Bachmann-Landau Notation

1 Introduction

Paul Bachmann (1837-1920) and Edmund Landau (1877-1930) were German mathematicians who are both of them remembered for their numerous contributions to number theory. It was in Bachmann's book *Die Analytische Zahlentheorie*[1] [Bac94] on analytic number theory, a branch of mathematics best characterised by the application of analytic methods to number theoretic problems, that big-O notation was first introduced to mathematics, which he used for handling the error terms in his asymptotic estimates. Inspired by Bachmann's idea, Landau introduced a stronger variant of big-O notation, which we now call little-o, in his two-volume treatise titled *Handbuch der Lehre von der Verteilung der Primzahlen*[2] [Lan09]. Today, these two notations, along with other similarly defined notions (like Θ-, ω-, or Ω-notation), are used in a wide variety of areas ranging, of course, from analytic fields like analytic number theory to even computer science.

I myself have often come across Bachmann-Landau notation in several contexts, but in each such instance, I have had to look up and struggle with their definitions since their meaning used to escape me whenever they appeared in equations. The usage of big-O and little-o seems to be intuitively clear to mathematicians yet communicating it properly often poses a difficulty. Thus this investigation is directed towards understanding big-O and little-o notation for what they are and especially how they are to be applied in appropriate mathematical contexts.

2 Theory

Both big-O and little-o notation describe the size of a function relative to another function "in the limit". Usually, this limit point is taken to be ∞ (for example in asymptotic number theory), called infinite asymptotics, or 0 (for example in analysis), called infinitesimal asymptotics. The definition given here, however, generalises this notion to any limit point $a \in \overline{\mathbb{R}}$. Here $\overline{\mathbb{R}}$ represents the *extended real numbers*, namely $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$. A good reference for the basic properties of the extended reals would be the beginning of [Rud53].

[1] Translates literally to *Analytic Number Theory*.

[2] Translates to *Handbook of the theory of distribution of prime numbers*.

2.1 Big-O

There are generally two ways to define each notation in the Bachmann-Landau family of notations: a direct limit-based approach, and a quantifier-based approach.

Definition 1. Given $a \in \overline{\mathbb{R}}$ and functions $f, g : \mathbb{R} \to \mathbb{R}$ such that g is nonzero in some neighbourhood of a, we say that $f(x) = O_a(g(x))$ or $f(x) = O(g(x))$ as $x \to a$ if one of the following equivalent conditions hold:

1. there exists $\delta \in (0, \infty)$ such that
$$\sup_{N_\delta(a)} \left| \frac{f(x)}{g(x)} \right| < \infty;$$
2. there exist $k, \delta \in (0, \infty)$ such that $x \in N_\delta(a)$ implies $|f(x)| \leq k\,|g(x)|$.

Here $N_\delta(d)$ denotes the neighbourhood of a of radius δ, which is $(a - \delta, a + \delta)$ for finite a. For infinite values, we can define it differently: set $N_\delta(\infty) = (\delta, \infty)$ and $N_\delta(-\infty) = (-\infty, -\delta)$.

The set implicit in the first supremum is

$$S_\delta = \left\{ \left| \frac{f(x)}{g(x)} \right| : x \in (a - \delta, a + \delta) \right\}.$$

Let $s_\delta = \sup S_\delta$. If $s_\delta < \infty$ for some δ, then as s_δ is an upper bound of S_δ, we have $|f(x)| \leq s_\delta\,|g(x)|$ for all x such that $x \in N_\delta$; thus the first condition implies the second. Conversely, suppose that there exist $k, \delta \in (0, \infty)$ such that $x \in N_\delta(a)$ implies $|f(x)| \leq k\,|g(x)|$. Then k is an upper bound of S_δ, and the least-upper-bound property of $\mathbb{R}$ then shows that S_δ has a finite supremum. Thus the two conditions are equivalent.

Note that g is required to be nonzero in order for the fraction $f(x)/g(x)$ to make sense, but that it is sufficient that it be nonzero over some neighbourhood of a for the definition to work: we can rewrite the definition to ask for such a δ that is less than the radius of the neighbourhood in which g is nonzero.

The definition found in [Lan09] is for $a = \infty$ only, since analytic number theory is concerned primarily with infinite limits; the cases with a finite, on the other hand, are useful in analysis, for example when truncating the higher order terms in a Taylor series.

When it comes to Bachmann-Landau notations, however, much more important than the rigorous definition is the understanding. The right way to understand big-O is that big-O is to functions what $\leq$ is to numbers:

$$f(x) = O_a(g(x)) \text{ means } f(x) \leq g(x) \text{ in the limit, up to a constant factor.}$$

Examples:

- Let us verify that $100x^2 = O_\infty(x^3)$ using both conditions. Firstly, for any $\delta \in (0, \infty)$ we have

$$\sup_{x>\delta} \left| \frac{100x^2}{x^3} \right| = \sup_{x>\delta} \frac{100}{x} = \frac{100}{\delta} < \infty.$$

For the second condition, we know that for $x > 1$ we have $100x^2 \leq 100x^3$; in fact, for any $k > 0$ there exists $\delta > 0$ such that $x > \delta$ implies $100x^2 \leq kx^3$.

- If f and g are functions that are each either even or odd, then $f = O_\infty(g)$ if and only if $f = O_{-\infty}(g)$. Thus $100x^2 = O_{-\infty}(x^3)$.
- But $100x^2 \neq O_0(x^3)$; otherwise, there would exist $\delta, k \in (0, \infty)$ such that $|x| < \delta$ implies $x \geq k/100$, which is absurd. We do have, however, that $x^3 = O_0(100x^2)$, as for $|x| < 100$ we have $x^3 \leq 100x^2$.
- We have $x^2 = O_\infty(x^{100})$ but $x^{100} = O_0(x^2)$. Basically, x^2 is "smaller" than x^{100} around ∞, while x^{100} is "smaller" around 0.

Note that the symbol O_a is not standard; the literature generally exhibits $f(x) = O(g(x))$ as $x \to a$, or even hides the "as $x \to a$" since the value of a is usually clear from the context. Nevertheless, we shall use it here for the sake of brevity.

2.2 Little-o

2.3 Equations with Landau notation

3 Applications of Landau notation

3.1 Merten's Theorems

3.2 Differentiation

4 Conclusion

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