1. Local dependence and layer dependence

Dependence between two variables generally varies with percentile. For example extreme movements in two stock markets are likely be highly related whereas minor fluctuations may be relatively independent. Natural catastrophes create significant insurance losses for several classes of business at the same time, while attritional losses between various classes are weakly dependent.

Local dependence measures aim to capture the dependence structure of a bivariate distribution. This contrasts with measures of overall dependence such as Pearson correlation, Spearman's ρ and Kendall's τ (Embrechts et al., 2002). Local dependence measures include the univariate tail concentration (Venter, 2002), correlation curve (Bjerve and Doksum, 1993), and bivariate measures by Bairamov et al. (2003), Jones (1996) and Holland and Wang (1987).

This paper introduces, illustrates and analyzes an alternate local dependence measure called "layer dependence." Layer dependence is the covariance between a random variable and a single "layer" of another. Layer dependence is also the "gap" between upper and lower conditional tail expectations. Random variables are replaced with their percentile rank transforms and layer dependence is calculated entirely from the copula underlying the joint distribution. Hence of interest is rank dependence rather than dependence between random variables in their original scale, as the latter is often distorted by marginal distributions.

Layer dependence satisfies "coherence" properties similar to Spearman's ρ: between -1 and 1, constant and equal to -1, 0 and 1 for countermonotonic, independent and comonotonic random variables, sign switching when ranking order reverses, and taking on higher values when dependence is stronger. Taking a weighted average of layer dependence values across the joint distribution yields Spearman's ρ and alternative coherent measures of overall dependence.

Layer dependence provides a more appropriate and accurate measure of local dependence compared to existing measures. Higher dispersion from various points of the $45°$ line reduces layer dependence and vice versa. Calculating layer dependence at the first instance from data or parametric copulas extracts essential and interpretable information – the dependence structure. For a parametric copula, the implication of its parametric form and parameters on the dependence structure is not always apparent. Similar problems apply when past data is scarce.

Layer dependence offers an alternative approach to copula modeling. First compute layer dependence values from past data, and apply parametric smoothing. Further adjust, if necessary, to incorporate expert opinion. A copula is then fitted to refined layer dependence values. The fitted copula overcomes the inflexibility of parametric copulas to closely capture the dependence structure in past data, whilst avoiding uncertainties of empirical copulas at the other extreme.

Remaining sections are structured as follows. Section 2 defines and analyzes layer dependence. Section 3 demonstrates the appropriateness of layer dependence as a local dependence measure using several copulas. Section 4 explains the behaviour of layer dependence by decomposing it into a negative function of discordance and dispersion. Section 5 describes coherence properties of layer dependence. Links to existing literature are highlighted in §6. Further properties of layer dependence are described in §7. Section 8 applies layer dependence to copula modeling, and uses historical stock returns as an illustration. Section 9 discuss alternative coherent measures of overall dependence based on weighted averages of layer dependence. Section 10 concludes.

2. Layer dependence

Suppose u and v are percentile ranks of continuous random variables x and y. Then (u, v) has standard uniform marginals and joint distribution C, a copula (Nelson, 1999).

The α–layer dependence between v and the α–layer of u is

$$\ell_\alpha \equiv \frac{\mathrm{cov}\{v, (u > \alpha)\}}{\mathrm{cov}\{u, (u > \alpha)\}} = \frac{\mathrm{cor}\{v, (u > \alpha)\}}{\mathrm{cor}\{u, (u > \alpha)\}} \ , \quad 0 \leq \alpha \leq 1 \ , \tag{1}$$

where cov and cor calculate covariance and correlation, respectively, and $(u > \alpha)$ is the indicator function of the event in brackets. The denominators in (1) are independent of C and implies $\ell_\alpha = 1$ if $u = v$ and $\ell_\alpha = -1$ if $u = 1 - v$. Further $-1 \leq \ell_\alpha \leq 1$. Independence implies $\ell_\alpha = 0$. Coherence properties of ℓ_α are explored in §5. The α–layer of u is further discussed below.

Expanding the covariances in (1) and manipulating yields

$$\ell_\alpha = \frac{\mathrm{E}(v|u > \alpha) - \mathrm{E}(v|u \leq \alpha)}{\mathrm{E}(u|u > \alpha) - \mathrm{E}(u|u \leq \alpha)} = 2\left\{\mathrm{E}(v|u > \alpha) - \mathrm{E}(v|u \leq \alpha)\right\} \ , \tag{2}$$

where E calculates expectations using C. The middle expression in (2) is the expected change in v relative to the expected change in u when u crosses α. The latter is 0.5 for all α, yielding the final expression in (2). Hence large ℓ_α implies v is sensitive to movements in u across α, indicating strong dependence between v and u at α. When $\ell_\alpha = 0$, v is unchanged on average when u crosses α, and u and v are independent at α.

The definition of layer dependence in (1) exploits the identity

$$u = \int_0^1 (u > \alpha)\mathrm{d}\alpha \ . \tag{3}$$

To prove (3), note the left hand side is $\int_0^u 1\mathrm{d}\alpha$ equaling the right hand side. In addition taking expectations of (3) yields the familiar result expressing expectation as the integral of the survival function. Informally, the derivative of the "α–layer" of u, $(u > \alpha)\mathrm{d}\alpha$ with respect to u is $\infty\mathrm{d}\alpha = 1$ when $u = \alpha$ and 0 otherwise. Thus the α–layer captures movements in u solely at α, and ignores movements elsewhere. In addition (3) shows u is formed from infinitely many

α–layers, with each layer capturing the variability of u at a different α. Calculating covariance between the α–layer of u and v and scaling by the same where $u = v$ yields layer dependence.

Using (1) and (3), Spearman's correlation between u and v is

$$\rho_S \equiv \mathrm{cor}(u,v) = \frac{\mathrm{cov}(u,v)}{\mathrm{var}(u)} = \frac{\int_0^1 \mathrm{cov}\{v,(u>\alpha)\}\mathrm{d}\alpha}{1/12} = \mathcal{E}(\ell_\alpha) \ . \tag{4}$$

where the expectation $\mathcal{E}$ is calculated over $0 \le \alpha \le 1$ using density $6\alpha(1-\alpha)$. The density integrates to 1, has minimum 0 at $\alpha = 0$ and 1, and increases symmetrically to maximum at $\alpha = 0.5$. Modifying the density leads to different emphasis on different areas of the relationship between u and v, and yields alternate measures of overall dependence. For example using the density $n\alpha^{n-1}$ where $n > 0$ puts increasing weight on upper tail dependence. This is further explored in §9.

3. Layer dependence curves for various copulas

The nine panels in Figure 1 display (u,v) scatterplots of exchangeable copulas, and their layer dependence curves ℓ_α for all $0 \le \alpha \le 1$. Each copula has Spearman's correlation $\rho_S = 0.6$.

Each ℓ_α curve reflects the dependence structure between u and v. Given α, ℓ_α is larger if points are more clustered around (α,α) and vice versa, as formalised in §4. In addition, ℓ_α increases to 1 in the tails if points converge to the 45° degree line indicating perfect dependence.

The nine panels in Figure 1 highlight the inadequacies of using Spearman's correlation to measure overall dependence, particularly in the tails. In contrast, layer dependence curves capture the dependence structure of each copula.

4. Layer dependence, discordance and dispersion

If (u,v) is exchangeable, $C(u,v) = C(v,u)$, then layer dependence ℓ_α measures the lack of discordance and dispersion at α:

$$\ell_\alpha = 1 - 2(1+\gamma_\alpha)\delta_\alpha \ , \tag{5}$$

where

$$\gamma_\alpha \equiv \mathrm{cor}\{(u \le \alpha),(v > \alpha)\} = \mathrm{cor}\{(u > \alpha),(v \le \alpha)\} \ ,$$
$$\delta_\alpha \equiv \mathrm{E}\left\{(|u-v|)|(u-\alpha)(v-\alpha) < 0\right\} \ .$$

A proof of (5) is below. The correlation $-1 \le \gamma_\alpha \le 1$ measures the tendency for (u,v) to be discordant at α: opposite signs on $u-\alpha$ and $v-\alpha$. The expectation $0 \le \delta_\alpha \le 1$ measures the dispersion between discordant points u and v at α.

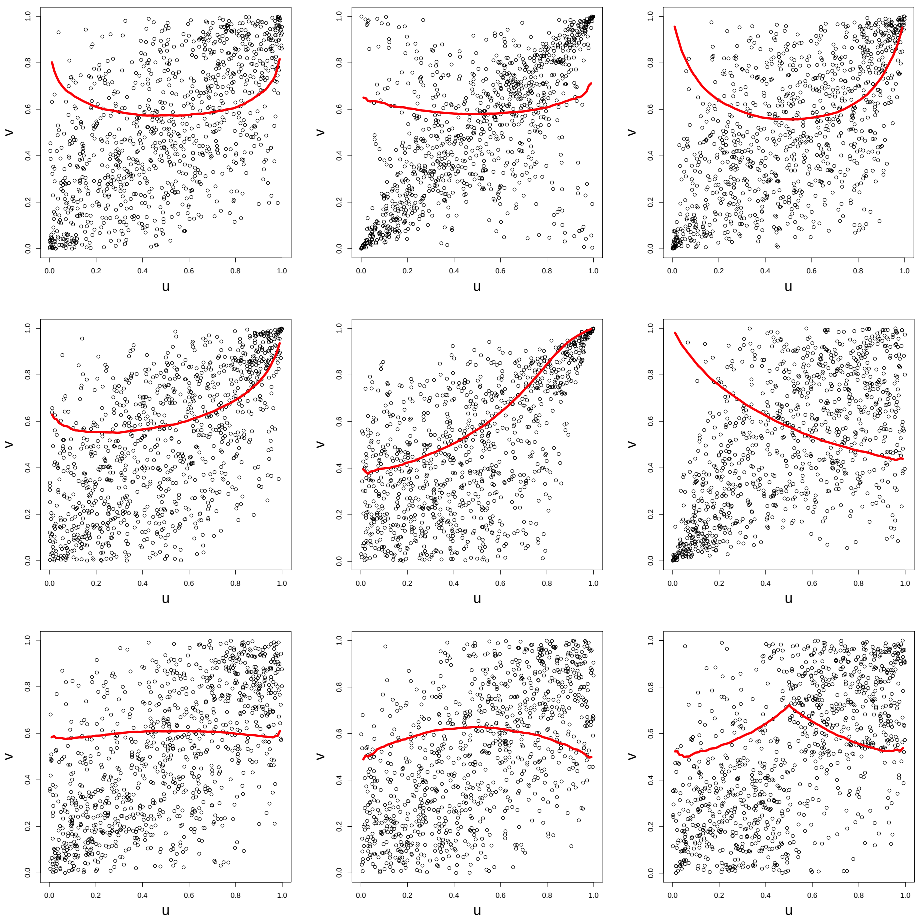

Figure 1: Copulas with the same $\rho_S = 0.6$ but different layer dependence curves ℓ_α over $0 \leq \alpha \leq 1$ (red curves) .

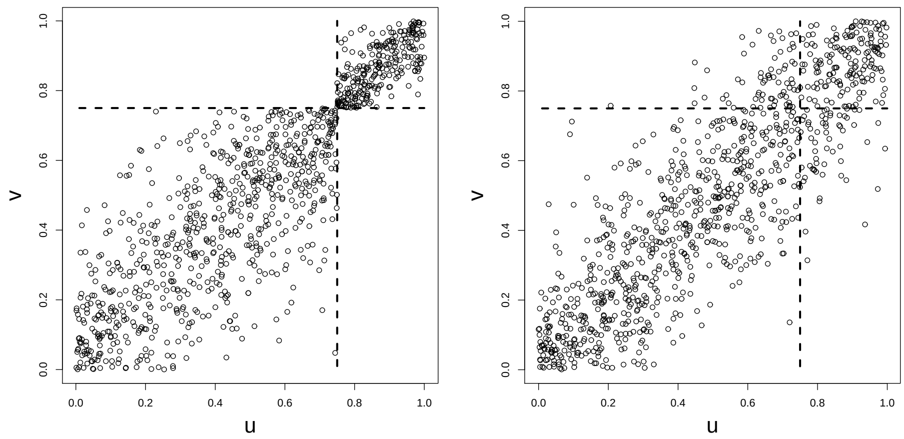

Figure 2: The left and right panel show $\ell_{0.75} = 1$ and $\ell_{0.75} = 0.86$, respectively. In the left panel, $\gamma_{0.75} = -1$ and $\delta_{0.75} = 0$. In the right panel, $\gamma_{0.75} = -0.65$ and $\delta_{0.75} = 0.21$.

The proof of (5) follows from

$$\gamma_\alpha = -\frac{\operatorname{cov}\{(u \le \alpha), (v \le \alpha)\}}{\operatorname{var}\{(u \le \alpha)\}} = \frac{\alpha^2 - C(\alpha, \alpha)}{\alpha(1-\alpha)} \ ,$$

and

$$\begin{aligned} \delta_\alpha &= 2\mathrm{E}\left\{(u-v)(u>v) | (u-\alpha)(v-\alpha) < 0\right\} \\ &= \frac{2\mathrm{E}\{(u-v)(u>v)(u>\alpha)(v \le \alpha)\}}{2\mathrm{E}\{(u>\alpha)(v\le\alpha)\}} = \frac{\mathrm{E}\{(u-v)(u>\alpha)(v\le\alpha)\}}{\alpha - C(\alpha,\alpha)} \\ &= \frac{\mathrm{E}\{(u-v)(u>\alpha)\} - \mathrm{E}\{(u-v)(u>\alpha)(v>\alpha)\}}{\alpha - C(\alpha,\alpha)} = \frac{\mathrm{E}\{(u-v)(u>\alpha)\}}{\alpha - C(\alpha,\alpha)} \ . \end{aligned}$$

Substituting the above expressions for γ_α and δ_α into the right hand side of (5) yields the expression for ℓ_α in (2), completing the proof.

Result (5) explains the behaviour of layer dependence curves in Figure 1. Layer dependence ℓ_α is larger if there are fewer discordant pairs at α, and discordant pairs at α are closer to the 45° degree line. The former indicates smaller γ_α and the latter indicates smaller δ_α. Vice versa for small ℓ_α. If $\ell_\alpha = 1$ then $\gamma_\alpha = -1$ or $\delta_\alpha = 0$, implying u and v are simultaneously below or above α and $u = v$ for discordant pairs. If $\ell_\alpha = 1$ in an interval, then $u = v$ in the same interval.

Figure 2 illustrates the relationship between layer dependence, discordance and dispersion in (5) using two copulas with $\ell_\alpha = 1$ and $\ell_\alpha = 0.86$ at $\alpha = 0.75$. When $\ell_\alpha = 1$, there is no discordance or dispersion at α. As ℓ_α decreases from 1, the number of discordant pairs and their dispersion increases.

5. Coherence properties of layer dependence

Layer dependence ℓ_α satisfies five "coherence" properties. These properties are extensions of properties applying to Spearman's correlation ρ_S.

- **Bounds**: Layer dependence lies between -1 and 1: $-1 \le \ell_\alpha \le 1$ for all α. Hence layer dependence is bounded in the same way as ρ_S.
- **Perfect dependence**: Constant layer dependence of -1 or 1 are equivalent to countermonotonicity and comonotonicity. Thus $\ell_\alpha = -1$ for all α if and only if $v = 1 - u$ while $\ell_\alpha = 1$ for all α if and only if $v = u$.
- **Independence**: If u and v are independent then $\ell_\alpha \equiv 0$. The converse is not true – zero layer dependence does not imply independence as shown by the following counterexample. Assume $v = u$ and $v = 1 - u$ with equal probability. Then $\mathrm{E}(v|u = t) = 0.5$ for all $0 \le t \le 1$ implying $\mathrm{E}(v|u > \alpha) = \mathrm{E}(v|u \le \alpha) = 0.5$. Hence $\ell_\alpha = 0$ from (2). However u and v are not independent.
- **Symmetry**: Ranking either u or v in the opposite direction switches the sign of layer dependence. Changing the ranking order of both variables preserves the sign of layer dependence.
- **Ordering**: Higher correlation order (Dhaene et al., 2009) leads to higher layer dependence. Consider bivariate uniform (u^*, v^*) exceeding (u, v) in correlation order: $C^*(a, b) \ge C(a, b)$ for all $0 \le a, b \le 1$, where C^* is the joint distribution of (u^*, v^*). Then $\ell^*_\alpha \ge \ell_\alpha$, $0 \le \alpha \le 1$ where ℓ^*_α denotes the α–layer dependence of (u^*, v^*). Hence greater dependence leads to higher layer dependence across all percentiles.

Independence, symmetry and ordering properties follow from the definition of layer dependence in (1). From (2), constant layer dependence of one implies $\mathrm{E}(v|u > \alpha) = (\alpha + 1)/2$ and $\mathrm{E}(v|u = \alpha) = \alpha$, for all $0 \le \alpha \le 1$, hence $v = u$. Similarly constant layer dependence of minus one implies $v = 1 - u$. The ordering property holds since higher correlation order implies larger covariances (Dhaene et al., 2009). Prove the bounds property by combining ordering and perfect dependence properties, and noting countermonotonicity and comonotonicity represent minimum and maximum correlation order, respectively.

6. Layer dependence and measures of tail behaviour

Measures have been proposed in the literature to capture the degree of tail dependence. The next two subsections describe the connection of these measures to layer dependence.

6.1. *Tail dependence*

Joe (1997) defines coefficients of lower and upper tail dependence in terms of the limiting tail probabilities

$$\lim_{\alpha\to 0} \mathrm{E}\{(v \le \alpha)|u \le \alpha\} \ , \quad \lim_{\alpha\to 1} \mathrm{E}\{(v > \alpha)|u > \alpha\} \ .$$

Limits of one indicate perfect positive tail dependence, and occur if and only if u and v converge simultaneously to zero (lower tail) or one (upper tail). Negative tail dependence definitions replace $(v \le \alpha)$ and $(v > \alpha)$ in the above expressions by $(v > 1-\alpha)$ and $(v \le 1-\alpha)$, respectively. Tail dependence has been used to model catastrophic events such as multiple bank failures and market crashes.

Perfect tail dependence by Joe (1997) is equivalent to perfect layer dependence in the tails. For example consider the upper tail. From (2), $\ell_1 = 2\mathrm{E}(v|u = 1) - 1$ implying $\ell_1 = 1$ if and only if $\mathrm{E}(v|u = 1) = 1$. Hence $\ell_1 = 1$ if and only if $u = 1$ implies $v = 1$. Similarly $\ell_1 = -1$ if and only if $u = 1$ implies $v = 0$. Similar remarks apply to the lower tail: $\ell_0 = 1$ if and only if $u = 0$ implies $v = 0$ while $\ell_0 = -1$ if and only if $u = 0$ implies $v = 1$. Hence perfect layer dependence in the tails implies variables simultaneously achieve their extreme values.

6.2. *Tail concentration*

Tail concentration (Venter, 2002) is a local dependence measure formed by combining lower and upper conditional tail probabilities at α:

$$\tau_\alpha \equiv (\alpha \le 0.5)\mathrm{P}(v \le \alpha|u \le \alpha) + (\alpha > 0.5)\mathrm{P}(v > \alpha|u > \alpha) \ .$$

Higher tail concentration τ_α implies u and v are more likely to fall in the same lower tail $(\alpha \le 0.5)$ or upper tail $(\alpha > 0.5)$.

To display the connection of layer dependence to tail concentration, rewrite the latter in terms of the copula C of (u, v):

$$\tau_\alpha = (\alpha \le 0.5)\frac{C(\alpha,\alpha)}{\alpha} + (\alpha > 0.5)\frac{1 - 2\alpha + C(\alpha,\alpha)}{1-\alpha} \ .$$

Standardising τ_α by subtracting the value under independence and dividing by the departure from independence under comonotonicity yields

$$\tau^*_\alpha \equiv \frac{\tau_\alpha - \{\alpha(\alpha \le 0.5) + (1-\alpha)(\alpha > 0.5)\}}{1 - \{\alpha(\alpha \le 0.5) + (1-\alpha)(\alpha > 0.5)\}} = \frac{C(\alpha,\alpha) - \alpha^2}{\alpha(1-\alpha)} = -\gamma_\alpha \ ,$$

where γ_α is defined below (5).

Hence, using (5), $\ell_\alpha = 1 - 2\delta_\alpha(1 - \tau^*_\alpha)$ where δ_α is the average dispersion between points (u, v) discordant at α. Note layer dependence and tail concentration have the same sign. Hence layer dependence refines tail concentration by standardising it and including further information on dispersion.

7. Further properties of layer dependence

This section lists and explores further properties of layer dependence.

7.1. Layer dependence does not uniquely characterise a copula

Suppose $v = u$ and $v = 1 - u$ with equal probability, implying $\mathrm{E}(v|u \leq \alpha) = \mathrm{E}(v|u > \alpha) = 0.5$ for all α. Then $\ell_\alpha = 0$, from (2). The copula is $C(u, v) = 0.5\{\min(u, v) + \max(u + v - 1, 0)\}$. The same ℓ_α results if u and v are independent: $C(u, v) = uv$.

This example emphasises layer dependence only captures first-order conditional tail expectations, based on (2). Hence two copulas with equal first-order conditional tail expectations have equal layer dependence.

7.2. Layer dependence for a non-exchangeable copula

If u and v are exchangeable, layer dependence in (1) is invariant when u and v are switched. Archimedean copulas (McNeil et al., 2005) for example are exchangeable. If u and v are not exchangeable then layer dependence differs when layers of v instead of u are applied, that is dependence between v and α–layer of u differs from dependence between u and α–layer of v.

An analogous property applies to least squares regression: regressing v on u is the same as regressing u on v, if the joint distribution is exchangeable. Otherwise the regressions differ.

7.3. Layer dependence preserves convex combination

Suppose (u, v) and (u^*, v^*) are both bivariate uniform with layer dependence ℓ_α and ℓ^*_α, respectively. Then $\pi\ell_\alpha + (1 - \pi)\ell^*_\alpha$ is the layer dependence of the random variable $x(u, v) + (1 - x)(u^*, v^*)$ where x is Bernoulli with success probability π. The proof is straightforward using (2). This result generalises to multiple and continuous convex combinations of a vector of bivariate uniform random variables.

7.4. One-sided conditional tail expectations

Since $\mathrm{E}(v) = \alpha\mathrm{E}(v|u \leq \alpha) + (1 - \alpha)\mathrm{E}(v|u > \alpha)$ it is straightforward to show from (2) that

$$\ell_\alpha = \frac{\mathrm{E}(v|u > \alpha) - \mathrm{E}(v)}{\mathrm{E}(u|u > \alpha) - \mathrm{E}(u)} = \frac{\mathrm{E}(v|u \leq \alpha) - \mathrm{E}(v)}{\mathrm{E}(u|u \leq \alpha) - \mathrm{E}(u)} ,$$

the gap between upper or lower conditional tail expectations of v and the unconditional expectation. As before denominators are scaling factors ensuring $\ell_\alpha = 1$ if u and v are comonotonic and -1 if countermonotonic.

7.5. Copula integration

Note $\mathrm{cov}\{(u > \alpha), (v > \beta)\} = C(\alpha, \beta) - \alpha\beta$ and apply (3) to v, to derive

$$\ell_\alpha = \frac{\int_0^1 \mathrm{cov}\{(u > \alpha), (v > \beta)\}\mathrm{d}\beta}{\alpha(1-\alpha)/2} = \frac{2\int_0^1 C(\alpha, \beta)\mathrm{d}\beta - \alpha}{\alpha(1-\alpha)} = \frac{2\int_0^1 C(\beta|\alpha)\mathrm{d}\beta - 1}{1-\alpha} \,,$$

where $C(\beta|\alpha) = \mathrm{P}(v \leq \beta | u \leq \alpha)$ is the conditional copula. Thus ℓ_α integrates copulas and conditional copulas to reduce their dimension from two and one, and scales the result to ensure it lies between ± 1.

With Archimedean copulas (McNeil et al., 2005) $C(\alpha, \beta) = \psi^{-}\{\psi(\alpha) + \psi(\beta)\}$ where ψ is the generator function and ψ^{-} its inverse. In this case closed form expressions for the integrals and hence for ℓ_α do not exist.

8. Copula fitting using layer dependence

This section proposes fitting a copula to data using layer dependence. The approach can incorporate expert opinion on the dependence structure. The approach is illustrated using historical NASDAQ and FTSE stock returns.

8.1. Fitting steps

A copula fitting procedure based on layer dependence is as follows:

1. Calculate layer dependence curve with desired granularity from percentile rank data.
2. Smooth calculated layer dependence curve either parametrically or semi-parametrically.
3. Refine layer dependence using expert knowledge, such as existence and level of tail dependence, and overall dependence.
4. Fit a copula given the fitted layer dependence curve, for example using the factor copula model described in the next subsection.

The layer dependence approach offers two advantages over fitting parametric copulas. Firstly layer dependence reflects the dependence structure exhibited in past data, whereas parametric copulas, with their relatively restricted dependence structures, may not properly fit past data. Secondly, layer dependence accommodates expert knowledge of local dependence, whereas parametric copulas permit limited changes to the dependence structure once a parametric form is selected.

In comparison to empirical copulas, layer dependence is more robust and less affected by data inadequacies. Layer dependence summarises past data into linear functions of conditional tail means, with a parametric curve fitted to calculated values. Hence layer dependence captures advantages of parametric and empirical copulas – the fitted copula utilises a smooth layer dependence curve, and the dependence structure underlying the fitted copula mirrors past data.

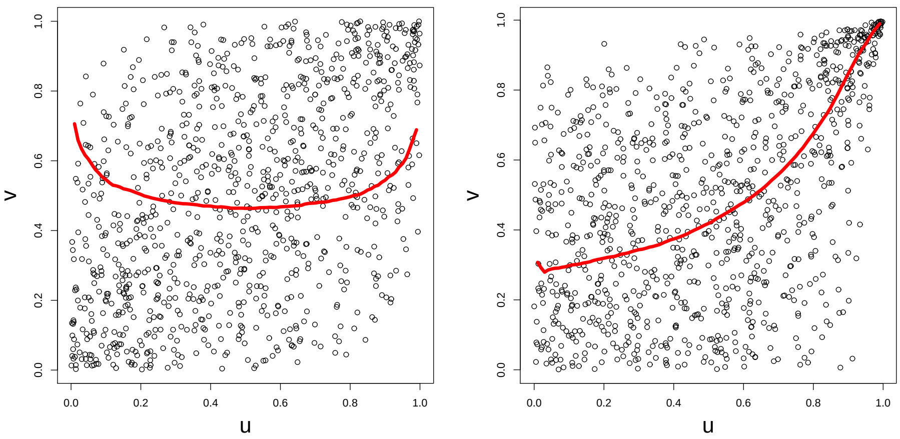

Figure 3: Simulated factor copulas and layer dependence curves assuming s is standard normal (left panel) and χ^2_1 (right panel).

8.2. *Factor copula model*

The following describes an approach to fit and simulate a copula given layer dependence ℓ_α for all $0 \le \alpha \le 1$. The copula is not unique to ℓ_α since ℓ_α, from (2), relates to first-order conditional expectations rather than the entire conditional distribution associated with the copula.

Suppose s is a random variable. The factor copula model is

$$u = F(s+\epsilon) \ , \quad v = F(s+\eta) \ , \quad \epsilon, \eta \sim N(0,1) \ , \tag{6}$$

where s, ϵ and η are independent and F is the distribution of the random variables in the brackets. Then both u and v are uniform and the joint distribution of (u,v) is exchangeable. Exchangeability is assumed since most commonly used copulas, such as Archimedean copulas (McNeil et al., 2005), are exchangeable. The factor copula model in (6) is made non-exchangeable for example by dropping either ϵ or η.

The common factor s in (6) generates and controls layer dependence between u and v. For example if s is highly volatile and right skewed relative to noise terms ϵ, η, then u and v are more dependent particularly in the upper tail. The two panels in Figure 3 simulate factor copulas and their layer dependence curves assuming s has standard normal and χ^2_1 distributions. Standard normal s generates a Gaussian copula, while a χ^2_1 distribution implies movements in u and v close to 1 are generated and controlled by s, implying upper tail dependence.

The factor copula model generates positive layer dependence. Negative layer dependence is created by first simulating the corresponding positive layer dependence, and then taking complements of either percentile rank.

To generate a sample of (u, v) of size n from the factor copula model with layer dependence ℓ_α, $0 \leq \alpha \leq 1$, first simulate $\epsilon_i, \eta_i \sim N(0,1)$ for $i = 1, \ldots, n$. Arbitrarily initialise s_i for $i = 1, \ldots, n$ by setting it equal to a normal sample with standard deviation chosen such that $s_i + \epsilon_i$ and $s_i + \eta_i$ have Spearman's correlation consistent with ℓ_α. Then repeat:

1. Compute "fitted" $\hat{\ell}_{j/n}$ between (u_i, v_i) for $j = 1, \ldots, n$, where u_i and v_i are calculated from the percentile ranks of $s_i + \epsilon_i$ and $s_i + \eta_i$, respectively.
2. Keep s_1 unchanged and update $s_2, \ldots, s_n$ according to

$$s_{i+1} \leftarrow s_i + \left(\frac{\ell_{i/n}}{\hat{\ell}_{i/n}} \right)^a (s_{i+1} - s_i) \ , \quad a > 0 \ ,$$

   in turn yielding an updated sample of (u_i, v_i).
3. Go to 1 unless $\|\ell_{i/n} - \hat{\ell}_{i/n}\|$ is less than some pre-specified small amount.

Step 1 can be simplified by computing $\hat{\ell}_{j/n}$ over a fewer number of points and then fitting a parametric curve to computed values. The resulting sample (u_i, v_i) for $i = 1, \ldots, n$ once the iteration is complete has layer dependence $\hat{\ell}_\alpha \approx \ell_\alpha$.

8.3. Illustration using stock returns

The following illustrates layer dependence copula fitting using NASDAQ and FTSE returns. The top left panel in Figure 4 plots percentile ranks of daily NASDAQ and FTSE returns from 1985 and 2014. Layer dependence at every integer percentile and Spearman's correlation are shown in the same panel. Spearman's correlation is 0.4, indicating moderate dependence between NASDAQ and FTSE returns. Layer dependence increases towards both tails, but more so in the lower tail. Hence major corrections in NASDAQ and FTSE tend to occur simultaneously, and negative corrections are more strongly dependent than positive corrections.

The top right panel in Figure 4 fits a factor copula to a smooth layer dependence curve passing through empirical values[1]. Simulated values of the fitted factor copula are shown in the same panel. The bottom left panel fits a Gaussian copula to past data by matching Spearman's correlation. The Gaussian copula has a symmetric layer dependence curve, and hence does not capture the stronger lower tail dependence exhibited in NASDAQ and FTSE data. Student-t and other elliptical copulas would suffer from the same inadequacy.

The bottom right panel in Figure 4 plots the density of the common factor s generating the fitted factor copula. Also shown is the density implicit in the fitted Gaussian copula. The density of s is more peaked and left skewed, reflecting the stronger lower tail dependence and weak dependence in the centre.

[1] The smoothing is performed by fitting a polynomial of order 4 using least squares estimation.

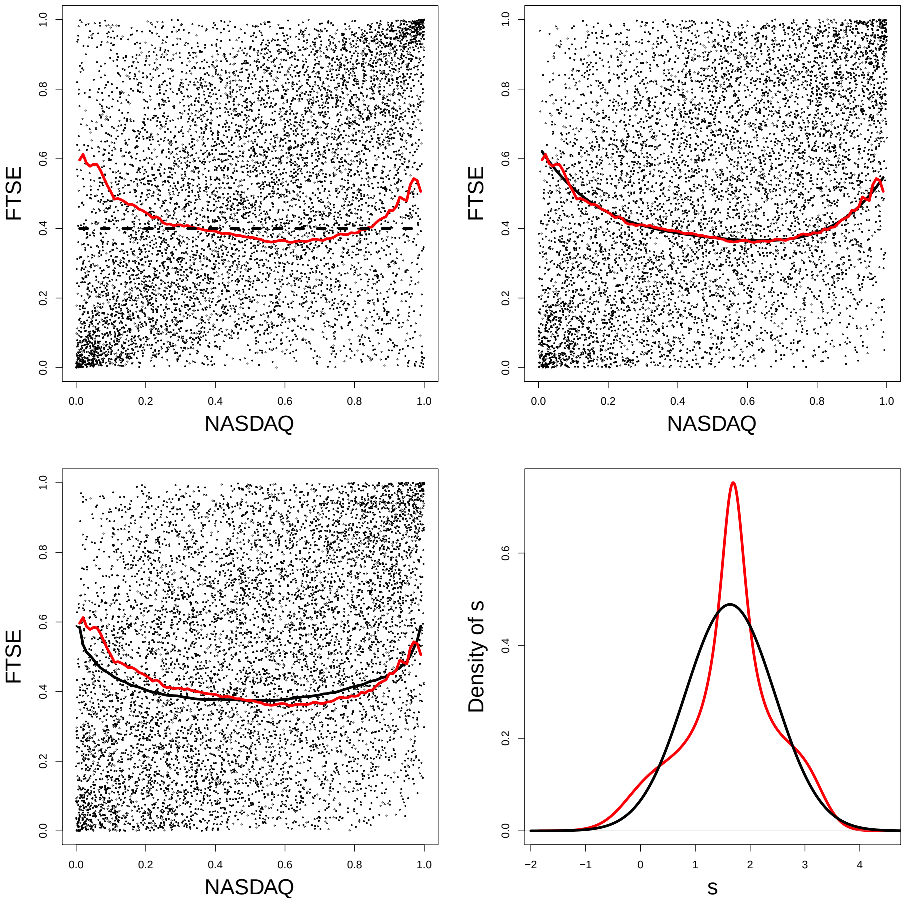

Figure 4: The top left panel plots percentile ranks of NASDAQ and FTSE daily returns from 1985 to 2014, and the empirical layer dependence curve. The top right panel plots a parametrically smoothed layer dependence curve and a fitted factor copula. The bottom left panel plots a Gaussian copula and its layer dependence curve. The Gaussian copula is fitted by matching Spearman's correlation. The red empirical layer dependence curve is plotted in both panels containing fitted factor and Gaussian copulas. The bottom right panel graphs the density of s underlying the fitted factor copula (red), and the implied density of s underlying the Gaussian copula (black).

9. Alternate measures of overall dependence

Equation (4) shows that Spearman's correlation is a weighted average of ℓ_α using weights $6\alpha(1-\alpha)$. Alternative rank dependence measures are attained by using different weights w_α:

$$\rho_W \equiv \int_0^1 w_\alpha \ell_\alpha \mathrm{d}\alpha = \int_0^1 w_\alpha \frac{\mathrm{cov}\{v,(u>\alpha)\}}{\mathrm{cov}\{u,(u>\alpha)\}}\mathrm{d}\alpha = \mathrm{cov}\{W(u), v\} \ ,$$

where

$$W(u) \equiv 2\int_0^u \frac{w_\alpha}{\alpha(1-\alpha)}\mathrm{d}\alpha \ ,$$

is the weighted cumulative of w_α. For Spearman's correlation ρ_S, $w_\alpha = 6\alpha(1-\alpha)$ and $W(u)$ is proportional to u.

Since ρ_W is a weighted average of ℓ_α, coherence properties of layer dependence described in §5 apply to ρ_W for any weight function w_α: bounded by ± 1, constant values of -1, 0 and 1 under countermonotonicity, independence and comonotonicity, respectively, sign reversal when ranking order switches, and higher values for stronger dependence.

The density w_α specifies "importance" of dependence at percentile α. With Spearman's correlation, $w_\alpha = 6\alpha(1-\alpha)$: importance is symmetric and highest at the median, and decreases to zero at both tails. Hence ρ_S emphasizes dependence at moderate percentiles and diminishes dependence in the tails. This weighting may be inappropriate when tail dependence is prominent and carries adverse consequences, such as multiple market crashes or catastrophe insurance losses. Alternate densities are described below.

The following densities w_α yield alternate rank dependence measures:

- Suppose dependence at different percentiles are equally important. Then w_α is the uniform density and

  $$\rho_W = 2\mathrm{cov}\left\{v, \log\left(\frac{u}{1-u}\right)\right\} = \sqrt{\frac{2}{3}}\ \mathrm{cor}\left\{v, \log\left(\frac{u}{1-u}\right)\right\} \ ,$$

  a multiple of the correlation between v and the logit of u. If tail dependence is pronounced, $\rho_W > \rho_S$ since ρ_W weights tail dependence more heavily compared to ρ_S.

- If $w_\alpha = 3\alpha^2$ then dependence at higher percentiles are considered more important. This density is applicable when upper tail dependence is critical, for example the simultaneous occurrence of large insurance losses in different lines of business. Then

  $$\rho_W = 6\mathrm{cov}\left\{v, \log\left(\frac{\mathrm{e}^{-u}}{1-u}\right)\right\} = \sqrt{3}\mathrm{cor}\left\{v, -\log(1-u)\right\} - \frac{\rho_S}{2} \ .$$

  If dependence is higher over percentiles above the median then $\rho_W > \rho_S$.

- If dependence over percentiles below the median is more important then for example $w_\alpha = 3(1-\alpha)^2$ yielding

$$\rho_W = 6\mathrm{cov}\left\{v, \log\left(u\mathrm{e}^{-u}\right)\right\} = \frac{\rho_S}{2} - \sqrt{3}\mathrm{cor}(v, \log u) \ .$$

- Suppose w_α is derived from V_α, the inverse marginal distribution of x with derivative V'_α:

$$w_\alpha = \frac{\mathrm{cov}\{u, (u > \alpha)\}V'_\alpha}{\int_0^1 \mathrm{cov}\{u, (u > \alpha)\}V'_\alpha \mathrm{d}\alpha} = \frac{\alpha(1-\alpha)V'_\alpha}{\mathrm{cov}(V_u, u)} = \frac{\alpha(1-\alpha)V'_\alpha}{\mathrm{cov}(x, u)} \ ,$$

  where $x = V_u$. This yields the Gini correlation (Schechtman and Yitzhaki, 1999)

$$\rho_W = \frac{\mathrm{cov}(V_u, v)}{\mathrm{cov}(V_u, u)} = \frac{\mathrm{cov}\{x, G(y)\}}{\mathrm{cov}\{x, F(x)\}} \ ,$$

  where $F \equiv V^-$ and G are distribution functions of x and y, respectively. In this example the density w_α depends on the marginal distribution of x. More skewness in x leads to more steeply increasing w_α hence greater emphasis on upper tail dependence.

10. Conclusion

Layer dependence captures dependence structures in bivariate copulas, and satisfies coherence properties. Taking weighted averages of layer dependence curves yields Spearman's correlation and alternate overall dependence measures.

Using layer dependence in copula fitting captures dependence structures in past data, whilst flexibly accommodating expert opinion. Layer dependence achieves a balance between parametric approaches (smooth fit, low flexibility) and empirical approaches (volatile fit, high flexibility).

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