12C3

Assignment not due today!

Test on Monday after break! So study hard!

Check website for test content!

Last Day: Diagonalization

If A is an $n \times n$ (square) matrix

& if geometric multiplicity = algebraic multiplicity

for all its eigenvalues (i.e. # of basis eigenvectors for all λ $= n =$ size of matrix!)

then A is diagonalizable

ie $A = PDP^{-1}$

$D =$ diagonal matrix of eigenvalues (each repeated to alg. multiplicity

P = matrix of corresponding basis eigenvectors

---

eg. Say A is a 3×3 matrix

with eigenvalues $\lambda = 2, -5$

with basis eigenvectors $\lambda = 2 \Rightarrow \vec{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

$\lambda = -5 \Rightarrow \vec{x}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \vec{x}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

notice total # of basis eigenvectors $= 3 = n$ ✓ $\Rightarrow$ Diagonalize!

So $A = PDP^{-1}$

Say $D = \begin{bmatrix} -5 & & 0 \\ & 2 & \\ 0 & & -5 \end{bmatrix}$

all orderings of λ's are valid diagonalizations! not unique!

P must have eigenvectors sorted to match D's eigenvalues

$\leftarrow \lambda = 2$ basis!

$$P = \left[\ \vec{x}\text{'s}\ \right] = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & -2 \\ 0 & 2 & 0 \end{bmatrix}$$

$\lambda = -5$ basis eigenvector!

an equivalent eigenvector to $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \leftarrow \lambda = -5$ eigenvec!

& Calculate $P^{-1} = \left(\cdots \right) \leftarrow$ exercise!

$$\Rightarrow A = PDP^{-1} = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & -2 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} -5 & & 0 \\ & 2 & \\ 0 & & -5 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & -2 \\ 0 & 2 & 0 \end{bmatrix}^{-1}$$

Properties of Diagonalization

- If A, an $n \times n$ matrix has n distinct eigenvalues

  $\Rightarrow$ geo. = alg. = 1 for all λ $\Rightarrow$ diagonalizable!

- Diagonalizability & Invertibility are not related

  (I can have $\lambda = 0$ on diag. $\Rightarrow$ no A^{-1}

  or no $\lambda = 0$ (i.e. all $\lambda \neq 0$) $\Rightarrow$ A^{-1} exists!)

- If D is ~~the~~ a diagonalization of A, then:

  1) $\det A = \det D$ ($\det(A) = \det(PDP^{-1})$

  $= \cancel{\det P} \cdot \det D \cdot \frac{1}{\cancel{\det P}}$)

  2) $C_A(\lambda) = C_D(\lambda)$ — same char. polynomial

  3) $tr(A) = tr\, D$

These true for any "<u>similar</u>" matrices

A is <u>Similar</u> to B if <u>Q matrix</u> exists such that

$$A = Q B Q^{-1}$$

"<u>conjugated by Q</u>".

<u>Cool</u> <u>Matrix</u> <u>App</u> : <u>Differential Equation</u>

Say $y' = ay$ $\Rightarrow$ $y = Ce^{ax}$

constant A

<u>C is arbitrary!</u>

<u>now</u> let $\vec{y}\,' = \begin{bmatrix} a \cdot y_1 \\ b \cdot y_2 \\ c \cdot y_3 \end{bmatrix} = \begin{bmatrix} a & & 0 \\ & b & \\ 0 & & c \end{bmatrix} \vec{y}$

ie. I have $\underline{\underline{3}}$ $\underline{\text{equation!}}$

$$\begin{cases} y_1' = a y_1 \\ y_2' = b y_2 \\ y_3' = c y_3 \end{cases}$$

$$\begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix} = \begin{bmatrix} a y_1 \\ b y_2 \\ c y_3 \end{bmatrix} \Rightarrow \begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$\Rightarrow \boxed{\vec{y}\,' = D\,\vec{y}} \qquad y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$D = \begin{bmatrix} a & & 0 \\ & b & \\ 0 & & c \end{bmatrix}$$

$$\vec{y} = \begin{bmatrix} k_1 e^{ax} \\ k_2 e^{bx} \\ k_3 e^{cx} \end{bmatrix}$$

k_i are arbitrary coeff.

eg $y_1' = y_1 + 2y_2$ ← a linear equation in y's.

$y_2' = 2y_1 + y_2$ } a linear system Differential Equations

$$\Rightarrow \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

In general: If $\vec{y}' = A\vec{y}$ & A diagonalizable

$\Rightarrow A = PDP^{-1}$

then $P^{-1}\vec{y}' = P^{-1}PDP^{-1}\vec{y}$

$$\underbrace{(P^{-1}\vec{y}')}_{\vec{u}'} = D\underbrace{(P^{-1}\vec{y})}_{\vec{u}}$$

$u' = D\vec{u}$, $\vec{u} = P^{-1}\vec{y}$

$$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \Rightarrow \vec{u} = \begin{bmatrix} k_1 e^{\lambda_1 x} \\ k_2 e^{\lambda_2 x} \\ \vdots \end{bmatrix} = \underline{\underline{P^{-1}\vec{y}}}$$

$$P^{-1}\vec{y} = \vec{u} \Rightarrow \vec{y} = P\vec{u}$$

<u>But</u> Remember

$$P\vec{e_i} = \vec{x_i}, \quad \vec{e_i} = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

all $\underline{\underline{0}}$ except entry i is $\underline{\underline{1}}$

$\vec{x_i}$ = ith eigenvector.

$$\Rightarrow \vec{y} = P\vec{u}$$

$$= P\, k_1 e^{\lambda_1 x} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + P\, k_2 e^{\lambda_2 x} \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix} + \dots$$

$$= k_1 e^{\lambda_1 x} \vec{x_1} + k_2 e^{\lambda_2 x} \vec{x_2} + \dots$$

$$= \sum_{i=1}^{n} k_i e^{\lambda_i x} \vec{x_i} = \text{Sum of}$$

(arbitrary const) $\cdot$ ($e^{\text{eigenvalue} \cdot x}$) $\cdot$ (eigenvector)

So, back to our problem:

$$y' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = Ay$$

$$C_A(\lambda) = (1-\lambda)^2 - 4 = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1)$$

$$\lambda = 3 \Rightarrow [A - \lambda I \mid \vec{0}\,] \Rightarrow \left[\begin{array}{cc|c} -2 & 2 & 0 \\ 2 & -2 & 0 \end{array}\right] \Rightarrow x = y \Rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix} = x_1$$

$$\lambda = -1 \Rightarrow [A - \lambda I \mid \vec{0}\,] \Rightarrow \left[\begin{array}{cc|c} 2 & 2 & 0 \\ 2 & 2 & 0 \end{array}\right] \Rightarrow x = -y \Rightarrow \begin{bmatrix} -1 \\ 1 \end{bmatrix} = x_2$$

eigenvectors

$$\Rightarrow \vec{y} = \sum k_i e^{\lambda_i x} \cdot x_i$$

$$= k_1 e^{3x} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + k_2 e^{-x} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\vec{y} = \begin{bmatrix} k_1 e^{3x} - k_2 e^{-x} \\ k_1 e^{3x} + k_2 e^{-x} \end{bmatrix} = \begin{pmatrix} \vec{y_1} \\ \vec{y_2} \end{pmatrix}$$