Problem 1

(a) By Markov's inequality, $0 \leq P(|X_n = c| \geq \varepsilon) = P((X_n = c)^2 \geq \varepsilon^2) \leq \frac{\mathbb{E}[(X_n-c)^2]}{\varepsilon^2}$. Take the limit as $n \to \infty$, we have $0 \leq \lim_{n\to\infty}((X_n = c)^2 \geq \varepsilon^2) \leq \lim_{n\to\infty}(\frac{\mathbb{E}[(X_n-c)^2]}{\varepsilon^2}) = 0$. Therefore, by the squeeze theorem, $\lim_{n\to\infty} P(|X_n = c| \geq \varepsilon) = 0$, which implies convergence in probability.

(b) Consider the random variable

$$X_n = \begin{cases} 1 - \frac{1}{n} & \text{if } x = 0 \\ \frac{1}{n} & \text{if } x = n \\ 0 & \text{otherwise} \end{cases}$$

Then this random variable converges in probability since $0 \leq \lim_{n\to\infty}(P(|Y_n - 0| > \varepsilon)) \leq \lim_{n\to\infty}(\frac{1}{n}) = 0$. However, $\mathbb{E}[(X_n - 0)^2] = \mathbb{E}[x_n^2] = 0^2 \cdot (1 - \frac{1}{n}) + n^2 \cdot \frac{1}{n} = n$, which implies $\lim_{n\to\infty} \mathbb{E}[x_n^2] = \infty$. Therefore, convergence in probability does not imply convergence in the mean square.

Problem 2

$|X_nY_n - ab| = |X_nY_n - aY_n + aY_n - ab| = |Y_n||X_n - a| + |a||Y_n - b|$. By triangle inequality, we know that $|X_nY_n - ab| \leq |Y_n||X_n - a| + |a||Y_n - b|$. To have $|X_nY_n - ab| < \varepsilon$, we can set $\varepsilon_1 = \frac{\varepsilon}{2(|b|+1)}$ and $\varepsilon_2 = \frac{\varepsilon}{2|a|}$, so that $|X_n - a| < \varepsilon_1$ and $|Y_n - b| < \varepsilon_2$, which ensures $|X_n - a|(|b| + 1) < \frac{\varepsilon}{2}$ and $|Y_n - b||a| < \frac{\varepsilon}{2}$. Using the above bounds, we have

$$P\left(|X_nY_n - ab| \geq \varepsilon\right) \leq P\left(|X_n - a| \geq \frac{\varepsilon}{2(|b|+1)}\right) + P\left(|Y_n - b| \geq \frac{\varepsilon}{2|a|}\right)$$

Since $X_n \xrightarrow{P} a$ and $Y_n \xrightarrow{P} b$, for any $\delta > 0$, $\lim_{n\to\infty} P\left(|X_n - a| \geq \delta\right) = 0$ and $\lim_{n\to\infty} P\left(|Y_n - b| \geq \delta\right) = 0$. By choosing $\delta = \frac{\varepsilon}{2(|b|+1)}$ and $\delta = \frac{\varepsilon}{2|a|}$ respectively, we ensure

$$\lim_{n\to\infty} P\left(|X_n - a| \geq \frac{\varepsilon}{2(|b|+1)}\right) = 0 \quad \text{and} \quad \lim_{n\to\infty} P\left(|Y_n - b| \geq \frac{\varepsilon}{2|a|}\right) = 0$$

Therefore, $\lim_{n\to\infty} P\left(|X_nY_n - ab| \geq \varepsilon\right) \leq 0 + 0 = 0$.

Problem 3

(a) By definition, $C_{n+1} = C_n \cdot X_{n+1}$, then $\mathbb{E}[C_{n+1}|C_n] = \mathbb{E}[C_n \cdot X_{n+1}|C_n] = C_n \cdot \mathbb{E}[X_{n+1}|C_n]$. Since the random variable x_{n+1} is independent of C_n, we can know that $\mathbb{E}[X_{n+1}|C_n] = \mathbb{E}[X_{n+1}] = 1$. Therefore, $\mathbb{E}[C_{n+1}|C_n] = C_n \cdot 1 = C_n$.

(b) Since $C_n = \prod_{k=1}^n X_k$, we have $\ln C_n = \sum_{k=1}^n \ln X_k$. We can define $Y_k = \ln X_k$, then $\ln C_n = \sum_{k=1}^n Y_k$. Since the logarithm is a concave function, we can apply Jensen's inequality to get $\mathbb{E}[\ln X_k] \leq \ln \mathbb{E}[X_k] = \ln 1 = 0$, hence $\mathbb{E}[Y_k] \leq 0$. Given $P(X_k = 1) < 1$, X_k is not almost surely 1. Therefore, $\mathbb{E}[\ln X_k] < \ln \mathbb{E}[X_k] = 0$, so $\mu := \mathbb{E}[Y_k] = \mathbb{E}[\ln X_k] < 0$. Applying SLLN we have $\frac{1}{n} \ln C_n = \frac{1}{n}\sum_{k=1}^n Y_k \xrightarrow{a.s.} \mu < 0 \quad$ as $n \to \infty$. Therefore, $\ln C_n \xrightarrow{a.s.} n\mu \quad$ as $n \to \infty$. Since $\mu < 0$, $n\mu \to -\infty$ as $n \to \infty$, then $\ln C_n \xrightarrow{a.s.} -\infty$. Thus $C_n = e^{\ln C_n} \xrightarrow{a.s.} e^{-\infty} = 0$.

Problem 4

(a) Each X_i is a Bernoulli random variable with $\mathbb{E}[X_i] = p$ and $\text{Var}(X_i) = p(1-p)$. So the mean of S_n is $\mathbb{E}[S_n] = \sum_{i=1}^{n} \mathbb{E}[X_i] = np$, and the variance of S_n is $\text{Var}(S_n) = \sum_{i=1}^{n} \text{Var}(X_i) = np(1-p)$. To apply the Central Limit Theorem, we standardize S_n to $Z_n = \frac{S_n - np}{\sqrt{np(1-p)}}$. Then by CLT, as $n \to \infty$, $Z_n = \frac{S_n - np}{\sqrt{np(1-p)}} \xrightarrow{d} N(0,1)$. Therefore, $S_n \sim N(np, np(1-p))$.

(b) $Q_n(x) = P(S_n = x) = \binom{n}{x} p^x (1-p)^{n-x} = \binom{n}{x} \left(\frac{\lambda}{n}\right)^x \left(1 - \frac{\lambda}{n}\right)^{n-x}$. For fixed x and large n, $\binom{n}{x} = \frac{n(n-1)(n-2)\cdots(n-x+1)}{x!} = \frac{n^x}{x!} \prod_{k=0}^{x-1} \left(1 - \frac{k}{n}\right) \approx \frac{n^x}{x!}$. Therefore, $Q_n(x) \approx \frac{n^x}{x!} \left(\frac{\lambda}{n}\right)^x \left(1 - \frac{\lambda}{n}\right)^{n-x} = \frac{\lambda^x}{x!} \left(1 - \frac{\lambda}{n}\right)^{n-x}$. Now we can analyze the limit of $Q_n(x)$ as $n \to \infty$. We have $\lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^n = e^{-\lambda}$ and $\lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^{-x} \approx \lim_{n\to\infty} \left(1 + \frac{x\lambda}{n}\right) = 1$. As a result, $\lim_{n\to\infty} Q_n(x) = \frac{\lambda^x}{x!} e^{-\lambda}$, which is the probability mass function of a Poisson distribution with parameter λ.

(c) The Central Limit Theorem (CLT) assumes that $\mathbb{E}[S_n] = np$, and $\text{Var}(S_n) = np(1-p)$, scale appropriately with n. However, when $p = \frac{\lambda}{n}$, $\mathbb{E}[S_n] = np = \lambda$ (constant), and $\text{Var}(S_n) = np(1-p) \approx \lambda(1 - \frac{\lambda}{n}) \to \lambda$ (constant as $n \to \infty$). Since the mean and variance of S_n are constant, this violates the CLT requirement that the variance grows with n. As a result, the distribution does not spread out enough to approach a normal distribution, making CLT inapplicable.

Problem 5

(a) The total number of ways to choose m students out of $2n$ is $\binom{2n}{m}$. The number of ways to choose m students that include two specific students is $\binom{2n-2}{m-2}$ because we have fixed 2 students and need to choose the remaining $m-2$ students from the remaining $2n-2$. Thus, the probability that both members of a given pair are chosen is $P(\text{both chosen} \mid M = m) = \frac{\binom{2n-2}{m-2}}{\binom{2n}{m}} = \frac{\frac{(2n-2)!}{(m-2)!(2n-m)!}}{\frac{(2n)!}{m!(2n-m)!}} = \frac{(2n-2)!(m!)}{(m-2)!(2n)!} = \frac{m(m-1)}{(2n)(2n-1)}$. Since there are n independent pairs, and for each pair the probability of both being chosen is the same, we have $E[X \mid M = m] = n \cdot P(\text{both chosen} \mid M = m) = \frac{m(m-1)}{2(2n-1)}$.

(b) From law of total expectation, $\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X \mid M]] = \sum_{m=0}^{2n} \frac{m(m-1)}{2(2n-1)} P(M = m) = \frac{1}{2(2n-1)} \mathbb{E}[M(M-1)]$. Since $M \sim \text{Binomial}(2n, p)$, $\mathbb{E}[M(M-1)] = 2n(2n-1)p^2$. Therefore, $\mathbb{E}[X] = \frac{1}{2(2n-1)} \mathbb{E}[M(M-1)] = \frac{1}{2(2n-1)} 2n(2n-1)p^2 = np^2$.