큰 수의 곱셈 : 분할 정복 알고리즘. — Karatsuba Algorithm

일반적인 $A * B$의 연산 : $O(1)$ 시간 (RAM)

하지만, 무수히 큰 두수의 곱은 $O(1)$이 아닐수있음.

$$A = a_{n-1} \ a_{n-2} \cdots a_1 \ a_0$$
$$\times \ B = b_{n-1} \ b_{n-2} \cdots b_1 \ b_0$$

$\square\square\square \cdots \square\square\square \quad (a_{n-1}\cdots a_0)\times b_0.$
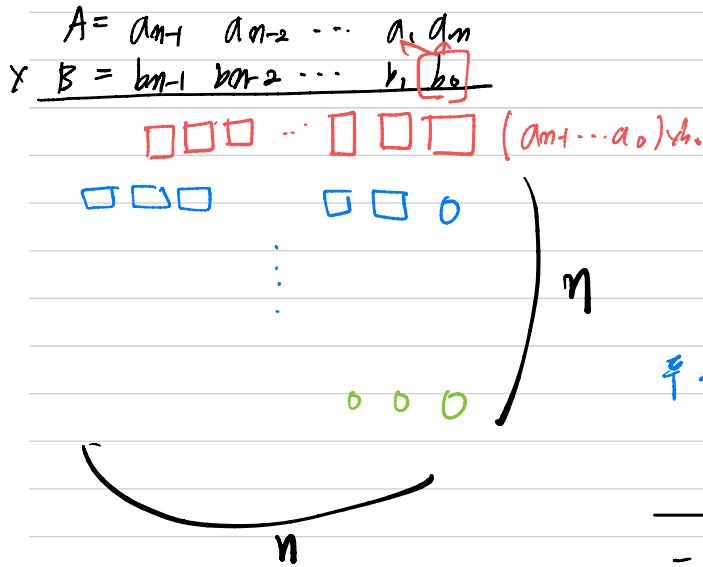
$= m \times m = m^2$의 기본 곱셈.

두 곱의 자리수 = 자리수 $(\underline{A} \times \underline{B}) = 2m$
$\qquad\qquad\qquad\qquad m \ \ m$자리수

$2m \times m = 2m^2$의 기본 덧셈

$= 3m^2$의 기본 덧셈과 기본 곱셈

$= O(m^2)$

— Karatsuba algorithm : U, V : m의 자리의 두수 (짝수)

$$U : a_{n-1} a_{n-2} \cdots a_{\frac{n}{2}} \ a_{\frac{n}{2}+1} \cdots a_1 a_0 = x \cdot 10^{\frac{n}{2}} + y$$
$\qquad\qquad \underbrace{\qquad\qquad}_{x} \ \underbrace{\qquad\qquad}_{y} \qquad\qquad\qquad\qquad \text{offset}$

$$V : b_{n-1} b_{n-2} \cdots b_{\frac{n}{2}} \ b_{\frac{n}{2}+1} \cdots b_1 b_0 = w \cdot 10^{\frac{n}{2}} + z$$
$\qquad\qquad \underbrace{\qquad\qquad}_{w} \ \underbrace{\qquad\qquad}_{z}$

$$U \times V = \left(x \cdot 10^{\frac{n}{2}} + y \right)\left(w \cdot 10^{\frac{n}{2}} + z \right)$$

$$U \times V = \underbrace{x \cdot w}_{\frac{n}{2} \times \frac{n}{2} \ \text{정수}} \boxed{10^n} + (\underbrace{x z}_{②} + \underbrace{y w}_{③}) \boxed{10^{\frac{n}{2}}} + \underbrace{y z}_{④}$$
$n \times n$
$\qquad\qquad\qquad\qquad n$자리 $\quad n$자리
$\qquad\qquad\qquad\qquad \frac{n}{2} \times \frac{n}{2}$
$\qquad\qquad\qquad\qquad n$자리$+ m$자리 $= O(n) \times 3$개

$$T(m) = 4T\left(\frac{n}{2}\right) + c \cdot m$$
$\qquad \uparrow$
$n \times m$ 자릿수를
곱하는데 필요한
총 연산수
$\qquad\qquad\qquad T(1) = C$

$$\begin{aligned}
n = 2^k &= 4^k = (2^k)^2 \\
T(m) &= 4\left(\frac{T}{2}\right) + cn \\
&= 4^2 \times \left(\frac{T}{2^2}\right) + 2cn + cn \\
&\vdots \\
&= 4^k T\left(\frac{n}{2^k}\right) + cn(1+2+\cdots 2^{k-1}) \\
&= O(4^k) = \boxed{O(m^2)}
\end{aligned}$$

$$U \times V = \underset{①\ \frac{n}{2} \times \frac{n}{2}\ \text{자리}}{\underset{n \times n}{\underline{x \cdot w}}} \boxed{10^n} + \underset{②}{(\underset{n\text{자리}}{\underline{xz}} + \underset{n\text{자리}}{\underline{yw}})} \boxed{10^{\frac{n}{2}}} + \underset{④}{yz}$$

$$\underset{n\text{자리} + n\text{자리} = O(n) \times 3\text{개}}{\frac{n}{2} \times \frac{n}{2}}$$

$$(xz + yw) = (x+y)(w+z) - xw - yz$$

$$U \times V = \boxed{x \cdot w} 10^n + \{\underset{\frac{n}{2}+\frac{n}{2}=\frac{n}{2}+1\ (\text{자리})}{\underset{}{\boxed{(x+y)(w+z)}}} \underset{\frac{n}{2}+1}{\boxed{-xw}} \boxed{-yz}\} 10^{\frac{n}{2}} + \boxed{yz}$$

$$n\text{자리} \times n\text{자리} = \left(\underset{xw}{\frac{n}{2}\text{자리}} \times \underset{yz}{\frac{n}{2}\text{자리}}\right) \times 2\text{개} + \underset{(x+y)(z+w)}{\left(\frac{n}{2}+1\right)\text{자리} \times \left(\frac{n}{2}+1\right)\text{자리}} + cn$$

$$T(n) = \tau\left(\frac{n}{2}\right) \times 2 + \tau\left(\frac{n}{2}+1\right) + cn$$

$$= \underset{\underset{\text{식에서 저것로 같군}}{\uparrow}}{3T\left(\frac{n}{2}\right) + cn} = O(n^{\log_2 3}) = O(n^{1.7\cdots})$$

$$4T\left(\frac{n}{2}\right) + cn = O(n^2) = O(n^{\log_2 4})$$