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10.1 COMPLEX NUMBERS

In this section we shall review the definition of a complex number and discuss the addition, subtraction, and multiplication of such numbers. We will also consider matrices with complex entries and explain how addition and subtraction of complex numbers can be viewed as operations on vectors.

Complex Numbers

Since $x^2 \geq 0$ for every real number *x*, the equation $x^2 = -1$ has no real solutions. To deal with this problem, mathematicians of the eighteenth century introduced the "imaginary" number,

$$i = \sqrt{-1}$$

which they assumed had the property

$$i^2 = (\sqrt{-1})^2 = -1$$

but which otherwise could be treated like an ordinary number. Expressions of the form

$$a + bi \tag{1}$$

where *a* and *b* are real numbers, were called "complex numbers," and these were manipulated according to the standard rules of arithmetic with the added property that $i^2 = -1$.

By the beginning of the nineteenth century it was recognized that a complex number 1 could be regarded as an alternative symbol for the ordered pair

$$(a, b)$$

of real numbers, and that operations of addition, subtraction, multiplication, and division could be defined on these ordered pairs so that the familiar laws of arithmetic hold and $i^2 = -1$. This is the approach we will follow.

DEFINITION

A ***complex number*** is an ordered pair of real numbers, denoted either by (a, b) or by $a + bi$, where $i^2 = -1$.

EXAMPLE 1 Two Notations for a Complex Number

Some examples of complex numbers in both notations are as follows:

Ordered Pair	Equivalent Notation
(3, 4)	$3+4i$
(−1, 2)	$-1+2i$
(0, 1)	$0+i$
(2, 0)	$2+0i$
(4, −2)	$4+(-2)i$

For simplicity, the last three complex numbers would usually be abbreviated as

$$0+i=i, \qquad 2+0i=2, \qquad 4+(-2)i=4-2i$$

Geometrically, a complex number can be viewed as either a point or a vector in the xy-plane (Figure 10.1.1).

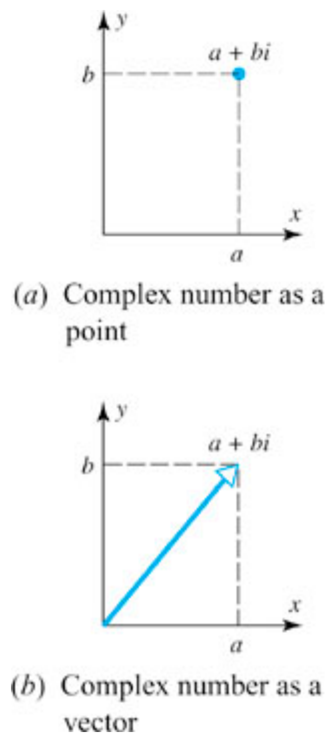


Figure 10.1.1

EXAMPLE 2 Complex Numbers as Points and as Vectors

Some complex numbers are shown as points in Figure 10.1.2*a* and as vectors in Figure 10.1.2*b*.

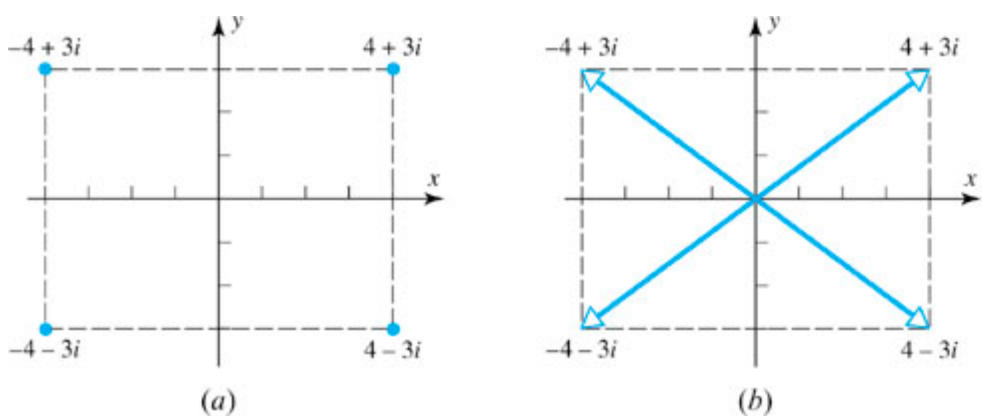


Figure 10.1.2

The Complex Plane

Sometimes it is convenient to use a single letter, such as z, to denote a complex number. Thus we might write

$$z = a + bi$$

The real number a is called the ***real part of z***, and the real number b is called the ***imaginary part of z***. These numbers are denoted by $\text{Re}(z)$ and $\text{Im}(z)$, respectively. Thus

$$\text{Re}(4 - 3i) = 4 \quad \text{and} \quad \text{Im}(4 - 3i) = -3$$

When complex numbers are represented geometrically in an xy-coordinate system, the x-axis is called the ***real axis***, the y-axis is called the ***imaginary axis***, and the plane is called the ***complex plane*** (Figure 10.1.3). The resulting plot is called an ***Argand diagram***.

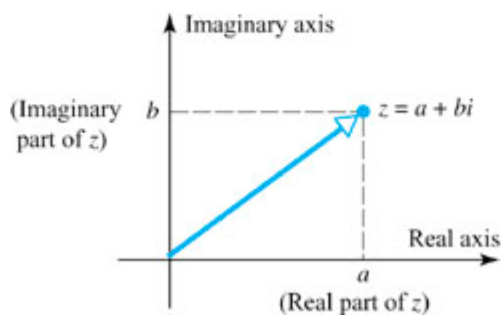


Figure 10.1.3
Argand diagram.

Operations on Complex Numbers

Just as two vectors in R^2 are defined to be equal if they have the same components, so we define two complex numbers to be equal if their real parts are equal and their imaginary parts are equal:

DEFINITION

Two complex numbers, $a + bi$ and $c + di$, are defined to be ***equal***, written

$$a + bi = c + di$$

if $a = c$ and $b = d$.

If $b = 0$, then the complex number $a + bi$ reduces to $a + 0i$, which we write simply as a. Thus, for any real number a,

$$a = a + 0i$$

so the real numbers can be regarded as complex numbers with an imaginary part of zero. Geometrically, the real numbers correspond to points on the real axis. If we have $a = 0$, then $a + bi$

reduces to $0 + bi$, which we usually write as bi. These complex numbers, which correspond to points on the imaginary axis, are called ***pure imaginary numbers***.

Just as vectors in R^2 are added by adding corresponding components, so complex numbers are added by adding their real parts and adding their imaginary parts:

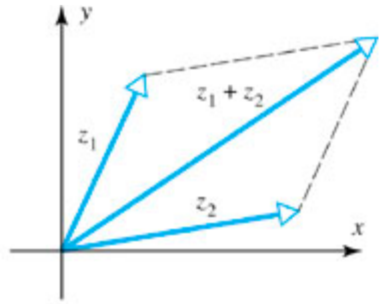
$$(a + bi) + (c + di) = (a + c) + (b + d)i \tag{2}$$

The operations of subtraction and multiplication by a *real* number are also similar to the corresponding vector operations in R^2:

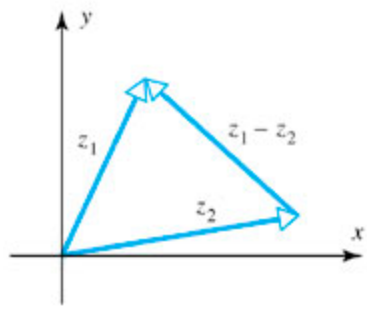
$$(a + bi) - (c + di) = (a - c) + (b - d)i \tag{3}$$

$$k(a + bi) = (ka) + (kb)i, \qquad k \text{ real} \tag{4}$$

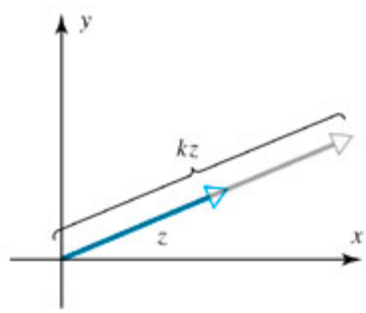
Because the operations of addition, subtraction, and multiplication of a complex number by a real number parallel the corresponding operations for vectors in R^2, the familiar geometric interpretations of these operations hold for complex numbers (see Figure 10.1.4).



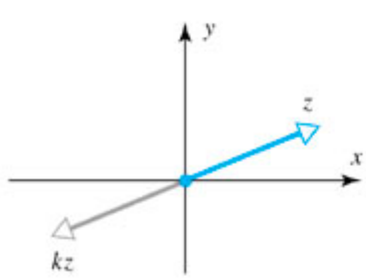
(*a*) The sum of two complex numbers



(*b*) The difference of two complex numbers



(*c*) The product of a complex number z and a positive real number k



(*d*) The product of a complex number z and a negative

Figure 10.1.4

It follows from 4 that $(-1)z + z = 0$ (verify), so we denote $(-1)z$ as $-z$ and call it the ***negative of z***.

EXAMPLE 3 Adding, Subtracting, and Multiplying by Real Numbers

If $z_1 = 4 - 5i$ and $z_2 = -1 + 6i$, find $z_1 + z_2$, $z_1 - z_2$, $3z_1$, and $-z_2$.

Solution

$$\begin{aligned} z_1 + z_2 &= (4 - 5i) + (-1 + 6i) = (4 - 1) + (-5 + 6)i = 3 + i \\ z_1 - z_2 &= (4 - 5i) - (-1 + 6i) = (4 + 1) + (-5 - 6)i = 5 - 11i \\ 3z_1 &= 3(4 - 5i) = 12 - 15i \\ -z_2 &= (-1)z_2 = (-1)(-1 + 6i) = 1 - 6i \end{aligned}$$

So far, there has been a parallel between complex numbers and vectors in R^2. However, we now define multiplication of complex numbers, an operation with no vector analog in R^2. To motivate the definition, we expand the product

$$(a + bi)(c + di)$$

following the usual rules of algebra but treating i^2 as −1. This yields

$$\begin{aligned} (a + bi)(c + di) &= ac + bdi^2 + adi + bci \\ &= (ac - bd) + (ad + bc)i \end{aligned}$$

which suggests the following definition:

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i \tag{5}$$

EXAMPLE 4 Multiplying Complex Numbers

$$(3+2i)(4+5i) = (3\cdot 4 - 2\cdot 5) + (3\cdot 5 + 2\cdot 4)i$$
$$= 2 + 23i$$
$$(4-i)(2-3i) = [4\cdot 2 - (-1)(-3)] + [(4)(-3) + (-1)(2)]i$$
$$= 5 - 14i$$
$$i^2 = (0+i)(0+i) = (0\cdot 0 - 1\cdot 1) + (0\cdot 1 + 1\cdot 0)i = -1$$

We leave it as an exercise to verify the following rules of complex arithmetic:

$$z_1 + z_2 = z_2 + z_1$$
$$z_1 z_2 = z_2 z_1$$
$$z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$$
$$z_1(z_2 z_3) = (z_1 z_2) z_3$$
$$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$$
$$0 + z = z$$
$$z + (-z) = 0$$
$$1 \cdot z = z$$

These rules make it possible to multiply complex numbers without using Formula 5 directly. Following the procedure used to motivate this formula, we can simply multiply each term of $a+bi$ by each term of $c+di$, set $i^2 = -1$, and simplify.

EXAMPLE 5 Multiplication of Complex Numbers

$$(3+2i)(4+i) = 12 + 3i + 8i + 2i^2 = 12 + 11i - 2 = 10 + 11i$$
$$\left(5 - \frac{1}{2}i\right)(2+3i) = 10 + 15i - i - \frac{3}{2}i^2 = 10 + 14i + \frac{3}{2} = \frac{23}{2} + 14i$$
$$i(1+i)(1-2i) = i(1 - 2i + i - 2i^2) = i(3 - i) = 3i - i^2 = 1 + 3i$$

Remark Unlike the real numbers, there is no size ordering for the complex numbers. Thus, the order symbols <, ≤, >, and ≥ are not used with complex numbers.

Now that we have defined addition, subtraction, and multiplication of complex numbers, it is possible to add, subtract, and multiply matrices with complex entries and to multiply a matrix by a complex number. Without going into detail, we note that the matrix operations and terminology discussed in

Chapter 1 carry over without change to matrices with complex entries.

EXAMPLE 6 Matrices with Complex Entries

If

$$A = \begin{bmatrix} 1 & -i \\ 1+i & 4-i \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} i & 1-i \\ 2-3i & 4 \end{bmatrix}$$

then

$$A+B = \begin{bmatrix} 1+i & 1-2i \\ 3-2i & 8-i \end{bmatrix}, \qquad A-B = \begin{bmatrix} 1-i & -1 \\ -1+4i & -i \end{bmatrix}$$

$$iA = \begin{bmatrix} i & -i^2 \\ i+i^2 & 4i-i^2 \end{bmatrix} = \begin{bmatrix} i & 1 \\ -1+i & 1+4i \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & -i \\ 1+i & 4-i \end{bmatrix} \begin{bmatrix} i & 1-i \\ 2-3i & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \cdot i + (-i) \cdot (2-3i) & 1 \cdot (1-i) + (-i) \cdot 4 \\ (1+i) \cdot i + (4-i) \cdot (2-3i) & (1+i) \cdot (1-i) + (4-i) \cdot 4 \end{bmatrix}$$

$$= \begin{bmatrix} -3-i & 1-5i \\ 4-13i & 18-4i \end{bmatrix}$$

Exercise Set 10.1

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1. In each part, plot the point and sketch the vector that corresponds to the given complex number.

(a) $2+3i$

(b) -4

(c) $-3-2i$

(d) $-5i$

2. Express each complex number in Exercise 1 as an ordered pair of real numbers.

3. In each part, use the given information to find the real numbers *x* and *y*.

(a) $x-iy=-2+3i$

(b) $(x+y)+(x-y)i=3+i$

4. Given that $z_1=1-2i$ and $z_2=4+5i$, find

(a) z_1+z_2

(b) z_1-z_2

(c) $4z_1$

(d) $-z_2$

(e) $3z_1+4z_2$

(f) $\frac{1}{2}z_1-\frac{3}{2}z_2$

5. In each part, solve for *z*.

(a) $z+(1-i)=3+2i$

(b) $-5z=5+10i$

(c) $(i-z)+(2z-3i)=-2+7i$

6. In each part, sketch the vectors z_1, z_2, z_1+z_2, and z_1-z_2.

(a) $z_1=3+i, z_2=1+4i$

(b) $z_1=-2+2i, z_2=4+5i$

7. In each part, sketch the vectors z and kz.

(a) $z=1+i, k=2$

(b) $z=-3-4i, k=-2$

(c) $z=4+6i, k=\frac{1}{2}$

8. In each part, find real numbers k_1 and k_2 that satisfy the equation.

(a) $k_1i+k_2(1+i)=3-2i$

(b) $k_1(2+3i)+k_2(1-4i)=7+5i$

9. In each part, find z_1z_2, z_1^2, and z_2^2.

(a) $z_1=3i, z_2=1-i$

(b) $z_1=4+6i, z_2=2-3i$

(c) $z_1=\frac{1}{3}(2+4i), z_2=\frac{1}{2}(1-5i)$

10. Given that $z_1=2-5i$ and $z_2=-1-i$, find

(a) $z_1-z_1z_2$

(b) $(z_1+3z_2)^2$

(c) $[z_1+(1+z_2)]^2$

(d) $iz_2-z_1^2$

In Exercises 11–18 perform the calculations and express the result in the form $a+bi$.

11. $(1+2i)(4-6i)^2$

12. $(2-i)(3+i)(4-2i)$

13. $(1-3i)^3$

14. $i(1+7i)-3i(4+2i)$

15. $\left[(2+i)\left(\frac{1}{2}+\frac{3}{4}i\right)\right]^2$

16. $(\sqrt{2}+i)-i\sqrt{2}(1+\sqrt{2}i)$

17. $(1+i+i^2+i^3)^{100}$

18. $(3-2i)^2-(3+2i)^2$

19. Let

$$A=\begin{bmatrix} 1 & i \\ -i & 3 \end{bmatrix}, \qquad B=\begin{bmatrix} 2 & 2+i \\ 3-i & 4 \end{bmatrix}$$

Find

(a) $A+3iB$

(b) BA

(c) AB

(d) B^2-A^2

20. Let

$$A=\begin{bmatrix} 3+2i & 0 \\ -i & 2 \\ 1+i & 1-i \end{bmatrix}, \qquad B=\begin{bmatrix} -i & 2 \\ 0 & i \end{bmatrix}, \qquad C=\begin{bmatrix} -1-i & 0 & -i \\ 3 & 2i & -5 \end{bmatrix}$$

Find

(a) $A(BC)$

(b) $(BC)A$

(c) $(CA)B^2$

(d) $(1+i)(AB)+(3-4i)A$

21. Show that

(a) $\operatorname{Im}(iz) = \operatorname{Re}(z)$

(b) $\operatorname{Re}(iz) = -\operatorname{Im}(z)$

22. In each part, solve the equation by the quadratic formula and check your results by substituting the solutions into the given equation.

(a) $z^2+2z+2=0$

(b) $z^2-z+1=0$

23.

(a) Show that if *n* is a positive integer, then the only possible values for i^n are 1, −1, *i*, and $-i$.

(b) Find i^{2509}.

24. Prove: If $z_1 z_2 = 0$, then $z_1 = 0$ or $z_2 = 0$.

25. Use the result of Exercise 24 to prove: If $zz_1 = zz_2$ and $z \neq 0$, then $z_1 = z_2$.

26. Prove that for all complex numbers z_1, z_2, and z_3,

(a) $z_1 + z_2 = z_2 + z_1$

(b) $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$

27. Prove that for all complex numbers z_1, z_2, and z_3,

(a) $z_1 z_2 = z_2 z_1$

(b) $z_1 (z_2 z_3) = (z_1 z_2) z_3$

28. Prove that $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$ for all complex numbers z_1, z_2, and z_3.

29. In quantum mechanics the ***Dirac matrices*** are

$$\beta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad \alpha_x = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

$$\alpha_y = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix}, \quad \alpha_z = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

(a) Prove that $\beta^2 = \alpha_x^2 = \alpha_y^2 = \alpha_z^2 = I$.

30. Describe the set of all complex numbers $z = a + bi$ such that $a^2 + b^2 = 1$. Show that if z_1, z_2 are such numbers, then so is $z_1 z_2$.