1ZC3 - Matlab Due! Today!

- TAs available in BSB (2nd floor) (see web!)

Last Day Eigenstuff!

Remember If A is an $n \times n$ (ie. square) matrix

& $\boxed{A\vec{x} = \lambda \vec{x}}$ for a non-zero vector $\vec{x}$

then λ is an eigenvalue of A & $\vec{x}$ is an eigenvector for A, for that λ.

Notice $A\vec{x} = \lambda \vec{x} \Rightarrow A^2 \vec{x} = \lambda^2 \vec{x}$ & in general $A^n \vec{x} = \lambda^n \vec{x}$

$\Rightarrow$ any λ eigenvector of A is a λ^n eigenvector of A^n.

Notice

If A^{-1} exists $\Rightarrow$ $A^{-1}A\vec{x} = A^{-1}\lambda\vec{x}$

$$I\vec{x} = \vec{x} = \lambda \cdot (A^{-1})(\vec{x})$$

so $\lambda \neq 0$

& $\frac{1}{\lambda}\vec{x} = A^{-1}\vec{x}$ or $\boxed{A^{-1}\vec{x} = \left(\frac{1}{\lambda}\right)\vec{x}}$

$\Rightarrow$ A^{-1} has same eigenvectors but now corresponds to $\frac{1}{\text{old } \lambda}$ ie new $\lambda = \frac{1}{\text{original } \lambda}$

Last Day we found eigenvalue & eigenvector of $\underline{\underline{A}}$

Eigenvalues = roots of $C_A(z) = \underbrace{|A - zI|}_{\text{"characteristic polynomial"}}$ = polynomial in z.

(equivalent, alternate form? $|zI - A|$ ⟸ same roots either way!)

& To get eigenvectors solve linear system

aug. matrix → $[\,A - \lambda I \mid \vec{0}\,]$ (ie $(A - \lambda I)\vec{x} = \vec{0}$.)

eg. Last day

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$C_A(z) = |A - zI| = \begin{vmatrix} 1-z & 0 & 0 \\ 1 & 2-z & 0 \\ 0 & 0 & 2-z \end{vmatrix}$$

$$= (1-z)(2-z)(2-z)$$

$\lambda = \underline{1, 2, 2}$

alg. mult. is $\underline{\underline{1}}$

alg. mult. $= 2$

Alg. mult. of λ is # repeated factors!

solve for eigenvector say $\lambda = 2$ $[A - 2I \mid \vec{0}]$

& solve!

$$\Rightarrow \vec{x} = \begin{bmatrix} 0 \\ t \\ s \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

two parameters $\Rightarrow$ geo. mult $=$ # params $= 2$

& 2 basis eigenvectors in $\lambda = 2$ eigenspace!

Always $1 \leq$ geo. mult $\leq$ alg. mult. $\leq n$

Properties 1) notice $C_A(z) = |A - zI|$ ← order n

$= \ldots \ldots \pm\ {}^n$

$=$

$-\ {}^n$

can be factored into monomials ↙ $= (\lambda_1 - z)(\lambda_2 - z)(\lambda_3 - z)\ldots(\lambda_n - z)$

$\Rightarrow$ roots $= \lambda$'s $= \lambda_1, \lambda_2$ etc $=$ eigenvalues!

$$\Rightarrow \quad \underset{|A - zI|}{C_A(z)} = \underbrace{(\lambda_1 \lambda_2 \lambda_3 \cdots \lambda_n)} + (\ \ )z + (\ \ )z^2 + \ldots$$

$$\ldots \pm z^n$$

product of the eigenvalues

each repeated to alg. multiplicity

So plug in $z = 0$

$$\det A = c_A(0) = \lambda_1 \lambda_2 \cdots \lambda_n$$

$\boxed{\text{So}}$ $\det A$ = product of eigenvalue (to their alg. multiplicity)

$$= \lambda_1 \lambda_2 \cdots \lambda_n$$

(& sum of all alg. multi is n)

And $\boxed{A^{-1} \text{ exists iff } \det A \neq 0 \text{ iff } \lambda \neq 0 \text{ for any eigenvalue of } A}$

Finally Cayley - Hamilton Theorem

If $\quad c_A(z) = |A - zI| =$ char. polynomial of A

then $\quad c_A(A) = 0$

eg. Say $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

$$\det(A - zI) = \begin{vmatrix} 1-z & 2 \\ 2 & 1-z \end{vmatrix}$$

$$= (1-z)^2 - 4 = z^2 - 2z - 3$$

$$= (z-3)(z+1) \Rightarrow \lambda = 3, -1.$$

by <u>Cayley - Hamilton.</u>

$$c_A(z) = z^2 - 2z - 3$$ ↙ plug in $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

we'll get $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}^2 - 2\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} - \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}}$$

Diagonalizability

Consider diag. matrices:

Pretty!

A diag. matrix has: – entries on diag are λ's

$$\begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

– basis eigenvectors are i, j, k, in general $\vec{e}_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$

1 only in spot i

eg

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \lambda_2 \end{bmatrix} = \lambda_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ etc.}$$

– geo. mult. = alg. mult. for all λ

Goal: Make matrix "act" diagonal