0.0.1 Basic State Space Model

We build upon the model proposed by Christensen et al. [**christensen2012forecasting**] and summarised in Equation 1. In the original Christensen model, the state of the traded commodity is modelled by 2 components: "value" x_1 and "trend" x_2. These components correspond to the the price and return of the traded commodity respectively.

In this model, the trend is specified as a mean-reverting random process subject to two different types of random innovations: Gaussian noise of constant volatility and random jumps. The value component(x_1) is obtained by integrating the trend process (x_2) with respect to time. Observed prices, y are modeled as observations of the value process x_1 corrupted by random Gaussian noise.

$$\begin{bmatrix} dx_{1,t} \\ dx_{2,t} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & \theta \end{bmatrix} \begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix} dt}_{\text{Mean Reverting Returns}} + \underbrace{\begin{bmatrix} 0 \\ \sigma \end{bmatrix} dW_t}_{\text{Brownian Motion}} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix} dJ_t}_{\text{Finite-activity Gauss-Poisson jump}} \tag{1}$$

However, sharp trend changes can cause difficulty for momentum strategies. This model attempts to address this issue by allowing for a finite number of discrete jumps in the trend process. This allows for modelling of sharp changes of sentiment in the market, and allow the filtered predictions to reflect these changes of sentiment more quickly than models that simply smooth the price series.

0.0.2 Proposed State-Space Model

Statistical testing has found that infinite-activity jumps (rather than finite-activity jumps) are present in high frequency stock returns [**ait2011testing**]. This might be due to significant price movements triggering stop-loss orders that prompt further trading, leading to a self-exciting series of high frequency small jumps [**osler2005stop**].

Thus, we propose modelling a tradable commodity's price using Lévy-type processes with infinite jump activity as given by Equation 3 below.

$$\begin{bmatrix} dx_{1,t} \\ dx_{2,t} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & \theta \end{bmatrix} \begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix} dt}_{\text{Mean Reverting Returns}} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix} dL_t}_{\text{Infinite-activity Lévy process}} \tag{2}$$

$$y_t = x_{1,t} + \sigma_{obs}\eta_t \tag{3}$$

where:

$$\eta_t \sim \mathcal{N}(0, 1)$$

0.0.3 Choice of Lévy process

We choose use an α-stable Lévy process to capture the characteristics of the noise terms of the basic state space model provided in Equation 1 – namely the Brownian motion term and the jump term. An α-stable Lévy process has both a continuous gaussian component and an infinite-activity jump process component, which allows us to capture the noise characteristics proposed in Equation 1 in a concise manner. This also allows us to leverage existing literature developed for α-stable Lévy processes to tackle the inference problem.

The family of α-stable distributions forms a rich class of distributions which allow for asymmetry and heavy tails. We define a real-valued random variable X to follow an stable distribution $S_\alpha(\sigma, \beta, \mu)$ if and only if its characteristic function is given by:

$$\mathbb{E}[e^{itX}] = \begin{cases} \exp\left(-\sigma^\alpha |t|^\alpha [1 - i\beta \text{sign}(t) \tan(\frac{\alpha\pi}{2})] + i\mu t\right) & \alpha \neq 1 \\ \exp\left(-\sigma^\alpha |t| [1 + i\beta \frac{2}{\pi} \text{sign}(t) \ln t] + i\mu t\right) & \alpha \neq 1 \end{cases} \tag{4}$$

We can then define the scalar α-stable process L_t that we will use as the driving Lévy process in our state-space model as a process which has:

- $L_0 = 0$
- Independent α-stable increments: $L_t - L_s \sim S_\alpha((t-s)^{1/\alpha}, \beta, 0), t > s$

For $1 \leq \alpha < 2$, this is a pure jump plus gaussian drift process as desired.

0.0.4 Solution for the proposed model

We consider a one-dimensional version of our proposed model which only considers the returns component (x_2).

$$dx_{2,t} = \theta x_{2,t} dt + \sigma dL_t \tag{5}$$

For the scalar case, this can be solved using results based on [**samoradnitsky2017stable**] to give a tractable state transition density:

$$f(x_{2,t}|x_{2,s}) \sim S_\alpha(\sigma_{t-s}, \beta, \exp(\theta(t-s)x_{2,s}), \qquad \sigma_{\delta t} = \left(\sigma \frac{\exp(\alpha\theta\delta t) - 1}{\alpha\theta}\right)^{1/\alpha} \tag{6}$$

0.0.5 Formulation of the Inference Problem in Discrete Time

Using the solution to the one-dimensional version of the proposed model in Equation 6, we can formulate our original continuous-time state space model as a discrete-time state space model where the parameters of the discrete-time state matrices $\boldsymbol{A}(\delta t)$ and $\boldsymbol{b}(\delta t)$ vary according to the continuous time solution and the gap to the previous time step, δt.

To reflect the discretised nature of time used in the inference problem, we change the time indexes using $k = t+\delta t, k-1 = t$. For brevity, we will also assume in all subsequent equations that $\boldsymbol{A}$ and $\boldsymbol{b}$ are implicitly dependent on δt, and will drop this explicit dependence.

This approach allows us to use established techniques used for discrete time inference.

$$\begin{aligned} \mathbf{x}_k &= \mathbf{A}(\delta t).\mathbf{x_{k-1}} + \mathbf{b}(\delta t).L_k \\ y_k &= \mathbf{C}.\mathbf{x}_k + \mathbf{d}.\eta_k \\ \text{where} \qquad \delta_t &= t_k - t_{k-1} \\ l_k &\sim S_{\alpha/2}(1,1,0) \text{ and } \eta_k \sim \mathcal{N}(0,1) \end{aligned} \tag{7}$$

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