Layer dependence as a measure of local dependence

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Abstract

A new measure of local dependence called "layer dependence" is analysed and illustrated. Layer dependence measures dependence between two variables at different percentiles in their joint distribution. Layer dependence satisfies coherence properties similar to Spearman's correlation, such as lying between -1 and 1, with -1, 0 and 1 corresponding to countermonotonicity, independence and comonotonicity, respectively. Spearman's correlation is a weighted average of layer dependence at different percentiles. Alternate overall correlation measures are arrived by varying the weights. Layer dependence allows copulas to be fitted and tailored to data and expert opinion on the dependence structure.

Keywords: Local dependence; rank dependence; conditional tail expectation; Spearman's correlation; concordance.

1. Local dependence and layer dependence

Dependence between two variables generally varies with percentile. For example extreme movements in two stock markets are likely to be highly related whereas minor fluctuations may be relatively independent. Natural catastrophes create significant insurance losses for several classes of business at the same time, while attritional losses between various classes are weakly dependent.

Local dependence measures aim to capture the dependence structure of a bivariate distribution. This contrasts with measures of overall dependence such as Pearson correlation, Spearman's ρ and Kendall's τ (Embrechts et al., 2002). Local dependence measures include the univariate tail concentration (Venter, 2002), correlation curve (Bjerve and Doksum, 1993), and bivariate measures by Bairamov et al. (2003), Jones (1996) and Holland and Wang (1987).

This paper introduces, illustrates and analyzes an alternate local dependence measure called "layer dependence." Layer dependence is the covariance between

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a random variable and a single "layer" of another. Layer dependence is also the "gap" between upper and lower conditional tail expectations. Random variables are replaced with their percentile rank transforms and layer dependence is calculated entirely from the copula underlying the joint distribution. Hence of interest is rank dependence rather than dependence between random variables in their original scale, as the latter is often distorted by marginal distributions.

Layer dependence satisfies "coherence" properties similar to Spearman's ρ: it is between -1 and 1, constant and equal to -1, 0 and 1 for countermonotonic, independent and comonotonic random variables, sign switching when the ranking order reverses, and taking on higher values when dependence is stronger. Taking a weighted average of layer dependence values across the joint distribution yields Spearman's ρ and alternative coherent measures of overall dependence.

Layer dependence provides a more appropriate and accurate measure of local dependence compared to existing measures. Higher dispersion from various points of the 45° line reduces layer dependence and vice versa. Calculating layer dependence at the first instance from data or parametric copulas extracts essential and interpretable information – the dependence structure. For a parametric copula, the implication of its parametric form and parameters on the dependence structure is not always apparent. Similar problems apply when past data is scarce.

Layer dependence offers an alternative approach to copula modeling. First compute layer dependence values from past data, and apply parametric smoothing. Further adjust, if necessary, to incorporate expert opinion. A copula is then fitted to refined layer dependence values. The fitted copula overcomes the inflexibility of parametric copulas to closely capture the dependence structure in past data, whilst avoiding uncertainties of empirical copulas at the other extreme.

Remaining sections are structured as follows. Section 2 defines and analyzes layer dependence. Section 3 demonstrates the appropriateness of layer dependence as a local dependence measure using several copulas. Section 4 explains the behaviour of layer dependence by decomposing it into a negative function of discordance and dispersion. Section 5 describes coherence properties of layer dependence. Links to existing literature are highlighted in §6. Further properties of layer dependence are described in §7. Section 9 applies layer dependence to copula modeling, and uses historical stock returns as an illustration. Section 8 discuss alternative coherent measures of overall dependence based on weighted averages of layer dependence. Section 10 concludes.

2. Layer dependence – motivation and definition

Suppose u and v are percentile ranks of continuous random variables x and y. Then (u, v) has standard uniform marginals and joint distribution C, a copula (Nelson, 1999). Of interest is the dependence structure underlying (x, y).

Spearman's ρ (Embrechts et al., 2002) between x and y is a common measure of overall rank dependence, the dependence between percentile ranks u and v.

Rank dependence is not distorted by marginal distributions unlike for example Pearson's correlation (McNeil et al., 2005) which is generally not ± 1 when x and y are perfectly dependent unless x and y are linear in one another. Spearman's ρ can also be applied to estimate copula parameters using the method of moments (Kojadinovic and Yan (2010), Bouyé et al. (2000)). However a drawback of Spearman's ρ is that it does not capture local dependence which may vary across the joint distribution as explained in section 1. This drawback is addressed by a generalisation of Spearman's ρ discussed in the following.

Spearman's ρ is defined and can be rewritten as

$$\rho \equiv \mathrm{cor}(v, u) = \frac{\mathrm{cov}(v, u)}{\mathrm{cov}(u, u)} = \frac{\int_0^1 \mathrm{cov}\{v, (u > \alpha)\}\mathrm{d}\alpha}{\int_0^1 \mathrm{cov}\{u, (u > \alpha)\}\mathrm{d}\alpha} \tag{1}$$

where cov and cor represent covariance and correlation, respectively. In addition $(u > \alpha)$ is a random variable equal to 1 if $u > \alpha$ and 0 otherwise. The final expression in (1) relies on the "layer decomposition" of u

$$u = \int_0^1 (u > \alpha)\mathrm{d}\alpha \ . \tag{2}$$

The left hand side of (2) is $\int_0^u 1\mathrm{d}\alpha = u$. (2) decomposes u into layers $(u > \alpha)\mathrm{d}\alpha$ across $0 \leq \alpha \leq 1$. The term layer is used since integrating $(u > \alpha)\mathrm{d}\alpha$ over a subset $[a, b]$ of the unit interval yields $\min\{\max(u - a, 0), b - a\}$ which corresponds to layers covered by excess-of-loss reinsurance (Wang, 1995) and debt tranches (Mandel et al., 2012). The infinitesimally small layer $(u > \alpha)\mathrm{d}\alpha$ captures movements in u at α since it jumps when u crosses α and is constant over other values of u.

Removing the integrals in the final expression of (1) yields the α–layer dependence between v and u:

$$\ell_\alpha \equiv \frac{\mathrm{cov}\{v, (u > \alpha)\}}{\mathrm{cov}\{u, (u > \alpha)\}} = \frac{\mathrm{cor}\{v, (u > \alpha)\}}{\mathrm{cor}\{u, (u > \alpha)\}} \ , \quad 0 \leq \alpha \leq 1 \ , \tag{3}$$

Denominators in (3) are independent of C and imply $\ell_\alpha = 1$ if $u = v$ and $\ell_\alpha = -1$ if $u = 1 - v$. Further $-1 \leq \ell_\alpha \leq 1$. Independence implies $\ell_\alpha = 0$. These and other properties of ℓ_α are explored in sections 5 and 7. Since $(u > \alpha)$ captures movements in u at α, ℓ_α generalises Spearman's ρ by measuring local dependence between v and u at α. This interpretation is further discussed below and in subsequent sections.

The local dependence interpretation of ℓ_α can be alternatively shown by expanding the covariances in (3) and manipulating to yield

$$\ell_\alpha = \frac{\mathrm{E}(v|u > \alpha) - \mathrm{E}(v|u \leq \alpha)}{\mathrm{E}(u|u > \alpha) - \mathrm{E}(u|u \leq \alpha)} = 2\left\{\mathrm{E}(v|u > \alpha) - \mathrm{E}(v|u \leq \alpha)\right\} \ , \tag{4}$$

where E calculates expectations using C. The middle expression in (4) is the expected change in v relative to the expected change in u when u crosses α.

The latter is 0.5 for all α, yielding the final expression in (4). Hence large ℓ_α implies v is sensitive to movements in u across α, indicating strong dependence between v and u at α. When $\ell_\alpha = 0$, v is unchanged on average when u crosses α, and u and v are independent at α.

Rewriting Spearman's ρ in (1) in terms of ℓ_α yields

$$\rho = \frac{\int_0^1 \mathrm{cov}\{u, (u > \alpha)\}\ell_\alpha \mathrm{d}\alpha}{\int_0^1 \mathrm{cov}\{u, (u > \alpha)\}\mathrm{d}\alpha} = \int_0^1 w_\alpha \ell_\alpha \mathrm{d}\alpha \ , \quad w_\alpha \equiv 6\alpha(1-\alpha) \ . \tag{5}$$

Hence Spearman's ρ is a weighted average of ℓ_α across $0 \le \alpha \le 1$. The weights are w_α and integrate to 1. Therefore whilst Spearman's ρ measures overall dependence between u and v, ℓ_α performs the same but locally and averaging values of ℓ_α yields ρ. Further note w_α has minimum 0 at $\alpha = 0$ and 1, and increases symmetrically to maximum at $\alpha = 0.5$. This implies Spearman's ρ places less emphasis on tail dependence which may be undesirable in finance or insurance where tail dependence is present. Modifying the weights leads to alternate measures of overall dependence. This is further explored in section 8.

3. Illustration of layer dependence curves for various copulas

The nine panels in Figure 1 display (u, v) scatterplots of exchangeable copulas $C(u, v) = C(v, u)$, and their layer dependence curves ℓ_α for all $0 \le \alpha \le 1$. ℓ_α and α are plotted on the same axes as v and u, respectively. Each copula has Spearman's $\rho = 0.6$. The intention of Figure 1 is to illustrate that copulas with the same overall dependence exhibit different local dependence structures which are captured by layer dependence.

Each ℓ_α curve reflects the dependence structure between u and v. Given α, ℓ_α is larger if points are more clustered around (α, α) and vice versa, as formalised in §4. In addition, ℓ_α increases to 1 in the tails if points converge to the 45° degree line indicating perfect dependence.

4. Layer dependence, discordance and dispersion

If (u, v) is exchangeable then layer dependence ℓ_α measures the lack of discordance and dispersion at α:

$$\ell_\alpha = 1 - 2(1 + \gamma_\alpha)\delta_\alpha \ , \tag{6}$$

where

$$\gamma_\alpha \equiv \mathrm{cor}\{(u \le \alpha), (v > \alpha)\} = \mathrm{cor}\{(u > \alpha), (v \le \alpha)\} \ ,$$
$$\delta_\alpha \equiv \mathrm{E}\left\{(|u - v|)|(u - \alpha)(v - \alpha) < 0\right\} \ .$$

A proof of (6) is below. The correlation $-1 \le \gamma_\alpha \le 1$ measures the tendency for (u, v) to be discordant at α: opposite signs on $u - \alpha$ and $v - \alpha$. The expectation $0 \le \delta_\alpha \le 1$ measures the dispersion between discordant points u and v at α.

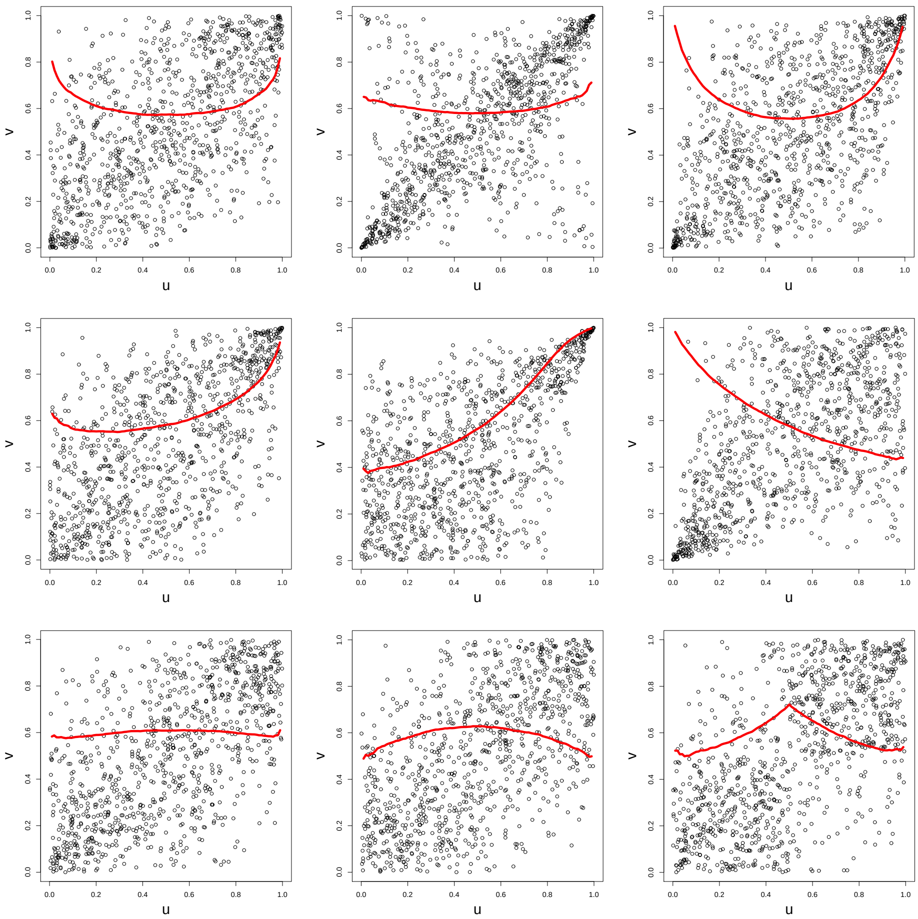

Figure 1: Copulas with the same $\rho_S = 0.6$ but different layer dependence curves ℓ_α over $0 \leq \alpha \leq 1$ (red curves) .

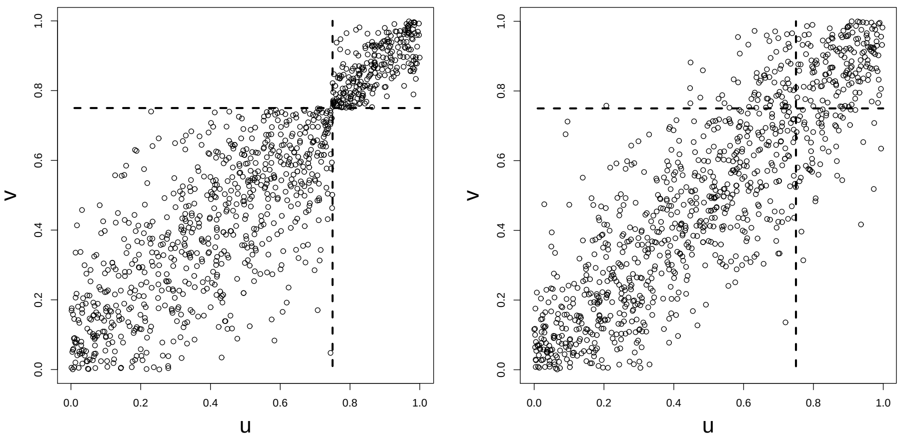

Figure 2: The left and right panel show $\ell_{0.75} = 1$ and $\ell_{0.75} = 0.86$, respectively. In the left panel, $\gamma_{0.75} = -1$ and $\delta_{0.75} = 0$. In the right panel, $\gamma_{0.75} = -0.65$ and $\delta_{0.75} = 0.21$.

The proof of (6) follows from

$$\gamma_\alpha = -\frac{\mathrm{cov}\{(u \le \alpha), (v \le \alpha)\}}{\mathrm{var}\{(u \le \alpha)\}} = \frac{\alpha^2 - C(\alpha,\alpha)}{\alpha(1-\alpha)} \ ,$$

and

$$\begin{aligned} \delta_\alpha &= 2\mathrm{E}\left\{(u-v)(u>v)|(u-\alpha)(v-\alpha)<0\right\} \\ &= \frac{2\mathrm{E}\{(u-v)(u>v)(u>\alpha)(v\le\alpha)\}}{2\mathrm{E}\{(u>\alpha)(v\le\alpha)\}} = \frac{\mathrm{E}\{(u-v)(u>\alpha)(v\le\alpha)\}}{\alpha - C(\alpha,\alpha)} \\ &= \frac{\mathrm{E}\{(u-v)(u>\alpha)\} - \mathrm{E}\{(u-v)(u>\alpha)(v>\alpha)\}}{\alpha - C(\alpha,\alpha)} = \frac{\mathrm{E}\{(u-v)(u>\alpha)\}}{\alpha - C(\alpha,\alpha)} \ . \end{aligned}$$

Substituting the above expressions for γ_α and δ_α into the right hand side of (6) yields the expression for ℓ_α in (4), completing the proof.

Result (6) explains the behaviour of layer dependence curves in Figure 1. Layer dependence ℓ_α is larger if there are fewer discordant pairs at α, and discordant pairs at α are closer to the 45° degree line. The former indicates smaller γ_α and the latter indicates smaller δ_α. Opposite observations apply for small ℓ_α. If $\ell_\alpha = 1$ then $\gamma_\alpha = -1$ or $\delta_\alpha = 0$, implying u and v are simultaneously below or above α and $u = v$ for discordant pairs. If $\ell_\alpha = 1$ in an interval, then $u = v$ in the same interval.

Figure 2 illustrates the relationship between layer dependence, discordance and dispersion in (6) using two copulas with $\ell_\alpha = 1$ and $\ell_\alpha = 0.86$ at $\alpha = 0.75$. When $\ell_\alpha = 1$, there is no discordance or dispersion at α. As ℓ_α decreases from 1, the number of discordant pairs and their dispersion increases.

5. Coherence properties of layer dependence

Layer dependence ℓ_α satisfies five "coherence" properties. These properties are extensions of properties applying to Spearman's ρ.

- **Bounds**: Layer dependence lies between -1 and 1: $-1 \leq \ell_\alpha \leq 1$ for all α. Hence layer dependence is bounded in the same way as ρ_S.
- **Perfect dependence**: Constant layer dependence of -1 or 1 are equivalent to countermonotonicity and comonotonicity. Thus $\ell_\alpha = -1$ for all α if and only if $v = 1 - u$ while $\ell_\alpha = 1$ for all α if and only if $v = u$.
- **Independence**: If u and v are independent then $\ell_\alpha \equiv 0$. The converse is not true – zero layer dependence does not imply independence as shown by the following counterexample. Assume $v = u$ and $v = 1 - u$ with equal probability. Then $\mathrm{E}(v|u = t) = 0.5$ for all $0 \leq t \leq 1$ implying $\mathrm{E}(v|u > \alpha) = \mathrm{E}(v|u \leq \alpha) = 0.5$. Hence $\ell_\alpha = 0$ from (4). However u and v are not independent.
- **Symmetry**: Ranking v in the opposite direction switches the sign of layer dependence. Doing the same to u (the random variable decomposed into layers) switches the sign of layer dependence and flips the layer dependence curve ℓ_α about $\alpha = 0.5$. Changing the ranking order of both u and v only flips the layer dependence curve.
- **Ordering**: Higher correlation order (Dhaene et al., 2009) leads to higher layer dependence. Consider bivariate uniform (u^*, v^*) exceeding (u, v) in correlation order: $C^*(a, b) \geq C(a, b)$ for all $0 \leq a, b \leq 1$, where C^* is the joint distribution of (u^*, v^*). Then $\ell^*_\alpha \geq \ell_\alpha$, $0 \leq \alpha \leq 1$ where ℓ^*_α denotes the α–layer dependence of (u^*, v^*). Hence greater dependence leads to higher layer dependence across all percentiles.

Independence, symmetry and ordering properties follow from the definition of layer dependence in (3). From (4), constant layer dependence of one implies $\mathrm{E}(v|u > \alpha) = (\alpha + 1)/2$ and $\mathrm{E}(v|u = \alpha) = \alpha$, for all $0 \leq \alpha \leq 1$, hence $v = u$. Similarly constant layer dependence of minus one implies $v = 1 - u$. The ordering property holds since higher correlation order implies larger covariances (Dhaene et al., 2009). Prove the bounds property by combining ordering and perfect dependence properties, and noting countermonotonicity and comonotonicity represent minimum and maximum correlation order, respectively. Detailed proofs of the coherence properties of layer dependence are discussed in the appendix in section **??**.

Most of the above layer dependence properties can be expressed using copulas. For the independence property, $C(u, v) = uv$ implies $\ell_\alpha = 0$ and for the perfect dependence property, $C(u, v) = \min(u, v)$ is equivalent to $\ell_\alpha = 1$ and $C(u, v) = \max(u+v-1, 0)$ is equivalent to $\ell_\alpha = -1$. For the symmetry property, the copula of $(u, 1-v)$ is $u - C(u, 1-v)$ and has layer dependence $-\ell_\alpha$. Similarly the copula of $(v, 1-u)$ is $v - C(1-u, v)$ and has layer dependence $-\ell_{1-\alpha}$. Lastly the copula of $(1-u, 1-v)$ is the survival copula $u+v-1+C(1-u, 1-v)$ of C and has layer dependence $\ell_{1-\alpha}$.

6. Connections to existing measures of tail dependence

Measures have been proposed to capture the degree of tail dependence. Tail dependence is dependence between extreme values of random variables, in this case values of u and v near 0 or 1. Strong tail dependence creates catastrophic events such as multiple bank failures and market crashes. Layer dependence is intimately connected to two existing tail dependence measures – coefficient of tail dependence and tail concentration function. Sweeting and Fotiou (2013) and Durante et al. (2014) further discusses tail dependence measures.

6.1. Tail dependence

Joe (1997) defines coefficients of lower and upper tail dependence in terms of the limiting tail probabilities

$$\lambda_L \equiv \lim_{\alpha \to 0} \mathrm{P}(v \le \alpha | u \le \alpha) \ , \quad \lambda_U \equiv \lim_{\alpha \to 1} \mathrm{P}(v > \alpha | u > \alpha) \ .$$

Unit coefficients indicate perfect positive tail dependence, and occur if and only if u and v converge simultaneously to 0 (lower tail) or 1 (upper tail). Coefficients of negative tail dependence replace $v \le \alpha$ and $v > \alpha$ in the above expressions with $v > 1-\alpha$ and $v \le 1-\alpha$, respectively. Sweeting and Fotiou (2013) discusses the drawback of these coefficients and suggests a modification by weakening the limits, yielding links to tail concentration functions discussed below.

The following shows $\lambda_L = 1$ is equivalent to $\ell_0 = 1$ and $\lambda_U = 1$ is equivalent to $\ell_1 = 1$. Similar links apply to negative tail dependence. Hence Joe (1997) and layer dependence characterise perfect tail dependence equivalently. From (4), $\ell_1 = 2\mathrm{E}(v|u=1) - 1$ implying $\ell_1 = 1$ if and only if $\mathrm{E}(v|u=1) = 1$. Hence $\ell_1 = 1$ if and only if $u = 1$ implies $v = 1$, which is equivalent to $\lambda_U = 1$. In addition $\ell_0 = 1$ if and only if $u = 0$ implies $v = 0$, which is equivalent to $\lambda_L = 1$. Similar proofs apply to perfect negative tail dependence.

6.2. Tail concentration function

Tail concentration (Venter, 2002) is a local dependence measure formed by combining lower and upper conditional tail probabilities at α:

$$\tau_\alpha \equiv (\alpha \le 0.5)\mathrm{P}(v \le \alpha | u \le \alpha) + (\alpha > 0.5)\mathrm{P}(v > \alpha | u > \alpha) \ .$$

Higher tail concentration τ_α implies u and v are more likely to fall in the same lower tail ($\alpha \le 0.5$) or upper tail ($\alpha > 0.5$). In addition taking the limits $\alpha \to 0$

or $\alpha \to 1$ yields coefficients of tail dependence λ_L and λ_U discussed above. Properties of tail concentration and its applications to distinguish families of copulas are further discussed in Durante et al. (2014).

To display the connection of layer dependence to tail concentration, rewrite the latter in terms of the copula C of (u, v):

$$\tau_\alpha = (\alpha \leq 0.5)\frac{C(\alpha,\alpha)}{\alpha} + (\alpha > 0.5)\frac{1-2\alpha+C(\alpha,\alpha)}{1-\alpha} \ .$$

Standardising τ_α by subtracting the value under independence and dividing by the departure from independence under comonotonicity yields

$$\tau_\alpha^* \equiv \frac{\tau_\alpha - \{\alpha(\alpha \leq 0.5) + (1-\alpha)(\alpha > 0.5)\}}{1 - \{\alpha(\alpha \leq 0.5) + (1-\alpha)(\alpha > 0.5)\}} = \frac{C(\alpha,\alpha) - \alpha^2}{\alpha(1-\alpha)} = -\gamma_\alpha \ ,$$

where γ_α is defined below (6).

Hence, using (6), $\ell_\alpha = 1 - 2\delta_\alpha(1 - \tau_\alpha^*)$ where δ_α is the average dispersion between points (u, v) discordant at α. Note ℓ_α and τ_α^* have the same sign. Hence layer dependence refines tail concentration by standardising it and including further information on dispersion.

7. Further properties of layer dependence

This section lists and explores further properties and results of layer dependence.

7.1. Copula integration

Note $\text{cov}\{(u > \alpha), (v > \beta)\} = C(\alpha, \beta) - \alpha\beta$ and apply (2) to v, to derive

$$\ell_\alpha = \frac{\int_0^1 \text{cov}\{(u>\alpha),(v>\beta)\}\text{d}\beta}{\alpha(1-\alpha)/2} = \frac{2\int_0^1 C(\alpha,\beta)\text{d}\beta - \alpha}{\alpha(1-\alpha)} \ . \tag{7}$$

Thus ℓ_α integrates copulas to reduce their dimension from two and one, and scales the result to ensure it lies between ± 1.

With Archimedean copulas (McNeil et al., 2005) $C(\alpha, \beta) = \psi^- \{\psi(\alpha) + \psi(\beta)\}$ where ψ is the generator function and ψ^- its inverse. In this case closed form expressions for the integrals and hence for ℓ_α do not exist.

7.2. Layer dependence preserves convex combination

A direct consequence of the copula integration result (7) is layer dependence preserves convex combinations of random variables, as follows.

Suppose (u^*, v^*) is bivariate uniform with α–layer dependence ℓ_α^* and copula C^*. Then a convex combination of (u, v) and (u^*, v^*) yielding the copula $\pi C + (1-\pi)C^*$ where $0 \leq \pi \leq 1$ is constant has layer dependence $\pi\ell_\alpha + (1-\pi)\ell_\alpha^*$. The proof follows directly from (7).

This result generalises to multiple and continuous convex combinations of bivariate uniform random variables and their copoulas.

7.3. One-sided conditional tail expectations

Since $\mathrm{E}(v) = \alpha\mathrm{E}(v|u \le \alpha) + (1-\alpha)\mathrm{E}(v|u > \alpha)$ it is straightforward to show from (4) that

$$\ell_\alpha = \frac{\mathrm{E}(v|u > \alpha) - \mathrm{E}(v)}{\mathrm{E}(u|u > \alpha) - \mathrm{E}(u)} = \frac{\mathrm{E}(v|u \le \alpha) - \mathrm{E}(v)}{\mathrm{E}(u|u \le \alpha) - \mathrm{E}(u)} \ ,$$

the gap between upper or lower conditional tail expectations of v and the unconditional expectation. Denominators are again scaling factors ensuring $\ell_\alpha = 1$ if u and v are comonotonic and $\ell_\alpha = -1$ if countermonotonic.

7.4. Layer dependence does not uniquely characterise a copula

Layer dependence curves extract and summarise dependence information in a copula. Therefore layer dependence curves do not uniquely specify the copula, as shown in the following counterexample.

Suppose $v = u$ and $v = 1-u$ with equal probability. The former and latter imply layer dependence of $= 1$ and -1, respectively. Hence $\ell_\alpha = 0$ for all α since layer dependence preserves convex combinations as discussed above. The copula of (u, v) is $C(u, v) = 0.5\{\min(u, v) + \max(u + v - 1, 0)\}$. However $\ell_\alpha = 0$ is also the case if u and v are independent: $C(u, v) = uv$. Hence layer dependence curves are not unique to the copula.

Non-uniqueness is seen from another perspective using (4): ℓ_α only captures first-order conditional tail expectations. Hence two copulas with equal first-order conditional tail expectations have equal layer dependence.

7.5. Layer dependence for a non-exchangeable copula

If u and v are exchangeable, such as in Archimedean copulas (McNeil et al., 2005), then layer dependence is invariant when u and v are switched:

$$\frac{\mathrm{cov}\{v, (u > \alpha)\}}{\mathrm{cov}\{u, (u > \alpha)\}} = \frac{\mathrm{cov}\{u, (v > \alpha)\}}{\mathrm{cov}\{v, (v > \alpha)\}} \ , \quad 0 \le \alpha \le 1 \ .$$

Hence it does not matter whether u or v is decomposed into layers.

If u and v are not exchangeable then layer dependence differs when layers of v instead of u are applied, that is dependence between v and α–layer of u differs from dependence between u and α–layer of v.

An analogous property applies to least squares regression: regressing v on u is the same as regressing u on v, if the joint distribution is exchangeable. The two regressions differ otherwise.

8. Alternate measures of overall dependence

Equation (5) expresses Spearman's ρ as a weighted average of ℓ_α over $0 \leq \alpha \leq 1$ using weights $w_\alpha = 6\alpha(1-\alpha)$ integrating to 1. As mentioned below (5), w_α may be inappropriate since it dismisses tail dependence. Averaging ℓ_α using different weights w^*_α yields alternate measures of overall rank dependence:

$$\rho^* \equiv \int_0^1 w^*_\alpha \ell_\alpha \mathrm{d}\alpha = \int_0^1 w^*_\alpha \frac{\mathrm{cov}\{v,(u>\alpha)\}}{\mathrm{cov}\{u,(u>\alpha)\}} \mathrm{d}\alpha = \mathrm{cov}\{W^*(u), v\} \ ,$$

where

$$W^*(u) \equiv 2\int_0^u \frac{w^*_\alpha}{\alpha(1-\alpha)} \mathrm{d}\alpha \ ,$$

is the weighted cumulative of w^*_α. Hence the alternate ρ^* is also a covariance, between v and a transformation of u. For Spearman's ρ, $w^*_\alpha = 6\alpha(1-\alpha)$ thus $W^*(u) = 12u$ yielding $\rho^* = \mathrm{cov}(12u, v) = \mathrm{cor}(u,v) = \rho$. Weights w^*_α indicate the importance of local dependence at various layers of u.

Since ρ^* averages ℓ_α, coherence properties of layer dependence described in section 5 apply to ρ^* for any $0 \leq w^*_\alpha \leq 1$ integrating to one. These properties are $-1 \leq \rho^* \leq 1$, $\rho^* = -1$, 0 and 1 under countermonotonicity, independence and comonotonicity, respectively, ρ^* switching its sign when u or v is replaced by its complement, and higher ρ^* when the correlation order of (u,v) increases. Identical properties apply to Spearman's ρ.

The following are examples of w^*_α yielding rank dependence measures alternate to Spearman's ρ:

- Suppose dependence at different percentiles are equally important. Then w^*_α is uniform yielding

  $$\rho^*_1 = 2\mathrm{cov}\left\{v, \log\left(\frac{u}{1-u}\right)\right\} = \sqrt{\frac{2}{3}}\ \mathrm{cor}\left\{v, \log\left(\frac{u}{1-u}\right)\right\} \ ,$$

  a multiple of the correlation between v and the logit of u. If tail dependence is pronounced, $\rho^*_1 > \rho$ since ρ^*_1 weights tail dependence more heavily compared to ρ. An illustration is shown below.

- If $w^*_\alpha = 3\alpha^2$ then dependence at higher percentiles are considered more important. This formulation is applicable when upper tail dependence is critical, for example the simultaneous occurrence of large insurance losses in different lines of business. Then

  $$\rho^*_2 = 6\mathrm{cov}\left\{v, \log\left(\frac{\mathrm{e}^{-u}}{1-u}\right)\right\} = \sqrt{3}\mathrm{cor}\left\{v, -\log(1-u)\right\} - \frac{\rho}{2} \ .$$

  If dependence is higher over percentiles above the median then $\rho^*_2 > \rho$.

- If dependence over percentiles below the median is more important then for example $w^*_\alpha = 3(1-\alpha)^2$ yielding

$$\rho^*_3 = 6\mathrm{cov}\left\{v, \log\left(u\mathrm{e}^{-u}\right)\right\} = \sqrt{3}\mathrm{cor}(v, \log u) - \frac{\rho}{2} \ .$$

- Suppose w^*_α is derived from V_α, the inverse marginal distribution of x with derivative V'_α:

$$w^*_\alpha = \frac{\mathrm{cov}\{u, (u > \alpha)\}V'_\alpha}{\int_0^1 \mathrm{cov}\{u, (u > \alpha)\}V'_\alpha \mathrm{d}\alpha} = \frac{\alpha(1-\alpha)V'_\alpha}{\mathrm{cov}(V_u, u)} = \frac{\alpha(1-\alpha)V'_\alpha}{\mathrm{cov}(x, u)} \ ,$$

  where $x = V_u$. This yields the Gini correlation (Schechtman and Yitzhaki, 1999)

$$\rho^*_3 = \frac{\mathrm{cov}(V_u, v)}{\mathrm{cov}(V_u, u)} = \frac{\mathrm{cov}\{x, G(y)\}}{\mathrm{cov}\{x, F(x)\}} \ ,$$

  where $F \equiv V^-$ and G are distribution functions of x and y, respectively. In this example the weights w^*_α depend on the marginal distribution of x. More skewness in x leads to more steeply increasing w^*_α hence greater emphasis on upper tail dependence.

8.1. Illustration of alternates to Spearman's ρ

Figure 3 illustrates the first three proposed alternates to Spearman's ρ listed above. Figure 3 reuses the nine copulas in Figure 1 with equal Spearman's $\rho = 0.6$. Spearman's ρ summarises lower and upper dependence symmetrically and dismisses tail dependence. These drawbacks are addressed by the alternate measures.

For example xxx...

9. Discussion of copula fitting using layer dependence

This section proposes fitting a copula to data using layer dependence. The approach can incorporate expert opinion on the dependence structure. The approach is illustrated using historical NASDAQ and FTSE stock returns.

9.1. Fitting steps

A copula fitting procedure based on layer dependence is as follows:

1. Calculate layer dependence curve with desired granularity from percentile rank data.
2. Smooth calculated layer dependence curve either parametrically or semi-parametrically.
3. Refine layer dependence using expert knowledge, such as existence and level of tail dependence, and overall dependence.

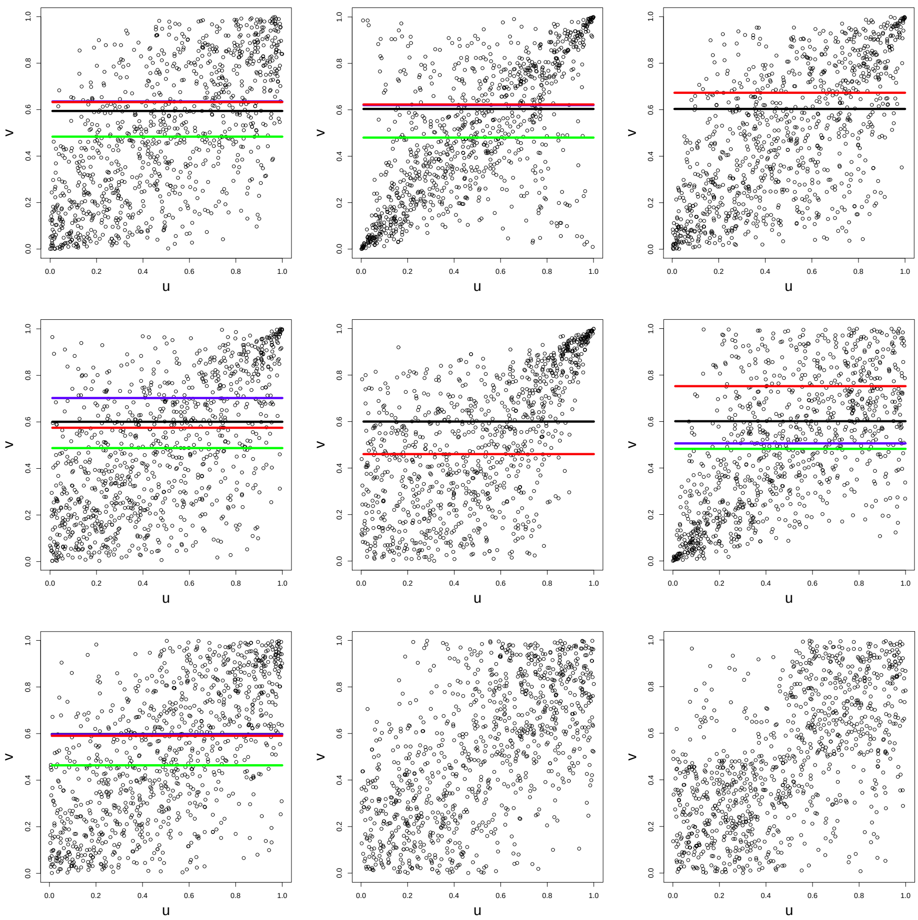

Figure 3: Copulas from Figure 1 with equal Spearman's $\rho = 0.6$ (black line) and different ρ_1^* (green), ρ_2^* (blue) and ρ_3^* (red).

4. Fit a copula given the fitted layer dependence curve, for example using the factor copula model described in the next subsection.

The layer dependence approach offers two advantages over fitting parametric copulas. Firstly layer dependence reflects the dependence structure exhibited in past data, whereas parametric copulas, with their relatively restricted dependence structures, may not properly fit past data. Secondly, layer dependence accommodates expert knowledge of local dependence, whereas parametric copulas permit limited changes to the dependence structure once a parametric form is selected.

In comparison to empirical copulas, layer dependence is more robust and less affected by data inadequacies. Layer dependence summarises past data into linear functions of conditional tail means, with a parametric curve fitted to calculated values. Hence layer dependence captures advantages of parametric and empirical copulas – the fitted copula utilises a smooth layer dependence curve, and the dependence structure underlying the fitted copula mirrors past data.

10. Conclusion

Layer dependence captures dependence structures in bivariate copulas, and satisfies coherence properties. Taking weighted averages of layer dependence curves yields Spearman's correlation and alternate overall dependence measures.

Using layer dependence in copula fitting captures dependence structures in past data, whilst flexibly accommodating expert opinion. Layer dependence achieves a balance between parametric approaches (smooth fit, low flexibility) and empirical approaches (volatile fit, high flexibility).

11. Appendix

11.1. Proof of equation (4)

Expanding the definition of layer dependence yields

$$\ell_\alpha = \frac{\operatorname{cov}\{v, (u > \alpha)\}}{\operatorname{cov}\{u, (u > \alpha)\}} = ...$$

11.2. Proof of coherence properties in section 5

Independence property

If u and v are independent, then layer dependence

$$\ell_\alpha = \frac{\operatorname{cov}\{v, (u > \alpha)\}}{\operatorname{cov}\{u, (u > \alpha)\}} = 0 \ , \quad 0 \le \alpha \le 1$$

since the numerator is zero.

Perfect dependence property

If u and v are comonotonic or $v = u$, then layer dependence

$$\ell_\alpha = \frac{\operatorname{cov}\{v, (u > \alpha)\}}{\operatorname{cov}\{u, (u > \alpha)\}} = \frac{\operatorname{cov}\{u, (u > \alpha)\}}{\operatorname{cov}\{u, (u > \alpha)\}} = 1 \ , \quad 0 \le \alpha \le 1 \ .$$

In addition $\ell_\alpha = 1$ implies from (4)

Symmetry

Replacing v with $1 - v$ yields layer dependence

$$\frac{\operatorname{cov}\{1 - v, (u > \alpha)\}}{\operatorname{cov}\{u, (u > \alpha)\}} = -\frac{\operatorname{cov}\{v, (u > \alpha)\}}{\operatorname{cov}\{u, (u > \alpha)\}} = -\ell_\alpha \ ,$$

whilst replacing u with $1 - u$ yields

$$\frac{\operatorname{cov}\{v, (1 - u > \alpha)\}}{\operatorname{cov}\{1 - u, (1 - u > \alpha)\}} = -\frac{\operatorname{cov}\{v, (u < 1 - \alpha)\}}{\operatorname{cov}\{u, (u < 1 - \alpha)\}} = -\frac{\operatorname{cov}\{v, 1 - (u > 1 - \alpha)\}}{\operatorname{cov}\{u, 1 - (u > 1 - \alpha)\}}$$

$$= -\frac{\operatorname{cov}\{v, (u > 1 - \alpha)\}}{\operatorname{cov}\{u, (u > 1 - \alpha)\}} = -\ell_{1-\alpha} \ .$$

Using a similar proof, replacing v and u with $1-v$ and $1-u$, respectively, yields layer dependence $\ell_{1-\alpha}$.

Correlation order

If (u_*, v_*) exceeds (u, v) in correlation order, then

$$\operatorname{cov}\{f(u_*), g(v_*)\} \ge \operatorname{cov}\{f(u), g(v)\}$$

for any non-decreasing functions f and g. Hence $\operatorname{cov}\{v_*, (u_* > \alpha)\} \ge \operatorname{cov}\{v, (u > \alpha)\}$ implying (u_*, v_*) has higher layer dependence than (u, v).

Bounds

Since layer dependence preserves correlation order, $\ell_\alpha \leq 1$ since $\ell_\alpha = 1$ if u and v are comonotonic and comonotonicity represents maximum correlation order. Similarly $\ell_\alpha \geq -1$ noting countermonotonicity represents minimum correlation order.

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