(Fall 2024) 515512 Probability **(Early Bird: 2023/12/15, 9pm; Normal: 2023/12/16, 9pm)**

Homework 4: LLNs, CLT, and Conditional Expectation

Submission Guidelines: Please compile all your write-ups into one .pdf file (photos/scanned copies are acceptable; please make sure that the electronic files are of good quality and reader-friendly). Please submit your deliverable via E3.

Problem 1 (Convergence in Probability) (12+12=24 points)

In this problem, we take a closer look at the concept of convergence in probability. To begin with, we introduce another relevant concept about convergence: A sequence of random variables $X_1, X_2, \cdots$ is said to converge to a number c **in the mean square**, if

$$\lim_{n\to\infty} \mathbb{E}[(X_n - c)^2] = 0.$$

(a) Show that convergence in the mean square implies convergence in probability. (Hint: For every $\varepsilon > 0$, consider $P(|X_n - c| \geq \varepsilon)$ and use Markov's inequality)

(b) Please construct an example that shows that "convergence in probability" does NOT imply "convergence in the mean square." (Hint: You may consider a random variable X which is 0 with probability $1 - \frac{1}{n}$ and is a large value with probability $\frac{1}{n}$)

Problem 2 (Weak Law of Large Numbers) (12 points)

Consider two sequences of random variables $X_1, X_2, \cdots$ and $Y_1, Y_2, \cdots$ defined on the same sample space. Suppose that X_n converges to a and Y_n converges to b in probability. Please show that the sequence $\{(X_n \cdot Y_n)\}_{n=1}^{\infty}$ converges to ab in probability. (Hint: You need to find the connection between $P(\{\omega : |(X_n(\omega) \cdot Y_n(\omega)) - ab| \geq \epsilon\})$ and $P(\{\omega : |X_n(\omega) - a| \geq \epsilon'\})$ and $P(\{\omega : |Y_n(\omega) - b| \geq \epsilon'\})$ under some proper ϵ')

Problem 3 (Strong Laws of Large Numbers) (8+14=22 points)

Remark: This problem is contributed by Nai-Chieh Huang, a former TA of this course.

Frieren had spent all her pocket money, so she decided to bet on her luck at gambling. Macht found this utterly amusing and, with a cunning smile, handed her 1 gold coin to gamble with, saying, "Let's see if you can turn everything to gold!" Accordingly, Frieren joined a magical gambling game with the following rules:

- **All-In Every Round:** At each n-th round, Frieren must wager her entire fortune (all the gold coins she has at that point).
- **Magical Multipliers:** The outcome of the round is determined by a magical multiplicative process, X_n, which is a sequence of i.i.d. random variables across all $n \geq 1$. The process evolves based on the following properties:
  1. The expected return per gold coin is exactly 1, i.e., $\mathbb{E}[X_n] = 1$.
  2. There is inherent variability in the process, meaning the result is not always 1, i.e., $\mathbb{P}[X_n = 1] < 1$.
  3. This gambling game guarantees non-zero outcomes, i.e., $\mathbb{P}[X_n > 0] = 1$.
  4. Moreover, $\mathbb{E}[|\ln X_n|] < \infty$, for all $n \geq 1$.

Accoridngly, at the end of the n-th round, Frieren's total wealth, measured in gold coins, is given by:

$$C_n = \prod_{k=1}^{n} X_n.$$

Moreover, $C_0 = 1$ is the initial wealth. Now, we want to analyze Frieren's wealth in this magical game:

(a) Show that Frieren's total wealth at each round, $\{C_n\}_{n\geq 1}$, satisfies that $\mathbb{E}[C_{n+1}|C_n] = C_n$. A sequence of random variables that satisfies this property is also known as a *martingale*.

(b) Prove that, with probability 1, Frieren will eventually lose all her gold coins, leaving Macht laughing at her. In other word, show that $C_n \to 0$ almost surely as $n \to \infty$. (Hint: Consider $\ln C_n$ and apply the Strong Law of Large Numbers)

Problem 4 (Central Limit Theorem) (10+10+12=32 points)

In this problem, we would like to get a deeper understanding about the Central Limit Theorem (CLT). Let $X_1, X_2 \cdots, X_n$ be a sequence of independent Bernoulli random variables with mean $p \in [0, 1]$. Define $S_n := X_1 + \cdots + X_n$.

(a) Suppose we would like to obtain an approximate distribution of S_n for large n. Then, by CLT, what could we say about the approximate distribution of S_n? (Hint: This is exactly what we learned about CLT approximation in Lecture 22)

(b) While CLT approximation is useful in practice, such approximation could be poor in some cases. Suppose now we consider $p = \lambda/n$, where $\lambda \in (0, 1)$. For $x \in \mathbb{N}$, define $Q_n(x) := P(X_1 + \cdots + X_n = x)$. Show that

$$\lim_{n\to\infty} Q_n(x) = \frac{e^{-\lambda}\lambda^x}{x!},$$

which is a Poisson distribution with rate λ and time window 1. (Hint: Directly write down the PMF of Binomial random variable and take the limit. This is something that we have also discussed in Lecture 8)

(c) Could you explain why the result in (b) does not contradict CLT? Moreover, in what way does the result in (b) show that CLT could indeed lead to a poor approximation in some cases?

Problem 5 (Conditional Expectation) (10+10=20 points)

Consider a group of n roommate pairs at a college (so there are $2n$ students). Each of these $2n$ students independently decides randomly whether to take a Reinforcement Learning (RL) course, with probability p of success (where "success" is defined as taking the course). Let M be the number of students among these $2n$ who take this RL course, and let X be the number of roommate pairs where both roommates in the pair take the course. Please find out the following two quantities:

(a) Please find out $E[X|M = m]$, for all possible m.

(b) Based on (a), please calculate $E[X]$.