In this section, we describe how to apply Bayesian parameter estimation for the Rao-Blackwellized Particle Filters developed in **??** to **??** using a Metropolis-within-Gibbs sampling scheme.

We build upon the discrete-time conditionally gaussian state space models given by **??**, with the extension that the matrices governing the model dynamics $\boldsymbol{A}(\boldsymbol{\theta})$, $\boldsymbol{b}(\boldsymbol{\theta})$, $\boldsymbol{C}(\boldsymbol{\theta})$ and $\boldsymbol{d}(\boldsymbol{\theta})$ all depend on some parameter vector $\boldsymbol{\theta}$ of fixed dimension.

In the context of our project, we define the parameter estimation problem as estimating the parameter vector $\boldsymbol{\theta} = \{\theta, \sigma, \sigma_{obs}\}$, given observations up to time T (i.e. $y_{1:T}$)

$$p(\boldsymbol{\theta}, \lambda_{0:k}|y_{1:k}) \tag{1}$$

We construct a Markov Chain to target the joint distribution of the sample parameters $\boldsymbol{\theta}$ and RBPF latent states $\lambda_{1:T}^{(i)}$ given by **??** using Gibbs Sampling by iteratively performing the following steps:

1. We draw $\boldsymbol{\theta}$ iteratively using a Metropolis Hastings step to target the required marginal distribution for θ_j given by Equation 2.

$$\begin{aligned} \theta_j &\sim p(\theta_j|\theta_{-j}, \lambda_{0:T}, y_{1:T}) \\ &\propto p(y_{1:T}|\boldsymbol{\theta}, \lambda^*_{0:T})p(\lambda^*_{0:T}|\boldsymbol{\theta})p(\theta_j|\theta_{-j}) \end{aligned} \tag{2}$$

   Here, we denote $\theta_{-j} = \boldsymbol{\theta} \setminus \theta_j$ and using $\lambda^*_{0:T}$ to make explicit the notion that a single realisation of $\lambda_{0:T}$ is being used.

   We can sample from this probability distribution using a Metropolis-Hastings step, with proposal density $q(\theta^*_j|\theta'_i)$, where θ'_j is the current sample of parameter θ_j and θ^*_j is the current proposal for θ_j.

   For the Metropolis-Hastings step, we accept the current proposal θ^*_j with acceptance probability $\min(1, \alpha)$, with α given by:

$$\alpha = \frac{p(y_{1:T}|\theta_{-j}, \theta^*_j, \lambda^*_{0:T})p(\lambda^*_{0:T}|\theta_{-j}, \theta^*_j)p(\theta^*_j|\theta_{-j})q(\theta'_j|\theta^*_j)}{p(y_{1:T}|\theta_{-j}, \theta'_j, \lambda^*_{0:T})p(\lambda^*_{0:T}|\theta_{-j}, \theta'_j)p(\theta'_j|\theta_{-j})q(\theta^*_j|\theta'_j)} \tag{3}$$

2. With the accepted parameter vector $\boldsymbol{\theta_{accepted}}$ from step 1, we can then look to sample another realisation of $\lambda^*_{0:T}$ by first drawing a set of weighted samples $\left\{w_T^{(i)}, \lambda_{0:T}^{(i)}\right\}$ from the RBPF, denoted by denoted by Equation 4.

$$\left\{w_T^{(i)}, \lambda_{0:T}^{(i)}\right\} \sim p(\lambda_{0:T}|y_{1:T}, \boldsymbol{\theta_{accepted}}) \tag{4}$$

Using the weighted samples $\left\{w_T^{(i)}, \lambda_{0:T}^{(i)}\right\}$ as an importance-sampling estimate of $p(\lambda_{0:T}|y_{1:T}, \boldsymbol{\theta_{accepted}})$, we then draw a particular realisation $\lambda^*_{0:k}$ using multinomial sampling by selecting $\lambda_{0:T}^{(i)}$ with probability $w_T^{(i)}$

For each time step, we thus obtain joint samples $\{\boldsymbol{\theta}, \lambda^*_{0:T}\}$, which can be marginalised to obtain samples from the posterior distribution of parameters.

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