MATHEMATICS 1LT3 TEST 1

Day Class
Duration of Test: 60 minutes
McMaster University

E. Clements

27 January 2016

FIRST NAME (please print): Sol^{NS}

FAMILY NAME (please print): ____________

Student No.: ____________

THIS TEST HAS 8 PAGES AND 7 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE.

Total number of points is 40. Marks are indicated next to the problem number in square brackets. You may use the McMaster standard calculator, Casio fx991 MS+, on this test.

USE PEN TO WRITE YOUR TEST. IF YOU USE A PENCIL, YOUR TEST WILL NOT BE ACCEPTED FOR REMARKING (IF NEEDED).

You need to show work to receive full credit, except for Multiple Choice.

Problem	Points	Mark
1	6	
2	6	
3	6	
4	6	
5	7	
6	5	
7	4	
TOTAL	40	

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1. **Multiple Choice.** Clearly **circle** the one correct answer.

(a) [3] Consider the differential equation $\frac{dy}{dx} = x + y$, where $y(0) = 1$. Using Euler's Method with step size $h = 1$, the approximate value of $y(2)$ is

(A) 4   (B) 4.5   (C) 5   (D) 5.5

(b) [3] Consider the basic model for the spread of a disease, $\frac{dI}{dt} = \alpha I(1 - I) - \mu I$, where $\alpha, \mu > 0$. Identify all correct statements.

(I) $I^* = 0$ is a stable equilibrium.
(II) If $\mu > \alpha$, then the disease will eventually die out.
(III) If $\mu < \alpha$ and $I(0) > 0$, then $I(t) \to 1$ as $t \to \infty$.

(A) none   (B) I only   (C) II only   (D) III only
(E) I and II   (F) I and III   (G) II and III   (H) all three

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2. A population of elephants is described by the logistic differental equation

$$\frac{dE}{dt} = 0.2E\left(1 - \frac{E}{350}\right)$$

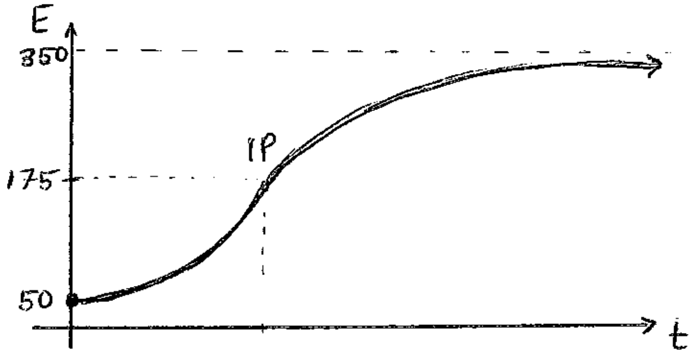
(a) [2] Find the equilibria of this equation. What does the larger equilibrium represent?

$\frac{dE}{dt} = 0$ when $E^* = 0$ or $E^* = 350$

equilibria

350 represents the carrying capacity of the population

(b) [2] The solution of this equation when $E(0) = 50$ is $E(t) = \frac{350}{1 + 6e^{-0.2t}}$, where t is measured in months. Sketch this solution curve.



(c) [2] Determine when the population will reach 95% of its carrying capacity.

Set $E(t) = 0.95 \times 350$:

$$0.95 \times 350 = \frac{350}{1 + 6e^{-0.2t}}$$

$$1 + 6e^{-0.2t} = \frac{1}{0.95}$$

$$e^{-0.2t} = \frac{\frac{100}{95} - 1}{6}$$

$$t = \frac{\ln\left(\frac{1}{114}\right)}{-0.2} \approx 23.7 \text{ months}$$

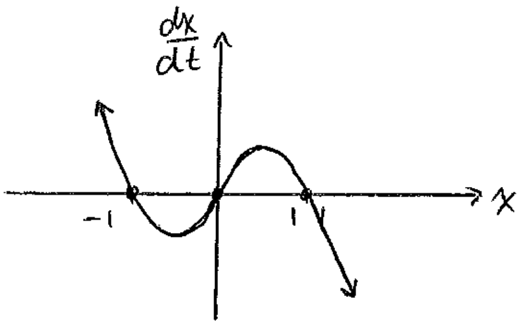
3. Consider the following autonomous differential equation $\frac{dx}{dt} = x - x^3$.

(a) [2] Determine the equilibria for this differential equation.

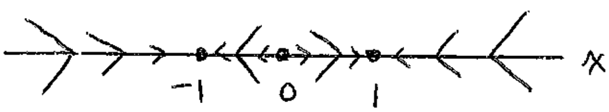
$\frac{dx}{dt} = 0$ when $x(1-x^2) = 0 \Rightarrow x^* = 0$ or $x^* = -1$ or $x^* = 1$

equilibria

(b) [2] Graph $\frac{dx}{dt}$ as a function of x.



(c) [2] Draw a phase-line diagram for $\frac{dx}{dt} = x - x^3$.



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4. The selection equation, $\frac{dp}{dt} = (\mu - \lambda)p(1-p)$, models the dynamics of two variants of a population, type A and type B, where p represents the proportion of type A and μ, $\lambda \geq 0$.

(a) [1] Find the equilibria for this differential equation.

$\frac{dp}{dt} = 0$ when $p^* = 0$ or $p^* = 1$

equilibria

(b) [3] Using the stability theorem, determine the stability of each equilibrium you found in part (a). (Note: Since stability depends on parameters μ and λ, you will need to consider different cases. Assume that $\mu \neq \lambda$.)

$f(p) = (\mu - \lambda)(p - p^2)$

$f'(p) = (\mu - \lambda)(1 - 2p)$

$f'(0) = \mu - \lambda \Rightarrow p^* = 0$ is stable when $\mu < \lambda$ and unstable when $\mu > \lambda$

$f'(1) = \lambda - \mu \Rightarrow p^* = 1$ is stable when $\mu > \lambda$ and unstable when $\mu < \lambda$.

(c) [2] Describe what will happen to the proportion of type A for each case in part (b).

When $\mu < \lambda$, 0 is a stable equilibrium and 1 is an unstable equilibrium

$\Rightarrow$ type B is "stronger" and the proportion of type A approaches 0

When $\mu > \lambda$, 0 is an unstable equilibrium and 1 is a stable equilibrium

$\Rightarrow$ type A is "stronger" and the proportion of type A approaches 1.

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5. Use the separation of variables technique to solve each differential equation.

(a) [3] $\frac{dy}{dx} = \frac{\ln x}{xy}$, where $y(1) = 2$

$$\int y\,dy = \int \frac{\ln x}{x}\,dx$$

$$\frac{y^2}{2} = \frac{(\ln x)^2}{2} + C$$

$$y^2 = (\ln x)^2 + 2C$$

Aside:
Let $u = \ln x$. Then $du = \frac{1}{x}dx$.
So, $\int \frac{\ln x}{x}dx = \int u\,du = \frac{u^2}{2} + C = \frac{(\ln x)^2}{2} + C$

when $x=1, y=2 \Rightarrow 2^2 = (\ln 1)^2 + 2C \Rightarrow 2C = 4$

$\therefore y^2 = (\ln x)^2 + 4$

$\therefore y = \sqrt{(\ln x)^2 + 4}$ (not $-\sqrt{(\ln x)^2+4}$ since $y(1) = +2$)

(b) (i) [3] $y' = x^2 y$

$$\frac{dy}{dx} = x^2 y$$

$$\int \frac{1}{y}\,dy = \int x^2\,dx$$

$$\ln|y| = \frac{x^3}{3} + C$$

$$|y| = e^C \cdot e^{x^3/3}$$

$$y = \pm e^C \cdot e^{x^3/3}$$

$\therefore y = Ke^{x^3/3}$, where $K = \pm e^C$

(ii) [1] Are there any other solutions to this differential equation **not** covered by the equation you found in part (i)?

YES! $y^* = 0$ is a sol^n of the DE.

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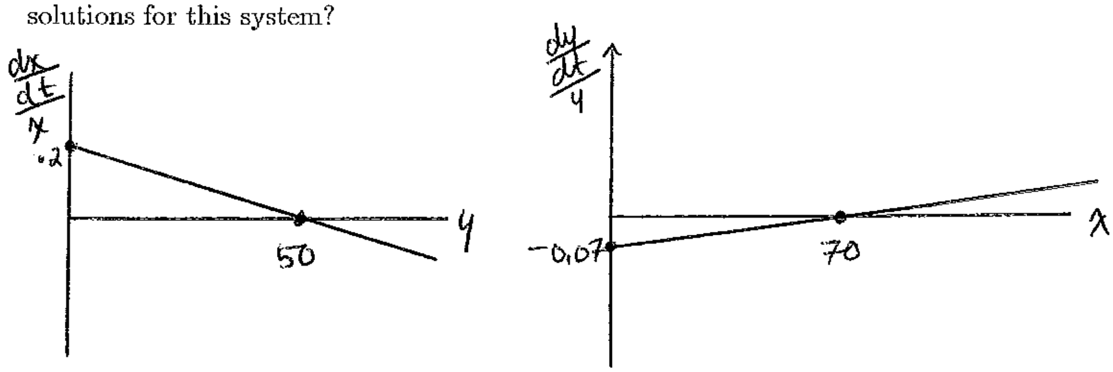
6. The following pair of equations represent the population growth of two different species where one is the predator and the other is the prey.

$$\frac{dx}{dt} = (0.2 - 0.004y)x \;= .2x - .004xy$$
$$\frac{dy}{dt} = (0.001x - 0.07)y \;= -0.07y + 0.001xy$$

(a) [2] Which of the variables represents the predator population? Explain.

y represents the predator population since in the absence of prey, $\frac{dy}{dt} = -0.07y$, ie the pop^n will die out. Also, encounters with prey are beneficial as indicated by the positive coefficient of the interaction term, $+0.001xy$.

(b) [3] Sketch the per capita production rates for both x and y. What are the equilibrium solutions for this system?



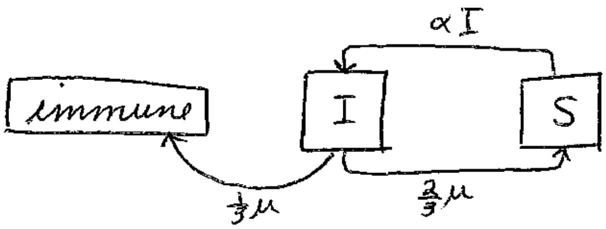
The equilibrium sol^ns are $x=0$ and $y=0$ or $x=70$ and $y=50$.

7. A system of differential equations for a disease model which measures how both the fraction of infected individuals, I, and the fraction of susceptible individuals, S, changes over time is given by

$$\frac{dI}{dt} = \alpha IS - \mu I$$
$$\frac{dS}{dt} = -\alpha IS + \mu I$$

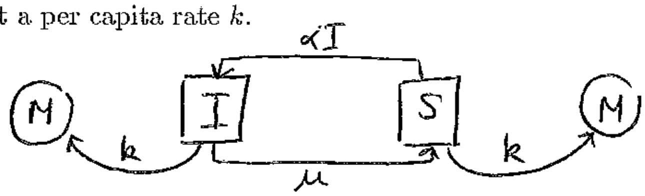
where α, $\mu > 0$. Modify this system of differential equations to reflect each situation described below.

(a) [2] Suppose that one-third of the individuals who leave the infected class through recovery become permanently immune and that the other two-thirds become susceptible again.



$$\frac{dI}{dt} = \alpha IS - \mu I$$
$$\frac{dS}{dt} = -\alpha IS + \frac{2}{3}\mu I$$

(b) [2] Suppose that all individuals become susceptible upon recovery (as in the basic model above) but that there is a source of mortality, so both infected and susceptible individuals die at a per capita rate k.



$$\frac{dI}{dt} = \alpha IS - \mu I - kI$$
$$\frac{dS}{dt} = -\alpha IS + \mu I - kS$$

THE END