Discrete Mathematics with Applications I

COMPSCI&SFWRENG 2DM3

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Anything Wrong?

Let the set Q be defined by the following:

(R) $\quad Q = \{S \mid S \notin S\}$

"The mother of all type errors"

$\Longrightarrow$ birth of type theory...

Then:

$$
\begin{array}{ll}
& Q \in Q \\
\equiv & \langle \text{ (R) } \rangle \\
& Q \in \{S \mid S \notin S\} \\
\equiv & \langle \text{ (11.3) Membership in set comprehension } \rangle \\
& (\exists S \mid S \notin S \bullet Q = S) \\
\equiv & \langle \text{ (9.19) Trading for } \exists\text{, (8.14) One-point rule } \rangle \\
& Q \notin Q \\
\equiv & \langle \text{ (11.0) Def. } \notin \rangle \\
& \neg(Q \in Q)
\end{array}
$$

With (3.15) $p \equiv \neg p \equiv \mathit{false}$, this proves:

(R′) $\quad \mathit{false}$

— "Russell's paradox"

Plan for Today

- Sets and Types
- Pairs, Cartesian products
- Relations

What is the Type of Set Complement $\sim\_$?

Consider:

- $\mathbb{N} : \textbf{set}\ \mathbb{Z}$
- $S_1 = \{1, 3, 8\}$
- $S_1 \in \mathbb{P}\,\mathbb{N}$
- $S_1\ :\ \textbf{set}\ \mathbb{Z}$
- $\sim S_1\ :\ \textbf{set}\ \mathbb{Z}$
- $\sim S_1 \notin \mathbb{P}\,\mathbb{N}$

Which of the following makes most sense?

- $\sim\_\ :\ \mathbb{P}\,S \to \mathbb{P}\,S$
- $\sim\_\ :\ \mathbb{P}\,S \to \mathbb{P}\,t$ — provided $S : \textbf{set}\ t$
- $\sim\_\ :\ \mathbb{P}\,S \to \textbf{set}\ t$ — provided $S : \textbf{set}\ t$
- $\sim\_\ :\ \textbf{set}\ t \to \textbf{set}\ t$

Note:

- $\textbf{set}\ :\ \mathsf{Type} \to \mathsf{Type}$
- $\mathbb{P}\ :\ \textbf{set}\ t \to \textbf{set}\ (\textbf{set}\ t)$
- $\mathbb{P}\,S :\ \textbf{set}\ (\textbf{set}\ t)$ — provided $S : \textbf{set}\ t$
- $\_{\to}\_\ :\ \mathsf{Type} \to \mathsf{Type} \to \mathsf{Type}$

Pairs and Cartesian Products

If b and c are expressions,
then $\langle b, c\rangle$ is their **2-tuple** or **ordered pair**

— "ordered" means that there is a **first** constituent (b) and a **second** constituent (c).

(14.2) **Axiom, Pair equality:** $\langle b, c\rangle = \langle b', c'\rangle \quad\equiv\quad b = b' \land c = c'$

(14.3) **Axiom, Cross product:** $S \times T \ =\ \{b, c \mid b \in S \land c \in T \bullet \langle b, c\rangle\}$

(14.4) **Membership:** $\langle b, c\rangle \in S \times T \quad\equiv\quad b \in S\ \land\ c \in T$

Cartesian product of types: $b : t_1\ ;\ c : t_2 \quad \text{iff} \quad \langle b, c\rangle : \langle\!\langle t_1, t_2\rangle\!\rangle$

(14.4p) **Axiom, Pair projections:**

$$\begin{array}{lcll} \mathsf{fst} & : & \langle\!\langle t_1, t_2\rangle\!\rangle \to t_1 & \qquad \mathsf{fst}\ \langle b, c\rangle = b \\ \mathsf{snd} & : & \langle\!\langle t_1, t_2\rangle\!\rangle \to t_2 & \qquad \mathsf{snd}\ \langle b, c\rangle = c \end{array}$$

(14.2p) **Pair equality:** For $p, q : t_1 \times t_2$, $\quad p = q \quad\equiv\quad \mathsf{fst}\ p = \mathsf{fst}\ q\ \land\ \mathsf{snd}\ p = \mathsf{snd}\ q$

Some Spice...

Converting between "different ways to take two arguments":

$$\begin{array}{lcl} \mathsf{curry} & : & (A \to B \to C) \to (A \times B \to C) \\ & & (A \to B \to C) \to (\langle\!\langle A, B\rangle\!\rangle \to C) \\ \mathsf{curry}\ f\ \langle x, y\rangle & := & f\ x\ y \\ \mathsf{uncurry} & : & (A \times B \to C) \to (A \to B \to C) \\ & & (\langle\!\langle A, B\rangle\!\rangle \to C) \to (A \to B \to C) \\ \mathsf{uncurry}\ g\ x\ y & := & g\langle x, y\rangle \end{array}$$

These functions correspond to the "Shunting" law:

(3.65) **Shunting:** $p \land q \Rightarrow r \quad\equiv\quad p \Rightarrow (q \Rightarrow r)$

The "currying" concept is named for Haskell Brooks Curry (1900–1982), but goes back to Moses Ilyich Schönfinkel (1889–1942) and Gottlob Frege (1848–1925).

Predicates and Tuple Types

$$\_called\_ \ : \ P \to P \to \mathbb{B}$$

$(\text{uncurry } \_called\_) \ : \ \langle P, P \rangle \to \mathbb{B}$ is the **characteristic function** of the set

$$R_C \ : \ \textbf{set}\ \langle P, P \rangle$$

$$R_C \ := \ \{p, q : P \mid C\ p\ q \bullet \langle p, q \rangle\}$$

R_C is a **(binary) relation**.

$$D \ : \ P \to City \to City \to \mathbb{B}$$

$$D\ p\ a\ b \ :\equiv \ p \text{ drove from } a \text{ to } b$$

$$R_D \ : \ \textbf{set}\ \langle P, City, City \rangle$$

$$R_D \ := \ \{p : P; a, b : City \mid D\ p\ a\ b \bullet \langle p, a, b \rangle\}$$

R_C is a **(ternary) relation**.

Relations

- LADM: A **relation** on $B_1 \times \cdots \times B_n$ is a subset of $B_1 \times \cdots \times B_n$
  — where $B_1, \ldots, B_n$ are sets, and $\_\times\_$ associates to the left???
- CalcCheck: A **relation** on $\langle t_1, \ldots, t_n \rangle$ is a subset of $\langle t_1, \ldots, t_n \rangle$,
  that is, an item of type **set** $\langle t_1, \ldots, t_n \rangle$
  — where $t_1, \ldots, t_n$ are types
- A relation on the tuple (Cartesian product) type $\langle t_1, \ldots, t_n \rangle$ is an ***n*-ary relation**.
  "Tables" in relational databases are n-ary relations.
- A relation on the pair (Cartesian product) type $\langle t_1, t_2 \rangle$ is a **binary relation**.
- The **type** of binary relations on $\langle t_1, t_2 \rangle$ is written $t_1 \leftrightarrow t_2$, with

$$t_1 \leftrightarrow t_2 \ := \ \textbf{set}\ \langle t_1, t_2 \rangle$$

What is a Relation?

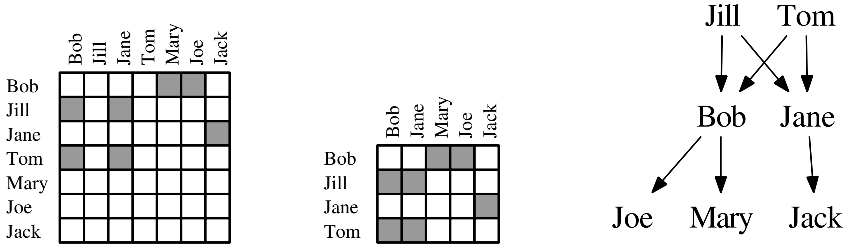
A **relation**
is a subset
of a Cartesian product.

What is a Binary Relation?

A **binary relation**
is a set of pairs.

Visualising Binary Relations

$$
\begin{aligned}
\mathit{parentOf} = \; & \{\langle \mathit{Jill}, \mathit{Bob}\rangle, \langle \mathit{Jill}, \mathit{Jane}\rangle, \langle \mathit{Tom}, \mathit{Bob}\rangle, \langle \mathit{Tom}, \mathit{Jane}\rangle, \\
& \langle \mathit{Bob}, \mathit{Mary}\rangle, \langle \mathit{Bob}, \mathit{Joe}\rangle, \langle \mathit{Jane}, \mathit{Jack}\rangle\}
\end{aligned}
$$



Formulae Expressing Relationship

Consider $R : B \leftrightarrow C$ and $x : B$ and $y : C$.

$$
\begin{aligned}
& R \;:\; B \leftrightarrow C \\
\mathit{iff} \quad & \langle \text{ Def. } \leftrightarrow \rangle \\
& R \;:\; \textbf{set}\,(B, C) \\
\mathit{iff} \quad & \langle \text{ Def. } \textbf{set}\text{, “types as sets” } \rangle \\
& R \subseteq B \times C
\end{aligned}
$$

“x is in relation R with y” — $x \,(\!(\, R \,)\!)\, y$

- explicit membership notation: $\langle x, y\rangle \in R$
- (traditional infix notation: $x \, R \, y$
- **both notations are interchangeable!**
- The traditional infix notation gives rise to ambiguities:
  CALCCHECK: Simple infix notation only for declared infix operators.
  For other relation expressions R: $x \,(\!(\, R \,)\!)\, y \quad \equiv \quad \langle x, y\rangle \in R$

Simple Binary Relations

- The **empty relation** on $\langle t_1, t_2 \rangle$ is $\{\} : t_1 \leftrightarrow t_2$

$$x \,(\!|\, \{\} \,|\!)\, y \equiv \mathit{false}$$
$$\langle x, y \rangle \in \{\} \equiv \mathit{false}$$

- The **identity relation** on $B : \mathbf{set}\, t$ is $\mathbb{I}\, B : t \leftrightarrow t$ with $\mathbb{I}\, B = \{x : t \mid x \in B \bullet \langle x, x \rangle\}$:

$$x \,(\!|\, \mathbb{I}\, B \,|\!)\, y \equiv x = y \in B$$
$$\langle x, y \rangle \in \mathbb{I}\, B \equiv x = y \wedge y \in B$$

- The **universal relation** on $B \times C$ is $B \times C$

(14.4)

$$x \,(\!|\, B \times C \,|\!)\, y \equiv x \in B \wedge y \in C$$
$$\langle x, y \rangle \in B \times C \equiv x \in B \wedge y \in C$$

- The **complement** of relation $R : t_1 \leftrightarrow t_2$ is $\sim R : t_1 \leftrightarrow t_2$:

$$x \,(\!|\, \sim R \,|\!)\, y \equiv \neg(x \,(\!|\, R \,|\!)\, y)$$