Layer dependence as a measure of local dependence

Abstract

A new measure of local dependence called "layer dependence" is proposed and analysed. Layer dependence measures the dependence between two random variables at different percentiles in their joint distribution. Layer dependence satisfies coherence properties similar to Spearman's correlation, such as lying between -1 and 1, with -1, 0 and 1 corresponding to countermonotonicity, independence and comonotonicity, respectively. Spearman's correlation is a weighted average of layer dependence across all percentiles. Alternate overall dependence measures are arrived by varying the weights. Layer dependence is an important input to copula modeling by extracting the dependence structure from past data and incorporating expert opinion if necessary.

Keywords: Local dependence; rank dependence; Spearman's correlation; layers; conditional tail expectation; concordance.

1. Local dependence and layer dependence

Dependence between two random variables generally varies with percentile. For example extreme movements in stock markets are likely to be highly related whereas minor fluctuations may be relatively independent. Catastrophes create significant insurance losses for several classes of business at the same time, while attritional losses are typically weakly dependent.

Local dependence measures aim to capture the dependence structure of a bivariate distribution. This contrasts with measures of overall dependence such as Pearson correlation, Spearman's ρ and Kendall's τ (Embrechts et al., 2002). Local dependence measures include the univariate tail concentration (Venter, 2002), correlation curve (Bjerve and Doksum, 1993), and bivariate measures by Bairamov et al. (2003), Jones (1996) and Holland and Wang (1987).

This paper introduces, illustrates and analyzes an alternate local dependence measure called "layer dependence." Layer dependence is the covariance between a random variable and a single "layer" of another. Layer dependence is also the "gap" between upper and lower conditional tail expectations. Layer dependence is calculated from the copula underlying the joint distribution. Hence of interest is rank dependence rather than dependence between random variables in their original scale: the latter is often distorted by marginal distributions.

Layer dependence satisfies "coherence" properties similar to Spearman's ρ: it is between -1 and 1, constant and equal to -1, 0 and 1 for countermonotonic, independent and comonotonic random variables, sign switching when the ranking order reverses, and taking on higher values when dependence is stronger. Taking a weighted average of layer dependence values across the joint distribution yields Spearman's ρ and alternate coherent measures of overall dependence.

Layer dependence provides a more appropriate and accurate measure of local dependence compared to existing measures. Higher dispersion between scatter points from the 45° line reduces layer dependence and vice versa. For a Gumbel copula exhibiting upper tail dependence, layer dependence starts from a lower value and increases to 1 while the opposite applies to a Clayton copula with lower tail dependence.

Calculating layer dependence at the first instance extracts essential and interpretable information – the dependence structure. This is often more informative than positing parametric copula, as the implication of its parametric form and parameters on the dependence structure is indirect. Computed layer dependence facilitates the selection and fitting of an appropriate copula.

Remaining sections are as follows. Section 2 discusses the concepts leading to definition of layer dependence. Section 3 demonstrates how layer dependence extracts the dependence structure from common copulas. Section 4 explains the behaviour of layer dependence by decomposing it into a negative function of discordance and dispersion. Section 5 describes coherence properties of layer dependence. Links to existing tail dependence measures are highlighted in §6. Further properties of layer dependence are described in §7. Section 8 forms alternate coherent measures of overall dependence apart from Spearman's ρ, using weighted averages of layer dependence. Section 9 discusses how layer dependence can be applied to copula modeling. Section 10 concludes.

2. Layer dependence – motivation and definition

A familiar construct in the study of bivariate dependence is Spearman's correlation (Embrechts et al., 2002) defined as the linear correlation between ranks of two random variables. Rank dependence avoids distortion arising from marginal distributions as with for example Pearson's correlation (McNeil et al., 2005) measuring the degree of linear relationship between random variables in their original scale. Spearman's correlation can also be applied to estimate copula parameters using the method of moments (Kojadinovic and Yan (2010), Bouyé et al. (2000)). However Spearman's correlation suffers from shortcomings and, as an aggregate measure, is inappropriate for assessing local dependence when dependence varies across the joint distribution including the tails.

Another familiar construct, in reinsurance, is a loss layer (Wang, 1995). For example the 95%–96% layer of a random continuous loss x is the portion of x between its 95th and 96th percentile

$$\min\left\{(x - x_{0.95})^+, x_{0.96} - x_{0.95}\right\} \ ,$$

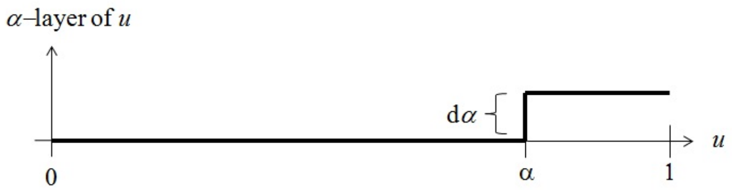

Figure 1: Illustration of α-layer of u, written as $I_\alpha(u)\mathrm{d}\alpha$. The α-layer of u is only sensitive to movements in u at α and ignores other movements.

where $()^+$ indicates the positive part of the expression inside the brackets and subscripts indicate the percentile. Layers also arise in the context of derivative payouts and debt tranches in collaterised debt obligations (Mandel et al., 2012). With rank dependence, the 95%–96% layer reduces to

$$\min\left\{(u-u_{0.95})^+, u_{0.96}-u_{0.95}\right\} = \min\left\{(u-\alpha)^+, \mathrm{d}\alpha\right\} \ , \tag{1}$$

where u is the percentile rank of x and in this example $\alpha = 0.95$ and $\mathrm{d}\alpha = 0.01$. Note $u_\alpha = \alpha$.

2.1. Percentile layer decomposition

The final expression in (1) can be written and approximated by, using the familiar infinitesimal notation,

$$-\mathrm{d}(u-\alpha)^+ \approx I_\alpha(u)\mathrm{d}\alpha \ , \quad I_\alpha(u) \equiv \begin{cases} 0 \ , \ u \leq \alpha \ , \\ 1 \ , \ u > \alpha \ . \end{cases} \tag{2}$$

The left hand side expression in (2) is called the α–layer of u. The approximation becomes exact as $\mathrm{d}\alpha \to 0$. The α–layer of u is an infinitesimally small increment $\mathrm{d}\alpha$ if u exceeds α and is zero otherwise. Hence the α–layer captures the movement in u at α and ignores movements elsewhere. Figure 1 illustrates the α–layer of u.

As often exploited in reinsurance, any non-negative random loss or variable can be thought of as a sum of layers:

$$u = \int_0^u 1\mathrm{d}\alpha = \int_0^1 I_\alpha(u)\mathrm{d}\alpha \ . \tag{3}$$

Hence u is formed from infinitely many α–layers, each capturing the movement of u at a different α.

2.2. Constructing layer dependence

Spearman's correlation is the correlation and also the standardized covariance between two percentile rank random variables v and u:

$$\rho \equiv \mathrm{cor}(u,v) = \frac{\mathrm{cov}(v,u)}{\sqrt{\mathrm{cov}(v,v)\mathrm{cov}(u,u)}} = \frac{\mathrm{cov}(v,u)}{\mathrm{cov}(u,u)} = 12\mathrm{cov}(v,u) \ , \tag{4}$$

where cor and cov indicate covariance and correlation, respectively. In this paper assume the copula of u and v is exchangeable (Nelson, 1999).

Using the decomposition of u in (3),

$$\mathrm{cov}(v,u) = \mathrm{cov}\left\{v, \int_0^1 I_\alpha(u)\mathrm{d}\alpha\right\} = \int_0^1 \mathrm{cov}\{v, I_\alpha(u)\}\mathrm{d}\alpha \; . \tag{5}$$

Hence the covariance $\mathrm{cov}(v,u)$ can be thought of as the sum of infinitely many covariances $\mathrm{cov}\{v, I_\alpha(u)\}$ for $0 \le \alpha \le 1$. Each covariance in the sum measures the dependence between v and the α–layer of u. In a reinsurance setting, this covariance measures dependence between a particular layer of a loss (in percentile rank terms) and another factor. Alternatively the covariance captures dependence between movements in u at α and v.

Similar to ρ, scaling the layer covariances $\mathrm{cov}\{v, I_\alpha(u)\}$ with the same when $v = u$ leads to the definition of layer dependence

$$\ell_\alpha \equiv \frac{\mathrm{cov}\{v, I_\alpha(u)\}}{\mathrm{cov}\{u, I_\alpha(u)\}} \; , \tag{6}$$

where the denominator

$$\mathrm{cov}\{u, I_\alpha(u)\} = \mathrm{E}[\{u - \mathrm{E}(u)\}I_\alpha(u)] = \int_\alpha^1 \left(u - \frac{1}{2}\right)\mathrm{d}\alpha = \frac{\alpha(1-\alpha)}{2} \; . \tag{7}$$

Combining (4), (5), (6) and (7) yields

$$\rho = 12\int_0^1 \ell_\alpha \mathrm{cov}\{u, I_\alpha(u)\}\mathrm{d}\alpha = \int_0^1 \ell_\alpha 6\alpha(1-\alpha)\mathrm{d}\alpha \; . \tag{8}$$

Hence ρ is a weighted average of ℓ_α for $0 \le \alpha \le 1$ with weights $w_\alpha = 6\alpha(1-\alpha)$ integrating to 1, and ℓ_α decomposes ρ into local dependence values. Note w_α has minimum 0 at $\alpha = 0$ and 1 and increases symmetrically to maximum at $\alpha = 0.5$. Hence Spearman's correlation places little emphasis on the tails which may be undesirable in finance or insurance where tail dependence is critical. Modifying the weights w_α leads to alternate measures of overall dependence further discussed in §8.

Layer dependence summarises the dependence structure of a copula. Layer dependence provides additional information compared to Spearman's correlation: how dependence varies across the joint distribution. As any summary measure, layer dependence can mislead but is less misleading than ρ and other measures of overall dependence. Layer dependence is a more meaningful characterisation of a copula compared to the parameters of copula families such as the Clayton or Gumbel (McNeil et al., 2005). Properties of and arguments for using layer dependence are explored below.

2.3. Key properties of layer dependence

In §3 and §4 it is illustrated and shown that $\ell_\alpha = 1$ occurs if and only if $I_\alpha(u)$ and $I_\alpha(v)$ are both simultaneously 1 or 0: that is u and v are both either greater than α or less than α. Additionally $\ell_\alpha = 1$ for $c \le \alpha \le d$ implies $u = v$ over $c \le u, v \le d$.

Layer dependence ℓ_α satisfies the following coherence properties for all $0 \le \alpha \le 1$. These properties are shared with Spearman's correlation and are formalised in §5. Similar to correlation, $0 \le \ell_\alpha \le 1$, and $\ell_\alpha = -1$, 0 and 1 if u and v are countermonotonic, independent and comonotonic, respectively. In addition ℓ_α increases with the correlation order of (u, v). Replacing u or v with their complement leads to straightforward changes in layer dependence.

The following is an alternative expression for ℓ_α:

$$\ell_\alpha = \frac{\mathrm{E}(v|u > \alpha) - \mathrm{E}(v|u \le \alpha)}{\mathrm{E}(u|u > \alpha) - \mathrm{E}(u|u \le \alpha)} = 2\left\{\mathrm{E}(v|u > \alpha) - \mathrm{E}(v|u \le \alpha)\right\} . \qquad (9)$$

Proofs are in an appendix. The middle expression in (9) is the expected change in v relative to the expected change in u when u crosses α. The latter is 0.5 for all α, yielding the final expression in (9). Hence large ℓ_α implies v is sensitive to movements in u across α, indicating strong dependence between v and u at α. When $\ell_\alpha = 0$, v is unchanged on average when u crosses α.

3. Illustration of layer dependence curves for various copulas

The nine panels in Figure 2 display (u, v) scatterplots of well known exchangeable copulas. Less standard copulas are also shown for illustration. All copulas are calibrated to have equal Spearman's correlation $\rho = 0.6$. Layer dependence curves ℓ_α for all $0 \le \alpha \le 1$ are plotted against α on (u, v) scatterplots to demonstrate the link between the scatter and ℓ_α.

The scatterplots in Figure 2 emphasize that copulas with the same overall dependence can exhibit a variety of local dependence structures and these structures are captured by layer dependence curves. Given α, ℓ_α is larger if scatter points are more clustered around (α, α) and vice versa. In particular ℓ_α increases to one in the tails of copulas exhibiting strong tail dependence. Hence ℓ_α tracks the clustering of scatter points across the 45° line. This is formalised in §4.

4. Discordance and dispersion

Layer dependence is intimately connected to discordance and dispersion. Again assuming the copula C of (u, v) is exchangeable then

$$\ell_\alpha = 1 - 2(1 + \gamma_\alpha)\delta_\alpha , \qquad (10)$$

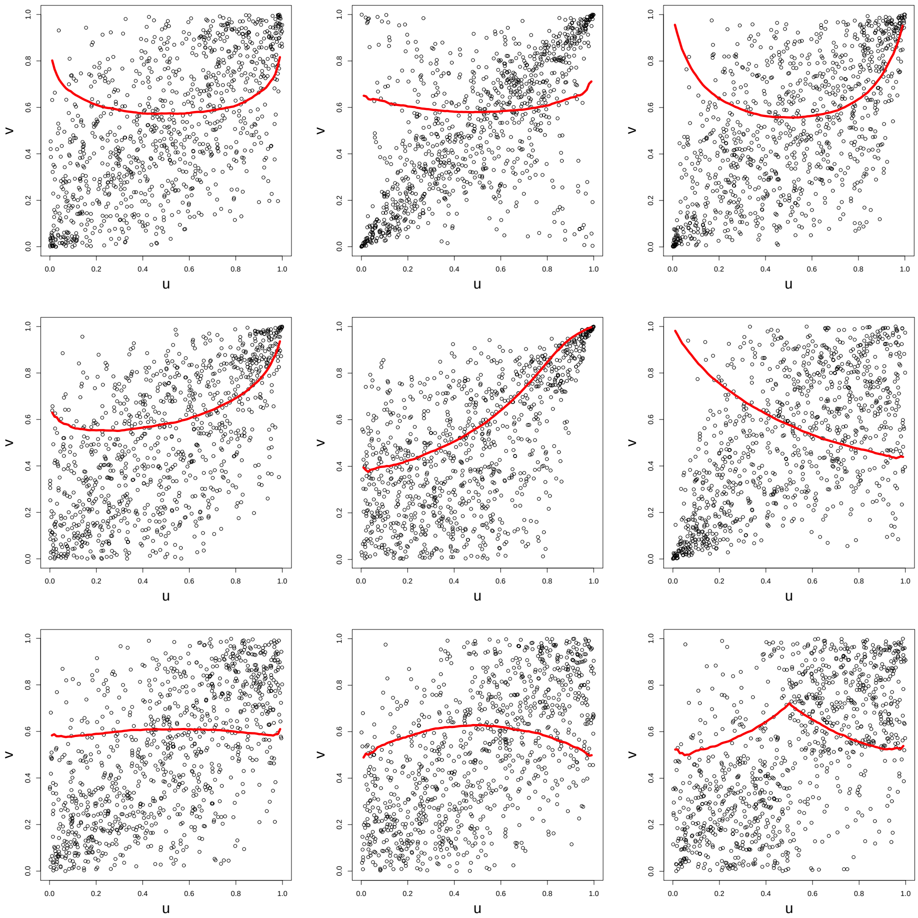

Figure 2: Copulas all with the same $\rho = 0.6$ but different layer dependence curves ℓ_α over $0 \leq \alpha \leq 1$ (in red). The top 3 copulas display relatively strong tail dependence in both lower and upper tail. The first two copulas in the middle three have relatively strong upper tail dependence and while the third has a high degree of lower tail dependence. The bottom three copulas have relatively high local dependence in the middle of the distribution and less correlation in the tails. Overall the panels illustrate how a given overall level of ρ can mask a range of dependence structures and that layer dependence is an appropriate tool for assessing local dependence. The top left panel displays a Gaussian copula, followed by a Student's t copula on the right. The leftmost panel in the middle three is a Gumbel copula, and the rightmost is a Clayton copula. The bottom left panel is a Frank copula. Remaining copulas are constructed from a factor model and are included to illustrate how layer dependence captures local dependence.

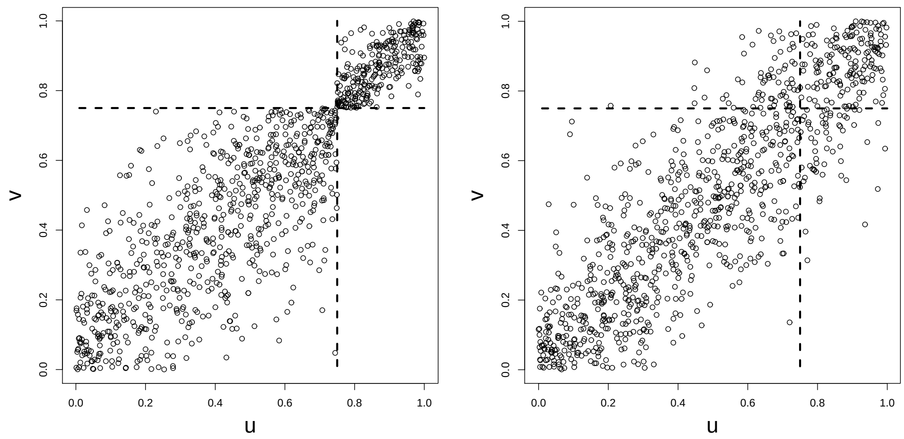

Figure 3: The left and right panel show copulas where $\ell_{0.75} = 1$ and $\ell_{0.75} = 0.86$, respectively. In the left panel, $\gamma_{0.75} = -1$ (no discordant points) and $\delta_{0.75} = 0$ (zero dispersion). In the right panel, $\gamma_{0.75} = -0.65$ (some discordant points) and $\delta_{0.75} = 0.21$ (some dispersion between discordant points).

where

$$\gamma_\alpha \equiv \mathrm{cor}\{1 - I_\alpha(u), I_\alpha(v)\} = \mathrm{cor}\{I_\alpha(u), 1 - I_\alpha(v)\} = \frac{\alpha^2 - C(\alpha, \alpha)}{\alpha(1-\alpha)} \ ,$$

and

$$\delta_\alpha \equiv \mathrm{E}\left\{(|u - v|)|(u-\alpha)(v-\alpha) < 0\right\} \ .$$

A proof of (10) is in the appendix. Equation (10), as shown below, explains the behaviour of layer dependence curves in Figure 2 where ℓ_α for any α increases as the scatter become more tightly concentrated around the point (α, α), that is lower discordance and dispersion relative to the same point.

The correlation $-1 \leq \gamma_\alpha \leq 1$ measures the tendency for (u, v) to be discordant at α: either u or v is above α and the other is below α. The expectation $0 \leq \delta_\alpha \leq 1$ measures the average dispersion between discordant u and v at α, noting $(u-\alpha)(v-\alpha) < 0$ is equivalent to $u > \alpha$ and $v \leq \alpha$ or $u \leq \alpha$ and $v > \alpha$.

Figure 3 illustrates (10) using two copulas specifically constructed such that, for $\alpha = 0.75$, $\ell_\alpha = 1$ for one and $\ell_\alpha = 0.86$ for the other. For the copula where $\ell_\alpha = 1$, there is no discordance or dispersion at α. This observation is confirmed by setting $\ell_\alpha = 1$ in (10) yielding $\gamma_\alpha = -1$ or $\delta_\alpha = 0$. In this case u and v are perfectly dependent at α, and are simultaneously below or above α. It is also straightforward to extend this result such that if $\ell_\alpha = 1$ over $c \leq \alpha \leq d$ then $u = v$ over $c \leq u, v \leq d$. As ℓ_α decreases from 1, as for the second copula, the extent of discordance γ_α and dispersion δ_α increases. Again this result can be confirmed from (10) noting ℓ_α is negatively related to γ_α and δ_α.

Applying (10) and generalising the illustration in Figure 3 explains the behaviour of layer dependence curves in Figure 2. Layer dependence ℓ_α is larger if scatter points are more clustered around (α, α), that is, fewer discordant points at α, and discordant points at α are closer to the 45° degree line. The former indicates smaller γ_α and the latter indicates smaller δ_α. Opposite observations apply for small ℓ_α.

5. Coherence properties of layer dependence

Layer dependence ℓ_α satisfies the following five "coherence" properties. These properties are extensions of properties applying to Spearman's correlation.

- **Bounds**: Layer dependence lies between -1 and 1: $-1 \leq \ell_\alpha \leq 1$ for all α. Hence layer dependence is bounded in the same way as ρ.
- **Perfect dependence**: Constant layer dependence of -1 or 1 are equivalent to countermonotonicity and comonotonicity, respectively. Thus $\ell_\alpha = -1$ for all α if and only if $v = 1 - u$ while $\ell_\alpha = 1$ for all α if and only if $v = u$.
- **Independence**: If u and v are independent then $\ell_\alpha = 0$ for all α. The converse is not true – zero layer dependence does not imply independence as shown by the following counterexample. Assume $v = u$ and $v = 1 - u$ with equal probability. Then $\mathrm{E}(v|u = \alpha) = 0.5$ for all $0 \leq \alpha \leq 1$ implying $\mathrm{E}(v|u > \alpha) = \mathrm{E}(v|u \leq \alpha) = 0.5$. Hence $\ell_\alpha = 0$ from (9). However u and v are not independent.
- **Symmetry**: Replacing v with $1 - v$ yields layer dependence curve $-\ell_\alpha$. Doing the same to u (the random variable decomposed into layers) yields layer dependence curve $-\ell_{1-\alpha}$ hence a flip is performed about $\alpha = 0.5$ in addition to a sign change. Replacing both u and v with their complements yields layer dependence curve $\ell_{1-\alpha}$.
- **Ordering**: Higher correlation order (Dhaene et al., 2009) leads to higher layer dependence. Specifically, consider bivariate uniform (u^*, v^*) exceeding (u, v) in correlation order: $C^*(a, b) \geq C(a, b)$ for all $0 \leq a, b \leq 1$, where C^* is the copula of (u^*, v^*). Then $\ell^*_\alpha \geq \ell_\alpha$, $0 \leq \alpha \leq 1$ where ℓ^*_α denotes the α–layer dependence of (u^*, v^*). Hence greater dependence leads to a higher layer dependence curve.

Independence and symmetry properties follow from the definition of layer dependence in (6). From (9), constant layer dependence of one implies $\mathrm{E}(v|u > \alpha) = (\alpha+1)/2$ and $\mathrm{E}(v|u = \alpha) = \alpha$, for all $0 \leq \alpha \leq 1$, hence $v = u$. Similarly constant layer dependence of minus one implies $v = 1 - u$. The ordering property holds since higher correlation order implies larger covariances (Dhaene et al., 2009). Prove the bounds property by combining ordering and perfect dependence properties, and noting countermonotonicity and comonotonicity represent minimum and maximum correlation order, respectively. Detailed proofs are shown in the appendix.

Most of the layer dependence coherence properties can be expressed using copulas shown in Table 1.

Copula	Description	LD
uv	Independent u and v	0
$\min(u, v)$	Comonotonic u and v	1
$\max(u + v - 1, 0)$	Countermonotonic u and v	-1
$u - C(u, 1 - v)$	Replace v with $1 - v$	$-\ell_\alpha$
$v - C(1 - u, v)$	Replace u with $1 - u$	$-\ell_{1-\alpha}$
$u + v - 1 + C(1 - u, 1 - v)$	Replace v and u with complements	$\ell_{1-\alpha}$

Table 1: Layer dependence (LD) for different copulas and transformations. Note the final copula is the survival copula of C.

6. Connections to measures of tail dependence

Measures have been proposed to capture the degree of tail dependence. Tail dependence is dependence between extreme values of random variables, in this case values of u and v near 0 or 1. Strong tail dependence creates catastrophic events such as multiple bank failures and market crashes. Layer dependence is intimately connected to two existing tail dependence measures – coefficient of tail dependence and tail concentration function. Sweeting and Fotiou (2013) and Durante et al. (2014) further discuss tail dependence measures.

6.1. Coefficients of tail dependence

Joe (1997) defines coefficients of lower and upper tail dependence as

$$\lambda_L \equiv \lim_{\alpha \downarrow 0} \mathrm{P}(v \leq \alpha | u \leq \alpha) \ , \quad \lambda_U \equiv \lim_{\alpha \uparrow 1} \mathrm{P}(v > \alpha | u > \alpha) \ .$$

Unit coefficients indicate perfect positive tail dependence, and occur if and only if u converging to 0 or 1 implies the same for v. Coefficients of negative tail dependence replace $v \leq \alpha$ and $v > \alpha$ in the above expressions with $v > 1-\alpha$ and $v \leq 1 - \alpha$, respectively. Sweeting and Fotiou (2013) discusses the drawback of these coefficients and suggests a modification by weakening the limits, yielding links to tail concentration functions discussed below.

From (9), $\ell_0 = \lim_{\alpha\downarrow 0} \ell_\alpha = 1 - 2\mathrm{E}(v|u = 0)$ which equals 1 if and only if $\mathrm{E}(v|u = 0) = 0$, or v approaching 0 if the same happens to u. Hence $\ell_0 = 1$ is equivalent to $\lambda_L = 1$. Similarly $\ell_1 = 2\mathrm{E}(v|u = 1) - 1 = 1$ if and only if $u = 1$ implies $v = 1$, which is equivalent to $\lambda_U = 1$. Similar results apply to perfect negative tail dependence.

Hence layer dependence characterises perfect tail dependence in the same way as the measures of Joe (1997): variables attaining their maximum or minimum values simultaneously.

6.2. *Tail concentration*

Tail concentration (Venter, 2002) is a local dependence measure formed from lower and upper conditional tail probabilities. Similar to layer dependence, tail concentration functions describe the dependence structure of copulas and are used to identify families of copulas. The following shows the connection between layer dependence and tail concentration and shows layer dependence refines tail concentration by incorporating additional information on dispersion defined in §4.

The tail concentration at α is

$$\tau_\alpha \equiv \begin{cases} \mathrm{P}(v \le \alpha | u \le \alpha) = \frac{C(\alpha,\alpha)}{\alpha} \ , & \alpha \le 0.5 \ , \\ \mathrm{P}(v > \alpha | u > \alpha) = \frac{1-2\alpha+C(\alpha,\alpha)}{1-\alpha} \ , & \alpha > 0.5 \ . \end{cases}$$

A larger τ_α implies u and v are more likely to fall in the same tail – lower tail if $\alpha \le 0.5$ and upper tail if $\alpha > 0.5$. Letting $\alpha \to 0$ and $\alpha \to 1$ yields the coefficients of tail dependence λ_L and λ_U discussed above. Properties of tail concentration and its applications to distinguish families of copulas are discussed in Durante et al. (2014).

To show the connection between layer dependence and tail concentration, first standardise τ_α by subtracting τ_α^0, its value under independence, and dividing by $\tau_\alpha^+ - \tau_\alpha^0$ where τ_α^+ is the value of τ_α if $u = v$, yielding:

$$\tau_\alpha^* = \frac{\tau_\alpha - \tau_\alpha^0}{\tau_\alpha^+ - \tau_\alpha^0} = \frac{C(\alpha,\alpha) - \alpha^2}{\alpha(1-\alpha)} \ , \quad \tau_\alpha^0 = \begin{cases} \alpha \ , & \alpha \le 0.5 \\ 1-\alpha \ , & \alpha > 0.5 \end{cases} \ , \quad \tau_\alpha^+ = 1 \ .$$

Then τ_α^* equals 1 or 0, if $u = v$ or if u and v are independent, respectively.

Hence $\tau_\alpha^* = -\gamma_\alpha$ where γ_α measures discordance between u and v at α as discussed in §4. Combining this result with (10) yields

$$\ell_\alpha = 1 - 2\delta_\alpha(1 - \tau_\alpha^*) = 1 - 2\delta_\alpha + 2\delta_\alpha\tau_\alpha^*$$

where δ_α as defined in §4 measures average dispersion between discordant u and v at α. Note ℓ_α is increasing in τ_α^* and τ_α. It is also straightforward to show that $\ell_\alpha = 1$ is equivalent to $\tau_\alpha^* = 1$ which is in turn equivalent to $\tau_\alpha = 1$. Hence layer dependence and tail concentration have equivalent characterisations of perfect local dependence at any point. This extends the above result that perfect tail dependence has identical implications on layer dependence and coefficients of tail dependence by Joe (1997).

Hence tail concentration and layer dependence are closely connected. Tail concentration (in its standardised form) is only one of two factors forming layer dependence. The other factor is the dispersion between discordant points, measured by δ_α. Using the right panel of Figure 3, τ_α or τ^*_α only reflects the number of discordant points at α, whereas layer dependence combines this information and average dispersion between discordant points. Hence layer dependence refines tail concentration by standardising and including information on dispersion.

7. Further properties of layer dependence

This section lists and explores further properties and results of layer dependence.

7.1. Copula integration

Layer dependence ℓ_α can be written as a standardised integral of the copula:

$$\ell_\alpha = \frac{\int_0^1 \operatorname{cov}\{I_\alpha(u), I_\beta(v)\}\mathrm{d}\beta}{\alpha(1-\alpha)/2} = \frac{2\int_0^1 C(\alpha,\beta)\mathrm{d}\beta - \alpha}{\alpha(1-\alpha)} \ . \tag{11}$$

The result follows from (6) by decomposing $v = \int_0^1 I_\beta(v)\mathrm{d}v$ similar to (3) and noting $\operatorname{cov}\{I_\alpha(u), I_\beta(v)\} = C(\alpha,\beta) - \alpha\beta$.

Thus ℓ_α summarises a copula: reducing the dimension from two and one, and scales the result to ensure it lies between ± 1. This computation extracts the dependence structure from the copula. In comparison, tail concentration performs the dependence extraction by computing the diagonal section $C(\alpha,\alpha)$.

7.2. Layer dependence preserves convex combination

Layer dependence is preserved under convex combinations of copulas. Suppose (u^*, v^*) has α–layer dependence ℓ^*_α and copula C^*. Then a mixture $b(u,v) + (1-b)(u^*,v^*)$ where b is Bernoulli with $\mathrm{E}(b) = \pi$ has copula $\pi C + (1-\pi)C^*$ and layer dependence $\pi\ell_\alpha + (1-\pi)\ell^*_\alpha$. Hence layer dependence preserves convex combinations of copulas. The proof follows directly from (11).

It is also straightforward to show that layer dependence preserves multiple and continuous convex combinations of copulas.

7.3. One-sided conditional tail expectations

Since $\mathrm{E}(v) = \alpha\mathrm{E}(v|u \le \alpha) + (1-\alpha)\mathrm{E}(v|u > \alpha)$ it follows from (9) that

$$\ell_\alpha = \frac{\mathrm{E}(v|u > \alpha) - \mathrm{E}(v)}{\mathrm{E}(u|u > \alpha) - \mathrm{E}(u)} = \frac{\mathrm{E}(v|u \le \alpha) - \mathrm{E}(v)}{\mathrm{E}(u|u \le \alpha) - \mathrm{E}(u)} \ ,$$

the gap between upper or lower conditional tail expectations of v and the unconditional expectation. Denominators are again scaling factors ensuring $\ell_\alpha = 1$ if u and v are comonotonic and $\ell_\alpha = -1$ if countermonotonic.

7.4. Layer dependence for a non-exchangeable copula

If the copula of u and v is not exchangeable, $C(u,v) \neq C(v,u)$, then layer dependence changes when v is decomposed into layers rather than u:

$$\frac{\mathrm{cov}\{v, I_\alpha(u)\}}{\mathrm{cov}\{u, I_\alpha(u)\}} \neq \frac{\mathrm{cov}\{u, I_\alpha(v)\}}{\mathrm{cov}\{v, I_\alpha(v)\}} ,$$

that is, the dependence between v and α–layer of u differs from the dependence between u and α–layer of v. This is similar to regression where the regression of y on x is not equivalent to the regression of x on y unless the joint distribution is exchangeable.

8. Alternate measures of overall dependence

It may still be convenient to use an overall dependence measure such as Spearman's correlation. However as discussed in §2, Spearman's correlation downplays tail dependence and may be inappropriate when tail dependence is critical. The following formulates alternative measures of overall dependence by taking different weighted averages of layer dependence:

$$\rho_W \equiv \int_0^1 w_\alpha \ell_\alpha \mathrm{d}\alpha = \int_0^1 w_\alpha \frac{\mathrm{cov}\{v, I_\alpha(u)\}}{\mathrm{cov}\{u, I_\alpha(u)\}} \mathrm{d}\alpha = \mathrm{cov}\{v, W(u)\} ,$$

where

$$W(u) \equiv \int_0^1 \frac{w_\alpha I_\alpha(u)}{\mathrm{cov}\{u, I_\alpha(u)\}} \mathrm{d}\alpha = 2\int_0^u \frac{w_\alpha}{\alpha(1-\alpha)} \mathrm{d}\alpha , \qquad \int_0^1 w_\alpha \mathrm{d}\alpha = 1 .$$

The weighting function w_α indicates the weight or importance of the α–layer of u and its dependence with v.

Hence ρ_W is the covariance between v and a transformation of u. For Spearman's correlation ρ, $w_\alpha = 6\alpha(1-\alpha)$ and $W(u) = 12u$. Even though the expression for ρ_W is asymmetric in u and v, the result is identical when u and v are switched if the copula C is exchangeable.

Since ρ_W averages ℓ_α using non-negative weights which integrate to 1, all coherence properties of ℓ_α described in §5 apply to ρ_W. Specifically $-1 \leq \rho_W \leq 1$, with $\rho_W = -1$, 0 and 1, under countermonotonicity, independence and comonotonicity, respectively. Further ρ_W switches its sign when u or v are replaced by their complement. These properties mimic those of Spearman's correlation.

The following are examples of w_α yielding alternate rank dependence measures ρ_W:

- Suppose dependence at different percentiles are equally important. Then $w_\alpha = 1$ yielding

$$\rho_1 = 2\mathrm{cov}\left\{v, \log\left(\frac{u}{1-u}\right)\right\} = \frac{\pi}{3}\ \mathrm{cor}\left\{v, \log\left(\frac{u}{1-u}\right)\right\}\ ,$$

a multiple of the correlation between v and the logit of u. The final expression follows by noting v and $\log\{u/(1-u)\}$ have standard deviations $1/\sqrt{12}$ and $\pi/\sqrt{3}$, respectively. If tail dependence is pronounced, $\rho_1 > \rho$ since ρ_1 weights tail dependence more heavily compared to ρ. An illustration ρ_1 and other alternates to ρ is shown below.

- If $w_\alpha = 3\alpha^2$ then dependence at higher percentiles are considered more important. This formulation is applicable when upper tail dependence is critical, for example the simultaneous occurrence of large insurance losses in different lines of business. Then

$$\rho_2 = 6\mathrm{cov}\left\{v, \log\left(\frac{\mathrm{e}^{-u}}{1-u}\right)\right\} = \sqrt{3}\mathrm{cor}\left\{v, -\log(1-u)\right\} - \frac{\rho}{2}\ ,$$

noting $-\log(1-u)$ has standard deviation 1. This dependence measure combines ρ and the correlation between v and logarithmic transform of u. If dependence is higher above the median then $\rho_2 > \rho$.

- If dependence over percentiles below the median is more important then for example $w_\alpha = 3(1-\alpha)^2$ yielding

$$\rho_3 = 6\mathrm{cov}\left\{v, \log\left(u\mathrm{e}^{-u}\right)\right\} = \sqrt{3}\mathrm{cor}(v, \log u) - \frac{\rho}{2}\ ,$$

similar to ρ_2 with an opposite logarithmic transform on u.

- Suppose w_α is derived from V_α, the inverse distribution of x with derivative V'_α:

$$w_\alpha = \frac{\mathrm{cov}\{u, I_\alpha(u)\}V'_\alpha}{\int_0^1 \mathrm{cov}\{u, I_\alpha(u)\}V'_\alpha \mathrm{d}\alpha} = \frac{0.5\alpha(1-\alpha)V'_\alpha}{\mathrm{cov}(u, V_u)} = \frac{0.5\alpha(1-\alpha)V'_\alpha}{\mathrm{cov}(u, x)}\ ,$$

where $\int_0^1 I_\alpha(u)V'_\alpha \mathrm{d}\alpha = \int_0^u V'_\alpha \mathrm{d}\alpha = V_u$ and $x = V_u$. Using these weights yields the Gini correlation (Schechtman and Yitzhaki, 1999)

$$\rho_4 = \frac{\mathrm{cov}(v, V_u)}{\mathrm{cov}(u, V_u)} = \frac{\mathrm{cov}\{G(y), x\}}{\mathrm{cov}\{F(x), x\}}\ ,$$

where $F \equiv V^-$ and G are distribution functions of observed random variables x and y with percentile ranks u and v, respectively. In this example the weights w_α depend on the marginal distribution of x. More skewness in x leads to more steeply increasing V'_α and w_α hence greater emphasis on upper tail dependence.

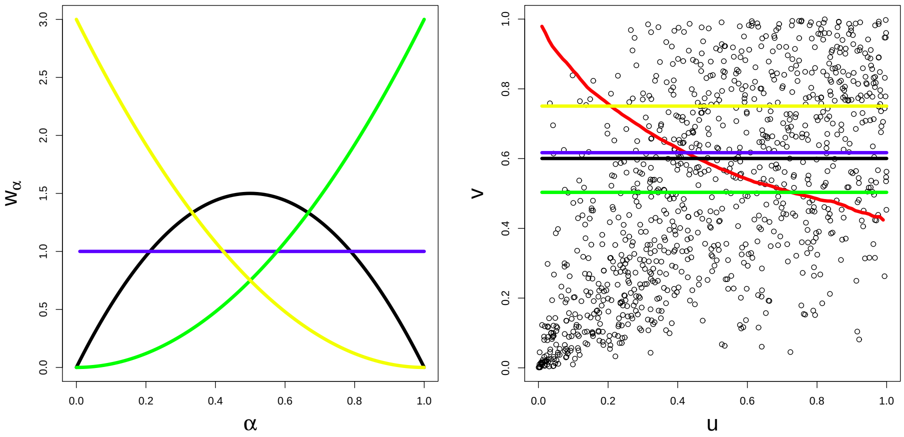

Figure 4: The left panel shows the weight functions $w_\alpha = 6\alpha(1-\alpha)$ (black), $w_\alpha = 1$ (blue), $w_\alpha = 3\alpha^2$ (green), $w_\alpha = 3(1-\alpha)^2$ (yellow). The right panel shows a Clayton copula with a red layer dependence curve and overall dependence measures ρ, ρ_1, ρ_2 and ρ_3 indicated using the same colors as corresponding weight functions in the left panel.

Figure 4 illustrates the first three alternates to Spearman's correlation listed above using a Clayton copula shown in Figure 2. Note ρ, ρ_1, ρ_2 and ρ_3 are all weighted averages of ℓ_α. However their values differ depending on the weight function used. As the Clayton copula exhibits strong lower tail dependence and weak upper tail dependence, ℓ_α is decreasing in α. Hence $\rho_1 > \rho$ as ρ_1 places relatively higher weight on large values of ℓ_α in the lower tail. ρ_3 is the largest as it places the most weight on the lower tail, whereas ρ_2 is the smallest as it focusses on upper tail dependence. Hence ρ_3 is a sensitive measure of lower tail dependence and is appropriate when lower tail dependence is of main concern, and vice versa for ρ_2.

9. Discussion of copula fitting involving layer dependence

This section discusses how layer dependence can be applied to select and fit copulas using past data and expert judgement.

The literature discusses copula fitting extensively. A common approach is to first select parametric copula and marginal distributions, then estimate parameters by maximising joint likelihood (Denuit et al., 2005). A semi-parametric approach replaces marginal distributions in joint likelihood with empirical ranks (Oakes, 1989). The method of moments involves setting up equations to match statistics computed from data and the parametric model (Kojadinovic and Yan, 2010). Kojadinovic and Yan (2010) also compares common copula fitting methods. The choice of parametric copula may be restricted to a specific class of copulas, such as Archimedean copulas (McNeil et al., 2005). Genest and Rivest (1993) suggests an approach to select the generator function of Archimedean copulas. Alternatively a visual assessment of data may suggest an appropriate parametric copula with similar dependence structure, for example the Gumbel or Clayton copulas if tail dependence is strong. At the other extreme is to use the empirical copula, if the volume of past data is sufficiently large. Czado (2010) discusses a semi-parametric approach to model multivariate copulas, based on vine copulas.

Layer dependence can be applied to the copula modeling process in several ways:

- Layer dependence is first computed from data at intervals of say 0.01, that is $\ell_{0.01}, \ell_{0.02}, \ldots$. The selected granularity depends on the volume of data. Layer dependence values may be smoothed parametrically or non-parametrically to reveal the dependence structure of the data.
- There may be an intermediate step where the computed layer dependence curve is adjusted to incorporate expert opinion. For example one may wish to increase layer dependence in the upper tail in anticipation of stronger upper tail dependence than observed historically.
- A parametric copula may then be selected by matching the shape of its layer dependence curve to the data. Shapes of layer dependence curves of typical copulas are shown in Figure 2. Parameters can be fitted using either maximum likelihood or method of moments.
- A mixture of parametric copulas, for example where the parameters follow a probability distribution, may be applied such that the layer dependence curve perfectly matches the data. The difficulty is closed form expressions may be unavailable.

Applying layer dependence in copula fitting refines existing approaches in several ways. Firstly layer dependence guides the selection of a copula family so that the dependence structure of past data is mirrored closely. In addition layer dependence is a convenient medium for incorporating expert opinion on the dependence structure. Further layer dependence is robust to data inadequacies as it summarises data into conditional tail mean values and smoothing is applied.

10. Conclusion

Layer dependence captures dependence structures in bivariate copulas, and shares the same practical properties as Spearman's correlation. Layer dependence is connected to, and refines, current approaches to measure tail dependence. Taking weighted averages of layer dependence curves yields Spearman's correlation and alternate measures of overall dependence which emphasize different areas of the dependence structure.

Using layer dependence in copula fitting enhances the process by capturing and reflecting the dependence structures in past data, whilst flexibly accommodating expert opinion.

11. Appendix

11.1. Proof of equation (9)

Since $\mathrm{E}(v) = \alpha\mathrm{E}(v|u \le \alpha) + (1-\alpha)\mathrm{E}(v|u > \alpha)$, the numerator of the middle expression is (9) is

$$\mathrm{E}(v|u > \alpha) - \mathrm{E}(v|u \le \alpha) = \mathrm{E}(v|u > \alpha) - \frac{\mathrm{E}(v) - (1-\alpha)\mathrm{E}(v|u > \alpha)}{\alpha}$$

$$= \frac{\mathrm{E}(v|u > \alpha) - \mathrm{E}(v)}{\alpha} = \frac{\mathrm{E}\{vI_\alpha(u)\} - \mathrm{E}(v)\mathrm{E}\{I_\alpha(u)\}}{\alpha(1-\alpha)} = \frac{\mathrm{cov}\{v, I_\alpha(u)\}}{\alpha(1-\alpha)} \ ,$$

and similarly the denominator is

$$\mathrm{E}(u|u > \alpha) - \mathrm{E}(u|u \le \alpha) = \frac{\mathrm{cov}\{u, I_\alpha(u)\}}{\alpha(1-\alpha)} = 0.5 \ ,$$

noting $\mathrm{cov}\{u, I_\alpha(u)\} = 0.5\alpha(1-\alpha)$ in (7). Combining these two results yields the first and third expressions of (9). This completes the proof.

11.2. Proof of equation (10)

Firstly the measure of discordance can be expressed as

$$\gamma_\alpha = -\frac{\mathrm{cov}\{1 - I_\alpha(u), 1 - I_\alpha(v)\}}{\mathrm{cov}\{I_\alpha(u), I_\alpha(u)\}} = \frac{\alpha^2 - C(\alpha,\alpha)}{\alpha(1-\alpha)} \ ,$$

and the measure of dispersion is (assuming the copula of (u, v) is exchangeable),

$$\delta_\alpha = 2\mathrm{E}\left\{(u-v)I_v(u)|(u-\alpha)(v-\alpha) < 0\right\}$$

$$= \frac{2\mathrm{E}[(u-v)I_v(u)I_\alpha(u)\{1 - I_\alpha(v)\}]}{2\mathrm{E}[I_\alpha(u)\{1 - I_\alpha(v)\}]} = \frac{\mathrm{E}[(u-v)I_\alpha(u)\{1 - I_\alpha(v)\}]}{\alpha - C(\alpha,\alpha)}$$

$$= \frac{\mathrm{E}\{(u-v)I_\alpha(u)\} - \mathrm{E}\{(u-v)I_\alpha(u)I_\alpha(v)\}}{\alpha - C(\alpha,\alpha)} = \frac{\mathrm{E}\{(u-v)I_\alpha(u)\}}{\alpha - C(\alpha,\alpha)} \ .$$

Substituting the above expressions for γ_α and δ_α into the right hand side of (10) yields the expression for ℓ_α in (9), completing the proof.

11.3. Proof of coherence properties in section 5

Independence property

If u and v are independent, then layer dependence is

$$\ell_\alpha = \frac{\mathrm{cov}\{v, I_\alpha(u)\}}{\mathrm{cov}\{u, I_\alpha(u)\}} = 0 \ , \quad 0 \le \alpha \le 1$$

since the numerator is zero.

Perfect dependence property

If u and v are comonotonic or $v = u$, then layer dependence

$$\ell_\alpha = \frac{\mathrm{cov}\{v, I_\alpha(u)\}}{\mathrm{cov}\{u, I_\alpha(u)\}} = \frac{\mathrm{cov}\{u, I_\alpha(u)\}}{\mathrm{cov}\{u, I_\alpha(u)\}} = 1 \ , \quad 0 \le \alpha \le 1 \ .$$

In addition $\ell_\alpha = 1$ implies from (9) $\mathrm{E}(v|u > \alpha) - \mathrm{E}(v|u \le \alpha) = 0.5$ for all α. Substituting $\mathrm{E}(v|u \le \alpha) = \{\mathrm{E}(v) - (1-\alpha)\mathrm{E}(v|u > \alpha)\}/\alpha$ yields $\mathrm{E}(v|u > \alpha) = 0.5(\alpha + 1)$. Hence the conditional expectation

$$\mathrm{E}(v|u = \alpha) = -\frac{\mathrm{d}}{\mathrm{d}\alpha}\{(1-\alpha)\mathrm{E}(v|u > \alpha)\} = \alpha \ , \quad 0 \le \alpha \le 1 \ ,$$

implying $v = u$, completing the proof. A similar proof holds for countermonotonicity where $v = 1 - u$.

Symmetry

Replacing v with $1 - v$ yields layer dependence

$$\frac{\mathrm{cov}\{1 - v, I_\alpha(u)\}}{\mathrm{cov}\{u, I_\alpha(u)\}} = -\frac{\mathrm{cov}\{v, I_\alpha(u)\}}{\mathrm{cov}\{u, I_\alpha(u)\}} = -\ell_\alpha \ ,$$

whilst replacing u with $1 - u$ yields

$$\frac{\mathrm{cov}\{v, I_\alpha(1-u)\}}{\mathrm{cov}\{1-u, I_\alpha(1-u)\}} = -\frac{\mathrm{cov}\{v, 1 - I_{1-\alpha}(u)\}}{\mathrm{cov}\{u, 1 - I_{1-\alpha}(u)\}} = -\frac{\mathrm{cov}\{v, I_{1-\alpha}(u)\}}{\mathrm{cov}\{u, I_{1-\alpha}(u)\}} = -\ell_{1-\alpha} \ .$$

Using a similar proof, replacing v and u with $1-v$ and $1-u$, respectively, yields layer dependence $\ell_{1-\alpha}$.

Correlation order

If (u^*, v^*) exceeds (u, v) in correlation order, then

$$\mathrm{cov}\{f(u^*), g(v^*)\} \ge \mathrm{cov}\{f(u), g(v)\}$$

for any non-decreasing functions f and g (Dhaene et al., 2009). Therefore the numerator of layer dependence in (6) $\mathrm{cov}\{v^*, I_\alpha(u^*)\} \ge \mathrm{cov}\{v, I_\alpha(u)\}$ since $I_\alpha(u)$ is increasing in u, and the denominators are the same. Thus (u^*, v^*) has higher layer dependence than (u, v).

Bounds

Since layer dependence preserves correlation order, $\ell_\alpha \le 1$ since $\ell_\alpha = 1$ if and only if u and v are comonotonic and comonotonicity represents maximum correlation order (Dhaene et al., 2009). Similarly $\ell_\alpha \ge -1$ noting countermonotonicity represents minimum correlation order and leads to $\ell_\alpha = -1$.

References

Bairamov, I., S. Kotz, and T. Kozubowski (2003). A new measure of linear local dependence. *Statistics: A Journal of Theoretical and Applied Statistics 37*(3), 243–258.

Bjerve, S. and K. Doksum (1993). Correlation curves: Measures of association as functions of covariate values. *The Annals of Statistics*, 890–902.

Bouyé, E., V. Durrleman, A. Nikeghbali, G. Riboulet, and T. Roncalli (2000). Copulas for finance-a reading guide and some applications. *Available at SSRN 1032533*.

Czado, C. (2010). Pair-copula constructions of multivariate copulas. In *Copula theory and its applications*, pp. 93–109. Springer.

Denuit, M., J. Dhaene, M. Goovaerts, and R. Kaas (2005). Actuarial theory for dependent risks: Measures, orders and models. *John Wiley&Sons*.

Dhaene, J., M. Denuit, and S. Vanduffel (2009). Correlation order, merging and diversification. *Insurance: Mathematics and Economics 45*(3), 325–332.

Durante, F., J. Fernández-Sánchez, and R. Pappadà (2014). Copulas, diagonals, and tail dependence. *Fuzzy Sets and Systems*.

Embrechts, P., A. McNeil, and D. Straumann (2002). Correlation and dependence in risk management: properties and pitfalls. *Risk management: value at risk and beyond*, 176–223.

Genest, C. and L.-P. Rivest (1993). Statistical inference procedures for bivariate archimedean copulas. *Journal of the American statistical Association 88*(423), 1034–1043.

Holland, P. W. and Y. J. Wang (1987). Dependence function for continuous bivariate densities. *Communications in Statistics-Theory and Methods 16*(3), 863–876.

Joe, H. (1997). *Multivariate models and dependence concepts*, Volume 73. CRC Press.

Jones, M. (1996). The local dependence function. *Biometrika 83*(4), 899–904.

Kojadinovic, I. and J. Yan (2010). Comparison of three semiparametric methods for estimating dependence parameters in copula models. *Insurance: Mathematics and Economics 47*(1), 52–63.

Mandel, B., D. Morgan, and C. Wei (2012). The role of bank credit enhancements in securitization. *Federal Reserve Bank of New York Economic Policy Review 18*(2), 35–46.

McNeil, A., R. Frey, and P. Embrechts (2005). *Quantitative risk management.* Princeton University Press.

Nelson, R. (1999). An Introduction to Copulas. *Lecture Notes in Statistics 139.*

Oakes, D. (1989). Bivariate survival models induced by frailties. *Journal of the American Statistical Association 84*(406), 487–493.

Schechtman, E. and S. Yitzhaki (1999). On the proper bounds of the gini correlation. *Economics letters 63*(2), 133–138.

Sweeting, P. and F. Fotiou (2013). Calculating and communicating tail association and the risk of extreme loss. *British Actuarial Journal 18*(01), 13–72.

Venter, G. G. (2002). Tails of copulas. In *Proceedings of the Casualty Actuarial Society*, Volume 89, pp. 68–113.

Wang, S. (1995). Insurance pricing and increased limits ratemaking by proportional hazards transforms. *Insurance: Mathematics and Economics 17*(1), 43–54.