Randomization

Algorithmic design patterns.

- Greed.
- Divide-and-conquer.
- Dynamic programming.
- Network flow.
- Randomization.

Often probability-based proofs
Large amt of data increases stability of proposition

in practice, access to a pseudo-random number generator

Randomization. Allow fair coin flip in unit time.

Why randomize? Can lead to simpler, faster, or only known algorithm for a particular problem.

Ex. Symmetry breaking protocols, graph algorithms, quicksort, hashing, load balancing, Monte Carlo integration, cryptography.

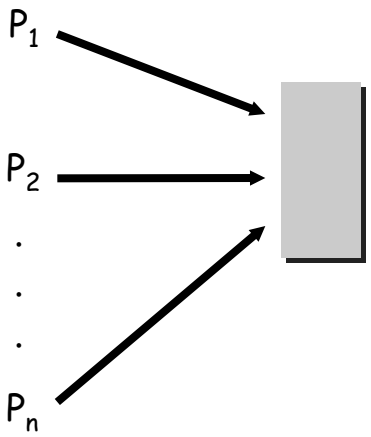
13.1 Contention Resolution

Contention Resolution in a Distributed System

Contention resolution. Given n processes P_1, …, P_n, each competing for access to a shared database. If two or more processes access the database simultaneously, all processes are locked out. Devise protocol to ensure all processes get through on a regular basis.

Restriction. Processes can't communicate.

Challenge. Need symmetry-breaking paradigm.



Contention Resolution: Randomized Protocol

Protocol. Each process requests access to the database at time t with probability p = 1/n.

Claim. Let S[i, t] = event that process i succeeds in accessing the database at time t. Then $1/(e \cdot n) \le \Pr[S(i, t)] \le 1/(2n)$.

Pf. By independence, $\Pr[S(i, t)] = p\,(1-p)^{n-1}$.

process i requests access

none of remaining n-1 processes request access

- Setting p = 1/n, we have $\Pr[S(i, t)] = 1/n\,(1 - 1/n)^{n-1}$. ▪

value that maximizes Pr[S(i, t)]

between 1/e (limit $n \to \infty$) and 1/2 (n=2)

Useful facts from calculus. As n increases from 2, the function:

- $(1 - 1/n)^n$ converges monotonically from 1/4 up to 1/e
- $(1 - 1/n)^{n-1}$ converges monotonically from 1/2 down to 1/e.

Contention Resolution: Randomized Protocol

Claim. The probability that process i fails to access the database in **e·n** rounds is at most **1/e**. After **e·n(c ln n)** rounds, the probability is at most n^{-c}.

Pf. Let F[i, t] = event that process i fails to access database in rounds 1 through t. By independence and previous claim, we have $\Pr[F(i, t)] \leq (1 - 1/(en))^t$.

- Choose $t = \lceil e \cdot n \rceil$: $\Pr[F(i,t)] \leq \left(1 - \frac{1}{en}\right)^{\lceil en \rceil} \leq \left(1 - \frac{1}{en}\right)^{en} \leq \frac{1}{e}$
- Choose $t = \lceil e \cdot n \rceil \lceil c \ln n \rceil$: $\Pr[F(i,t)] \leq \left(\frac{1}{e}\right)^{c \ln n} = n^{-c}$

Contention Resolution: Randomized Protocol

Claim. The probability that all processes succeed within **2e · n ln n** rounds is at least **1 - 1/n**.

Pf. Let F[t] = event that at least one of the n processes fails to access database in any of the rounds 1 through t.

$$\Pr[F[t]] = \Pr\left[\bigcup_{i=1}^{n} F[i,t]\right] \le \sum_{i=1}^{n} \Pr[F[i,t]] \le n\left(1-\frac{1}{en}\right)^{t}$$

union bound — previous slide

- Choosing $t = \lceil en \rceil \lceil 2 \ln n \rceil$ yields $\Pr[F[t]] \le n \cdot n^{-2} = 1/n$. ▪

Union bound. Given events $E_1, \ldots, E_n$, $\Pr\left[\bigcup_{i=1}^{n} E_i\right] \le \sum_{i=1}^{n} \Pr[E_i]$

13.2 Global Minimum Cut

Global Minimum Cut

Global min cut. Given a connected, undirected graph $G = (V, E)$ find cut (A, B) of minimum cardinality (= number of edges connecting A & B).

Applications. Partitioning items in a database, identify clusters of related documents, network reliability, network design, circuit design.

Network flow solution.

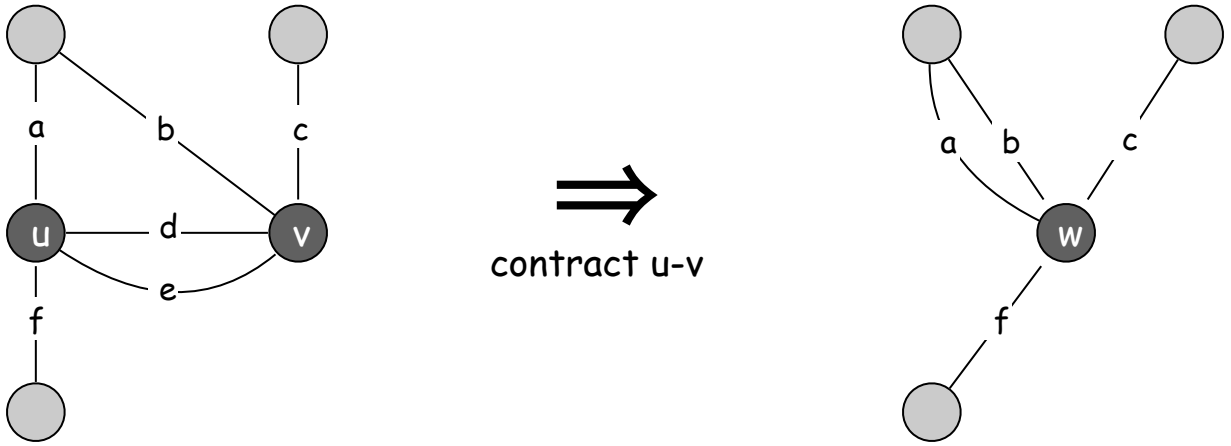
- Replace every edge (u, v) with two antiparallel edges (u, v) and (v, u).
- Pick some vertex s and compute min s-v cut separating s from each other vertex $v \in V$.

Resulting False intuition. Global min-cut is harder than min s-t cut.

Contraction Algorithm

Contraction algorithm. [Karger 1995]

- Pick an edge e = (u, v) uniformly at random.
- Contract edge e.
  - replace u and v by single new super-node w
  - preserve edges, updating endpoints of u and v to w
  - keep parallel edges, but delete self-loops
- Repeat until graph has just two nodes v_1 and v_2.
- Return the cut (all nodes that were contracted to form v_1).

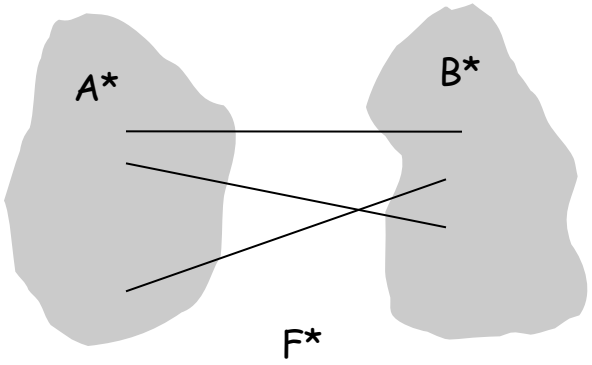


Contraction Algorithm

Claim. The contraction algorithm returns a min cut with **prob** $\geq$ **$2/n^2$**.

Pf. Consider a global min-cut (A^*, B^*) of G. Let F^* be edges with one endpoint in A^* and the other in B^*. Let $k = |F^*|$ = size of min cut.

- In first step, algorithm contracts an edge in F^* with probability $k / |E|$.
- Every node has degree $\geq k$ since otherwise (A^*, B^*) would not be min-cut. $\Rightarrow$ $|E| \geq \frac{1}{2}kn$.
- Thus, algorithm contracts an edge in F^* with probability $\leq 2/n$.



Contraction Algorithm

Claim. The contraction algorithm returns a min cut with **prob $\geq$ $2/n^2$**.

Pf. Consider a global min-cut (A*, B*) of G. Let F* be edges with one endpoint in A* and the other in B*. Let k = |F*| = size of min cut.

- Let G' be graph after j iterations. There are n' = n-j supernodes.
- Suppose no edge in F* has been contracted. The min-cut in G' is still k.
- Since value of min-cut is k, $|E'| \geq \frac{1}{2}kn'$.
- Thus, algorithm contracts an edge in F* with probability $\leq 2/n'$.

- Let E_j = event that an edge in F* is not contracted in iteration j.

$$
\begin{aligned}
\Pr[E_1 \wedge E_2 \cdots \wedge E_{n-2}] &= \Pr[E_1] \times \Pr[E_2 \mid E_1] \times \cdots \times \Pr[E_{n-2} \mid E_1 \wedge E_2 \cdots \wedge E_{n-3}] \\
&\geq \left(1-\tfrac{2}{n}\right)\left(1-\tfrac{2}{n-1}\right)\cdots\left(1-\tfrac{2}{4}\right)\left(1-\tfrac{2}{3}\right) \\
&= \left(\tfrac{n-2}{n}\right)\left(\tfrac{n-3}{n-1}\right)\cdots\left(\tfrac{2}{4}\right)\left(\tfrac{1}{3}\right) \\
&= \frac{2}{n(n-1)} \\
&\geq \frac{2}{n^2}
\end{aligned}
$$

Contraction Algorithm

Amplification. To amplify the probability of success, run the contraction algorithm many times.

Claim. If we repeat the contraction algorithm $\mathbf{n^2 \ln n}$ times with independent random choices, the probability of failing to find the global min-cut is at most $\mathbf{1/n^2}$.

Pf. By independence, the probability of failure is at most

$$\left(1 - \frac{2}{n^2}\right)^{n^2 \ln n} = \left(\left(1 - \frac{2}{n^2}\right)^{\frac{1}{2}n^2}\right)^{2\ln n} \leq \left(e^{-1}\right)^{2\ln n} = \frac{1}{n^2}$$

$(1 - 1/x)^x \leq 1/e$

Global Min Cut: Context

Remark. Overall running time is slow since we perform $\Theta(n^2 \log n)$ iterations and each takes $\Omega(m)$ time.

Improvement. [Karger-Stein 1996] $O(n^2 \log^3 n)$.

- Early iterations are less risky than later ones: probability of contracting an edge in min cut hits **50%** when $n/\sqrt{2}$ nodes remain.
- Run contraction algorithm until $n/\sqrt{2}$ nodes remain.
- Run contraction algorithm twice on resulting graph, and return best of two cuts.

Extensions. Naturally generalizes to handle positive weights.

Best known. [Karger 2000] $O(m \log^3 n)$.

faster than best known max flow algorithm or deterministic global min cut algorithm

13.3 Linearity of Expectation

Expectation

Expectation. Given a discrete random variables X, its expectation E[X] is defined by:

$$E[X] = \sum_{j=0}^{\infty} j \Pr[X = j]$$

Waiting for a first success. Coin is heads with probability p and tails with probability 1-p. How many independent flips X until first heads?

$$E[X] = \sum_{j=0}^{\infty} j \cdot \Pr[X = j] = \sum_{j=0}^{\infty} j (1-p)^{j-1} p = \frac{p}{1-p} \sum_{j=0}^{\infty} j (1-p)^{j} = \frac{p}{1-p} \cdot \frac{1-p}{p^2} = \frac{1}{p}$$

j-1 tails 1 head

Expectation: Two Properties

Useful property. If X is a 0/1 random variable, E[X] = Pr[X = 1].

Pf. $$E[X] = \sum_{j=0}^{\infty} j \cdot \Pr[X = j] = \sum_{j=0}^{1} j \cdot \Pr[X = j] = \Pr[X = 1]$$

not necessarily independent

Linearity of expectation. Given two random variables X and Y defined over the same probability space, E[X + Y] = E[X] + E[Y].

Decouples a complex calculation into simpler pieces.

Guessing Cards

Game. Shuffle a deck of n cards; turn them over one at a time; try to guess each card.

Memoryless guessing. No psychic abilities; can't even remember what's been turned over already. Guess a card from full deck uniformly at random.

Claim. The expected number of correct guesses is 1.

Pf. (surprisingly effortless using linearity of expectation)

- Let $X_i = 1$ if i^{th} prediction is correct and 0 otherwise.
- Let X = number of correct guesses = $X_1 + \ldots + X_n$.
- $E[X_i] = \Pr[X_i = 1] = 1/n$.
- $E[X] = E[X_1] + \ldots + E[X_n] = 1/n + \ldots + 1/n = 1$. ▪

linearity of expectation

Guessing Cards

Game. Shuffle a deck of n cards; turn them over one at a time; try to guess each card.

Guessing with memory. Guess a card uniformly at random from cards not yet seen.

Claim. The expected number of correct guesses is $\Theta(\log n)$.

Pf.

- Let $X_i = 1$ if i^{th} prediction is correct and 0 otherwise.
- Let X = number of correct guesses = $X_1 + \ldots + X_n$.
- $E[X_i] = \Pr[X_i = 1] = 1 / (n - i - 1)$.
- $E[X] = E[X_1] + \ldots + E[X_n] = 1/n + \ldots + 1/2 + 1/1 = H(n)$. ▪

linearity of expectation

$\ln(n+1) < H(n) < 1 + \ln n$

Coupon Collector

Coupon collector. Each box of cereal contains a coupon. There are n different types of coupons. Assuming all boxes are equally likely to contain each coupon, how many boxes before you have ≥ 1 coupon of each type?

Claim. The expected number of steps is Θ**(n log n)**.

Pf.

- Phase j = time between j and j+1 distinct coupons.
- Let X_j = number of steps you spend in phase j.
- Let X = number of steps in total = $X_0 + X_1 + \ldots + X_{n-1}$.

$$E[X] = \sum_{j=0}^{n-1} E[X_j] = \sum_{j=0}^{n-1} \frac{n}{n-j} = n \sum_{i=1}^{n} \frac{1}{i} = n H(n)$$

prob of success = (n-j)/n
⇒ expected waiting time = n/(n-j)

13.5 Randomized Divide-and-Conquer

Quicksort

Sorting. Given a set of n distinct elements S, rearrange them in ascending order.

```
RandomizedQuicksort(S) {
   if |S| = 0 return

   choose a splitter a_i ∈ S uniformly at random
   foreach (a ∈ S) {
      if      (a < a_i) put a in S^-
      else if (a > a_i) put a in S^+
   }
   RandomizedQuicksort(S^-)
   output a_i
   RandomizedQuicksort(S^+)
}
```

Remark. Can implement in-place.

↑

O(log n) extra space

Quicksort

Running time.

- [Best case.] Select the median element as the splitter: quicksort makes $\Theta(n \log n)$ comparisons.
- [Worst case.] Select the smallest element as the splitter: quicksort makes $\Theta(n^2)$ comparisons.

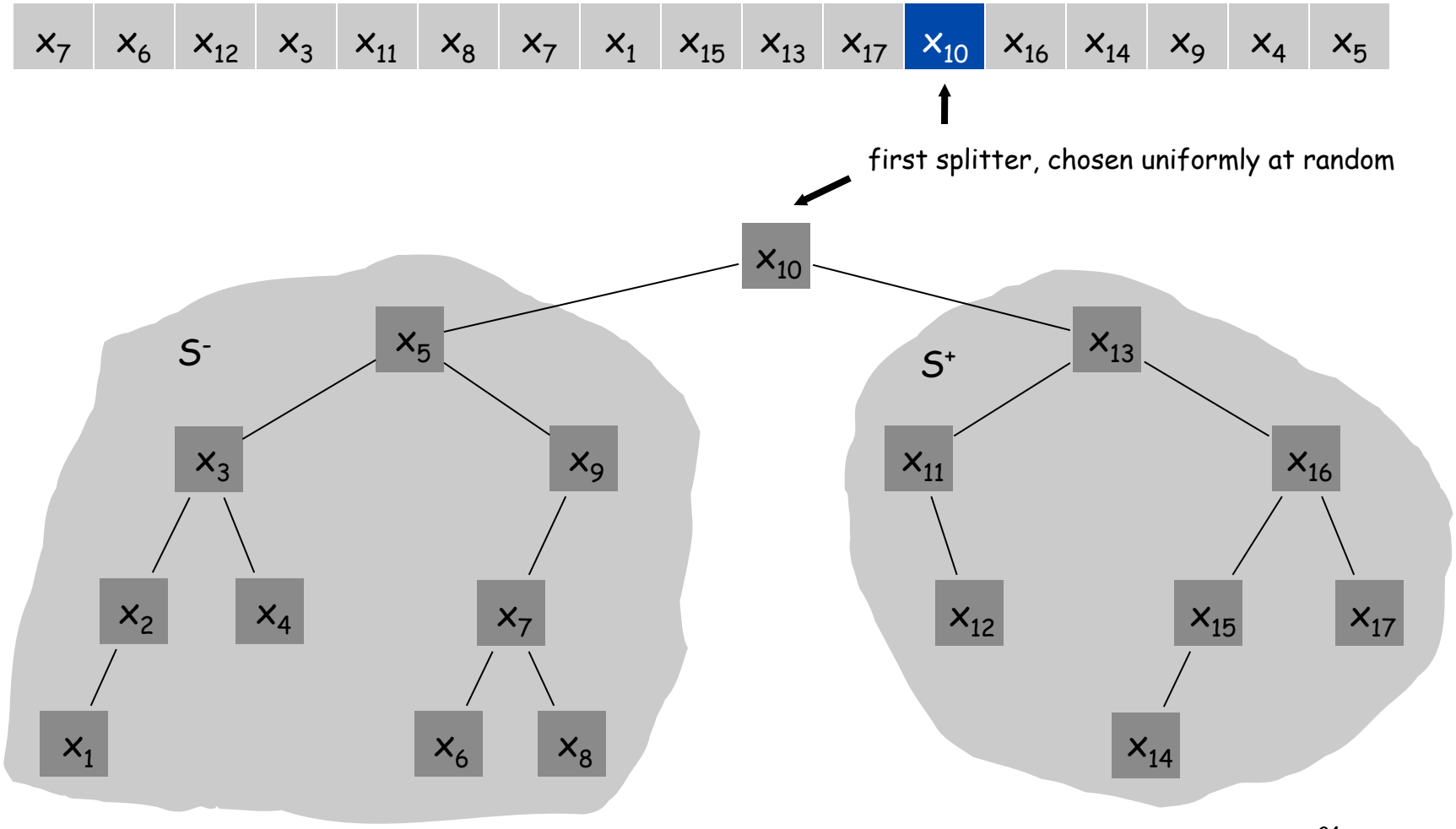
Randomize. Protect against worst case by choosing splitter at random.

Intuition. If we always select an element that is bigger than 25% of the elements and smaller than 25% of the elements, then quicksort makes $\Theta(n \log n)$ comparisons.

Notation. Label elements so that $x_1 < x_2 < \ldots < x_n$.

Quicksort: BST Representation of Splitters

BST representation. Draw recursive BST of splitters.



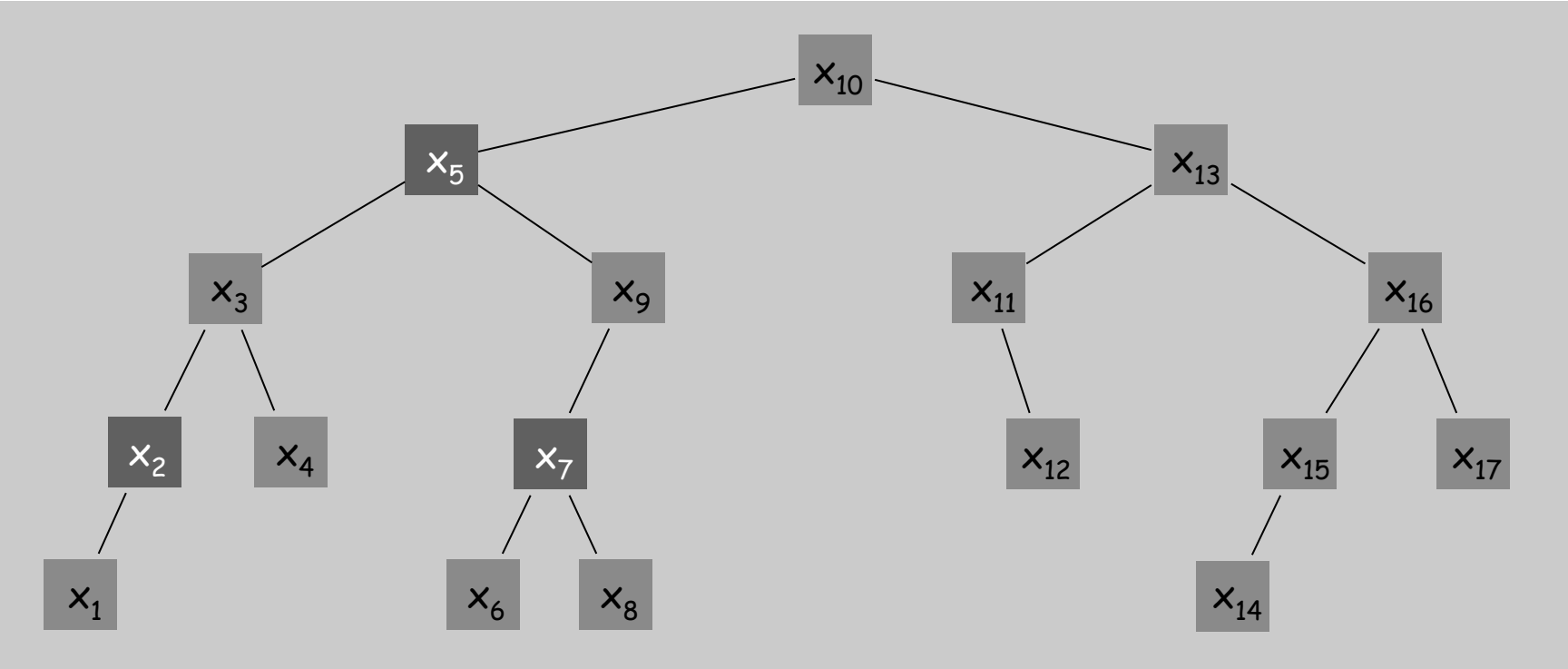
Quicksort: BST Representation of Splitters

Observation. Element only compared with its ancestors and descendants.

- x_2 and x_7 are compared if their lca = x_2 or x_7.
- x_2 and x_7 are not compared if their lca = x_3 or x_4 or x_5 or x_6.

Claim. Pr[x_i and x_j are compared] = 2 / (j - i + 1).

Let C_{ij} be the indicator Rand.Var. of the event "x_i and x_j are compared".



Quicksort: Expected Number of Comparisons

Theorem. Expected # of comparisons is $O(n \log n)$.

Pf.

Let C be the Rand.Var. of the # of comparisons.

$$E[C] = E[C_{12}] + E[C_{13}] + E[C_{23}] + \ldots + E[C_{n-2,n-1}] =$$

$$\sum_{1 \le i < j \le n} \frac{2}{j - i + 1} = 2 \sum_{i=1}^{n} \sum_{j=2}^{i} \frac{1}{j} \le 2n \sum_{j=1}^{n} \frac{1}{j} \le 2n \int_{x=1}^{n} \frac{1}{x} dx = 2n \ln n$$

probability that i and j are compared

Theorem. [Knuth 1973] Stddev of number of comparisons is ~ 0.65n.

Ex. If n = 1 million, the probability that randomized quicksort takes less than 4n ln n comparisons is at least 99.94%.

Chebyshev's inequality. $\Pr[|X - \mu| \ge k\delta] \le 1 / k^2$.

Quicksort: Expected Number of Comparisons

The expected number of comparisons in a randomized Quicksort of n elements is (γ is Euler's constant near 0.577) :

$$q_n = 2n \ln n - (4 - 2\gamma)n + 2 \ln n + O(1).$$

In 1996, McDiarmid and Hayward have formulated an exact expression for the probability that the number of comparisons $\boldsymbol{Q_n}$ be far from its average $\boldsymbol{q_n}$

$$\Pr\left[\left|\frac{Q_n}{q_n} - 1\right| > \varepsilon\right] = n^{-(2+o(1))\varepsilon \ln^{(2)} n}$$

Let c be a positive constant. McDiarmid and Hayward's formula imply that there exists another positive constant a smaller than 1 such that

$$\Pr[\, \boldsymbol{Q_n} \in \Theta(n^{1+c}) \,] < a^{n^c}.$$

13.6 Universal Hashing

Dictionary Data Type

Dictionary. Given a universe U of possible elements, maintain a subset $S \subseteq U$ so that inserting, deleting, and searching in S is efficient.

Dictionary interface.

- `Create()`: Initialize a dictionary with $S = \emptyset$.
- `Insert(u)`: Add element $u \in U$ to S.
- `Delete(u)`: Delete u from S, if u is currently in S.
- `Lookup(u)`: Determine whether u is in S.

Challenge. Universe U can be extremely large so defining an array of size |U| is infeasible.

Applications. File systems, databases, Google, compilers, checksums P2P networks, associative arrays, cryptography, web caching, etc.

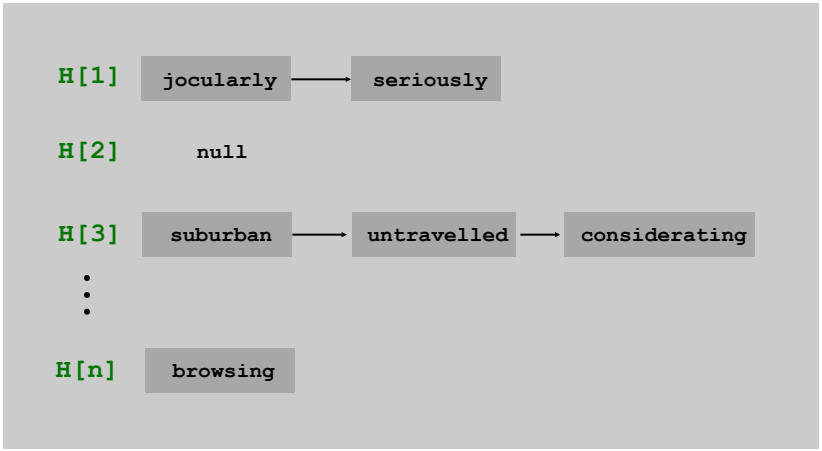
Hashing

Hash function. $h : U \rightarrow \{ 0, 1, \ldots, n-1 \}$.

Hashing. Create an array H of size n. When processing element u, access array element H[h(u)].

Collision. When $h(u) = h(v)$ but $u \neq v$.

- A collision is expected after $\Theta(\sqrt{n})$ random insertions. This phenomenon is known as the "birthday paradox."
- Separate chaining: H[i] stores linked list of elements u with h(u) = i.



Ad Hoc Hash Function

Ad hoc hash function.

```
int h(String s, int n) {
   int hash = 0;
   for (int i = 0; i < s.length(); i++)
      hash = (31 * hash % n) + s[i];
   return hash % n;
}
```

hash function à la Java string library

Deterministic hashing. If $|U| \geq n^2$, then for any fixed hash function h, there is a subset $S \subseteq U$ of n elements that all hash to same slot. Thus, $\Theta(n)$ time per search in worst-case.

Q. But isn't ad hoc hash function good enough in practice?

Algorithmic Complexity Attacks

When can't we live with ad hoc hash function?

- Obvious situations: aircraft control, nuclear reactors.
- Surprising situations: denial-of-service attacks.

malicious adversary learns your ad hoc hash function (e.g., by reading Java API) and causes a big pile-up in a single slot that grinds performance to a halt

Real world exploits. [Crosby-Wallach 2003]

- Bro server: send carefully chosen packets to D.O.S. the server, using less bandwidth than a dial-up modem
- Perl 5.8.0: insert carefully chosen strings into associative array.
- Linux 2.4.20 kernel: save files with carefully chosen names.

Hashing Performance

Idealistic hash function. Maps m elements uniformly at random to n hash slots.

- Running time depends on length of chains.
- Average length of chain = $\alpha = m / n$.
- Choose $n \approx m \Rightarrow$ on average $O(1)$ per insert, lookup, or delete.

Challenge. Achieve idealized randomized guarantees, but with a hash function where you can easily find items where you put them.

Approach. Use randomization in the choice of h.

adversary knows the randomized algorithm you're using, but doesn't know random choices that the algorithm makes

Universal Hashing

Universal class of hash functions. [Carter-Wegman 1980s]

- For any pair of elements u≠v ∈ U, $\Pr_{h \in H}[\, h(u) = h(v)\,] \le 1/n$
- Can select random h efficiently.
- Can compute h(u) efficiently.

chosen uniformly at random

Ex. U = { a, b, c, d, e, f }, n = 2.

	a	b	c	d	e	f
$h_1(x)$	0	1	0	1	0	1
$h_2(x)$	0	0	0	1	1	1

$H = \{h_1, h_2\}$

$\Pr_{h \in H}[h(a) = h(b)] = 1/2$

$\Pr_{h \in H}[h(a) = h(c)] = 1$

$\Pr_{h \in H}[h(a) = h(d)] = 0$

. . .

not universal

	a	b	c	d	e	f
$h_1(x)$	0	1	0	1	0	1
$h_2(x)$	0	0	0	1	1	1
$h_3(x)$	0	0	1	0	1	1
$h_4(x)$	1	0	0	1	1	0

$H = \{h_1, h_2, h_3, h_4\}$

$\Pr_{h \in H}[h(a) = h(b)] = 1/2$

$\Pr_{h \in H}[h(a) = h(c)] = 1/2$

$\Pr_{h \in H}[h(a) = h(d)] = 1/2$

$\Pr_{h \in H}[h(a) = h(e)] = 1/2$

$\Pr_{h \in H}[h(a) = h(f)] = 0$

. . .

universal

Universal Hashing

Universal hashing property. Let H be a universal class of hash functions; let $h \in H$ be chosen uniformly at random from H; and let $u \in U$. For any subset $S \subseteq U$ of size at most n, the expected number of items in S that collide with u is at most 1.

Pf. For any element $s \in S$, define indicator random variable $X_s = 1$ if h(s) = h(u) and 0 otherwise. Let X be a random variable counting the total number of collisions with u.

$$E_{h \in H}[X] = E[\sum_{s \in S} X_s] = \sum_{s \in S} E[X_s] = \sum_{s \in S} \Pr[X_s = 1] \leq \sum_{s \in S} \frac{1}{n} = |S| \frac{1}{n} \leq 1$$

linearity of expectation

X_s is a 0-1 random variable

universal (assumes $u \notin S$)

Designing a Universal Family of Hash Functions

Theorem. [Bertrand-Chebyshev (1845|1850)]
There exists a prime between n and 2n.

Modulus. Choose a prime number p ≈ n. ← no need for randomness here

Integer encoding. Identify each element u ∈ U with a base-p integer of r digits: $x = (x_1, x_2, \ldots, x_r)$.

Hash function. Let A = set of all r-digit, base-p integers. For each $a = (a_1, a_2, \ldots, a_r)$ where $0 \le a_i < p$, define

$$h_a(x) = \sum_{i=1}^{r} a_i x_i \mod p$$

Hash function family. $H = \{ h_a : a \in A \}$.

Designing a Universal Class of Hash Functions

Theorem. $H = \{ h_a : a \in A \}$ is a universal class of hash functions.

Pf. Let $x = (x_1, x_2, \ldots, x_r)$ and $y = (y_1, y_2, \ldots, y_r)$ be two distinct elements of U. We need to show that $\Pr[h_a(x) = h_a(y)] \le 1/n$.

- Since $x \neq y$, there exists an integer j such that $x_j \neq y_j$.
- We have $h_a(x) = h_a(y)$ iff

$$a_j \underbrace{(y_j - x_j)}_{z} = \underbrace{\sum_{i \neq j} a_i (x_i - y_i)}_{m} \bmod p$$

- Can assume a was chosen uniformly at random by first selecting all coordinates a_i where $i \neq j$, then selecting a_j at random. Thus, we can assume a_i is fixed for all coordinates $i \neq j$.
- Since p is prime, $a_j z = m \bmod p$ has at most one solution among p possibilities. ← see lemma on next slide
- Thus $\Pr[h_a(x) = h_a(y)] = 1/p \le 1/n$. ▪

Number Theory Facts

Fact. Let p be prime, and let $z \neq 0 \bmod p$. Then $\alpha z = m \bmod p$ has at most one solution $0 \leq \alpha < p$.

Pf.

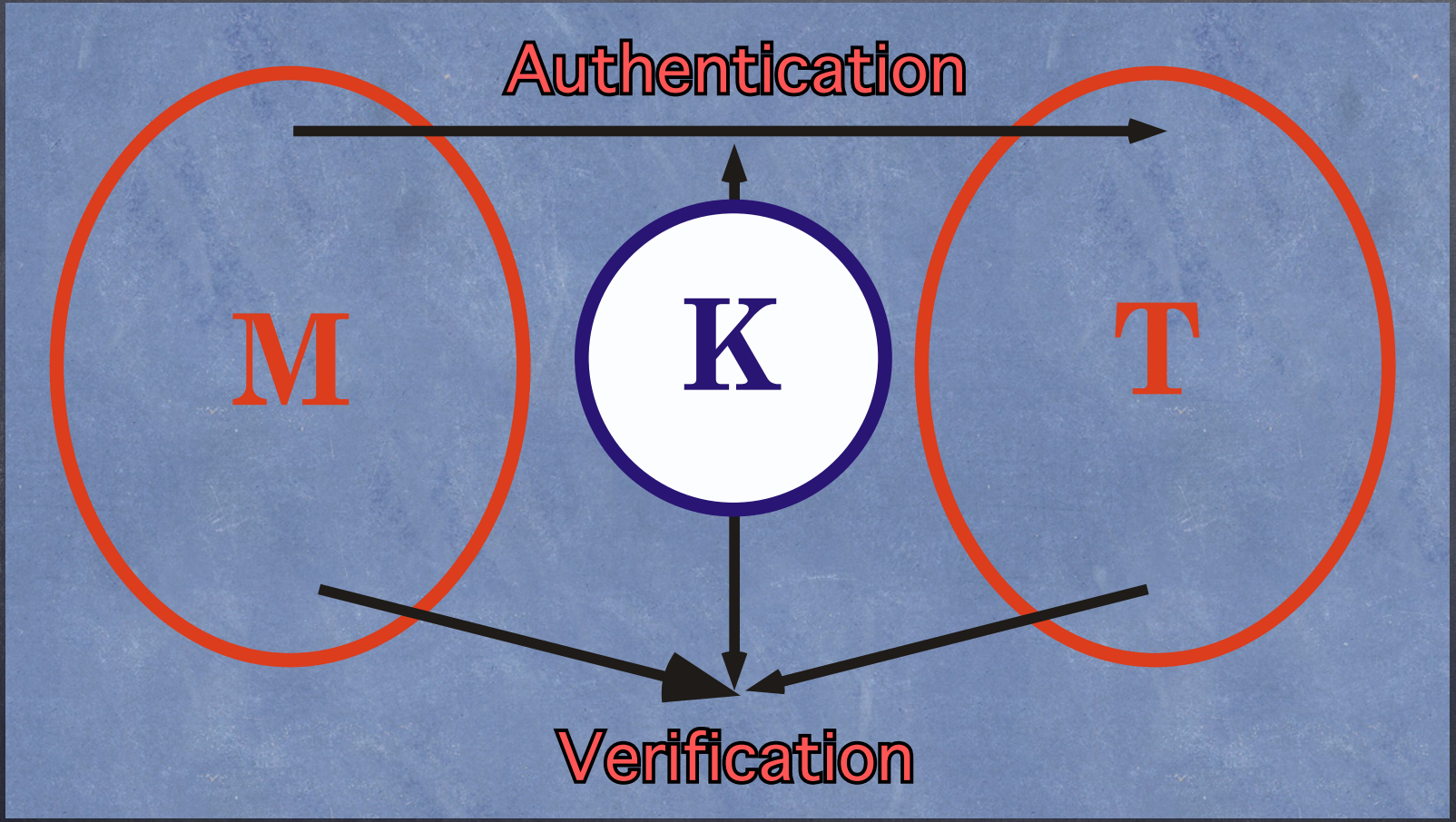
- Suppose α and β are two different solutions.
- Then $(\alpha - \beta)z = 0 \bmod p$; hence $(\alpha - \beta)z$ is divisible by p.
- Since $z \neq 0 \bmod p$, we know that z is not divisible by p; it follows that $(\alpha - \beta)$ is divisible by p.
- This implies $\alpha = \beta$. ▪

Bonus fact. Can replace "at most one" with "exactly one" in above fact.

Pf idea. Extended Euclid's algorithm.

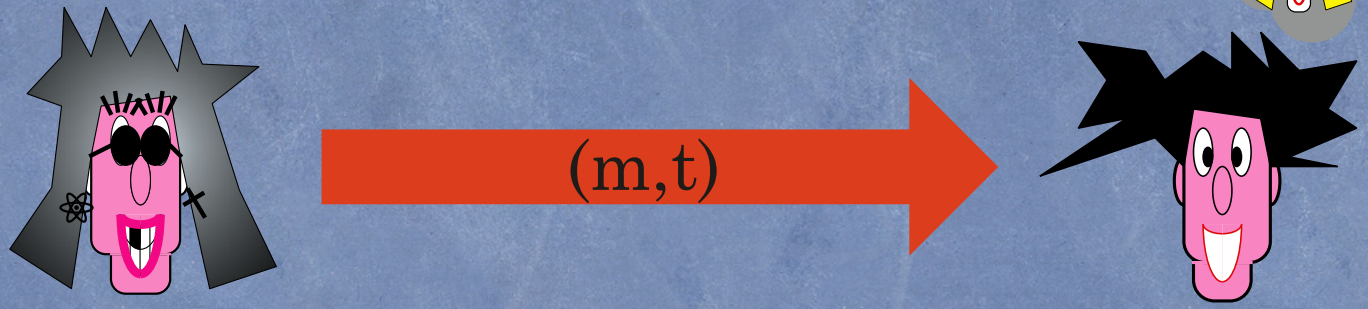
Authentication

Symmetric Authentication

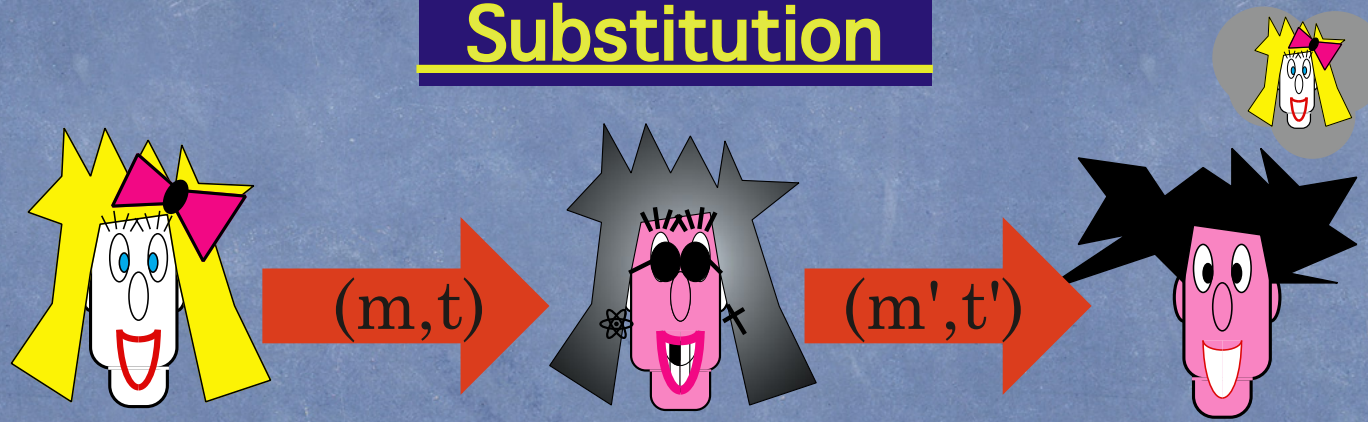


Information Theoretical Security

Impersonation

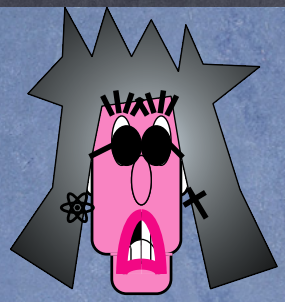


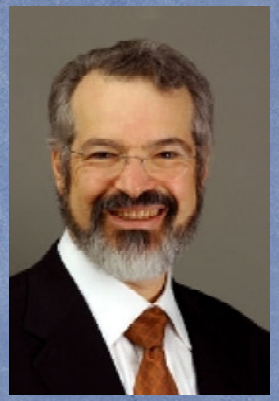
Substitution

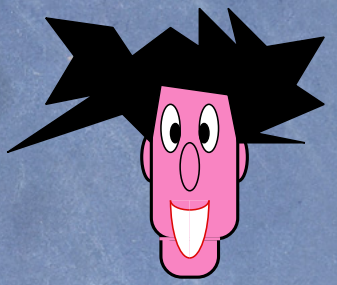


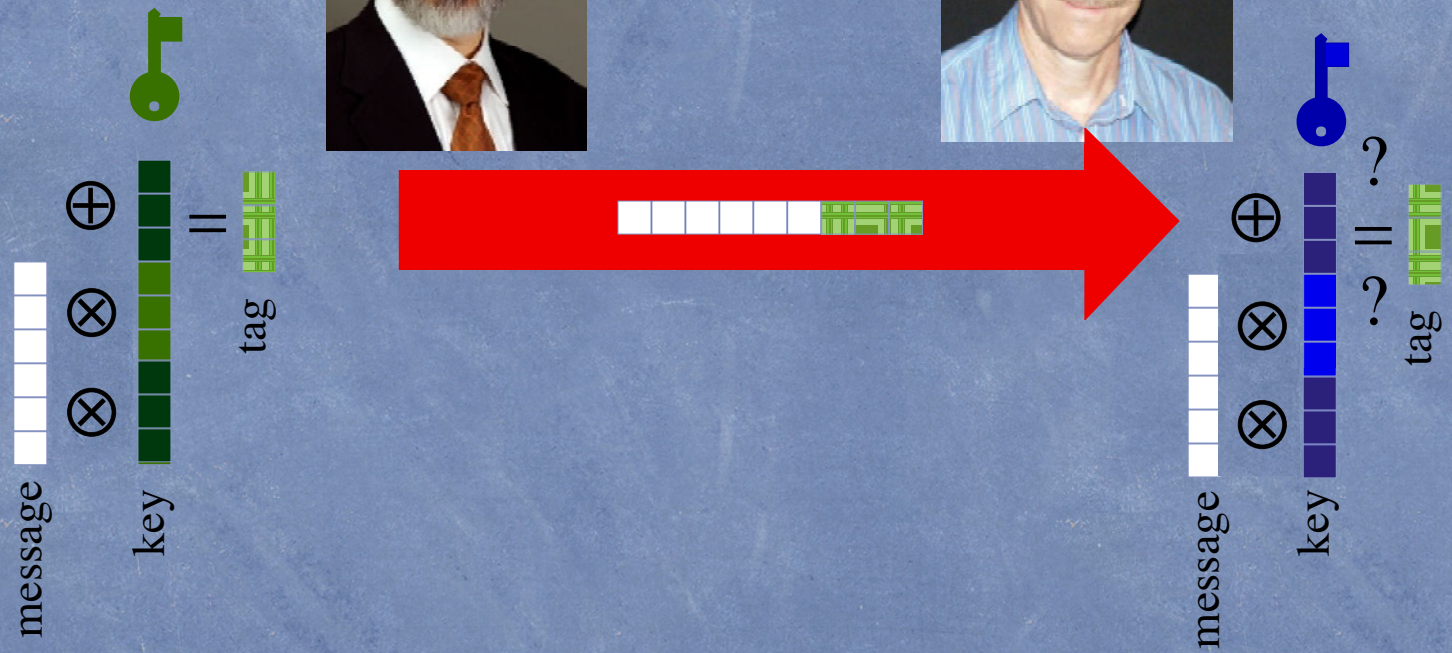
Information Theoretical Security

Wegman-Carter One-Time Authentication









$$m \neq m' \implies \Pr[h_{a,b}(m)=h_{a,b}(m')]=1/p$$

Gemmel-Naor One-Time Authentication

Authentication

$t := A_{k}(m)$

$|k| \approx 5|t| + \log(|m|)$

Monte Carlo vs. Las Vegas Algorithms

Monte Carlo algorithm. Guaranteed to run in poly-time, likely to find correct answer.

Ex: Contraction algorithm for global min cut.

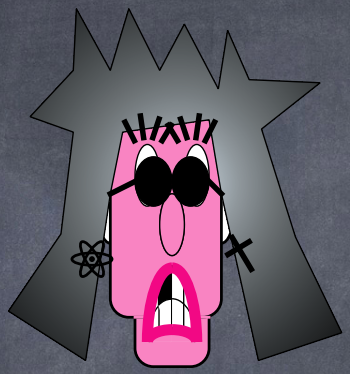
Las Vegas algorithm. Guaranteed to find correct answer, likely to run in poly-time.

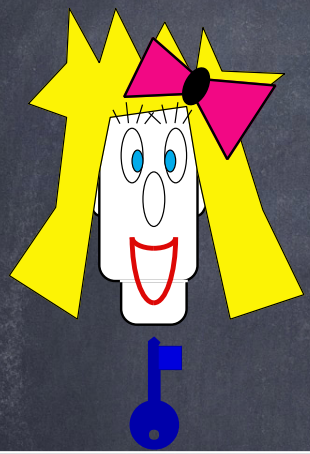
Ex: Randomized quicksort.

stop algorithm after a certain point

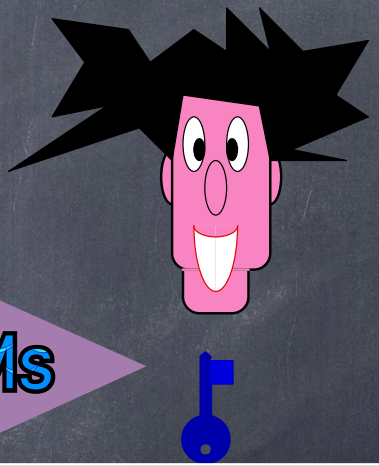
Remark. Can always convert a Las Vegas algorithm into Monte Carlo, but no known method to convert the other way.

Encryption





8RdewtU5qkLa$es!T9@



I(D%eXhDqIiykI#2cV7dEwnMs

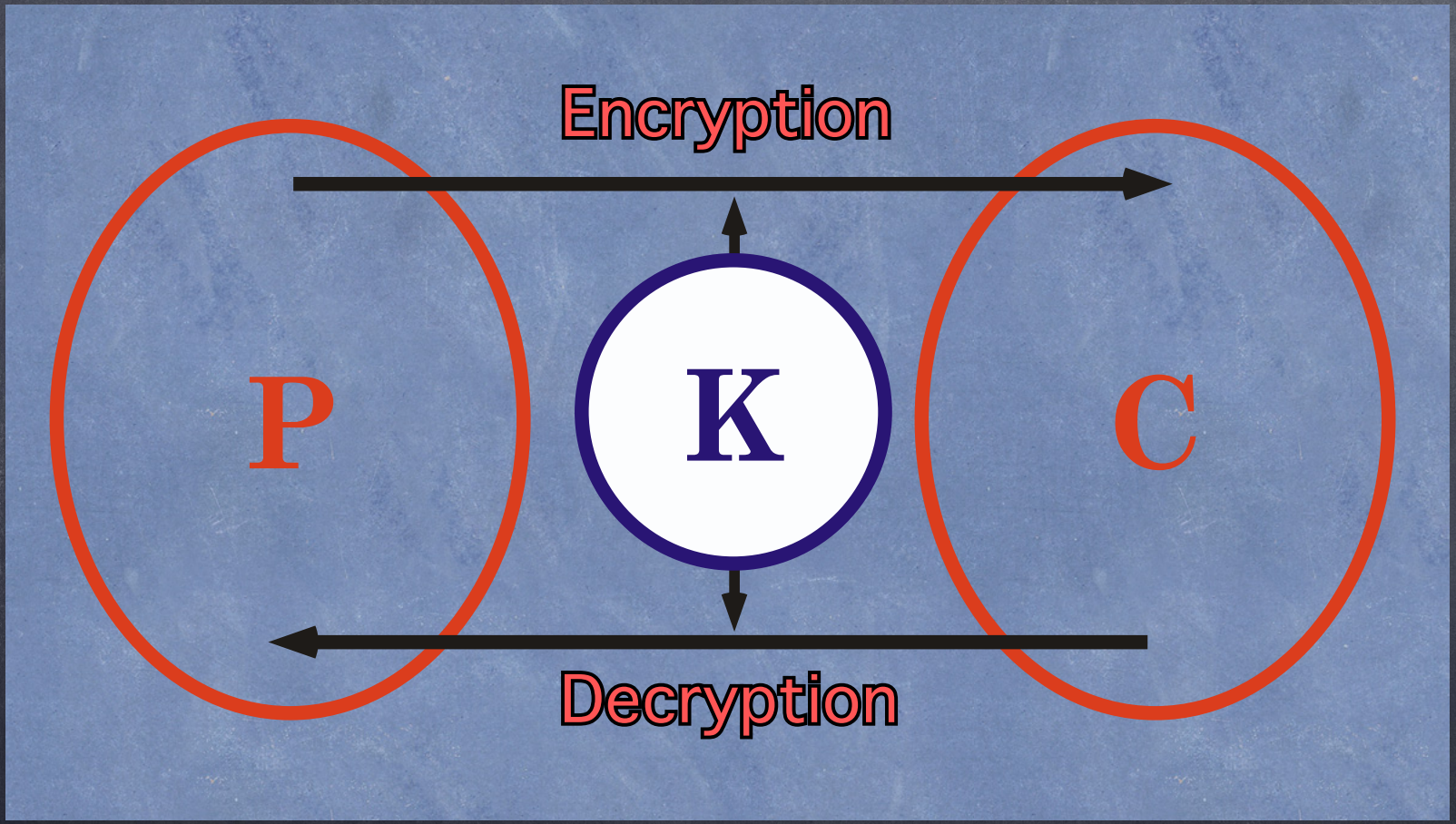
Encryption

Divorce your wif 2cV7dEwnMs

Decryption

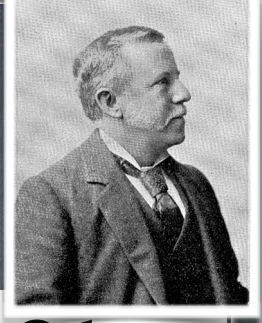
I(D%eXhDqIiykI#2cV7 ur wife first !

Symmetric Encryption

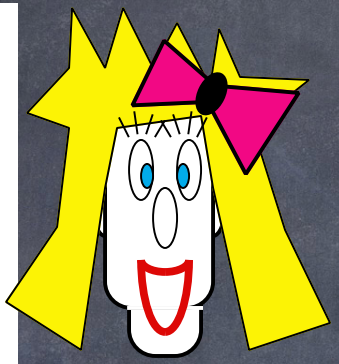


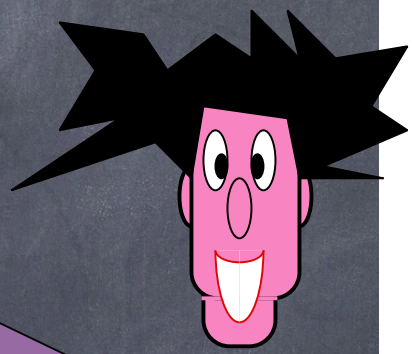
Information Theoretical Security

VERNAM's Cipher

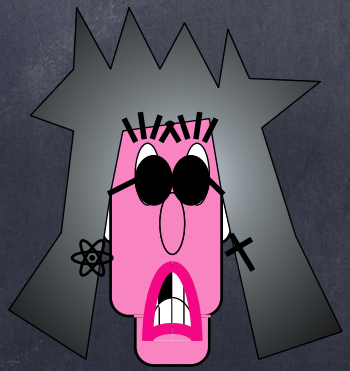


m	⊕	k	=	c
1		1		0
0		1		1
1		1		0
0		0		0
0		0		0
1		1		0
0		1		1
0		0		0
1		1		0
1		1		0
1		0		1
1		1		0
1		0		1
0		1		1
0		1		1
1		1		0





c



c	⊕	k	=	m
0		1		1
1		1		0
0		1		1
0		0		0
0		0		0
0		1		1
1		1		0
0		0		0
0		1		1
0		1		1
1		0		1
0		1		1
1		0		1
1		1		0
1		1		0
0		1		1

M VERNAM

⊕

K [redacted]

=

C [redacted]

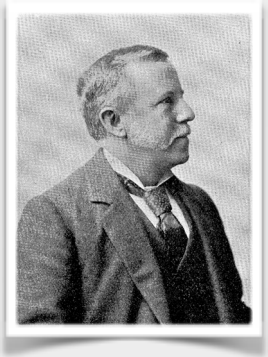
C [redacted]

⊕

K [redacted]

=

M VERNAM



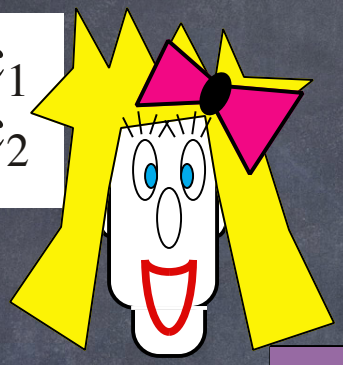
C [redacted] K

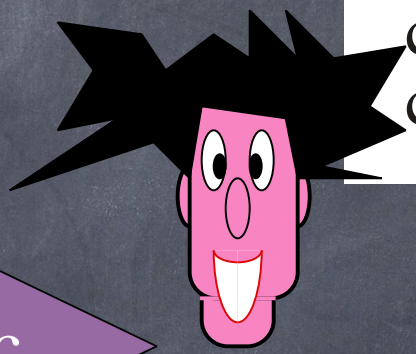
=

M' VERNAM

VERNAM's One-Time Pad

$m_1 \oplus k = c_1$
$m_2 \oplus k = c_2$



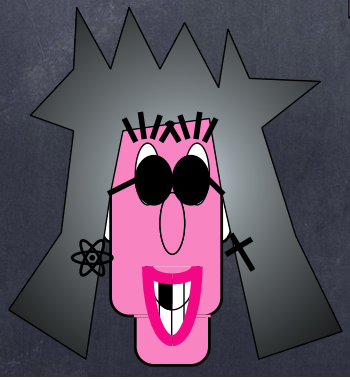


$c_1 \oplus k = m_1$
$c_2 \oplus k = m_2$

c_1

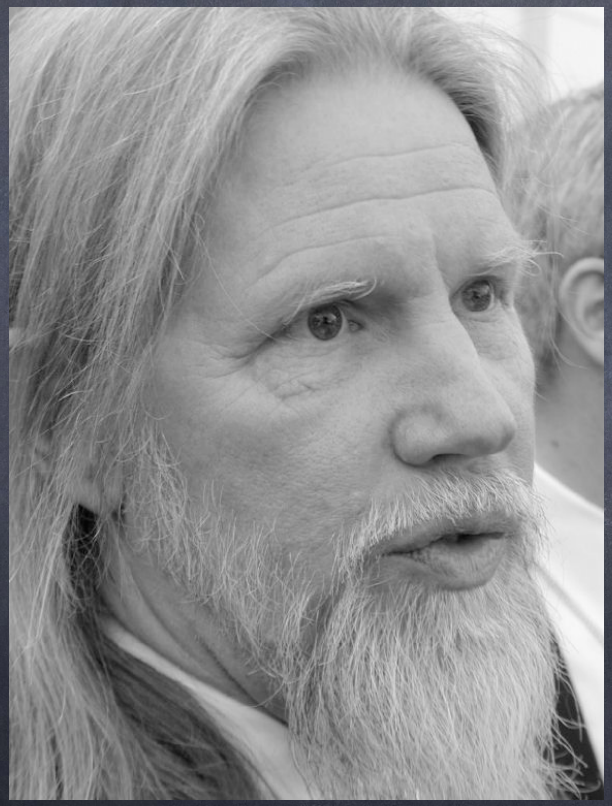
c_2

ie cannot use key multiple times as exploit becomes possible



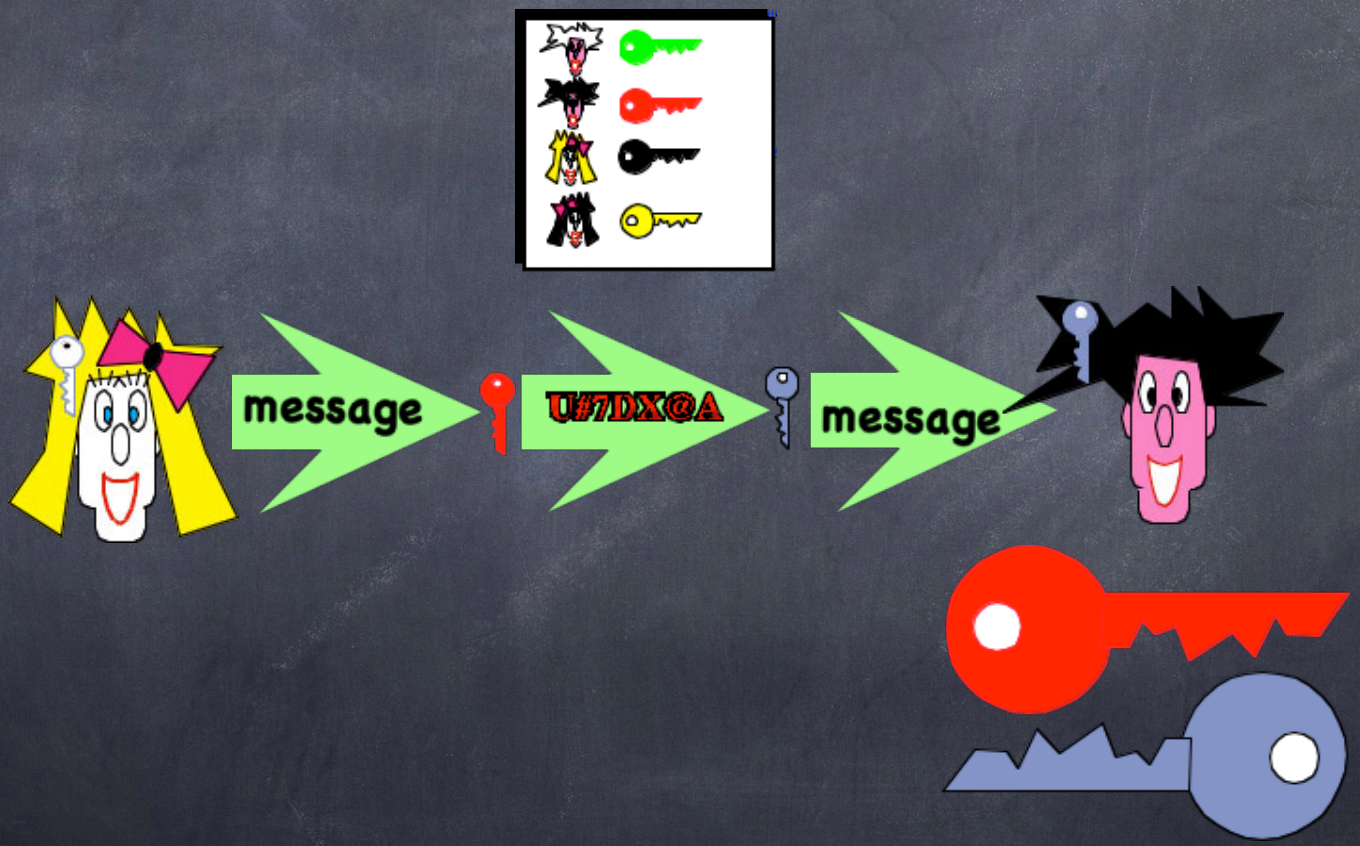
$c_1 \oplus c_2 = m_1 \oplus m_2$

The Public-Key Revolution



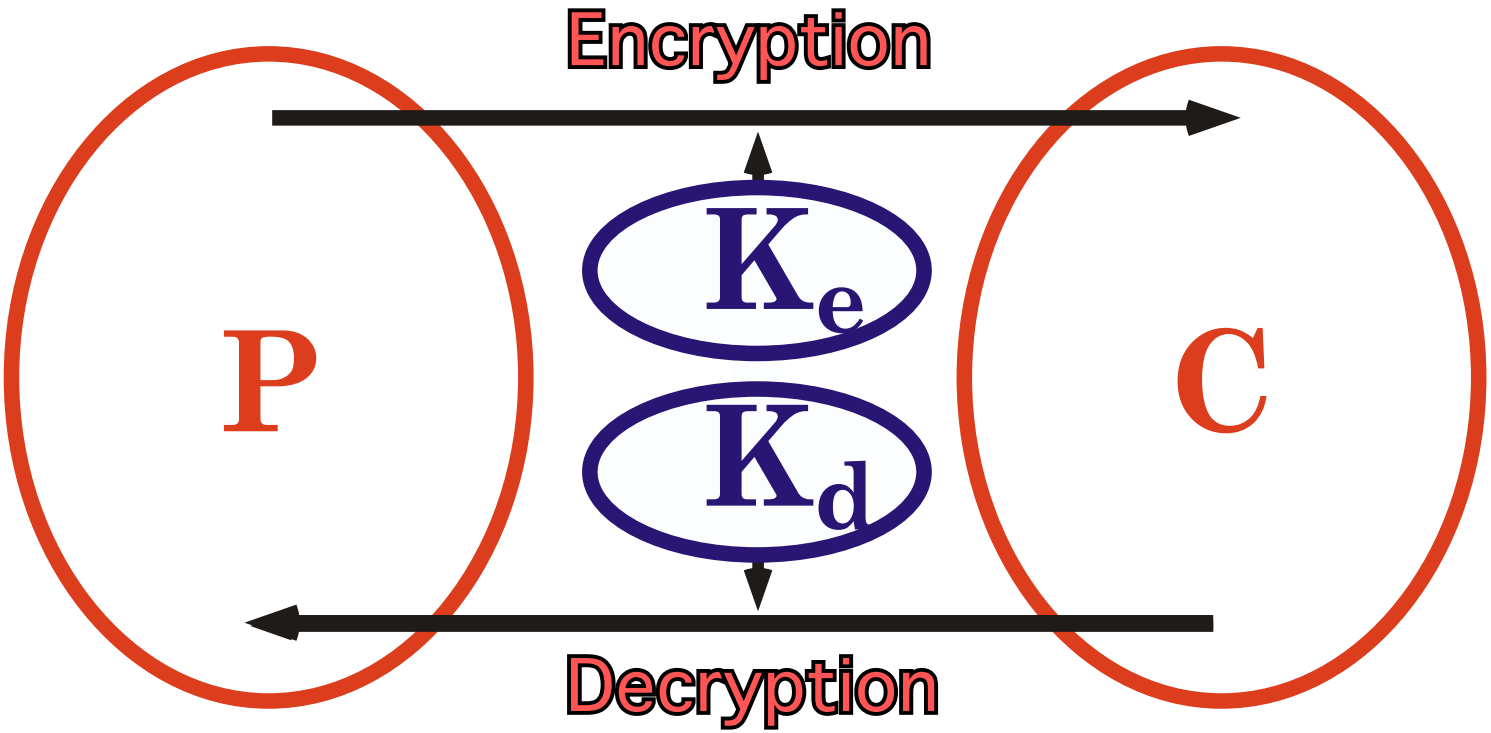
Whitfield Diffie and Martin Hellman

The Public-Key Revolution



Asymmetric Encryption

(Public-Key Cryptography)

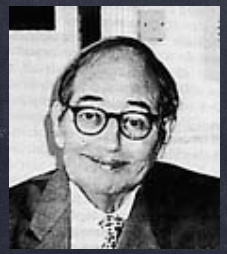


Complexity Theoretical Security

RSA Encryption

Public inventors

Private inventors

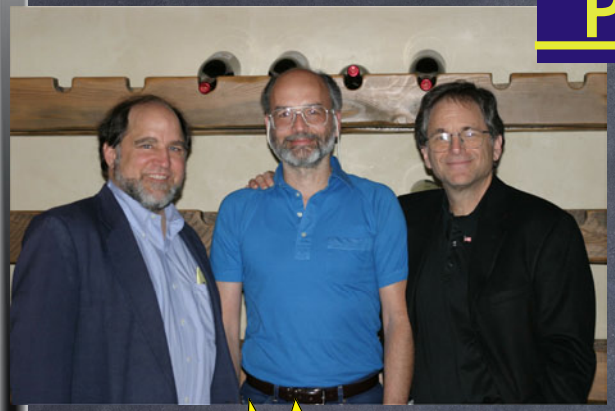


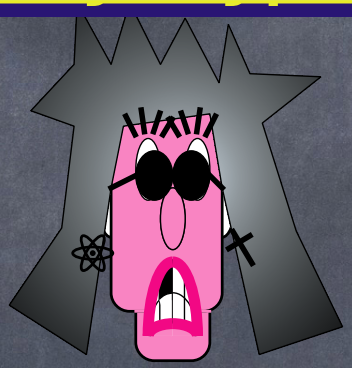
Ellis,

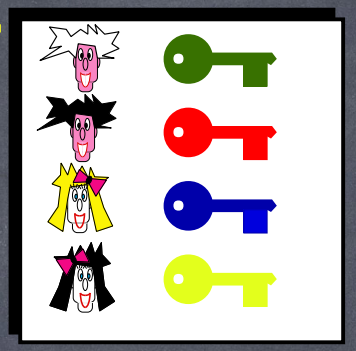
Cocks,

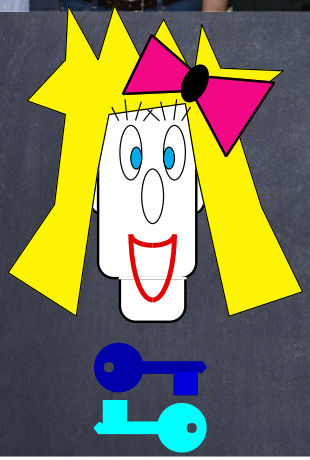
Williamson

Public-Key Cryptography

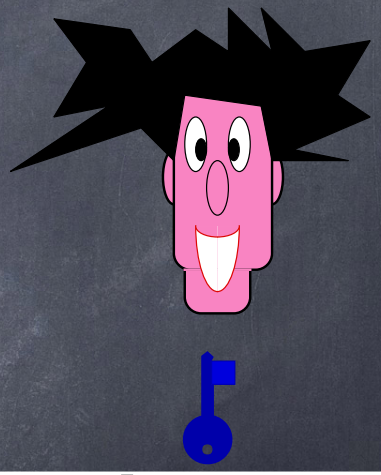








8RdewtU5qkLa$es!T9@



Decryption

Will you marry m es!T9@

Encryption

8RdewtU5qkLa$ y me ?

13.99 Primality Testing

Computing mod N

Elementary Operations. Let N,a,b be n-bit integers.

- a+b mod N is computable in time $O(n)$.
- axb mod N is computable in time $O(n^2)$ and asymptotically $O(n^{1+\delta})$.
- a^b mod N is computable in time $O(n^3)$ and asymptotically $O(n^{2+\delta})$.
- gcd(a,b) is computable in time $O(n^2)$.
- [AKS2002]
  Deciding if a number N is prime or not is computable in time $O(n^{12})$.
  Way too slow in practice.
- [PL2005]
  Deciding if a number N is prime or not is computable in time $O(n^6)$.
  Still too slow in practice.

Computing mod N

Rabin-Miller pseudo-primality test. Let N,1<a<N be n-bit integers.

Let $N-1 = 2^s t$ where t is odd.	$O(n)$
Let a be a random element such that $1 < a < N$.	$O(n)$
If gcd(a,N) > 1 **then** fail.	$O(n^2)$
Compute $x_0 := N-1$; $x_1 := a^t \bmod N$.	$O(n^3)$
Compute $x_{i+1} := x_i^2 \bmod N$, for $1 \le i \le s$.	$O(n^2)$
If $x_{s+1} > 1$ **then** fail.	$O(1)$
Let m be such that $x_m > 1$ and $x_{m+1} = 1$.	$O(n)$
If $x_m = N-1$ **then** succeed **else** fail.	$O(1)$

- Rabin theorem[1977]. Let N,a be n-bit integers.

- **If** N is prime **then** all a such that gcd(a,N)=1 lead to success
- **else** at least 3/4 of all a such that gcd(a,N)=1 lead to failure.

Computing mod N

Rabin theorem[1977]. Let N,a be n-bit integers.

- **If** N is prime **then** all a such that gcd(a,N)=1 lead to success
- **else** at least 3/4 of all a such that gcd(a,N)=1 lead to failure.

Corollary. If this test is executed k times with random independent a's, then if N is prime then Pr[k success] = 1 else Pr[k success] < $1/4^k$.

Running time = $O(kn^{2+\delta})$

RSA Encryption

RSA key generation GenRSA

Input: Security parameter 1^n
Output: N, e, d as described in the text

$(N, p, q) \leftarrow \mathsf{GenModulus}(1^n)$
$\phi(N) := (p-1)(q-1)$
choose e such that $\gcd(e, \phi(N)) = 1$
compute $d := [e^{-1} \bmod \phi(N)]$
return N, e, d

In Cocks' variation, $e=N$ and therefore $d=N^{-1} \bmod \varphi(N)$.

RSA Encryption

- Enc: on input a public key $pk = \langle N, e \rangle$ and a message $m \in \mathbb{Z}_N^*$, compute the ciphertext

$$c := [m^e \bmod N].$$

- Dec: on input a private key $sk = \langle N, d \rangle$ and a ciphertext $c \in \mathbb{Z}_N^*$, compute the message

$$m := [c^d \bmod N].$$

The "textbook RSA" encryption scheme.

The RSA Assumption

The **RSA** problem can be described informally as:

- a modulus N,
- an exponent $e > 0$ that is relatively prime to $\varphi(N)$, and
- an element $c \in \mathbb{Z}_N^*$,
- compute $\sqrt[e]{c} \bmod N$;

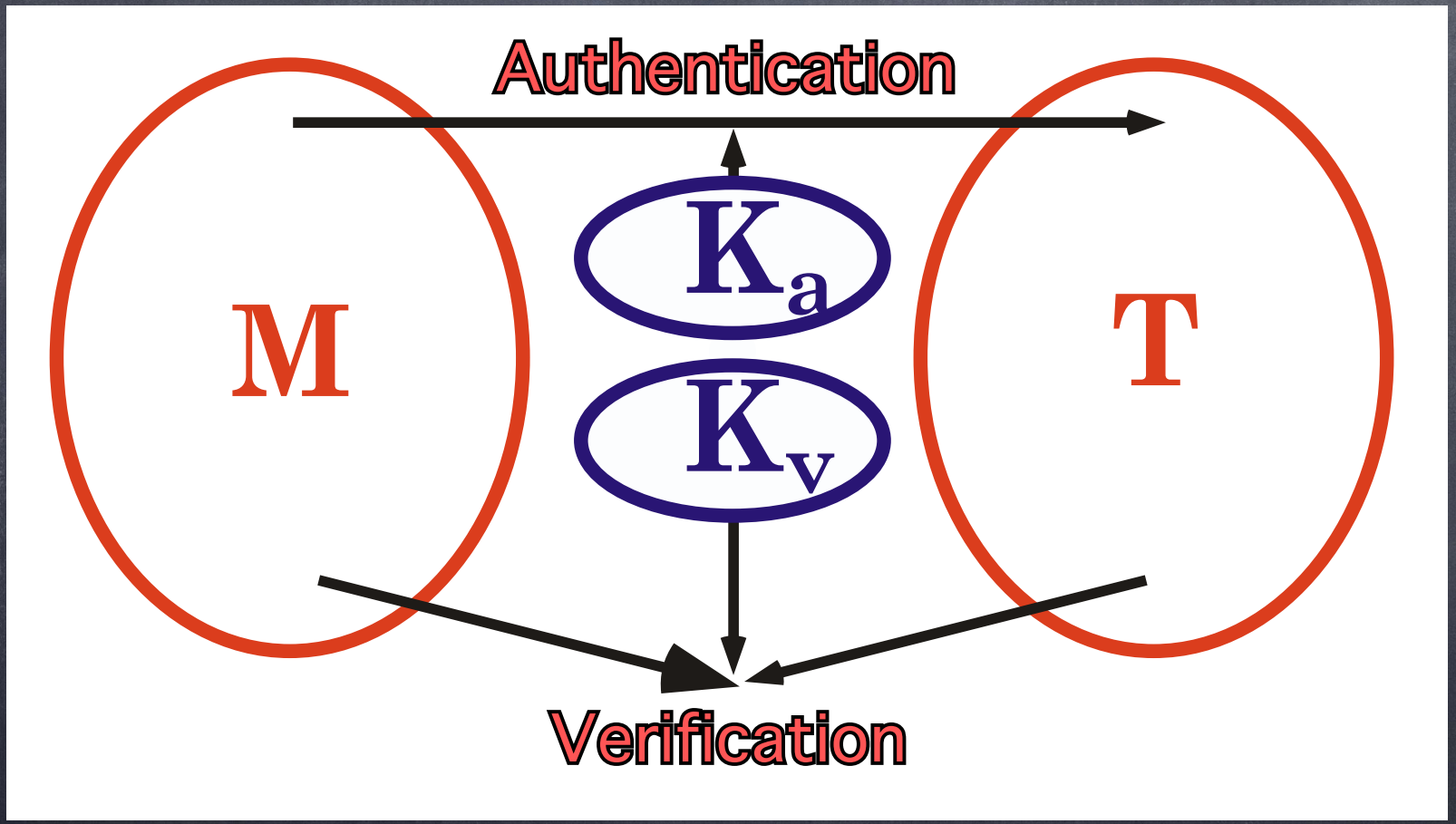
  or

  Given N,e,c find m such that $m^e = c \bmod N$.

Digital Signatures

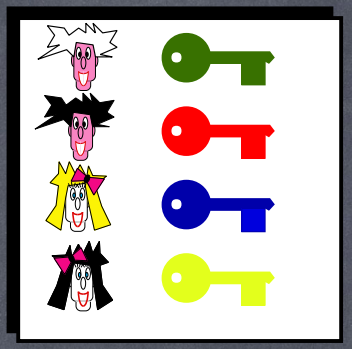
Asymmetric Authentication

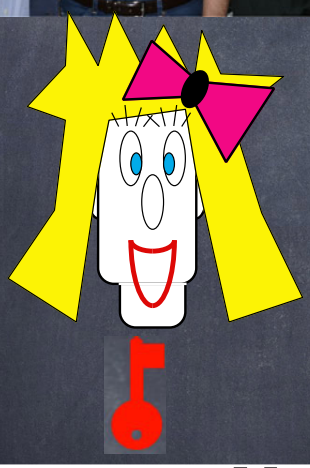
(Digital Signature Scheme)



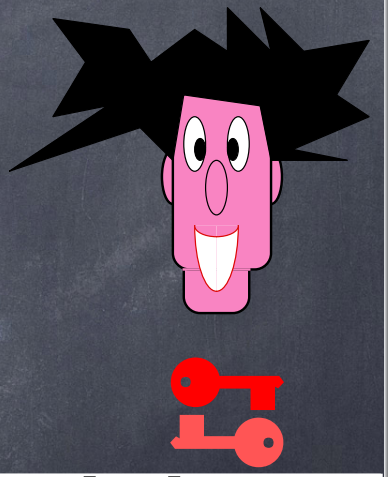
Complexity Theoretical Security

Digital Signature





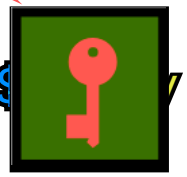
8RdewtU5qkLa$es!T9@
Will you marry me ?



Verification

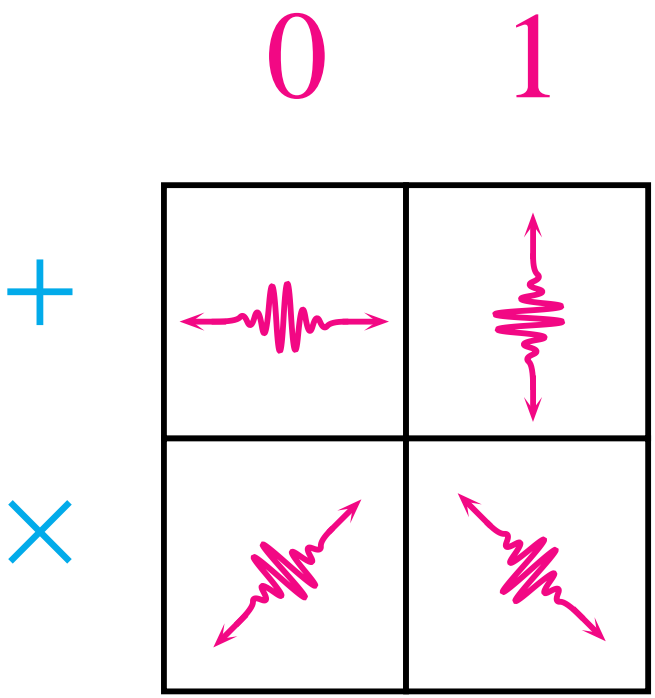
VALID  es!T9@
ry me ?

Authentication

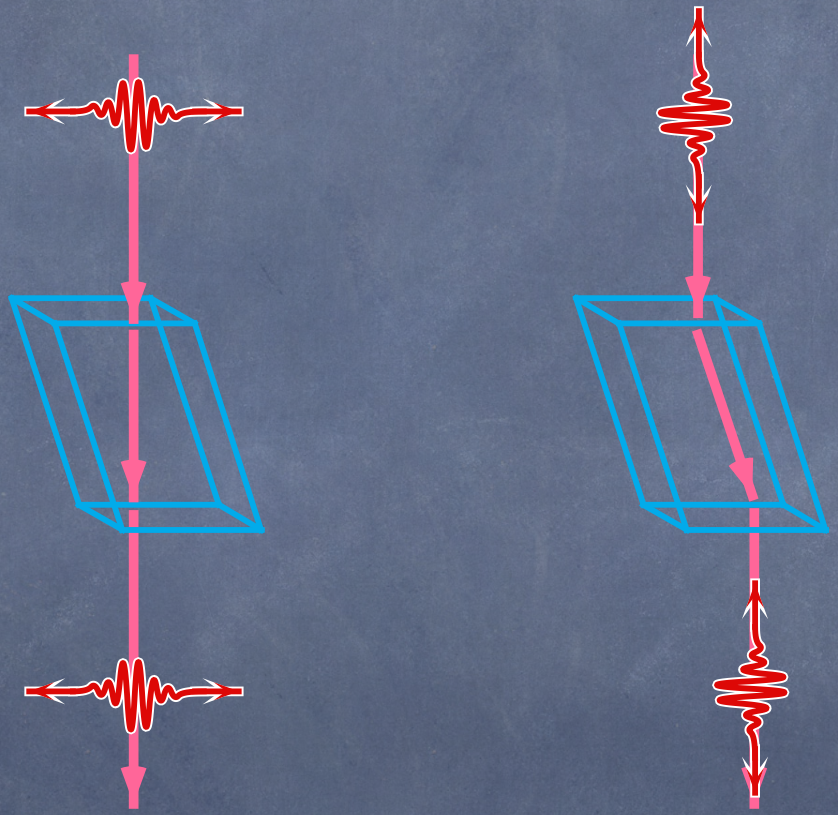
8RdewtU5qkLa  y me ?

Quantum Cryptography

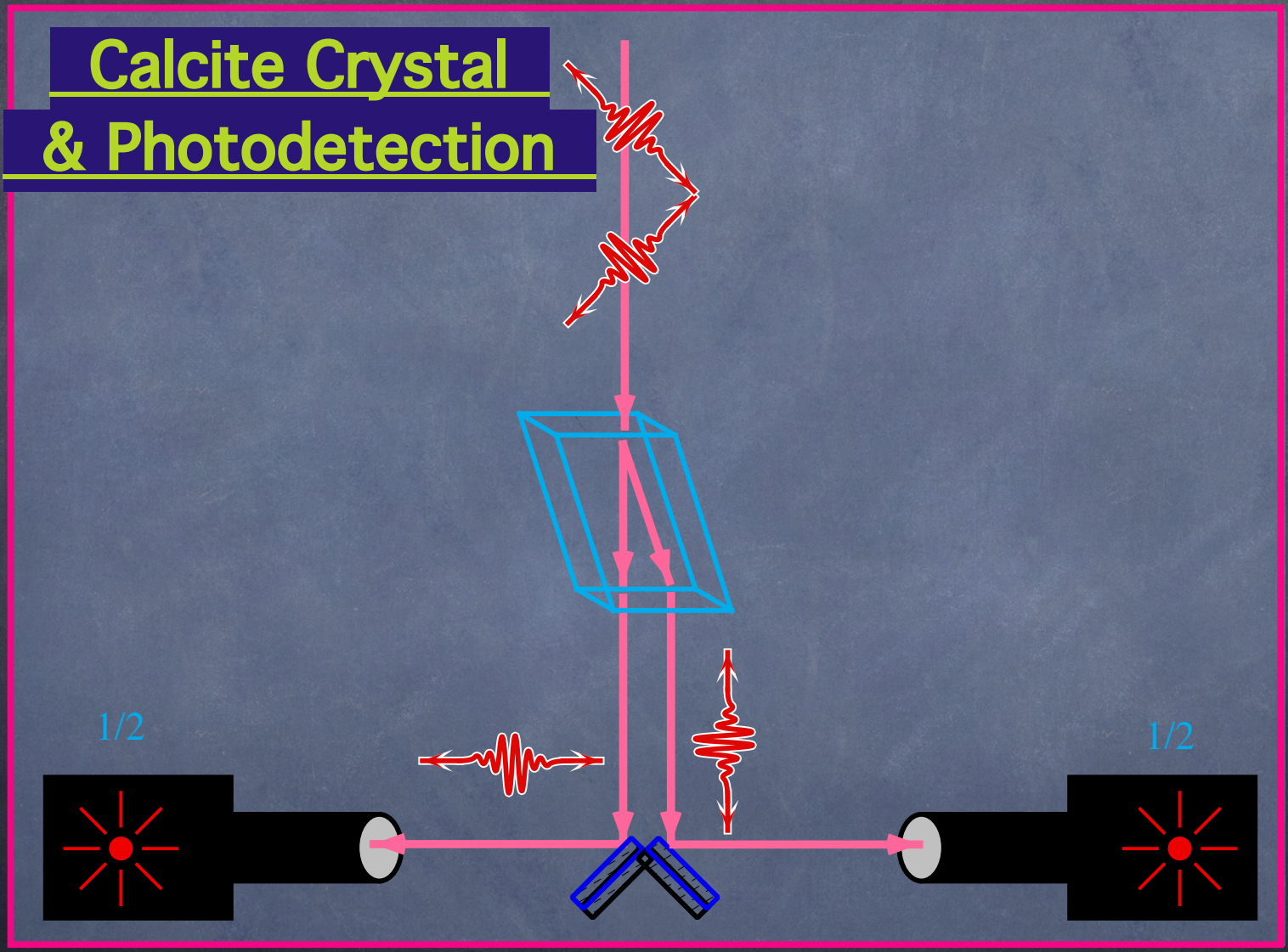
Ambiguous Coding Scheme



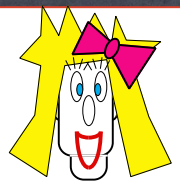
Calcite Crystal

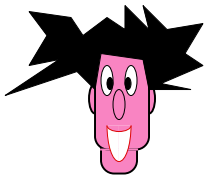


Calcite Crystal & Photodetection



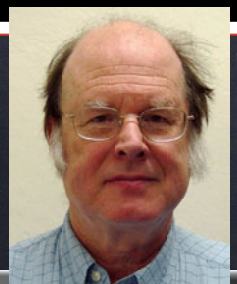
Quantum Key Distribution





A:	0	1	1	0	0	1	0	0	1	1	0	1	0	0	0	1	1	1	0	1	1	0	0	0
	×	+	×	+	+	+	×	×	×	×	+	+	+	+	×	×	×	+	×	+	+	+	×	+
B:	×	×	+	+	×	+	+	+	×	+	+	×	×	×	+	×	×	×	+	+	×	+	×	+
	0	0	1	0	0	1	0	0	1	0	0	0	0	1	1	1	0	0	0	1	1	0	0	0
A:	×	+	×	+	+	+	×	×	×	×	+	+	+	+	×	×	×	+	×	+	+	+	×	+
B:	0	↓	↓	0	↓	1	↓	↓	1	↓	0	↓	↓	↓	↓	1	0	↓	↓	1	↓	0	0	0
B:	0			0		1			1		0					1	0			1		0	0	0
A:	0			0		1			1		0					1	1			1		0	0	0
A:	0					1					0						1							0
B:	=					=					=					≠								=
B:				0					1							1				1		0	0	
A:				0					1							1				1		0	0	

20%



Bennett-Brassard

Quantum Key Distribution

- Produces raw classical key
- Observed error rate indicates amount of eavesdropper information
- Error-correction is used to fix errors
- Random hash function is used to distill a smaller very secret classical key