Problem 1

(a) For $x = 0$, $P(X = x) = P(Z = 0) + P(Z = 1) \times P(Y = 0) = (1 - p) + pe^{-\lambda}$
For $x = 1, 2, 3, \ldots$, $P(X = x) = P(Z = 1) \times P(Y = x) = p \times \frac{e^{-\lambda}\lambda^x}{x!}$
Therefore, $P(X = x) = \begin{cases} 1 - p + pe^{-\lambda} & x = 0 \\ p \times \frac{e^{-\lambda}\lambda^x}{x!} & x \in \mathbb{N} \\ 0 & otherwise \end{cases}$

(b) For $x = 0$, $P(\tilde{X} = x) = P(1 - I = 0) + P(1 - I = 1) \times P(Y = 0) = P(I = 1) + P(I = 0) \times P(Y = 0) = (1 - p) + pe^{-\lambda}$
For $x = 1, 2, 3, \ldots$, $P(\tilde{X} = x) = P(1 - I = 1) \times P(Y = x) = P(I = 0) \times P(Y = x) = p \times \frac{e^{-\lambda}\lambda^x}{x!}$
Therefore, $P(\tilde{X} = x) = \begin{cases} 1 - p + pe^{-\lambda} & x = 0 \\ p \times \frac{e^{-\lambda}\lambda^x}{x!} & x \in \mathbb{N} \\ 0 & otherwise \end{cases}$, which is the same as $P(X = x)$.

(c) Let $p_U(u) = P(U = u)$, $p_V(v) = P(V = v)$. Since U and V are independent, $p_{UV}(u, v) = p_U(u) \times p_V(v)$. Therefore, $E[UV] = \sum_{u \in S_U} \sum_{v \in S_v} uv \times p_{UV}(u, v) = \sum_{u \in S_U} \sum_{v \in S_V} uv \times p_U(u) \times p_V(v) = \sum_{u \in S_U} u \times p_U(u) \times \sum_{v \in S_V} v \times p_V(v) = E[U] \times E[V]$.

Problem 2

(a) The MGF of a Poisson random variable is $e^{\lambda(e^t - 1)}$, so we can know $M_{3X}(t) = M_X(3t) = e^{\lambda(e^{3t} - 1)}$, $M_{4Y}(t) = M_Y(4t) = e^{\lambda(e^{4t} - 1)}$. We also know that $M_{3X+4Y}(t) = M_{3X}(t) \times M_{4Y}(t) = e^{\lambda(e^{3t} - 1)} \times e^{\lambda(e^{4t} - 1)} = e^{\lambda(e^{3t} + e^{4t} - 2)}$, which is not the MGF of a Poisson random variable. Therefore, $3X + 4Y$ is not a Poisson random variable.

(b) $M_Y(t) = \sum_{k=1}^{\infty} e^{tk} \cdot \frac{6}{\pi^2 k^2} = \frac{6}{\pi^2} \sum_{k=1}^{\infty} \frac{e^{tk}}{k^2}$. For $t > 0$, the term $\frac{e^{tk}}{k^2}$ grows exponentially, which makes the series $\sum_{k=1}^{\infty} \frac{e^{tk}}{k^2}$ diverges. Therefore, $M_Y(t)$ does not exist since it is not finite for any $t > 0$.

Problem 3

(a) Marginal CDF of X: $F_X(t) = \lim_{u \to \infty} F_{XY}(t, u) = \lim_{u \to \infty} (1 - e^{-t} - e^{-u} + e^{-(t+u+\theta tu)}) = 1 - e^{-t}$
Marginal CDF of Y: $F_Y(t) = \lim_{t \to \infty} F_{XY}(t, u) = \lim_{t \to \infty} (1 - e^{-t} - e^{-u} + e^{-(t+u+\theta tu)}) = 1 - e^{-u}$
If X and Y are independent, $F_{XY}(t, u) = F_X(t) \times F_Y(u) = (1 - e^{-t}) \times (1 - e^{-u}) = 1 - e^{-t} - e^{-u} + e^{-(t+u)}$, which means $e^{-(t+u+\theta tu)} = e^{-(t+u)}$. Therefore, X and Y are independent only if $\theta = 0$.

(b) $\frac{\partial F_{X,Y}(t,u)}{\partial t} = \frac{\partial}{\partial t}(1 - e^{-t} - e^{-u} + e^{-(t+u+\theta tu)}) = e^{-t} + (1 + \theta u)e^{-(t+u+\theta tu)}$
Joint PDF of X and Y: $f_{X,Y}(t, u) = \frac{\partial}{\partial u}(e^{-t} + (1 + \theta u)e^{-(t+u+\theta tu)}) = \theta e^{-(t+u+\theta tu)} + (1 + \theta u)(1 + \theta t)e^{-(t+u+\theta tu)} = [1 + \theta(1 + u + t) + \theta^2 ut]e^{-(t+u+\theta tu)}$

Problem 4

(a) Since $R(T_a) > R(T_b)$, we can know that each labeler votes for T_a with probability $p = \sigma(R(T_a) - R(T_b)) > 0.5$. We can define the indicator random variable X_i for the i-th labeler, where $X_i = 1$ if the labeler votes for T_a and $X_i = 0$ if the labeler votes for T_b. Therefore, $X_i \sim Bernoulli(p)$.
Define the sample mean $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$, then the event E that T_a gets more votes than T_b is equivalent to $E = \{\bar{X} > \frac{1}{2}\}$. Using Hoeffding's inequality for bounded independent random variables, $P(|\bar{X}-\mu| \geq \epsilon) \leq 2e^{-\frac{2n\epsilon^2}{(b-a)^2}}$, where $a = 0$ and $b = 1$ in this case, so $P(E) = P(\bar{X} > \frac{1}{2}) = 1-P(\bar{X} \leq \frac{1}{2}) = 1 - P(\bar{X} - \mu \leq -(\mu - 0.5)) \geq 1 - e^{-2n(\mu-0.5)^2}$. Therefore, $P(E) \geq 1 - e^{-2n(p-\frac{1}{2})^2}$.

(b) 1. We should ensure that $P(E) = P(\bar{X} > \frac{1}{2}) \geq 1 - \delta$, which means $P(\bar{X} \leq \frac{1}{2}) \leq \delta$. Applying Chebyshev's inequality, $P(\bar{X} \leq \frac{1}{2}) = P(|\bar{X} - \mu| \geq \mu - \frac{1}{2}) \leq \frac{p(1-p)}{n(\mu-\frac{1}{2})^2}$. Therefore, we require $\frac{p(1-p)}{n(\mu-\frac{1}{2})^2} \leq \delta$, which means $n \geq \frac{p(1-p)}{\delta(p-\frac{1}{2})^2}$.
2. $P(E) \geq 1 - e^{-2n(p-\frac{1}{2})^2} \geq 1 - \delta$. This requires $e^{-2n(p-\frac{1}{2})^2} \leq \delta$, which can be simplified to $2n(p - \frac{1}{2})^2 \geq \ln(\frac{1}{\delta})$. Therefore, $n \geq \frac{1}{2(p-\frac{1}{2})^2} \ln(\frac{1}{\delta})$.

(c) We need to find an upper bound on $P(S > 90)$, where $S = \sum_{j=1}^{100} I_j$. For each I_j,
$\mathbb{E}[e^{tI_j}] = (1 - p_j)e^{t\cdot 0} + p_j e^t = 1 + p_j(e^t - 1)$. Since I_j are independent, for any $t > 0$, we have $P(S \geq a) \leq e^{-ta} \cdot M_X(t) = e^{-ta}\prod_{j=2}^{100}(1 + p_j(e^t - 1))$, where $a = 91$ since $S > 90$. Using the inequality $1 + x \leq e^x$, we can simplify the expression to $e^{-ta}\prod_{j=2}^{100} e^{p_j(e^t-1)} = e^{-ta} \cdot e^{\sum_{j=2}^{100} p_j(e^t-1)}$. Now we have $P(S \geq a) \leq e^{-ta+\mu(e^t-1)}$, where $\mu = \sum_{j=2}^{100} p_j$. To find the tightest bound, set $\phi(t) = -ta + \mu(e^t - 1)$, then $\phi'(t) = -a + \mu e^t$. Set $\phi'(t) = 0$, we can find the optimal $t = \ln\frac{a}{\mu}$. Therefore, $P(S \geq a) \leq e^{-a\ln\frac{a}{\mu}+a-\mu} = e^{-91\ln\frac{91}{\mu}+91-\mu}$.

Problem 5

(a)
1. Since N is the smallest interget such that $S_N > 1$, therefore $N = n$ occurs when $S_{n-1} < 1$ and $S_n > 1$. So $P(N = n) = P(S_{n-1} < 1 \text{ and } S_n > 1) = P(S_{n-1} < 1) - P(S_n \leq 1)$. Since $P(S_n = 1) = 0$, $P(N = n) = P(S_{n-1} < 1) - P(S_n < 1)$.
2. Proof $P(S_k < v) = \frac{v^k}{k!}$ by induction:
   Base case: $P(S_1 < 1) = 1 = \frac{1^1}{1!}$
   Inductive step: Assume $n = k$, $P(S_k < 1) = \frac{v^k}{k!}$. Then for $n = k+1$, $P(S_{k+1} < 1) = \int_{u_{k+1}=0}^{1} P(S_k < 1 - u_{k+1})\, du_{k+1}$. Let $v = 1 - u_{k+1}$, then $P(S_{k+1} < 1) = \int_{v=0}^{1} P(S_k < v)\, dv = \int_{v=0}^{1} \frac{v^k}{k!}\, dv = \frac{1}{k!}\int_0^1 v^k\, dv = \frac{1}{k!}\left[\frac{v^{k+1}}{k+1}\right]_{v=0}^{1} = \frac{1}{k!} \cdot \frac{1}{k+1} = \frac{1}{(k+1)!}$.
   Therefore, $P(S_k < 1) = \frac{1^k}{k!} = \frac{1}{k!}$.
3. $P(N = n) = P(S_{n-1} < 1) - P(S_n < 1) = \frac{1}{(n-1)!} - \frac{1}{n!} = \frac{n}{n!} - \frac{1}{n!} = \frac{n-1}{n!}$.
   $E[N] = \sum_{n=1}^{\infty} n \times P(N = n) = \sum_{n=1}^{\infty} \frac{n(n-1)}{n!} = \frac{1 \cdot 0}{1!} + \sum_{n=2}^{\infty} \frac{1}{(n-2)!} = \sum_{m=0}^{\infty} \frac{1}{m!} = e$

(b) Procedure of simulation:
1. Define the number of trials N_{trials} of each simulation run.
2. Generate a sequence of i.i.d. $Uniform(0, 1)$ random variables $U_1, U_2, \ldots, U_{N_{trials}}$.
3. For each $i = 1, 2, \ldots, N_{trials}$, record the value n such that $S_{n-1} < 1$ and $S_n > 1$.
4. After all trials, compute the average of all recorded N_i values. This average is an estimate of e.

```
All Estimates of Euler's Number e:
Trials:       10, Estimated e: 2.8000000000
Trials:     1000, Estimated e: 2.7240000000
Trials:   100000, Estimated e: 2.7153000000
Trials: 10000000, Estimated e: 2.7183091000
```

Problem 6

(a) Observations:

1. For lower ϵ values ($\epsilon = 0.01$), the performance of Epsilon Greedy is closer to Empirical Means.
2. For moderate ϵ values ($\epsilon = 0.03$), a balance between exploration and exploitation is achieved.
3. For higher ϵ values ($\epsilon = 0.1$), Epsilon Greedy outperforms Empirical Means in environments with sparse reward distributions (e.g., Environment 2).
4. Larger ϵ values ($\epsilon = 0.3$) result in higher regrets due to excessive exploration.

	$\epsilon = 0.01$	$\epsilon = 0.03$	$\epsilon = 0.1$	$\epsilon = 0.3$
Environment 0	0.048430	0.041721	0.040679	0.040719
Environment 1	0.018111	0.008061	0.009311	0.007883
Environment 2	0.040201	0.039511	0.042640	0.042674

Mean Regrets for Empirical Means

	$\epsilon = 0.01$	$\epsilon = 0.03$	$\epsilon = 0.1$	$\epsilon = 0.3$
Environment 0	0.059363	0.030785	0.044694	0.121840
Environment 1	0.031996	0.022477	0.039259	0.103480
Environment 2	0.046880	0.027960	0.015047	0.022627

Mean Regrets for Epsilon Greedy

(b) Observations:

1. Smaller α values ($\alpha = 0.1$) result in slower decay of exploration, leading to excessive regret in some environments.
2. Moderate α values ($\alpha = 0.5$) achieve the best trade-off, minimizing regret effectively across all environments.
3. Larger α values ($\alpha = 2.0$) lead to insufficient exploration, causing suboptimal performance.
4. Sparse reward environments (e.g., Environment 2) are more sensitive to the choice of α, as insufficient exploration can quickly result in higher regret.

	$\alpha = 0.1$	$\alpha = 0.5$	$\alpha = 1.0$	$\alpha = 2.0$
Environment 0	0.044749	0.024617	0.048363	0.036975
Environment 1	0.010909	0.005492	0.009362	0.009503
Environment 2	0.043087	0.042800	0.046280	0.044781

Mean Regrets for Empirical Means

	$\alpha = 0.1$	$\alpha = 0.5$	$\alpha = 1.0$	$\alpha = 2.0$
Environment 0	0.177120	0.015667	0.113410	0.417050
Environment 1	0.149890	0.011907	0.030241	0.381130
Environment 2	0.031097	0.021433	0.078094	0.087898

Mean Regrets for Epsilon Greedy