COMPSCI/SFWRENG 2FA3

Discrete Mathematics with Applications II

Winter 2020

Assignment 1 with Solutions

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Assignment 1 consists of some background definitions, two sample problems, and two required problems. You must write your solutions to the required problems using LaTeX. Use the solutions of the sample problems as a guide.

Please submit Assignment 1 as two files, `Assignment_1_`*`YourMacID`*`.tex` and `Assignment_1_`*`YourMacID`*`.pdf`, to the Assignment 1 folder on Avenue under Assessments/Assignments. *`YourMacID`* must be your personal MacID (written without capitalization). The `Assignment_1_`*`YourMacID`*`.tex` file is a copy of the LaTeX source file for this assignment (`Assignment_1.tex` found on Avenue under Contents/Assignments) with your solution entered after each required problem. The `Assignment_1_`*`YourMacID`*`.pdf` is the PDF output produced by executing

```
pdflatex Assignment_1_YourMacID
```

This assignment is due **Sunday, January 26, 2020 before midnight.** You are allow to submit the assignment multiple times, but only the last submission will be marked. **Late submissions and files that are not named exactly as specified above will not be accepted!** It is suggested that you submit your preliminary `Assignment_1_`*`YourMacID`*`.tex` and `Assignment_1_`*`YourMacID`*`.pdf` files well before the deadline so that your mark is not zero if, e.g., your computer fails at 11:50 PM on January 26.

Although you are allowed to receive help from the instructional staff and other students, your submission must be your own work. Copying will be treated as academic dishonesty! If any of the ideas used in your submission were obtained from other students or sources outside of the lectures and tutorials, you must acknowledge where or from whom these ideas were obtained.

Background

1. The notation $\sum_{i=m}^{n} f(i)$ is defined by:

$$\sum_{i=m}^{n} f(i) = \begin{cases} 0 & \text{if } m > n \\ f(n) + \sum_{i=m}^{n-1} f(i) & \text{if } m \leq n \end{cases}$$

2. The notation $\prod_{i=m}^{n} f(i)$ is defined by:

$$\prod_{i=m}^{n} f(i) = \begin{cases} 1 & \text{if } m > n \\ f(n) * \prod_{i=m}^{n-1} f(i) & \text{if } m \leq n \end{cases}$$

3. The factorial function $\mathsf{fact} : \mathbb{N} \to \mathbb{N}$ is defined by:

$$\mathsf{fact}(n) = \begin{cases} 1 & \text{if } n = 0 \\ n * \mathsf{fact}(n-1) & \text{if } n > 0 \end{cases}$$

4. $\mathsf{base\text{-}2\text{-}exp} : \mathbb{N} \to \mathbb{N}$ is defined by:

$$\mathsf{base\text{-}2\text{-}exp}(n) = \begin{cases} 1 & \text{if } n = 0 \\ 2 & \text{if } n = 1 \\ \mathsf{base\text{-}2\text{-}exp}(n-1) + (2 * \mathsf{base\text{-}2\text{-}exp}(n-2)) & \text{if } n \geq 2 \end{cases}$$

Sample Problems

1. Prove $\sum_{i=0}^{n-1} 2^i = 2^n - 1$ for all $n \in \mathbb{N}$.

   Proof. Let $P(n) \equiv \sum_{i=0}^{n-1} 2^i = 2^n - 1$. We will prove $P(n)$ for all $n \in \mathbb{N}$ by weak induction.

   Base case 1: $n = 0$. We must show $P(0)$.

$$\begin{aligned} & \sum_{i=0}^{0-1} 2^i && \langle \text{LHS of } P(0) \rangle \\ = & \sum_{i=0}^{-1} 2^i && \langle \text{arithmetic} \rangle \\ = & \ 0 && \langle \text{definition of } \textstyle\sum_{i=m}^{n} f(i) \text{ when } m > n \rangle \\ = & \ 1 - 1 && \langle \text{arithmetic} \rangle \\ = & \ 2^0 - 1 && \langle \text{arithmetic; RHS of } P(0) \rangle \end{aligned}$$

   So $P(0)$ holds.

Induction step: $n \geq 0$. Assume $P(n)$. We must show $P(n+1)$.

$$
\begin{aligned}
& \sum_{i=0}^{(n+1)-1} 2^i && \langle \text{LHS of } P(n+1) \rangle \\
&= \sum_{i=0}^{n} 2^i && \langle \text{arithmetic} \rangle \\
&= 2^n + \sum_{i=0}^{n-1} 2^i && \langle \text{definition of } \textstyle\sum_{i=m}^{n} f(i) \rangle \\
&= 2^n + 2^n - 1 && \langle \text{induction hypothesis: } P(n) \rangle \\
&= 2 * 2^n - 1 && \langle \text{arithmetic} \rangle \\
&= 2^{n+1} - 1 && \langle \text{arithmetic; RHS of } P(n+1) \rangle
\end{aligned}
$$

So $P(n+1)$ holds.

Therefore, $P(n)$ holds for all $n \in \mathbb{N}$ by weak induction. □

2. Prove that, if $n \in \mathbb{N}$ with $n \geq 2$, then n is a product of prime numbers.

Proof. Let $P(n)$ hold iff n is a product of prime numbers. We will prove $P(n)$ for all $n \in \mathbb{N}$ with $n \geq 2$ by strong induction.

Base case: $n = 2$. We must show $P(2)$. Since 2 is a prime number, 2 is obviously a product of prime numbers. So $P(2)$ holds.

Induction step: $n > 2$. Assume $P(2), P(3), \ldots, P(n-1)$ hold. We must show $P(n)$.

Case 1: n is a prime number. Then n is obviously a product of prime numbers.

Case 2: n is not a prime number. Then $n = x * y$ where $x, y \in \mathbb{N}$ with $2 \leq x, y \leq n-1$. Thus, by the induction hypothesis ($P(x)$ and $P(y)$),

$$x = p_0 * \cdots * p_i$$

and

$$y = q_0 * \cdots * q_j$$

where $p_0, \ldots, p_i, q_0, \ldots, q_j$ are prime numbers. Then

$$n = x * y = p_0 * \cdots * p_i * q_0 * \cdots * q_j$$

and so $P(n)$ holds.

Therefore, $P(n)$ holds for all $n \in \mathbb{N}$ with $n \geq 2$ by strong induction. □

Required Problems

1. **[10 points]** Prove

$$\prod_{i=1}^{n} \frac{i^2}{i+1} = \frac{\mathsf{fact}(n)}{n+1}.$$

for all $n \in \mathbb{N}$.

Put your name, MacID, and date here.

Solution:

Proof. Let

$$P(n) \equiv \prod_{i=1}^{n} \frac{i^2}{i+1} = \frac{\mathsf{fact}(n)}{n+1}.$$

We will prove $P(n)$ for all $n \in \mathbb{N}$ by weak induction.

Base case: $n = 0$. We must show $P(0)$.

$$\begin{aligned}
&\prod_{i=1}^{0} \frac{i^2}{i+1} && \langle \text{LHS of } P(0) \rangle \\
&= 1 && \langle \text{definition of } \textstyle\prod_{i=m}^{n} f(i) \text{ when } m > n \rangle \\
&= \frac{1}{0+1} && \langle \text{arithmetic} \rangle \\
&= \frac{\mathsf{fact}(0)}{0+1} && \langle \text{def. of } \mathsf{fact}\text{; RHS of } P(0) \rangle
\end{aligned}$$

So $P(0)$ holds.

Induction step: $n \geq 0$. Assume $P(n)$. We must show $P(n+1)$.

$$\begin{aligned}
&\prod_{i=1}^{n+1} \frac{i^2}{i+1} && \langle \text{LHS of } P(n+1) \rangle \\
&= \frac{(n+1)^2}{(n+1)+1} * \prod_{i=1}^{n} \frac{i^2}{i+1} && \langle \text{definition of } \textstyle\prod_{i=m}^{n} f(i) \rangle \\
&= \frac{(n+1)^2}{(n+1)+1} * \frac{\mathsf{fact}(n)}{n+1} && \langle \text{induction hypothesis: } P(n) \rangle \\
&= \frac{(n+1) * \mathsf{fact}(n)}{(n+1)+1} && \langle \text{arithmetic} \rangle \\
&= \frac{\mathsf{fact}(n+1)}{(n+1)+1} && \langle \text{definition of } \mathsf{fact}\text{; RHS of } P(n+1) \rangle
\end{aligned}$$

So $P(n+1)$ holds.

Therefore, $P(n)$ holds for all $n \in \mathbb{N}$ by weak induction. $\square$

2. **[10 points]** Prove

$$\mathsf{base\text{-}2\text{-}exp}(n) = 2^n$$

for all $n \in \mathbb{N}$.

Put your name, MacID, and date here.

Solution:

Proof. Let $P(n) \equiv \mathsf{base\text{-}2\text{-}exp}(n) = 2^n$. We will prove $P(n)$ for all $n \in \mathbb{N}$ by strong induction.

Base case 1: $n = 0$. We must show $P(0)$.

$$\begin{aligned}
& \mathsf{base\text{-}2\text{-}exp}(0) && \langle \text{LHS of } P(0) \rangle \\
= {} & 1 && \langle \text{definition of } \mathsf{base\text{-}2\text{-}exp} \rangle \\
= {} & 2^0 && \langle \text{arithmetic; RHS of } P(0) \rangle
\end{aligned}$$

So $P(0)$ holds.

Base case 2: $n = 1$. We must show $P(1)$.

$$\begin{aligned}
& \mathsf{base\text{-}2\text{-}exp}(1) && \langle \text{LHS of } P(1) \rangle \\
= {} & 2 && \langle \text{definition of } \mathsf{base\text{-}2\text{-}exp} \rangle \\
= {} & 2^1 && \langle \text{arithmetic; RHS of } P(1) \rangle
\end{aligned}$$

So $P(1)$ holds.

Induction step: $n \geq 2$. Assume $P(0), P(1), P(2), \ldots, P(n-1)$ hold. We must show $P(n)$.

$$\begin{aligned}
& \mathsf{base\text{-}2\text{-}exp}(n) && \langle \text{LHS of } P(n) \rangle \\
= {} & \mathsf{base\text{-}2\text{-}exp}(n-1) + {} && \\
& (2 * \mathsf{base\text{-}2\text{-}exp}(n-2)) && \langle \text{definition of } \mathsf{base\text{-}2\text{-}exp} \rangle \\
= {} & 2^{n-1} + (2 * 2^{n-2}) && \langle \text{ind. hypo.: } P(n-1) \text{ and } P(n-2) \rangle \\
= {} & 2^{n-1} + 2^{n-1} && \langle \text{arithmetic} \rangle \\
= {} & 2 * 2^{n-1} && \langle \text{arithmetic} \rangle \\
= {} & 2^n && \langle \text{arithmetic; RHS of } P(n) \rangle
\end{aligned}$$

So $P(n)$ holds.

Therefore, $P(n)$ holds for all $n \in \mathbb{N}$ by strong induction. $\square$