1ZC3

Last Day: Symmetric Matrices

A matrix A is symmetric if A is a $n \times n$ matrix & $A = A^T$

note if A & B both $n \times n$ symmetric matricies, then:

1) $(A+B)^T = A+B$ (sum is symmetric!)
2) $(kA)^T = kA^T = kA$ (scalar mult. is symmetric.
3) $(A^{-1})^T = (A^T)^{-1} = A^{-1}$ (inverse, if exists is symmetric
4) $(AB)^T = B^TA^T = BA \neq AB$ (in general)

AB is symmetric if A symm, B symm & $AB = BA$ (ie commute)

Otherwise not symmetric

eg.

$$\begin{bmatrix} 2 & 3 & 5 \\ 3 & 7 & 4 \\ 5 & 4 & 2 \end{bmatrix}$$ } Symmetric!
Same across principal diagonal!

Determinants Remember if A is 2×2

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow A^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \cdot \frac{1}{ad-bc}$$

$ad-bc = \det(A) =$ determinant of A

$\det(A) = 0 \Rightarrow$ no inverse
$\det(A) \neq 0 \Rightarrow$ inverse exists

Also alternate notation
$\det(A) = |A|$

not an absolute value!

Determinants for $n \times n$ matrices

Define $\det(A)$ = sum of (product of elements one from each col/row of the matrix) $\cdot$ (permutation sign)

never use this! **Awkward**

eg if $A :=$
$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \Rightarrow \det A$$

$$= aei \quad \} \text{ row } 1, 2, 3$$

$$-ahf \quad \} \text{ row } 1, 3, 2$$

neg. multiplie per swap

$$+(-1)^2 gbf \quad \} \text{ row } 3, 1, 2$$

$$\vdots$$

$$= \det(A)$$

Vastly

Instead Let's get some terminology!

M_{ij} = minor (minor det.) of position row i, col. j

= det. of matrix with row i, col. j removed

eg Let $B = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 3 & 1 \\ 0 & 2 & 5 \end{bmatrix}$

For this B, let's calculate minors!

$M_{23} = \det\left(\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}\right)$

$= \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix}$ straight bars ⇒ det

$= (1)2 - 2(0) = 2$

$M_{33} = \begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix} = 1(3) - (2)(-1)$

$= 3 + 2 = 5$

In general: det(A) = sum of (entries)(minors)$(-1)^{i+j}$ ← position sign

along any 1 row or 1 col

$$= \sum_{i=1}^{n} a_{ij} (-1)^{i+j} M_{ij}$$

$$= \sum_{j=1}^{n} a_{ij} (-1)^{i+j} M_{ij}$$

} for any $\underline{\underline{i \text{ or } j}}$

e.g. Let $A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 5 \\ 2 & 1 & 0 \end{bmatrix}$ let's calc. $\det(A)$ by "expanding along col. $\underline{\underline{\#3}}$"

$$\det A = a_{13}(-1)^{1+3} M_{13} + a_{23}(-1)^{2+3} M_{23} + a_{33}(-1)^{3+3} M_{33}$$

$$= 3(+1) \begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} + 5(-1) \begin{vmatrix} 2 & -1 \\ 2 & 1 \end{vmatrix} + \cancel{0 \left(\text{who cares!} \right)}$$

-2

$2 + 2 = 4$

ad - bc

$2(1) - (-1)2 = \underline{\underline{4}}$

$$= -6 - 20 = -26$$

or say expand along row #2

$$\det(A) = \cancel{a_{21}(-1)^{2+1}M_{21}} + a_{22}(-1)^{2+2}M_{22} + a_{23}(-1)^{2+3}M_{23}$$

$a_{21} = 0$, so that term $\to 0$

$a_{22} = 1$, $(-1)^{2+2} = 1$, $M_{22} = \begin{vmatrix} 2 & 3 \\ 2 & 0 \end{vmatrix} = -6$

$a_{23} = 5$, $(-1)^{2+3} = -1$, $M_{23} = 4$

$$= 0 - 6 - 20 = -26 \checkmark$$

Small notes.

1) $(-1)^{i+j}$ = "position sign"

can be found by inspection or counting!

eg

$$A = \begin{bmatrix} 2^{+} & -1^{-} & 3^{+} \\ 0^{-} & 1^{+} & 5^{-} \\ 2^{+} & 1^{-} & 0^{+} \end{bmatrix}$$

memory trick for $(-1)^{i+j}$

$(-1)^{i+j} = \pm 1$

$(-1)^{3+2} = (-1)$

$(-1)^{3+3} = +1$

2) (position sign) $\cdot$ (Minor) $= (-1)^{i+j} \cdot M_{ij} = C_{ij}$

C_{ij} = cofactor of position i,j

$\Rightarrow \det A = \sum_{i \text{ or } j=1}^{n} a_{ij} C_{ij}$

cofactor $= (-1)^{i+j} M_{ij}$

i.e. a cofactor expansion of determinant

Use cofactor expansion to calculate $\det(B)$ if B is:

$$B = \begin{bmatrix} 1 & 3 & 5 & 2 & \pi \\ 0 & -1 & 2 & 1 & e \\ 0 & 0 & 6 & 3 & 5 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 10 \end{bmatrix}$$

<u>Solution</u> Let's expand on <u>col # 1</u>

$$|B| = b_{11} C_{11} + \underbrace{\underset{0}{\cancel{b_{21}}} C_{21} \cdots + \underset{0}{\cancel{b_{51}}} C_{51}}_{=0}$$

$$= 1 \cdot \underset{1}{\cancel{(-1)^{1+1}}} \cdot M_{11} = M_{11} = \begin{vmatrix} -1 & 2 & 1 & e \\ 0 & 6 & 3 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 10 \end{vmatrix}$$

& we expand along its col. #1

$$|B| = b_{11} C_{11} + \underbrace{\cancel{b}_{21} C_{21} \cdots \cancel{b}_{41} C_{41}}_{\text{all 0 entries}} \Big\} \leftarrow \text{new entries, new cofact.!}$$

$$= (-1)\,\underset{1}{\cancel{(-1)^{1+1}}} \begin{vmatrix} 6 & 3 & 5 \\ 0 & 2 & 1 \\ 0 & 0 & 10 \end{vmatrix}$$

& expand again!

$$= (-1)\,(6) \cdot \begin{vmatrix} 2 & 1 \\ 0 & 10 \end{vmatrix} + 0$$

$$= 1\,(-1)\,(6) \cdot (2) \cdot (10)$$