For subsets $R \subseteq T$, we also define:

$$
\begin{aligned}
w(R) &= \sum_{i \in R} w_i = \text{number of wins of the teams in } R \\
g(R) &= \sum_{i,j \in R, i<j} g_{ij} = \text{number of games to be played where both teams are in } R \\
a(R) &= \frac{w(R) + g(R)}{|R|}.
\end{aligned}
$$

Claim 1 *Some team $i \in R$ wins at least $a(R)$ games.*

Proof: The total number of wins by all teams in R must be at least the total of their current wins plus the number of games played within set R, which is $w(R)+g(R)$. Therefore, the average number of wins by teams in R is $a(R)$, so some team must win at least $a(R)$ games. $\square$

Corollary 2 *For a team $i \in T$ and any $R \subseteq T - \{i\}$, if $a(R) > w_i + g_i$, then team i is eliminated.*

Example. Let $R = \{\text{New York, Toronto}\}$ and $i =$ Boston. Then $a(R) = \frac{(93+88)+6}{2} = 93.5 > 93 = 89 + 4 = w_i + g_i$. So Boston is eliminated, as we saw above.

Now let x_{ij} be the number of times team i defeats team j in the remaining games. Then team k is *not* eliminated if there exists $\{x_{ij}\}$ such that the following conditions hold:

$$
\begin{aligned}
x_{ij} + x_{ji} &= g_{ij}, \quad \forall i, j \in T \\
w_k + \sum_{j \in T} x_{kj} &\geq w_i + \sum_{j \in T} x_{ij}, \quad \forall i \in T \\
x_{ij} &\geq 0, \quad x_{ij} \text{ integer}, \quad \forall i, j \in T
\end{aligned}
$$

The first condition states that exactly one of team i or team j must win each game played between teams i and j. The second states that, at the end of the season, team k must have won at least as many games as any other team. The third simply guarantees that the x_{ij} must have nonnegative integer values.

If such x_{ij} exist, then team k is not eliminated, so it wins its division (possibly in a tie with another team). So then we could change some of the x_{kj} to make team k win all of its remaining games, and team k would still not be eliminated. This means we could find x'_{ij} to satisfy the following three criteria:

$$
\begin{aligned}
x'_{ij} + x'_{ji} &= g_{ij}, \quad \forall i, j \in T && (1) \\
w_k + g_k &\geq w_i + \sum_{j \in T} x'_{ij}, \quad \forall i \in T && (2) \\
x'_{ij} &\geq 0, \quad x'_{ij} \text{ integer}, \quad \forall i, j \in T && (3)
\end{aligned}
$$

We can create a network to determine whether team k is not eliminated. To do this, we create source and sink nodes, s and t; a node for every team $i \in T - \{k\}$; and a pair node for each $\{i, j\} \subseteq T - \{k\}$ with $i < j$ to avoid double counting. We make edges from the source s to the pair node $\{i, j\}$ and give these capacities of g_{ij}. We make edges from team node i to the sink t and give them capacities of $w_k + g_k - w_i$. We also create edges from each pair $\{i, j\}$ to teams i and j with infinite capacity. This is shown in Figure 1.

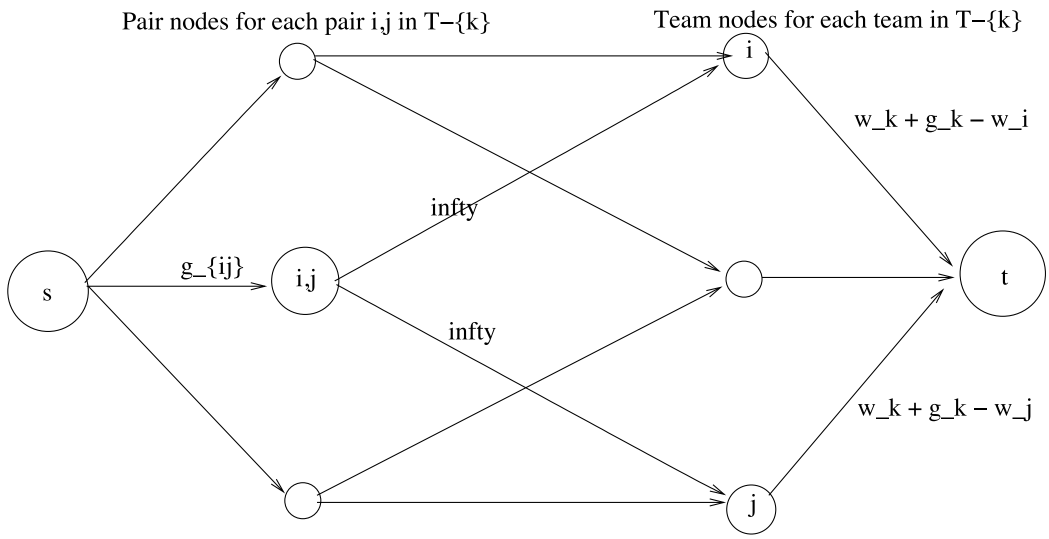

Figure 1: Flow instance for deciding if team k is not eliminated.

Note that we can assume that the capacities on the arcs going from the team nodes to the sink t are non-negative, since if $w_k + g_k - w_j < 0$, then $w_j > g_k + w_k$, and we know that team k is eliminated.

Now let $G = g(T - \{k\})$ be the sum of capacities on the arcs out of s, which gives the number of games to be played not involving team k. Then we can state the following lemma:

Lemma 3 *If a flow of value G exists, then team k is not eliminated.*

Proof: Notice that if a flow of value G exists, it must saturate all of the arcs out of s. Let x_{ij} be the flow from pair node $\{i, j\}$ to team node i. We want to show that the three conditions given above hold.

1. $x_{ij} + x_{ji} = g_{ij}$ is satisfied since the flow to pair node $\{i, j\}$ is g_{ij}, so flow conservation guarantees that the flow out, which is $x_{ij} + x_{ji}$ equals g_{ij}.
2. $\sum_{j \in T-\{k\}} x_{ij} \leq w_k + g_k - w_i$ is satisfied because of flow conservation and capacity constraints on arcs into t.
3. All of the x_{ij} are nonnegative, and we can assume they are all integers because of the integrality property of flow.

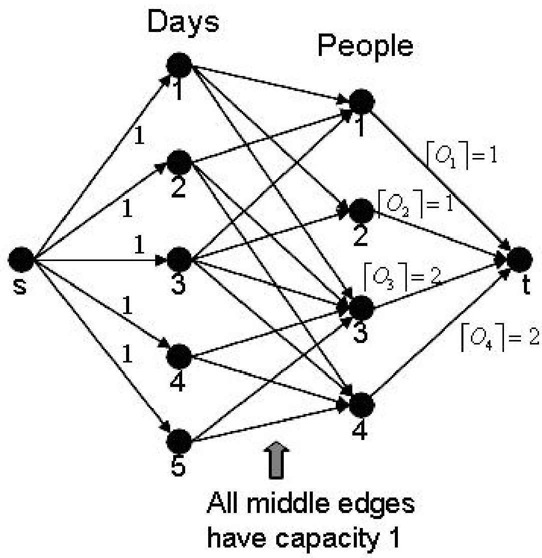

Figure 2: Flow instance for determining a fair carpool.

1.2 Carpool Fairness

Description: In this scenario, n people are sharing a carpool for m days. Each person may choose whether to participate in the carpool on each day.

Example. The following table describes a carpool in which 4 people share a carpool 5 days. X's indicate days when people participate in the carpool.

Person	**Days:** 1	2	3	4	5
1	X	X	X		
2	X		X		
3	X	X	X	X	X
4		X	X	X	X

Our goal is to allocate the daily driving responsibilities 'fairly.' One possible approach is to split the responsibilities based on how many people use the car. So, on a day when k people use the carpool, each person incurs a responsibility of $\frac{1}{k}$. That is, for each person i, we calculate his or her driving obligation O_i as shown below. We can then require that person i drives no more than $\lceil O_i \rceil$ times every m days. Table 1.2 shows the calculation of these O_i and their ceilings.

Person	**Days:** 1	2	3	4	5	O_i	$\lceil O_i \rceil$
1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{4}$			1	1
2	$\frac{1}{3}$		$\frac{1}{4}$			$\frac{7}{12}$	1
3	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{7}{4}$	2
4		$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{19}{12}$	2
$\sum$	1	1	1	1	1	-	-

Table 1: Driver Responsibilities

To determine whether such an assignment is possible, we formulate the problem as a network, as shown in Figure 2.

We use this network to prove a claim for an m day carpool.

Claim 7 *If flow of value m exists, then a fair driving schedule exists.*

Proof: Note that all capacities are integer and if a flow of value m exists, then an integral flow of value m also exists. So, for each day, exactly one arc pointing outward has a flow of 1. This arc points to some person, and this is the person who should drive for the day. By flow conservation and the capacity of the arcs into t, no one will have to drive more than their obligation. $\square$

Note that we do not have to compute the maximum flow to conclude that there always exists a fair driving schedule.

Claim 8 *A flow of value m always exists.*

Proof: We can always give a fractional flow of value m, where each person present on a given day drives $\frac{1}{k}$ on a day when k people participate in the carpool. $\square$