1 Iterated Integral Operators

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In the problem statement, we can define the following integral operators:

$$F\phi(x) = \int_a^x k_1(x,y)\phi(y)dy \tag{1}$$

$$G\phi(x) = \int_a^x k_2(x,y)\phi(y)dy \tag{2}$$

Since we are relating these integral operators to upper/triangular matrices, I make the following assumption based on the integral operator bounds:

$$k_1(x,y) = k_2(x,y) = 0 \;\; \forall y > x \tag{3}$$

Given the above operators and properties, the goal is to find $k_3(x,y) \ni$

$$G(F\phi)(x) = (G \circ F)\phi(x) = \int_a^x k_3(x,y)\phi(y)dy \tag{4}$$

Result 1.1. *The resulting kernel within $(G \circ F)\phi(x)$ is:*

$$k_3(x,y) = \int_y^x k_2(x,z)k_1(z,y)dz$$

Proof.

$$\begin{aligned}
(G \circ F)\phi(x) &= \int_a^x k_2(x,z)F\phi(z)dz \\
&= \int_a^x k_2(x,z)\int_a^z k_1(z,y)\phi(y)dydz \\
&= \int_a^x \phi(y)\int_a^x k_2(x,z)k_1(z,y)dzdy - \int_a^x k_2(x,z)\cancelto{0}{\int_z^x k_1(z,y)\phi(y)dydz} \\
&= \int_a^x \phi(y)\cancelto{0}{\int_a^y k_2(x,z)k_1(z,y)dzdy} + \int_a^x \phi(y)\int_y^x k_2(x,z)k_1(z,y)dzdy \\
&= \int_a^x \phi(y)\int_y^x k_2(x,z)k_1(z,y)dzdy \\
&= \int_a^x k_3(x,y)\phi(y)dy \\
\therefore k_3(x,y) &= \int_y^x k_2(x,z)k_1(z,y)dz
\end{aligned}$$

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