Homework 4 - Problem 1
Systems of integral equations: Theory

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Contents

1 Normed Spaces

Let X_i $\forall i = 1, \cdots, n$ be complete Banach spaces with norms $\|\cdot\|_i$. For this first problem, the goal is to show that the product space $X := X_1 \times \cdots \times X_n$, with n-tuple elements of the form $\phi = (\phi_1, \cdots, \phi_n)$, are a normed space given this product space has a maximum norm defined in the following manner:

$$\|\phi\|_\infty = \max_{i=1,\cdots,n} \|\phi_i\|_i \tag{1}$$

To show this, we need to show that the following properties exist for some $x, y \in X$:

1. $\|x\|_\infty \geq 0$ and $\|x\|_\infty = 0 \iff x = 0$
2. $\|\alpha x\|_\infty = |\alpha| \, \|x\|_\infty$ for some scalar α
3. $\|x + y\|_\infty \leq \|x\|_\infty + \|y\|_\infty$

1.1 Property 1

We can easily show that 1 holds. First, let's plug in $\phi = (0, \cdots, 0)$ and see the following:

$$\begin{aligned}
\|\phi\|_\infty &= \max_{i=1,\cdots,n} \|\phi_i\|_i \\
&= \max_{i=1,\cdots,n} \|0\|_i \\
&= 0
\end{aligned}$$

The above work trivially shows the above norm can only be zero if ϕ is zero in all of its elements. Now let's assume there exists some index subset $J \subset \{1, 2, \cdots, n\}$ such that $\phi_k \neq 0$ $\forall k \in J$, and all other elements are 0. We can find the following:

$$\begin{aligned}
\|\phi\|_\infty &= \max_{i=1,\cdots,n} \|\phi_i\|_i \\
&= \max_{i \in J} \|\phi_i\|_i \\
&\geq 0
\end{aligned}$$

The above results show that $\|\phi\|_\infty$ satisfies property 1.

1.2 Property 2

Let us assume $\hat{\phi} = \alpha\phi = (\alpha\phi_1, \cdots, \alpha\phi_n)$. Then we can work out the following:

$$
\begin{aligned}
\left\|\hat{\phi}\right\|_\infty = \|\alpha\phi\|_\infty &= \max_{i=1,\cdots,n} \|\alpha\phi_i\|_i \\
&= \max_{i=1,\cdots,n} |\alpha| \, \|\phi_i\|_i \\
&= |\alpha| \max_{i=1,\cdots,n} \|\phi_i\|_i \\
&= |\alpha| \, \|\phi\|_\infty
\end{aligned}
$$

Thus we can see that property 2 is satisfied.

1.3 Property 3

Let us show that $\|x+y\|_\infty \leq \|x\|_\infty + \|y\|_\infty$ for $x, y \in X$.

$$
\begin{aligned}
\|x+y\|_\infty &= \max_i \|x_i + y_i\|_i \\
&\leq \max_i \|x_i\|_i + \|y_i\|_i \\
&\leq \left(\leq \max_i \|x_i\|_i\right) + \left(\max_i \|y_i\|_i\right) \\
&= \|x\|_\infty + \|y\|_\infty
\end{aligned}
$$

Thus, $\|x+y\|_\infty \leq \|x\|_\infty + \|y\|_\infty$ holds and Properties 1, 2, and 3 are satisfied, showing that $(X, \|\cdot\|_\infty)$ together represent a normed vector space.

2 Composite Operators

The goal of this part is to show that $(A\phi)_i = \sum_{k=1}^n A_{ik}\phi_k$ is compact $\forall i$ iff $A_{ik} : X_k \to X_i$ is compact $\forall i, k$. Let us define the below statements:

$$
(A\phi)_i = \sum_{k=1}^n A_{ik}\phi_k \text{ is compact } \forall i \tag{2}
$$

$$
A_{ik} : X_k \to X_i \text{ is compact } \forall i, k \tag{3}
$$

If (3) is true, then it holds that for each bounded sequence $(\phi_k(m))$ in X_k, some subsequence of $(A_{ik}\phi_k(m))$ is convergent in X_i. This means $(A\phi)_i = \sum_{k=1}^n A_{ik}\phi_k$ must be convergent $\forall i, k$, since each term of the summation converges, and in turn implies (2) is true.

Let us assume that (2) holds and that $\exists j, l \ni A_{jl}$ is not compact. This means that $(A\phi)_j = \sum_{k=1}^n A_{jk}\phi_k$ is not compact for some j because the sum is not convergent based on some bounded sequence $(\phi(m))$. Since $(A\phi)_j$ is not convergent for some j, then $A\phi$ is not compact. Due to this contradiction, we see that (2) implies that (3) must be true.