Assignment Time Series

A a single pdf-file containing solutions to the problems below is to be uploaded in Blackboard, due no later than the deadline for this assignment. Submitted report should be well-organized and include step-by-step explanations, relevant plots, tables, R code and conclusions.

I. Computational problems

1. The time series dataset `sunspot.month` (included in the standard `R` installation) provides the monthly count of observed sunspots from 1749 to 1997.

   (i) Try to identify trend and seasonal components for this time series.

   (ii) According to astrophysicists, the cycle of sunspot activity has a period of about 11 years. State whether the period of the seasonal component in your analysis is in agreement.

   (iii) Obtain and plot a detrended and deseasonalized version of the original time series. For this resulting time series, obtain and plot an estimate of the autocorrelation function.

   (iv) Comment on your findings.

2. (i) Generate an ARMA(p, q) time series of length n, using reasonable values of n, p, q and σ^2 of your choice. Provide a plot of the time series you generated. Fit an ARMA(p, q) model to the generated time series. Present the estimates of the model parameters, along with approximate 95% confidence intervals for the coefficients $\alpha_1, \ldots, \alpha_p$ and $\beta_1, \ldots, \beta_q$. Comment on the quality of the estimates. Obtain and plot the residuals. Use a Portmanteau test to evaluate the quality of the fit. Comment on your findings.

   (ii) Perform the same analysis as in (i) for an AR(p) model. Use Yule-Walker estimates for the parameters of your model. Compare the quality of the estimates with that from (i).

3. The *cross-correlation* between two stationary time series $\{X_t\}$ and $\{Y_t\}$ at *lag* k is defined as

$$\gamma_{XY}(k) = \frac{E[(X_t - \mu_X)(Y_{t+k} - \mu_Y)]}{\sigma_X \sigma_Y} ,$$

   where μ_X, μ_Y, σ_X and σ_Y are the respective means and standard deviations of the two series. This is a measurement of the synchrony between two time series when one lags the other by k time steps. Given n observations $\{x_t\}$ from $\{X_t\}$ and n observations $\{y_t\}$ from $\{Y_t\}$, the cross-correlation between $\{X_t\}$ and $\{Y_t\}$ at lag k is estimated as

$$\hat{\gamma}_{XY}(k) = \frac{\sum_{t=p}^{q}(x_t - \bar{x})(y_{t+k} - \bar{y})}{\sqrt{\sum_{t=1}^{n}(x_t - \bar{x})^2 \sum_{t=1}^{n}(y_t - \bar{y})^2}} ,$$

   where $p = \max\{1, 1-k\}$, $q = \min\{n, n-k\}$, $\bar{x} = n^{-1}\sum_{t=1}^{n} x_t$ and $\bar{y} = n^{-1}\sum_{t=1}^{n} y_t$. The `R` function `ccf` can compute this estimate (consult `R` help file for details).

   The multivariate time series `EuStockMarkets` (included in the standard `R` installation) gives the daily closing prices of four major European stock indices (German DAX, Swiss SMI, French CAC and British FTSE) from 1991 to 1998, in business time (excluding weekends and holidays). Plot these four time series together and describe any interesting similarities and differences you observe. For each pair of stock indices, compute the maximum cross-correlation and determine at which lag the maximum occurs. Which pair of stock indices were the most correlated? The least? Does the maximum occur at any lag other than 0 for any pair? What inferences about the relationships among the stock indices can you make from this?

II. Theoretical problems

1. (i) Let $X_t = \cos(\lambda t + U)$, where $\lambda \in \mathbb{R}$ and U is a random variable uniformly distributed on $(-\pi, \pi]$. Determine the mean function $m_t = EX_t$ and its autocovariance function $\gamma_X(s,t) = \mathrm{Cov}(X_s, X_t)$. Is $\{X_t\}$ (weakly) stationary?

   (ii) Let $X_t = A\cos(\lambda t) + B\sin(\lambda t)$, where $\lambda \in \mathbb{R}$, A and B are uncorrelated random variables with mean zero and variance σ^2. Determine the mean function $m_t = EX_t$ and its autocovariance function $\gamma_X(s,t) = \mathrm{Cov}(X_s, X_t)$. Is X_t (weakly) stationary?

2. Let $\{Y_t\}$ be an arbitrary time series and $a, b \in \mathbb{R}$.

   (i) Suppose $\{X_t\}$ is a time series given by $X_t = a + bt + s_t + Y_t$, where s_t is a seasonal component with period d. Express $\nabla\nabla_d X_t$ in terms of $\{Y_t\}$ and constants a, b and d.

   (ii) Suppose $\{W_t\}$ is a time series given by $W_t = at^2 + bts_t + Y_t$, where s_t is a seasonal component with period d. Express $\nabla_d^2 W_t$ in terms of $\{Y_t\}$ and constants a, b and d.

3. Suppose $\{V_t\}$ is a stationary time series with autocovariance function $\gamma_V(h)$, $h \in \mathbb{Z}$, such that $\sum_{h\in\mathbb{Z}} |\gamma_V(h)| < \infty$. Then the series $f_V(\lambda) = \frac{1}{2\pi}\sum_{h\in\mathbb{Z}} \gamma_V(h)e^{-ih\lambda}$ is absolutely convergent, uniformly in $\lambda \in \mathbb{R}$. The real valued (since $\gamma_V(h)$ is even) 2π-periodic function $f_V(\lambda)$, $\lambda \in (-\pi, \pi]$, is called the *spectral density* of the time series $\{V_t\}$. Introduce a linear transform $\{W_t\}$ of $\{V_t\}$: $W_t = \sum_{k\in\mathbb{Z}} \psi_k V_{t-k}$ (called the *linear filter* with filter coefficients ψ_k) such that $\sum_k |\psi_k| < \infty$. The (complex-valued) function $\psi(\lambda) = \sum_{k\in\mathbb{Z}} \psi_k e^{-ik\lambda}$ is called the *transfer function* of the filter. Then the time series $\{W_t\}$ is also stationary and has the following spectral density

$$f_W(\lambda) = |\psi(\lambda)|^2 f_V(\lambda). \qquad (*)$$

   (i) Let $\{Z_t\}$ be a white noise time series, see Example 4.1 in the lecture notes. Determine its spectral density $f_Z(\lambda)$.

   (ii) Consider a moving average time series of order 1: $X_t = \alpha Z_t + \beta Z_{t-1}$, see Example 4.2 in the lecture notes. By using the expression for the autocovariance function $\gamma_X(h)$ of $\{X_t\}$ (see (4.6) in the lecture notes), determine the spectral density $f_X(\lambda)$ of $\{X_t\}$.

   (iii) Show that $\{X_t\}$ from 3(ii) is a linear filter of $\{Z_t\}$, determine the corresponding filter coefficients ψ_k and the transfer function $\psi(\lambda)$. Demonstrate that the formula $(*)$ holds in this case, i.e., show that $f_X(\lambda) = |\psi(\lambda)|^2 f_Z(\lambda)$.

   Hint: make use of the famous Euler formula $e^{ia} = \cos(a) + i\sin(a)$, $a \in \mathbb{R}$, and the fact that $|z|^2 = a^2 + b^2$ for a complex number $z = a + ib$, with $a, b \in \mathbb{R}$.