Numerical Methods
Assignment 01

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February 26, 2017

Exercise 1

The function

The matlab function `sineapprox = Chouliaras_assignment1_exercise1(x)` calculates the approximation of sin(x) using Taylor's series around 0 with $n = 48$ terms and x in radians. The reason that this specific n was selected, is due to the fact that it approximates the sine efficiently for the largest possible interval of x. This was concluded after testing many possible values for n.

The input of the function is a scalar or vector x which denotes radians and the output `sineapprox` is the approximation of the sine function for this specific x. In the case that the input is a vector, it returns the approximation of all the values in the vector. For example, in order to approximate the sine of $x = 3$ radians, one should type:

```
>> Chouliaras_assignment1_exercise1(3)

ans =

   0.141120008059867
```

and in this case the output is 0.14112. In case that the input is a vector $x = [1\ 5\ 9\ 10]$, then the function should be called as:

```
>> Chouliaras_assignment1_exercise1([1 5 9 10])

ans =

   0.841470984807897  -0.958924274663136   0.412118485241901  -0.544021110889270
```

where in this case the output consists of a vector of the same size as the input. The function works for both row and column vectors as inputs and produces the output in the corresponding form.

The method

The sine function can be approximated as a Taylor series around 0 as follows:

$$sin(x) \approx \sum_{n=0}^{48} (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \text{h.o.t.} \tag{1}$$

where n denotes the number of terms in the series. When it comes to the largest term in the series the following holds. We can express the next term in the series in terms of the previous one as follows:

$$\frac{x^{2n+1}}{(2n+1)!} = \frac{x^{2n-1}}{(2n-1)!} \frac{x^2}{(2n+1)2n} \tag{2}$$

From equation (2) it can be seen that the term that determines whether the next term will be larger or smaller than the previous one, is $\frac{x^2}{(2n+1)2n}$. In other words, if this number is larger or smaller than 1. From this, we can derive two cases:

Case 1

If $x > \sqrt{4n^2 + 2n}$ then the next term is larger than the previous.

Case 2

If $x < \sqrt{4n^2 + 2n}$ then the next term is smaller than the previous.

An example is the following: For $x = 10$ it is found using the aforementioned formulas that the maximum term is for $n = 5$. After the 5^{th} iteration, the absolute value of the terms is getting smaller. The first 7 terms produced by the function for x = 10 are the following:

$$10 - 166.7 + 833.3 - 1984.1 + \mathbf{2755.7} - 2505.2 + 1605.9 + \ldots$$

From these terms it can be seen that indeed, the first terms are increasing, the 5^{th} is the largest one, and after this, the terms are getting smaller.

Discussion and Illustration

Concerning the precision of the method, it can be seen from figure 1 that the function approximates the real sine precisely when $x \in [-35, 35]$. When the number of the approximation's terms is smaller, this interval gets also smaller. However, by increasing the number of iterations ($n > 49$) this interval does not increase. In fact, when the number of iterations is very large ($n > 150$), the function returns NaN values in the series' terms. In order to measure the precision of the method the relative error was calculated for several values of x:

x	$sin(x)$	approximation	relative error
-30	0.98803	0.98789	$1.4 \cdot 10^{-4}$
2	0.90929	0.90929	0
10	-0.54402	-0.54402	$1.8 \cdot 10^{-13}$
5	-0.95892	-0.95892	$2.8 \cdot 10^{-15}$

Table 1: Real $sin(x)$ and its approximation for several values of x.

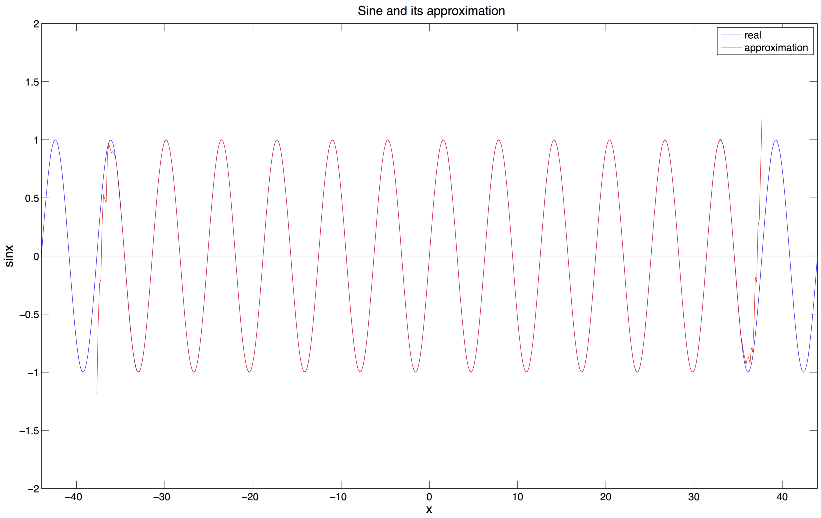

Figure 1: Plot of the real sine function against the approximated.

Exercise 2

The function

The matlab function `[t,x] = Chouliaras_assignment1_exercise2(m,b,a,x0,v0)` calculates the solution of the differential equation

$$m\frac{d^2x}{dt^2}(t) + \beta\frac{dx}{dt}(t) + \alpha x(t) = 0 \tag{3}$$

with initial conditions $x(0) = x_0$ and $\frac{dx}{dt}(0) = v_0$. The input of the function are the mass `m` > 0, the friction coefficient `b` (≥ 0), the spring constant `a` (≥ 0), the initial position `x0` ($\in \mathbb{R}$) and the initial velocity `v0` ($\in \mathbb{R}$). In case that `m, b` or `a` are not appropriate the functions returns an error message. Furthermore, `x0` and `v0` can be scalars or row/column vectors of the same size. The output `[t,x]` consists of a vector `t` with the time in $[0, T]$ and `x` which contains the corresponding values of the solution is a vector, if `x0` and `v0` are scalars, otherwise is a matrix.

The method

For the solution of this 2^{nd} order differential equation with constant coefficients, three different cases can be identified, depending on the values of the discriminant $D = \beta^2 - 4m\alpha$. The three cases are presented below.

Case $D > 0$

In the case that $D > 0$ equation (3) has two different real roots $r_{1,2} = \frac{-\beta \pm \sqrt{D}}{2m}$ and the solution is:

$$x(t) = Ae^{r_1 t} + Be^{r_2 t} \tag{4}$$

for constants A, B which can be determined by the initial conditions. In this case the system is called *overdamped*.

Case $D = 0$

When $D = 0$ equation (3) has two equal roots $r_1 = r_2 = r = \frac{-\beta}{2m}$ and the solution is:

$$x(t) = Ae^{rt} + Bte^{rt} \tag{5}$$

In this case the system is called *critically damped*.

Case $D < 0$

If $D < 0$ equation (3) has complex conjugate roots and the solution is given by:

$$x(t) = Ae^{lt}cos(\omega t) + Be^{lt}sin(\omega t) \tag{6}$$

where $l = -\frac{\beta}{2m}$ and $\omega = \frac{\sqrt{-D}}{2m}$. In this case the system is called *underdamped*.

Discussion and Illustration

In order to have a better view in the differences between each case, the corresponding graphs are presented below. In figure 2 the case when $D > 0$ is presented for constant parameters `m, b, a, x0` and varying parameters `v0`. The code that calculates the solution is the following:

```
>> [t,x] = Chouliaras_assignment1_exercise2(2,1,1,[2 2 2], [5 15 30]);
```

It needs to be noted that in this case the solution is depicted in $t \in [0, 20]$ since after this point the solution is almost zero since the oscillation is damping. This interval holds for this specific selection of the parameters. In case that the parameter `m` gets larger, the oscillation also lasts longer. Furthermore, as the initial velocity gets larger, the oscillation becomes also more intense. In figure 3 can be seen the illustration of the second case where $D = 0$ again for fixed paramaters `m, b, a, x0` and varying `v0`. In this case the oscillator does not take negative values. The solution is depicted this time in $t \in [0, 10]$ since after this point the solution is zero. It should be added here, that also larger values for the parameters have been tried. It was found for larger parameters that the height of the oscillation gets larger, however it damps to zero at the same time interval as the one which is depicted.
Lastly, in figure 4 the third case $D < 0$ is depicted for fixed parameters `m, b, a` and varying `x0, v0`. In this case, the oscillation lasts until $t = 40$, so the solution is depicted in the time interval $[0, 40]$. It needs to be noted here that the duration of the oscillation depends on the parameters `m, b, a`. For example, as `b` gets smaller, the oscillation lasts longer.

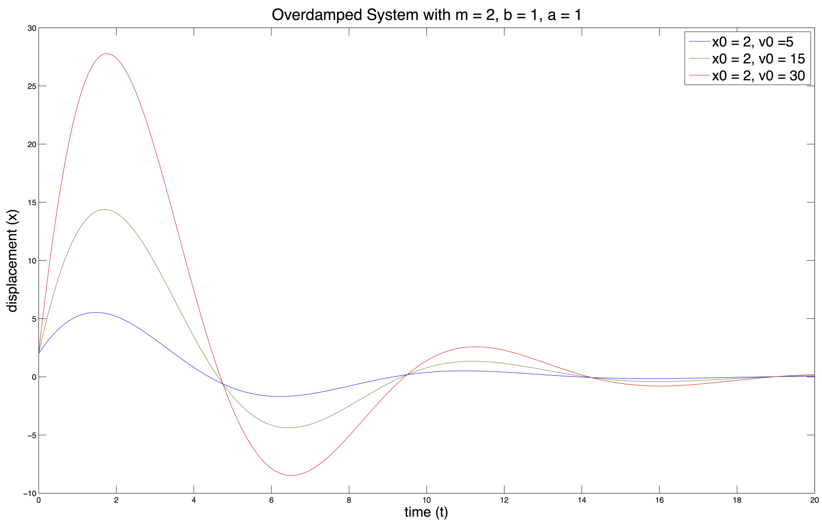

Figure 2: Illustration of the solution when $m = 2, b = 1, a = 1, x_0 = 2$ and varying values of v_0.

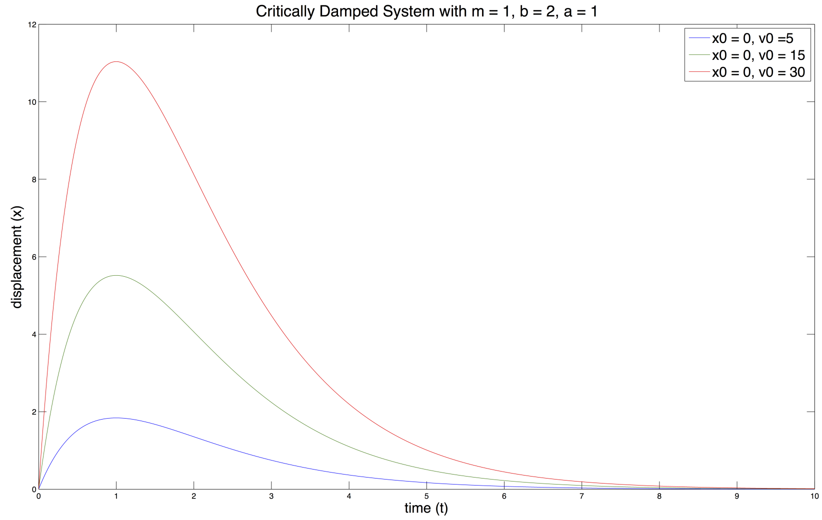

Figure 3: Illustration of the solution when $m = 1, b = 1, a = 1, x_0 = 0$ and varying values of v_0.

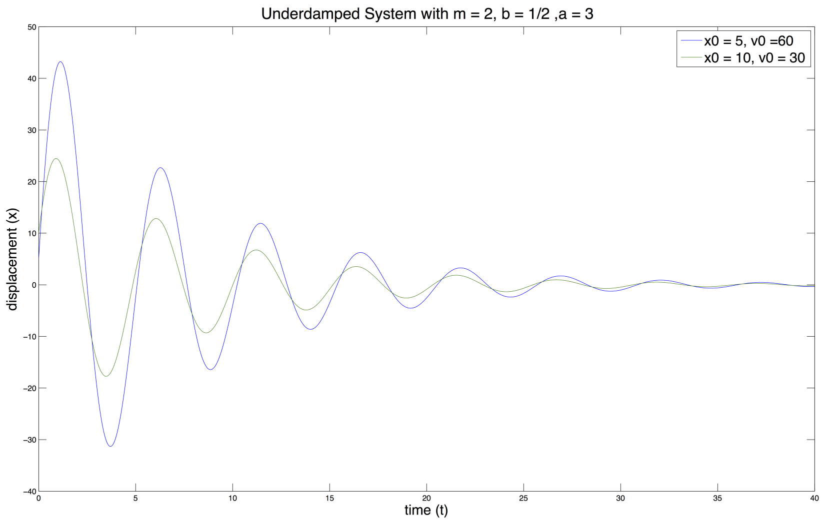

Figure 4: Illustration of the solution when $m = 2, b = 1/2, a = 3$ and varying values of x_0, v_0.

Exercise 3

The function

The function `Chouliaras_assignment1_exercise3(lmin,lmax,N)` produces the bifurcation diagram for the logistic map $x_{n+1} = \lambda x_n(1 - x_n)$ for $\lambda \in [0, 4]$ and $x_0 \in (0, 1)$. The inputs are the minimum and maximum values of λ, `lmin` and `lmax`, and optionally `N`, the number of values of λ between λ_{min} and λ_{max} for which the attractor is calculated. In case that the values of `lmin, lmax` are not appropriate the function returns an error. Also, the default `N` is set to 1000.

Illustration

The function calculates for every value of λ, $n = 1000$ values but only depicts the last 50 in the picture, as in this way the picture looks much nicer. Figure 5 depicts the picture plotted by the function for $x_0 = 0.8$, $\lambda_{min} = 0$ and $\lambda_{max} = 4$ as follows:

`Chouliaras_assignment1_exercise3(0,4)`

Note that in this case `N` is not specified, thus the default $N = 1000$ is used in this case. Concerning x_0 it should be noted that when $x_0 = 0$ or $x_0 = 1$ then the bifurcation diagram is not produced. On the other hand, when $0 < x < 1$ the function produces the same diagram as in figure 5. In other words, selecting x_0 as 0.3 or 0.9 does not make any difference in the picture of the attractor.

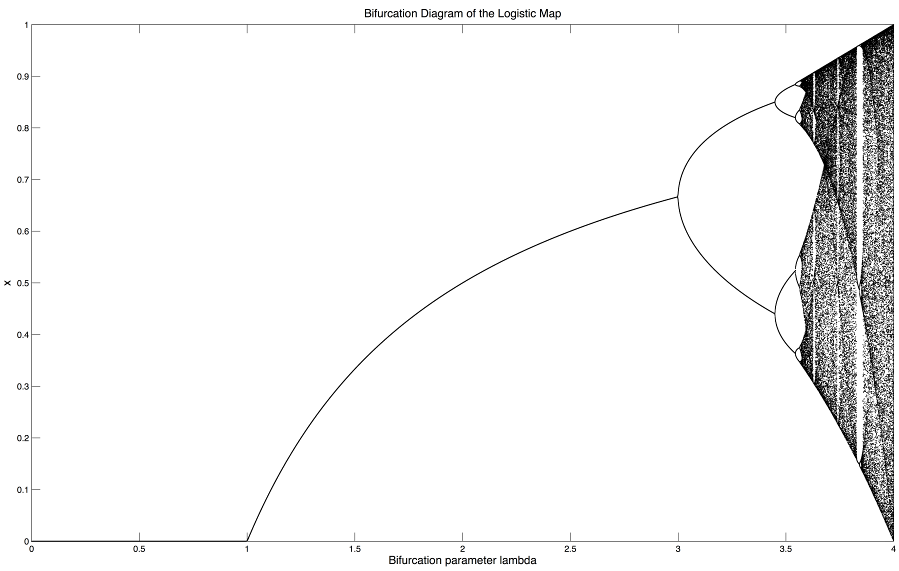

Figure 5: Bifurcation diagram for the logistic map, for $x_0 = 0.8$, λ in $[0, 4]$ and N = 1000. Only the 50 last values of each iteration of the sequence are shown.