Partial Fractions

If you have $$\frac{1}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$$

One term for each monomial

If you have $$\frac{1}{(x-a)(x-b)^2} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{(x-b)^2}$$

One term for each power

If you have an Irreducible Quadratic

$$\frac{1}{(x-a)(x^2+b^2)} = \frac{A}{x-a} + \frac{Bx+C}{x^2+b^2}$$

$$\frac{1}{(x^2+b^2)(x^2+a^2)} = \frac{Ax+B}{x^2+b^2} + \frac{Cx+D}{x^2+a^2}$$

One for each irred. quadratic

$$\frac{1}{(x-a)(x^2+x+1)^2} = \frac{A}{x-a} + \frac{Bx+C}{x^2+x+1} + \frac{Dx+E}{(x^2+x+1)^2}$$

One term for each power of irred. quadratic!

If Order of Top $\geq$ Order of Bottom then

Use (Synthetic) Division, **Then** Separate:

e.g. $\frac{x^4+1}{x^2-x} \Rightarrow$

$$\begin{array}{r|l}
 & x^2+x+1 \\ \hline
x^2-x & x^4+0x^3+0x^2+0x+1 \\
 & \underline{x^4-x^3} \\
 & \quad x^3+0x^2 \\
 & \quad \underline{x^3-x^2} \\
 & \qquad x^2+0x \\
 & \qquad \underline{x^2-x} \\
 & \qquad\quad x+1
\end{array}$$

$$= x^2+x+1 + \frac{x+1}{\underbrace{x^2-x}}$$

$$\frac{A}{x} + \frac{B}{x-1} = \frac{x+1}{x(x-1)}$$

$$A \cdot x - A + Bx = x+1$$

$$\begin{cases} x: & A+B=1 \\ 1: & -A=1 \end{cases} \Rightarrow A=-1 \ \& \ B=2$$

$$= \boxed{x^2+x+1 - \frac{1}{x} + \frac{2}{x-1}}.$$

How do we get the coefficients?

Fast Way Plug in x-values to zero terms

eg $$\frac{1+x}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$$

$$1+x = A(x+2) + B(x-1)$$

$$x = -2 \Rightarrow -1 = 0 + B(-3)$$

$$B = 1/3$$

$$x = 1 \Rightarrow 2 = 3A, \ A = 2/3$$

So $$\frac{1+x}{(x-1)(x+2)} = \frac{2/3}{x-1} + \frac{1/3}{x+2}.$$

Good: Fast! Easy to Do!

Bad: Won't work on irred. quadratics or higher power of monomial

Reliable Way

Compare coefficients of powers of x on left & right sides.

eg. $$\frac{2x+1}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

$$0x^2 + 2x + 1 = Ax^2 + 4A + Bx^2 + Cx$$

x^2-coeffs: $0 = A + B$

x-coeffs: $2 = C$.

1-coeff: $1 = 4A \Rightarrow \underline{A = \frac{1}{4}} \Rightarrow \underline{B = \frac{-1}{4}}$ & $\underline{C = 2}$

(constants)

$\underline{\underline{\text{so}}}$ $$\frac{2x+1}{x(x^2+4)} = \frac{1/4}{x} + \frac{-\frac{1}{4}x+2}{x^2+4}.$$

<u>Good</u>: Always works

<u>Bad</u>: Slow, weird.

Integrating the Results

$$\int \frac{A}{x-a}\,dx = A\ln|x-a| + c$$

$$\int \frac{B}{x^2+a^2}\,dx = B\cdot\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$$

$$\int \frac{Ax}{x^2+a^2}\,dx \Rightarrow \text{let } u = x^2+a^2,\ du = 2x\,dx$$

$$\hookrightarrow \frac{A}{2}\int \frac{1}{u}\,du = \frac{A}{2}\ln|u| + c$$

$$= \frac{A}{2}\ln(x^2+a^2) + c$$

$$\int \frac{Ax+B}{\text{irres. Quadratic}}\,dx \Rightarrow \text{complete the square \& subsitute.}$$

eg. $$\int \frac{x-1}{2x^2+4x+8}\,dx = \frac{1}{2}\int \frac{x-1}{\boxed{x^2+2x+4}}\,dx$$

$$\swarrow$$

$$x^2+2x+4 = (x\pm\beta)^2+\gamma$$

$$x^2 + \underline{2}x + 4 = x^2 + \underline{\underline{2\beta}}x + \beta^2 + \gamma$$

$$\beta = 1 \Rightarrow \quad 4 = 1 + \gamma$$

$$\underline{\underline{\gamma = 3}}$$

$$x^2 + 2x + 4 = \underline{\underline{(x+1)^2}} + 3$$

so

$$\int \frac{x-1}{2x^2+4x+8}\,dx = \frac{1}{2}\int \frac{x-1}{(x+1)^2+3}\,dx$$

let $u = x+1 \Rightarrow du = dx\,,\ x = u-1$

$$= \frac{1}{2}\int \frac{u-2}{u^2+3}\,dx$$

$$= \frac{1}{2}\int \frac{u}{u^2+3}\,dx - \int \frac{1}{u^2+3}\,dx$$

$$= \underline{\frac{1}{4}\ln|u^2+3| - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{u}{\sqrt{3}}\right) + C}$$

$$= \boxed{\frac{1}{4}\ln(x^2+2x+4) - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x-1}{\sqrt{3}}\right) + C}$$