbol{2}}\mathbb{1}_-(\bar{d}_3 - \bar{d}_2)\Big]$$
$$\mathcal{D}_i^2(\mathbf{Z}_{-\boldsymbol{i}}) - \mathcal{D}_i^1(\mathbf{Z}_{-\boldsymbol{i}}) = \sum_{\boldsymbol{\omega}_{-i}} \Pr(\boldsymbol{\omega}_{-\boldsymbol{i}})\Big[\boldsymbol{\Theta}_{\boldsymbol{i}+}^{\boldsymbol{2}}(\mathbb{1}_+ - \boldsymbol{a}_+^{\boldsymbol{2}})(\bar{d}_2 - \bar{d}_1) + \boldsymbol{\Theta}_{\boldsymbol{i}-}^{\boldsymbol{2}}\mathbb{1}_-(\bar{d}_2 - \bar{d}_1)\Big]$$

The monotonicity and concavity result follows.

Case IV: $\mathcal{F}_1^+(\boldsymbol{\omega}) \subset \mathcal{F}_2^+(\boldsymbol{\omega}) \subset \mathcal{F}_3^+(\boldsymbol{\omega})$ for some $\boldsymbol{\omega}$.

The proof is a combination of case 2 and case 3:

$$\mathcal{D}_i^3(\mathbf{Z}_{-\boldsymbol{i}}) - \mathcal{D}_i^2(\mathbf{Z}_{-\boldsymbol{i}}) < \sum_{\boldsymbol{\omega}_{-i}} \Pr(\boldsymbol{\omega}_{-\boldsymbol{i}})\Big[\boldsymbol{\Theta}_{\boldsymbol{i}+}^{\boldsymbol{2}}(\mathbb{1}_+ - \boldsymbol{a}_+^{\boldsymbol{2}})(\bar{d}_3 - \bar{d}_2) + \boldsymbol{\Theta}_{\boldsymbol{i}-}^{\boldsymbol{2}}\mathbb{1}_-(\bar{d}_3 - \bar{d}_2)\Big]$$
$$\mathcal{D}_i^2(\mathbf{Z}_{-\boldsymbol{i}}) - \mathcal{D}_i^1(\mathbf{Z}_{-\boldsymbol{i}}) > \sum_{\boldsymbol{\omega}_{-i}} \Pr(\boldsymbol{\omega}_{-\boldsymbol{i}})\Big[\boldsymbol{\Theta}_{\boldsymbol{i}+}^{\boldsymbol{2}}(\mathbb{1}_+ - \boldsymbol{a}_+^{\boldsymbol{2}})(\bar{d}_2 - \bar{d}_1) + \boldsymbol{\Theta}_{\boldsymbol{i}-}^{\boldsymbol{2}}\mathbb{1}_-(\bar{d}_2 - \bar{d}_1)\Big]$$

The monotonicity and concavity result follows.

Because $\mathcal{F}_1^+(\boldsymbol{\omega}) \subseteq \mathcal{F}_2^+(\boldsymbol{\omega}) \subseteq \mathcal{F}_3^+(\boldsymbol{\omega})$ for all $\boldsymbol{\omega} \in \Omega$, Case I-IV (or some combination of them) exhaust all the possibilities. Intuitively, the proof shows that the network risk-taking distortion is increasing in $\bar{d}$, but at a slower rate. This is because the change of $\bar{d}$ makes some insolvent banks "solvenet" and this decreases the marginal effect of $\bar{d}$. □

PROOF OF PROPOSITION 3: By lemma 1, the Nash Equilibrium for risk exposure $\mathbf{Z}^*$ is a supermodular game. By lemma 3, bank's expected profit exhibits an increasing difference in Z_i and $\bar{d}$. Then the Pareto-dominant equilibrium risk exposure is increasing in $\bar{d}$ (Milgrom and Roberts (1990), theorem 6). $\square$

PROOF OF PROPOSITION 4: Let's separately analyze the two types of networks.

(a) Complete Network

In a complete network, failed banks are either all solvent or all insolvent. That means we have either $\mathcal{F}^+(\boldsymbol{\omega}) = \mathcal{F}(\boldsymbol{\omega})$ or $\mathcal{F}^+(\boldsymbol{\omega}) = \varnothing$. Let's solve the payment equilibrium (equation 16 and 17) in those two types of states of nature.

1. For $\boldsymbol{\omega}$ where $\mathcal{F}^+(\boldsymbol{\omega}) = \mathcal{F}(\boldsymbol{\omega})$ (i.e., failed banks are solvent),

   If $\omega_i = f$, then $a_i(\boldsymbol{\omega}) = 1$ and $b_i(\boldsymbol{\omega}) = 1/(1 - \sum_{j\in\mathcal{F}_\omega} \theta_{ij})$.

2. For $\boldsymbol{\omega}$ where $\mathcal{F}^+(\boldsymbol{\omega}) = \varnothing$ (i.e., failed banks are insolvent),

   If $\omega_i = f$, then $a_i(\boldsymbol{\omega}) = \sum_{\omega_j=s} \theta_{ij}$ and $b_i(\boldsymbol{\omega}) = 1$.

By definition, a bank is solvent if $a_i\bar{d} - b_i v \geqslant 0$. Plugging the solution in case 1, we know $\mathcal{F}(\boldsymbol{\omega})^+ = \mathcal{F}(\boldsymbol{\omega})$ if and only if $\bar{d} \geqslant 1/(1 - \sum_{j\in\mathcal{F}_\omega} \theta_{ij}) \cdot v$. We can hence solve the payment equilibrium as

$$d_i^C(\boldsymbol{\omega}) = \begin{cases} \bar{d} & \forall \quad \omega_i = s \\ \left(\bar{d} - \frac{1}{\sum_{\omega_j\in s}\theta_{ij}} v\right)^+ & \forall \quad \omega_i = f \end{cases}$$

where $1/\sum_{\omega_j=s}\theta_{ij} = (N-1)$ / # of successful banks. We can rewrite the network risk-taking distortion as

$$\begin{aligned}\mathcal{D}_i^C(\mathbf{Z}_{-i}) &= \sum_{m=1}^{N-1} \Bigg(\bar{d} - \underbrace{\left(\bar{d} - \frac{N-1}{N-m} v\right)^+ \cdot \frac{m}{N-1}}_{\text{payment from failed banks}} - \underbrace{\bar{d} \cdot \frac{N-1-m}{N-1}}_{\text{payment from successful banks}}\Bigg) \cdot \Pr(m \text{ banks failed}) \\ &= \sum_{m=1}^{N-1} \min\left(\frac{m \cdot v}{N-m}, \frac{m \cdot \bar{d}}{N-1}\right) \cdot \Pr(m \text{ banks failed}) \qquad (23)\end{aligned}$$

where

$$\Pr(m \text{ banks failed}) = \binom{N-1}{m} \Big(1 - P(Z_{-i})\Big)^m \Big(P(Z_{-i})\Big)^{N-1-m}$$

(b) Ring Network

For a failed bank, there are three scenarios: (1) its debtor succeeds, (2) its debtor failed but solvent, and (3) its debtor failed and insolvent. Let's solve the payment equilibrium (equation 16 and 17) in those three types of states of nature.

1. For $i \in \mathcal{F}$ with $\omega_{i-1} \in \mathcal{S}(\boldsymbol{\omega})$,

$$a_i(\boldsymbol{\omega}) = 1 \text{ and } b_i(\boldsymbol{\omega}) = 1.$$

2. For $i \in \mathcal{F}$ with $\omega_{i-1} \in \mathcal{F}^+(\boldsymbol{\omega})$,

$$a_i(\boldsymbol{\omega}) = a_{i-1}(\boldsymbol{\omega}) \text{ and } b_i(\boldsymbol{\omega}) = b_{i-1}(\boldsymbol{\omega}) + 1.$$

3. For $i \in \mathcal{F}$ with $\omega_{i-1} \in \mathcal{F}^-(\boldsymbol{\omega})$,

$$a_i(\boldsymbol{\omega}) = 0 \text{ and } b_i(\boldsymbol{\omega}) = 1.$$

By induction, we have

$$d_i^R(\boldsymbol{\omega}) = \begin{cases} \bar{d} & \forall \quad \omega_i = s \\ \left(\bar{d} - K_i(\boldsymbol{\omega})v\right)^+ & \forall \quad \omega_i = f \end{cases}$$

where $K_i(\boldsymbol{\omega}) \equiv \min\{o : \omega_{i-o} = s\}$ is the number of failed debtors in the chain before reaching the first successful bank. Conditioning on m banks fail, the total interbank payments received by bank i is

$$\sum_j \theta_{ij} d_j^R(\boldsymbol{\omega}) = \begin{cases} \bar{d} & \text{w.p.} \quad \binom{N-2}{N-2-m} \Big/ \binom{N-1}{m} \\ \left(\bar{d} - v\right)^+ & \text{w.p.} \quad \binom{N-3}{N-2-m} \Big/ \binom{N-1}{m} \\ \ldots\ldots & \\ \left(\bar{d} - mv\right)^+ & \text{w.p.} \quad \binom{N-2-m}{N-2-m} \Big/ \binom{N-1}{m} \end{cases} \tag{24}$$

Equation 24 has a clean interpretation. The first line corresponds to the scenario where i's direct debtor succeeded. In this case, bank i will receive an interbank payment of $\bar{d}$. Conditioning on m banks failed, the probability of this scenario is $\binom{N-2}{N-2-m} \Big/ \binom{N-1}{m}$. Similarly, the second line corresponds to the scenario where i's direct debtor failed but its debtor's debtor succeeded. In this case, bank i will receive an interbank payment of $(\bar{d} - v)^+$. The probability of this scenario is $\binom{N-3}{N-2-m} \Big/ \binom{N-1}{m}$. The same logic applies till all m banks failed. It is easy to confirm by Hockey-stick identity (Lemma I.A) that the total probability in equation 24 is one. Taking the expectation, the network risk-taking distortion of a ring network is

$$\mathcal{D}_i^R(\mathbf{Z}_{-i}) = \sum_{m=1}^{N-1} \left[\bar{d} - \sum_{l=0}^{m} \left(\bar{d} - lv\right)^+ \binom{N-2-l}{N-2-m} \Big/ \binom{N-1}{m} \right] \cdot \Pr(m \text{ banks failed})$$

Let's compare it with the network distortion of a bank in a complete network:

$$\begin{aligned}
\mathcal{D}_i^R(\mathbf{Z}_{-i}) \leqslant & \sum_{m=1}^{N-1} \left[\bar{d} - \left(\sum_{l=0}^{m} \left(\bar{d} - lv\right) \binom{N-2-l}{N-2-m} \Big/ \binom{N-1}{m} - \bar{d} \cdot \frac{N-1-m}{N-1} \right)^+ - \bar{d} \cdot \frac{N-1-m}{N-1} \right] \\
& \cdot \Pr(m \text{ banks failed}) \qquad \textbf{(\textit{By lemma I.B})} \\
= & \sum_{m=1}^{N-1} \left(\bar{d} - \left(\bar{d} - \frac{N-1}{N-m} v \right)^+ \cdot \frac{m}{N-1} - \bar{d} \cdot \frac{N-1-m}{N-1} \right) \cdot \Pr(m \text{ banks failed}) \qquad \textbf{(\textit{By lemma I.A})} \\
= & \mathcal{D}_i^C(\mathbf{Z}_{-i})
\end{aligned}$$

It's worth noting that $\mathcal{D}_i^R(\mathbf{Z}_{-i}) = \mathcal{D}_i^C(\mathbf{Z}_{-i}) = \mathcal{D}_i^{max}(\mathbf{Z}_{-i})$ if $\bar{d} - mv \geqslant 0$ for all m. A necessary and sufficient condition is $\bar{d} \geqslant (N-1)v$. This observation confirms Proposition 5: there is an upper bound for the network risk-taking distortion and the upper bound is independent of the network topology.

Finally, by monotone selection theorem, the equilibrium risk exposure of banks in a complete network is larger than that of banks in a ring network. □

PROOF OF PROPOSITION 5:

Part (a): Let $\widetilde{\mathbf{\Theta}}$ denote the largest path-connected sub-network of $\mathbf{\Theta}$ where bank i belongs to. Suppose when $\bar{d} = \bar{d}_1$, all failed banks in this sub-network are solvent in any state of nature. This means $\widetilde{\mathbf{\Theta}}_{++}\mathbb{1}_+ + \widetilde{\mathbf{\Theta}}_{+s}\mathbb{1}_s = \mathbb{1}_+$. As a result, equation 20 becomes $\boldsymbol{a}_+(\boldsymbol{\omega}) = (\boldsymbol{I}_+ - \widetilde{\mathbf{\Theta}}_{++})^{-1}\widetilde{\mathbf{\Theta}}_{+s}\mathbb{1}_s = \mathbb{1}_+$ for all $\boldsymbol{\omega}$. If this is the case, equation 21 implies $\mathcal{D}_i(\boldsymbol{Z}_{-\boldsymbol{i}}; \bar{d}_2) - \mathcal{D}_i(\boldsymbol{Z}_{-\boldsymbol{i}}; \bar{d}_1) = 0$ for all $\bar{d}_2 > \bar{d}_1$.

To show the upper bound exists, it remains to prove that $\bar{d}_1$ exits: i.e., there exists a $\bar{d}_1$ such that all failed banks in $\widetilde{\mathbf{\Theta}}$ are solvent in any state of nature. Because $\widetilde{\mathbf{\Theta}}$ is path-connected by construction, there is a chain $\{j, a, b, c, ..., i\}$ from any failed bank j to the successful bank i. Then consider

$$\bar{d}_j^{max} = \frac{1}{\widetilde{\theta}_{bc}}(\frac{1}{\widetilde{\theta}_{ab}}(\frac{1}{\widetilde{\theta}_{ja}} \cdot v_j + v_a) + v_b) + v_c + ...$$

Clearly, $\bar{d}_j^{max}$ is finite because the network is path-connected ($\widetilde{\theta}_{ja}$, $\widetilde{\theta}_{ab}$, $\widetilde{\theta}_{bc}$.. are all strictly positive). Suppose $\bar{d} = \bar{d}_j^{max}$, then even when any bank outside this chain failed and insolvent (i.e., unable to contribute to the chain), bank j can fulfill its deposits and become solvent. Intuitively, that means $\bar{d}$ is so large that bank i can itself bail out bank j even though they may not be directly connected. Then let's define

$$\bar{d}^{max} = \max_j \bar{d}_j^{max}$$

When $\bar{d}_1 = \bar{d}^{max}$, then in any state of nature, all failed bank are solvent. This completes the proof. □

Part (b): From part (a), $\mathcal{D}_i(\boldsymbol{Z}_{-\boldsymbol{i}})$ reaches the maximum when every failed banks are 'solvent" in all possible states of nature. In this case, we can rewrite failed banks' equilibrium payment (equation 13) as

$$\boldsymbol{d}_{\boldsymbol{f}}^*(\boldsymbol{\omega}) = \mathbf{\Theta}_{\boldsymbol{ff}}\boldsymbol{d}_{\boldsymbol{f}}^*(\boldsymbol{\omega}) + \mathbf{\Theta}_{\boldsymbol{fs}}\mathbb{1}_s\bar{d} - \mathbb{1}_f v \qquad \forall \boldsymbol{\omega}$$

It implies

$$\boldsymbol{d}_{\boldsymbol{f}}^*(\boldsymbol{\omega}) = (\mathbf{I}_f - \mathbf{\Theta}_{\boldsymbol{ff}})^{-1}(\mathbf{\Theta}_{\boldsymbol{fs}}\mathbb{1}_s\bar{d} - \mathbb{1}_f v) = \mathbb{1}_f\bar{d} - (\mathbf{I}_f - \mathbf{\Theta}_{\boldsymbol{ff}})^{-1}\mathbb{1}_f v \qquad \forall \boldsymbol{\omega}$$

The interbank payments received by the successful banks are

$$\mathbf{\Theta}_{\boldsymbol{sf}}\boldsymbol{d}_{\boldsymbol{f}}^*(\boldsymbol{\omega}) + \mathbf{\Theta}_{\boldsymbol{ss}}\mathbb{1}_s\bar{d} = \mathbb{1}_s\bar{d} - \mathbf{\Theta}_{\boldsymbol{sf}}(\mathbf{I}_f - \mathbf{\Theta}_{\boldsymbol{ff}})^{-1}\mathbb{1}_f v \qquad \forall \boldsymbol{\omega}$$

That means successful banks' network distortion vector in state $\boldsymbol{\omega}$ is $\vec{\mathcal{D}}(\boldsymbol{\omega}) = \mathbf{\Theta}_{\boldsymbol{sf}}(\mathbf{I}_f - \mathbf{\Theta}_{\boldsymbol{ff}})^{-1}\mathbb{1}_f v$. By the network symmetry, the expected distortion conditional on the set $\boldsymbol{f}$ fails will be the ratio of column sum of $\vec{\mathcal{D}}(\boldsymbol{\omega})$ and the number of columns. That is

$$\mathbb{E}[\mathcal{D}_i^{max}|\text{set } f \text{ fails}] = \frac{\mathbb{1}_s'\mathbf{\Theta}_{\boldsymbol{sf}}(\mathbf{I}_f - \mathbf{\Theta}_{\boldsymbol{ff}})^{-1}\mathbb{1}_f v}{\mathbb{1}_s'\mathbb{1}_s} = \frac{\mathbb{1}_f'\mathbb{1}_f}{\mathbb{1}_s'\mathbb{1}_s} v$$

Then a bank's unconditional expected network distortion is $\sum_{\boldsymbol{f}} \frac{\mathbb{1}_f'\mathbb{1}_f}{\mathbb{1}_s'\mathbb{1}_s} v \cdot \Pr(\mathcal{F} = \boldsymbol{f})$. Again due to the symmetry, the permutation among the failed banks is irrelevant. Therefore, the maximum network risk-taking distortion is

$$\mathcal{D}_i^{max}(\boldsymbol{Z_{-i}}) = \sum_{f=1}^{N-1} \frac{f}{N-f} \cdot v \cdot \binom{N-1}{f} \Big[P(Z_{-i})\Big]^{N-1-f} \Big[1 - P(Z_{-i})\Big]^f$$

The expression has a clear interpretation. Suppose that f banks fail their projects in some state. The amount of money that needs to be cross-subsidized in the network is $f \cdot v$, which is the amount of failed banks' deposits. If the network is symmetric, each successful bank will cross-subsidize $f \cdot v/(N-f)$. The probability with which f banks fail is $\binom{N-1}{f}[P(Z_{-i})]^{N-1-f}[1-P(Z_{-i})]^f$. By binomial theorem, we can rewrite the above equation as

$$\mathcal{D}_i^{max}(\boldsymbol{Z_{-i}}) = \frac{1 - P(Z_{-i}) - [1 - P(Z_{-i})]^N}{P(Z_{-i})} \cdot v$$

It is immediate that $\mathrm{d}\mathcal{D}_i^{max}(\boldsymbol{Z_{-i}})/\,\mathrm{d}N > 0$. By monotone selection theorem, each bank's maximum risk exposure Z_i^* is increasing in the number of banks N in the network. □

PROOF OF PROPOSITION 6: Denote the central clearing counterparty (CCP) as bank 0. Because the CCP has no outside liabilities, it's always solvent. The payment equilibrium when m banks fail can be represented as

$$\begin{aligned} d_s^* &= \bar{d} \\ d_f^* &= (d_0^*/N - v)^+ \\ d_0^* &= (N-m) \cdot d_s^* + m \cdot d_f^* \end{aligned}$$

The solution to the above fixed point is

$$d_i^{CCP}(\boldsymbol{\omega}) = \begin{cases} \bar{d} & \forall \quad \omega_i = s \\ \left(\bar{d} - \frac{N}{N-m} v\right)^+ & \forall \quad \omega_i = f \end{cases}$$

As a result, the risk-taking distortion of a bank in any financial network with a CCP is

$$\begin{aligned} \mathcal{D}_i^{CCP}(\boldsymbol{Z_{-i}}) &= \sum_{m=1}^{N-1} \Bigg(\bar{d} - \underbrace{\left(\bar{d} - \frac{N}{N-m} v\right)^+ \cdot \frac{m}{N}}_{\text{payment from failed banks}} - \underbrace{\bar{d} \cdot \frac{N-m}{N}}_{\substack{\text{payment from} \\ \text{successful banks}}}\Bigg) \cdot \Pr(m \text{ banks failed}) \\ &= \sum_{m=1}^{N-1} \min\left(\frac{m \cdot v}{N-m}, \frac{m \cdot \bar{d}}{N}\right) \cdot \Pr(m \text{ banks failed}) \end{aligned} \tag{25}$$

Compare equation 25 with 23, we have $\mathcal{D}_i^{CCP}(\boldsymbol{Z_{-i}}; \bar{d}) = \mathcal{D}_i^{C}(\boldsymbol{Z_{-i}}; \frac{N-1}{N}\bar{d})$. As a result, the risk-taking equilibrium of a network with a CCP is equivalent to that of a complete network with $(\frac{N-1}{N}\bar{d}, \boldsymbol{\Theta^C})$. □

PROOF OF PROPOSITION 7: With the cash flow distribution in equation 7, Lemma 2 still holds: the payment equilibrium $\boldsymbol{d}^*(\boldsymbol{\omega}; \boldsymbol{Z})$ is constant in the risk exposure vector $\boldsymbol{Z}$. To see why, reordering $d_1^*(\boldsymbol{\omega}; \boldsymbol{Z}), ...d_N^*(\boldsymbol{\omega}; \boldsymbol{Z})$ in equation 2 gives us,

$$
\begin{aligned}
&d_i^*(\boldsymbol{\omega};\boldsymbol{Z}) = \bar{d} && \forall \omega_i = \text{good}\\
&d_i^*(\boldsymbol{\omega};\boldsymbol{Z}) = \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega};\boldsymbol{Z}) + \beta - v && \forall \omega_i = \text{middle}\\
&d_i^*(\boldsymbol{\omega};\boldsymbol{Z}) = \Big(\sum_j \theta_{ij} d_j^*(\boldsymbol{\omega};\boldsymbol{Z}) + \gamma - v\Big)^+ && \forall \omega_i = \text{bad}
\end{aligned}
$$

As a result, the fixed point solution $\boldsymbol{d}^*(\boldsymbol{\omega};\boldsymbol{Z})$ is constant in $\boldsymbol{Z}$.

From the payment equilibrium (equation 2), we can find that the expected profit in the contagion state $\big[\beta - v - \big(d_i^*(\boldsymbol{\omega}^{\boldsymbol{i}=\boldsymbol{\beta}}) - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}^{\boldsymbol{i}=\boldsymbol{\beta}})\big)\big]^+ = \beta - v - \big(d_i^*(\boldsymbol{\omega}^{\boldsymbol{i}=\boldsymbol{\beta}}) - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}^{\boldsymbol{i}=\boldsymbol{\beta}})\big)$. Intuitively, $d_i^*(\boldsymbol{\omega}^{\boldsymbol{i}=\boldsymbol{\beta}})$ will adjust endogenously in the payment equilibrium such that the previous expression is always non-negative. Define

$$
\begin{aligned}
\mathcal{D}_\alpha(\boldsymbol{Z}_{-\boldsymbol{i}}) &\equiv \sum_{\boldsymbol{\omega}_{-\boldsymbol{i}}} \Big(\bar{d} - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}^{\boldsymbol{i}=\boldsymbol{\alpha}})\Big) \cdot \Pr(\boldsymbol{\omega}_{-\boldsymbol{i}})\\
\mathcal{D}_\beta(\boldsymbol{Z}_{-\boldsymbol{i}}) &\equiv \sum_{\boldsymbol{\omega}_{-\boldsymbol{i}}} \Big(d_i^*(\boldsymbol{\omega}^{\boldsymbol{i}=\boldsymbol{\beta}}) - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}^{\boldsymbol{i}=\boldsymbol{\beta}})\Big) \cdot \Pr(\boldsymbol{\omega}_{-\boldsymbol{i}})
\end{aligned}
$$

Proposition 1 proved $\mathrm{d}\mathcal{D}_\alpha(\boldsymbol{Z}_{-\boldsymbol{i}})/\,\mathrm{d}Z_m > 0$. To find the sign of $\mathrm{d}\mathcal{D}_\beta(\boldsymbol{Z}_{-\boldsymbol{i}})/\,\mathrm{d}Z_m$, let's examine $\mathcal{D}_\beta(\boldsymbol{Z}_{-\boldsymbol{i}})$ when we change Z_m to Z'_m with $Z'_m > Z_m$. Let $\boldsymbol{Z'}_{-\boldsymbol{i}}$ denote the new risk-exposure vector that differs from $\boldsymbol{Z}_{-\boldsymbol{i}}$ only in Z_m. We have

$$
\begin{aligned}
&\mathcal{D}_\beta(\boldsymbol{Z'}_{-\boldsymbol{i}}) - \mathcal{D}_\beta(\boldsymbol{Z}_{-\boldsymbol{i}})\\
&= \sum_{\boldsymbol{\omega}_{-\boldsymbol{i}-\boldsymbol{m}}} \Pr(\boldsymbol{\omega}_{-\boldsymbol{i}-\boldsymbol{m}}) \cdot \bigg\{ P_\alpha(Z'_m)\Big(d_i^*(\boldsymbol{\omega}^{\boldsymbol{m}=\boldsymbol{\alpha}}) - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}^{\boldsymbol{m}=\boldsymbol{\alpha}})\Big)\\
&\qquad + P_\beta(Z'_m)\Big(d_i^*(\boldsymbol{\omega}^{\boldsymbol{m}=\boldsymbol{\beta}}) - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}^{\boldsymbol{m}=\boldsymbol{\beta}})\Big) + P_\gamma(Z'_m)\Big(d_i^*(\boldsymbol{\omega}^{\boldsymbol{m}=\boldsymbol{\gamma}}) - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}^{\boldsymbol{m}=\boldsymbol{\gamma}})\Big)\bigg\}\\
&\qquad - \sum_{\boldsymbol{\omega}_{-\boldsymbol{i}-\boldsymbol{m}}} \Pr(\boldsymbol{\omega}_{-\boldsymbol{i}-\boldsymbol{m}}) \cdot \bigg\{ P_\alpha(Z_m)\Big(d_i^*(\boldsymbol{\omega}^{\boldsymbol{m}=\boldsymbol{\alpha}}) - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}^{\boldsymbol{m}=\boldsymbol{\alpha}})\Big)\\
&\qquad\quad + P_\beta(Z_m)\Big(d_i^*(\boldsymbol{\omega}^{\boldsymbol{m}=\boldsymbol{\beta}}) - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}^{\boldsymbol{m}=\boldsymbol{\beta}})\Big) + P_\gamma(Z_m)\Big(d_i^*(\boldsymbol{\omega}^{\boldsymbol{m}=\boldsymbol{\gamma}}) - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}^{\boldsymbol{m}=\boldsymbol{\gamma}})\Big)\bigg\}\\
&= \sum_{\boldsymbol{\omega}_{-\boldsymbol{i}-\boldsymbol{m}}} \Pr(\boldsymbol{\omega}_{-\boldsymbol{i}-\boldsymbol{m}}) \cdot\\
&\qquad \bigg\{ \underbrace{\Big(P_\alpha(Z'_m) - P_\alpha(Z_m)\Big)}_{<0} \cdot \underbrace{\Big[\Big(d_i^*(\boldsymbol{\omega}^{\boldsymbol{m}=\boldsymbol{\alpha}}) - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}^{\boldsymbol{m}=\boldsymbol{\alpha}})\Big) - \Big(d_i^*(\boldsymbol{\omega}^{\boldsymbol{m}=\boldsymbol{\gamma}}) - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}^{\boldsymbol{m}=\boldsymbol{\gamma}})\Big)\Big]}_{<0}\\
&\qquad + \underbrace{\Big(P_\beta(Z'_m) - P_\beta(Z_m)\Big)}_{<0} \cdot \underbrace{\Big[\Big(d_i^*(\boldsymbol{\omega}^{\boldsymbol{m}=\boldsymbol{\beta}}) - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}^{\boldsymbol{m}=\boldsymbol{\beta}})\Big) - \Big(d_i^*(\boldsymbol{\omega}^{\boldsymbol{m}=\boldsymbol{\gamma}}) - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}^{\boldsymbol{m}=\boldsymbol{\gamma}})\Big)\Big]}_{<0}\bigg\}\\
&> 0
\end{aligned}
$$

The inequality $\Big(d_i^*(\boldsymbol{\omega}^{\boldsymbol{m}=\boldsymbol{\alpha}}) - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}^{\boldsymbol{m}=\boldsymbol{\alpha}})\Big) - \Big(d_i^*(\boldsymbol{\omega}^{\boldsymbol{m}=\boldsymbol{\gamma}}) - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}^{\boldsymbol{m}=\boldsymbol{\gamma}})\Big) < 0$ means that bank

i's cross-subsidies to other banks are smaller when bank m succeeds compared with when it fails. To see why this is true, define $H(\omega_m) \equiv d_i^*(\boldsymbol{\omega}_{-m}; \omega_m) - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}_{-m}; \omega_m)$. From equation 1, $H(\omega_m) = \min[\beta - v, \bar{d} - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}_{-m}; \omega_m)]$. By auxiliary lemma, we find $H(\omega_m = \alpha) < H(\omega_m = \gamma)$ and $H(\omega_m = \beta) < H(\omega_m = \gamma)$. From here, we proved $\mathrm{d}\mathcal{D}_\beta(\mathbf{Z}_{-i})/\,\mathrm{d}Z_m > 0$. Finally,

$$\frac{\mathrm{d}^2\mathbb{E}\Big[\Pi_i(\boldsymbol{\omega}; \mathbf{Z})\Big]}{\mathrm{d}Z_i\,\mathrm{d}Z_m} = \underbrace{\frac{\mathrm{d}^2}{\mathrm{d}Z_i\,\mathrm{d}Z_m}\Big\{P_\alpha(Z_i)\Big(Z_i - v - \mathcal{D}_\alpha(\mathbf{Z}_{-i})\Big)\Big\}}_{>0,\text{ by proposition 1}} + P_\beta'(Z_i)\Big(-\frac{\mathrm{d}\mathcal{D}_\beta(\mathbf{Z}_{-i})}{\mathrm{d}Z_m}\Big) > 0$$

The game is supermodular and hence $\mathbf{Z}$ is strategically complementary. □

PROOF OF PROPOSITION 8: Define $\boldsymbol{\Omega}_{-i}^{f+}$ as the set of counterparty state $\boldsymbol{\omega}_{-i}$ such that bank i does not default while failing its project: superscript f denotes that bank i fails its project, and superscript "+" denotes that bank i does not default. Formally,

$$\boldsymbol{\Omega}_{-i}^{f+} \equiv \Big\{\boldsymbol{\omega}_{-i}\Big|\sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}^{i=f}) - d_i^*(\boldsymbol{\omega}^{i=f}) - v > 0\Big\}$$

where $\boldsymbol{\omega}^{i=f} \equiv (\omega_1 ..., \omega_{i-1}, f, \omega_{i+1}, ...\omega_N)$, a vector that appends bank i's failure to other banks' states of nature $\boldsymbol{\omega}_{-i}$. Because $d_i^*(\boldsymbol{\omega})$ is constant in $\mathbf{Z}$ for all $\boldsymbol{\omega}$, $\boldsymbol{\Omega}_{-i}^{f+}$ is constant in $\mathbf{Z}$.[18] We can rewrite bank i's expected profit as

$$\begin{aligned}
\mathbb{E}\Big[\Pi_i(\boldsymbol{\omega}; \mathbf{Z})\Big] = \quad & P(Z_i)\cdot(Z_i - v) \quad + \\
& P(Z_i) \sum_{\boldsymbol{\omega}_{-i} \notin \boldsymbol{\Omega}_{-i}^{f+}} -\Big(\bar{d}_i - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}^{i=s})\Big)\cdot \Pr(\boldsymbol{\omega}_{-i}) \quad + \\
& P(Z_i) \sum_{\boldsymbol{\omega}_{-i} \in \boldsymbol{\Omega}_{-i}^{f+}} -\Big(\bar{d}_i - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}^{i=s})\Big)\cdot \Pr(\boldsymbol{\omega}_{-i}) \quad + \\
& \Big(1 - P(Z_i)\Big) \sum_{\boldsymbol{\omega}_{-i} \in \boldsymbol{\Omega}_{-i}^{f+}} \Big[0 - v - \Big(d_i^*(\boldsymbol{\omega}^{i=f}) - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}^{i=f})\Big)\Big]\cdot \Pr(\boldsymbol{\omega}_{-i})
\end{aligned}$$

The first line represents the expected profit of a stand-alone bank. The second and third lines split bank i's cross-subsidies (which can be negative here) by whether $\boldsymbol{\omega}_{-i} \in \boldsymbol{\Omega}_{-i}^{f+}$. The last line represents bank i's expected downside payoff, which is unique to nonregular networks and is new here. Equation 2, combining with the definition of $\boldsymbol{\Omega}_{-i}^{f+}$, implies that $d_i^*(\boldsymbol{\omega}^{i=f}) = \bar{d}_i$ for all $\boldsymbol{\omega}_{-i} \in \boldsymbol{\Omega}_{-i}^{f+}$. Intuitively, bank profit is junior to interbank debt. Hence, bank i will pay its interbank debt in full if it does not default. This also implies $\sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}^{i=f}) = \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}^{i=s})$ for all $\boldsymbol{\omega}_{-i} \in \boldsymbol{\Omega}_{-i}^{f+}$, because bank i's state of nature

[18]Note that the probability $\Pr(\boldsymbol{\omega}_{-i} \in \boldsymbol{\Omega}_{-i}^{f+})$ is a function of $\mathbf{Z}_{-i}$. The proof that $d_i^*(\boldsymbol{\omega})$ is constant in $\mathbf{Z}$ is nearly identical to lemma 2, with the only exception that the second line of equation 13 is replaced with $d_i^*(\boldsymbol{\omega}; \mathbf{Z}) = \{\min[\sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}; \mathbf{Z}) - v, \bar{d}_i]\}^+$.

becomes irrelevant for the payment equilibrium. With those results, we have

$$
\begin{aligned}
\frac{\mathrm{d}^2\mathbb{E}\Big[\Pi_i(\boldsymbol{\omega};\mathbf{Z})\Big]}{\mathrm{d}Z_i\,\mathrm{d}Z_m} = \;& \underbrace{\frac{\mathrm{d}^2}{\mathrm{d}Z_i\,\mathrm{d}Z_m}\Big\{P(Z_i)\cdot Z_i\Big\}}_{=0} \\
& + \frac{\mathrm{d}^2}{\mathrm{d}Z_i\,\mathrm{d}Z_m}\Big\{P(Z_i)\sum_{\boldsymbol{\omega}_{-i}\notin\boldsymbol{\Omega}^{f+}_{-i}}\Big[-v-\Big(\bar{d}_i-\sum_j\theta_{ij}d^*_j(\boldsymbol{\omega}^{i=s})\Big)\Big]\cdot\Pr(\boldsymbol{\omega}_{-i})\Big\} \\
& \underbrace{\begin{aligned}+ \frac{\mathrm{d}^2}{\mathrm{d}Z_i\,\mathrm{d}Z_m}\Bigg\{&P(Z_i)\sum_{\boldsymbol{\omega}_{-i}\in\boldsymbol{\Omega}^{f+}_{-i}}\Big[-v-\Big(\bar{d}_i-\sum_j\theta_{ij}d^*_j(\boldsymbol{\omega}^{i=s})\Big)\Big]\cdot\Pr(\boldsymbol{\omega}_{-i}) \\
&+\Big(1-P(Z_i)\Big)\sum_{\boldsymbol{\omega}_{-i}\in\boldsymbol{\Omega}^{f+}_{-i}}\Big[-v-\Big(\bar{d}_i-\sum_j\theta_{ij}d^*_j(\boldsymbol{\omega}^{i=s})\Big)\Big]\cdot\Pr(\boldsymbol{\omega}_{-i})\Bigg\}\end{aligned}}_{=0}
\end{aligned}
\tag{26}
$$

Note that $\sum_{\boldsymbol{\omega}_{-i}\in\boldsymbol{\Omega}^{f+}_{-i}}\Big[-v-\Big(\bar{d}_i-\sum_j\theta_{ij}d^*_j(\boldsymbol{\omega}^{i=s})\Big)\Big]\cdot\Pr(\boldsymbol{\omega}_{-i})$ is constant in Z_i. As a result, the last two lines cancel out. Intuitively, in a world where bank i earns a positive profit regardless of whether its project succeeds or not, its interbank claims become an independent asset that does not affect its risk-taking incentive. Finally,

$$
\frac{\mathrm{d}^2\mathbb{E}\Big[\Pi_i(\boldsymbol{\omega};\mathbf{Z})\Big]}{\mathrm{d}Z_i\,\mathrm{d}Z_m} = -P'(Z_i)\cdot\underbrace{\frac{\mathrm{d}}{\mathrm{d}Z_m}\Big\{\sum_{\boldsymbol{\omega}_{-i}\notin\boldsymbol{\Omega}^{f+}_{-i}}\Big[v+\Big(\bar{d}_i-\sum_j\theta_{ij}d^*_j(\boldsymbol{\omega}^{i=s})\Big)\Big]\cdot\Pr(\boldsymbol{\omega}_{-i})\Big\}}_{>0} > 0
$$

The proof that the second part is greater than zero is similar to equation 15, but with a slight twist to account for the fact that that the counterparty risk Z_m can increase the probability of bank i defaulting (i.e., $\boldsymbol{\omega}_{-i}\notin\boldsymbol{\Omega}^{f+}_{-i}$). To formally prove it, define $\mathcal{D}^*_i(\mathbf{Z}_{-i})\equiv\sum_{\boldsymbol{\omega}_{-i}\notin\boldsymbol{\Omega}^{f+}_{-i}}\Big[v+\Big(\bar{d}_i-\sum_j\theta_{ij}d^*_j(\boldsymbol{\omega}^{i=s})\Big)\Big]\cdot\Pr(\boldsymbol{\omega}_{-i})$. We have

$$
\begin{aligned}
&\mathcal{D}^*_i(\mathbf{Z}_{-i}) \\
&= \underbrace{\sum_{\substack{\boldsymbol{\omega}^{m=s}_{-i}\notin\boldsymbol{\Omega}^{f+}_{-i}\\ \boldsymbol{\omega}^{m=f}_{-i}\notin\boldsymbol{\Omega}^{f+}_{-i}}}\Pr(\boldsymbol{\omega}_{-i-m})\Big[\Big(1-P(Z_m)\Big)\Big(v+\bar{d}_i-\sum_j\theta_{ij}d^*_j(\boldsymbol{\omega}^{\substack{i=s\\m=f}})\Big)+P(Z_m)\Big(v+\bar{d}_i-\sum_j\theta_{ij}d^*_j(\boldsymbol{\omega}^{\substack{i=s\\m=s}})\Big)\Big]}_{\boldsymbol{\omega}_{-i-m}\text{ is such that bank } i\text{, if failed, will default regardless of } \omega_m=s\text{ or }\omega_m=f} \\
&+ \underbrace{\sum_{\substack{\boldsymbol{\omega}^{m=s}_{-i}\in\boldsymbol{\Omega}^{f+}_{-i}\\ \boldsymbol{\omega}^{m=f}_{-i}\notin\boldsymbol{\Omega}^{f+}_{-i}}}\Pr(\boldsymbol{\omega}_{-i-m})\Big[\Big(1-P(Z_m)\Big)\Big(v+\bar{d}_i-\sum_j\theta_{ij}d^*_j(\boldsymbol{\omega}^{\substack{i=s\\m=f}})\Big)\Big]}_{\boldsymbol{\omega}_{-i-m}\text{ is such that bank } i\text{, if failed, will default when } \omega_m=f\text{ and not default when }\omega_m=s.}
\end{aligned}
$$

where $\boldsymbol{\omega}^{m=s}_{-i}$ ($N-1$ length) is the appended vector of $\boldsymbol{\omega}_{-i-m}$ ($N-2$ length) and $\omega_m=s$; and $\boldsymbol{\omega}^{\substack{i=s\\m=s}}$ (N length) is the appended vector of $\boldsymbol{\omega}_{-i-m}$, $\omega_m=s$, and $\omega_i=s$. Let $Z'_m>Z_m$, and let $\mathbf{Z'}_{-i}$ denotes the vector that differs from $\boldsymbol{Z_{-i}}$ only in Z_m. We have

$$
\begin{aligned}
&\mathcal{D}_i^*(\boldsymbol{Z'_{-i}}) - \mathcal{D}_i^*(\boldsymbol{Z_{-i}}) = \\
&\qquad \underbrace{\sum_{\substack{\boldsymbol{\omega}_{-i}^{m=s} \notin \boldsymbol{\Omega}_{-i}^{f+} \\ \boldsymbol{\omega}_{-i}^{m=f} \notin \boldsymbol{\Omega}_{-i}^{f+}}} \Pr(\boldsymbol{\omega_{-i-m}}) \Big[\Big(P(Z'_m) - P(Z_m) \Big) \Big(\sum_j \theta_{ij} d_j^* (\overset{\boldsymbol{i=s}}{\boldsymbol{\omega}^{\boldsymbol{m=f}}}) - \sum_j \theta_{ij} d_j^* (\overset{\boldsymbol{i=s}}{\boldsymbol{\omega}^{\boldsymbol{m=s}}}) \Big) \Big]}_{>0 \text{ (proof is identical to equation 15)}} \\
&\qquad + \sum_{\substack{\boldsymbol{\omega}_{-i}^{m=s} \in \boldsymbol{\Omega}_{-i}^{f+} \\ \boldsymbol{\omega}_{-i}^{m=f} \notin \boldsymbol{\Omega}_{-i}^{f+}}} \Pr(\boldsymbol{\omega_{-i-m}}) \underbrace{\Big(P(Z_m) - P(Z'_m) \Big)}_{>0} \underbrace{\Big(v + \bar{d}_i - \sum_j \theta_{ij} d_j^* (\overset{\boldsymbol{i=s}}{\boldsymbol{\omega}^{\boldsymbol{m=f}}}) \Big)}_{>0}
\end{aligned}
$$

The first line of the above equation, similar to Proposition 1, shows that Z_m increases bank i's upside distortion. For the scenarios in which bank i does not default if it fails, Z_m's effects on the bank's upside and downside payoffs cancel out, as Equation 26 has already shown. The second line represents the fact that Z_m can increase the probability of bank i defaulting. This further increases bank i's risk-taking distortion.[19] Intuitively, to the extent that an increase in counterparty risk makes bank i defaults (i.e., bank i's downside payoff becomes zero), the downside effect is actually smaller than the upside effect because the distortion at downside is capped at bank i's profit.

Thus, we have established $\mathrm{d}^2\mathbb{E}[\Pi_i(\boldsymbol{\omega}; \boldsymbol{Z})] / \mathrm{d}Z_i \, \mathrm{d}Z_m > 0$. The game is supermodular and $\boldsymbol{Z}$ is strategically complementary. $\square$

PROOF OF PROPOSITION 9: From equation 13, the payment vector $\boldsymbol{d}^*$ is still independent of the risk vector $\boldsymbol{Z}$ or the correlation matrix $\boldsymbol{\Phi}$. Let's compare bank i's expected profit when it chooses between ϕ_{ij} and $\widetilde{\phi}_{ij}$ with $\widetilde{\phi}_{ij} > \phi_{ij}$

$$
\begin{aligned}
&\mathbb{E}\Big[\Pi_i(\boldsymbol{\omega}; Z_i, \widetilde{\boldsymbol{\phi}}_{\boldsymbol{ij}})\Big] - \mathbb{E}\Big[\Pi_i(\boldsymbol{\omega}; Z_i, \boldsymbol{\phi_{ij}})\Big] = \\
&\qquad - \sum_{\boldsymbol{\omega}_{-i-j}} \Big(\bar{d} - \sum_l \theta_{il} d_l^* (\boldsymbol{\omega}^{\boldsymbol{i=s,j=s}}) \Big) \cdot \Pr(\boldsymbol{\omega}_{-i-j} | \omega_i = s, \omega_j = s) \cdot P(Z_j) \cdot (\widetilde{\phi}_{ij} - \phi_{ij}) \\
&\qquad + \sum_{\boldsymbol{\omega}_{-i-j}} \Big(\bar{d} - \sum_l \theta_{il} d_l^* (\boldsymbol{\omega}^{\boldsymbol{i=s,j=f}}) \Big) \cdot \Pr(\boldsymbol{\omega}_{-i-j} | \omega_i = s, \omega_j = f) \cdot P(Z_j) \cdot (\widetilde{\phi}_{ij} - \phi_{ij})
\end{aligned}
$$

Suppose $\phi_{j,k}^* = 1$ for all $k \neq i$. That implies $\Pr(\boldsymbol{\omega}_{-i-j} | \omega_i = s, \omega_j = s) = 1$ if and only if every element of $\boldsymbol{\omega}_{-i-j}$ is s. Similarly, $\Pr(\boldsymbol{\omega}_{-i-j} | \omega_i = s, \omega_j = f) = 1$ if and only if every element of $\boldsymbol{\omega}_{-i-j}$ is f.

By Auxiliary Lemma in the appendix above, $\sum_l \theta_{il} d_l^* (\boldsymbol{\omega}^{\boldsymbol{i=s,-i=s}}) \geqslant \sum_l \theta_{il} d_l^* (\boldsymbol{\omega}^{\boldsymbol{i=s,-i=f}})$. This implies bank i's expected profit is increasing in its project's dependence ϕ_{ij} with other banks. Therefore, for all $\boldsymbol{Z}$, bank i's choices of conditional dependence with bank j won't deviate from $\phi_{i,j}^* = 1$. With perfect correla-

[19] The second line being greater than zero can be proved by contradiction. Suppose $v + \bar{d}_i - \sum_j \theta_{ij} d_j^* (\overset{\boldsymbol{i=s}}{\boldsymbol{\omega}^{\boldsymbol{m=f}}}) < 0$, then d_i^* equaling $\bar{d}_i$ is consistent with the payment equilibrium (equation 2) in the state $\overset{\boldsymbol{i=f}}{\boldsymbol{\omega}^{\boldsymbol{m=f}}}$. To see why, suppose $d_i^* = \bar{d}_i$, then $d_j^* (\overset{\boldsymbol{i=s}}{\boldsymbol{\omega}^{\boldsymbol{m=f}}}) = d_j^* (\overset{\boldsymbol{i=f}}{\boldsymbol{\omega}^{\boldsymbol{m=f}}})$ for all j. This implies that $\sum_j \theta_{ij} d_j^* (\overset{\boldsymbol{i=f}}{\boldsymbol{\omega}^{\boldsymbol{m=f}}}) - v = \sum_j \theta_{ij} d_j^* (\overset{\boldsymbol{i=s}}{\boldsymbol{\omega}^{\boldsymbol{m=f}}}) - v > \bar{d}_i$. Hence by equation 2, d_i^* equaling to $\bar{d}_i$ is consistent with the payment equilibrium in the state $\overset{\boldsymbol{i=f}}{\boldsymbol{\omega}^{\boldsymbol{m=f}}}$. This contradicts $\boldsymbol{\omega}_{-i}^{m=f} \notin \boldsymbol{\Omega}_{-i}^{f+}$.

tion, the network risk-taking distortion disappears: $\mathcal{D}_i(\boldsymbol{Z}^*_{-\boldsymbol{i}}, \mathbb{1}) = 0$ for all $\boldsymbol{Z}^*_{-\boldsymbol{i}}$. Hence, the equilibrium is characterized by

$$\begin{aligned}\phi^*_{ij} &= 1 \quad \forall i, j \in \mathcal{N} \\ P'(Z^*_i)(Z^*_i - v) + P(Z^*_i) &= 0 \quad \forall i \in \mathcal{N}\end{aligned}$$

And $\rho^*_{ij} = 1$ for all i, j. □

PROOF OF PROPOSITION 10: From bank i's expected payoff, it will prefer to join the network $(\bar{d}, \Theta)$ (over stand-alone) if

$$P(Z^*_i)\Big[Z^*_i - v - \mathcal{D}_i(\boldsymbol{Z}^*_{-i})\Big] + c_i - \Big[1 - P(Z^*_i)\Big]\Pr\Big(i \in \mathcal{F}^-_{\boldsymbol{\omega}} | \omega_i = f\Big) \cdot c_i > P(Z^{**}_i)\Big[Z^{**}_i - v + c_i\Big] \tag{27}$$

where Z^*_i is equilibrium risk-taking of a bank in the network: $Z^*_i = \operatorname{argmax} \mathbb{E}\Big[\Pi_i(\boldsymbol{\omega}; \boldsymbol{Z}^*)\Big]$, and Z^{**}_i is the optimal risk-taking of a stand-alone bank: $Z^{**}_i = \operatorname{argmax} P(Z_i)(Z_i - v + c_i)$. From definition of the Nash equilibrium, the LHS of equation 27 is greater than

$$A \equiv P(Z^{**}_i)\Big[Z^{**}_i - v - \mathcal{D}_i(\boldsymbol{Z}^*_{-i})\Big] + c_i - \Big[1 - P(Z^{**}_i)\Big]\Pr\Big(i \in \mathcal{F}^-_{\boldsymbol{\omega}} | \omega_i = f\Big) \cdot c_i$$

Define the RHS of equation 27 as $B \equiv P(Z^{**}_i)\Big[Z^{**}_i - v + c_i\Big]$

$$A - B = \Big[1 - P(Z^{**}_i)\Big]\Big[1 - \Pr\Big(i \in \mathcal{F}^-_{\boldsymbol{\omega}} | \omega_i = f\Big)\Big] \cdot c_i - P(Z^{**}_i) \cdot \mathcal{D}_i(\boldsymbol{Z}^*_{-i})$$

If $\bar{d} > v$, $\Pr(i \in \mathcal{F}^-_{\boldsymbol{\omega}} | \omega_i = f) < 1$. This means that it is possible that bank i's deposits get fully fulfilled from counterparties' cross subsidies. Since Z^* and Z^{**} are bounded, there exists $\bar{c} \in \mathbb{R}^+$ such that if all $c_i > \bar{c}$, $A - B > 0$. □

PROOF OF PROPOSITION 11

Part (a): The proof is similar to that of lemma 3. In any state of nature $\boldsymbol{\omega}$, the payment vector for solvent failed banks is $\boldsymbol{d}^*_+ = \boldsymbol{\Theta}_{++} \cdot \boldsymbol{d}^*_+ + \boldsymbol{\Theta}_{+\boldsymbol{s}} \cdot \mathbb{1}_s \cdot \bar{d} - \mathbb{1}_+ \cdot v + \boldsymbol{\epsilon}_+$, or

$$\boldsymbol{d}^*_+ = (\mathbf{I}_+ - \boldsymbol{\Theta}_{++})^{-1}(\boldsymbol{\Theta}_{+\boldsymbol{s}} \cdot \mathbb{1}_s \cdot \bar{d} - \mathbb{1}_+ \cdot v + \boldsymbol{\epsilon}_+)$$

To conserve space, I suppress the state $\boldsymbol{\omega}$ in $\boldsymbol{d}^*_+(\boldsymbol{\omega})$, $\boldsymbol{\Theta}_{++}(\boldsymbol{\omega})$, $\boldsymbol{\Theta}_{+\boldsymbol{s}}(\boldsymbol{\omega})$, $\mathbb{1}_s(\boldsymbol{\omega})$ and $\boldsymbol{\epsilon}_+(\boldsymbol{\omega})$. We can again rewrite bank i's network risk-taking distortion in a matrix form as

$$\mathcal{D}_i(\boldsymbol{Z}_{-\boldsymbol{i}}) = \sum_{\boldsymbol{\omega}_{-\boldsymbol{i}}} \Pr(\boldsymbol{\omega}_{-\boldsymbol{i}})\Big[\boldsymbol{\Theta}_{\boldsymbol{i}+}(\mathbb{1}_+\bar{d} - \boldsymbol{d}^*_+) + \boldsymbol{\Theta}_{\boldsymbol{i}-}\mathbb{1}_-\bar{d}\Big] \tag{28}$$

Compare three financial systems with different sizes of equity for bank j: ϵ_{j1}, ϵ_{j2}, ϵ_{j3} where $\epsilon_{j3} - \epsilon_{j2} = \epsilon_{j2} - \epsilon_{ji} = \xi$. Denote $\boldsymbol{\epsilon_1}/\boldsymbol{\epsilon_2}/\boldsymbol{\epsilon_3}$ as the vectors of banks' equities with the j's item being $\epsilon_{j1}/\epsilon_{j2}/\epsilon_{j3}$. We need to consider the following four cases.

Case I: $\mathcal{F}^+_1(\boldsymbol{\omega}) = \mathcal{F}^+_2(\boldsymbol{\omega}) = \mathcal{F}^+_3(\boldsymbol{\omega})$ for all $\boldsymbol{\omega}$.

For all $\boldsymbol{\omega}$, $\boldsymbol{d}_+^*$ is linearly increasing in ϵ_j. This is because $\boldsymbol{d}_+^{\mathbf{3}*} - \boldsymbol{d}_+^{\mathbf{2}*} = (\mathbf{I}_+ - \boldsymbol{\Theta}_{++})^{-1}(\boldsymbol{\epsilon}_{3+} - \boldsymbol{\epsilon}_{2+})$ equals $\boldsymbol{d}_+^{\mathbf{2}*} - \boldsymbol{d}_+^{\mathbf{1}*} = (\mathbf{I}_+ - \boldsymbol{\Theta}_{++})^{-1}(\boldsymbol{\epsilon}_{2+} - \boldsymbol{\epsilon}_{1+})$. From here, we know that the network risk-taking distortion is linearly decreasing in ϵ_j.

$$\mathcal{D}_i^3(\mathbf{Z}_{-i}) - \mathcal{D}_i^2(\mathbf{Z}_{-i}) = \mathcal{D}_i^2(\mathbf{Z}_{-i}) - \mathcal{D}_i^1(\mathbf{Z}_{-i}) = \sum_{\boldsymbol{\omega}_{-i}} \Pr(\boldsymbol{\omega}_{-i})\Big[\boldsymbol{\Theta}_{i+}(\boldsymbol{d}_+^{\mathbf{1}*} - \boldsymbol{d}_+^{\mathbf{2}*})\Big] \leqslant 0$$

Case II: $\mathcal{F}_1^+(\boldsymbol{\omega}) \subset \mathcal{F}_2^+(\boldsymbol{\omega}) = \mathcal{F}_3^+(\boldsymbol{\omega})$ for some $\boldsymbol{\omega}$.

We first compare the equity $\boldsymbol{\epsilon_2}$ with $\boldsymbol{\epsilon_1}$. The condition $\mathcal{F}_1^+(\boldsymbol{\omega}) \subset \mathcal{F}_2^+(\boldsymbol{\omega})$ implies that, in $\boldsymbol{\omega}$, some otherwise insolvent failed banks for $(\bar{d}, \boldsymbol{\Theta}; \boldsymbol{\epsilon_1})$ become solvent for $(\bar{d}, \boldsymbol{\Theta}; \boldsymbol{\epsilon_2})$. Denote those banks as bank $t_1, t_2, ..., t_T$, where $T \geqslant 1$. Because payment equilibrium is continuous in ϵ_j (equation 10), there exists $\epsilon_{j1} < \tilde{\epsilon}_{j_1} < \tilde{\epsilon}_{j_2} ... < ... < \tilde{\epsilon}_{j_S} < \epsilon_{j2}$ (where $1 \leqslant S \leqslant T$) between ϵ_{j1} and ϵ_{j2}, such that when $\epsilon_j = \tilde{\epsilon}_{j_s}$, some banks are exactly solvent (i.e., just able to fulfill their deposits). In other words, those margin banks are solvent when $\epsilon_j \in (\tilde{\epsilon}_{j_s}, \tilde{\epsilon}_{j_{s+1}})$ and insolvent when $\epsilon_j \in (\tilde{\epsilon}_{j_{s-1}}, \tilde{\epsilon}_{j_s})$ respectively. Denote $\widetilde{\mathcal{D}}_i^s(\mathbf{Z}_{-i})$ as the network risk-taking distortion when $\epsilon_j = \tilde{\epsilon}_{j_s}$. We have

$$\begin{aligned}
\mathcal{D}_i^2(\mathbf{Z}_{-i}) - \widetilde{\mathcal{D}}_i^S(\mathbf{Z}_{-i}) &= \sum_{\boldsymbol{\omega}_{-i}} \Pr(\boldsymbol{\omega}_{-i})\Big[\boldsymbol{\Theta}_{i+}^{\mathbf{2}}(\tilde{\boldsymbol{d}}_+^{\mathbf{S}*} - \boldsymbol{d}_+^{\mathbf{2}*})\Big] \leqslant 0 \\
\widetilde{\mathcal{D}}_i^{s+1}(\mathbf{Z}_{-i}) - \widetilde{\mathcal{D}}_i^s(\mathbf{Z}_{-i}) &= \sum_{\boldsymbol{\omega}_{-i}} \Pr(\boldsymbol{\omega}_{-i})\Big[\widetilde{\boldsymbol{\Theta}}_{i+}^{\boldsymbol{s}}(\tilde{\boldsymbol{d}}_+^{\boldsymbol{s}*} - \tilde{\boldsymbol{d}}_+^{\boldsymbol{s+1}*})\Big] \leqslant 0 \qquad \forall s = 1, ..., S-1 \\
\widetilde{\mathcal{D}}_i^1(\mathbf{Z}_{-i}) - \mathcal{D}_i^1(\mathbf{Z}_{-i}) &= \sum_{\boldsymbol{\omega}_{-i}} \Pr(\boldsymbol{\omega}_{-i})\Big[\boldsymbol{\Theta}_{i+}^{\mathbf{1}}(\boldsymbol{d}_+^{\mathbf{1}*} - \tilde{\boldsymbol{d}}_+^{\mathbf{1}*})\Big] \leqslant 0
\end{aligned} \tag{29}$$

Summing the above equations, we can find that $\mathcal{D}_i^2(\mathbf{Z}_{-i}) - \mathcal{D}_i^1(\mathbf{Z}_{-i}) \leqslant 0$. Using the technique in case I, we can find that $\mathcal{D}_i^3(\mathbf{Z}_{-i}) - \mathcal{D}_i^2(\mathbf{Z}_{-i}) \leqslant 0$. Hence the monotonicity result follows. The proof of concavity is identical to the proof of Lemma 3.

Case III: $\mathcal{F}_1^+(\boldsymbol{\omega}) = \mathcal{F}_2^+(\boldsymbol{\omega}) \subset \mathcal{F}_3^+(\boldsymbol{\omega})$: The proof is identical to proof of Case II except we use the technique of equation 29 to prove $\mathcal{D}_i^3(\mathbf{Z}_{-i}) - \mathcal{D}_i^2(\mathbf{Z}_{-i}) \leqslant 0$.

Case IV: $\mathcal{F}_1^+(\boldsymbol{\omega}) \subset \mathcal{F}_2^+(\boldsymbol{\omega}) \subset \mathcal{F}_3^+(\boldsymbol{\omega})$: The proof is a combination of case 2 and case 3.

Because $\mathcal{F}_1^+(\boldsymbol{\omega}) \subseteq \mathcal{F}_2^+(\boldsymbol{\omega}) \subseteq \mathcal{F}_3^+(\boldsymbol{\omega})$ for all $\boldsymbol{\omega} \in \Omega$, Case I to IV or some combination of them exhaust all the possibilities. $\square$

Part (b): Taking the partial derivative of bank i's expected payoff $\mathbb{E}\Big[\Pi_i(\boldsymbol{\omega}; \mathbf{Z}, \boldsymbol{\epsilon})\Big]$ respect to Z_i and ϵ_i:

$$\frac{\mathrm{d}^2\mathbb{E}\Big[\Pi_i(\boldsymbol{\omega}; \mathbf{Z}, \boldsymbol{\epsilon})\Big]}{\mathrm{d}Z_i \, \mathrm{d}\epsilon_i} = P'(Z_i) < 0$$

Similarly,

$$\frac{\mathrm{d}^2\mathbb{E}\Big[\Pi_i(\boldsymbol{\omega}; \mathbf{Z}, \boldsymbol{\epsilon})\Big]}{\mathrm{d}Z_i \, \mathrm{d}\epsilon_j} = -P'(Z_i) \cdot \frac{\mathrm{d}\mathcal{D}_i(\mathbf{Z}_{-i}; \boldsymbol{\epsilon})}{\mathrm{d}\epsilon_j} < 0$$

The above two equations imply that $\mathbb{E}\big[\Pi_i(\boldsymbol{\omega};\mathbf{Z},\boldsymbol{\epsilon})\big]$ exhibits a decreasing difference in Z_i and ϵ_i and also a decreasing difference in Z_i and ϵ_j. By the Monotone Selection Theorem, the equilibrium Z_i^* is decreasing in ϵ_i and ϵ_j. $\square$

PROPOSITION 12: The payment vector for solvent failed banks is

$$\boldsymbol{d}^*_+(\boldsymbol{\omega};\tau) = \begin{cases} (\mathbf{I}_+ - \boldsymbol{\Theta}_{++})^{-1}(\boldsymbol{\Theta}_{+\boldsymbol{s}}\mathbb{1}_s\bar{d} + \mathbb{1}_+(\tau - v)) & \text{if} \quad \#\{l|\omega_l = f\} \geqslant \eta \\ (\mathbf{I}_+ - \boldsymbol{\Theta}_{++})^{-1}(\boldsymbol{\Theta}_{+\boldsymbol{s}}\mathbb{1}_s\bar{d} + \mathbb{1}_+(0 - v)) & \text{otherwise} \end{cases}$$

The first line corresponds to the state of nature where a bailout occurs. The second line corresponds to the other cases. Compare two bailout amount τ_1 and τ_2 with $\tau_2 - \tau_1 = \xi > 0$. We again have two cases: (1) $\mathcal{F}_2^+(\boldsymbol{\omega}) = \mathcal{F}_1^+(\boldsymbol{\omega})$ for all $\boldsymbol{\omega}$. (2) $\mathcal{F}_1^+(\boldsymbol{\omega}) \subset \mathcal{F}_2^+(\boldsymbol{\omega})$ for some $\boldsymbol{\omega}$. Let $\mathcal{B}(\boldsymbol{\omega})$ denote the bailout indicator $\mathbb{1}\big[\#\{l|\omega_l = f\} > \eta\big]$. Since $\eta < N$, $\Pr\big(\mathcal{B}(\boldsymbol{\omega}) = 1\big) > 0$.

For Case I,

$$\boldsymbol{d}^*_+(\boldsymbol{\omega};\tau_2) - \boldsymbol{d}^*_+(\boldsymbol{\omega};\tau_1) = \mathcal{B}(\boldsymbol{\omega})(\mathbf{I}_+ - \boldsymbol{\Theta}_{++})^{-1}\mathbb{1}_+\xi \quad \forall \boldsymbol{\omega} \in \boldsymbol{\Omega}$$

From equation 28,

$$\begin{aligned} &\mathcal{D}_i(\mathbf{Z}_{-\boldsymbol{i}};\tau_2) - \mathcal{D}_i(\mathbf{Z}_{-\boldsymbol{i}};\tau_1) \\ &\qquad = \sum_{\boldsymbol{\omega}_{-i}} \Pr(\boldsymbol{\omega}_{-\boldsymbol{i}})\Big[\boldsymbol{\Theta}_{\boldsymbol{i}+}(\boldsymbol{d}^{\mathbf{1}*}_+ - \boldsymbol{d}^{\mathbf{2}*}_+)\Big] = \sum_{\boldsymbol{\omega}_{-i}} -\mathcal{B}(\boldsymbol{\omega}^{i=s})\Pr(\boldsymbol{\omega}_{-\boldsymbol{i}})\Big[\boldsymbol{\Theta}_{\boldsymbol{i}+}(\mathbf{I}_+ - \boldsymbol{\Theta}_{++})^{-1}\mathbb{1}_+\xi\Big] < 0 \end{aligned}$$

The proof of case 2 is identical to case 2 of lemma 3 or proposition 11.(a), hence omitted here to avoid repetition. From here, we proved that $\mathcal{D}_i(\mathbf{Z}_{-\boldsymbol{i}})$ is decreasing in τ. If $\tau \leqslant v$, a failed bank earns zero profit, and this will preserve $\mathbb{E}\big[\Pi_i(\boldsymbol{\omega};\mathbf{Z})\big] = P(Z_i)\big(Z_i - v - \mathcal{D}_i(\mathbf{Z}_{-\boldsymbol{i}},\tau)\big)$. As a result $\mathbb{E}\big[\Pi_i(\boldsymbol{\omega};\mathbf{Z},\tau)\big]$ exhibits a decreasing difference in Z_i and τ. From the Monotone Selection Theorem, banks' equilibrium risk exposure is decreasing in τ. $\square$

Online Appendix

A. Omitted Proofs

LEMMA I.A [Hockey-stick Identity]
For all $n > r$, we have

$$\text{(i)} \quad \sum_{l=r}^{n} \binom{l}{r} = \binom{n+1}{r+1} \qquad \text{and} \qquad \text{(ii)} \quad \sum_{l=r}^{n} \binom{l}{r}(n-l) = \binom{n+1}{r+1}\frac{n-r}{r+2}$$

PROOF
We proceed by induction. For an initial $n = r+1$

$$\text{(i)} \quad \binom{r}{r} + \binom{r+1}{r} = \binom{r+2}{r+1}$$

$$\text{(ii)} \quad \binom{r}{r} * 1 + \binom{r+1}{r} * 0 = \binom{r+2}{r+1} * \frac{1}{r+2} = 1$$

The above equations are to confirm the initial conditions hold. Now suppose that for $n = k$, the two equality holds. For $n = k+1$, we have

$$\text{(i)} \quad \sum_{l=r}^{k+1} \binom{l}{r} = \sum_{l=r}^{k} \binom{l}{r} + \binom{k+1}{r} = \binom{k+1}{r+1} + \binom{k+1}{r} = \binom{k+2}{r+1}$$

$$\text{(ii)} \quad \sum_{l=r}^{k+1} \binom{l}{r}(k+1-l) = \sum_{l=r}^{k} \binom{l}{r}(k+1-l) = \binom{k+1}{r+1}\frac{k-r}{r+2} + \binom{k+1}{r+1} = \binom{k+2}{r+1}\frac{k+1-r}{r+2}$$

Q.E.D by induction. $\square$

LEMMA I.B [Triangle Inequality]
For any sequence $\{A_i\}$ and $B \in \mathbb{R}$ with $B < \max_i(A_i)$, we have

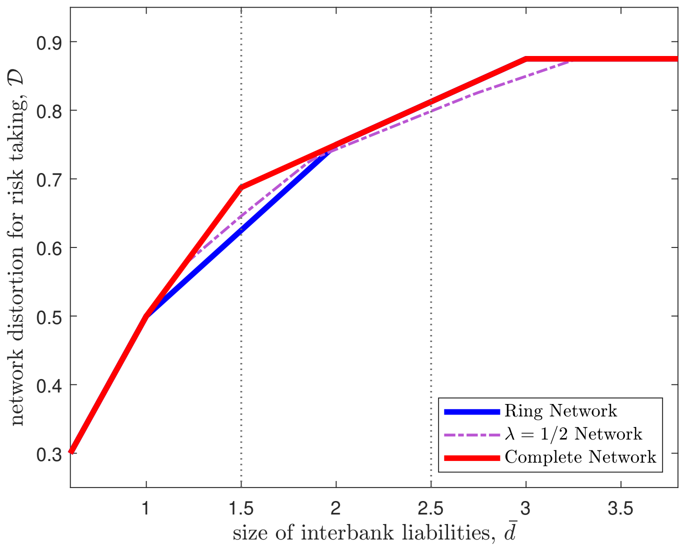
$$\sum_i \Big(A_i\Big)^+ \geqslant \Big(\sum_i A_i - B\Big)^+ + B$$

PROOF Without loss of generality, let $A_0 = \max_i(A_i)$

$$\sum_i \Big(A_i\Big)^+ - B = \sum_{i \neq 0} \Big(A_i\Big)^+ + (A_0 - B)^+ \geqslant \Big(\sum_i A_i - B\Big)^+$$

$\square$

B. Numerical Example: intermediately-connected networks



We consider a ring, a $\lambda = 0.5$, and a complete network with four banks. Let the bank of interest be bank $i = 4$. The purpose of this section is to numerically solve the network risk-taking distortion for the three kinds of networks. Let $P(Z_j) = 0.5 \quad \forall j \neq i$.

$$, \boldsymbol{\Theta}^R = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad \boldsymbol{\Theta}^\lambda = \begin{bmatrix} 0 & 1/6 & 1/6 & 2/3 \\ 2/3 & 0 & 1/6 & 1/6 \\ 1/6 & 2/3 & 0 & 1/6 \\ 1/6 & 1/6 & 2/3 & 0 \end{bmatrix}$$

$$\boldsymbol{\Theta}^C = \begin{bmatrix} 0 & 1/3 & 1/3 & 1/3 \\ 1/3 & 0 & 1/3 & 1/3 \\ 1/3 & 1/3 & 0 & 1/3 \\ 1/3 & 1/3 & 1/3 & 0 \end{bmatrix}$$

define $\boldsymbol{\theta}_4^R$, $\boldsymbol{\theta}_4^\lambda$, and $\boldsymbol{\theta}_4^C$ as vectors that represent the last row of each $\boldsymbol{\Theta}$.

(i) Small $\bar{d}$ ($\bar{d} = 1.5$)

$\boldsymbol{\omega} = (s, s, s, s)$:

$$\mathcal{D}^R(\boldsymbol{\omega}) = -\begin{pmatrix} 1.5 & 1.5 & 1.5 & 1.5 \end{pmatrix} \cdot \boldsymbol{\theta}_4^R + 1.5 = 0$$
$$\mathcal{D}^\lambda(\boldsymbol{\omega}) = -\begin{pmatrix} 1.5 & 1.5 & 1.5 & 1.5 \end{pmatrix} \cdot \boldsymbol{\theta}_4^\lambda + 1.5 = 0$$
$$\mathcal{D}^C(\boldsymbol{\omega}) = -\begin{pmatrix} 1.5 & 1.5 & 1.5 & 1.5 \end{pmatrix} \cdot \boldsymbol{\theta}_4^C + 1.5 = 0$$

$\boldsymbol{\omega} = (s, s, f, s)$:

$$\mathcal{D}^R(\boldsymbol{\omega}) = -\begin{pmatrix} 1.5 & 1.5 & 0.5 & 1.5 \end{pmatrix} \cdot \boldsymbol{\theta}_4^R + 1.5 = 1$$
$$\mathcal{D}^\lambda(\boldsymbol{\omega}) = -\begin{pmatrix} 1.5 & 1.5 & 0.5 & 1.5 \end{pmatrix} \cdot \boldsymbol{\theta}_4^\lambda + 1.5 = 2/3$$
$$\mathcal{D}^C(\boldsymbol{\omega}) = -\begin{pmatrix} 1.5 & 1.5 & 0.5 & 1.5 \end{pmatrix} \cdot \boldsymbol{\theta}_4^C + 1.5 = 1/3$$

$\boldsymbol{\omega} = (s, f, s, s)$:

$$\mathcal{D}^R(\boldsymbol{\omega}) = -\begin{pmatrix} 1.5 & 0.5 & 1.5 & 1.5 \end{pmatrix} \cdot \boldsymbol{\theta}_4^R + 1.5 = 0$$
$$\mathcal{D}^\lambda(\boldsymbol{\omega}) = -\begin{pmatrix} 1.5 & 0.5 & 1.5 & 1.5 \end{pmatrix} \cdot \boldsymbol{\theta}_4^\lambda + 1.5 = 1/6$$
$$\mathcal{D}^C(\boldsymbol{\omega}) = -\begin{pmatrix} 1.5 & 0.5 & 1.5 & 1.5 \end{pmatrix} \cdot \boldsymbol{\theta}_4^C + 1.5 = 1/3$$

$\boldsymbol{\omega} = (f, s, s, s)$:

$$\mathcal{D}^R(\boldsymbol{\omega}) = -\begin{pmatrix} 0.5 & 1.5 & 1.5 & 1.5 \end{pmatrix} \cdot \boldsymbol{\theta}_4^R + 1.5 = 0$$
$$\mathcal{D}^\lambda(\boldsymbol{\omega}) = -\begin{pmatrix} 0.5 & 1.5 & 1.5 & 1.5 \end{pmatrix} \cdot \boldsymbol{\theta}_4^\lambda + 1.5 = 1/6$$
$$\mathcal{D}^C(\boldsymbol{\omega}) = -\begin{pmatrix} 0.5 & 1.5 & 1.5 & 1.5 \end{pmatrix} \cdot \boldsymbol{\theta}_4^C + 1.5 = 1/3$$

$\boldsymbol{\omega} = (s, f, f, s)$:

$$\mathcal{D}^R(\boldsymbol{\omega}) = -\begin{pmatrix} 1.5 & 0.5 & 0 & 1.5 \end{pmatrix} \cdot \boldsymbol{\theta}_4^R + 1.5 = 1.5$$
$$\mathcal{D}^\lambda(\boldsymbol{\omega}) = -\begin{pmatrix} 1.5 & 0.25 & 0 & 1.5 \end{pmatrix} \cdot \boldsymbol{\theta}_4^\lambda + 1.5 = 29/24$$
$$\mathcal{D}^C(\boldsymbol{\omega}) = -\begin{pmatrix} 1.5 & 0 & 0 & 1.5 \end{pmatrix} \cdot \boldsymbol{\theta}_4^C + 1.5 = 1$$

$\boldsymbol{\omega} = (f, s, f, s)$:

$$\mathcal{D}^R(\boldsymbol{\omega}) = -\begin{pmatrix} 0.5 & 1.5 & 0.5 & 1.5 \end{pmatrix} \cdot \boldsymbol{\theta}_4^R + 1.5 = 1$$
$$\mathcal{D}^\lambda(\boldsymbol{\omega}) = -\begin{pmatrix} 0.3 & 1.5 & 0.3 & 1.5 \end{pmatrix} \cdot \boldsymbol{\theta}_4^\lambda + 1.5 = 1$$
$$\mathcal{D}^C(\boldsymbol{\omega}) = -\begin{pmatrix} 0 & 1.5 & 0 & 1.5 \end{pmatrix} \cdot \boldsymbol{\theta}_4^C + 1.5 = 1$$

$\boldsymbol{\omega} = (f, f, s, s)$:

$$\mathcal{D}^R(\boldsymbol{\omega}) = -\begin{pmatrix} 0.5 & 0 & 1.5 & 1.5 \end{pmatrix} \cdot \boldsymbol{\theta}_4^R + 1.5 = 0$$
$$\mathcal{D}^\lambda(\boldsymbol{\omega}) = -\begin{pmatrix} 0.25 & 0 & 1.5 & 1.5 \end{pmatrix} \cdot \boldsymbol{\theta}_4^\lambda + 1.5 = 11/24$$
$$\mathcal{D}^C(\boldsymbol{\omega}) = -\begin{pmatrix} 0 & 0 & 1.5 & 1.5 \end{pmatrix} \cdot \boldsymbol{\theta}_4^C + 1.5 = 1$$

$\boldsymbol{\omega} = (f, f, f, s)$:

$$\mathcal{D}^R(\boldsymbol{\omega}) = -\begin{pmatrix} 0.5 & 0 & 0 & 1.5 \end{pmatrix} \cdot \boldsymbol{\theta}_4^R + 1.5 = 1.5$$
$$\mathcal{D}^\lambda(\boldsymbol{\omega}) = -\begin{pmatrix} 0 & 0 & 0 & 1.5 \end{pmatrix} \cdot \boldsymbol{\theta}_4^\lambda + 1.5 = 1.5$$
$$\mathcal{D}^C(\boldsymbol{\omega}) = -\begin{pmatrix} 0 & 0 & 0 & 1.5 \end{pmatrix} \cdot \boldsymbol{\theta}_4^C + 1.5 = 1.5$$

Let m denotes the number failed banks. Conditioning on bank i succeeds, $\Pr(m = 0) = \frac{1}{8}$, $\Pr(m = 1) = \frac{3}{8}$, $\Pr(m = 2) = \frac{3}{8}$, and $\Pr(m = 3) = \frac{1}{8}$. From here we can calculate the network risk-taking distortion as

$$\mathcal{D}^R = \Pr(m = 0) \cdot 0 + \Pr(m = 1) \cdot \tfrac{1}{3} + \Pr(m = 2) \cdot \tfrac{5}{6} + \Pr(m = 3) \cdot \tfrac{3}{2} = \tfrac{5}{8}$$
$$\mathcal{D}^\lambda = \Pr(m = 0) \cdot 0 + \Pr(m = 1) \cdot \tfrac{1}{3} + \Pr(m = 2) \cdot \tfrac{8}{9} + \Pr(m = 3) \cdot \tfrac{3}{2} = \tfrac{31}{48}$$
$$\mathcal{D}^C = \Pr(m = 0) \cdot 0 + \Pr(m = 1) \cdot \tfrac{1}{3} + \Pr(m = 2) \cdot 1 + \Pr(m = 3) \cdot \tfrac{3}{2} = \tfrac{11}{16}$$

(ii) Large $\bar{d}$ ($\bar{d} = 2.5$)

$\boldsymbol{\omega} = (s, s, s, s)$:

$\mathcal{D}^R(\boldsymbol{\omega}) = -\begin{pmatrix}1.5 & 1.5 & 1.5 & 1.5\end{pmatrix} \cdot \boldsymbol{\theta}_4^R + 2.5 = 0$

$\mathcal{D}^\lambda(\boldsymbol{\omega}) = -\begin{pmatrix}1.5 & 1.5 & 1.5 & 1.5\end{pmatrix} \cdot \boldsymbol{\theta}_4^\lambda + 2.5 = 0$

$\mathcal{D}^C(\boldsymbol{\omega}) = -\begin{pmatrix}1.5 & 1.5 & 1.5 & 1.5\end{pmatrix} \cdot \boldsymbol{\theta}_4^C + 2.5 = 0$

$\boldsymbol{\omega} = (s, s, f, s)$:

$\mathcal{D}^R(\boldsymbol{\omega}) = -\begin{pmatrix}2.5 & 2.5 & 1.5 & 2.5\end{pmatrix} \cdot \boldsymbol{\theta}_4^R + 2.5 = 1$

$\mathcal{D}^\lambda(\boldsymbol{\omega}) = -\begin{pmatrix}2.5 & 2.5 & 1.5 & 2.5\end{pmatrix} \cdot \boldsymbol{\theta}_4^\lambda + 2.5 = 2/3$

$\mathcal{D}^C(\boldsymbol{\omega}) = -\begin{pmatrix}2.5 & 2.5 & 1.5 & 2.5\end{pmatrix} \cdot \boldsymbol{\theta}_4^C + 2.5 = 1/3$

$\boldsymbol{\omega} = (s, f, s, s)$:

$\mathcal{D}^R(\boldsymbol{\omega}) = -\begin{pmatrix}2.5 & 1.5 & 2.5 & 2.5\end{pmatrix} \cdot \boldsymbol{\theta}_4^R + 2.5 = 0$

$\mathcal{D}^\lambda(\boldsymbol{\omega}) = -\begin{pmatrix}2.5 & 1.5 & 2.5 & 2.5\end{pmatrix} \cdot \boldsymbol{\theta}_4^\lambda + 2.5 = 1/6$

$\mathcal{D}^C(\boldsymbol{\omega}) = -\begin{pmatrix}2.5 & 1.5 & 2.5 & 2.5\end{pmatrix} \cdot \boldsymbol{\theta}_4^C + 2.5 = 1/3$

$\boldsymbol{\omega} = (f, s, s, s)$:

$\mathcal{D}^R(\boldsymbol{\omega}) = -\begin{pmatrix}1.5 & 2.5 & 2.5 & 2.5\end{pmatrix} \cdot \boldsymbol{\theta}_4^R + 2.5 = 0$

$\mathcal{D}^\lambda(\boldsymbol{\omega}) = -\begin{pmatrix}1.5 & 2.5 & 2.5 & 2.5\end{pmatrix} \cdot \boldsymbol{\theta}_4^\lambda + 2.5 = 1/6$

$\mathcal{D}^C(\boldsymbol{\omega}) = -\begin{pmatrix}1.5 & 2.5 & 2.5 & 2.5\end{pmatrix} \cdot \boldsymbol{\theta}_4^C + 2.5 = 1/3$

$\boldsymbol{\omega} = (s, f, f, s)$:

$\mathcal{D}^R(\boldsymbol{\omega}) = -\begin{pmatrix}2.5 & 1.5 & 0.5 & 2.5\end{pmatrix} \cdot \boldsymbol{\theta}_4^R + 2.5 = 2$

$\mathcal{D}^\lambda(\boldsymbol{\omega}) = -\begin{pmatrix}2.5 & 19/16 & 5/8 & 2.5\end{pmatrix} \cdot \boldsymbol{\theta}_4^\lambda + 2.5 = 47/32$

$\mathcal{D}^C(\boldsymbol{\omega}) = -\begin{pmatrix}2.5 & 1 & 1 & 2.5\end{pmatrix} \cdot \boldsymbol{\theta}_4^C + 2.5 = 1$

$\boldsymbol{\omega} = (f, s, f, s)$:

$\mathcal{D}^R(\boldsymbol{\omega}) = -\begin{pmatrix}1.5 & 2.5 & 1.5 & 2.5\end{pmatrix} \cdot \boldsymbol{\theta}_4^R + 2.5 = 1$

$\mathcal{D}^\lambda(\boldsymbol{\omega}) = -\begin{pmatrix}1.3 & 2.5 & 1.3 & 2.5\end{pmatrix} \cdot \boldsymbol{\theta}_4^\lambda + 2.5 = 1$

$\mathcal{D}^C(\boldsymbol{\omega}) = -\begin{pmatrix}1.5 & 2.5 & 1 & 2.5\end{pmatrix} \cdot \boldsymbol{\theta}_4^C + 2.5 = 1$

$\boldsymbol{\omega} = (f, f, s, s)$:

$\mathcal{D}^R(\boldsymbol{\omega}) = -\begin{pmatrix}1.5 & 0.5 & 2.5 & 2.5\end{pmatrix} \cdot \boldsymbol{\theta}_4^R + 2.5 = 0$

$\mathcal{D}^\lambda(\boldsymbol{\omega}) = -\begin{pmatrix}19/16 & 5/8 & 2.5 & 2.5\end{pmatrix} \cdot \boldsymbol{\theta}_4^\lambda + 2.5 = 17/32$

$\mathcal{D}^C(\boldsymbol{\omega}) = -\begin{pmatrix}1 & 1 & 2.5 & 2.5\end{pmatrix} \cdot \boldsymbol{\theta}_4^C + 2.5 = 1$

$\boldsymbol{\omega} = (f, f, f, s)$:

$\mathcal{D}^R(\boldsymbol{\omega}) = -\begin{pmatrix}1.5 & 0.5 & 0 & 2.5\end{pmatrix} \cdot \boldsymbol{\theta}_4^R + 2.5 = 2.5$

$\mathcal{D}^\lambda(\boldsymbol{\omega}) = -\begin{pmatrix}2/3 & 0 & 0 & 2.5\end{pmatrix} \cdot \boldsymbol{\theta}_4^\lambda + 2.5 = 43/18$

$\mathcal{D}^C(\boldsymbol{\omega}) = -\begin{pmatrix}0 & 0 & 0 & 2.5\end{pmatrix} \cdot \boldsymbol{\theta}_4^C + 2.5 = 2.5$

$$\mathcal{D}^R = \Pr(m=0)\cdot 0 + \Pr(m=1)\cdot \tfrac{1}{3} + \Pr(m=2)\cdot 1 + \Pr(m=3)\cdot 2.5 = \tfrac{13}{16}$$
$$\mathcal{D}^\lambda = \Pr(m=0)\cdot 0 + \Pr(m=1)\cdot \tfrac{1}{3} + \Pr(m=2)\cdot 1 + \Pr(m=3)\cdot \tfrac{43}{18} = \tfrac{115}{144}$$
$$\mathcal{D}^C = \Pr(m=0)\cdot 0 + \Pr(m=1)\cdot \tfrac{1}{3} + \Pr(m=2)\cdot 1 + \Pr(m=3)\cdot 2.5 = \tfrac{13}{16}$$

As we see from this example, if $\bar{d} = 2.5$, bank 4's risk-taking distortion is not monotonic to the degree of connectedness λ: the distortion of a $\lambda = 0.5$ network is lower than that of a complete and a ring network.