12A3

Today: Partial Fractions.

eg. $\int \frac{1}{x+3}\,dx = \ln|x+3| + C$

eg. $\int \frac{1}{x^2-4}\,dx$

Most basic case of Partial Fraction Decomposition

$\frac{1}{x^2-4} = \frac{A}{x-2} + \frac{B}{x+2}$

x^2-4
$=(x-2)(x+2)$

To get $A, B \Rightarrow$ kill fractions!

$$\frac{1}{x^2-4} = \frac{A}{x-2} + \frac{B}{x+2}$$ } mult. by $(x-2)\cdot(x+2)$

$$1 = A(x+2) + B(x-2)$$

plug in values: $x=-2 \Rightarrow 1 = A(0) - 4B$

$B = -1/4$.

$x = +2 \Rightarrow 1 = 4A + (0)B$

$A = 1/4$

so $$\frac{1}{x^2-4} = \frac{1/4}{x-2} + \frac{(-1/4)}{x+2}$$

$\stackrel{so}{=}$ $$\int \frac{1}{x^2-4}\,dx = \frac{1}{4}\int \frac{1}{x-2}\,dx - \frac{1}{4}\int \frac{1}{x+2}\,dx$$

$$= \frac{1}{4}\ln|x-2| - \frac{1}{4}\ln|x+2| + C$$

$$= \frac{1}{4}\ln\left|\frac{x-2}{x+2}\right| + C$$

$$= \ln\left\{\left(\left|\frac{x-2}{x+2}\right|\right)^{1/4}\right\} + C.$$

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eg. $$\int \frac{x+2}{\underbrace{x(x+1)(x+3)}_{\downarrow}}\,dx$$

$$\frac{x+2}{x(x+1)(x+3)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+3}$$

Kill fraction!

$$x+2 = A(x+1)(x+3) + Bx(x+3) + Cx(x+1).$$

we now could plug in $x = -1, -3, 0$ & solve for A, B, C.

Alternate (more powerful) method: Collect like powers!

$$\underline{\underline{0x^2}} + x + 2 = Ax^2 + Bx^2 + Cx^2 + 4Ax + 3Bx + Cx + 3A$$

"Two polynomials are equal iff coefficients are equal"

$x^2 \Rightarrow 0 = A + B + C$

$x \Rightarrow 1 = 4A + 3B + C$

const $\Rightarrow 2 = 3A.$

$A = \cancel{4}\,\frac{2}{3}$

$$B + C = -2/3$$

$$C = -B - \frac{2}{3}$$

$$1 = \frac{8}{3} + \underbrace{3B - B}_{2B} - \frac{2}{3}$$

$$2B = -1 \Rightarrow B = -\frac{1}{2}$$

$$C = \frac{1}{2} - \frac{2}{3} = -1/6$$

So

$$\int \frac{x+2}{x(x+1)(x+3)}\,dx$$

$$= \int \frac{(2/3)}{x}\,dx + \int \frac{(-1/2)}{x+1}\,dx + \int \frac{(-1/6)}{x+3}\,dx$$

$\leftarrow A$; B ; C

$$= \frac{2}{3}\ln|x| - \frac{1}{2}\ln|x+1| - \frac{1}{6}\ln|x+3| + C$$

9. $\displaystyle\int \frac{x^3 - x + 3}{x^2 + 2x}\, dx$

~~x^2~~ $3 \geq 2$

order of top $\geq$ order denom!

$\Rightarrow$ divide!

Note

$$\frac{?}{x^2+2x} = \frac{?}{x(x+2)} = \frac{A}{x} + \frac{B}{x+2}$$

$$= \frac{A(x+2) + Bx}{x(x+2)}$$

In general

order of numerator < denom. order!

$$\begin{array}{r} x \quad -2 \\ x^2 + 2x + 0 \,\overline{)\, x^3 + 0x^2 - x + 3} \\ \underline{x^3 + 2x^2 \qquad\qquad} \\ -2x^2 - x \quad \end{array}$$

$$\frac{-2x^2-4x}{3x+3}$$

Remainder!

order $1 < 2$ ✓

$$\frac{x^3-x+3}{x^2+2x} = x-2+\frac{3x+3}{x^2+2x}$$

PF this!

$$\frac{3x+3}{x^2+2x} = \frac{A}{x} + \frac{B}{x+2}$$

$$3x+3 = Ax + Bx + 2A$$

$x \rightsquigarrow A+B = 3 \xrightarrow{\text{const}} A = 3/2$, $B = 3 - \frac{3}{2} = 3/2$.

Sol:

$$\int \frac{x^3 - x + 3}{x^2 + 2x}\,dx = \int x - 2\,dx + \int \frac{(3/2)}{x}\,dx + \int \frac{3/2}{x+2}\,dx$$

$$= \boxed{\frac{1}{2}x^2 - 2x + \frac{3}{2}\ln|x| + \frac{3}{2}\ln|x+2| + C.}$$

eg.

$$\int \frac{1}{x^2(x+1)}\,dx$$

"repeated root".

$$\frac{1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$

In general.

$$\frac{1}{(x-a)^{n}(x-b)} = \overbrace{\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_n}{(x-a)^n}}^{\text{One term for each power!}} + \frac{B}{x-b}.$$

So $\frac{1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$

$$1 = Ax(x+1) + B(x+1) + C(x^2)$$

$$1 = Ax^2 + Ax + Bx + B + Cx^2$$

$x^2 \Rightarrow 0 = A + C$ $\quad x \Rightarrow 0 = A + B$

const $\Rightarrow \underline{\underline{1 = B}} \Rightarrow \underline{\underline{A = -1}} \Rightarrow C = 1.$

$$\int \frac{1}{x^2(x+1)}\,dx = \int \frac{-1}{x}\,dx + \int \frac{1}{x^2}\,dx + \int \frac{1}{x+1}\,dx$$

$$= -\ln|x| - \underline{\underline{\frac{1}{x}}} + \ln(x+1) + c$$

$$= \ln\left|\frac{x+1}{x}\right| - \frac{1}{x} + C$$