MATH 1B03/1ZC3 Winter 2019

Lecture 16: Vectors in $\mathbb{R}^n$

Instructor: Dr Rushworth March 5th

Vectors in $\mathbb{R}^n$

(from Chapter 3.1 of Anton-Rorres)

In earlier sections of the course we have considered row and column vectors: $1 \times n$ and $n \times 1$ matrices, respectively. We will now look at vectors in more details, and consider some of their geometric properties.

Definition 16.1: A vector in $\mathbb{R}^n$

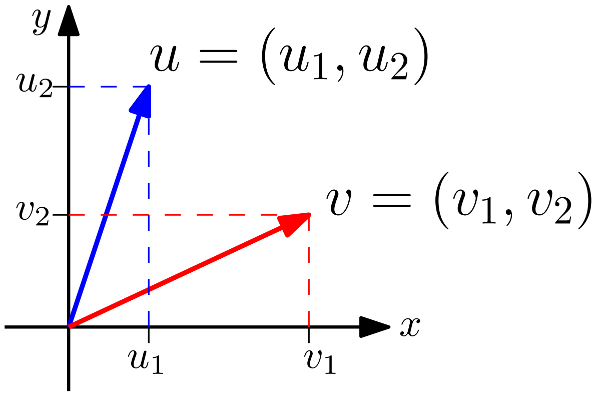
Given a positive integer n, a vector in $\mathbb{R}^n$ is an ordered list of n real numbers

$$\mathbf{u} = (u_1, u_2, \ldots, u_n)$$

The real number u_i is the i-th component of $\mathbf{u}$.
The collection of all vectors is $\mathbb{R}^n$, known as n-space.

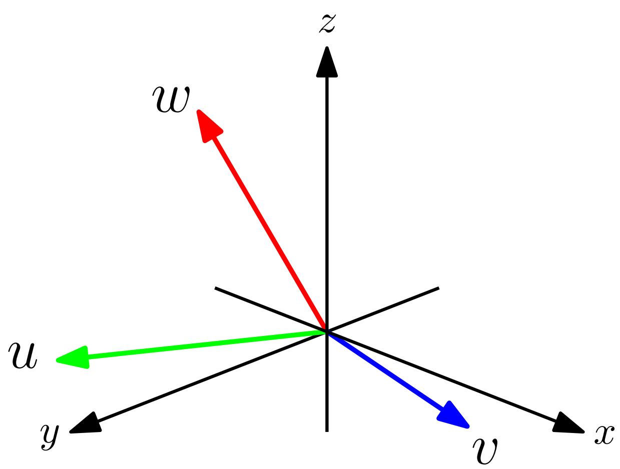
This definition is completely equivalent to the definition as $1 \times n$ and $n \times 1$ matrices. You can think of each entry in the list of number as a co-ordinate point on a set of axes.

The space $\mathbb{R}^2$ is the familiar (x, y)-plane with directed arrows:



We see that each vector has a direction, given by the arrowhead. We refer to the point at which the vector starts as the initial point, and the point at which it ends as the terminal point (these terms may be applied to vectors in any space $\mathbb{R}^n$).

The space $\mathbb{R}^3$ is also familiar: it is three dimensional space.



Pictures such as those above are not very helpful when $n \geq 4$. In these dimensions we must rely on the algebra. Much of the geometric interpretation is still valid in higher dimensions, however.

Why would we be interested in studying higher dimensional space anyway?

Much like complex numbers, the higher dimensional spaces $\mathbb{R}^n$ present new algebraic structures that are different to the usual numbers we have become used to. As we will see later, they are an example of a mathematical structure known as a *vector space*.

In engineering and physics, many complex systems can be conveniently described using vectors, for example: a 1080p monitor has 1920 rows of pixels and 1080 columns of pixels: this yields $2, 073, 600$ pixels in total. Each pixel has a *hue* and a *saturation* value. Therefore the image displayed on the monitor can be described by vector in

$$4, 147, 200 = \underbrace{(2, 073, 600)}_{\text{number of pixels}} \times \underbrace{(1 + 1)}_{\text{hue + saturation}}$$

dimensional space. In other words, the space $\mathbb{R}^{4, 147, 200}$ describes every possible image a 1080p montior can display.

Operations on vectors

These definitions are exactly equivalent to those given with respect to matrices earlier in the course.

Definition 16.2: Equality of vectors

Two vectors $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are equal, written $\mathbf{u} = \mathbf{v}$ if

$$\begin{aligned} u_1 &= v_1 \\ u_2 &= v_2 \\ &\vdots \\ u_n &= v_n \end{aligned}$$

That is, if their components are equal.

As we did with matrices, we define the zero vector as $\mathbf{0} = (0, 0, \ldots, 0)$.

Definition 16.3: Operations on vectors

Let $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ be vectors in $\mathbb{R}^n$. Define

1. Addition:

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n)$$

2. Subtraction:

$$\mathbf{u} - \mathbf{v} = (u_1 - v_1, u_2 - v_2, \ldots, u_n - v_n)$$

3. Scalar multiplication: let k be a real number, known as a *scalar*

$$k\mathbf{u} = (ku_1, ku_2, \ldots, ku_n)$$

Fact 16.4

Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ be vectors in $\mathbb{R}^n$, and k, m scalars. Then

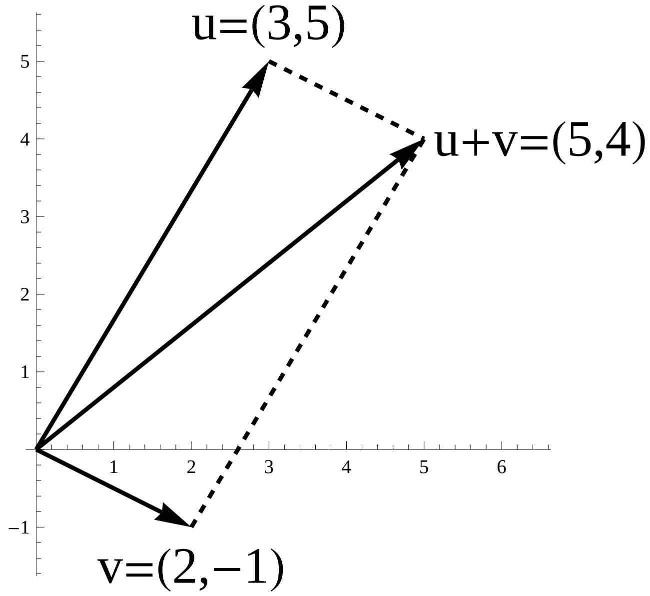
1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
3. $\mathbf{u} + \mathbf{0} = \mathbf{u}$
4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
5. $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$
6. $(k + m)\mathbf{u} = k\mathbf{u} + m\mathbf{u}$
7. $k(m\mathbf{u}) = (km)\mathbf{u}$
8. $1\mathbf{u} = \mathbf{u}$

All of these facts can be proved directly from the definitions given above. As usual, use these facts to speed up your calculations with vectors. It is often quick to use these properties and work with the vectors themselves, rather than their individual components.

Vector operations in $\mathbb{R}^2$ and $\mathbb{R}^3$

When we are working in $\mathbb{R}^2$ or $\mathbb{R}^3$, we have the benefit of being able to draw useful pictures. In fact, the intuition gained in this way extends into the higher dimensions also.

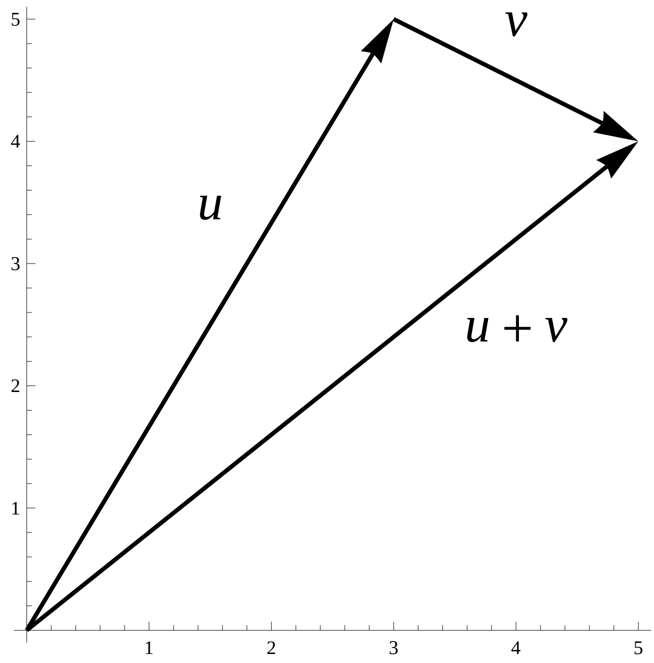
When we are in $\mathbb{R}^2$ we can add two vectors using the parallelogram rule. If the vectors have the same initial point, use them to form a parallelogram: their vector sum is then the vector bisecting this parallelogram. For example, if $\mathbf{u} = (3, 5)$ and $\mathbf{v} = (2, -1)$ then



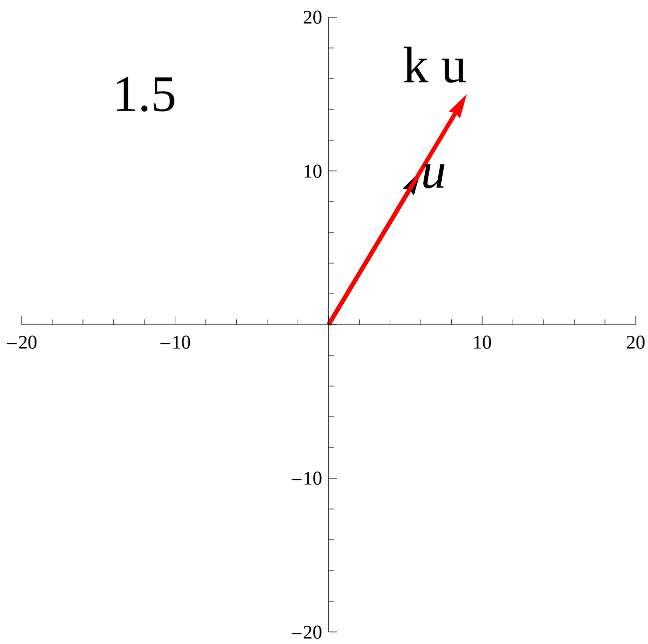
This is equivalent to vector addition defined above, so that

$$\begin{aligned}\mathbf{u} + \mathbf{v} &= (3, 5) + (2, -1)\\ &= (5, 4)\end{aligned}$$

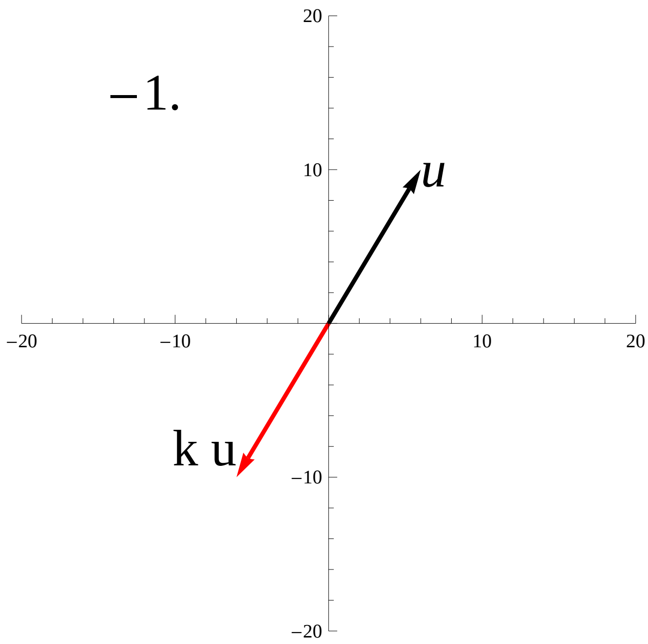
If $\mathbf{u}$ and $\mathbf{v}$ are end to end, we can also express $\mathbf{u} + \mathbf{v}$ as



We can also understand scalar multiplication this way. Let $\mathbf{u} = (6, 10)$ and $k = 1.5$, then



and when $k = -1$



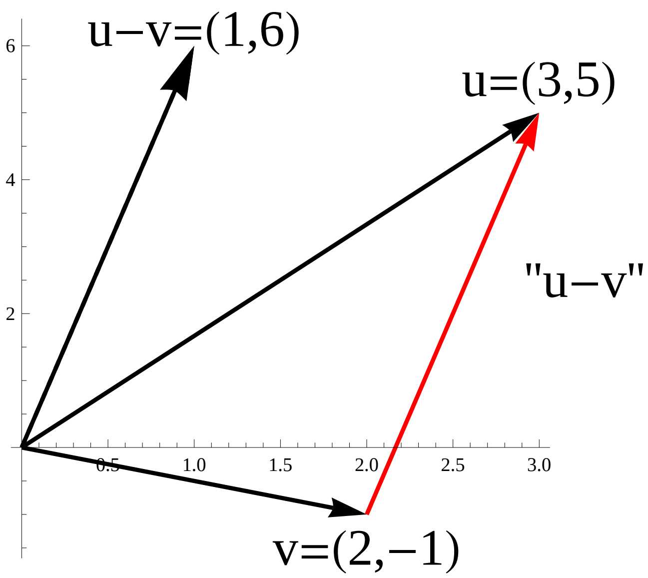
That is, multiplying a vector by a scalar does not change its direction, only its length.

Vector subtraction is similar, but here we need to be careful. When **writing a vector in co-ordinates we always place its initial point at the origin** i.e. at the zero vector. However, in certain situations we are permitted to think of vectors which do not have their initial point at the origin.

One such situation is as follows. Given vectors $\mathbf{u}$ and $\mathbf{v}$, we can think of the vector $\mathbf{u}-\mathbf{v}$ as the vector which points from the terminal point of $\mathbf{v}$ to the terminal point of $\mathbf{u}$, even though the vector $\mathbf{u}-\mathbf{v}$ actually has its initial point at the origin. This is justified as the two vectors have the **same direction and length**.

Therefore, if we were asked to find the distance between the terminal point of $\mathbf{u}$ and the terminal point of $\mathbf{v}$, we could compute the length of the vector $\mathbf{u}-\mathbf{v}$, even though it does not lie between the desired points.

For example, let $\mathbf{u}=(3,\,5)$ and $\mathbf{v}=(2,\,-1)$ again, then



Example 16.5

Question: Given $\mathbf{u} = (3, 5)$ and $\mathbf{v} = (2, -1)$ compute the distance between the terminal point of $\mathbf{u}$ and the terminal point of $\mathbf{v}$
Answer: The picture we need is given above. Although $\mathbf{u} - \mathbf{v}$ does not strictly lie between the terminal points of $\mathbf{u}$ and $\mathbf{v}$, is has the same length and direction of the vector that we are interested in.
Therefore the distance is given by the length of $\mathbf{u} - \mathbf{v} = (1, 6)$

$$\sqrt{1^2 + 6^2} = \sqrt{37}$$

These notions of vector addition, subtraction, and scalar multiplication can also be used in $\mathbb{R}^3$: when answering questions involving vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$, draw pictures!

Vectors which do not start at the origin

Related to the discussion above, given a vector whose initial point **is not** at the origin, we sometimes abuse notation and write this vector in terms of components (even though its initial point is not the origin).

Definition 16.6: Vectors via their endpoints

Let P and Q be points in $\mathbb{R}^n$. The vector with initial point Q and terminal point P is denoted by

$$\mathbf{u} = \overrightarrow{QP}$$

Recipe 16.7: Finding the components of a vector away from the origin

Given two points in $\mathbb{R}^n$, Q and P, use this recipe to find the components of the vector $\mathbf{u} = \overrightarrow{QP}$.

Step 1: Let $P = (p_1, p_2, \ldots, p_n)$ and $Q = (q_1, q_2, \ldots, q_n)$. Then

$$P - Q = (p_1 - q_1, p_2 - q_2, \ldots, p_n - q_n)$$

Step 2: The components of the vector $\mathbf{u} = \overrightarrow{QP}$ are then

$$\mathbf{u} = (p_1 - q_1, p_2 - q_2, \ldots, p_n - q_n)$$

Unless you are in this exact situation, and are specifically asked to compute the components of a vector with initial point away from the origin, you can assume that any vector written in terms of components, such as

$$\mathbf{u} = (u_1, u_2, \ldots, u_n)$$

has its initial point at the origin.

Example 16.8

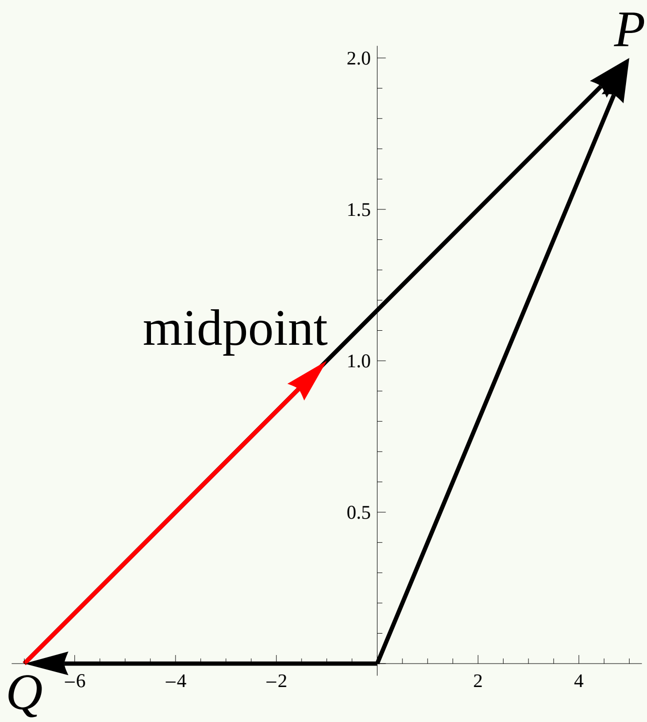
Question: Find the components of the vector starting at $P = (9, 0, -4)$ and ending at $Q = (-5, 2, 4)$.

Answer: We have

$$\begin{aligned}\mathbf{u} &= \overrightarrow{QP} \\ &= (9, 0, -4) - (-5, 2, 4) \\ &= (14, -2, -8)\end{aligned}$$

Example 16.9

Question: Let P be the point $(5, 2)$, and Q the point $(-7, 0)$.
Find the midpoint of the line joining P and Q.
Answer: Draw a picture:



Let O denote the origin, and let $\mathbf{p} = \overrightarrow{OP}$, and $\mathbf{q} = \overrightarrow{OQ}$.
The vector from Q to P is given by

$$\begin{aligned}\mathbf{p} - \mathbf{q} &= (5, 2) - (-7, 0) \\ &= (12, 2)\end{aligned}$$

and the mid point is given by the end point of the red vector, which is found by

$$\frac{1}{2}(12, 2) = (6, 1)$$

The midpoint of the line joining P and Q is therefore

$$\mathbf{q} + (6, 1) = (-1, 1)$$

Generalizing distance: length of vectors in $\mathbb{R}^n$

(from Chapter 3.2 of Anton-Rorres)

Recall that the magnitude (a.k.a. the length) of the vector $\mathbf{u} = (x, y)$ in $\mathbb{R}^2$ is given by

$$||\mathbf{u}|| = \sqrt{x^2 + y^2}.$$

A very similar formula allows us to compute the length of a vector in $\mathbb{R}^n$:

Definition 16.10: Norm of a vector

Let $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ be a vector in $\mathbb{R}^n$.
The norm of $\mathbf{u}$ is denoted $||\mathbf{u}||$ and is defined to be

$$||\mathbf{u}|| = \sqrt{u_1^2 + u_2^2 + \cdots u_n^2}$$

Example 16.11

If

$$\mathbf{u} = (1, -1, 4, 02)$$

then

$$\begin{aligned} ||\mathbf{u}|| &= \sqrt{1 + 1 + 16 + 4} \\ &= \sqrt{22} \end{aligned}$$

Notice that the norm takes a vector as input, and outputs a scalar. When working with vectors it is crucial to keep track of which quantities are scalars, and which quantities are vectors.

Question 16.12

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$. Which of the following are well-defined?

$$\mathbf{u} + \mathbf{v}$$
$$\mathbf{u} + ||\mathbf{v}||$$
$$||\mathbf{u}|| + ||\mathbf{v}||$$

Fact 16.13: Properties of the norm

Let $\mathbf{u}$ be a vector in $\mathbb{R}^n$, and k a scalar. Then

- $||\mathbf{u}|| \geq 0$
- $||\mathbf{u}|| = 0$ if and only if $\mathbf{u} = \mathbf{0}$
- $||k\mathbf{u}|| = k||\mathbf{u}||$

Definition 16.14: Unit vector

A unit vector is a vector $\mathbf{u}$ in $\mathbb{R}^n$ such that

$$||\mathbf{u}|| = 1$$

Given any vector $\mathbf{u}$ in $\mathbb{R}^n$ we can always produce a unit vector which points in the same direction as $\mathbf{u}$: the vector

$$\mathbf{v} = \frac{1}{||\mathbf{u}||}\mathbf{u}$$

is a unit vector, but as it is a scalar multiple of $\mathbf{u}$ it must point in the same direction. The process of producing the vector $\mathbf{v}$ is known as normalizing $\mathbf{u}$.

Question 16.15

Why is

$$\mathbf{v} = \frac{1}{||\mathbf{u}||}\mathbf{u}$$

a unit vector for any vector $\mathbf{u}$?

Definition 16.16: The standard unit vectors

The vectors

$$\mathbf{i} = (1, 0), \quad \mathbf{j} = (0, 1)$$

are the **standard unit vectors in** $\mathbb{R}^2$ (do not confuse with the complex number i).

The vectors

$$\mathbf{i} = (1, 0, 0), \quad \mathbf{j} = (0, 1, 0), \quad \mathbf{k} = (0, 0, 1)$$

are the **standard unit vectors in** $\mathbb{R}^3$

The vectors

$$\mathbf{e_1} = (1, 0, \ldots, 0), \quad \mathbf{e_1} = (0, 1, \ldots, 0), \ldots, \mathbf{e_n} = (0, 0, \ldots, 1)$$

are the **standard unit vectors in** $\mathbb{R}^n$.

Definition 16.17: Linear combination

Let $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}$ be a collection of vectors in $\mathbb{R}^n$. A linear combination of these vectors is a new vector

$$\mathbf{v} = a_1\mathbf{v_1} + a_2\mathbf{v_2} + \cdots + a_k\mathbf{v_k} \tag{1}$$

for $a_1, a_2, \ldots, a_k$ scalars.

Notice that every vector in $\mathbb{R}^n$ can be written as a linear combination of the standard unit vectors. For example

$$\begin{aligned}(4, -2, 0, 1) &= 4(1, 0, 0, 0) - 2(0, 1, 0, 0) + 0(0, 0, 1, 0) + (0, 0, 0, 1)\\ &= 4\mathbf{e_1} - 2\mathbf{e_2} + \mathbf{e_4}\end{aligned}$$

Distance between two points

We can use the norm to compute the distance between the endpoint points of two vectors.

Fact 16.18: Distance between two points

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}n$. The distance between the endpoints of $\mathbf{u}$ and $\mathbf{v}$ is given by

$$||\mathbf{u} - \mathbf{v}|| \tag{2}$$

Example 16.19

Question: Find the distance between the points $(0, -1, 3, -3)$ and $(2, -1, -2, 4)$.
Answer: We have

$$\begin{aligned}||\mathbf{u} - \mathbf{v}|| &= ||(0, -1, 3, -3) - (2, -1, -2, 4)||\\ &= ||(-2, 0, 5, -7)||\\ &= \sqrt{4 + 25 + 49}\\ &= \sqrt{78}\end{aligned}$$

The dot product

The norm takes a vector as input and outputs a scalar. We can define another operation which takes as input **two vectors**, and outputs a scalar. This operation is known as the dot product.

Definition 16.20: The dot product

Let $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ be vectors in $\mathbb{R}^n$. The dot product of $\mathbf{u}$ and $\mathbf{v}$ is denoted $\mathbf{u} \bullet \mathbf{v}$ and defined

$$\mathbf{u} \bullet \mathbf{v} = u_1v_1 + u_2v_2 + \ldots + u_nv_n$$

Example 16.21

Let $\mathbf{u} = (4, -1, 1, 4)$ $\mathbf{v} = (0, 8, -3, 2)$. Then

$$\begin{aligned} \mathbf{u} \bullet \mathbf{v} &= (4)(0) + (-1)(8) + (1)(-3) + (4)(2) \\ &= -8 - 3 + 8 \\ &= -3 \end{aligned}$$

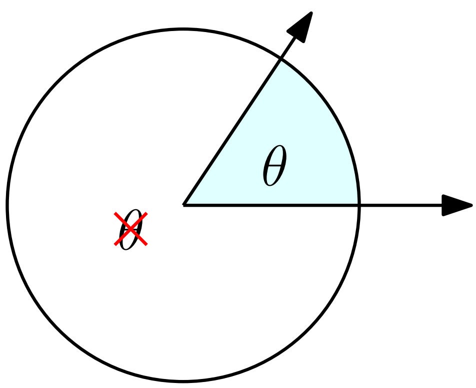
Fact 16.22

Let $\mathbf{u}$ be a vector in $\mathbb{R}^n$. Then

$$\mathbf{u} \bullet \mathbf{u} = ||\mathbf{u}||^2$$

What is the dot product of $\mathbf{u}$ and $\mathbf{v}$? It measures how much the vectors “overlap". This can be made precise as follows.

Given two vectors $\mathbf{u}$ and $\mathbf{v}$ we can compute the angle between them, denoted θ: we will always pick the angle so that $0 \leq \theta \leq \pi$. That is,



Fact 16.23

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors and θ the angle between them. Then

$$\mathbf{u} \bullet \mathbf{v} = ||\mathbf{u}||||\mathbf{v}|| \cos(\theta)$$

Proof: This can be proved in $\mathbb{R}^2$ using Pythagoras. The proof in $\mathbb{R}^n$ can then be reduced to the $\mathbb{R}^2$ case. ∎

As mentioned previously, when working with vectors is it very important to keep track of what is a vector and what is a scalar, to make sure what you are computing is well-defined. The formula given in Fact 16.23 is well defined as on the left, we have $\mathbf{u} \bullet \mathbf{v}$, which is defined to be a scalar. On the right we have the product of three scalars $||\mathbf{u}||$, $||\mathbf{v}||$, and $\cos(\theta)$, which is still a scalar. Therefore what we have written is well-defined.

By rearranging the formula in Fact 16.23 we can compute the angle between two given vectors.

Example 16.24

Question: Given $\mathbf{u} = (3, 0, 1)$ and $\mathbf{v} = (-1, -1, 0)$ compute the angle between $\mathbf{u}$ and $\mathbf{v}$.
Answer: Compute the required quantities

$$\begin{aligned} ||\mathbf{u}|| &= \sqrt{9+1} = \sqrt{10} \\ ||\mathbf{v}|| &= \sqrt{1+1} = \sqrt{2} \\ \mathbf{u} \bullet \mathbf{v} &= (3)(-1) + (0)(1) + (1)(0) \\ &= -3 \end{aligned}$$

Then by Fact 16.23 we have

$$\begin{aligned} \cos(\theta) &= \frac{\mathbf{u} \bullet \mathbf{v}}{||\mathbf{u}||||\mathbf{v}||} \\ &= -\frac{3}{\sqrt{2}\sqrt{10}} \\ &= -\frac{3}{2\sqrt{5}} \end{aligned}$$

Then

$$\begin{aligned}\theta &= \arccos\left(-\frac{3}{2\sqrt{5}}\right) \\ &\approxeq 2.31 \text{ rad} \\ &\approxeq 132.1 \text{ degrees}\end{aligned} \tag{3}$$

Suggested Problems

Practice the material covered in this lecture by attempting the following questions from Chapter 3.1 of Anton-Rorres, starting on page 141

- Questions $7, 13, 17, 19, 21, 23, 27$

and from Chapter 3.3 of Anton-Rorres, starting on page 153

- Questions $7, 13, 15, 17, 27$
- True/False (d), (e), (g), (h)