Insights to systematic risk and diversification across a joint probability distribution

Abstract

This paper analyses and develops insights to systematic risk and diversification when random, imperfectly dependent, losses are aggregated. Systematic risk and diversification are shown to vary across layers of component losses according to local dependence and volatility structures. Systematic risk is high and diversification is weak overall if high risk layers are heavily dependent on the aggregate loss. This result explains weak diversification observed in financial markets despite weak to moderate correlations overall. A coherent risk setup is assumed in this paper, where risks are measured using distortion and allocated using the Euler principle.

Keywords: Distortion risk; spectral risk; Euler allocation; systematic risk; diversification; layer; value-at-risk.

1. Introduction to systematic risk and diversification

Suppose x is one of several continuous, non-negative and random component losses aggregating to x_+. For example x may be the loss from an insurance class and x_+ is the loss aggregated across all classes. Or x may be the credit loss on a portfolio of loans and x_+ is the aggregate credit loss across all portfolios. Component losses are imperfectly dependent leading to risk diversification as formalised below.

This paper applies the following risk setup. Suppose $\phi \geq 0$ is an increasing risk aversion function integrating to 1. Standalone risk of x, systematic risk of x and aggregate risk of x_+ are respectively

$$r = \text{cov}\{x, \phi(u)\}, \quad \bar{r} = \text{cov}\{x, \phi(u_+)\}, \quad r_+ = \text{cov}\{x_+, \phi(u_+)\} \ , \qquad (1)$$

where cov denotes covariance and u and u_+ are percentile ranks of x and x_+. If F and F_+ are distribution functions of x and x_+ then $u \equiv F(x)$ and $u \equiv F_+(x_+)$. Standalone and systematic risks of other component losses forming x_+ are measured using similar covariance expressions.

The risk setup (1) is well established in the literature. Choo and De Jong (2009) refers to r and r_+ as loss aversion risks of x and x_+ and establishes their equivalence with distortion risks (Wang, 1996) and spectral risks (Acerbi, 2002).

Examples of ϕ leading to different risk measures are discussed in Choo and De Jong (2009). In addition $\overline{r}$ is an allocation of r_+ to x by applying the Euler principle (McNeil et al., 2005) or game theory (Tasche, 2007). The allocation is extensively discussed in Choo and De Jong (2010), Buch and Dorfleitner (2008) and Tsanakas and Christofides (2006).

The risk setup (1) is coherent in the sense of Artzner et al. (1999), Denault (2001) and Kalkbrener (2005): positively homogenous, translation invariant, monotonic and subadditive. Further the allocation is complete: adding $\overline{r}$ across component losses yields r_+, and there is no undercut: $\overline{r} \leq r$ no matter how x is carved out from x_+.

The term "systematic risk" is consistent with the capital asset pricing model in finance (Luenberger, 1998), where the volatility or risk of an asset return is divided into diversifiable and non-diversifiable or systematic risk. Systematic risk increases with the correlation between asset and market returns, and is a key driver of risk premiums: expected asset returns above market return.

This paper compares r, the risk of x before aggregation, with $\overline{r}$, the risk of x after aggregation. Since x and u_+ are imperfectly dependent, $r \leq \overline{r}$ and the difference $r - \overline{r}$ is a diversification benefit. Diversification benefits are critical in risk management by enabling risks to be managed viably and efficiently as a group, a classic case being insurance. The systematic risk ratio is

$$\theta \equiv \frac{\overline{r}}{r} = \frac{\operatorname{cov}\{x, \phi(u_+)\}}{\operatorname{cov}\{x, \phi(u)\}} = \frac{\operatorname{cor}\{x, \phi(u_+)\}}{\operatorname{cor}\{x, \phi(u)\}} \leq 1 \,, \tag{2}$$

and a lower ratio indicates greater diversification. The final expression holds since u and u_+ are both uniform hence $\phi(u)$ and $\phi(u_+)$ have equal standard deviations. Stronger dependence between x and x_+ leads to weaker diversification due to higher numerators in (2) hence larger θ. Conversely weaker dependence leads to stronger diversification.

Systematic risk and diversification benefit are shown to vary across the distribution of x based on the local dependence structure underlying x and x_+. This result is important when formulating risk management strategies to maximise diversification, as demonstrated by a case study using historical stock returns. The analysis focusses on x, however equivalent results apply to other component losses forming x_+.

The remaining paper is structured as follows. Section 2 gives an overview of mean and risk densities and Value-at-Risk (VaR) loss layers established in **?**. Section 3 defines systematic risk densities analogous to standalone risk densities, and establishes links to local dependence structures underlying component and aggregate losses. Section 4 uses systematic risk densities to explore drivers of systematic risk and diversification such as in financial markets. A theoretical example of the proposed analytical framework is shown in §5. Section 6 attributes aggregate mean and risk densities to component losses. The attribution is important when risk management strategies target the aggregate loss.

Section 7 examines the case where component losses are comonotonic, a benchmark for measuring diversification. Section 8 concludes with an illustration of the framework using historical stock returns.

2. Overview of VaR layers, mean and risk densities

This section outlines VaR layers, mean and risk densities developed in **?**. Mean and risk densities track the mean and standalone risk of a loss across its VaR layers. Risk densities are critical to the analysis of systematic risk and diversification as shown in subsequent sections.

The α–Value–at–Risk (VaR_α) of a continuous random loss $x \geq 0$ with distribution function F is the percentile $V_\alpha \equiv F^-(\alpha)$, where F^- is the inverse distribution function of x. In addition define $L_\alpha \equiv \min(x, V_\alpha)$, the loss capped at its VaR_α. Consider the following two expressions:

$$L_\beta - L_\alpha = \min\{\max(x - V_\alpha, 0), V_\beta - V_\alpha\}, \quad x = L_1 - L_0 = \int_0^1 \mathrm{d}L_\alpha \ .$$

The first expression above is the VaR layer $[V_\alpha, V_\beta]$ of x: the excess of x over V_α with the excess capped at $V_\beta - V_\alpha$. The second expression decomposes x into infinitesimal VaR layers $\mathrm{d}L_\alpha$ over $0 \leq \alpha \leq 1$. The infinitesimal V_α–layer of x is

$$\mathrm{d}L_\alpha = L'_\alpha \mathrm{d}\alpha = I_\alpha(u) V'_\alpha \mathrm{d}\alpha \ ,$$

where L'_α and V'_α are derivatives of L_α and V_α with respect to α, and $I_\alpha(u)$ is an indicator equal to 1 if $u > \alpha$ or equivalently $x > V_\alpha$, and 0 otherwise. Hence the V_α–layer of x is an increment proportional to V'_α if $x > V_\alpha$ and 0 otherwise.

Layers are standard insurance and financial constructs, and are also called tranches. VaR layers self-adjust to the shape and scale of the loss distribution and are hence comparable across loss distributions. Refer to **?** for further discussion of layers and reasons for defining layers using VaRs.

The risk density indicates the standalone risk of infinitesimal VaR layers of x and based on (1) is the covariance

$$r_\alpha = \text{cov}\{I_\alpha(u) V'_\alpha, \phi(u)\} = \{\alpha - \Phi(\alpha)\} V'_\alpha, \quad \Phi(\alpha) \equiv \int_0^\alpha \phi(u) \mathrm{d}u \ ,$$

where Φ cumulates ϕ. Integrating r_α yields standalone risks of larger layers:

$$\int_\alpha^\beta r_\pi \mathrm{d}\pi = \text{cov}\left\{\int_\alpha^\beta L'_\pi \mathrm{d}\pi, \phi(u)\right\} = \text{cov}\{L_\beta - L_\alpha, \phi(u)\} \ ,$$

and the entire area under r_α is the overall standalone risk of x: $\int_0^1 r_\alpha \mathrm{d}\alpha = r$.

The mean density indicates the mean of infinitesimal VaR layers and is

$$m_\alpha = \text{E}(L'_\alpha) = \text{E}\{I_\alpha(u) V'_\alpha\} = (1 - \alpha) V'_\alpha \ .$$

Similar to r_α, integrating m_α yields the mean of larger layers. Hence mean and risk densities are akin to probability densities: integrating yields quantities over larger regions. Refer to **?** for further properties, illustrations and applications of mean and risk densities.

3. Systematic risk densities and links to local dependence

Systematic risk densities are akin to standalone risk densities in §2, and describe systematic risks of infinitemisal VaR layers forming a random loss. Comparing systematic and standalone risk densities reveals the extent of diversification at various layers of the loss distribution. The comparison also leads to the dependence structure involving the aggregate loss.

The systematic risk density of x according to (1) is the covariance

$$\overline{r}_\alpha \equiv \mathrm{cov}\{L'_\alpha, \phi(u_+)\} = \mathrm{cov}\{I_\alpha(u), \phi(u_+)\} V'_\alpha \ .$$

Similar to r_α, integrating $\overline{r}_\alpha$ yields systematic risks of larger layers and the overall systematic risk of x:

$$\int_\alpha^\beta \overline{r}_\pi \mathrm{d}\pi = \mathrm{cov}\left\{L_\beta - L_\alpha, \phi(u_+)\right\} , \qquad \int_0^1 \overline{r}_\alpha \mathrm{d}\alpha = \overline{r} \ .$$

The ratio between systematic and standalone risk densities is the proportion of remaining risk in each VaR layer after diversification, and is given by

$$\theta_\alpha \equiv \frac{\overline{r}_\alpha}{r_\alpha} = \frac{\mathrm{cov}\{I_\alpha(u)V'_\alpha, \phi(u_+)\}}{\mathrm{cov}\{I_\alpha(u)V'_\alpha, \phi(u)\}} = \frac{\mathrm{cov}\{I_\alpha(u), \phi(u_+)\}}{\mathrm{cov}\{I_\alpha(u), \phi(u)\}} \ , \tag{3}$$

where the last expression follows since the constant V'_α in $\overline{r}_\alpha$ and r_α cancel with division. If x and x_+ are comonotonic or $u = u_+$ then $\theta_\alpha = 1$ for all $0 \le \alpha \le 1$. Otherwise $\theta_\alpha < 1$.

For any α, large θ_α implies weak diversification at V_α–layer of x since $\overline{r}_\alpha$ is close to r_α and the diversification benefit $r_\alpha - \overline{r}_\alpha$ is small relative to r_α. Vice versa diversification at V_α–layer is strong if θ_α is small or even negative. An analysis of θ_α against α hence reveals layers with weak diversification, which can be mitigated using reinsurance or hedging, and layers with strong diversification which should be preserved since they reduce overall risk. In addition θ_α varies over α in line with local dependence between x and x_+. The notion is explored in the rest of this section.

The systematic risk ratio θ_α is calculated entirely from the joint distribution of (u, u_+) or equivalently the copula C of (x, x_+). Marginal distributions are not directly involved in θ_α, although the marginal distribution of x affects C[1].

[1]For example suppose x dominates other losses forming x_+. Then x and x_+ are strongly dependent compared to the case where x is dominated by other losses.

The denominator of the last term in (3) is $\alpha - \Phi(\alpha)$ and the numerator is

$$\int_0^1 \mathrm{cov}\{I_\alpha(u), I_\beta(u_+)\mathrm{d}\phi(\beta) = \int_0^1 \{C(\alpha,\beta) - \alpha\beta\}\phi'(\beta)\mathrm{d}\beta \ ,$$

a partial integration of the copula C weighted by the derivative ϕ'.

The set of θ_α values over $0 \le \alpha \le 1$ reflects and characterises the dependence structure of (x, x_+). Given α, θ_α is the local dependence between V_α–layer of x and a function of the aggregate loss. Of interest is rank dependence since θ_α is defined in terms of percentile ranks u and u_+. If (x, x_+) exhibits strong upper tail dependence and weak lower tail dependence, then θ_α starts small at $\alpha = 0$ and increases to one as α approaches one. Example systematic risk ratios given ϕ are discussed in §5.

Systematic risk ratios have similar properties as linear and Spearman's correlation which are measures of overall dependence. For all $0 \le \alpha \le 1$, $\theta_\alpha \le 1$, $\theta_\alpha = 1$ if $u = u_+$ and $\theta_\alpha = 0$ if u and u_+ are independent. Negative dependence yields $\theta_\alpha \le 0$, and $u_+ = 1 - u$ leads to $\theta_\alpha = -1$ if $\phi(u)$ is symmetric about $u = 0.5$ such that $\phi(u - 0.5) = \phi(0.5 - u)$. In addition stronger correlation order (Dhaene et al., 2009) between x and x_+ increases θ_α for all α. Proofs are straightforward from (3).

Systemic risk ratios are similar to layer dependence **?** which measures local dependence between arbitrary percentile rank random variables u and v:

$$\ell_\alpha \equiv \frac{\mathrm{cov}\{I_\alpha(u), v\}}{\mathrm{cov}\{I_\alpha(u), u\}} \ .$$

Layer dependence ℓ_α assumes a linear ϕ in systematic risk ratio θ_α.

4. Explaining and exploring systematic risk and diversification

High overall systematic risk ratio in (2), or low overall diversification, arises when strong local dependence coincides with high local skewness or volatility, such as in financial markets. High systematic risk or low diversification overall may arise even when overall correlation is weak to moderate.

Noting $\overline{r}_\alpha = \theta_\alpha r_\alpha$ from (3), overall systematic risk and risk ratio of x defined in (1) and (2) are respectively written as

$$\overline{r} = \int_0^1 r_\alpha \theta_\alpha \mathrm{d}\alpha, \quad \theta = \frac{\overline{r}}{r} = \frac{\int_0^1 r_\alpha \theta_\alpha \mathrm{d}\alpha}{\int_0^1 r_\alpha \mathrm{d}\alpha} \ . \tag{4}$$

Similar expressions apply when considering sytematic risk over $[V_\alpha, V_\beta]$ VaR layer of x: change integration limits in (4) from $[0, 1]$ to $[\alpha, \beta]$.

From (4), the overall systematic risk ratio is a risk weighted average of individual systematic risk ratios across VaR layers. As mentioned in the previous section, individual systematic risk ratios characterise the dependence structure of (x, x_+). Risk weights are formed by the standalone risk density $r_\alpha = \{\alpha - \Phi(\alpha)\}V'_\alpha$, and are large for a loss distribution if V'_α is large. The derivative V'_α measures local volatility since it is the relative gap between successive VaRs. Suppose a sample of infinite size is drawn from the loss distribution and hence ordered observations are close to VaRs. Large V'_α implies observations around V_α are highly spread out, or volatile.

The result in (4) explains high systematic risk ratios and low diversification observed in financial markets, for example during the 2008 global financial crisis (Kolb, 2010), despite moderate correlations across time. From (4), large θ arises when strong local dependence θ_α coincides with large risk weights or local volatility reflected in r_α. Financial returns are heavy tailed (Cont (2001), Hsieh (1988)) and exhibit strong tail dependence (Rodriguez (2007), Hartmann et al. (2004)). The former implies large r_α and the latter implies large θ_α, both over large α. These two implications combine to create large θ, even though θ_α may be small across most other smaller values of α. Hence risk weighting of individual systematic risk ratios leads to diversification being heavily influenced by local dependence in volatile layers.

Conversely diversification is strong when risks of x are concentrated in layers which are weakly dependent on the aggregate loss. Large r does not necessarily imply x is undesirable if other component losses are structured to yield low θ and $\bar{r}$. For example suppose x is heavy-tailed, such as Pareto, yielding a large standalone risk concentrated in high layers. Constructing other losses so that they have favourable values when x is large leads to high layers of x being weakly dependent on the aggregate loss. As a result most of the risk of x is diversified upon aggregation.

5. Theoretical example using the conditional-tail-expectation

Assume a stepped risk aversion function $\phi(u) = I_t(u)/(1-t)$ where t is a parameter in the unit interval. Then according to Choo and De Jong (2009) and Choo and De Jong (2010),

$$r = \mathrm{E}(x|u > t) - \mathrm{E}(x), \quad \bar{r} = \mathrm{E}(x|u_+ > t) - \mathrm{E}(x), \quad \theta = \frac{\mathrm{E}(x|u_+ > t) - \mathrm{E}(x)}{\mathrm{E}(x|u > t) - \mathrm{E}(x)} .$$

Risk in this case is the gap between the conditional–tail–expectation (Rockafellar and Uryasev, 2002) of x and its unconditional expectation. Standalone risk considers the tail event $u > t$ of x whereas systematic risk considers the aggregate tail event $u_+ > t$. The overall systematic risk ratio θ of x captures the relative impact of the two tail events.

The systematic risk ratio at V_α–layer using (3) is

$$\theta_\alpha = \frac{\mathrm{cov}\{I_\alpha(u), I_t(u_+)/(1-t)\}}{\mathrm{cov}\{I_\alpha(u), I_t(u)/(1-t)\}} = \frac{\mathrm{cov}\{I_\alpha(u), I_t(u_+)\}}{\mathrm{cov}\{I_\alpha(u), I_t(u)\}}$$

$$= \frac{\mathrm{P}(u > \alpha, u_+ > t) - \mathrm{P}(u > \alpha)\mathrm{P}(u_+ > t)}{\mathrm{P}\{u > \max(\alpha, t)\} - \mathrm{P}(u > \alpha)\mathrm{P}(u > t)} = \frac{\mathrm{P}(u > \alpha | u_+ > t) - \mathrm{P}(u > \alpha)}{\mathrm{P}(u > \alpha | u > t) - \mathrm{P}(u > \alpha)} ,$$

where P calculates probability. Given α, θ_α increases with the probability of $u > \alpha$ jointly with, or conditional on, $u_+ > t$. Other above probabilities are marginal probabilities, and are scaling factors such that $\theta_\alpha = 1$ when $u = u_+$ and $\theta_\alpha = 0$ when u and u_+ are independent. In addition results are unchanged when inequalities are reversed. Therefore individual systematic risk ratios in this example reflect the likelihood of joint tail events in x and x_+.

For any α, $\theta_\alpha = 1$ implies $\mathrm{P}(u > \alpha | u_+ > t) = \mathrm{P}(u > \alpha | u > t) = \min\{1, (1 - \alpha)/(1 - t)\}$, whilst $\theta_\alpha = 0$ implies $\mathrm{P}(u > \alpha | u_+ > t) = \mathrm{P}(u > \alpha) = 1 - \alpha$. The former is the maximum conditional probability and the latter is the unconditional probability assuming independence locally.

An alternative expression for θ_α in terms of the copula C of (x, x_+) is

$$\theta_\alpha = \frac{C(\alpha, t) - \alpha t}{\min(\alpha, t) - \alpha t} .$$

Suppose θ_α is specified for all $0 \leq \alpha \leq 1$ and for all thresholds $0 \leq t \leq 1$. Then C is completely specified. Hence the set of individual systematic risk ratios over all parameters completely characterises the dependence structure of (x, x_+).

The panels in Figure 1 plot θ_α against α for $t = 0.5$, 0.75 and 0.9, assuming Gumbel and Clayton copulas for (u, u_+). Apart from a kink at $\alpha = t$, θ_α traces local dependence between u and u_+, indicated by the extent of clustering along the 45° line. For the Gumbel copula, systematic risk ratios are higher at both tails due to tail dependence, and lower in the middle. For the Clayton copula, systematic risk ratios are higher at the left tail and lower at the right tail.

6. Sub-aggregate mean and risk densities

Write the mean and risk densities of aggregate loss x_+ as respectively

$$m_{\beta,+} = (1 - \beta)V'_{\beta,+}, \quad r_{\beta,+} = \{\beta - \Phi(\beta)\}V'_{\beta,+}$$

where $V_{\beta,+}$ is the VaR_β of x_+ and $V'_{\beta,+}$ is the derivative of $V_{\beta,+}$ with respect to β. Aggregate risk reflects diversification and is therefore composed entirely of systematic risks of component losses forming x_+. In addition the VaR_β layer of x_+ is $I_\beta(u_+)V'_{\beta,+}\mathrm{d}\beta$ where u_+ is the percentile rank of x_+.

The following allocates aggregate mean and risk densities and aggregate VaR layers to component losses including x. The allocation is critical when risk management strategies are applied to layers of x_+, and the impact and cost of the strategies are attributed to component losses. Example of such strategies are stop–loss reinsurance and aggregate hedging of a portfolio of losses. The proposed allocation shown below is unbiased and aligns with the overall mean and systematic risk of component losses. The approach relies on conditional mean sharing by Denuit and Dhaene (2012).

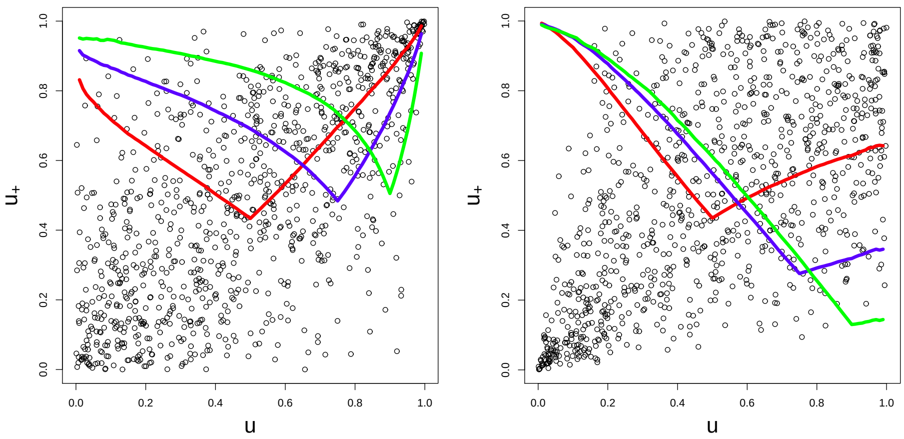

Figure 1: Left and right panels plot θ_α against α for Gumbel and Clayton copulas respectively. θ_α is computed assuming a risk aversion function $\phi(u) = (u > t)/(1-t)$ for $t = 0.5$ (red), 0.75 (blue) and 0.9 (green). The calculation is performed over 100 intervals of β forming $[0, 1]$.

The aggregate loss is $x_+ = \sum_i x_i$ where x_i represents a component loss such as x. Using iterated expectations, write x_+ as

$$x_+ = \sum_i g_i(x_+), \quad g_i(x_+) \equiv \mathrm{E}(x_i|x_+) \ ,$$

where g_i calculates the mean value of x_i conditional on x_+. Denuit and Dhaene (2012) allocates $g_i(x_+)$ to x_i, called conditional mean sharing. The allocation is unbiased since $\mathrm{E}\{g_i(x_+)\} = \mathrm{E}\{\mathrm{E}(x_i|x_+)\} = \mathrm{E}(x_i)$. Substituting $x_+ = V_{\beta,+}$ into the above result and taking derivatives with respect to β yields

$$V'_{\beta,+} = \sum_i \{g_i(V_{\beta,+})\}' = V'_{\beta,+} \sum_i g'_i(V_{\beta,+}), \quad \sum_i g'_i(V_{\beta,+}) = 1 \ . \tag{5}$$

(5) allocates a fraction $g'_i(V_{\beta,+})$ of $V'_{\beta,+}$ to x_i, and fractions sum to one across i. Apply this fractional allocation to aggregate VaR layers and aggregate mean and risk densities, since all are proportional to $V'_{\beta,+}$. Hence the allocated or "sub-aggregate" mean and risk densities of x are respectively

$$\dot{m}_{\beta,+} \equiv m_{\beta,+} g'(V_{\beta,+}), \quad \dot{r}_{\beta,+} \equiv r_{\beta,+} g'(V_{\beta,+}) \ ,$$

where $g(x_+) = \mathrm{E}(x|x_+)$. Identical densities are obtained by performing ground-up calculations on $g(x_+)$ assuming g is increasing and VaR_β of $g(x_+)$ is $g(V_{\beta,+})$. Given β, $\dot{m}_{\beta,+}$ and $\dot{r}_{\beta,+}$ are portions of the mean and risk of VaR_β layer of x_+ attributable to x.

Sub-aggregate mean and risk densities of x are aligned to the overall mean and systematic risk of x. Integrating $\dot{m}_{\beta,+}$ over all β yields

$$\int_0^1 \dot{m}_{\beta,+}\mathrm{d}\beta = \int_0^1 (1-\beta)\{g(V_{\beta,+})\}'\mathrm{d}\beta = \mathrm{E}\{g(x_+)\} = \mathrm{E}(x) ,$$

and similarly

$$\int_0^1 \dot{r}_{\beta,+}\mathrm{d}\beta = \int_0^1 \mathrm{cov}\{I_\beta(u_+), \phi(u_+)\}\{g(V_{\beta,+})\}'\mathrm{d}\beta$$
$$= \mathrm{cov}\{g(x_+), \phi(u_+)\} = \mathrm{cov}\{x, \phi(u_+)\} = \overline{r} .$$

Integrating $\dot{m}_{\beta,+}$ and $\dot{r}_{\beta,+}$ over a subset of the unit interval yields an allocation of the mean and risk of the corresponding VaR layer of x_+ to x.

Sub-aggregate mean and risk densities are different from mean and systematic risk densities: $\dot{m}_{\beta,+} \neq m_\alpha$ and $\dot{r}_\beta \neq \overline{r}_\alpha$ for $\alpha = \beta$, although they integrate to the same result. The former relates to VaR layers of x_+ whereas the latter relates to VaR layers of x. However equality applies when component losses are comonotonic. This special case is discussed in the next section.

7. Comonotonicity as a diversification benchmark

Comonotonic x and x_+ implies $u = u_+$: x and x_+ are always at equal percentiles, and $\mathrm{E}(x|x_+) = x$: x_+ pinpoints the value of x. Dhaene et al. (2002) and Wang and Dhaene (1998) further discuss the concept of comonotonicity. Comonotonicity yields maximum systematic risk across all layers and is hence a benchmark for assessing diversification as shown below.

Comonotonicity between x and x_+ implies:

- Systematic and standalone risk densities of x are equal, and there is no diversification at any VaR layer of x. Since $u = u_+$,

  $$\overline{r}_\alpha = \mathrm{cov}\{I_\alpha(u), \phi(u_+)\}V'_\alpha = \mathrm{cov}\{I_\alpha(u), \phi(u)\}V'_\alpha = r_\alpha ,$$

  implying $\theta_\alpha = 1$. In addition $\overline{r} = r$ and $\theta = 1$. This result is discussed in section 3. Hence $\overline{r}_\alpha$ is maximised across all α under comonotonicity.

- Sub-aggregate densities are also equal to standalone densities. Since $x = \mathrm{E}(x|x_+) = g(x_+)$, $V_\beta = g(V_{\beta,+})$, hence $V'_\beta = \{g(V_{\beta,+})\}'$ and

  $$\dot{m}_{\beta,+} = m_\beta, \quad \dot{r}_{\beta,+} = r_\beta .$$

  Since comonotonic x_+ and x reach their VaR_β layers simultaneously, the attributed mean and risk of the VaR_β layer of x_+ to x is the mean and risk of the VaR_β layer of x.

The standalone risk density of x thus indicates maximum systematic risk across all layers, and comparing it with systematic and sub-aggregate risk densities of x reveals the extent of diversification in each layer. Section 3 already compares $\bar{r}_\alpha$ with r_α and the ratio characterises the dependence structure of (x, x_+). The following shows a similar interpretation when $\dot{r}_{\beta,+}$ is compared with r_β. Define the ratio

$$\gamma_\beta \equiv \frac{\dot{r}_{\beta,+}}{r_\beta} = \frac{\{\beta - \Phi(\beta)\}\{g(V_{\beta,+})\}'}{\{\beta - \Phi(\beta)\}V'_\beta} = \frac{\{g(V_{\beta,+})\}'}{V'_\beta} = \frac{g'(V_{\beta,+})V'_{\beta,+}}{V'_\beta}$$

$$= \frac{\frac{\mathrm{d}}{\mathrm{d}\beta}\mathrm{E}(x|u_+ = \beta)}{\frac{\mathrm{d}}{\mathrm{d}\beta}\mathrm{E}(x|u = \beta)} = \frac{\mathrm{cov}\left\{x, \frac{\mathrm{d}}{\mathrm{d}\beta}\delta_\beta(u_+)\right\}}{\mathrm{cov}\left\{x, \frac{\mathrm{d}}{\mathrm{d}\beta}\delta_\beta(u)\right\}} \ . \tag{6}$$

where $\delta_\beta(u)$ is the Dirac delta function which approaches ∞ when $u = \beta$ and is 0 otherwise. Similar to θ_α, $\gamma_\beta = 1$ if $u = u_+$ and $\gamma_\beta = 0$ if u and u_+ are independent. Unlike θ_α, γ_β is computed entirely from the joint distribution of (x, x_+) and does not involve the risk aversion function ϕ. In addition γ_β may exceed one since the numerator in its definition relates to the VaR_β layer of x_+ whereas the denominator relates to the VaR_β layer of x and there is no strict inequality between the two.

As per θ_α, γ_β measures local dependence between x and x_+ but in a different manner. The second last expression in (6) measures the sensitivity in the conditional expectation of x to changes in x_+ at VaR_β, relative to the same calculation when $u = u_+$. The last expression in (6) computes the covariance between x and a function of u_+, again relative to the comonotonic case.

Figure 2 plots γ_β against β using copulas in Figure 1. Assume exponential and normal distributions for x. Note γ_β is independent of location and scale of x. Calculations show $\gamma_\beta > 1$ over some values of β hence γ_β is scaled by its maximum value over β so that resulting values do not exceed 1. Similar to θ_α, γ_β traces the local dependence structure of (x, x_+): the dispersion between scatter points along the 45° line. However calculated values of γ_β are more volatile than θ_α as the former involve conditional expectations over narrow windows whereas the latter use conditional tail expectations.

8. Case study using historical stock returns

This section applies the proposed systematic risk and diversification analytical framework to daily NASDAQ, S&P and FTSE returns between 1985 and 2015. Form a hypothetical portfolio of \$100 in each market index, and assume the empirical joint probability distribution. To be consistent with the proposed framework, focus on investment losses rather than gains. Hence switch the signs of investment returns, with gains being negative losses. Use the risk aversion function $\phi(\alpha) = 4I_{0.75}(\alpha)$, hence risk is the mean loss above $\mathrm{VaR}_{0.75}$ compared to the overall mean.

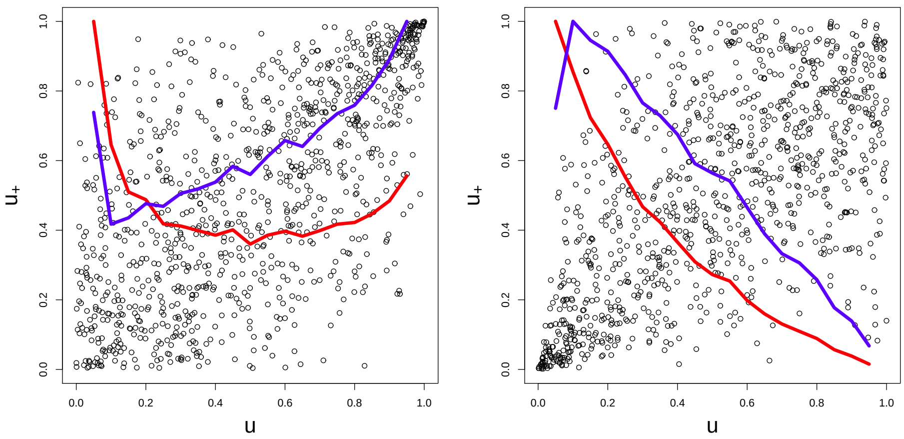

Figure 2: Left and right panels plot $\gamma_\beta / \max_\beta(\gamma_\beta)$ against β for Gumbel and Clayton copulas respectively. γ_β is computed assuming x is exponential (red) and normal (blue). The calculation is performed over 20 intervals of β forming $[0, 1]$ to produce smoother curves. Note how the curves trace the dependence structure of (u, u_+) represented by the scatter of points along the $45°$ line.

Top four panels in Figure 3 display empirical marginal probability distributions and copulas of hypothetical index losses. NASDAQ lossess are the most skewed, while S&P losses are the most peaked. In addition NASDAQ and S&P losses are highly dependent, and both are less dependent overall on FTSE losses (presumably due to different geographical location). All three market indices exhibit significant upper and lower tail dependence: extreme losses and gains are highly dependent across markets.

Bottom two panels in Figure 3 graph standalone and systematic risk densities for each market index, calculated from the empirical joint probability distribution. Density values are smoothed to reduce volatility across VaR layers. Before aggregation, NASDAQ has the highest risk density due to greater skewness of its probability distribution. S&P and FTSE have similar risk densities, although S&P has a slightly lower density in middle VaR layers due to greater peakedness of its probability density. All three risk densities are reduced after diversification, most notably FTSE. Subsequent figures compare and analyse standalone and systematic risk densities.

Figure 4 compares standalone and systematic risk densities for each market index. Systematic risk ratios, and copulas between index and aggregate losses, are also shown. Note from §3 that systematic risk ratios are computed entirely from the copulas and summarise the dependence structure. Also note the kink in systematic risk ratios at $\text{VaR}_{0.75}$ layers due to the selection of the risk aversion function. FTSE has lower systematic risk ratios than NASDAQ and S&P across all VaR layers due to its weaker dependence with the aggregate loss.

Hence FTSE experiences stronger diversification. All three market indices have systematic risk ratios close to 1 in extreme VaR layers, due to tail dependence. Therefore there is minimal risk diversification in extreme VaR layers of all three indices.

Table 1 shows overall risks for each market index, before and after diversification. Overall systematic risk ratios are also shown. As expected NASDAQ has the highest standalone and systematic risk. S&P and FTSE have similar standalone risks, although FTSE has higher diversification and therefore lower systematic risk. NASDAQ and S&P have similar overall systematic risk ratios.

Portfolio	r	$\bar{r}$	θ
NASDAQ	\$2.00	\$1.86	0.93
S&P	\$1.32	\$1.22	0.92
FTSE	\$1.31	\$0.87	0.66
Overall	\$4.63	\$3.95	0.85

Table 1: Overall standalone risk, systematic risk and risk ratio for each market index.

Figure 5 shows mean and risk densities of the aggregate portfolio loss and its breakdown into sub-aggregate densities. NASDAQ generally has a higher sub-aggregate mean and risk density, notably in high VaR layers of the aggregate loss, due to its higher skewness and systematic risk contribution. Note from §6 that sub-aggregate mean and risk densities integrate to the overall mean and systematic risk of each market index.

Figures 3, 4 and 5 guide the formulation of effective risk management strategies. For example standalone and systematic risks are concentrated in high VaR layers and can be significantly reduced with put options. In addition, from Figure 4, dependence is strong at high VaR layers and weaker at lower VaR layers, hence diversification is improved when high VaR layers are eliminated. Consider two scenarios: put options on each market index and an aggregate put option on the portfolio. Assume an exercise price of $\mathrm{VaR}_{0.95}$ of the loss referenced in the put option. Risks after put options are areas under risk densities up to the exercise price. Summary results of the first scenario are shown in table 2 in a similar format as table 1. Under the second scenario, aggregate risk after diversification is 3.55 hence the first scenario is more risk-effective without considering option prices.

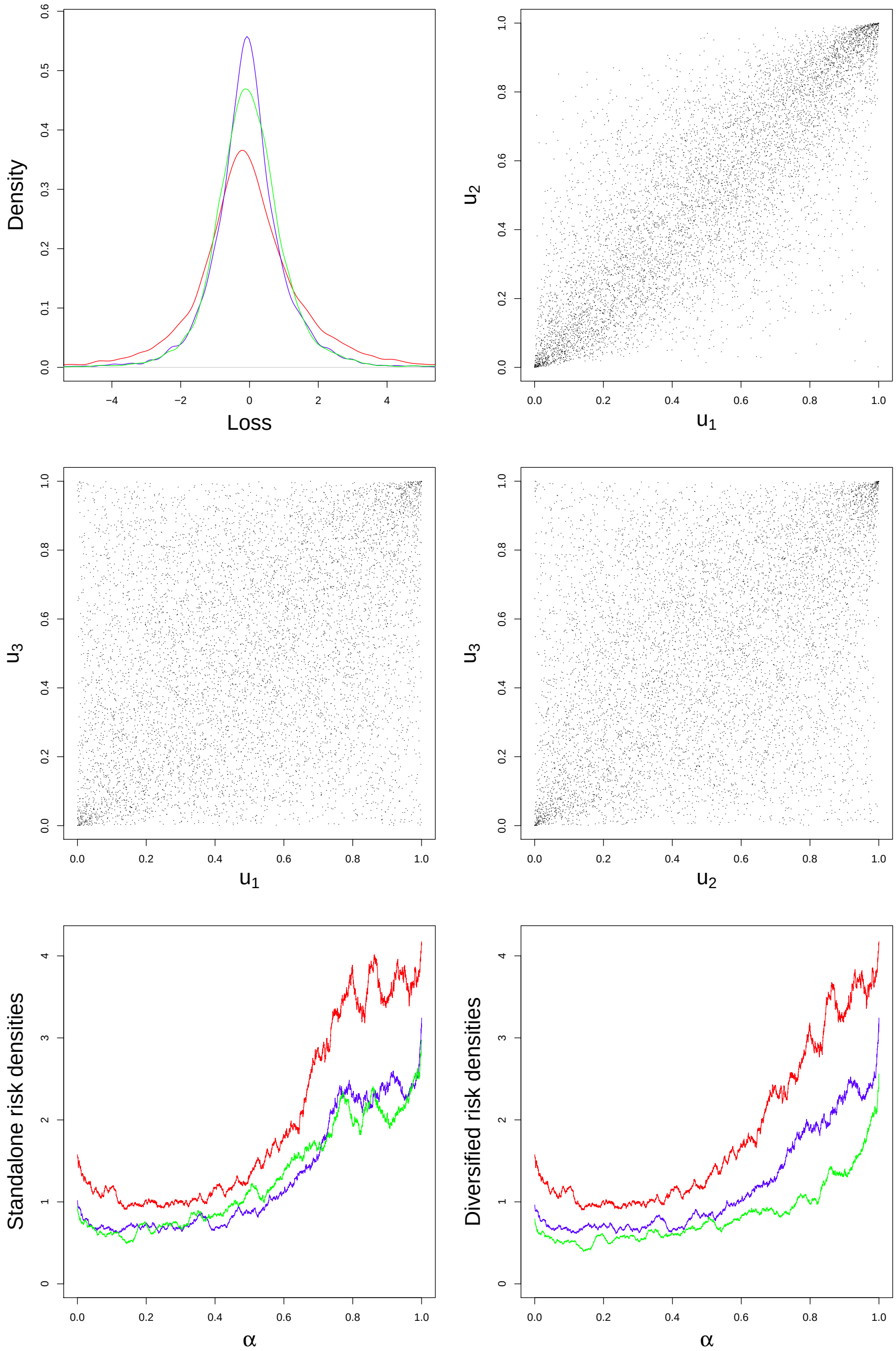

Figure 3: Top left panel plots empirical probability densities. Next 3 top panels plot empirical copulas (u_1:NASDAQ, u_2: S&P, u_3: FTSE). Bottom left and right panels plot calculated standalone and systematic risk densities, respectively. Red, blue and green represent NASDAQ, S&P and FTSE, respectively.

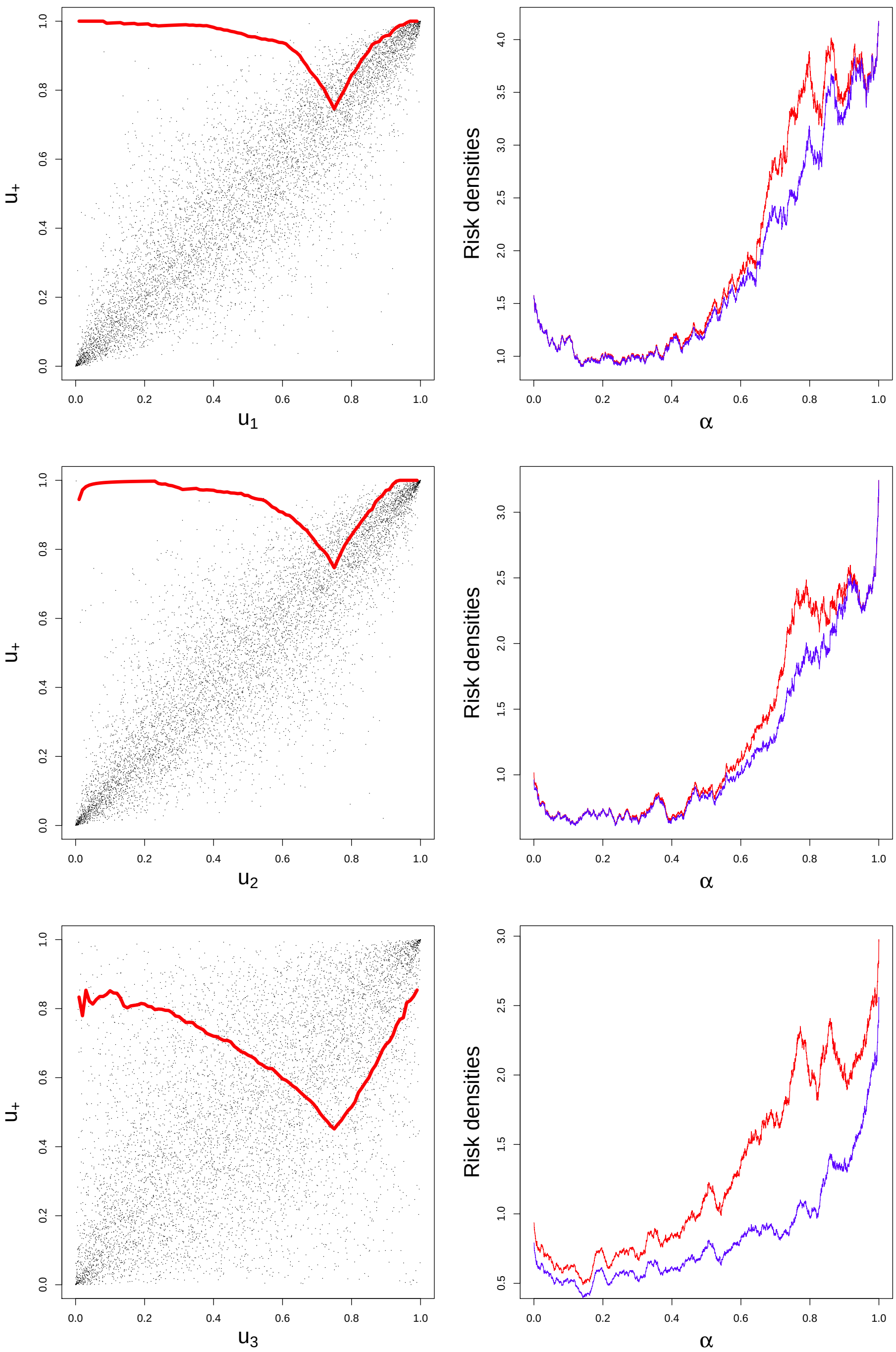

Figure 4: Left panels plot historical (u_i, u_+) and, in red, (α, θ_α). Right panels plot calculated (α, r_α) in red and $(\alpha, \overline{r}_\alpha)$ in blue. Note $\theta_\alpha = \overline{r}_\alpha / r_\alpha$. Top, middle and bottom panels are for NASDAQ, S&P and FTSE, respectively.

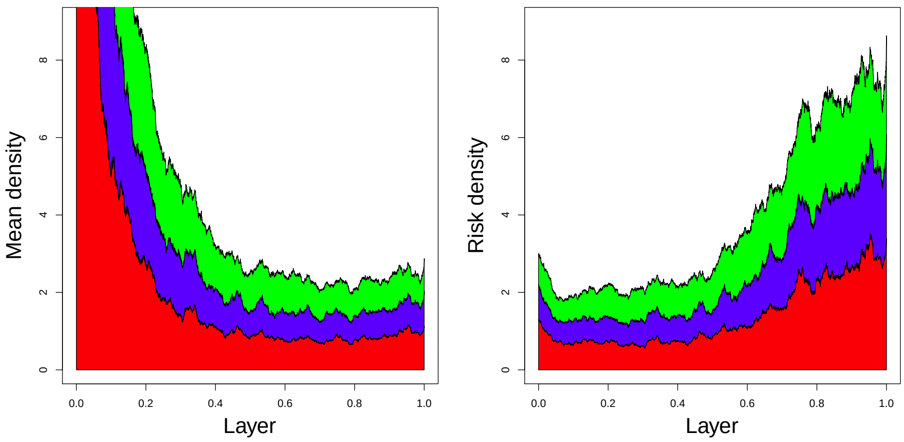

Figure 5: Left panel shows aggregate mean density and its breakdown into sub-aggregate mean densities. Right panel shows the same, for risk densities. Red, blue and green represent NASDAQ, S&P and FTSE, respectively.

Portfolio	r	$\overline{r}$	θ
NASDAQ	\$1.81	\$1.67	0.92
S&P	\$1.17	\$1.07	0.91
FTSE	\$1.17	\$0.75	0.64
Overall	\$4.15	\$3.49	0.84

Table 2: Overall standalone risk, systematic risk and risk ratio for each market index, after purchasing put options at exercise price $V_{0.95}$ for each index.

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