

1 Iterated Integral Operators

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In the problem statement, we can define the following integral operators:

$$F\phi(x) = \int_a^x k_1(x, y)\phi(y)dy \quad (1)$$

$$G\phi(x) = \int_a^x k_2(x, y)\phi(y)dy \quad (2)$$

Given the above operators, the goal is to find $k_3(x, y) \ni$

$$G(F\phi)(x) = (G \circ F)\phi(x) = \int_a^x k_3(x, y)\phi(y)dy \quad (3)$$

Result 1.1. *The resulting kernel within $(G \circ F)\phi(x)$ is:*

$$k_3(x, y) = \int_y^x k_2(x, z)k_1(z, y)dz$$

Proof.

$$\begin{aligned} (G \circ F)\phi(x) &= \int_a^x k_2(x, z)F\phi(z)dz \\ &= \int_a^x k_2(x, z) \int_a^z k_1(z, y)\phi(y)dydz \\ &= \int_a^x \phi(y) \int_a^x k_2(x, z)k_1(z, y)dzdy - \int_a^x k_2(x, z) \int_z^x k_1(z, y)\phi(y)dydz \xrightarrow{0} \\ &= \int_a^x \phi(y) \int_a^y k_2(x, z)k_1(z, y)dzdy + \int_a^x \phi(y) \int_y^x k_2(x, z)k_1(z, y)dzdy \xrightarrow{0} \\ &= \int_a^x \phi(y) \int_y^x k_2(x, z)k_1(z, y)dzdy \\ &= \int_a^x k_3(x, y)\phi(y)dy \\ \therefore k_3(x, y) &= \int_y^x k_2(x, z)k_1(z, y)dz \end{aligned}$$

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