MATHEMATICS 1LT3E TEST 2

Evening Class
Duration of Examination: 75 minutes
McMaster University, 18 July 2012

E. Clements

FIRST NAME (please print): SOL^{NS}

FAMILY NAME (please print): __________

Student No.: __________

THIS TEST HAS 8 PAGES AND 11 QUESTIONS. QUESTIONS BEGIN ON PAGE 2. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE.

Total number of points is 50. Marks are indicated next to the problem number. Any non-graphing calculator is allowed.

USE PEN TO WRITE YOUR TEST. IF YOU USE A PENCIL YOUR TEST WILL NOT BE ACCEPTED FOR REMARKING (IF NEEDED).

You need to show work to receive full credit.

Page	Points	Mark
2	7	
3	4	
4	8	
5	8	
6	8	
7	8	
8	7	
TOTAL	50	

Continued on next page

Name:________________

Student No.: ________________

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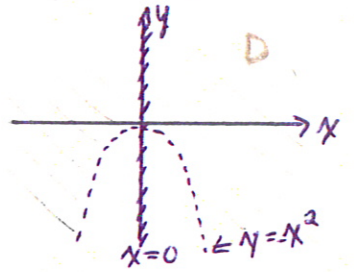
1. Find and sketch the domain of the following functions.

(a) [3] $f(x,y) = x^{-3}\ln(x^2+y)$

① $x \neq 0$

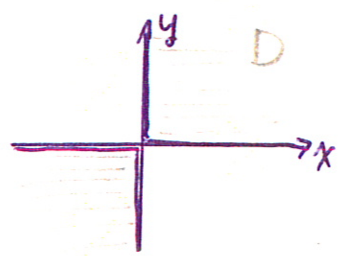
② $x^2+y>0 \Rightarrow y>-x^2$

$D = \{(x,y) \in \mathbb{R}^2 \mid x \neq 0 \text{ and } y > -x^2\}$



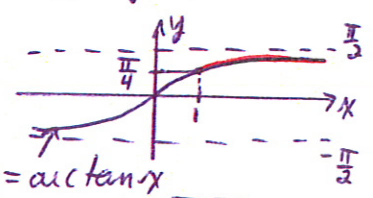
(b) [2] $g(x,y) = \sqrt{xy}$

$xy \geq 0 \Rightarrow$ ① $x \geq 0$ and $y \geq 0$ or ② $x \leq 0$ and $y \leq 0$



2. [2] Determine the range of $h(x,y) = \arctan(1+x^2+y^2)$. (Note: You do not have to provide a formal proof like we did in class, just explain your reasoning.)

$x^2+y^2 \geq 0 \Rightarrow x^2+y^2+1 \geq 1$



$y = \arctan x$

range: $y \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$\arctan 1 = \frac{\pi}{4}$

$\frac{\pi}{4} \leq \arctan(1+x^2+y^2) < \frac{\pi}{2}$

$\therefore$ range: $z \in [\frac{\pi}{4}, \frac{\pi}{2})$

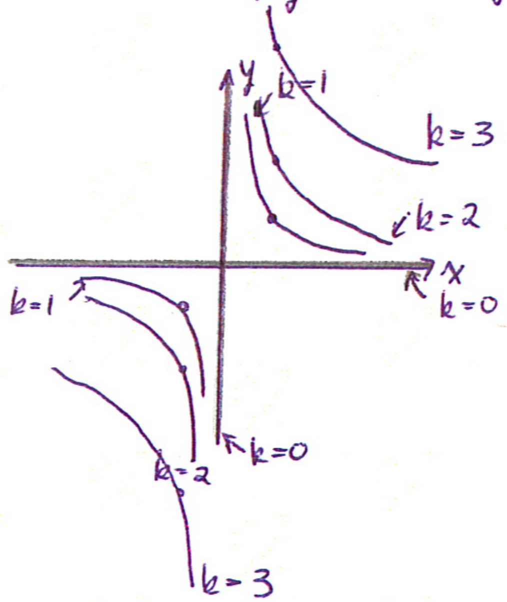
3. Consider the function $g(x, y) = \sqrt{xy}$.

(a) [3] State the range of g. Create a contour map for g showing at least 4 level curves.

$\sqrt{xy} \geq 0$ ... range : $z \geq 0$

level curves: $\sqrt{xy} = k$, where $k \geq 0$

$xy = k^2 \Rightarrow y = \frac{k^2}{x}$



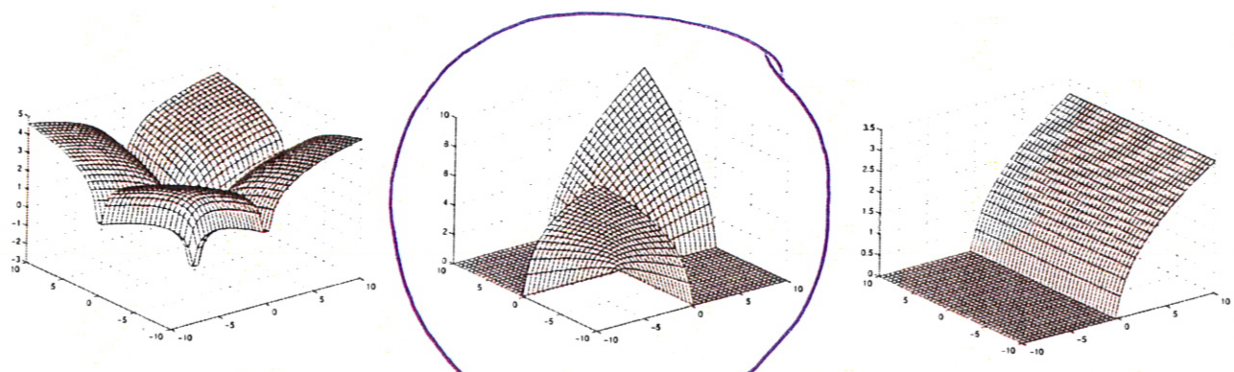
when $k=0 \Rightarrow xy = 0 \Rightarrow x=0$ or $y=0$

$k=1 \Rightarrow y = \frac{1}{x}$

$k=2 \Rightarrow y = \frac{4}{x}$

$k=3 \Rightarrow y = \frac{9}{x}$

(b) [1] Circle the graph which best represents the graph of the function $g(x, y) = \sqrt{xy}$.



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4. (a) [2] Explain why we cannot show that $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exists by computing limits along various paths to $(0,0)$.

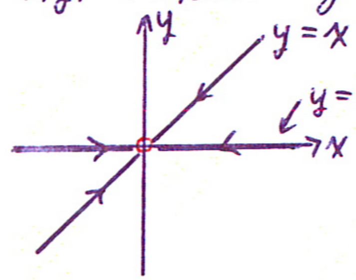
We would have to compute the limit along EVERY path to $(0,0)$ in the domain of f (infinitely many paths!) and this is impossible to do!

(b) [3] Show that $\lim\limits_{(x,y)\to(0,0)} \overbrace{\frac{(x-y)^2}{x^2+y^2}}^{f}$ does not exist. Sketch the domain of the function and the paths involved.

$f(x,0) = \dfrac{x^2}{x^2+0^2} = 1 \Rightarrow f \to 1$ as $(x,y)\to(0,0)$ along $y=0$

$f(x,x) = \dfrac{0^2}{x^2+x^2} = \dfrac{0}{2x^2} = 0 \Rightarrow f \to 0$ as $(x,y)\to(0,0)$ along $y=x$

$\therefore \lim\limits_{(x,y)\to(0,0)} f(x,y)$ D.N.E.



5. [3] Use the definition of continuity to show that $f(x,y) = \begin{cases} \dfrac{1-y}{x^2} & \text{if } (x,y) \neq (1,-1) \\ 2 & \text{if } (x,y) = (1,-1) \end{cases}$

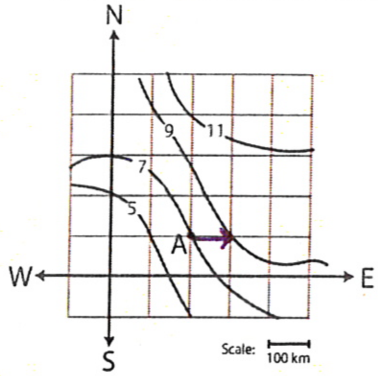
is continuous at $(1,-1)$.

① $\lim\limits_{(x,y)\to(1,-1)} f(x,y) = \lim\limits_{(x,y)\to(1,-1)} \dfrac{1-y}{x^2} = \dfrac{1-(-1)}{1^2} = 2$

② $f(1,-1) = 2$

③ Since $\lim\limits_{(x,y)\to(1,-1)} f(x,y) = f(1,-1)$, by the def^N of continuity, f is continuous at $(1,-1)$.

6. (a) [2] Given the contour map for the temperature function, T, in degrees Celsius, for a certain area, approximate the rate of change in temperature at the point A as we travel east. Remember to include units!



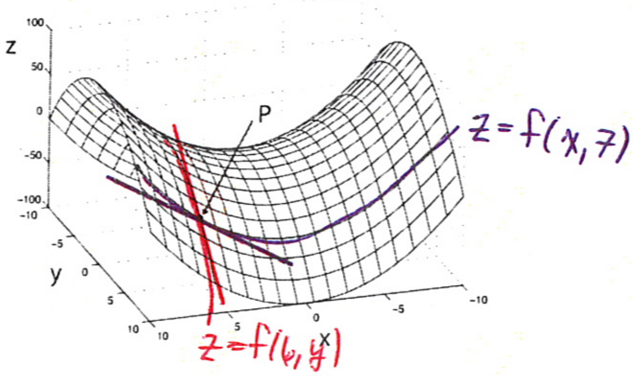
$$\text{avg rate of change} = \frac{9-7}{100} = 0.02\ \frac{°C}{km}$$

At the point A, the temperature is increasing at about 0.02°C / km as we move east.

(b) [2] In which direction, north or east, is the rate of change in temperature the greatest at A? Explain.

In both directions, the temperature is increasing but the next level curve is closer when we move to the east. So, the rate of change in temperature is greatest when we move east.

7. Consider the graph of $z = x^2 - y^2$ below.



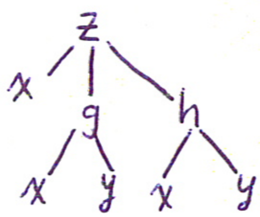
(a) [2] Draw and label the curves $z = f(x,7)$ and $z = f(6,y)$ and the tangent lines to these curves at $P(6,7,-13)$.

(b) [2] Determine the signs of $f_x(\overset{(6,7)}{\cancel{1,2}})$ and $f_y(\overset{(6,7)}{\cancel{1,2}})$. at (6,7).

$f_x(6,7) > 0$

$f_y(6,7) < 0$

8. [3] Suppose that $z = F(x, g(x, y), h(x, y))$. Sketch a tree diagram and find formulas for $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.



$$\frac{\partial z}{\partial x} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial g} \cdot \frac{\partial g}{\partial x} + \frac{\partial F}{\partial h} \cdot \frac{\partial h}{\partial x}$$

$$\frac{\partial z}{\partial y} = \frac{\partial F}{\partial g} \cdot \frac{\partial g}{\partial y} + \frac{\partial F}{\partial h} \cdot \frac{\partial h}{\partial y}$$

9. The number of wolves W in a certain fixed region depends on both the availability of food F and the distance L from urban areas. Suppose that $\frac{\partial W}{\partial F} = 0.4$ and $\frac{\partial W}{\partial L} = 0.6$.

(a) [2] What is the significance of the signs of these partial derivatives?

$\frac{\partial W}{\partial F} = 0.4 > 0 \Rightarrow$ as food availability increases, # of wolves increases

$\frac{\partial W}{\partial L} = 0.6 > 0 \Rightarrow$ as distance from urban areas increases, # of wolves increases

(b) [3] Suppose that over time, the availability of food decreases at a rate of 2 units per year, and the urban areas grow, shortening the distance to the wolves' habitat by 0.4 km per year. Estimate the current change $\frac{dW}{dt}$ in the population of wolves. Remember to include units!

$\frac{dF}{dt} = -2$ $\qquad$ $\frac{dL}{dt} = -0.4$

$$\frac{dW}{dt} = \frac{\partial W}{\partial F} \cdot \frac{dF}{dt} + \frac{\partial W}{\partial L} \cdot \frac{dL}{dt}$$

$$= (0.4)(-2) + (0.6)(-0.4)$$

$$= -1.04 \text{ wolves / year}$$

$\therefore$ The wolve pop^n is decreasing by about one wolve per year

10. (a) [2] True or False: If $f_x(x,y)$ and $f_y(x,y)$ exist at (a,b), then $f(x,y)$ is differentiable at (a,b). Explain.

FALSE!

. If f_x and f_y are continuous in $B_r(a,b)$ for some $r>0$, then f is differentiable at (a,b).

(b) [2] Compute the partial derivatives of $f(x,y) = \arctan\left(\frac{x}{y}\right)$. $y \neq 0$

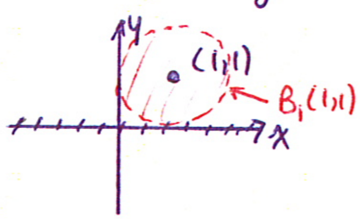
$$f_x = \frac{1}{1+\left(\frac{x}{y}\right)^2}\left(\frac{1}{y}\right) = \frac{1}{y + \frac{x^2}{y}} = \frac{y}{y^2+x^2}$$

$$f_y = \frac{1}{1+\left(\frac{x}{y}\right)^2}\left(-\frac{x}{y^2}\right) = \frac{-x}{y^2+x^2}$$

(c) [2] Find and sketch the domain of $f_x(x,y)$ and $f_y(x,y)$.

domain of f_x and $f_y \subseteq$ domain f

restriction: $y \neq 0$



(d) [2] Explain why the function $f(x,y) = \arctan\left(\frac{x}{y}\right)$ is differentiable at (1,1). What is the largest open disk centred at (1,1) that you can use?

f_x and f_y are continuous on their domains and $B_1(1,1)$ belongs to their domain so f_x and f_y are continuous on $B_1(1,1) \Rightarrow f$ is differentiable at $(1,1)$.

$= 2x \cdot \ln y$

11. (a) [4] Find the directional derivative of the function $f(x, y) = x \ln y^2 + \frac{x}{y}$ at the point (2,1) in the direction of the vector $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j}$.

$\|\vec{v}\| = \sqrt{3^2 + 4^2} = 5$

$\hat{u} = \frac{3}{5}\hat{i} + \frac{4}{5}\hat{j}$

$f_x = \ln y^2 + \frac{1}{y} \quad \ldots \quad f_x(2,1) = 1$

$f_y = \frac{2x}{y} - \frac{x}{y^2} \quad \ldots \quad f_y(2,1) = 2$

$D_{\vec{v}} f(2,1) = 1 \cdot \frac{3}{5} + 2 \cdot \frac{4}{5} = \frac{11}{5}$

(b) [1] What does this number tell us about the function f at the point (2,1)?

f is increasing at (2,1) when we move in the direction defined by $\vec{v} = 3\hat{i} + 4\hat{j}$.

(c) [2] What is the maximum rate of change of f at (2,1)? In which direction does this occur?

max rate of change $= \|\nabla f(2,1)\|$

$\nabla f(2,1) = 1\hat{i} + 2\hat{j}$

$\|\nabla f(2,1)\| = \sqrt{1^2 + 2^2} = \sqrt{5}$

The max. rate of change is $\sqrt{5}$ and occurs in the direction of the gradient at (2,1), ie $\nabla f(2,1) = 1\hat{i} + 2\hat{j}$.

THE END