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A parameter-uniform implicit difference scheme for solving time-dependent Burgers' equations

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Abstract

A numerical study is made for solving one dimensional time dependent Burgers' equation with small coefficient of viscosity. Burgers' equation is one of the fundamental model equations in the fluid dynamics to describe the shock waves and traffic flows. For high coefficient of viscosity a number of solution methodology exist in the literature [6–9] and [14] but for the sufficiently low coefficient of viscosity, the exist solution methodology fail and a discrepancy occurs in the literature. In this paper, we present a numerical method based on finite difference which works nicely for both the cases, i.e., low as well as high viscosity coefficient. The method comprises a standard implicit finite difference scheme to discretize in temporal direction on uniform mesh and a standard upwind finite difference scheme to discretize in spacial direction on piecewise uniform mesh. The quasilinearzation process is used to tackle the non-linearity. An extensive amount of analysis has been carried out to obtain the parameter uniform error estimates which show that the resulting method is uniformly convergent with respect to the parameter.

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To illustrate the method, numerical examples are solved using the presented method and compare with exact solution for high value of coefficient of viscosity.

Keywords: Burgers' equation; Euler implicit method; Quasilinearization; Shishkin mesh; Uniform convergence

1. Introduction

In this paper, we examine the one dimensional time dependent Burgers' equation with small coefficient of viscosity. This is a homogeneous quasilinear parabolic parabolic partial differential equation which is encountered in the mathematical modeling of turbulent fluid and shock waves and considered as one of the fundamental model equation in the fluid dynamics to describe the shock waves and traffic flows. Such type of equations can be found in the paper of Batman [1] in 1915. In 1948, Burgers [2] introduced this equation to capture some features of turbulent fluid in a channel caused by the interaction of the opposite effects of convection and diffusion.

The numerical solution of the convection dominating Burgers' equation in one space dimension has been studied extensively. In 1972, Benton and Platzman [3] published a number of distinct solutions to the initial value problems for the Burgers' equation in the infinite domain as well as in the finite domain. In 1996, Ozis and Ozdes [13] applied the variational method, from which they succeeded to generate an approximate solution in the form of sequence solution which converges to the exact solution. In 1997, Mazzia and Mazzia [12] converted Burgers' equation to a system of ordinary differential equations, and then applied the transverse scheme in combination with boundary value methods. In 1996, Mittal and Singhal [11] presented a numerical approximation of the one dimensional Burgers' equations, they truncated one dimensional Fourier expansion with time dependent coefficients and formulated as the approximated problem which consisted of a system of non-linear ordinary differential equations for the coefficients. They also extended their work to study Burgers' equation with periodic boundary conditions.

For small values of the coefficient of viscosity ϵ, one of the major difficulties encountered is due to inviscid boundary layers produced by the steeping effect of the non-linear advection term in Burgers' equation. In case where the coefficient of viscosity $\leqslant 10^{-5}$, the exact solution is not available and a discrepancy exits in the literature see [16].

Here we construct the numerical method for solving the Burgers' equation with small ϵ, i.e., the coefficient of viscosity, whose convergence behavior in the global maximum norm is parameter-uniform, i.e. ϵ-uniform. We first semi discretize the original non-linear Burgers'equation in the temporal direction by backward Euler scheme with the constant time step which produces a set of stationary Burgers' equations. Then use the quasilinearization process [4] to linearize the stationary Burgers' equation obtained from semi discretization and shown that the sequence of solutions of linearized problem converges quadratically to the solution of the original non-linear problem at each time step. For totally discrete scheme, we discretize the set of linear problems resulting from the time semidiscretization using the simple upwind finite difference scheme defined on an appropriate piecewise uniform mesh of Shishkin type. We decompose the global error in two components, which are analyzed separately and obtain parameter uniform error estimates for the totally discrete method.

Throughout the paper we denote C (sometimes subscripted), a generic positive constant independent of ϵ and of the mesh.

2. The continuous problem

Consider the Burgers' equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \epsilon\frac{\partial^2 u}{\partial x^2}, \quad (x,t) \in \Omega \times (0,T], \tag{2.1a}$$

where

$$\Omega = (0,1),$$

With initial condition

$$u(x,0) = f(x) \quad \text{for } 0 \leqslant x \leqslant 1, \tag{2.1b}$$

and boundary conditions

$$u(0,t) = 0 \quad u(1,t) = 0 \quad \text{for } 0 \leqslant t \leqslant T, \tag{2.1c}$$

where $\epsilon > 0$ is the coefficient of kinematic viscosity and the prescribed function $f(x)$ is sufficiently smooth.

We impose the compatibility conditions

$$f(0) = 0, \quad f(1) = 0, \tag{2.2}$$

so that the data matches at the two corners $(0,0)$ and $(1,0)$ of the domain $\overline{\Omega} \times [0,T]$.

These conditions guarantee that there exits a constant C such that for $0 < \epsilon \leqslant 1$ and for all $(x,t) \in \overline{\Omega} \times [0,T]$, (see Bobisud [5]).

$$|u(x,t) - f(x)| \leqslant Ct, \quad t \in (0,T], \tag{2.3}$$

$$|u(x,t) - 0| \leqslant Cx, \quad x \in (0,1), \tag{2.4}$$

Using the properties of norm, Eq. (2.3) is written as

$$|u(x,t) - f(x)| \leqslant |u(x,t) - f(x)| \leqslant Ct, \quad t \in (0,T] \tag{2.5}$$

i.e.

$$|u| \leqslant Ct + |f|$$

since $f(x)$ is sufficiently smooth and x and t also lie in the bounded interval, it is clear with the above relations that the solution u is bounded i.e.

$$|u(x,t)| \leqslant C, \quad (x,t) \in \Omega \times [0,T]. \tag{2.6}$$

Lemma 1. *By keeping x fixed along the line* $\{(x,t):0 \leqslant t \leqslant T\}$ *the bound of* u_t *is given by*

$$|u_t(x,t)| \leqslant C \tag{2.7}$$

Proof. We assume that the solution $u(x,t)$ is sufficiently smooth in the domain $\overline{\Omega} \times [0,T]$ and by mean value theorem there exits a t^* in the interval $(t, t+k)$ along the line $\{(x,t):0 \leqslant t \leqslant T\}$ such that

$$\begin{aligned} u_t(x,t^*) &= \frac{u(x,t+k) - u(x,t)}{k}, \\ |u_t(x,t^*)| &\leqslant \frac{2|u(x,t)|}{k} \end{aligned} \tag{2.8}$$

using Eq. (2.6), we get

$$|u_t(x,t)| \leqslant C. \tag{2.9}$$

Similarly we get the bounds of the $u_{tt}(x,t)$ and $u_{ttt}(x,t)$ along the line, $\{(x,t):0 \leqslant t \leqslant T\}$, *i.e.* we have

$$\left|\frac{\partial^i u(x,t)}{\partial t^i}\right| \leqslant C \quad \text{for } i = 0,1,2,3. \qquad \square \tag{2.10}$$

3. The time semidiscretization

We can write Eq. (2.1) in the following form:

$$u_t = \epsilon u_{xx} - u u_x \ \text{for} (x,t) \in \Omega \times (0,T].$$

Discretizing the time variable by means of Euler implicit method with uniform size Δt, we obtain

$$u_0 = f(x), \tag{3.11a}$$

$$\frac{u_{j+1} - u_j}{\Delta t} = [\epsilon u_{xx} - u u_x]_{j+1}, = [\epsilon u_{j+1}]_{xx} - u_{j+1}[u_{j+1}]_x \quad 0 \leqslant j \leqslant M-1 \tag{3.11b}$$

with the boundary conditions

$$u_{j+1}(0) = 0, \quad u_{j+1}(1) = 0 \quad \text{for } j = 0, 1, \ldots, M-1. \tag{3.11c}$$

where u_{j+i} is the solution of the above differential equation at the $(j+1)$th time step. The differential Eq. (3.11) can be written in the following form:

$$u_0 = f(x), \tag{3.12a}$$

$$-\epsilon\Delta t[u_{j+1}]_{xx} + [1 + \Delta t[u_{j+1}]_x]u_{j+1} = u_j, \quad 0 \leqslant j \leqslant M-1 \tag{3.12b}$$

with the boundary conditions,

$$u_{j+1}(0) = 0, \quad u_{j+1}(1) = 0 \quad \text{for } j = 0, 1, \ldots, M-1. \tag{3.12c}$$

Eq. (3.12) is non-linear ordinary differential equation for each j lying between 0 to $M-1$.

4. Quasilinearization

Using quasilinearization process we linearize the above non-linear ordinary differential equations and followed by simplification yields

$$u_0^{(n+1)} = f(x), \tag{4.13a}$$

$$-\epsilon\Delta t\left[u_{j+1}^{(n+1)}\right]_{xx} + \Delta t(u_{j+1}^{(n+1)})_x u_{j+1}^{(n)} + \left[1 + \Delta t(u_{j+1}^{(n)})_x\right]u_{j+1}^{(n+1)}$$
$$= u_j^{(n+1)} + \Delta t(u_{j+1}^{(n)})_x u_{j+1}^{(n)}, 0 \leqslant j \leqslant M-1, \tag{4.13b}$$

with boundary conditions

$$u_{j+1}^{(n+1)}(0) = 0, \quad u_{j+1}^{(n+1)}(1) = 0. \tag{4.13c}$$

with initial guess $u^{(0)}$ and n is the iteration index.

Using the notation $u^{(n+1)} = U$ the above equation can be written as

$$U_0 = f(x) \tag{4.14a}$$

$$-\epsilon\Delta t(U_{j+1})_{xx} + \Delta t(u_{j+1}^{(n)})(U_{j+1})_x + \left[1 + \Delta t(u_{j+1}^{(n)})\right]U_{j+1}$$
$$= U_j + \Delta t(u_x^{(n)})_{j+1}u_{j+1}^{(n)}. \tag{4.14b}$$

For the simplicity we use the following notations:

$$\left[\frac{1}{\Delta t} + (u_{j+1}^{(n)})_x\right] = a^{(n)}(x),$$

$$u_{j+1}^{(n)} = b^{(n)}(x),$$

$$\left[\frac{U_j}{\Delta t} + \left(u_{j+1}^{(n)}\right)_x (u_{j+1}^{(n)})\right] = F^{(n)}(x).$$

Using the above notations Eq. (4.14) can be written in the following form:

$$U_0 = f(x) \tag{4.15a}$$

$$-\epsilon(U_{j+1}(x))_{xx} + b^{(n)}(x)(U_{j+1}(x))_x + a^{(n)}(x)U_{j+1}(x) = F^{(n)}(x),$$
$$j = 0(1)M-1. \tag{4.15b}$$

We assume that α is the lower bound of the $b^{(n)}(x)$, i.e.

$$b^{(n)}(x) \geqslant \alpha > 0, \quad x \in \overline{\Omega}. \tag{4.16}$$

Writing Eq. (4.15) in the following form:

$$\begin{aligned} &U_0 = f(x), \\ &L_\epsilon U_{j+1}(x) = F^{(n)}(x), \quad 0 \leqslant j \leqslant M-1, \end{aligned} \tag{4.17}$$

where

$$L_\epsilon U_{j+1}(x) \equiv -\epsilon(U_{j+1})_{xx} + b^{(n)}(x)(U_{j+1})_x(x), + a^{(n)}(x)U_{j+1}(x)$$

with boundary conditions

$$U_{j+1}(0) = 0, \quad U_{j+1}(1) = 0.$$

4.1. Convergence of quasilinearization process

For the sake of simplicity we consider the following form of equation:

$$(U_{j+1})_{xx}(x) = H(U_{j+1}) \tag{4.18}$$

with boundary conditions

$$U_{j+1}(0) = 0, \quad U_{j+1}(1) = 0.$$

We assume that $U_{j+1}^{(0)}(x)$ be the initial guess. Consider the sequence obtained by the following recurrence relation:

$$(U_{j+1}^{(n+1)})_{xx} = H(U_{j+1}^{(n)}) + (U_{j+1}^{(n+1)} - U_{j+1}^{(n)})H_{U_{j+1}}(U_{j+1}^{(n)}) \tag{4.19}$$

with boundary conditions

$$U_{j+1}^{(n+1)}(0) = 0, \quad U_{j+1}^{(n+1)}(1) = 0.$$

Subtracting the $(n+1)$th equation from $(n+2)$th equation, we have

$$\left(U_{j+1}^{(n+2)} - U_{j+1}^{(n+1)}\right)_{xx} = H(U_{j+1}^{(n+1)}) - H(U_{j+1}^{(n)}) - (U_{j+1}^{(n+1)} - U_{j+1}^{(n)})H_{U_{j+1}}(U_{j+1}^{(n)}) + H_{U_{j+1}}(U_{j+1}^{(n+1)}) \times (U_{j+1}^{(n+2)} - U_{j+1}^{(n+1)}). \tag{4.20}$$

The above equation is a differential equation for $(U_{j+1}^{(n+2)} - U_{j+1}^{(n+1)})$, and converting into an integral equation, we have

$$(U_{j+1}^{(n+2)} - U_{j+1}^{(n+1)}) = \int_0^1 G(x,s)[H(U_{j+1}^{(n+1)}) - H(U_{j+1}^{(n)}) \times (U_{j+1}^{(n+1)} - U_{j+1}^{(n)})H_{U_{j+1}}(U_{j+1}^{(n)}) + H_{U_{j+1}} \times (U_{j+1}^{(n+1)})(U_{j+1}^{(n+2)} - U_{j+1}^{(n+1)})]\,\mathrm{d}s, \tag{4.21}$$

where the Green's function

$$G(x,s) = \begin{cases} x(s-1), & 0 \leqslant x \leqslant s \leqslant 1, \\ (x-1)s, & 0 \leqslant s \leqslant x \leqslant 1, \end{cases} \qquad \max_{x,s} G(x,s) = \frac{1}{4}. \tag{4.22}$$

The mean-value theorem gives us

$$H(U_{j+1}^{(n+1)}) = H(U_{j+1}^{(n)}) + (U_{j+1}^{(n+1)} - U_{j+1}^{(n)})H_{U_{j+1}}(U_{j+1}^{(n)}) + \frac{(U_{j+1}^{(n+1)} - U_{j+1}^{(n)})^2}{2} H_{U_{j+1}U_{j+1}}(\theta), \tag{4.23}$$

where θ lies between $U_{j+1}^{(n)}$ and $U_{j+1}^{(n+1)}$.

Using Eq. (4.23), Eq. (4.21) become

$$(U^{(n+2)} - U_{j+1}^{(n+1)}) = \int_0^1 G(x,s) \times \left[\frac{(U_{j+1}^{(n+1)} - U_{j+1}^{(n)})^2}{2} H_{U_{j+1}U_{j+1}}(\theta) + H_{U_{j+1}}(U_{j+1}^{(n+1)})(U_{j+1}^{(n+2)} - U_{j+1}^{(n+1)})\right] \mathrm{d}s. \tag{4.24}$$

Define

$$k = \max_{|U_{j+1}| \leqslant 1} |H_{U_{j+1}U_{j+1}}(U_{j+1})| \quad \text{and } m = \max_{|U_{j+1}| \leqslant 1} |H_{U_{j+1}}(U_{j+1})|. \tag{4.25}$$

Now Eq. (4.24) become

$$\left|\left(U_{j+1}^{(n+2)} - U_{j+1}^{(n+1)}\right)\right| \leqslant \frac{1}{4}\int_0^1 \left[\frac{k}{2}\left(U_{j+1}^{(n+1)} - U_{j+1}^{(n)}\right)^2 + m\left|U_{j+1}^{(n+2)} - U_{j+1}^{(n+1)}\right|\right] \mathrm{d}s. \tag{4.26}$$

Taking the maximum over x on both sides of the above inequality and after simplification, we have

$$\max_x \left|(U_{j+1}^{(n+2)} - U_{j+1}^{(n+1)})\right| \leqslant \frac{\frac{k}{8}}{\left(1-\frac{m}{4}\right)} \max_x (U_{j+1}^{(n+1)} - U_{j+1}^{(n)})^2. \tag{4.27}$$

The inequality (4.27) shows that the convergence of quasilinearization process is quadratic.

4.2. Maximum principle [15]

Assume that any function $\Psi \in C^2(\overline{\Omega})$ satisfies $\psi(0) \geqslant 0$ and $\psi(1) \geqslant 0$. Then, $L_\epsilon\psi(x) \geqslant 0$ for all $x \in \Omega$ implies that $\psi(x) \geqslant 0$ for all $x \in \overline{\Omega}$.

Proof. Let x^* be such that $\psi(x*) = \min_{\overline{\Omega}}\psi(x)$ and suppose that $\psi(x^*) < 0$. It is clear that $x^* \notin \{0, 1\}$. Therefore $\psi_x(x^*) = 0, \psi_{xx}(x^*) \geqslant 0$, and

$$L_\epsilon\psi(x^*) = -\epsilon\psi_{xx}(x^*) + b^{(n)}(x^*)\psi_x(x^*) + a^{(n)}(x^*)\psi(x^*), \tag{4.28}$$

since $a^{(n)}(x) \geqslant 0$ for all $x \in \overline{\Omega}$ therefore

$$L_\epsilon\psi(x^*) < 0,$$

which contradicts the assumption, therefore it follows that $\psi(x^*) \geqslant 0$ and thus $\psi(x) \geqslant 0$ for all $x \in \overline{\Omega}$. □

4.3. Local error

The local truncation error is given by the following relation:

$$e_{j+1} = U(t_{j+1}) - U_{j+1},$$

where U_{j+i} is the solution of the semidiscretized Eq. (4.14). With the help of this local error at each time step, we will find the global error of the time discretization which is defined as

$$E_j \equiv U(x, t_j) - U_j(x).$$

$\|\cdot\|_h$ denotes the discrete maximum norm, given by

$$\|x\|_h = \max_{1\leqslant i\leqslant N} \|x_i\|$$

Theorem 1. *If*

$$\left|\frac{\partial^k u(x,t)}{\partial^k t}\right| \leqslant C, (x,t) \in \overline{\Omega} \times [0,T], \quad 0 \leqslant k \leqslant 2 \tag{4.29}$$

then the local and global error estimates satisfy the following estimates

$$\|e_{j+1}\|_h \leqslant C(\Delta t)^2. \tag{4.30}$$

$$\|E_{j+1}\|_h \leqslant C\Delta t \quad \textit{for all } j \leqslant T/\Delta t. \tag{4.31}$$

Therefore, the time discretisation process is uniformly convergent of first-order.

Proof. The function U_{j+1} satisfies

$$L_\epsilon U_{j+1} = F^{(n)}(x). \tag{4.32}$$

Linearize the original problem by quasilinearization process, we have

$$\frac{\partial U}{\partial t} = -\epsilon U_{xx} + u^{(n)} U_x + u_x^{(n)} U + u^{(n)}(u^{(n)})x, \tag{4.33}$$

where

$$U = u^{(n+1)}$$

Since the solution of Eq. (4.33) is smooth enough, it holds

$$U(t_j) = U(t_{j+1}) - \Delta t \frac{\partial U(t_{j+1})}{\partial t} + \int_{t_{j+1}}^{t_j} (t_j - s) \frac{\partial^2 u}{\partial t^2}(s)\,\mathrm{d}s \tag{4.34}$$

Using Eq. (4.33), we have

$$\begin{aligned} U(t_j) = U(t_{j+1}) &- \Delta t\big(\epsilon U_{xx} - u^{(n)} U_x - u_x^{(n)} U - u^{(n)}(u^{(n)})_x\big)(t_{j+1}) \\ &+ \int_{t_{j+1}}^{t_j} (t_j - s) \frac{\partial^2 U(s)}{\partial t^2}\,\mathrm{d}s. \end{aligned} \tag{4.35}$$

Subtracting Eq. (4.14) from Eq. (4.35), we get

$$\Delta t L_\epsilon e_{j+1} = \mathrm{O}(\Delta t)^2 \tag{4.36}$$

Since the operator $(\Delta t L_\epsilon)$ satisfies the maximum principle therefore

$$\|(\Delta t L_\epsilon)^{-1}\|_h \leqslant C_1 \tag{4.37}$$

where C_1 is a positive constant independent on Δt. Using the inequality (4.37) we get

$$\|e_{j+1}\|_h \leqslant C(\Delta t)^2. \tag{4.38}$$

Using the local error estimates up to the jth time step we get the following global error estimate at $(j + 1)$th time step:

$$\begin{aligned}
\|E_{j+1}\|_h &= \left\|\sum_{l=1}^{j} e_l\right\|_h, \quad j \leqslant T/\Delta t, \\
&\leqslant \|e_1\|_h + \|e_2\|_h + \ldots + \|e_j\|_h, \\
&\leqslant C_1(j.\Delta t).\Delta t, \quad \text{using Eq.(4.38)}, \\
&\leqslant C_1 T\Delta t \quad \text{since } j\Delta t \leqslant T, \\
&= C\Delta t,
\end{aligned} \tag{4.39}$$

where C is a positive constant independent of ϵ and Δt.

4.4. Bounds for U and its derivatives

Lemma 2. *Let $U_{j+1}(x)$ be the solution of Eq. (4.15), there exits a constant C such that*

$$|U_{j+1}(x)| \leqslant C \quad \textit{for all } x \in \overline{\Omega}. \tag{4.40}$$

Proof. By the construction of barrier functions we find the bounds of $U_{j+1}(x)$,

$$\psi^{\pm}(x) = M(1+x) \pm U_{j+1}(x), \tag{4.41}$$

where M is a constant chosen sufficiently large such that

$$\psi^{\pm}(0) \geqslant 0, \quad \psi(1) \geqslant 0,$$

$$L_\epsilon \psi^{\pm}(x) = Mb^{(n)}(x) + a^{(n)}(x)(1+x) \pm F^{(n)}(x), \quad x \in \Omega.$$

Since $x \in \Omega$, and $a^{(n)}(x) \geqslant 0$ therefore

$$L_\epsilon \psi^{\pm}(x) \geqslant 0, \quad x \in \Omega. \tag{4.42}$$

Using maximum principle for L_ϵ we have,

$$\begin{aligned}
\psi^{\pm}(x) &\geqslant 0, \quad x \in \overline{\Omega} \\
M(1-x) \pm U_{j+1}(x) &\geqslant 0, \\
|U_{j+1}(x)| &\leqslant M(1-x), \quad x \in \overline{\Omega}
\end{aligned} \tag{4.43}$$

and so

$$|U_{j+1}(x)| \leqslant C \text{ for all } x \in \overline{\Omega}.$$

Theorem 2. *Let U_{j+i} be the the solution of Eq. (4.17), the bounds of its derivatives are given by*

$$|U_{j+1}^{(k)}| \leqslant C(1 + \epsilon^{-k} \mathrm{e}^{-\alpha(1-x)/\epsilon}) \quad \textit{for all } x \in \overline{\Omega}, \quad k = 1, 2, 3. \tag{4.44}$$

Proof. We find the bounds of the derivatives of U_{j+i} with the help of induction.

By the differentiating Eq. (4.15) k times w. r. t. x we have

$$L_\epsilon U_{j+1}^{(k)}(x) = F_k(x) \quad \text{for } 0 \leqslant k \leqslant 3,$$

where

$$F_0 = F$$

$$F_k = F^{(k)} - \sum_{s=0}^{k-1}\left[\binom{k}{s} b^{(k-s)}U^{(s+1)} + \binom{k}{s} a^{(k-s)}U^{(s)}\right] \quad \text{for } 1 \leqslant k \leqslant 3.$$

Assume that for all l, $0 \leqslant l \leqslant k$, the following estimates hold:

$$|U_{j+1}^{(l)}(x)| \leqslant C\left(1 + \epsilon^{-l} e^{\frac{-\alpha(1-x)}{\epsilon}}\right), \quad x \in \overline{\Omega},$$

where α is the lower bound of $b^{(n)}(x)$.

From the above assumption we have

$$L_\epsilon U_{j+1}^{(k)} = F_k,$$

where

$$|U_{j+1}^{(k)}(x)| \leqslant C\left(1 + \epsilon^{-k} e^{\frac{-\alpha(1-x)}{\epsilon}}\right),$$
$$F_k(x) \leqslant C\left(1 + \epsilon^{-k} e^{\frac{-\alpha(1-x)}{\epsilon}}\right).$$

And at the boundaries we have

$$|U_{j+1}^{(k)}(0)| \leqslant C\left(1 + \epsilon^{-k} e^{\frac{-\alpha}{\epsilon}}\right), \leqslant C(1 + \epsilon^{-(k-1)}) \text{ since } e^{\frac{-\alpha}{\epsilon}} \leqslant 1 \leqslant \epsilon^{-1}, \tag{4.45}$$

$$|U_{j+1}^{(k)}(1)| \leqslant C(1 + \epsilon^{-k}). \tag{4.46}$$

Since $\epsilon \leqslant 1$, this implies that

$$\epsilon^{-(k-1)} \geqslant 1 \text{ for } 1 \leqslant k \leqslant 3.$$

Using the above fact in Eqs. (4.45) and (4.46),

$$|U_{j+1}^{(k)}(0)| \leqslant C\epsilon^{-(k-1)}, \quad |U_{j+1}^{(k)}(1)| \leqslant C\epsilon^{-k}.$$

Defining

$$\theta_k(x) = \frac{1}{\epsilon}\int_x^1 F_k(s) e^{-(A(x)-A(s))}\, ds,$$

where

$$A(x) = \int_x^1 b^{(n)}(s)\, ds.$$

The particular solution of

$$L_\epsilon U_{j+1}^{(k)} = F_k,$$

is given by

$$(U_{j+1})_p(x) = -\int_x^1 \theta_k(s)\,\mathrm{d}s.$$

Its general solution can be written in the form of

$$U_{j+1}^{(k)} = (U_{j+1})_p^{(k)} + (U_{j+1})_h^{(k)},$$

where the homogeneous solution $(U_{j+1})_h^{(k)}$ satisfies

$$L_\epsilon (U_{j+1})_h^{(k)} = 0,$$

$$(U_{j+1})_h^{(k)}(0) = U_{j+1}^{(k)}(0) - (U_{j+1})_p^{(k)}(0),$$

$$(U_{j+1})_h^{(k)}(1) = U_{j+1}^{(k)}(1).$$

Introducing the function

$$\phi(x) = \frac{\int_x^1 \mathrm{e}^{-A(s)/\epsilon}\,\mathrm{d}s}{\int_0^1 \mathrm{e}^{-A(s)/\epsilon}\,\mathrm{d}s},$$

we have, $L_\epsilon\phi = 0, \phi(0) = 1, \phi(1) = 0$ and $0 \leqslant \phi(x) \leqslant 1$.

Now $(U_{j+1})_h^k$ can be written as

$$(U_{j+1})_h^{(k)}(x) = U_{j+1}^{(k)}(0) - (U_{j+1})_p^{(k)}(0)\phi(x) + U_{j+1}^{(k)}(1 - \phi(x)) \tag{4.47}$$

The above leads the following expression for $U_{j+1}^{(k+1)}$:

$$\begin{aligned} U_{j+1}^{(k+1)} &= (U_{j+1})_p^{(k+1)} + (U_{j+1})_h^{(k+1)} \\ &= \theta_k(x) + (U_{j+1}(0) - (U_{j+1})_p^{(k)}(0))\phi_x(x) + U_{j+1}^{(k)}(1)(-\phi_x(x)). \end{aligned}$$

Since

$$\phi_x(x) = \frac{-\mathrm{e}^{-A(x)/\epsilon}}{\int_0^1 \mathrm{e}^{A(s)/\epsilon}\,\mathrm{d}s},$$

the upper and lower bound of $b^{(n)}(x)$ lead the estimate

$$|\phi_x(x)| \leqslant C\epsilon^{-1}\mathrm{e}^{-\alpha(1-x)/\epsilon}.$$

Furthermore

$$|\theta_k(x)| \leqslant C\epsilon^{-1}\int_x^1 (1 + \epsilon^{-k}\mathrm{e}^{-\alpha(1-s)/\epsilon})\mathrm{e}^{-\alpha(s-x)/\epsilon}\,\mathrm{d}s.$$

Evaluating the above integral exactly and estimating the terms in the resulting expression gives

$$|\theta_{(k)}(x)| \leqslant C(1+\epsilon^{-(k+1)}\mathrm{e}^{-\alpha(1-x)/\epsilon}). \tag{4.48}$$

Since

$$(U_{j+1})_p^{(k)}(0) = -\int_0^1 \theta_k(s)\,\mathrm{d}s,$$

Using Eq. (4.48) in the above expression and followed by a simplification gives

$$\begin{aligned}|(U_{j+1})_p^{(k)}(0)| &\leqslant \int_0^1 |\theta_k(s)|\,\mathrm{d}s, \leqslant C\int_0^1 (1+\epsilon^{-(k+1)}\mathrm{e}^{-\alpha(1-s)/\epsilon})\,\mathrm{d}s,\\ &= C\left(1+\frac{\epsilon^{-k}}{\alpha}(1-\mathrm{e}^{-\alpha/\epsilon})\right),\\ &\leqslant C\left(1+\frac{\epsilon^{-k}}{\alpha}\right), \quad \text{since } (1-\mathrm{e}^{-\alpha/\epsilon}) \leqslant 1,\end{aligned}$$

Since $\epsilon \leqslant 1$, this implies that $\epsilon^{-k} \geqslant 1$, $1 \leqslant k \leqslant 3$, using this fact in the above expression we have

$$|(U_{j+1})_p^{(k)}(0)| \leqslant C\epsilon^{-k}, \quad |(U_{j+1})_p^{(k)}(1)| = 0. \tag{4.49}$$

Differentiating Eq. (4.47) we have

$$|U_{j+1}^{(k+1)}| \leqslant |\theta_{(k)}| + \left(|U_{j+1}^{(k)}(0)| + |(U_{j+1})_p^{(k)}(0)| + |U_{j+1}^{(k)}(1)|\right)|\theta_x|. \tag{4.50}$$

Using the estimates of Eq. (4.49) yields

$$|U_{j+1}^{(k+1)}| \leqslant C(1+\epsilon^{-(k+1)}\mathrm{e}^{-\alpha(1-x)/\epsilon}). \tag{4.51}$$

5. The spatial discretization

5.1. Shishkin mesh

Shishkin meshes are piecewise-uniform meshes which condense approximately in the boundary layer regions as $\epsilon \to 0$; this is accomplished by the use of transition parameter τ, which depend naturally on ϵ, crucially on N.

Thus for a given N and ϵ, the interval $[0,1]$, is divided into parts, $[0, 1-\tau]$, $[1-\tau, 1]$ where the transition point τ is given by

$$\tau \equiv \min\left\{\frac{1}{2}, C\epsilon \log N\right\}$$

the value of the constant C depends on which scheme is used.

Define

$$h_i = \begin{cases} \frac{2(1-\tau)}{N}, & \text{if } i = 1, 2, \ldots, \frac{N}{2}, \\ \frac{2\tau}{N}, & \text{if } i = \frac{N}{2}+1, \ldots, N. \end{cases} \tag{5.52}$$

where N is the no discretization points and the set of mesh points $\Omega^N = \{x_i\}_{i=0}^N$ with

$$x_i = \begin{cases} 2(1-\tau)N^{-1}i, & \text{if } i = 0, 2, \ldots, \frac{N}{2}, \\ 1-\tau+2\tau N^{-1}(i-\frac{N}{2}) & \text{if } i = \frac{N}{2}+1, \ldots, N. \end{cases} \tag{5.53}$$

Define

$$D_x^- U_{i,j} = \frac{U_{i,j} - U_{i-1,j}}{h_i},$$

$$\delta_x^2 U_{i,j} = \frac{1}{h_i}\left[\frac{U_{i+1,j} - U_{i,j}}{h_{i+1}} - \frac{U_{i,j} - U_{i-1,j}}{h_i}\right],$$

$$\bar{h}_i = (h_i + h_{i+1})/2.$$

Discretizing Eq. (4.15) by using standard upwind finite difference operator on the piecewise uniform mesh $\overline{\Omega}^N = \{x_i\}_{i=0}^N$ with $x_N = 1$, we get

$$U_{i,0} = f(x_i), \quad x_i \in \Omega^N, \tag{5.54a}$$

$$-\epsilon\delta_x^2 U_{i,j+1} + b_i^{(n)} D_x^- U_{i,j+1} + a_i^{(n)} U_{i,j+1} = F^{(n)}(x_i), \quad x_i \in \Omega^N \tag{5.54b}$$

with boundary conditions

$$U_{0,j+1} = 0, \quad U_{N,j+1} = 0. \tag{5.54c}$$

The finite difference operator in Eq. (5.54) defined by

$$L_\epsilon^N \equiv -\epsilon\delta_x^2 + b_i^{(n)} D_x^- + a_i^{(n)} I. \tag{5.55}$$

5.2. Discrete maximum principle

Assume that the mesh function ψ_i satisfies $\psi_0 \geqslant 0$ and $\psi_N \geqslant 0$. Then $L_\epsilon^N \psi_i \geqslant 0$ for $1 \leqslant i \leqslant N-1$ implies that $\psi_i \geqslant 0$ for all $0 \leqslant i \leqslant N$.

Proof. Let there exits an integer k such that $\psi_k = \min_i \psi_i$ and suppose $\psi_k < 0$. We have $\psi_0 \geqslant 0$ and, $\psi_N \geqslant 0$ therefore $k \notin \{0, N\}$.

Now we have

$$\psi_{k+1} - \psi_k \geqslant 0, \quad \psi_k - \psi_{k-1} < 0.$$

Therefore

$$
\begin{aligned}
L_\epsilon^N \psi_k &= -\epsilon \delta_x^2(\psi_k) + b^{(n)}(x_k) D_x^- \psi_k + a^{(n)}(x_k)\psi_k \\
&= -\frac{\epsilon}{\bar{h}_k}\left[\frac{\psi_{k+1}-\psi_k}{h_{k+1}} - \frac{\psi_k - \psi_{k-1}}{h_k}\right] + b^{(n)}(x_k)\left[\frac{\psi_k - \psi_{k-1}}{h_k}\right] + a^{(n)}(x_k)\psi_k \\
&< 0,
\end{aligned}
$$

which contradicts the hypothesis, thus we have

$$\psi_i \geqslant 0 \quad \text{for all } 0 \leqslant i \leqslant N. \qquad \square$$

6. Error estimates and convergence analysis

To derive ϵ-uniform error estimates, we need sharper bounds on the derivatives of the solution U. We derive these using the following decomposition of the solution into smooth and singular components

$$U_{j+1}(x) = V_{j+1}(x) + W_{j+1}(x),$$

where V_{j+1} can be written in the form

$$V_{j+1} = V_0 + \epsilon V_1 + \epsilon^2 V_2$$

and V_0, V_1 and V_2 are defined, respectively, to be the solutions of the problems

$$b^{(n)}(x)V_0 + a^{(n)}(x)(V_0)_x = F^{(n)}(x), \quad V_0(0) = U_{j+1}(0), \tag{6.56}$$

$$b^{(n)}(x)V_1 + a^{(n)}(x)(V_1)_x = -(V_0)_{xx}, \quad V_1(0) = 0, \tag{6.57}$$

$$-\epsilon(V_2)_{xx} + b^{(n)}(x)V_2 + a^{(n)}(x)(V_2)_x = -(V_1)_{xx}, \quad V_2(0) = 0, \quad V_2(1) = 0. \tag{6.58}$$

thus the smooth component V_{j+1} is the solution of

$$L_\epsilon V_{j+1} = F^{(n)}(x), \quad V_{j+1}(0) = V_0(0) + \epsilon V_1(0), \quad V_{j+1}(1) = U_{j+1}(1) \tag{6.59}$$

and consequently the singular component W_{j+1} is the solution of the homogeneous problem

$$L_\epsilon W_{j+1} = 0, \quad W_{j+1}(0) = 0, \quad W_{j+1}(1) = U_{j+1}(1) - V_{j+1}(1). \tag{6.60}$$

Lemma 3. *The bounds of V_{j+1}, W_{j+1} and their derivatives are given as follows*:

$$\|V_{j+1}^{(k)}\| \leqslant C(1 + \epsilon^{(2-k)}), \quad k = 0, 1, 2, 3, \tag{6.61a}$$

$$\|W_{j+1}(x)\| \leqslant C\mathrm{e}^{-\alpha(1-x)/\epsilon} \quad \textit{for all } x \in \overline{\Omega}, \tag{6.61b}$$

$$\|W_{j+1}^{(k)}(x)\| = C\epsilon^{-k}, \quad k = 1, 2, 3, \tag{6.61c}$$

where $\|\cdot\|$ *denotes the maximum norm taken over the appropriate domain of the independent variable*, *given by*

$$\|f\| = \max_{x\in\overline{\Omega}} |f(x)|.$$

Proof. Since V_0 is the solution of the reduced problem which is the first order linear differential Eq. (6.56), with bounded coefficients i.e. $b^{(n)}(x)$, $a^{(n)}(x)$ and $F^{(n)}(x)$, therefore V_0 is the bounded and the bound is independent of the ϵ.

Using the bounds of V_0, $a^{(n)}$ and $F^{(n)}$ in Eq. (6.56) we get $(V_0)_x$ is bounded and independent of ϵ.

Differentiating Eq. (6.56) w.r.t. x we get the second order differential equation, by using the bounds of V_0, $(V_0)x$, $b_x^{(n)}$, $a_x^{(n)}$ and $F_x^{(n)}$ we get the estimate for $(V_0)_{xx}$.

Similarly twice differentiating of Eq. (6.56) gives the third order differential equation, by using the bounds of V_0, $(V_0)_x$ and $(V_0)_{xx}$ we get the estimate for $(V_0)_{xxx}$.

Thus we have

$$V_0^{(k)} \leqslant C_1, \quad k = 0, 1, 2, 3, \tag{6.62}$$

where C_1 is a positive constant independent on the ϵ.

In the same fashion we get the bounds on V_1 and its derivatives $V_1^{(k)}$ for $k = 1, 2, 3$. Thus we have

$$V_1^{(k)} \leqslant C_2, \quad k = 0, 1, 2, 3, \tag{6.63}$$

where C_2 is the positive constant independent on the ϵ.

Since V_2 is the solution of Eq. (6.58), which is similar to Eq. (4.15) therefore the estimates for the bounds on V_2 and its derivatives are given by

$$V_2^{(k)} \leqslant C_3\epsilon^{-k}\mathrm{e}^{-\alpha(1-x)/\epsilon}, \quad k = 0, 1, 2, 3, \tag{6.64}$$

where C_3 is positive constant independent on ϵ.

$$V_{j+1}^{(k)} = V_0^{(k)} + V_1^{(k)} + V_2^{(k)} \tag{6.65}$$

with the help of Eqs. (6.62), (6.63) and (6.64) the above equation can be written as

$$V_{j+1}^{(k)} \leqslant C(1 + \epsilon^{-k}\mathrm{e}^{-\alpha(1-x)/\epsilon}), \quad k = 0, 1, 2, 3. \tag{6.66}$$

To obtain the required bounds on W_{j+1} and its derivatives, we consider the barrier functions defined as

$$\psi^{\pm}(x) = C\mathrm{e}^{-\alpha(1-x)/\epsilon} \pm W_{j+1}(x). \tag{6.67}$$

Now we have

$$\begin{aligned}\psi^{\pm}(0) &= C\mathrm{e}^{-\alpha/\epsilon} \pm W_{j+1}(0),\\ &= C\mathrm{e}^{-\alpha/\epsilon}, \quad \text{since } W_{j+1}(0)=0,\\ &\geqslant 0,\end{aligned}$$

$$\psi^{\pm}(1) = C \pm W_{j+1}(1). \tag{6.68}$$

We can choose the value C such that

$$\psi^{\pm}(1) = C \pm W_{j+1}(1) \geqslant 0, \tag{6.69}$$

$$\begin{aligned}L_\epsilon\psi^{\pm}(x) &= -\epsilon\psi^{\pm}_{xx}(x) + b^{(n)}(x)\psi^{\pm}_{x}(x) + a^{(n)}(x)\psi^{\pm}(x),\\ &= -\frac{\alpha^2}{\epsilon}\mathrm{e}^{-\alpha(1-x)/\epsilon} + b^{(n)}(x)\frac{\alpha}{\epsilon}\mathrm{e}^{-\alpha(1-x)/\epsilon} + a^{(n)}(x)\mathrm{e}^{-\alpha(1-x)/\epsilon},\\ &= \frac{\alpha}{\epsilon}[-\alpha + b^{(n)}(x)]\mathrm{e}^{-\alpha(1-x)/\epsilon} + a^{(n)}(x)\mathrm{e}^{-\alpha(1-x)/\epsilon},\end{aligned} \tag{6.70}$$

since α is the minimum of $b^{(n)}(x)$, therefore

$$-\alpha + b^{(n)}(x) \geqslant 0$$

and $a^{(n)}(x)$ is non-negative, thus we have

$$L_\epsilon\psi^{\pm}(x) \geqslant 0.$$

Since L_ϵ satisfies the maximum principle and we have shown that $L_\epsilon\psi^{\pm}(x) \geqslant 0$, therefore

$$\psi^{\pm}(x) = C\mathrm{e}^{-\alpha(1-x)/\epsilon} \pm W_{j+1}(x) \geqslant 0 \quad \text{for all } x \in \overline{\Omega}$$

or

$$\|W_{j+1}(x)\| \leqslant C\mathrm{e}^{-\alpha(1-x)/\epsilon} \quad \text{for all } x \in \overline{\Omega}$$

$$\begin{aligned}\left|\int_0^x (b(W_{j+1})_x)(s)\,\mathrm{d}s\right| &= \left|(b^{(n)}_x W_{j+1})_0^x - \int_0^x (b^{(n)}_x (W_{j+1})_x)(s)\,\mathrm{d}s\right|,\\ &= \left|b^{(n)}_x W_{j+1}(x) - \int_0^x (b^{(n)}_x (W_{j+1})_x)(s)\,\mathrm{d}s\right|\\ &\leqslant C\mathrm{e}^{-\alpha(1-x)/\epsilon}.\end{aligned} \tag{6.71}$$

By mean-value theorem in the interval $(0,\epsilon)$ there exist $z \in (0,\epsilon)$ such that

$$\begin{aligned}|W_{j+1}(z)| &= \left|\frac{W_{j+1}(\epsilon) - W_{j+1}(0)}{\epsilon}\right|,\\ &= \left|\frac{W_{j+1}(\epsilon)}{\epsilon}\right|\\ &\leqslant C\epsilon^{-1}\mathrm{e}^{-\alpha(1-\epsilon)/\epsilon}\\ &\leqslant C\epsilon^{-1}.\end{aligned} \tag{6.72}$$

Integrating Eq. (6.60) w.r.t. x from 0 to x we have

$$-\epsilon(W_{j+1})_x(x)+\epsilon(W_{j+1})_x(0)+\int_0^x b^{(n)}(W_{j+1})_x(s)\,\mathrm{d}s + \int_0^x a^{(n)}W_{j+1}(s)\,\mathrm{d}s=0, \tag{6.73}$$

the above equation gives

$$|\epsilon(W_{j+1})_x(0)|=\left|\epsilon(W_{j+1})_x(z)-\int_0^z b^{(n)}(W_{j+1})_x(s)\,\mathrm{d}s-\int_0^z a^{(n)}W_{j+1}(s)\,\mathrm{d}s\right|, \tag{6.74}$$

making use of Eqs. (6.71) and (6.72) we have

$$|\epsilon(W_{j+1})_x(0)|\leqslant C\mathrm{e}^{-\alpha(1-\epsilon)/\epsilon}\leqslant C. \tag{6.75}$$

Now Eq. (6.73) gives the following estimate with the help of the estimates give by (6.71) and (6.75)

$$|(W_{j+1}(x))_x|\leqslant C\epsilon^{-1},\quad x\in\overline{\Omega}. \tag{6.76}$$

using the above estimate in Eq. (6.60) we get the estimate for $(W_{j+1}(x))_{xx}$ *i.e.*

$$|(W_{j+1})_{xx}|\leqslant C\epsilon^{-2}. \tag{6.77}$$

Similarly by differentiating Eq. (6.60) w.r.t. x and using the estimates given by Eqs. (6.76) and (6.77) we get the estimate for $(W_{j+1})_{xxx}$. Thus we have

$$|W_{j+1}^{(k)}|\leqslant C\epsilon^{-k}\quad\text{for } k=1,2,3.\qquad\square \tag{6.78}$$

Theorem 3. *Let $U_{j+1}(x_i)$ be the solution of Eq. (4.15) and $U_{i,j+1}$ is the solution of the discrete problem (5.54) at the point x_i, at the $(j+1)$th time label, there is a constant C such that*

$$\|U_{i,j+1}-U_{j+1}(x_i)\|_h\leqslant CN^{-1}(\log N)^2\quad \textit{for } i=1,2,\ldots,N,$$

where $\|\cdot\|_h$ denotes the discrete maximum norm.

Proof. The solution $U_{i,j+1}$ of the discrete problem (5.54) at the ith point on the $(j+1)$th time level is decomposed as follows:

$$U_{i,j+1}=V_{i,j+1}+W_{i,j+1},$$

where $V_{i,j+1}$ is the solution of the in non-homogeneous problem,

$$\begin{aligned}&L_\epsilon^N V_{i,j+1}=F^{(n)},\quad 1\leqslant i\leqslant N-1,\\&V_{0,j+1}=V_{j+1}(0),\quad V_{N,j+1}=V_{j+1}(1)\end{aligned} \tag{6.79}$$

and $W_{i,j+1}$ is the solution of the homogeneous problem

$$\begin{aligned} &L_\epsilon^N W_{i,j+1} = 0, \quad 1 \leqslant i \leqslant N-1, \\ &W_{0,j+1} = W_{j+1}(0), \quad W_{N,j+1} = W_{j+1}(1). \end{aligned} \tag{6.80}$$

The error estimate can be written in the following form:

$$U_{i,j+1} - U_{j+1}(x_i) = (V_{i,j+1} - V_{j+1}(x_i)) + (W_{i,j+1} - W_{j+1}(x_i)), \quad x_i \in \overline{\Omega}^N.$$

The error of smooth and singular components can be estimated separately. □

Estimate the error of smooth component as follows:

$$\begin{aligned} L_\epsilon^N(V_{i,j+1} - V_{j+1}(x_i)) &= F^{(n)} - L_\epsilon^N(V_{j+1}(x_i)), \quad x_i \in \overline{\Omega}^N \\ &= (L_\epsilon - L_\epsilon^N)V_{j+1}(x_i) \\ &= -\epsilon\left(\frac{\partial^2}{\partial x^2} - \delta_x^2\right)V_{j+1}(x_i) + b^{(n)}(x_i)\left(\frac{\partial}{\partial x} - D_x^-\right)V_{j+1}(x_i). \end{aligned} \tag{6.81}$$

Let $x_i \in \overline{\Omega}^N$. Then for any function $\psi \in C^2(\overline{\Omega})$

$$\left|\left(D_x^- - \frac{\partial}{\partial x}\right)\psi(x_i)\right| \leqslant (x_i - x_{i-1})\|\psi^{(2)}\|/2$$

and for any function $\psi \in C^3(\overline{\Omega})$

$$\left|\left(\delta_x^2 - \frac{\partial^2}{\partial x^2}\right)\psi(x_i)\right| \leqslant (x_{i+1} - x_{i-1})\|\psi^{(3)}\|/3$$

for the proof of the above results are given in Lemma 4.1 [p. 21], of [10].

Using these results in Eq. (6.81) followed by simplification gives

$$|L_\epsilon^N(V_{i,j+1} - V_{j+1}(x_i))| \leqslant C(x_{i+1} - x_{i-1})\left(\epsilon|V_{j+1}^{(3)}| + |V_{j+1}^{(2)}|\right). \tag{6.82}$$

Since $x_{i+1} - x_{i-1} \leqslant 2N^{-1}$ and using Lemma 3 for the estimates of $V_{j+1}^{(3)}$ and $V_{j+1}^{(2)}$ then we have

$$|L_\epsilon^N(V_{i,j+1} - V_{j+1}(x_i))| \leqslant CN^{-1} \tag{6.83}$$

and using Lemma 3 [p. 60] of [10] we have

$$|(V_{i,j+1} - V_{j+1}(x_i))| \leqslant CN^{-1}. \tag{6.84}$$

The estimation of the singular component of the error $(W_{i,j+1} - W_{j+1}(x_i))$, depends on the argument whether the value of transition parameter $\tau = 1/2$ or $\tau = (C\epsilon \log N)/\alpha$.

Case (i) ($C\epsilon \log N \geqslant 1/2$, i.e., when the mesh is uniform).

The classical argument used in the estimation of the smooth component of the error leads to

$$|L_\epsilon^N(W_{i,j+1} - W_{j+1}(x_i))| = C(x_{i+1} - x_{i-1})\left(\epsilon|W_{j+1}^{(3)}| + |W_{j+1}^{(2)}|\right)$$

Since $(x_{i+1} - x_{i-1}) \leqslant 2N^{-1}$, using Lemma 3 for the estimates of bounds of $W_{j+1}^{(3)}$, $W_{j+1}^{(2)}$ lead to

$$|L_\epsilon^N(W_{i,j+1} - W_{j+1}(x_i))| \leqslant C\epsilon^{-2}N^{-1}.$$

In this case $\epsilon^{-1} \leqslant 2C\log N$. Using the above inequality we obtained

$$|L_\epsilon^N(W_{i,j+1} - W_{j+1}(x_i))| \leqslant CN^{-1}(\log N)^2.$$

Using the Lemma 3 [p. 60] of [10] we get

$$|(W_{i,j+1} - W_{j+1}(x_i))| \leqslant CN^{-1}(\log N)^2.$$

Case (ii) ($C\epsilon\log N < 1/2$, i.e., when the mesh is piecewise uniform with the mesh spacing $2(1-\tau)/N$ in the subinterval $[0, 1-\tau)$. and $2\tau/N$ in the subinterval $[1-\tau, 1]$). Since by the Lemma 3 we have

$$|W_{j+1}(x_i)| \leqslant Ce^{-\alpha(1-x_i)/\epsilon}, \quad 1 \leqslant i \leqslant N/2$$

Since $\exp\{-\alpha(1-x)\}$ is a increasing function for x in $[0, 1-\tau]$. Using this fact in the above inequality we have

$$|W_{j+1}(x_i)| \leqslant |W_{j+1}(1-\tau), \quad 1 \leqslant i \leqslant N/2$$

But in this case we have $\tau = Ce\log N$. Using this value of τ in the above inequality we get

$$|W_{j+1}(x_i)| \leqslant CN^{-1}, \quad x_i, \quad 1 \leqslant i \leqslant N/2 \tag{6.85}$$

To establish the similar bound on $W_{i,j+1}$, we construct a mesh function $\widehat{W}_{i,j+1}$ with the help of Lemma 5 [p. 53] of [10] we get

$$|W_{i,j+1}| \leqslant |\widehat{W}_{i,j+1}|, \quad 0 \leqslant i \leqslant N. \tag{6.86}$$

where $\widehat{W}_{i,j+1}$ is the solution of the difference equation

$$-\epsilon\delta_x^2\widehat{W}_{i,j+1} + \alpha D_x^-\widehat{W}_{i,j+1} + a^{(n)}(x_i)\widehat{W}_{i,j+1} = F^{(n)}(x_i). \tag{6.87}$$

By Lemma 3 [p. 51] of [10] we have

$$|\widehat{W}_{i,j+1}| \leqslant CN^{-1}, \quad 0 \leqslant i \leqslant N/2. \tag{6.88}$$

Combining the inequalities (6.86) and (6.88) we have

$$|W_{i,j+1}| \leqslant CN^{-1}, \quad 0 \leqslant i \leqslant N/2. \tag{6.89}$$

Using the inequalities (6.85) and (6.89) we get

$$|(W_{i,j+1} - W_{j+1}(x_i))| \leqslant CN^{-1}, \quad 0 \leqslant i \leqslant N/2. \tag{6.90}$$

In the second subinterval $[1-\tau, 1]$, using the classical argument yields the following estimate of local truncation error for $N/2+1 \leqslant i \leqslant N-1$:

$$|L_\epsilon^N(W_{i,j+1} - W_{i,j+1}(x_i))| \leqslant C\epsilon^{-2}|(x_{i+1} - x_{i-1})|. \tag{6.91}$$

Since $(x_{i+1} - x_{i-1}) = 4\tau/N$, using this fact in the above inequality we get

$$|L_\epsilon^N(W_{i,j+1} - W_{j+1}(x_i))| \leqslant 4C\epsilon^{-2}\tau N^{-1}. \tag{6.92}$$

Also we have

$$|W_{N,j+1} - W_{j+1}(1)| = 0, \tag{6.93}$$

$$|W_{N/2,j+1} - W_{j+1}(x_{N/2})| \leqslant |W_{N/2,j+1}| + |W_{j+1}(x_{N/2})| \tag{6.94}$$

Using inequalities (6.85) and (6.89)in the above inequality we get

$$|W_{N/2,j+1} - W_{j+1}(x_{N/2})| \leqslant CN^{-1}. \tag{6.95}$$

Now introducing the barrier function

$$\Phi_i = (x_i - (1-\tau))C_1\epsilon^{-2}\tau N^{-1} + C_2N^{-1}, \quad \frac{N}{2} \leqslant i \leqslant N.$$

For the suitable choice of C_1 and C_2, the mesh functions

$$\psi_i^{\pm} = \Phi_i \pm (W_{i,j+1} - W_{j+1}(x_i)), \quad \frac{N}{2} \leqslant i \leqslant N. \tag{6.96}$$

$$\Phi_{N/2} = (x_{N/2} - (1-\tau))C_1\epsilon^{-2}\tau N^{-1} + C_2N^{-1} = C_2N^{-1}, \text{ since } x_{N/2} = 1-\tau \tag{6.97}$$

$$\begin{aligned}
\psi_{N/2}^{\pm} &= \Phi_{N/2} \pm (W_{N/2,j+1} - W_{j+1}(x_{N/2})) \\
&\geqslant C_2N^{-1} \pm (\mp CN^{-1}) \quad \text{using (6.97) and (6.95),} \\
&= (C_2 - C)N^{-1}, \text{ choose } C_2 \geqslant C \\
&\geqslant C_3N^{-1}, C_3 = (C_2 - C) \quad \text{a positive constant} \\
&\geqslant 0
\end{aligned}$$

$$\psi_N^{\pm} = \Phi_N \pm (W_{N,j+1} - W_{j+1}(x_N)), \tag{6.98}$$

where

$$\begin{aligned}
\Phi_N &= (x_N(1-\tau))C_1\epsilon^{-2}\tau N^{-1} + C_2N^{-1} \\
&= C_1\epsilon^{-2}\tau^2N^{-1} + C_2N^{-1}, \text{ since } x_N = 1 \\
&\geqslant 0
\end{aligned} \tag{6.99}$$

using the inequalities (6.93) and (6.99), in the inequality (6.98) we have

$$\psi_N^\pm \geqslant 0, \tag{6.100}$$

$$\begin{aligned} L_\epsilon^N \psi_i^\pm &= L_\epsilon^N(\Phi_i \pm W_{i,j+1} W_{j+1}(x_i)), \quad \frac{N}{2}+1 \leqslant i \leqslant N-1, \text{ by (6.96)} \\ &= L_\epsilon^N \Phi_i \pm L_\epsilon^N(W_{i,j+1} - W_{j+1}(x_i)) \\ &\geqslant \alpha(x_i - \tau) C_1 \epsilon^{-2} \tau N^{-1} + C_2 N^{-1} \pm (\mp 4C\epsilon^{-2}\tau N^{-1}), \\ &= (\alpha(x_i-\tau)C_1 - 4C)\epsilon^{-2}\tau N^{-1} + C_2 N^{-1} \\ &\geqslant 0, \end{aligned} \tag{6.101}$$

since $x_i \geqslant \tau$ and choose C_1 very large such that $(\alpha(x_i - \tau)C_1 - 4C) \geqslant 0$. Now we have

$$\begin{aligned} &\psi_{N/2}^\pm \geqslant 0, \quad \psi_N^\pm \geqslant 0, \\ &L_\epsilon^N \psi_i^\pm \geqslant 0, \quad N/2+1 \leqslant i \leqslant N-1. \end{aligned}$$

Using discrete maximum principle for L_ϵ^N on the interval $[1 - \tau,1]$ gives

$$\psi_i^\pm \geqslant 0, \quad N/2 \leqslant i \leqslant N \tag{6.102}$$

Using the inequality (6.96)in the above inequality we have

$$\Phi_i \pm (W_{i,j+1} - W_{j+1}(x_i)) \geqslant 0, N/2 \leqslant i \leqslant N \tag{6.103}$$

The above inequality implies that

$$|(W_{i,j+1} - W_{j+1}(x_i)| \leqslant \Phi_i \leqslant C_1\epsilon^{-2}\tau^2 N^{-1} + C_2 N^{-1}. \tag{6.104}$$

Since $\tau = C\epsilon\log N$, using this fact in the above inequality, we have

$$\begin{aligned} |(W_{i,j+1} - W_{j+1}(x_i))| &\leqslant C_1(\log N)^2 N^{-1} + C_2 N^{-1} \\ &\leqslant (C_1 + C_2)(\log N)^2 N^{-1}, \log N \geqslant 1. \end{aligned} \tag{6.105}$$

Combining the error estimates in the subintervals $[0, 1 - \tau]$ and $[1 - \tau, 1]$ as given by the inequalities (6.90) and (6.105), we have

$$|(W_{i,j+1} - W_{j+1}(x_i))| \leqslant C(\log N)^2 N^{-1}, \quad N/2 \leqslant i \leqslant N \tag{6.106}$$

Since

$$|U_{i,j+1} - U_{j+1}(x_i)| \leqslant |V_{i,j+1} - V_{j+1}(x_i)| + |W_{i,j+1} - W_{j+1}(x_i)|. \tag{6.107}$$

Using the inequalities (6.84) and (6.106) we get

$$|(U_{i,j+1} - U_{j+1}(x_i))| \leqslant CN^{-1}(\log N)^2. \tag{6.108}$$

The uniform convergence of the totally discrete scheme is obtained as follows.

Theorem 4. *Let* $U(x_i, t_{j+1})$ *be the solution of the linearized problem (4.33) of Eq. (2.1),* $U_{j+1}(x_i)$ *be the solution of the differential* Eq. (4.15) *and the* $U_{i,j+1}$ *be the solution of the totally discrete Eq. (5.54). By Theorems* 1 *and* 3, *there exist a constant C such that*

$$\|U(x_i, t_{j+1}) - U_{i,j+1}\|_h \leqslant C(\Delta t + N^{-1}(\log N)^2).$$

Proof. The proof of the above Theorem is just the consequence of the Theorems 1 and 3,

$$\begin{aligned}\|U(x_i, t_{j+1}) - U_{i,j+1}\|_h &\leqslant \|U(x_i, t_{j+1}) - U_{j+1}(x_i)\|_h + \|U_{j+1}(x_i) - U_{i,j+1}\|_h \\ &\leqslant C_1\Delta t + C_2 N^{-1}(\log N)^2 \text{ using Theorems 1 and 3} \\ &\leqslant C(\Delta t + N^{-1}(\log N)^2) \quad \text{where } C = \max\{C_1, C_2\}.\end{aligned} \tag{6.109}$$

7. Numerical results

In this section to illustrate the performance of the proposed method, we consider the some test examples and compare the computed results with the exact solution for the considered examples.

Example 1

$$u_t + uu_x = u_{xx}, \quad (x, t) \in (0, 1) \times (0, T] \tag{7.110a}$$

with initial condition

$$u(x, 0) = \sin(\pi x), \quad 0 \leqslant t \leqslant T \tag{7.110b}$$

and boundary conditions

$$u(0, t) = 0, \quad u(1, t) = 0, \quad 0 \leqslant x \leqslant 1. \tag{7.110c}$$

The numerical computations were done by using the uniform mesh. For the comparison we compute the analytical and numerical solution at some mesh points for the given time step, $\Delta t = 0.01$. Tables 1 and 2 give the numerical and exact values of the solution u for $\epsilon = 1$ and 0.1. The results by the proposed method are in good agreement with exact solution.

In Figs. 1 and 3, numerical results with uniform mesh are shown for Example 1 at different times for $\Delta x = 0.02$, $\Delta t = 0.001$, $\epsilon = 1.0$ and $\Delta x = 0.02$, $\Delta t = 0.001$, $\epsilon = 0.1$. These numerical predictions exhibit good physical behaviour.

Table 1
Comparison with exact solution for Example 1 with $\epsilon = 1.0$, $T = 0.1$ and time step $\Delta t = 0.01$

x	Computed solution for different values of N				Exact solution
	8	16	32	64	
0.000	0.000000	0.000000	0.000000	0.000000	0.000000
0.125	0.121487	0.125104	0.127346	0.128578	0.135829
0.250	0.225959	0.233046	0.237412	0.239809	0.253638
0.375	0.298627	0.308467	0.314521	0.317851	0.336742
0.500	0.328236	0.339392	0.346287	0.350090	0.371577
0.625	0.308567	0.319105	0.325680	0.329320	0.350123
0.750	0.240241	0.248160	0.253236	0.256060	0.272582
0.875	0.131624	0.135849	0.138571	0.140092	0.149239
1.000	0.000000	0.000000	0.000000	0.000000	0.000000

Table 2
Comparison with exact solution for Example 1 with $\epsilon = 0.1$, $T = 0.1$ and time step $\Delta t = 0.01$

x	Computed solution for different values of N				Exact solution
	8	16	32	64	
0.000	0.000000	0.000000	0.000000	0.000000	0.000000
0.125	0.257761	0.265351	0.269246	0.271187	0.278023
0.250	0.484196	0.504004	0.514352	0.519585	0.534143
0.375	0.657091	0.692249	0.711395	0.721350	0.743852
0.500	0.754675	0.804818	0.833364	0.848611	0.877280
0.625	0.754136	0.813231	0.848258	0.867467	0.897099
0.750	0.633137	0.687764	0.721028	0.739558	0.761797
0.875	0.376760	0.408496	0.427982	0.438783	0.447836
1.000	0.000000	0.000000	0.000000	0.000000	0.000000

The point wise errors by using non-uniform mesh of Shishkin type is shown in Table 3. We show the error tables with $N = 8$ and $\Delta t = 0.01$ and we multiply N and Δt remains same.

We compute point wise errors by

$$\mathrm{e}_\epsilon^N(i,j) = |u^N(x_i,t_j) - u^{2N}(x_i,t_j)|,$$

where superscript indicates the number of mesh points used in the spatial direction, and $t_j = j\Delta t$ and Δt is the time step.

For each ϵ, the maximum nodal error is given by

$$E_{\epsilon,N,\Delta t} = \max_{i,j} e_\epsilon^{N,\Delta t}(i,j)$$

and for each N and Δt, the the ϵ uniform maximum error is defined as

$$E_{N,\Delta t} = \max_\epsilon E_{\epsilon,N,\Delta t}.$$

The results are shown in Table 3.

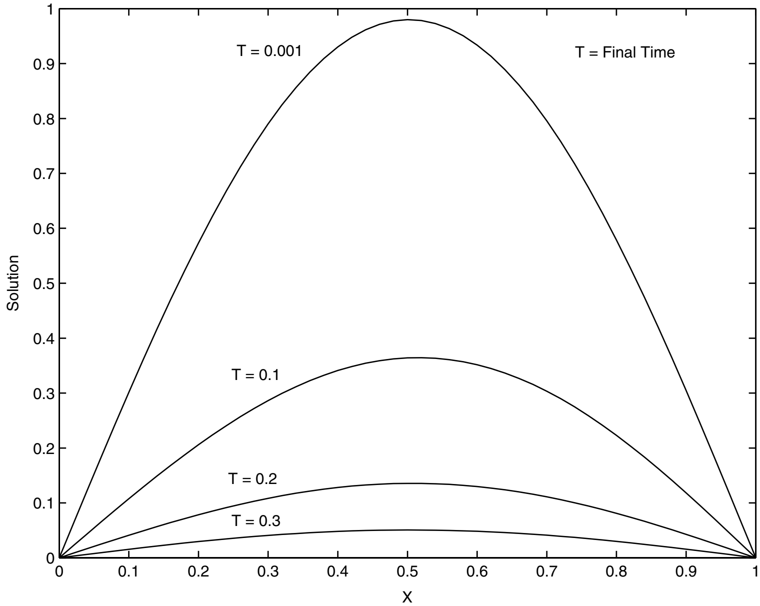

Fig. 1. Numerical results at different times for Example 1 for $\epsilon = 1.0$, $\Delta x = 0.02$ and $\Delta t = 0.001$.

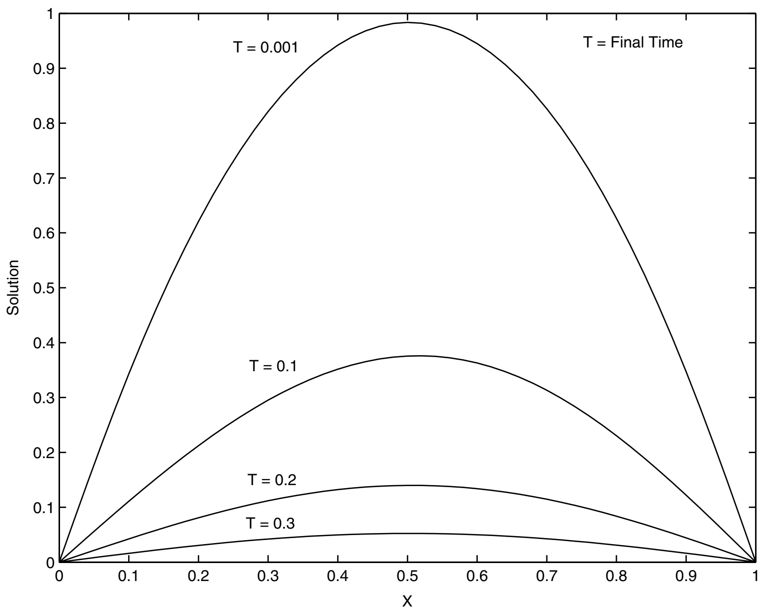

Fig. 2. Numerical results at different times for Example 2 for $\epsilon = 1.0$, $\Delta x = 0.02$ and $\Delta t = 0.001$.

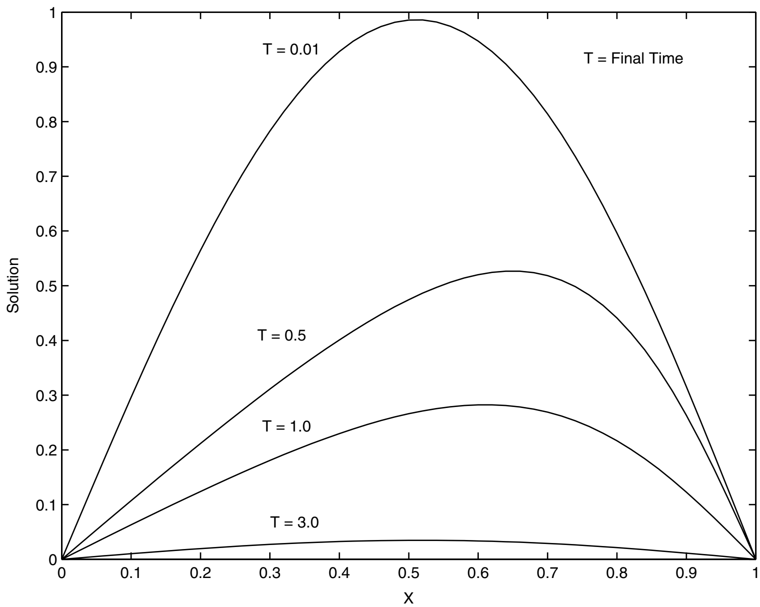

Fig. 3. Numerical results at different times for Example 1 for $\epsilon = 0.1$, $\Delta x = 0.02$ and $\Delta t = 0.001$.

Table 3
Maximum nodal errors for Example 1

ϵ	N				
	8	16	32	64	128
2^{-1}	0.126549	0.070806	0.036909	0.018840	0.009518
2^{-2}	0.123205	0.079446	0.050444	0.029792	0.015093
2^{-3}	0.319113	0.177713	0.084627	0.036491	0.015093
2^{-4}	0.460880	0.282160	0.149309	0.074455	0.035970
2^{-5}	0.542130	0.344630	0.185675	0.098233	0.049515
2^{-6}	0.582309	0.379252	0.211558	0.113234	0.058150
2^{-7}	0.599244	0.397179	0.226394	0.119052	0.060987
2^{-8}	0.604842	0.405218	0.234203	0.126925	0.065276
2^{-9}	0.605793	0.407901	0.238113	0.131401	0.067698
2^{-10}	0.605253	0.408168	0.239992	0.133786	0.069086
2^{-11}	0.606854	0.411069	0.240708	0.134967	0.069745
$E_{N,\Delta t}$	**0.606854**	**0.411069**	**0.240708**	**0.134967**	**0.069745**

Example 2

$$u_t + uu_x = u_{xx}, \quad (x,t) \in (0,1) \times (0,T] \tag{7.111a}$$

with initial condition

$$u(x,0) = 4x(1-x), \quad 0 \leqslant x \leqslant 1 \tag{7.111b}$$

and boundary conditions

$$u(0,t) = 0, \quad u(1,t) = 0, \quad 0 \leqslant t \leqslant T. \tag{7.111c}$$

Table 4
Maximum nodal error for Example 2

ϵ	N				
	8	16	32	64	128
2^{-1}	0.068049	0.038832	0.019986	0.010134	0.009518
2^{-2}	0.098964	0.055833	0.032670	0.019375	0.015093
2^{-3}	0.265789	0.142863	0.067190	0.028797	0.016726
2^{-4}	0.410171	0.245469	0.129438	0.064377	0.035970
2^{-5}	0.507933	0.319100	0.173349	0.091323	0.049515
2^{-6}	0.565204	0.369275	0.205817	0.112251	0.058150
2^{-7}	0.608845	0.420887	0.242355	0.140985	0.068855
2^{-8}	0.621333	0.439466	0.261480	0.145203	0.076823
2^{-9}	0.623685	0.444969	0.269322	0.152896	0.082167
2^{-10}	0.622498	0.444803	0.271893	0.157022	0.085812
2^{-11}	0.620764	0.443056	0.271884	0.159089	0.088103
$E_{N,\Delta t}$	**0.623685**	**0.444969**	**0.271893**	**0.159089**	**0.088103**

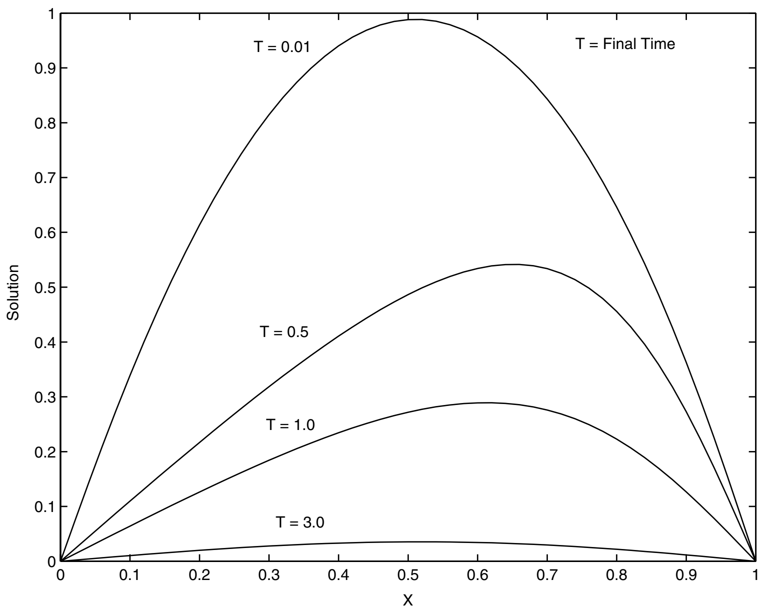

Fig. 4. Numerical results at different times for the Example 2 for $\epsilon = 0.1$, $\Delta x = 0.02$ and $\Delta t = 0.001$.

The points wise errors with non-uniform mesh of Shishkin type is given in Table 4 with various values of ϵ and N.

In Figs. 2 and 4, numerical results with uniform mesh are shown for Example 2 at different times for $\Delta x = 0.02$, $\Delta t = 0.001$, $\epsilon = 1.0$ and $\Delta x = 0.02$, $\Delta t = 0.001$, $\epsilon = 0.1$. These numerical predictions exhibit good physical behaviour.

8. Conclusions

In this paper, we propose a numerical scheme for solving time dependent Burgers' equation for low viscosity coefficient, i.e. for high Reynold's number. The Burgers' equation is a non-linear parabolic partial differential equation. To tackle the non-linearity, quasilinearization is used. The sequence of solutions of the linear equations obtained after applying quasilinearization is shown to converge quadratically to the solution of the original non-linear problem. An extensive amount of analysis has been carried out to establish the parameter uniform error estimates for time as well as spatial discretization.

In support of predicted theory some test examples are solved using the presented method. The computed solution is compared with exact solution for high viscosity coefficient. Since for low viscosity coefficient or high Reynold's number, the exact solution of the Burgers' equation [16] is not available, so to illustrate the performance of the proposed method for low viscosity coefficient, the maximum error is calculated in Tables 3 and 4 using half mesh principle. The error Tables 3 and 4 show that the method converges independently of perturbation parameter. The solution is also plotted with uniform meshes for $\epsilon = 1.0$ and $\epsilon = 0.1$ in Figs. 1–4.

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