112550013 周廷威

Problem 1

(a) For any set A, B, we have $A \cup B = (A-B) \cup (A \cap B) \cup (B-A)$. Then, by the axiom, $P(A \cup B) = P(A-B) + P(A \cap B) + P(B-A)$.

Also, $\begin{matrix} P(A-B) = P(A) - P(A \cap B) \\ P(B-A) = P(B) - P(A \cap B) \end{matrix}$. Thus, $P(A \cup B) + P(A \cap B) = P(A) + P(B)$.

(b) In using non-negativity part, $P(A \cup B) \ge P(A) + P(B)$ is not correct. We know $P(A \cup B) + P(A \cap B) = P(A) + P(B)$, which means $P(A) + P(B) \ge P(A \cup B)$. Also, in the last part of inductive step, it should be $\sum_{i=1}^{k} P(A_i) + P(A_{k+1}) - \sum_{i=1}^{k} (A_i \cap A_{k+1}) \le \sum_{i=1}^{k+1} P(A_i)$

Problem 2

(a) Let $S = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} S_n$, $T = \{ x \mid x \in S_n \text{ for infinitely many } n \}$

$(\Rightarrow)$ $S \subseteq T$

This means for every $k \ge 1$, $x \in \bigcup_{n=k}^{\infty} S_n$. Therefore, for each $k \ge 1$, there exists $n_k \ge k$ s.t. $x \in S_{n_k}$. Since $n_k \ge k$ and $k \in \mathbb{N}$, the sequence $\{n_k\}$ tends to infinity. Thus, x belongs to S_n for infinitely many n, which means $x \in T$.

$(\Leftarrow)$ $T \subseteq S$

Let $x \in T$, then $x \in S_n$ for infinitely many n. Let $N = \{ n \in \mathbb{N} \mid x \in S_n \}$. Since N is infinite, therefore for every $k \ge 1$, $x \in \bigcup_{n=k}^{\infty} S_n$. Since this holds for all k, we have $x \in \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} S_n = S$. Thus, $x \in S$.

Since $S \subseteq T$ and $T \subseteq S$, we conclude that $\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} S_n = \{ x \mid x \in S_n \text{ for infinitely many } n \}$

(b) Let $S = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} S_n$, $H = \lim_{m \to \infty} \bigcap_{k=1}^{m} \bigcup_{n=k}^{m} S_n$

$(\Rightarrow)$ $S \subseteq H$

If $x \in S$, then $x \in S_n$ for infinitely many n. This is equivalent that if $m \to \infty$, we can choose a $k \ge 1$, and there exists an i s.t. $k \le i \le m$ and $x \in S_i$, which means $x \in H$.

$(\Leftarrow)$ $H \subseteq S$

If $x \in H$, then $x \in \bigcup_{n=m}^{m} S_n = S_m$. Since we can choose a $k \ge 1$, and there exists an $n_k \ge k$ s.t. $x \in S_{n_k}$, which means $x \subseteq S$.

Since $S \subseteq T$ and $T \subseteq S$, we conclude that $\lim_{m \to \infty} \bigcap_{k=1}^{m} \bigcup_{n=k}^{m} S_n$ is equivalent to $\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} S_n$.

(c) $A_1 = A_2 = A_3 = B - C$, $A_4 = C - B$

(1) $\bigcap_{n=1}^{\infty} A_n = (B-C) \cap (B-C) \cap (B-C) \cap (C-B) \ldots = (B-C) \cap (C-B) = \phi$

(2) $\bigcup_{n=1}^{\infty} A_n = (B-C) \cup (B-C) \cup (B-C) \cup (C-B) \ldots = (B-C) \cup (C-B) = (B \cup C) - (B \cap C)$

(3) $\bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} = (\bigcap_{n=1}^{\infty}) \cup (\bigcap_{n=2}^{\infty}) \cup (\bigcap_{n=3}^{\infty}) \cup \ldots = \phi \cup \ldots = \phi$

(4) $\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} = (\bigcup_{n=1}^{\infty}) \cap (\bigcup_{n=2}^{\infty}) \cap (\bigcup_{n=3}^{\infty}) \cap \ldots = ((B \cup C) - (B \cap C)) \cup ((B \cup C) - (B \cap C)) \cup ((B \cup C) - (B \cap C)) \cup \ldots = (B \cup C) - (B \cap C)$

(d) Assume there are countably infinite real numbers in $(0,1)$, which means we can list all of them in sequence $x_1, x_2, x_3 \ldots$

Each number x_i in $(0,1)$ can be represent by $x_i = 0.d_{i1} d_{i2} d_{i3} \ldots$, where $d_{ij} \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

Let $y = 0.y_1 y_2 y_3$, where $y_i = \begin{cases} 1 & \text{if } x_i = 1 \\ 2 & \text{if } x_i \ne 1 \end{cases}$. Suppose that $y = x_k$ for some k, but at the k-th decimal place, by our construction, $y_k \ne d_{kk}$, which means y is a real number in $(0,1)$ but not included in $\{x_i\}$. Therefore, the set of real number in $(0,1)$ is uncountable.

Problem 3

(a) (1) We know that $B_k = \bigcup_{n \geq k}^{\infty} A_n$ is decreasing sequence, $B_1 \supseteq B_2 \supseteq B_3 \ldots$, which means $P(\bigcap_{k=1}^{\infty} B_k) = \lim_{k\to\infty} P(B_k)$. Next, apply the union bound $P(B_k) = P(\bigcup_{n\geq k}^{\infty} A_n) \leq \sum_{n=k}^{\infty} P(A_n)$. Since $\sum_{n=1}^{\infty} P(A_n) < \infty$, $\sum_{n=k}^{\infty} P(A_n)$ approach 0 as $k \longrightarrow \infty$. Therefore, $\lim_{k\to\infty} P(B_k) \leq \lim_{k\to\infty} \sum_{n=k}^{\infty} P(A_n) = 0$. Thus, $P(\limsup_{n\to\infty} A_n) = \lim_{k\to\infty} P(B_k) = 0$.

(2) Assume that $P(\limsup_{k\to\infty} A_k) > 0$, then $\sum_{k=1}^{\infty} P(\bigcup_{n\geq k}^{\infty} A_n) \geq P(\limsup_{k\to\infty} A_k) > 0$. However, if $\sum_{n=1}^{\infty} P(A_n) < \infty$, then $\sum_{n=k}^{\infty} P(A_n)$ approach 0 as $k \longrightarrow \infty$. Also, by the union bond $P(\bigcup_{n\geq k}^{\infty} A_n) \leq \sum_{n=k}^{\infty} P(A_n) = 0$. This contradiction implies $\sum_{n=1}^{\infty} P(A_n) = \infty$.

(b) (1) $N > 1$

$\sum_{k=1}^{\infty} P(A_k) = \sum_{n=1}^{\infty} \frac{1}{10} \cdot (2n)^{-N} = \frac{1}{10} \cdot 2^{-N} \cdot \sum_{n=1}^{\infty} n^{-N}$. Since $N > 1$, $\sum_{n=1}^{\infty} n^{-N}$ converges. Therefore, $\sum_{k=1}^{\infty} P(A_k) < \infty$, which means $P(I) = 0$.

(2) $0 < N \leq 1$

Compute $\sum_{k=1}^{\infty} P(A_k) = \frac{1}{10} \cdot 2^{-N} \cdot \sum_{n=1}^{\infty} n^{-N}$. Since $N \leq 1$, $\sum_{n=1}^{\infty} n^{-N}$ diverges. Since $\sum_{k=1}^{\infty} P(A_k) = \infty$ and A_k are independent, $P(\limsup_{n\to\infty} A_n) = 1$. Therefore, $P(I) = 1$.

Problem 4

(a) $\Omega = \{\text{Option\#1}, \text{Option\#2}, \text{Option\#3}, \ldots, \text{Option\#}N\}$

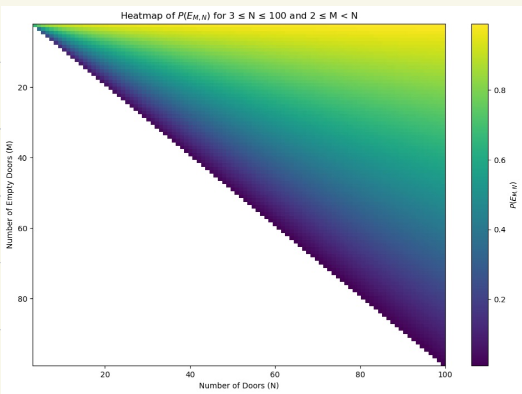
(b) Let A: Bill's initial choice is a prize; W: Bill wins.

$P(A) = \frac{N-M}{N}$. $P(W \mid A) = \frac{\text{Remaining Prizes}}{\text{Remaining Doors}} = \frac{N-M-1}{N-2}$

$P(A') = \frac{M}{N}$. $P(W \mid A') = \frac{\text{Remaining Prizes}}{\text{Remaining Doors}} = \frac{N-M}{N-2}$

$\therefore P(E_{M,N}) = P(A) \cdot P(W \mid A) + P(A') + P(W \mid A') = \frac{N-M}{N} \cdot \frac{N-M-1}{N-2} + \frac{M}{N} \cdot \frac{N-M}{N-2}$

(c)



For a fixed N, as the number of empty door M increases, $P(E_{M,N})$ decreases. Also, for large N and fixed M, $P(E_{M,N})$ approaches a certain value.

Problem 5

(a) $P(C \mid A_1) = (0.25)^6$ $\qquad N_1 = P(C \mid A_1) \cdot P(A_1) = \frac{1}{4096} \times \frac{1}{2} \approx 0.000122207$

$P(C \mid A_2) = (0.5)^3 \cdot (0.2)^2 \cdot (0.1)$ $\qquad N_2 = P(C \mid A_2) \cdot P(A_2) = 0.0005 \times \frac{1}{4} = 0.000125$

$P(C \mid A_3) = (0.2)^3 \cdot (0.5) \cdot (0.2) \cdot (0.1)$ $\qquad N_3 = P(C \mid A_3) \cdot P(A_3) = 0.00008 \times \frac{1}{4} = 0.00002$

$P(C) = N_1 + N_2 + N_3 = 0.000122207 + 0.000125 + 0.00002 = 0.000267207$

$P(A_1 \mid C) = \frac{N_1}{P(C)} = \frac{0.000122207}{0.000267207} \approx 0.45699$

$P(A_2 \mid C) = \frac{N_2}{P(C)} = \frac{0.000125}{0.000267207} \approx 0.46828$

$P(A_3 \mid C) = \frac{N_3}{P(C)} = \frac{0.00002}{0.000267207} \approx 0.07473$

(b) Since $P(A_2 \mid C)$ is the highest, the most probable value for θ is $(0.5, 0.2, 0.2, 0.1)$.