MA373: Quiz - 1 Feb - 2025 Answers

Q1: $22.50 = 20e^{r/2} \Rightarrow r = 2\ln 1.125 \Rightarrow F(0,1) = 20e^{r} = 25.3125$

Q2: By put-call parity, $C_t > P_t$ if $S_t > Ke^{-(r-a)(T-t)}$

Here, $S_{1/2} > 200e^{-0.05(1/2)} = 195.0620$

Q3: Given $a = 0.01$, $\alpha = 0.05 + a = 0.06$, $\sigma = 0.25$.

MPR is $\theta = \dfrac{\alpha - r}{\sigma} = 0.24 - 4r$.

Use this θ to define Q s.t. $W_t^{Q} = W_t + \theta t$ is a BM

Thus, $E^{Q}[W_t^{Q}] = 0 = E^{Q}(W_t) + \theta t$.

Given $E^{Q}(W_{0.5}) = -0.03$ and hence, with $t = 1/2$,

$$-0.03 + (0.24 - 4r)(0.5) = 0 \Rightarrow r = 0.045 \text{ (or } 4.5\%)$$

Q4 We have $dX_t = \alpha'(t)\left[x_0 + \int_0^t \beta(s)\,dW_s\right]dt + \alpha(t)\beta(t)\,dW_t$

$$= \frac{\alpha'(t)}{\alpha(t)} X_t\,dt + \alpha(t)\beta(t)\,dW_t$$

Comparing the coefficients, we have $a(t) = \dfrac{\alpha'(t)}{\alpha(t)}$

and $b(t) = \alpha(t)\beta(t)$. Note $\alpha(0) = 1$.

For the given SDE, $a(t) = t$, $b(t) = 3e^{t^2/2}$, $x_0 = 2$.

$\Rightarrow t = \dfrac{\alpha'(t)}{\alpha(t)}$, $\alpha(0) = 1$ and $3e^{t^2/2} = \alpha(t)\beta(t)$

$\Rightarrow \alpha(t) = e^{t^2/2}$, $\beta(t) = 3$.

Hence, $X_t = e^{t^2/2}\left[2 + 3W_t\right]$.

Now $EX_t = 2e^{t^2/2}$, $Var(X_t) = 9te^{t^2}$.

Q5: Consider $J = \frac{1}{\sqrt{2\pi T}} \int_K^b e^{\beta + \gamma w - \frac{1}{2T} w^2} \, dw$

$$= e^{\frac{1}{2}\gamma^2 T + \beta} \frac{1}{\sqrt{2\pi}} \int_{\frac{1}{\sqrt{T}}(K-\gamma T)}^{\frac{1}{\sqrt{T}}(b-\gamma T)} e^{-\frac{1}{2}y^2} \, dy$$

$$= e^{\frac{1}{2}\gamma^2 T + \beta}\left[N\left(\frac{b-\gamma T}{\sqrt{T}}\right) - N\left(\frac{K-\gamma T}{\sqrt{T}}\right)\right]$$

$$= e^{\frac{1}{2}\gamma^2 T + \beta}\left[N\left(\frac{1}{\sigma\sqrt{T}}\left[\ln\frac{S}{K} + \gamma\sigma T\right]\right) - N\left(\frac{1}{\sigma\sqrt{T}}\left(\ln\frac{S}{B} + \gamma\sigma T\right)\right)\right]$$

The given integral I is ~~of~~ the form above for J with $\beta = -rT - \frac{1}{2}\alpha^2 T$ and $\gamma = \alpha + \sigma$, so $\frac{1}{2}\gamma^2 T + \beta = 0$ and $\gamma\sigma = r + \frac{1}{2}\sigma^2$.

Therefore, $\underline{I = N\left(d\left(\frac{S}{K}\right)\right) - N\left(d\left(\frac{S}{B}\right)\right)}$

where $d(s)$ is as defined in the question.