12C3 Last Day C-S Inequality:

$$|\langle \vec{u}, \vec{v} \rangle| \leq \|\vec{u}\|\|\vec{v}\|$$

From this we see if $\vec{u}, \vec{v} \neq \vec{0} \Rightarrow \dfrac{|\langle \vec{u}, \vec{v} \rangle|}{\|\vec{u}\|\|\vec{v}\|} \leq 1$

$\Rightarrow$ Let $\boxed{\cos\theta = \dfrac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\|\|\vec{v}\|}}$ } geometric intepetation.

& in all cases define if $\vec{u}, \vec{v} \neq 0$, $\langle \vec{u}, \vec{v} \rangle = 0$

$\Rightarrow$ $\vec{u}, \vec{v}$ are orthogonal

Defn

$\operatorname{proj}_{\vec{v}} \vec{u}$ = "orthogonal projection of $\vec{u}$ onto $\vec{v}$"

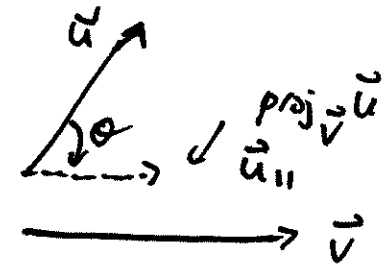
" orthog. component of $\vec{u}$ on $\vec{v}$"

" component parallel to $\vec{v}$" = $\vec{u}_{\parallel}$

$$\operatorname{proj}_{\vec{v}} \vec{u} = \|\vec{u}\| \cos\theta \, \frac{\vec{v}}{\|\vec{v}\|}$$

$\uparrow$ unit vector of $\vec{v}$

$= \hat{v}$



but $\cos\theta = \dfrac{\langle u, v \rangle}{\|\vec{u}\| \, \|\vec{v}\|}$

$$\Rightarrow \operatorname{proj}_{\vec{v}} \vec{u} = \cancel{\|\vec{u}\|} \frac{\langle \vec{u}, \vec{v} \rangle}{\cancel{\|\vec{u}\|} \, \|\vec{v}\|} \cdot \frac{\vec{v}}{\|\vec{v}\|} = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{v}\|^2} \vec{v}$$

eg. Let $\vec{u} = (1, 3, 0)$ $\vec{v} = (2, 2, 1)$

Find $\text{proj}_{\vec{v}} \vec{u}$ (by <u>default</u> we'll be using <u>usual</u> <u>dot</u> <u>product</u>)

<u>Solution</u>

$$\text{proj}_{\vec{v}} \vec{u} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2} \vec{v} = \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \vec{v}$$

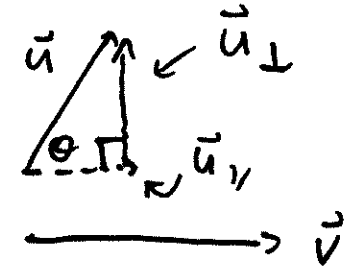
$$= \frac{1(2) + 3(2) + 0(1)}{2^2 + 2^2 + 1^2} \vec{v}$$

$$= \frac{8}{9} \vec{v} = \frac{8}{9}(2, 2, 1) = \left(\frac{16}{9}, \frac{16}{9}, \frac{8}{9}\right)$$

$$= \vec{u}_{\parallel}$$

"Component of $\vec{u}$ <u>orthogonal</u> to $\vec{v}$" $= \vec{u}_{\perp} = \text{orth}_{\vec{v}} \vec{u}$

$$\vec{u}_{\perp} = \vec{u} - \text{proj}_{\vec{v}} \vec{u} = \text{orth}_{\vec{v}} \vec{u}$$



Check orthogonality

$$\vec{v}\circ\vec{u}_\perp = \vec{v}\cdot\vec{u} - \vec{v}\cdot \mathrm{proj}_{\vec{v}}\vec{u}$$

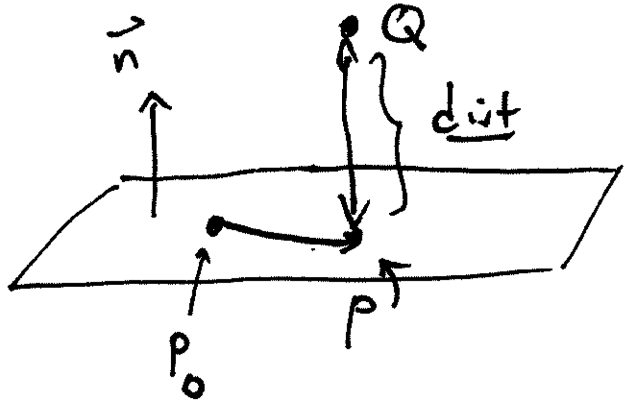
$$= \vec{v}\circ\vec{u} - \vec{v}\cdot\left(\frac{\langle \vec{u},\vec{v}\rangle}{\|\vec{v}\|^2}\vec{v}\right)$$

$$= \vec{u}\cdot\vec{v} - \frac{\vec{u}\cdot\vec{v}}{\cancel{\|\vec{v}\|^2}}\cancel{(\vec{v}\cdot\vec{v})} = \vec{u}\cdot\vec{v} - \vec{u}\cdot\vec{v} = 0$$

Yes! Orthogonal!

Common projection app: Point-Plane & Point-Line distances.

Point-Plane: Say I have a plane in $\mathbb{R}^3$ & a point, Q



If P_0 is any chosen particular point in plane, & P is any other point

$\Rightarrow$ $\overrightarrow{P_0P} \perp \vec{n}$

Alternate plane form

Parametric eqn. of plane

$$\vec{p} = \vec{p}_0 + t\vec{u} + s\vec{v}$$

t, s, parameters!

Distance from Q to plane means minimum distance between Q & points on the plane

Plane Recap

in $\mathbb{R}^3$

Plane: $ax + by + cz = d$

$(a,b,c) = \vec{n}$ normal $\perp$ to plane

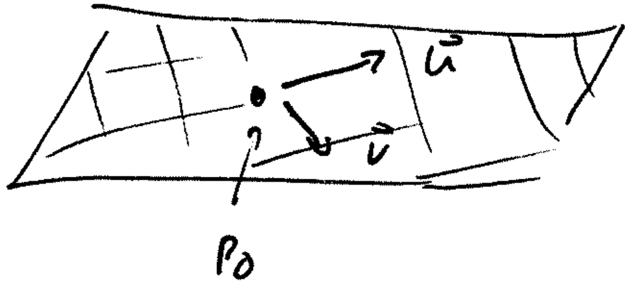
$$\vec{n} \cdot (\overrightarrow{P_0P}) = 0$$

$$\vec{n} \cdot (\vec{p} - \vec{p}_0) = 0$$

$$\vec{n} \cdot \vec{p} = \cancel{\vec{n} \cdot \vec{p}_0}\ d$$

(x, y, z)

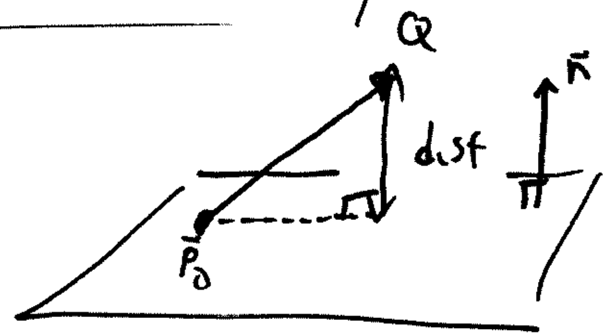
$$\vec{n} \cdot (x, y, z) = d$$



in $\mathbb{R}^3$

$\vec{u} \times \vec{v} = \vec{n}$

Back to our problem



Find the distance!

In general Pick a $\vec{P_0}$ dist = $\| \text{proj}_{\vec{n}} \vec{PQ} \|$

e.g. How far is $(1,3,2)$ from $3x + z = 5$

Solution $Q = (1,3,2)$ $\vec{n} = (3,0,1)$

Pick a P_0 that satisfies $3x + z = 5$ & is easy!

e.g. $P_0 = (0,0,5)$

$$\vec{PQ} = \vec{Q} - \vec{P} = (1,3,2) - (0,0,5)$$
$$= \underline{\underline{(1,3,-3)}}$$

$$\left\| \text{proj}_{\vec{n}} \vec{PQ} \right\| = \left\| \frac{(\vec{n} \cdot \vec{PQ})}{\|\vec{n}\|^2} \vec{n} \right\| = \frac{|\vec{n} \cdot \vec{PQ}|}{\|\vec{n}\|^{\cancel{2}}} \cancel{\|\vec{n}\|}$$

$$= \frac{\vec{n} \cdot \vec{PQ}}{\|\vec{n}\|} = \frac{|(3,0,1)\cdot(1,3,-3)|}{\sqrt{3^2+0^2+1^2}}$$

$$= \frac{3+0-3}{\sqrt{10}} = \boxed{0} \leftarrow Q \text{ in plane!}$$

<u>Oops!</u>

<u>Point - Line Distance</u>

<u>First</u>: <u>Line recap</u>

in $\mathbb{R}^2$ $\quad y = mx + b$, <u>or</u> $x = c$

<u>or</u> $\quad \underline{ax + by = c}$

<u>note</u> $\quad \underline{\underline{(a,b) \perp \text{ to line!}}}$

Other notation: (in any $\mathbb{R}^n$)

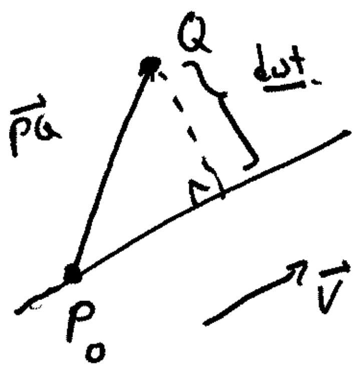
parameter!

$$(x, y, z) = \vec{p} = \vec{P_0} + t\vec{v}$$

"initial" point

$\vec{v}$ = direction vector

Now say I draw a line in $\mathbb{R}^n$ & I have a point Q.



Find the (minimum) dist. from Q to the line.

$$\text{dist.} = \| \text{orth}_{\vec{v}} \vec{PQ} \|$$

$$= \| \vec{PQ} - \text{proj}_{\vec{v}} \vec{PQ} \|$$

Next day I'll finish