1 Wrapping Many Integrations into One

Author: Christian Howard
Using the problem statement, we can define the following integral operators:

$$D^{-1}\phi(x) = \int_0^x \phi(y)dy \tag{1}$$

$$D^{-n}\phi(x) = \frac{1}{(n-1)!}\int_0^x (x-y)^{n-1}\phi(y)dy \tag{2}$$

Given the above operators, the goal is to prove $D^{-n}\phi(x)$ is correct for all positive integers n. To do this, let us first recall the following based on previous work:

Lemma 1.1 *Given two integral operators, $F\phi(x)$ and $G\phi(x)$, and their corresponding kernels, $k_1(\cdot,\cdot)$ and $k_2(\cdot,\cdot)$, the kernel $k_3(\cdot,\cdot)$ within $(G \circ F)\phi(x)$ can be found using:*

$$k_3(x,y) = \int_y^x k_2(x,z)k_1(z,y)dz$$

Next, let us define $P(n,x) = D^{-n}\phi(x)$. Using this, we can first check that $P(n,x)$ satisfies the base case, where $n = 1$, by doing the following:

$$\begin{aligned} P(1,x) &= \frac{1}{(1-1)!}\int_0^x (x-y)^{1-1}\phi(y)dy \\ &= \int_0^x \phi(y)dy \\ &= D^{-1}\phi(x) \end{aligned}$$

Now let us assume that $P(n,x)$ holds for $0 \le n \le k$. We can then find $P(k+1,x)$ by first noting the following relationship:

$$P(k+1,x) = (D^{-1} \circ D^{-k})\phi(x) = \int_0^x K(x,y)\phi(y)dy$$

Using our inductive hypothesis that $P(k,x)$ holds and Lemma 1.1, we can find the resulting kernel, $K(\cdot,\cdot)$, for $(D^{-1} \circ D^{-k})\phi(x)$ to be the following:

$$\begin{aligned} K(x,y) &= \int_y^x \frac{(x-z)^{k-1}}{(k-1)!}dz \\ &= \left(-\frac{(x-z)^k}{k(k-1)!}\right]_y^x \\ &= \frac{(x-y)^k}{k!} \end{aligned}$$

With the above kernel, we can find the final form for $P(k+1,x)$ to be:

$$\begin{aligned} P(k+1,x) &= \frac{1}{k!}\int_0^x (x-y)^k \phi(y)dy \\ &= D^{-(k+1)}\phi(x) \end{aligned}$$

Thus, the form of $P(k+1,x)$ matches the result for $P(n,x)$ when $n = k+1$, proving by induction that $P(n,x)$ holds $\forall n \in \mathbb{N}^+$.