DATA 609 HW Week 12 & 13

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Week 12

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Verify that the given function pair is a solution to the first-order system.

$x = -e^t, \quad y = e^t$

$\frac{dx}{dt} = -y, \quad \frac{dy}{dt} = -x$

Solution

We can verigy the given function pair is a solution to the first-order system by taking derivative of x and y with respect to t.

$$\frac{dx}{dt} = -e^t, y = e^t, \frac{dx}{dt} = -y$$

$$\frac{dy}{dt} = e^t, x = -e^t, \frac{dy}{dt} = -x$$

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Find and classify the rest points of the given autonomous system.

$\frac{dx}{dt} = -(y-1), \quad \frac{dy}{dt} = x - 2$

Solution

Rest point is determined by putting, $\frac{dx}{dt} = \frac{dy}{dt} = 0$

the given autonomus system,

$$-(y-1) = 0$$
$$\Rightarrow y = 1$$

$$x - 2 = 0$$
$$\Rightarrow x = 2$$

The rest point $(2, 1)$

Classify:

Write differential equation in $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{x-2}{-(y-1)}$$

$$\Rightarrow \frac{dy}{dx} - \frac{x-2}{1-y} = 0$$

$$\Rightarrow (1-y)dy - (x-2)dx = 0$$

$$\Rightarrow \int (1-y)dy - \int (x-2)dx = 0$$

$$\Rightarrow -2y + y^2 + x^2 - 4x - C = 0 \quad (multiplying\, bothsides\, by\; -1)$$

$$\Rightarrow y^2 - 2y = -x^2 + 4x + C$$

Applying $(y-1)^2 = y^2 - 2y + 1 \Rightarrow y^2 - 2y = (y-1)^2 - 1$

We get,

$$(y-1)^2 - 1 = -x^2 + 4x + C \quad Equation\, 1$$

From Equation 1,

$C = (y-1)^2 - 1 + x^2 - 4x$

$\Rightarrow c = -1 + 4 - 8 = -5$

$$(y-1)^2 = -x^2 + 4x - 4$$

$$\Rightarrow y = \sqrt{-x^2 + 4x - 4} - 1$$

From Equation 1,

$C = (y-1)^2 - 1 + x^2 - 4x$

$\Rightarrow c = -1 + 4 - 8 = -5$

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Apply the first and second derivative tests to the function $f(y) = \frac{y^a}{e^{by}}$ to show that $y = \frac{a}{b}$ is a unique critical point that yields the relative maximum f(a/b). Show also that f(y) approaches 0 as y tends to infinity.

Solution

$$f(y) = \frac{y^a}{e^{by}}$$

$$\Rightarrow f(y) = y^a e^{-by}$$

$$\Rightarrow f'(y) = ay^{a-1}e^{-by} - by^a e^{-by} = 0$$

$$\Rightarrow ay^{a-1}e^{-by} = by^a e^{-by}$$

$$\Rightarrow a = by$$

$$\Rightarrow y = \frac{a}{b}$$

$$f''(y) = \frac{d}{dy}\left(\frac{y^{a-1}(a-by)}{e^{by}}\right)$$

$$= (a^2 - a)y^{a-2} - aby^{a-1}$$

$$= \frac{y^{a-2}(a^2 - a - aby)e^{by} - by^{a-1}(a-by)e^{by}}{e^{2by}}$$

$$= \frac{y^{a-2}(a^2 - a - 2aby + b^2y^2)}{e^{by}}$$

$$= \frac{(\frac{a}{b})^{a-2}(a^2 - a - 2ab\frac{a}{b} + b^2\frac{a^2}{b^2})}{e^{b\frac{a}{b}}} \quad (Replace\ y\ with\ \frac{a}{b})$$

$$= \frac{-a^{a-1}}{b^{a-2}e^{a}} \quad (The\ sign\ of\ this\ can\ not\ be\ determined)$$

Show that f(y) approaches 0 as y tends to infinity.

$$\lim_{y\to\infty} f(y) = \lim_{y\to\infty} y^a/e^{by} \lim_{y\to\infty} \frac{e^{a\ lny}}{e^{by}}$$

We can see both y and e^y approach infinity as y approaches infinity.

Week 13

Page 584: #2

Find the local minimum value of the function

$$f(x, y) = 3x^2 + 6xy + 7y^2 - 2x + 4y$$

Solution

First, lets calculate the derivative:

$$\frac{\partial f}{\partial x} = 6x + 6y - 2$$

$$\frac{\partial f}{\partial y} = 6x + 14y + 4$$

Setting the first derivative with respect to x to 0, we get,

$$\frac{\partial f}{\partial x} = 6x + 6y - 2$$
$$\Rightarrow 6x + 6y - 2 = 0$$

$$\Rightarrow x = \frac{1}{3} - y$$

Setting the first derivative with respect to y to 0, substituting x we get,

$$\frac{\partial f}{\partial y} = 6x + 14y + 4$$

$$\Rightarrow 6x + 14y + 4 = 0$$
$$\Rightarrow y = \frac{-3}{4}$$

Substituting y into for x we get,

$$x = \frac{1 + \frac{9}{4}}{3} = \frac{13}{12}$$

The local minimum would be at the point $(\frac{13}{12}, \frac{-3}{4})$

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Find the hottest point (x, y, z) along the elliptical orbit

$$4x^2 + y^2 + 4z^2 = 16$$

where the temperature function is

$$T(x, y, z) = 8x^2 + 4yz - 16z + 600$$

Solution

Using Lagrange multipliers:

$$L(x, y, z, \lambda) = 8x^2 + 4yz - 16z + 600 - \lambda\big[4x^2 + y^2 + 4z^2 - 16\big]$$

Partial Derivatives Set Equal to Zero

$$\frac{\partial L}{\partial x} = 16x - 8\lambda x = 0$$

$$\Rightarrow 16x = 8\lambda x$$

$$\Rightarrow \lambda = 2$$

$$\frac{\partial L}{\partial y} = 4z - 2\lambda y = 0$$

$$\Rightarrow z = y$$

$$\frac{\partial L}{\partial z} = 4y - 16 - 8\lambda z = 0$$

$$\Rightarrow y = \frac{-4}{3}$$

We also have z = y. Therefore $z = \frac{-4}{3}$.

$$\frac{\partial L}{\partial \lambda} = -4x^2 - y^2 - 4z^2 + 16 = 0$$

$$\Rightarrow x = \frac{4}{3}$$

Therefore, the hottest point (x, y, z) is: $(\frac{4}{3}, \frac{-4}{3}, \frac{-4}{3})$.