MATHEMATICS 1LT3E TEST 1

Evening Class
Duration of Test: 75 minutes
McMaster University, 29 June 2011

E. Clements

$\overline{}$
45

FIRST NAME (please print) : Sol^{NS}
FAMILY NAME (please print) :
Student No.:

THIS TEST HAS 8 PAGES AND 6 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE.

Total number of points is 45. Marks are indicated next to the problem number in square brackets. Any Casio fx991 (or lower, non-graphing) calculator is allowed.

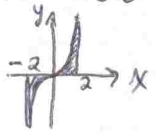
USE PEN TO WRITE YOUR TEST. IF YOU USE A PENCIL, YOUR TEST WILL NOT BE ACCEPTED FOR REMARKING (IF NEEDED).

You need to show work to receive full credit.

---

1. [2] (a) Without computing the integral, explain why $\int_{-2}^{2} x^3\, dx = 0$.

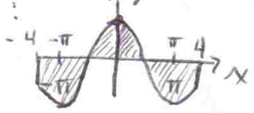
$f(x) = x^3$ is an odd f^N so it is symmetric about the origin:



Since the area above the x-axis matches the area below, the net area is $\underline{0}$.
$\int_{-2}^{2} x^3 dx$

[2] (b) Without computing the integral, explain why $\int_{-4}^{4} \cos x\, dx = 2\int_{0}^{4} \cos x\, dx$.

$f(x) = \cos x$ is an even f^N so it is symmetric about the y-axis:



Since the net area to the left of the y-axis is the same as the net area on the right, we could just calculate the area on the right, $\int_0^4 \cos x dx$ and multiply by 2.

2. [4] (a) Find the average value of $f(x) = 10xe^{-x}$ on the interval $[0, 5]$.

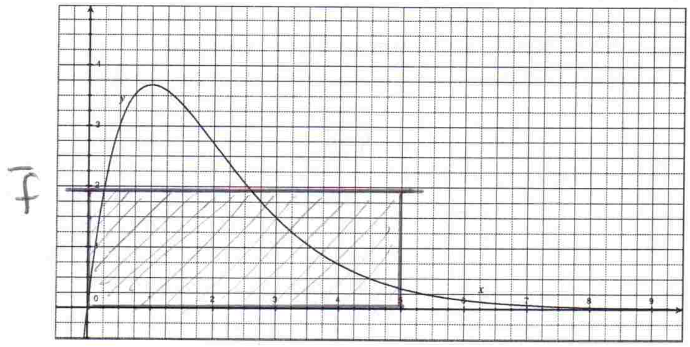
$$\bar{f} = \frac{1}{5-0}\int_0^5 10xe^{-x}\,dx$$
$$= 2\int_0^5 xe^{-x}\,dx$$
$$= 2\left[e^{-x}(-x-1)\right|_0^5$$
$$= 2\left[e^{-5}(-6) - e^{-0}(-1)\right]$$
$$= 2\left[1 - 6e^{-5}\right]$$
$$\approx 1.92$$

$u = x \quad dv = e^{-x}dx$

$du = dx \quad v = -e^{-x}$

$$\int xe^{-x}dx = -xe^{-x} - \int -e^{-x}dx$$
$$= -xe^{-x} - e^{-x} + C$$
$$= e^{-x}(-x-1) + C$$

[2] (b) On the diagram below, draw a rectangle that has the same area as the region between the curve $f(x) = 10xe^{-x}$ and the x-axis on the interval $[0, 5]$.



3. [4] A pipe bursts in a room and water starts filling the room at a **rate** of $\dfrac{2t}{\sqrt{1+t^2}}$ litres per minute. How much water is there in the room after 10 minutes? (Assume that the time at which the pipe bursts corresponds to $t = 0$).

Assume there is no water in the room before the pipe breaks, ie $V(0)=0$.

$$V(10) - V(0) = \int_0^{10} \frac{dV}{dt}\,dt$$
$$= \int_0^{10} \frac{2t}{\sqrt{1+t^2}}\,dt$$
$$= 2\sqrt{1+t^2}\,\Big|_0^{10}$$
$$= 2\sqrt{101} - 2\sqrt{1}$$
$$\approx 18.1 \text{ L}$$

$u = 1 + t^2$

$du = 2t\,dt$

$$\Rightarrow \int \frac{2t}{\sqrt{1+t^2}}\,dt = \int u^{-1/2}\,du$$
$$= 2\sqrt{u} + C$$
$$= 2\sqrt{1+t^2} + C$$

4. Compute the following integrals.

(a) [5] $\int_3^4 \frac{5}{x^3 - 2x^2}\,dx$

$$\frac{5}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}$$

$$= \frac{Ax(x-2)+B(x-2)+Cx^2}{x^2(x-2)}$$

$$\Rightarrow 5 = A(x-2)x + B(x-2) + Cx^2$$

$x=0 \Rightarrow 5 = B(-2) \Rightarrow B = -\frac{5}{2}$

$x=2 \Rightarrow 5 = C(2)^2 \Rightarrow C = \frac{5}{4}$

$x=1 \Rightarrow 5 = A(-1) + B(-1) + C$

so, $A = -5 - (-\frac{5}{2}) + \frac{5}{4}$

$\Rightarrow A = -\frac{5}{4}$

So, $\int_3^4 \frac{5}{x^3-2x^2}\,dx = \ldots$

$$= \int_3^4 \left(\frac{-\frac{5}{4}}{x} - \frac{\frac{5}{2}}{x^2} + \frac{\frac{5}{4}}{x-2} \right) dx$$

$$= \left[-\frac{5}{4}\ln|x| + \frac{5}{2}\cdot\frac{1}{x} + \frac{5}{4}\ln|x-2| \right]_3^4$$

$$= \left[\frac{5}{4}\ln\left|\frac{x-2}{x}\right| + \frac{5}{2x} \right]_3^4$$

$$= \left[\frac{5}{4}\ln\frac{1}{2} + \frac{5}{8} \right] - \left[\frac{5}{4}\ln\frac{1}{3} + \frac{5}{6} \right]$$

$$= \frac{5}{4}\ln\frac{3}{2} - \frac{5}{24}$$

(b) [5] $\int \frac{x^3 + 4x^2 - 1}{x^2 + 5x + 6}\,dx$

$$\begin{array}{r} x - 1 \\ x^2+5x+6 \overline{)\, x^3 + 4x^2 + 0x - 1} \\ -\ (x^3 + 5x^2 + 6x) \downarrow \\ \hline -x^2 - 6x - 1 \\ -\ (-x^2 - 5x - 6) \\ \hline -x + 5 \end{array}$$

$$\frac{x^3+4x^2-1}{x^2+5x+6} = x - 1 + \frac{-x+5}{x^2+5x+6}$$

$$\frac{-x+5}{x^2+5x+6} = \frac{A}{x+2} + \frac{B}{x+3}$$

$$\Rightarrow -x+5 = A(x+3) + B(x+2)$$

$x=-3 \Rightarrow 8 = B(-1) \Rightarrow B = -8$

$x=-2 \Rightarrow 7 = A(1) \Rightarrow A = 7$

So, $\int \frac{x^3+4x^2-1}{x^2+5x+6}\,dx = \ldots$

$$= \int \left(x - 1 + \frac{7}{x+2} - \frac{8}{x+3} \right) dx$$

$$= \frac{x^2}{2} - x + 7\ln|x+2| - 8\ln|x+3| + C$$

5. Let $f(x) = \arctan x$.

[3] (a) Determine the 3rd degree Taylor polynomial of $f(x)$ with base point $a = 1$.

$f'(x) = \frac{1}{1+x^2}$

$f''(x) = \frac{-2x}{(1+x^2)^2}$

$f'''(x) = \frac{-2(1+x^2)^2 + 2x(2)(1+x^2)'(2x)}{(1+x^2)^4}$

$= \frac{-2(1+x^2)+8x^2}{(1+x^2)^3}$

$= \frac{6x^2-2}{(1+x^2)^3}$

$f(1) = \arctan 1 = \frac{\pi}{4}$

$f'(1) = \frac{1}{1+1^2} = \frac{1}{2}$

$f''(1) = \frac{-2}{(1+1)^2} = -\frac{1}{2}$

$f'''(1) = \frac{6-2}{(1+1)^3} = \frac{1}{2}$

$\therefore P_3(x) = \frac{\frac{1}{2}}{3!}(x-1)^3 - \frac{\frac{1}{2}}{2!}(x-1)^2 + \frac{\frac{1}{2}}{1!}(x-1) + \frac{\pi}{4}$

$= \frac{1}{12}(x-1)^3 - \frac{1}{4}(x-1)^2 + \frac{1}{2}(x-1) + \frac{\pi}{4}$

[2] (b) Use part (a) to estimate arctan 0.5 and arctan 2. Which estimation is closer to the actual value? Explain.

$\arctan 0.5 \approx P_3(0.5) \approx \frac{1}{12}(0.5-1)^3 - \frac{1}{4}(0.5-1)^2 + \frac{1}{2}(0.5-1) + \frac{\pi}{4} \approx 0.4625$

Actual: $\arctan 0.5 \approx 0.4636$

This estimate is close to the actual value because 0.5 is close to the base point 1 ~~set~~ around which the estimate is best.

$\arctan 2 \approx \frac{1}{12}(2-1)^3 - \frac{1}{4}(2-1)^2 + \frac{1}{2}(2-1) + \frac{\pi}{4} \approx 1.119$

Actual: $\arctan 2 \approx 1.107$

[3] (c) Suppose you want to find the approximate the value of $\int_3^4 \sqrt{1+x^2}\, dx$ using a 3rd degree Taylor polynomial. What would be a reasonable base point for you to use and why? How could you improve your estimate?

A reasonable basepoint would be any x-value between 3 and 4 (or even 3 or 4).

The estimate could be improved by increasing the degree of the Taylor polynomial.

6. Try to compute the following integrals to determine whether they converge or diverge. (Even if you know the integral is divergent, work through the calculations to show you obtain an answer of infinity).

[4] (a) $\displaystyle\int_0^\infty \frac{1}{(1+3x)^2}\,dx$

$$\int_0^\infty \frac{1}{(1+3x)^2}\,dx = \lim_{T\to\infty}\int_0^T (1+3x)^{-2}\,dx$$

$$= \lim_{T\to\infty}\left[-\frac{1}{3}\cdot\frac{1}{1+3x}\right]_0^T$$

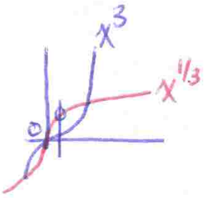
$$= \lim_{T\to\infty}\left[-\frac{1}{3}\cdot\frac{1}{1+3T}+\frac{1}{3}\cdot\frac{1}{1+3\cdot(0)}\right]$$

$$= -\frac{1}{3}\cdot\cancel{\frac{1}{1+3\cdot\infty}}^{\to 0}+\frac{1}{3}$$

$$= \frac{1}{3}$$

∴ CONVERGENT

[4] (b) $\displaystyle\int_0^1 \underbrace{\frac{1}{x^3+x^{\frac{1}{3}}}}_{f}\,dx$



$$f_0 = \frac{1}{(x^3+x^{1/3})_0} = \frac{1}{x^{1/3}}$$

$$\int_0^1 \frac{1}{x^{1/3}}\,dx = \lim_{\varepsilon\to 0^+}\int_\varepsilon^1 x^{-1/3}\,dx$$

$$= \lim_{\varepsilon\to 0^+}\left[\frac{3}{2}x^{2/3}\right]_\varepsilon^1$$

$$= \lim_{\varepsilon\to 0^+}\left[\frac{3}{2}1^{2/3}-\frac{3}{2}\varepsilon^{2/3}\right]$$

$$= \frac{3}{2}-\frac{3}{2}\cdot 0^{2/3 \to 0}$$

$$= \frac{3}{2}$$

∴ $\displaystyle\int_0^1 \frac{1}{x^{1/3}}\,dx$ converges

∴ $\displaystyle\int_0^1 \frac{1}{x^3+x^{1/3}}\,dx$ converges also.

6. continued...

[3] (c) $\int_1^{\infty} 2e^{-0.8x}\,dx$ $= \lim_{T\to\infty} \int_1^T 2e^{-0.8x}\,dx$

$$= \lim_{T\to\infty} \left[\frac{2}{-0.8}e^{-0.8x}\right|_1^T$$

$$= \lim_{T\to\infty} \left[-\frac{5}{2}e^{-0.8T} + \frac{5}{2}e^{-0.8}\right]$$

$$= \cancel{-\frac{5}{2}e^{-0.8(\infty)}}^{\,0} + \frac{5}{2}e^{-0.8}$$

$$= \frac{5}{2}e^{-0.8}$$

∴ CONVERGENT

[2] (d) Even though we cannot find the exact value of $\int_1^{\infty} 2e^{-0.8x^2}\,dx$ how do we know that this integral is convergent?

The graph of $f(x) = 2e^{-0.8x^2}$ lies below the graph of $f(x) = 2e^{-0.8x}$ on $[1, \infty)$ and since $\int_1^{\infty} 2e^{-0.8x}\,dx$ is convergent (the area under the graph on 1 to ∞ is finite) $\int_1^{\infty} 2e^{-0.8x^2}\,dx$ is also convergent ∵ the area under its graph on 1 to ∞ is even smaller + is ∴ finite as well.

$$0 < \underbrace{\int_1^{\infty} 2e^{-0.8x^2}\,dx}_{\text{real \#}} < \int_1^{\infty} 2e^{-0.8x}\,dx = \frac{5}{2e^{0.8}}$$

∴ convergent.