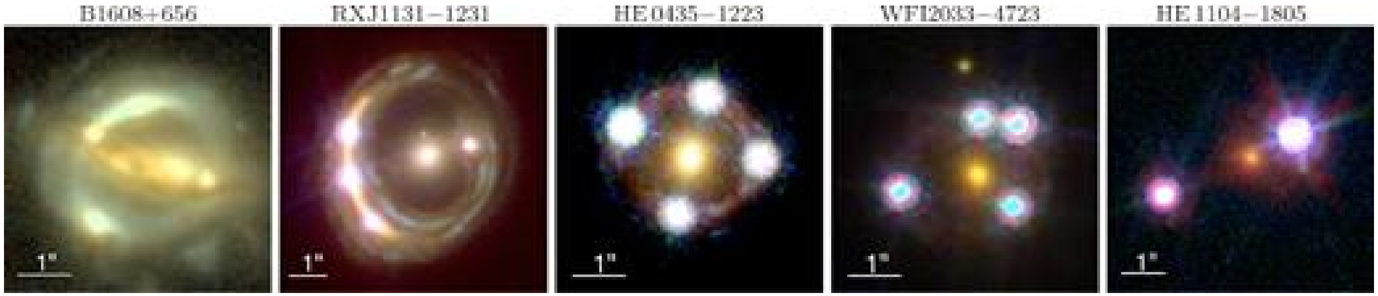
浅析引力透镜

何显龙　　2020102020001
武汉大学 天体物理中心

Figure 1.



Open in new tab   Download slide

H0LiCOW lens sample, consisting of four quadruply lensed quasar systems in various configurations and one doubly lensed quasar system. The lens name is indicated above each panel. The colour images are composed using two (for B1608+656) or three (for other lenses) *HST* imaging bands in the optical and near-infrared. North is up and east is left.

多重像，Einstein十字
S. H. Suyu2017：
https://doi.org/10.1093/mnras/stx483

Einstein环：
Wambsganss1998,Gravitati-onal Lensing in Astronomy

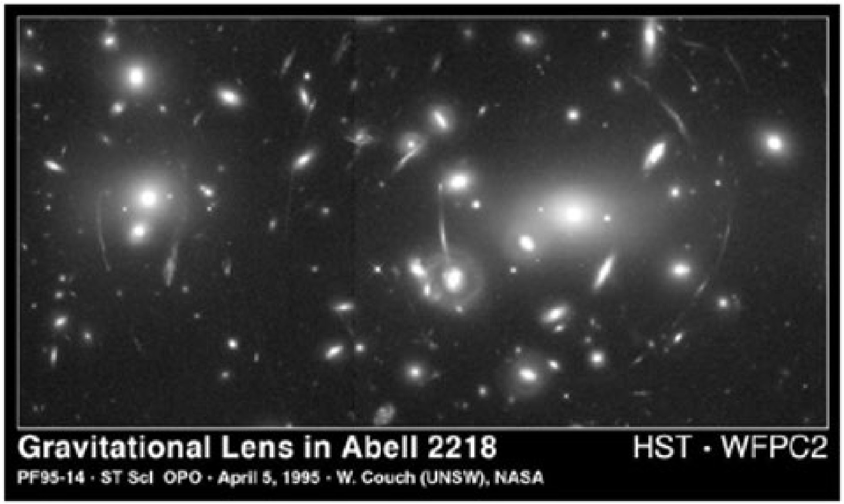


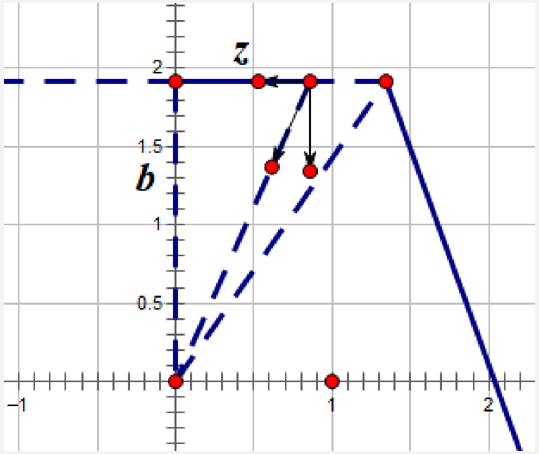
Figure 16: Galaxy Cluster Abell 2218 with Giant Luminous Arcs and many arclets, imaged with the Hubble Space Telescope. The original picture can be found in [98]. (Credits: W. Couch, R. Ellis and NASA.)

是个啥

NEWTON经典星光偏折

- 光线轴向radial加速度积分

$$a_\perp = \frac{GM}{r^2}\cos\theta = \frac{GMb}{\left(b^2+z^2\right)^{3/2}} \qquad \alpha = \frac{v_\perp}{v_\parallel} \cong \color{red}{\frac{2GM}{bc^2}} = \frac{R_S}{b}$$

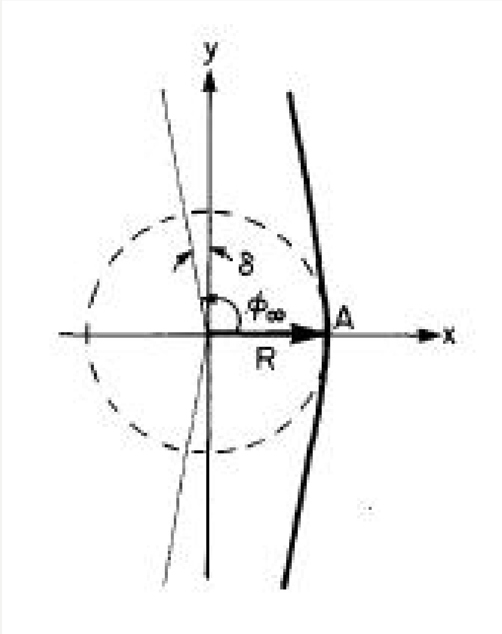


$$v_\perp = \int a_\perp \mathrm{d}t = \int_{-\infty}^{+\infty} \frac{GMb}{\left(b^2+z^2\right)^{3/2}} \frac{\mathrm{d}z}{c} \xrightarrow{z=b\cdot\tan\theta} = \frac{2GM}{bc} \ll c$$

- “光子”的双曲线轨道速度

$$\frac{1}{r} := \mu(\varphi) = \frac{GMm^2}{L^2}\left(1+e\cos\varphi\right) := \frac{1+e\cos\varphi}{R\left(1+e\right)}$$

$$\vec{v} = \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}t}\vec{e}_r + r\frac{\mathrm{d}\vec{e}_r}{\mathrm{d}t} \Rightarrow v^2 = \frac{GM}{R(1+e)}\left(1+2e\cos\varphi+e^2\right)$$



粗暴的取$c^2 = v^2\big|_{\varphi=\varphi_\infty}$ 或机械能$E\big|_\infty = \frac{1}{2}mc^2$

反解出偏心率e，进而可得偏转角

$$\alpha = 2\delta \cong \frac{2}{e} = \frac{2}{\frac{c^2 R}{GM}+1} \cong \color{red}{\frac{2GM}{c^2 R}}$$

EISTEIN广义相对论星光偏折

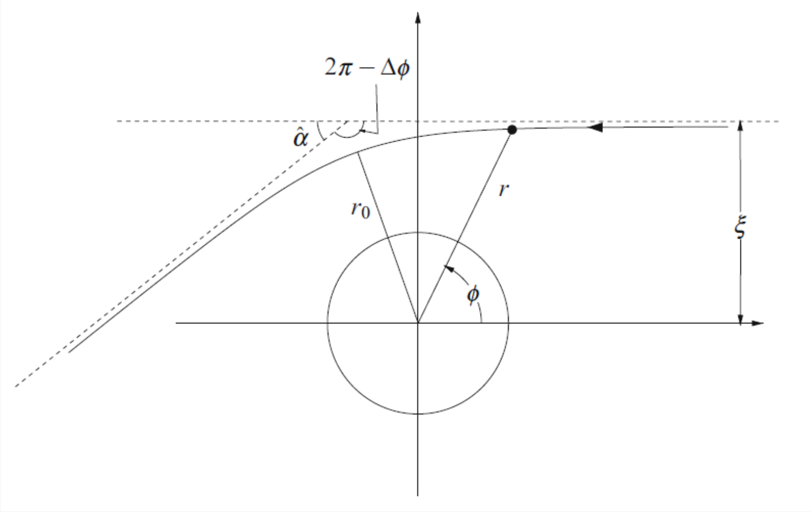
- **施瓦西度规，测地线级数解**

施瓦西时空：$\mathrm{d}s^2=-\left(1-\dfrac{2M}{r}\right)\mathrm{d}t^2+\left(1-\dfrac{2M}{r}\right)^{-1}\mathrm{d}r^2+r^2\left(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\varphi^2\right)$，类光性条件：$\mathrm{d}s^2=0$

结合测地线方程:$\dfrac{\mathrm{d}^2x^\mu}{\mathrm{d}\lambda^2}+\Gamma^\mu_{\nu\sigma}\dfrac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\dfrac{\mathrm{d}x^\sigma}{\mathrm{d}\lambda}=0$，给出轨道方程（非线性）：$\mu:=r^{-1}\Rightarrow\dfrac{\mathrm{d}^2\mu}{\mathrm{d}\varphi^2}+\mu=3M\mu^2$

一阶近似解：$\mu(\varphi)\approx\mu_0+\mu_1=\dfrac{1}{\xi}\sin\varphi+\dfrac{M}{\xi^2}(1-\cos\varphi)^2$，及边界条件给出:$0=\dfrac{1}{r_\infty}=\mu(\pi+\alpha)$

解得$\alpha\cong\dfrac{4M}{\xi}\triangleq\dfrac{4GM}{c^2\xi}=\dfrac{2R_s}{\xi}$



- **测地线方程直接积分**

直接对测地线给出的方程作积分：

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\varphi}\right)^2-\frac{E^2r^4}{L^2}+r^2\left(1-\frac{2M}{r}\right)=0\xrightarrow{u=r_0/r}$$

$$\Delta\phi=\int\frac{\mathrm{d}\varphi}{\mathrm{d}r}\mathrm{d}r=2\int_0^1\left[\left(1-u^2\right)-\frac{2M}{r_0}\left(1-u^3\right)\right]^{-1/2}\mathrm{d}u$$

$$\cong2\int_0^1\left[\frac{1}{\left(1-u^2\right)^{1/2}}+\frac{1-u^3}{\left(1-u^2\right)^{3/2}}\frac{M}{r_0}+\mathcal{O}\left(\frac{M}{r_0}\right)^2\right]\mathrm{d}u=\pi+\frac{4M}{r_0}+\mathcal{O}\left(\frac{M}{r_0}\right)^2\Rightarrow\Rightarrow\alpha\cong\frac{4M}{r_0}\triangleq\frac{4GM}{c^2\xi}$$

EISTEIN广义相对论星光偏折

- 推广，度规扰动下的变分法

线性引力论中的牛顿极限，

借用“光速变化”与“折射率”的概念：

$$n = c/c' = \frac{1}{1+\frac{2\Phi}{c^2}} \approx 1-\frac{2\Phi}{c^2}$$

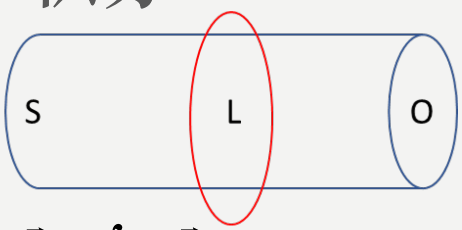
$$\eta_{\mu\nu} \to g_{\mu\nu} = \begin{pmatrix} 1+\frac{2\Phi}{c^2} & 0 & 0 & 0 \\ 0 & -\left(1-\frac{2\Phi}{c^2}\right) & 0 & 0 \\ 0 & 0 & -\left(1-\frac{2\Phi}{c^2}\right) & \\ 0 & 0 & 0 & -\left(1-\frac{2\Phi}{c^2}\right) \end{pmatrix}$$

欧拉－拉格朗日方程，近似沿未散射路径积分，最终给出：$\hat{\vec{\alpha}}(b) = \frac{2}{c^2}\int_{-\infty}^{+\infty} \vec{\nabla}_{\perp}\Phi \cdot \mathrm{d}z$

- 例：点透镜体Newton势 $\Phi = -\frac{GM}{r}, r = \sqrt{\left(x^2+y^2\right)+z^2} = \sqrt{b^2+z^2}$
- 是经典偏角的2倍

$$\hat{\vec{\alpha}}(b) = \frac{2}{c^2}\begin{pmatrix} x \\ y \end{pmatrix}\int_{-\infty}^{+\infty} \frac{GM \cdot \mathrm{d}z}{\left(b^2+z^2\right)^{3/2}} = \frac{2}{c^2}\cdot\frac{2GM}{b^2}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{4GM}{c^2}\frac{\vec{b}}{b^2}$$

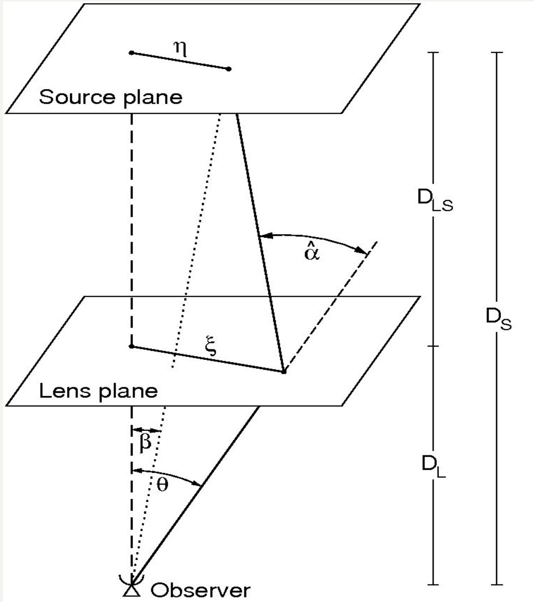
- 线性依赖于M。对扩展源，线性叠加->积分
- 薄透镜近似，投影（面）质量密度：



$$\hat{\vec{\alpha}}(\vec{\xi}) = \sum_i \hat{\vec{\alpha}}_i\left(\vec{\xi}-\vec{\xi}_i\right) = \frac{4G}{c^2}\sum_i M_i \frac{\vec{\xi}-\vec{\xi}_i}{\left|\vec{\xi}-\vec{\xi}_i\right|^2} = \frac{4G}{c^2}\int \frac{\left(\vec{\xi}-\vec{\xi}'\right)\Sigma\left(\vec{\xi}'\right)}{\left|\vec{\xi}-\vec{\xi}'\right|^2}\mathrm{d}^2\xi', \quad \Sigma(\vec{\xi}) = \int \rho(\vec{\xi}, z)\mathrm{d}z$$

透镜方程（再无广相）

- 非常trivial（角度✖距离✔）



角度：$\vec{\beta}=\vec{\theta}-\vec{\alpha}(\vec{\theta})$，正弦定理给出$\vec{\alpha}(\vec{\theta}):=\dfrac{D_{\mathrm{LS}}}{D_{\mathrm{S}}}\hat{\vec{\alpha}}(\vec{\theta})$

或由径向距离：$D_{os}\tan\theta=D_{ls}\tan\tilde{\alpha}+D_{os}\tan\beta$

$\Rightarrow\vec{\theta}\cdot D_{os}=\vec{\beta}\cdot D_{os}+\tilde{\vec{\alpha}}\cdot D_{ls}$，其中$\hat{\vec{\alpha}}(b)=\dfrac{2}{c^2}\int_{-\infty}^{+\infty}\vec{\nabla}_{\perp}\Phi\cdot \mathrm{d}z$

归一化：$\vec{y}=\vec{x}-\vec{\alpha}(\vec{x})$，$\vec{\alpha}(\vec{x})=\dfrac{D_{\mathrm{L}}D_{\mathrm{LS}}}{\xi_0 D_{\mathrm{S}}}\hat{\vec{\alpha}}\left(\xi_0\vec{x}\right)\Rightarrow$透镜映射Lens Mapping，奇函数

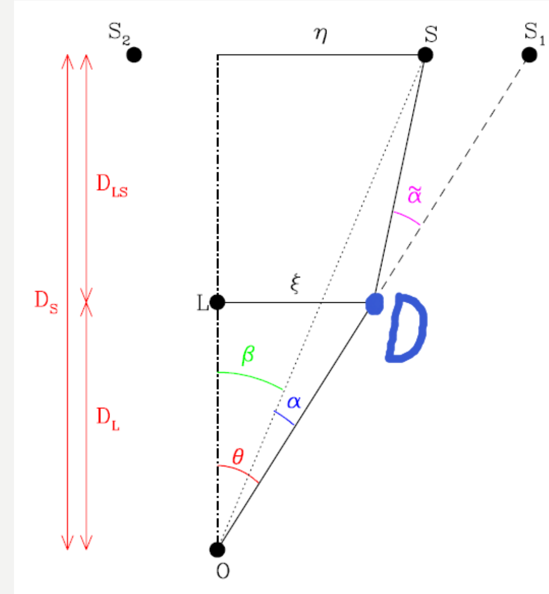
透镜方程（再无广相）

- 非常trivial（角度✖距离✔）

角度：$\color{red}{\vec{\beta}=\vec{\theta}-\vec{\alpha}(\vec{\theta})}$，正弦定理给出$\vec{\alpha}(\vec{\theta}):=\dfrac{D_{\mathrm{LS}}}{D_{\mathrm{S}}}\hat{\vec{\alpha}}(\vec{\theta})$

或由径向距离：$D_{os}\tan\theta=D_{ls}\tan\tilde{\alpha}+\ D_{os}\tan\beta$

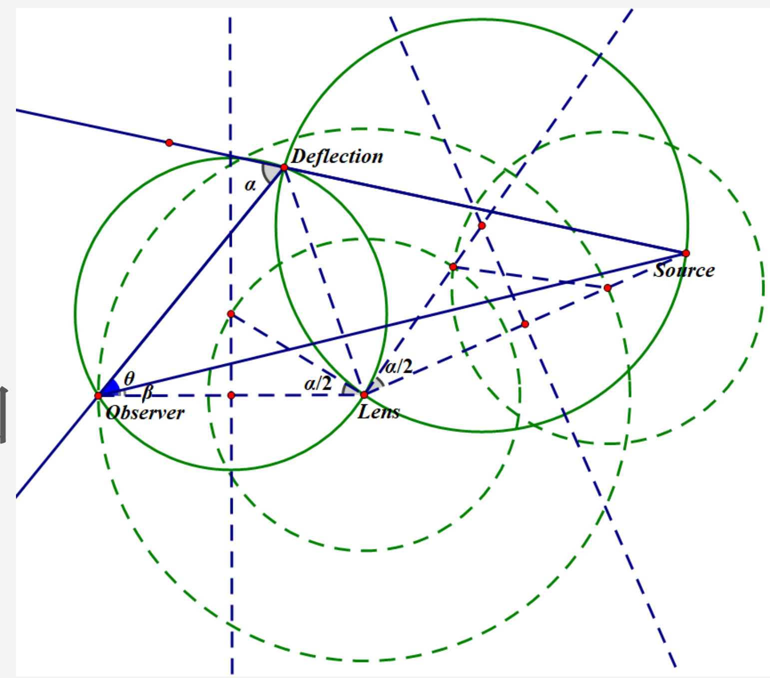
$\Rightarrow\vec{\theta}\cdot D_{os}=\vec{\beta}\cdot D_{os}+\tilde{\vec{\alpha}}\cdot D_{ls}$，其中$\hat{\vec{\alpha}}(b)=\dfrac{2}{c^2}\int_{-\infty}^{+\infty}\vec{\nabla}_{\perp}\Phi\cdot\mathrm{d}z$



归一化：$\color{red}{\vec{y}=\vec{x}-\vec{\alpha}(\vec{x})}$，$\vec{\alpha}(\vec{x})=\dfrac{D_{\mathrm{L}}D_{\mathrm{LS}}}{\xi_0 D_{\mathrm{S}}}\hat{\vec{\alpha}}\left(\xi_0\vec{x}\right)\Rightarrow$透镜映射Lens Mapping，奇函数

- 尺规作图
  - 已知三点O、L、S
  - 已知偏角α≅4GM/(c^2 l)
  - 光路可逆，两等角(π-α)/2
  - “圆周角相等”，绿色实线圆

  - 但l=DL*cosα/2也是一个限制



有些啥

“距离”与宇宙学

$$\hat{\alpha}(b)=\frac{4GM}{c^2 b}\Rightarrow \hat{\alpha}\uparrow, M\uparrow b\downarrow$$

- “远区弱场小角度近似”
  - 艾丁顿1919年日食实验，利用星光偏折验证广义相对论
  - Observer、Lens、Source近似三点一线，事件率极低
  - “弱场”，光子不会被黑洞俘获绕圈，黑洞照片
- 距离的不可加性　宇宙学距离：$D_{OS}\neq D_{OL}+D_{LS}$（角直径）

膨胀的宇宙，FRW度规：$\mathrm{d}s^2=-\mathrm{d}t^2+a^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2}+r^2\mathrm{d}\theta^2+r^2\sin^2\theta\mathrm{d}\varphi^2\right]$

共动距离的积分$D_H=a(t_0)\eta_0=a(t_0)\int_0^x \mathrm{d}x=a(t_0)\int_0^{t_0}\frac{c\cdot dt}{a(t)}=\int_0^{a(t_0)=1}\frac{c}{a}\frac{da}{aH(a)}$

Fridemann方程：$\left(\frac{H(a)}{H_0}\right)^2=\Omega_{M0}a^{-3}+\Omega_{R0}a^{-4}+\Omega_k a^{-2}+\Omega_\Lambda$，宇宙学红移：$\frac{\omega_1}{\omega_2}=\frac{a(t_2)}{a(t_1)}\Rightarrow 1+z=\frac{1}{a(t)}$

Planck卫星宇宙学参数$\Omega_{M0}=0.3089,\Omega_{R0}\cong 0,\Omega_k=0.0,\Omega_\Lambda=0.6911,H_0=67.74\mathrm{km}/(\mathrm{s}\cdot\mathrm{Mpc})$

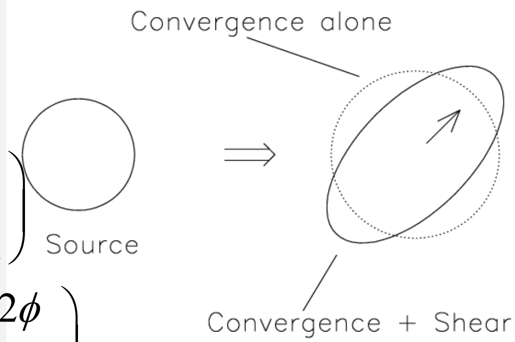
最终，得到距离-红移关系：$D_H(z_1,z_2)\approx\frac{c}{H_0}\int_{z(t_0)=0}^{\infty}\frac{dz}{\sqrt{0.3(1+z)^3+0.7}}=46.5\times10^9$光年$=4.31\times10^{26}\mathrm{m}$

$D_{OL}(z_L=0.5)=8.665$ Gly，$D_{OS}(z_S=2.0)=23.935$ Gly，而积分上下限从0.5~2给出$D_{LS}=11.12$Gly

JACOBI矩阵数学意义

- 一阶Taylor展开

透镜方程：$\vec{x}-\vec{\alpha}(\vec{x})=\vec{y}=\cancel{\vec{y}|_{\vec{x}=0}}+\dfrac{\partial\vec{y}}{\partial\vec{x}}\cdot\vec{x}+\cancel{\cdots}$，即$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}=\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$



其中$\dfrac{\partial\vec{y}}{\partial\vec{x}}:=\mathbf{A}:=\begin{pmatrix} 1-\kappa-\gamma_1 & -\gamma_2 \\ -\gamma_2 & 1-\kappa+\gamma_1 \end{pmatrix}:=(1-\kappa)\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}-\gamma\begin{pmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{pmatrix}$

对半径为 r 的圆形源，可以重新配方：

$$r^2=y_1^2+y_2^2=\cdots=(1-\kappa-\gamma)^2(x_1\cos 2\phi+x_2\sin 2\phi)^2+(1-\kappa+\gamma)^2(-x_1\sin 2\phi+x_2\cos 2\phi)^2$$

这恰是旋转$+2\phi$后的椭圆：（2倍，如轴对称SIS分布解释）

$\left(\dfrac{x_1'}{a}\right)^2+\left(\dfrac{x_2'}{b}\right)^2=1$，其中$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix}=\begin{pmatrix} \cos 2\phi & \sin 2\phi \\ -\sin 2\phi & \cos 2\phi \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$，半长轴、半短轴$a=\dfrac{r}{1-\kappa-\gamma},b=\dfrac{r}{1-\kappa+\gamma}$

面积缩放$:S':=\pi ab=\pi r^2\dfrac{1}{(1-\kappa)^2-\gamma^2}:=S\cdot\mu$，其中$\mu:=\dfrac{1}{\det\mathbf{A}}$可解释为放大率

- 放大率
  - 二（多）重积分换元法：

源$\iint f(\vec{\beta})\mathrm{d}\vec{\beta}=$像$\iint f(\vec{\theta})|\mathrm{A}|\mathrm{d}\vec{\theta}$

即立体角$\mathrm{d}\vec{\beta}=|\mathrm{A}|\mathrm{d}\vec{\theta}$，依据光度守恒，

自然定义放大率$\mu:=\left|\dfrac{\mathrm{d}\vec{\theta}}{\mathrm{d}\vec{\beta}}\right|=\left|\dfrac{\mathrm{d}\vec{x}}{\mathrm{d}\vec{y}}\right|=\dfrac{1}{\det\mathbf{A}}$

- 二阶引力透镜效应（略）

透镜势Ψ

- 一阶导α（梯度），矢量->标量

透镜方程：$\vec{y}=\vec{x}-\vec{\alpha}(\vec{x}),\frac{\partial\vec{y}}{\partial\vec{x}}:=\mathbf{A}=\left[\delta_{ij}-\frac{\partial\alpha_i}{\partial x_j}\right]:=\mathbf{I}-\left[\frac{\partial\Psi}{\partial x_i x_j}\right]$，定义$\vec{\alpha}(\vec{x}):=\vec{\nabla}_\perp\Psi=\frac{\partial\Psi}{\partial x_1}\vec{e}_1+\frac{\partial\Psi}{\partial x_2}\vec{e}_2$

这要求混合偏导相等：$\frac{\partial\alpha_1}{\partial x_2}=\frac{\partial^2\Psi}{\partial x_1\partial x_2}=\gamma_2=\frac{\partial^2\Psi}{\partial x_2\partial x_1}=\frac{\partial\alpha_2}{\partial x_1}$，所幸星光偏折角：$\hat{\vec{\alpha}}(b)=\frac{2}{c^2}\int_{-\infty}^{+\infty}\hat{\vec{\nabla}}_\perp\Phi\cdot\mathrm{d}z$

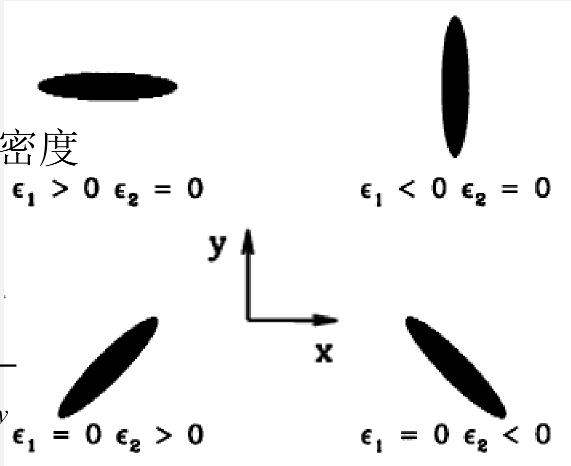
透镜势&引力势：$\hat{\Psi}(\vec{\theta})=\frac{D_{\mathrm{LS}}}{D_{\mathrm{L}}D_{\mathrm{S}}}\frac{2}{c^2}\int_{-\infty}^{+\infty}\Phi\left(D_{OL}\vec{\theta},z\right)\cdot\mathrm{d}z$ （有量纲），球对称$\Phi(r)=-\int_0^r\frac{GM(r)}{r^2}\mathrm{d}r$

- 二阶导κ（散度），汇聚参数convergence

定义式：Jacobi矩阵的参数$\kappa=\frac{1}{2}(2-\mathbf{A}_{11}-\mathbf{A}_{22})=\frac{1}{2}\vec{\nabla}_\perp\cdot\vec{\alpha}=\frac{1}{2}\nabla_\perp^2\Psi=\frac{D_L D_{LS}}{\xi_0^2 D_S}\frac{1}{c^2}\int_{-\infty}^{+\infty}\nabla_\perp^2\Phi\cdot dz$

牛顿引力势的二阶偏导，Poisson方程：$\hat{\nabla}_\perp^2\Phi+\frac{\partial^2}{\partial z^2}\Phi=\hat{\nabla}^2\Phi=4\pi G\rho$

计算式：$\kappa(\vec{x})=\frac{D_L D_{LS}}{c^2 D_S}\int_{-\infty}^{+\infty}4\pi G\rho\cdot dz=\frac{\Sigma(\vec{x})}{\Sigma_{cr}}$，恰为无量纲化面质量密度



- 切变参数shear

质量四极矩$q_{ij}:=\int d^2\theta I_{\mathrm{obs}}(\theta)\theta_i\theta_j$，椭率$\epsilon_1:=\frac{q_{xx}-q_{yy}}{q_{xx}+q_{yy}}$，$\epsilon_2:=\frac{2q_{xy}}{q_{xx}+q_{yy}}$

$\epsilon_1=\frac{\left(A_{xx}^{-1}\right)^2-\left(A_{yy}^{-1}\right)^2}{\left(A_{xx}^{-1}\right)^2+\left(A_{yy}^{-1}\right)^2+2\left(A_{xy}^{-1}\right)^2}=\frac{(1-\kappa+\gamma_1)^2-(1-\kappa-\gamma_1)^2}{(1-\kappa+\gamma_1)^2+(1-\kappa-\gamma_1)^2+2\gamma_2^2}\simeq 2\gamma_1$，类似计算$\epsilon_2\simeq 2\gamma_2$

透镜体质量分布模型

点透镜体---0维

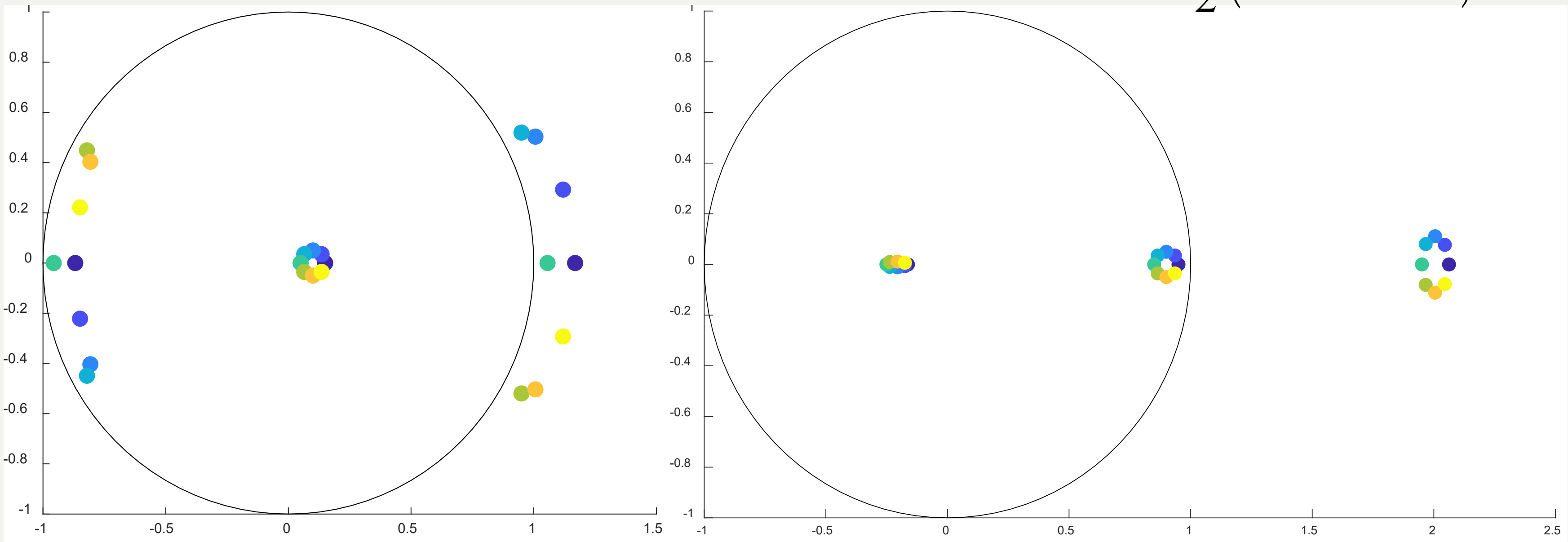
- 透镜方程，Einstein半径θE

星光偏折：$\hat{\alpha}=\frac{4GM}{c^2 b}=\frac{4GM}{c^2 D_L\theta}$，正弦定理：$\alpha(\theta):=\frac{D_{\rm LS}}{D_{\rm S}}\hat{\alpha}(\theta)$，$\theta_E:=\sqrt{\frac{4GM}{c^2}\frac{D_{LS}}{D_L D_S}}$

透镜方程：$\beta=\theta-\alpha(\theta)=\theta-\frac{\theta_E^2}{\theta}$，无量纲化：$y=x-\frac{1}{x}$。$\theta=\theta_E$对应$y=0$

- 透镜映射Lens Mapping

透镜方程的解：$x_\pm=\frac{1}{2}\left(y\pm\sqrt{y^2+4}\right)$



- 放大率

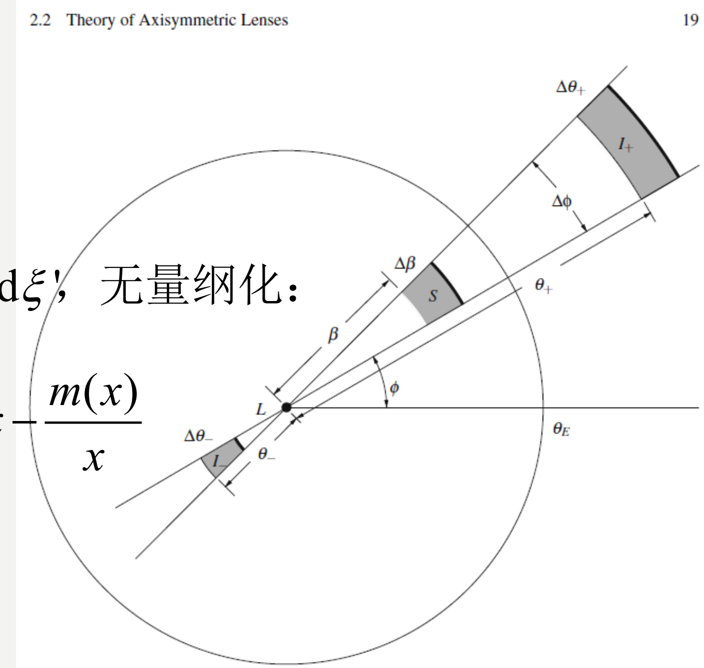
$$\det\mathbf{A}=\left|\frac{\mathrm{d}\vec{y}}{\mathrm{d}\vec{x}}\right|=\frac{y}{x}\frac{\mathrm{d}y}{\mathrm{d}x}=1-\left(\frac{1}{x}\right)^4\Rightarrow\mu:=\frac{1}{\det\mathbf{A}}=\left[1-\left(\frac{1}{x}\right)^4\right]^{-1}$$ 。$\theta=\theta_E$或$y=0$恰对应$\mu\to\infty$

轴对称分布---1维

- **轴对称**性（上页Einstein环）

可以证明：$\hat{\alpha}(\xi)=\dfrac{4GM(\xi)}{c^2\xi}$，$M(\xi)=2\pi\int_0^{\xi}\Sigma(\xi')\cdot\xi'\mathrm{d}\xi'$，无量纲化：

$\alpha(x)=\dfrac{m(x)}{x}$，$m(x)=2\int_0^{|x|}\kappa(x')\cdot x'\mathrm{d}x'$，透镜方程$y=x-\dfrac{m(x)}{x}$



- 放大率μ，tangential，radial

$$\mu^{-1}=\det\mathbf{A}=\left|\frac{\mathrm{d}\vec{\beta}}{\mathrm{d}\vec{\theta}}\right|=\left|\frac{\mathrm{d}\vec{y}}{\mathrm{d}\vec{x}}\right|=\left|\frac{y\cdot\mathrm{d}y\,\cancel{\mathrm{d}\varphi}}{x\cdot\mathrm{d}x\,\cancel{\mathrm{d}\varphi}}\right|$$ 对∀轴对称都成立

$$=\mu_t^{-1}\cdot\mu_r^{-1}=\frac{y}{x}\cdot\frac{\mathrm{d}y}{\mathrm{d}x}=\left(1-\frac{\alpha(x)}{x}\right)\left(1-\frac{\mathrm{d}\alpha(x)}{\mathrm{d}x}\right)$$

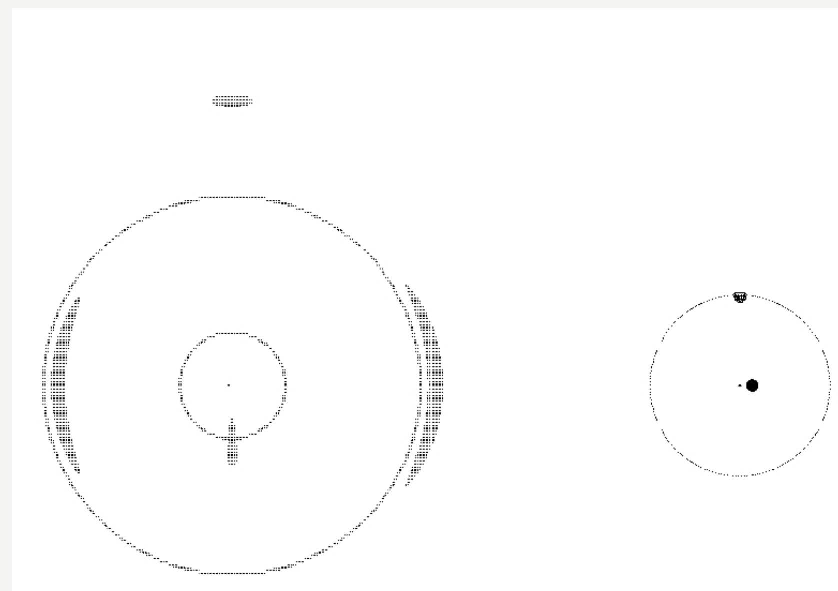
- Critical Line， Caustic Line

放大率无穷$\mu\to\infty\Rightarrow$

Critical: $\mu^{-1}(x)=0\Rightarrow y$平面，Caustic

μ_t^{-1}、$\mu_r^{-1}\Rightarrow$ tangential、radial Critial Line

轴对称：Line $\to$ Ring

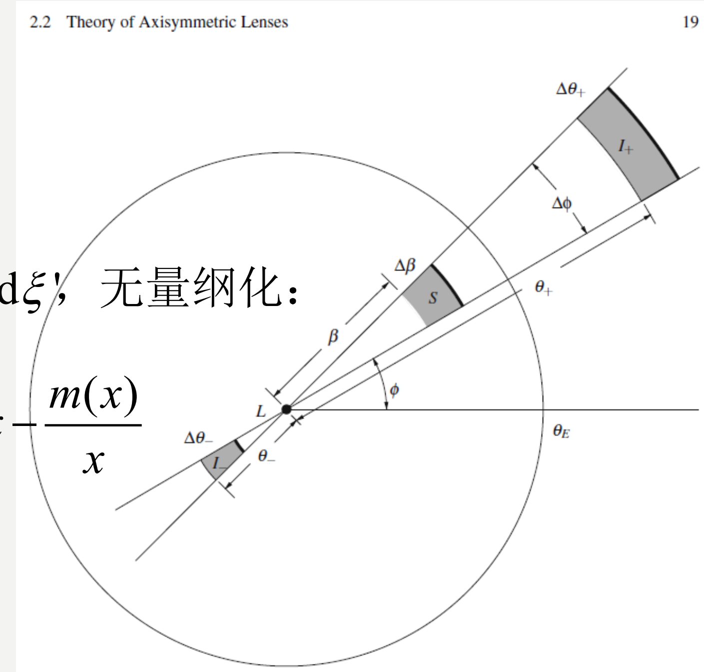


轴对称分布---1维

- 轴对称性

可以证明：$\hat{\alpha}(\xi)=\dfrac{4GM(\xi)}{c^2\xi}$，$M(\xi)=2\pi\int_0^{\xi}\Sigma(\xi')\cdot\xi'\mathrm{d}\xi'$，无量纲化：

$\alpha(x)=\dfrac{m(x)}{x}$，$m(x)=2\int_0^{|x|}\kappa(x')\cdot x'\mathrm{d}x'$；透镜方程$y=x-\dfrac{m(x)}{x}$



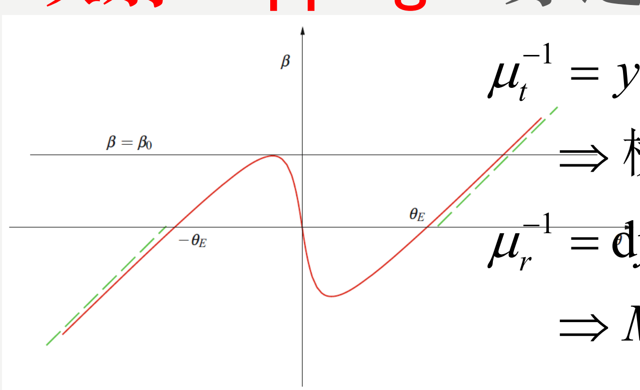
- 放大率μ，tangential，radial

$$\mu^{-1}=\det\mathbf{A}=\left|\frac{\mathrm{d}\vec{\beta}}{\mathrm{d}\vec{\theta}}\right|=\left|\frac{\mathrm{d}\vec{y}}{\mathrm{d}\vec{x}}\right|=\left|\frac{y\cdot\mathrm{d}y\,\cancel{\mathrm{d}\varphi}}{x\cdot\mathrm{d}x\,\cancel{\mathrm{d}\varphi}}\right| \text{对}\forall\text{轴对称都成立}$$

$$=\mu_t^{-1}\cdot\mu_r^{-1}=\frac{y}{x}\cdot\frac{\mathrm{d}y}{\mathrm{d}x}=\left(1-\frac{\alpha(x)}{x}\right)\left(1-\frac{\mathrm{d}\alpha(x)}{\mathrm{d}x}\right)$$

- Critical Line，Caustic Line

- 映射Mapping：穿过Caustic，±2个像

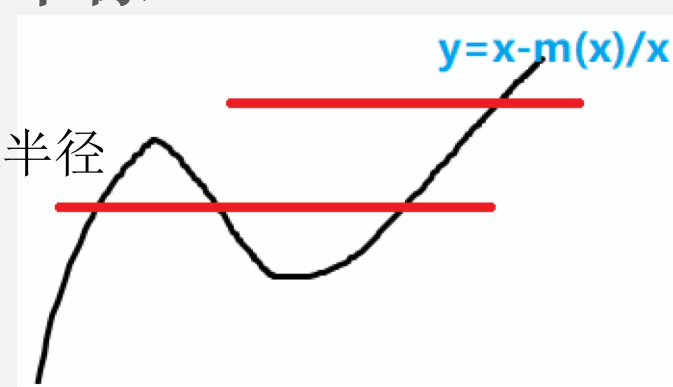


$$\mu_t^{-1}=y/x=0$$

$\Rightarrow$ 横轴交点，*Einstein*半径

$$\mu_r^{-1}=\mathrm{d}y/\mathrm{d}x=0$$

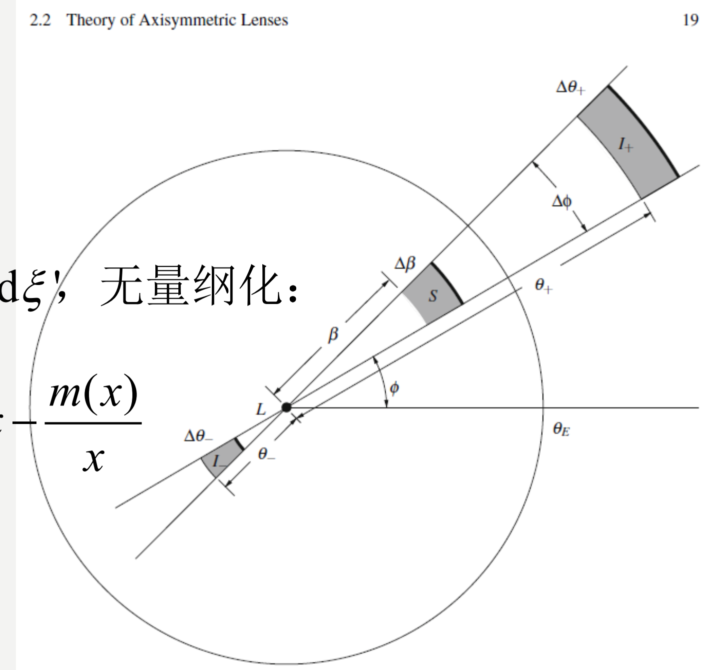
$\Rightarrow$ *M*apping极值点



轴对称分布---1维

- 轴对称性

可以证明：$\hat{\alpha}(\xi)=\dfrac{4GM(\xi)}{c^2\xi}$，$M(\xi)=2\pi\int_0^{\xi}\Sigma(\xi')\cdot\xi'\mathrm{d}\xi'$，无量纲化：



$\alpha(x)=\dfrac{m(x)}{x}$，$m(x)=2\int_0^{|x|}\kappa(x')\cdot x'\mathrm{d}x'$；透镜方程$y=x-\dfrac{m(x)}{x}$

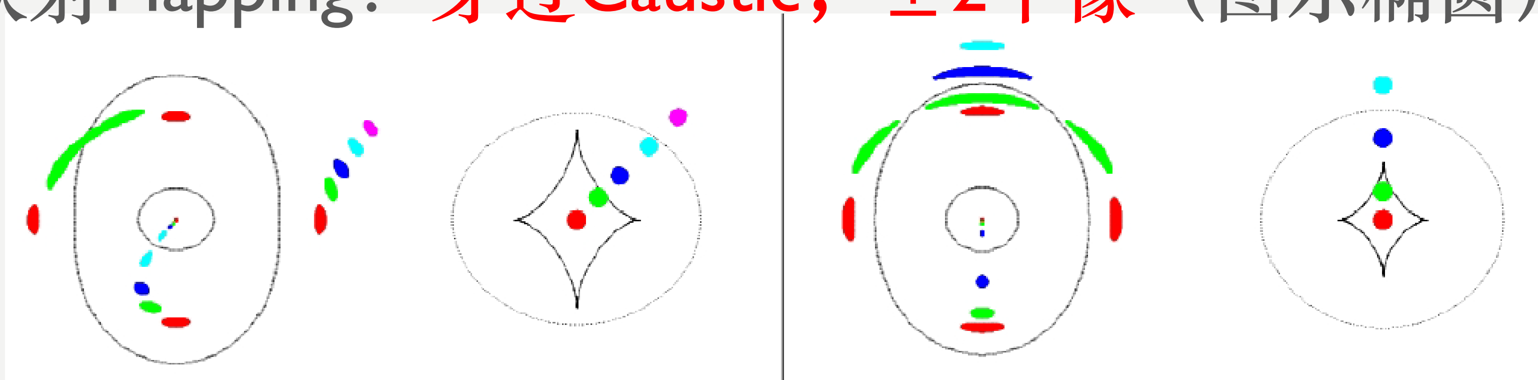
- 放大率μ，tangential，radial

$$\mu^{-1}=\det\mathbf{A}=\left|\frac{\mathrm{d}\vec{\beta}}{\mathrm{d}\vec{\theta}}\right|=\left|\frac{\mathrm{d}\vec{y}}{\mathrm{d}\vec{x}}\right|=\left|\frac{y\cdot\mathrm{d}y\,\cancel{\mathrm{d}\varphi}}{x\cdot\mathrm{d}x\,\cancel{\mathrm{d}\varphi}}\right| \text{对}\forall\text{轴对称都成立}$$

$$=\mu_t^{-1}\cdot\mu_r^{-1}=\frac{y}{x}\cdot\frac{\mathrm{d}y}{\mathrm{d}x}=\left(1-\frac{\alpha(x)}{x}\right)\left(1-\frac{\mathrm{d}\alpha(x)}{\mathrm{d}x}\right)$$

- Critical Line，Caustic Line Critical: $\mu^{-1}(x)=0\Rightarrow y$平面，Caustic

- 映射Mapping：穿过Caustic，±2个像（图示椭圆）



轴对称分布---1维

$$\mathbf{I}-\frac{\partial\vec{\alpha}}{\partial\vec{x}}=\frac{\partial\vec{y}}{\partial\vec{x}}:=\mathbf{A}:=\mathbf{I}-\begin{pmatrix}\kappa+\gamma_1 & \gamma_2\\ \gamma_2 & \kappa-\gamma_1\end{pmatrix}$$

$$=(1-\kappa)\begin{pmatrix}1 & \\ & 1\end{pmatrix}-\gamma\begin{pmatrix}\cos 2\phi & \sin 2\phi\\ \sin 2\phi & -\cos 2\phi\end{pmatrix}$$

- 平均面质量密度，Einstein半径

定义 $\dfrac{m(x)}{x^2}=\dfrac{2\pi\cdot\int_0^{|x|}\kappa(x')\cdot x'\mathrm{d}x'}{\pi x^2}:=\bar{\kappa}(x)$，$\bar{\kappa}(x)=1$给出$y=0$，即Einstein半径

- 汇聚与切变

向量形式：$\vec{\alpha}(\vec{x})=\dfrac{m(x)}{x^2}\vec{x}$，即$\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}=\dfrac{m(\sqrt{x_1^2+x_2^2})}{x^2}\cdot\begin{pmatrix}x_1\\ x_2\end{pmatrix}$（$\therefore$ 散度$\vec{\nabla}_\perp\cdot$）

可以计算：$\kappa(x):=\dfrac{1}{2}\vec{\nabla}_\perp\cdot\vec{\alpha}=\dfrac{1}{2}\left(\dfrac{\partial\alpha_1}{\partial x_1}+\dfrac{\partial\alpha_2}{\partial x_2}\right)=\dfrac{1}{2x}\dfrac{\mathrm{d}m(x)}{\mathrm{d}x}=\dfrac{1}{2}\left(\dfrac{\mathrm{d}\alpha(x)}{\mathrm{d}x}+\dfrac{\alpha(x)}{x}\right)$

$$\gamma_1(x)=\frac{1}{2}\left(x_1^2-x_2^2\right)\left(\frac{\mathrm{d}m(x)}{\mathrm{d}x}\frac{1}{x^3}-\frac{2m(x)}{x^4}\right),\quad \gamma_2(x)=x_1x_2\left(\frac{\mathrm{d}m(x)}{\mathrm{d}x}\frac{1}{x^3}-\frac{2m(x)}{x^4}\right)$$

容易发现，$\dfrac{\gamma_2}{\gamma_1}=\dfrac{2x_1x_2}{x_1^2-x_2^2}=\dfrac{2\cos\phi\cdot\sin\phi}{\cos^2\phi-\sin^2\phi}=\tan 2\phi\Rightarrow$ Jacobi矩阵中的2ϕ

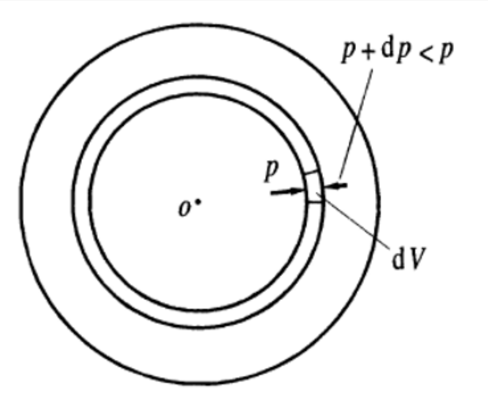
$$\gamma(x)=\sqrt{\gamma_1^2+\gamma_2^2}=\sqrt{\left(x_1^2-x_2^2\right)^2+\left(2x_1x_2\right)^2}\left|\frac{\mathrm{d}m(x)}{\mathrm{d}x}\frac{1}{2x^3}-\frac{m(x)}{x^4}\right|=\frac{m(x)}{x^2}-\kappa(x)=\bar{\kappa}(x)-\kappa(x)$$

- 透镜势

$$\frac{m(x)}{x^2}\cdot\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\vec{\alpha}:=\vec{\nabla}_\perp\Psi=\begin{pmatrix}\partial\Psi/\partial x_1\\ \partial\Psi/\partial x_2\end{pmatrix}=\frac{\mathrm{d}\Psi}{\mathrm{d}x}\cdot\frac{1}{x}\begin{pmatrix}x_1\\ x_2\end{pmatrix}\Rightarrow\Psi(x)=\int_0^x\frac{m(x')}{x'}\mathrm{d}x'=\int_0^x\alpha(x')\mathrm{d}x'$$

其中最后一步：$\dfrac{\partial}{\partial x_1}\Psi\left(\sqrt{x_1^2+x_2^2}\right)=\dfrac{\mathrm{d}\Psi(x)}{\mathrm{d}x}\cdot\dfrac{\partial x}{\partial x_1}=\dfrac{\mathrm{d}\Psi}{\mathrm{d}x}\cdot\dfrac{x_1}{x}$

SIS奇异等温球模型



- Singular Isothermal Sphere物理背景

经典流体静力学平衡方程：$\mathrm{d}p\cdot\mathrm{d}S+\dfrac{GM(r)}{r^2}\cdot\rho(\mathrm{d}r\mathrm{d}S)\equiv 0$

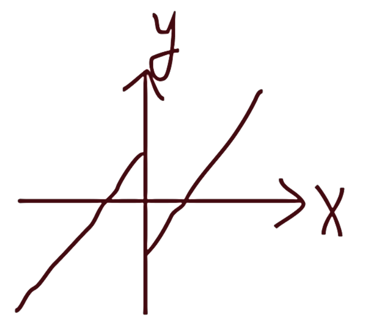
理想气体$p=\dfrac{\rho kT}{m}$，弥散速度热平衡$\dfrac{3}{2}m\sigma_v^2=\dfrac{3}{2}kT$，及$\dfrac{\mathrm{d}M(r)}{\mathrm{d}r}=4\pi r^2\cdot\rho$

$\Rightarrow -\dfrac{G}{\sigma_v^2}M'=\dfrac{r^2M''-M'\cdot 2r}{M}$。 “解”得$M(r)=\dfrac{2\sigma_v^2}{G}\cdot r$，及$\rho(r)=\dfrac{\sigma_v^2}{2\pi G}\cdot\dfrac{1}{r^2}$

- 透镜参数

$$\Sigma(\xi)=\int_{-\infty}^{+\infty}\rho(\xi,z)dz=\frac{\sigma_v^2}{2\pi G}\int_{-\infty}^{+\infty}\frac{dz}{\xi^2+z^2}=\frac{\sigma_v^2}{2G\xi},\quad M(\xi)=2\pi\int_0^{\xi}\Sigma(\xi')\cdot\xi'\mathrm{d}\xi'=\frac{\pi\sigma_v^2}{G}\xi$$

$\alpha(\theta):=\dfrac{D_{\mathrm{LS}}}{D_{\mathrm{S}}}\hat{\alpha}(\theta)=\dfrac{D_{\mathrm{LS}}}{D_{\mathrm{S}}}\dfrac{4GM(\xi)}{c^2\xi}=\dfrac{D_{\mathrm{LS}}}{D_{\mathrm{S}}}\dfrac{\sigma_v^2}{c^2}4\pi$，常数，$\alpha(\theta)\equiv\theta_E$



其中θ_E类比定义$\theta_E:=\sqrt{\dfrac{4GM(\theta_E D_L)}{c^2}\dfrac{D_{LS}}{D_L D_S}}\Rightarrow$ 透镜方程: $\beta=\theta\pm\theta_E$

无量纲化：$\kappa(x)=\dfrac{\Sigma(x)}{\Sigma_{cr}}=\dfrac{1}{2|x|}$，$\alpha(x)=\dfrac{|x|}{x}$，$\Psi(x)=|x|$，$\gamma(x)=\dfrac{1}{2|x|}$，$\mu(x)=\dfrac{|x|}{|x|-1}$

幂律质量分布模型

- M(r)与M(ξ)

设$\rho(r)=C_0\cdot r^{-n}$，显然$M_{3D}(r)=4\pi\int_0^r\rho(r')\cdot r'^2\,\mathrm{d}r'=4\pi C_0\dfrac{r^{3-n}}{3-n}$，$n\geq3$ 时发散

而$\Sigma(\xi)=\int_{-\infty}^{+\infty}\rho(\xi,z)\,\mathrm{d}z=2C_0\int_0^{\infty}\dfrac{\mathrm{d}z}{\left(\xi^2+z^2\right)^{n/2}}\xrightarrow[\mathrm{d}z=\xi\cdot\mathrm{d}\theta/\cos^2\theta]{\text{设}z=\xi\cdot\tan\theta}=2C_0\xi^{1-n}\int_0^{\pi/2}(\cos\theta)^{n-2}\,\mathrm{d}\theta$

故$M_{2D}(\xi)=2\pi\int_0^{\xi}\Sigma(\xi)\cdot\xi'\mathrm{d}\xi'=4\pi C_0\cdot\int_0^{\xi}\xi'^{2-n}\,\mathrm{d}\xi'\cdot\int_0^{\pi/2}(\cos\theta)^{n-2}\,\mathrm{d}\theta=4\pi C_0\dfrac{\xi^{3-n}}{3-n}\cdot I_{n-2}$

即$M_{2D}(\xi)$与$M_{3D}(r)$仅相差一个积分常数，数值积分测试$n\leq1$ 时发散

- 推广的归一化参数

透镜方程：$\beta=\theta-\alpha(\xi)=\theta-\dfrac{D_{\mathrm{LS}}}{D_{\mathrm{S}}}\dfrac{4GM(\xi)}{c^2\cdot\theta D_{\mathrm{L}}}:=\theta-\dfrac{\theta_E^2(\theta D_{\mathrm{L}})}{\theta}$

定义$\theta_E(\theta D_{\mathrm{L}}):=\sqrt{C_2M(\xi)}$，带入$M(\xi)=C_1\xi^{3-n}$，

由$\theta_E^2(\theta_E D_{\mathrm{L}}):=C_2M(\xi_E)=C_1C_2(\theta_E D_{\mathrm{L}})^{3-n}$反解$C_1C_2\sim\theta_E$，带回总有：

$\beta=\theta-\alpha(\xi)=\theta-\dfrac{C_1C_2(\theta D_{\mathrm{L}})^{3-n}}{\theta}=\theta-\dfrac{\theta^{2-n}}{\theta_E^{1-n}}$，即$\dfrac{\beta}{\theta}=1-\left(\dfrac{\theta_E}{\theta}\right)^{n-1}$

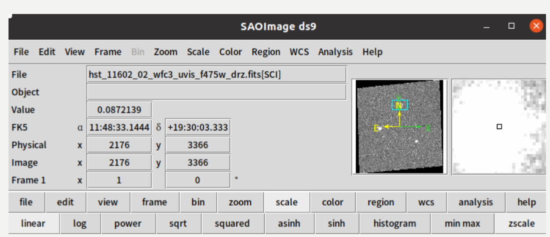
$n=1$ ，均匀面质量密度；$n=2$ ，奇异等温球；$n=3$ ，类似点透镜体

小结：EINSTEIN半径与环

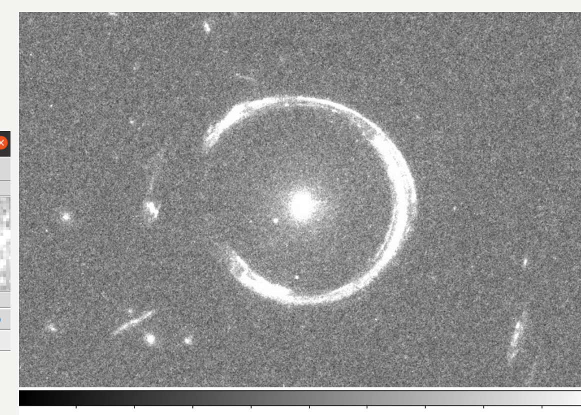
- 为归一化点透镜体方程 $\alpha(\theta) := \dfrac{D_{\mathrm{LS}}}{D_{\mathrm{S}}}\hat{\alpha}(\theta) = \dfrac{D_{\mathrm{LS}}}{D_{\mathrm{S}}}\dfrac{4GM}{c^2 D_L \theta}$，$\beta = \theta - \alpha(\theta) = \theta - \dfrac{\theta_E^2}{\theta}$
- 可推广定义至轴对称透镜体 $\theta_E := \sqrt{\dfrac{4GM\left(\theta_E D_L\right)}{c^2}\dfrac{D_{LS}}{D_L D_S}}$，反解$\theta_E$
- 幂律分布的归一化参数 $\rho\left(r\right) = C_0 \cdot r^{-n}$，$M\left(\xi\right) = C_1 \xi^{3-n} \Rightarrow \dfrac{\beta}{\theta} = 1 - \left(\dfrac{\theta_E}{\theta}\right)^{n-1}$
- 透镜映射lens mapping横轴交点 $\beta = \theta - \dfrac{m\left(\theta\right)}{\theta}$：$\theta = \theta_E \Rightarrow \beta = 0$
- 平均面质量密度 $1 = \dfrac{m\left(\theta_E\right)}{\theta_E^2} := \dfrac{2\pi \cdot \int_0^{\theta_E} \kappa\left(\theta'\right) \cdot \theta' \mathrm{d}\theta'}{\pi\theta^2} := \bar{\kappa}\left(\theta_E\right)$
- “平均”的物理意义 $\Sigma_{cr} := \dfrac{c^2}{4\pi G}\dfrac{D_S}{D_L D_{LS}} \Rightarrow \pi \cdot \left(\theta_E D_L\right)^2 \cdot \Sigma_{cr} = M\left(\theta_E D_L\right)$
- 切向放大率无穷

$$\mu^{-1} = \mu_t^{-1} \cdot \mu_r^{-1} = \frac{\beta}{\theta} \cdot \frac{\mathrm{d}\beta}{\mathrm{d}\theta}$$

$$\beta = 0 \Rightarrow \mu_t \to \infty$$



Hubble Legacy Archive (stsci.edu)



NIS（SIS）非奇异等温球

- Nonsingular（Softened）I. S. 物理背景

$\rho(r)=\frac{\sigma_v^2}{2\pi G}\cdot\frac{1}{r^2}$，或$\rho(r)=C_0\cdot r^{-n}$，显然$r\to 0$时$\rho\to\infty$，引入$r_c$ ：$\rho(r)=\frac{\sigma_v^2}{2\pi G}\cdot\frac{1}{r^2+r_c^2}$

- 透镜方程

$$\Sigma(\xi)=\frac{\sigma_v^2}{2\pi G}\int_{-\infty}^{+\infty}\frac{dz}{\xi^2+z^2+r_c^2}=\frac{\sigma_v^2}{2G\sqrt{\xi^2+r_c^2}},\quad M(\xi)=2\pi\int_0^{\xi}\Sigma(\xi')\cdot\xi'\mathrm{d}\xi'=\frac{\pi\sigma_v^2}{G}\left(\sqrt{\xi^2+r_c^2}-r_c\right)$$

$$\alpha(\theta):=\frac{D_{LS}}{D_S}\frac{4GM(\xi)}{c^2\xi}:=\frac{\theta_E^{SIS}}{\theta}\left(\sqrt{\theta^2+\theta_c^2}-\theta_c\right),\text{ 其中}\theta_c:=\frac{r_c}{D_L},\quad \theta_E^{SIS}:=4\pi\frac{D_{LS}}{D_S}\frac{\sigma_v^2}{c^2}$$

尽管原则上也可由$0\equiv\beta(\theta_E)=\theta_E-\alpha(\theta_E)$，解得$\theta_E=\sqrt{\left(\theta_E^{SIS}\right)^2-2\theta_c\cdot\theta_E^{SIS}}$

但对透镜方程的化简并无实质性帮助：$\alpha(\theta)=\left(\sqrt{\theta_E^2+\theta_c^2}+\theta_c\right)\frac{\sqrt{\theta^2+\theta_c^2}-\theta_c}{\theta}$

- 透镜参数

无量纲化：$y=x-\alpha(x),\alpha(x)=\frac{\sqrt{x^2+x_c^2}-x_c}{x}$，$x_c:=\frac{\theta_c}{\theta_E^{SIS}}$，$m(x)=x\cdot\alpha=\sqrt{x^2+x_c^2}-x_c$

$\Psi(x)=\int_0^x\alpha(x')\mathrm{d}x'$，$\kappa(x)=\frac{\Sigma(x)}{\Sigma_{cr}}=\frac{1}{2\sqrt{x^2+x_c^2}}$，$\gamma(x)=\frac{m(x)}{x^2}-\kappa(x)$，$\mu^{-1}=\left(1-\frac{\alpha}{x}\right)\left(1-\frac{\mathrm{d}\alpha}{\mathrm{d}x}\right)$

NIS（SIS）非奇异等温球

- Einstein半径与多重像

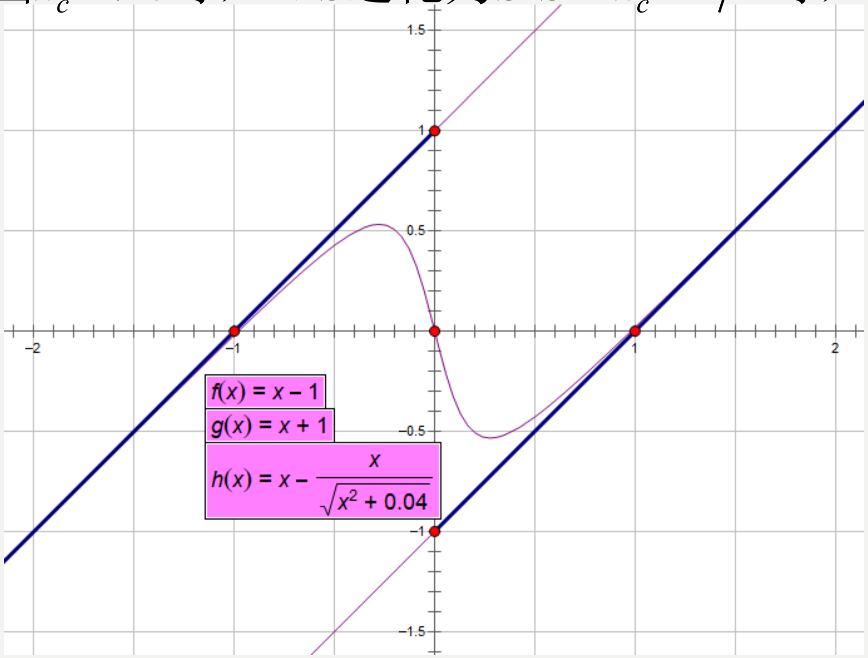
由$0 \equiv \beta(\theta_E) = \theta_E - \alpha(\theta_E)$解出$\theta_E = \sqrt{\left(\theta_E^{SIS}\right)^2 - 2\theta_c \cdot \theta_E^{SIS}}$

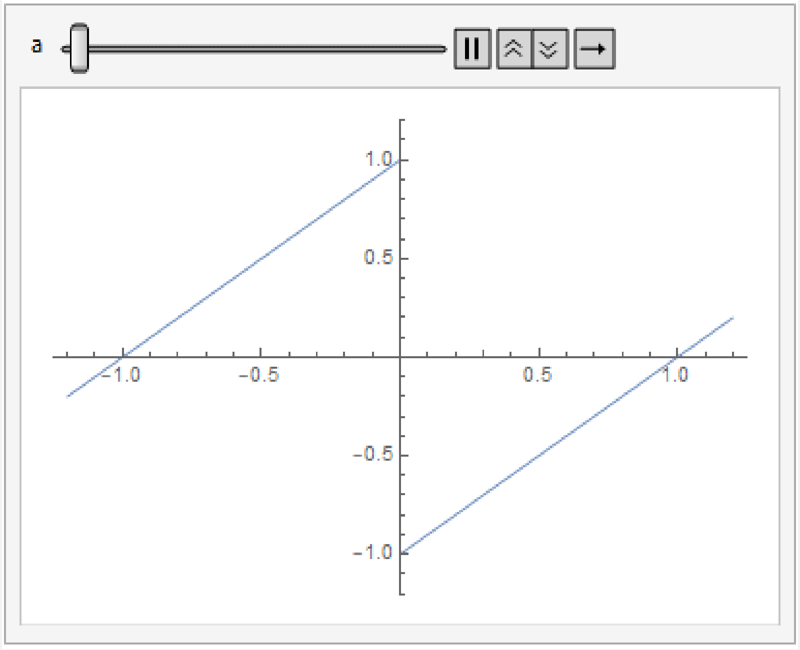
显然,$\left(\theta_E^{SIS}\right)^2 - 2\theta_c \cdot \theta_E^{SIS} < 0$即$\dfrac{\theta_c}{\theta_E^{SIS}} > 1/2$时，不能定义Einstein半径，没有多重像

- Lens Mapping

无量纲化透镜方程：$y = x - \alpha(x) := x - \left(\sqrt{x^2 + x_c^2} - x_c\right)/x$，$x_c := \theta_c / \theta_E^{SIS}$

当$x_c \to 0$时，NIS退化为SIS。$x_c > 1/2$时，恒不存在多重像





小结：超临界密度与多重像

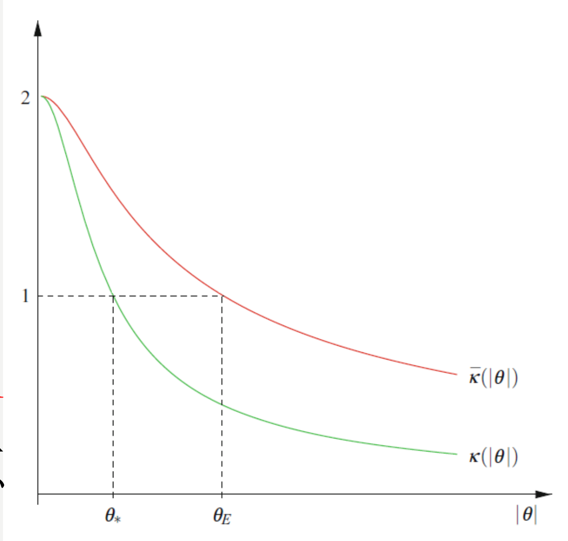
- 充分性

$\kappa(x)=\Sigma(x)/\Sigma_{cr}$，随$x$单调递减，渐进向0

$\bar{\kappa}(x):=\dfrac{m(x)}{x^2}=\dfrac{2\pi\cdot\int_0^{|x|}\kappa(x')\cdot x'\mathrm{d}x'}{\pi x^2}$，通常也单调递减

（中心）超（过）临界密度时：$\kappa(0)=\bar{\kappa}(0)>\bar{\kappa}(x_E)=1$

可定义Einstein半径$x_E \Rightarrow f(x):=x-\alpha(x)$存在横轴交点



- 必要性

透镜方程（映射）：$y=x-\alpha(x):=f(x)$

可以计算：$f(x)=x-m(x)/x=x\cdot\bar{\kappa}(x)$

$f'(x)=1-2\kappa(x)+\bar{\kappa}(x)$，$f'(0)=1-\kappa(0)<0$

结合$f(0)\equiv 0$，$f'(\infty)=1 \Rightarrow$ 存在极值 $\Rightarrow$ 多重像

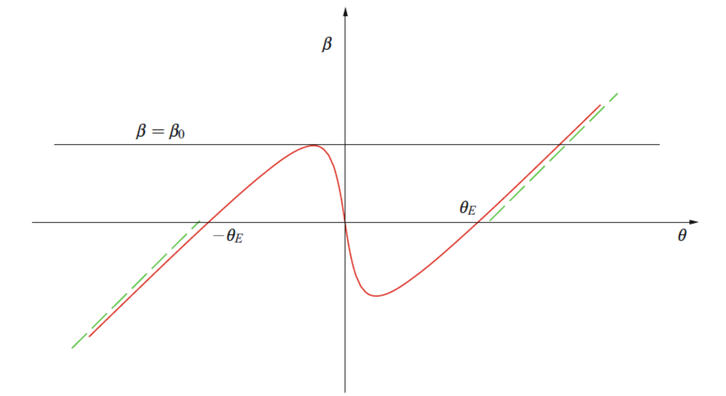


Fig. 2.9 Example of a continuous lens mapping $\beta=f(\theta)$ that produces one image when $\beta>\beta_0$ and three images when $0<\beta<\beta_0$. An Einstein ring of angular radius θ_E appears when $\beta=0$. Note that the slope $f'(\theta)$ of the lens mapping asymptotically approaches unity

- 轴对称，充要；非轴对称，充分不必要

- NIS的验证：

$$\kappa(0)=\frac{1}{2\sqrt{0^2+x_c^2}}>1\Rightarrow x_c<\frac{1}{2}$$

非轴对称：椭圆

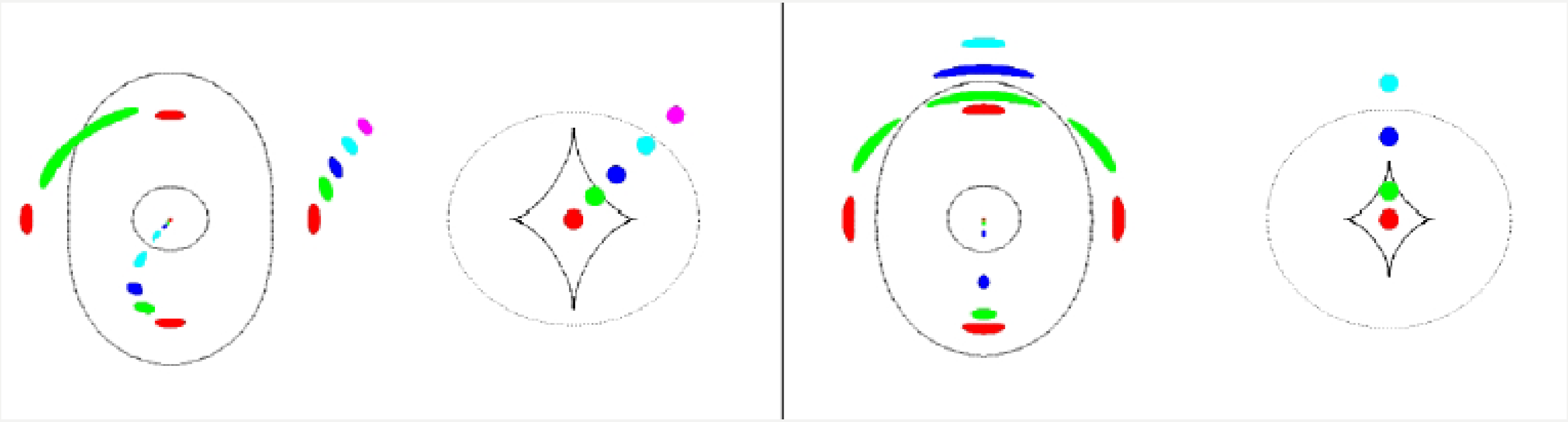
- 轴对称中间变量代换

投影质量$\Sigma(x)=f(x)$，其中$x\rightarrow X=\sqrt{x_1^2/(1-e)+x_2^2\cdot(1-e)}$

透镜势$\Psi(x)=g(x)\Rightarrow \alpha_1=\frac{\partial\Psi}{\partial x_1}=\frac{x_1}{(1-e)X}\tilde{\alpha}(X)$，$\alpha_2=\frac{\partial\Psi}{\partial x_2}=\frac{x_2(1-e)}{X}\tilde{\alpha}(X)$

$$\left(\frac{\partial\Psi}{\partial x_i x_j}\right)=\begin{pmatrix}-\kappa-\gamma_1 & -\gamma_2\\ -\gamma_2 & -\kappa+\gamma_1\end{pmatrix},\quad \mu=\frac{1}{(1-\kappa)^2-\gamma^2}$$

- 源穿过Fold caustic和Cusp caustic，±2个像
  - Einstein cross十字（像平面critical line，源平面caustic）



时间延迟

$$\Delta t = \frac{1+z_l}{c}\frac{D_{ol}D_{os}}{D_{ls}}\left[\frac{(\theta-\beta)^2}{2}-\phi(\theta)\right],$$

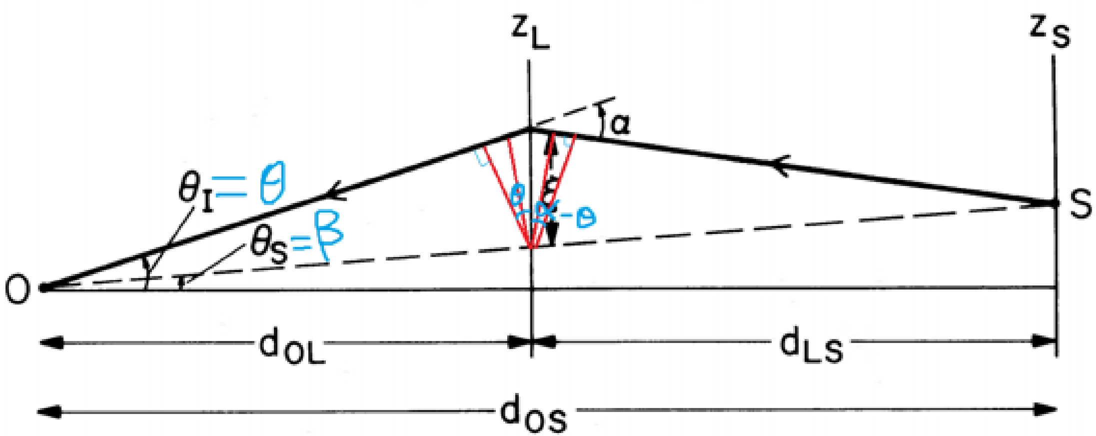
- 尺度因子，几何路径延迟、引力势延迟（尚未完全理解）

几何关系确实可以验证：
ΔI=ξ*sinθ-ξ*cosθ*tan(θ-β)/2+ξ*sin(α-θ)-ξ*cos(α-θ)*tan(α-θ+β)/2=ξ*α/2
由此容易计算几何路径延迟

the observer along the direction θ_I. In the small angle approximation, the geometrical time delay due to the extra path length of the ray as compared with the direct ray from S to O is measured by an observer in the lens plane to be $\alpha\xi/2$ (Fig. 1), where here and henceforth we set $G = c = 1$. Noting that $\xi = (\theta_I - \theta_S)d_{OL}$ and inserting a redshift factor, we find that the observed time delay is

$$t_{\rm geom}(\theta_I;\theta_S) = \frac{(1+z_L)d_{OL}\,d_{OS}}{2d_{LS}}(\theta_I-\theta_S)^2\,. \tag{2.1}$$

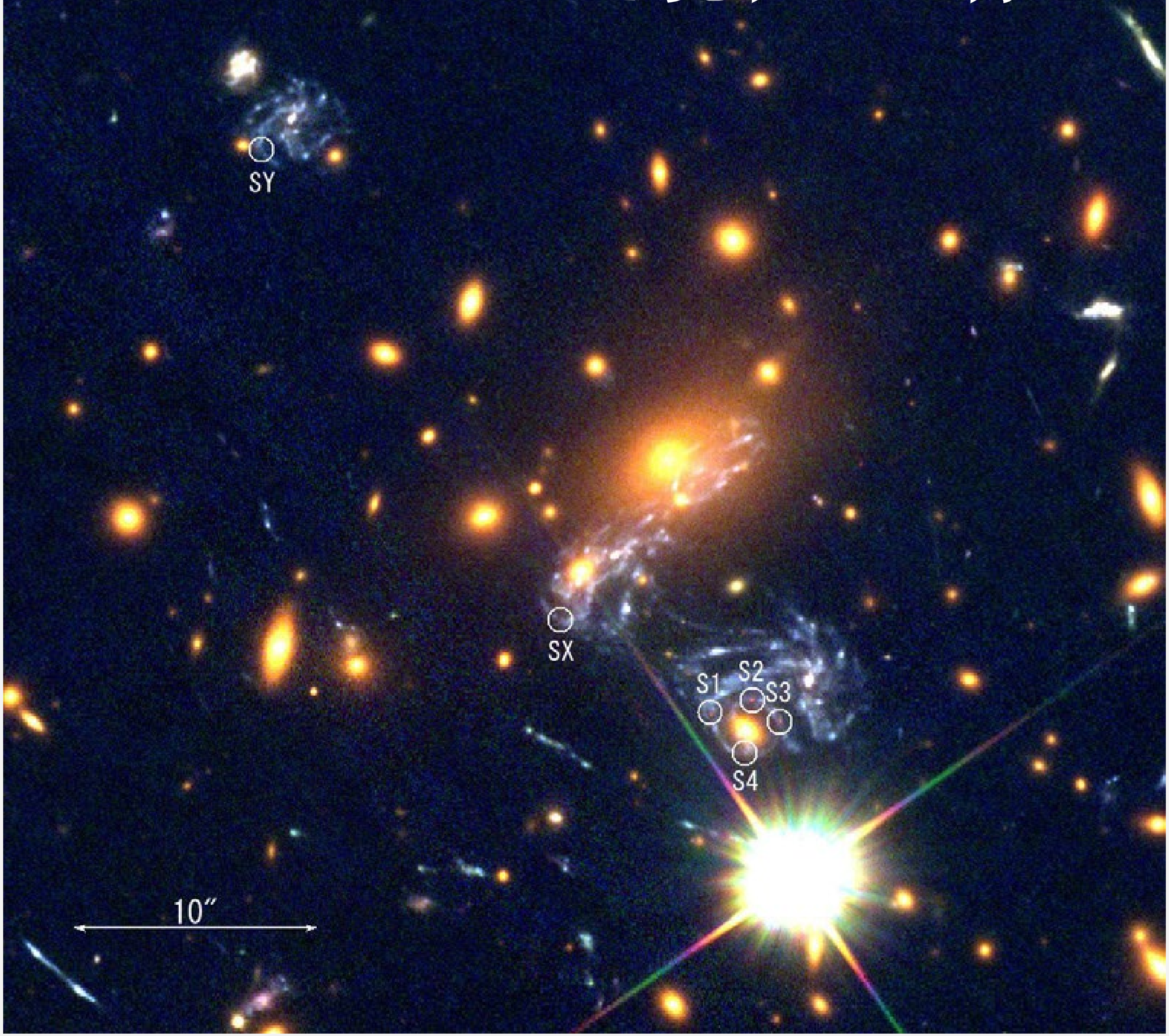
Blandford, Roger 1986APJ



对θ求导恰为透镜方程。
此即，
Fermat原理：光路为光程取极值的路径

能干啥

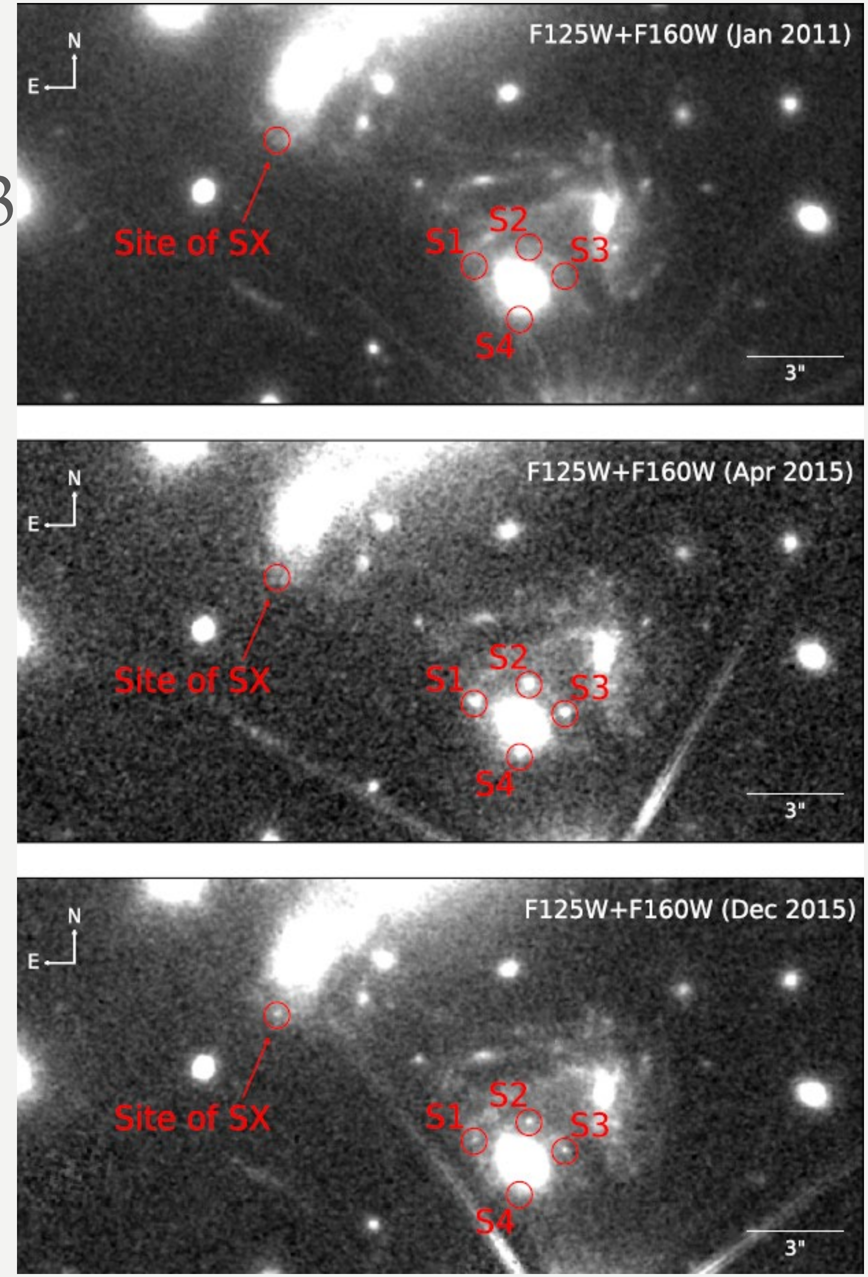
SPUERNOVA透镜化6重像



Oguri M 2019, Strong gravitational lensing of explosive transients, *Rep. Prog. Phys. 82 126901*

SN透镜化6重像

- 星系团MACS J1149.6+2223
- 旋涡星系z=1.49多重像
- 超新星：II-SN Refsdal
- 被椭圆星系z=0.54透镜化
- 2014.11.11观测到S1-S4
- 2015.12.11观测到SX



Kelly P L *et al* 2015 Multiple images of a highly magnified supernova formed by an early-type cluster galaxy lens *Science* **347** 1123–6
Kelly P L *et al* 2016 Deja vu all over again: the reappearance of supernova Refsdal *Astrophys. J.* **819** L8

主要应用

Oguri M 2019, Strong gravitational lensing of explosive transients, *Rep. Prog. Phys.* 82 126901

- 多重像，重构质量分布->重子物质、暗物质。但鲁棒性严重不足，需要光通量比、透镜弧、速度弥散
- 放大效果->看得更远更暗，看清事件本身。如（利用时延）提前等待并探测超新星光变曲线前段
  - Gravitational lens：现象phenomenon ✖ 工具tool ✔
- 时延宇宙学->距离，测H0->缓解H0 tension
  - 超新星标准烛光、CMB各向异性分别给出H0=74、67
- 波动光学强透镜->小尺寸源导致的干涉、更小的衍射，主要对GWs，可探测致密暗物质

REFENCE

- [Massimo Meneghetti]Gravitational Lensing-Lecture scripts
- 梁灿彬，微分几何入门与广义相对论，2006
- Wambsganss_1998_Gravitational Lensing in Astronomy
- [Dodelson, S. 2003] Modern Cosmology, Academic Press, San Diego.
- [A. B. Congdon2018] Principles Of Gravitational Lensing