Announcements

Topics:

- **Section 4:** Partial Derivatives
- **Section 5:** Tangent Plane, Linearization, and Differentiability
- **Section 7:** Second-Order Partial Derivatives

To Do:

- Read sections 4, 5, and 7 in the "Functions of Several Variables" module
- Work on Assignments and Suggested Practice Problems assigned on the webpage under the SCHEDULE + HOMEWORK link

Partial Derivatives

Recall:
<u>Definition of the Derivative in Single Variable Calculus</u>:

$$\frac{df}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

instantaneous rate of change of *f* with respect to *x*

The **partial derivative** of a function of several variables is a way to measure the rate of change of the function as **<u>one</u>** of its variables changes.

Partial Derivatives

Example: Body Mass Index

$$BMI(w,h) = \frac{w}{h^2}$$

(a) Determine the rate at which *BMI* is changing with respect to changes in height for a 56kg person who is currently 1.7m tall.

(b) Determine the rate at which *BMI* is changing with respect to changes in weight for a 1.7m tall person who currently weighs 56kg.

Partial Derivatives of z=f(x,y)

The **partial derivative of *f* with respect to *x*** is the real-valued function $\partial f / \partial x$ defined by

$$\frac{\partial f}{\partial x}(x,y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x,y)}{h}$$

provided that the limit exists.

This function tells us the rate of change of *f* in the *x*-direction at all points (*x*,*y*) for which the limit exists.

Partial Derivatives of z=f(x,y)

The **partial derivative of *f* with respect to *y*** is the real-valued function $\partial f/\partial y$ defined by

$$\boxed{\frac{\partial f}{\partial y}(x,y)=\lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}}$$

provided that the limit exists.

This function tells us the rate of change of *f* in the *y*-direction at all points (*x*,*y*) for which the limit exists.

Partial Derivatives of z=f(x,y)

Example:

Using the definitions, compute $\partial f / \partial x$ and $\partial f / \partial y$ for $f(x) = x^2 - y$.

Partial Derivatives of z=f(x,y)

Rule for finding partial derivatives of $z=f(x,y)$:

1. To find f_x, treat y as a constant and differentiate $f(x,y)$ with respect to x.

2. To find f_y, treat x as a constant and differentiate $f(x,y)$ with respect to y.

Partial Derivatives of z=f(x,y)

Example:

Find the first partial derivatives of the following functions.

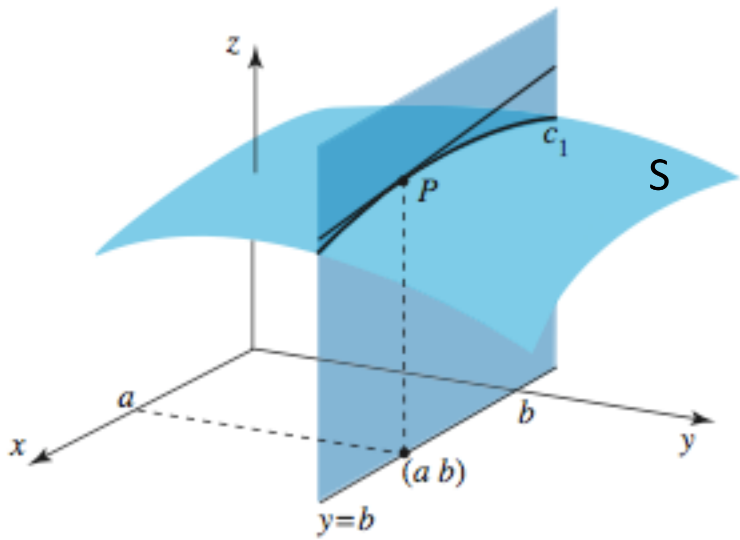
(a) $f(x,y) = x^4y^3 + 8x^2y$

(b) $z = x^y$

(c) $f(x,y) = \dfrac{e^x}{y + x^2}$

(d) $z = \arctan\left(\dfrac{y}{x}\right)$

Geometric Interpretation of the Partial Derivatives of z=f(x,y)

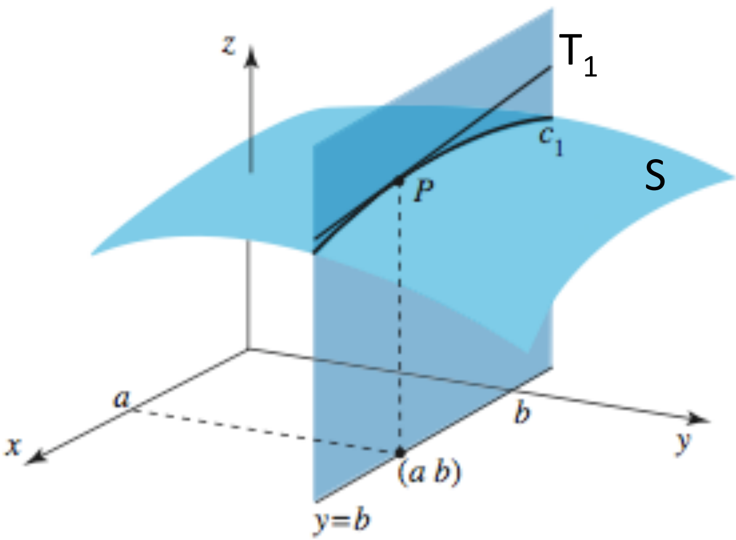


Let $z=f(x,y)$ be a function of two variables whose graph is the surface S.

Fix $y=b$ (constant) and let x vary.

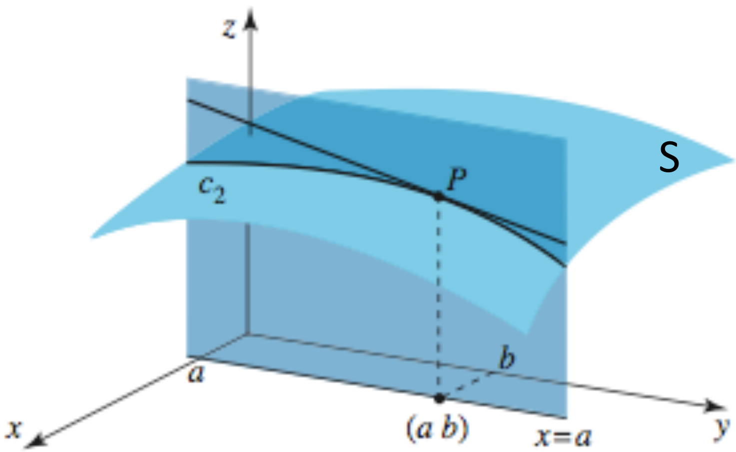
The curve c_1 on the surface S is defined by $z=f(x,b)$. (Note: this is now only a function of the variable x)

Geometric Interpretation of the Partial Derivatives of z=f(x,y)



The partial derivative of *f* with respect to *x* at (*a*,*b*) is the slope of the tangent T_1 to the curve c_1 at the point P.

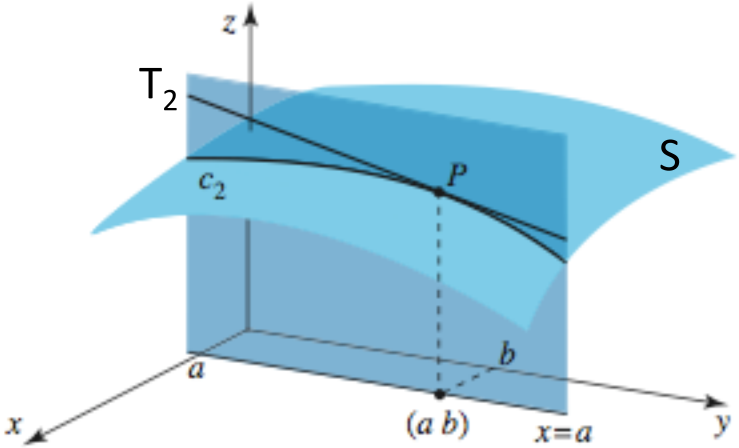
Geometric Interpretation of the Partial Derivatives of z=f(x,y)



Now, fix $x=a$ (constant) and let y vary.

The curve c_2 on the surface S is defined by $z=f(a,y)$. (Note: this is now only a function of the variable y)

Geometric Interpretation of the Partial Derivatives of z=f(x,y)

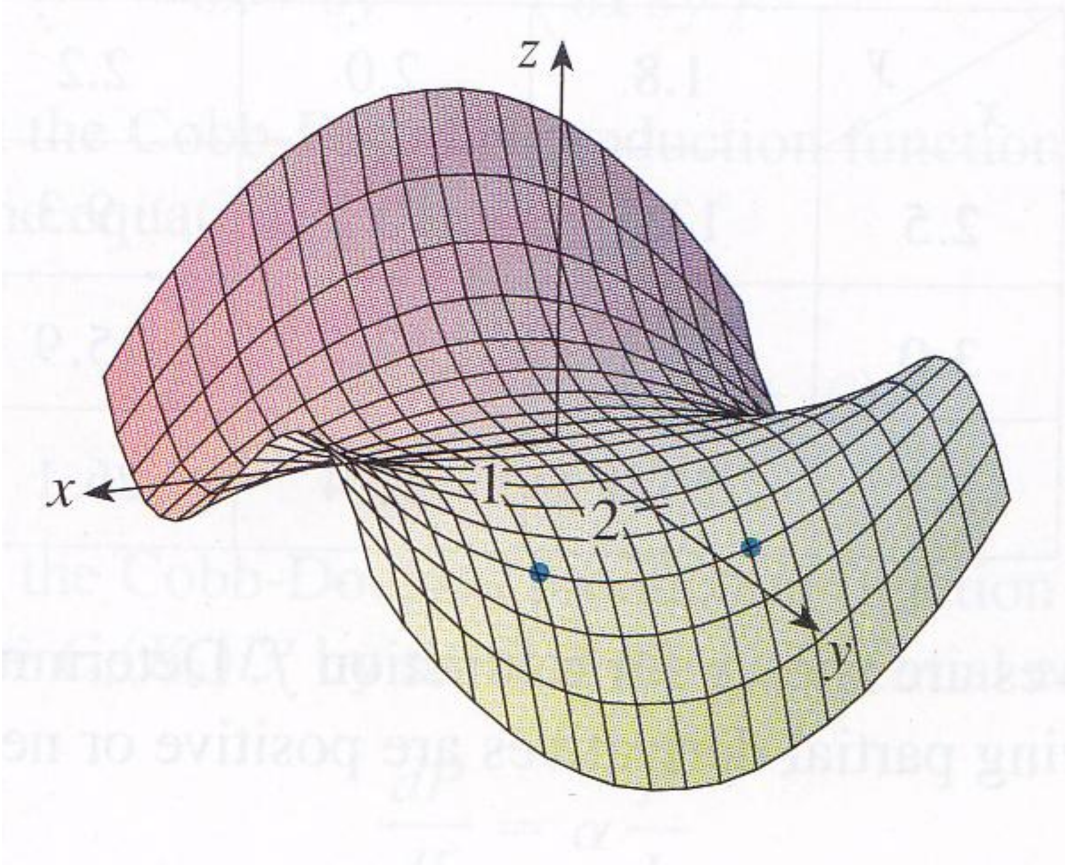


The partial derivative of *f* with respect to *y* at (*a*,*b*) is the slope of the tangent T_2 to the curve c_2 at the point P.

Geometric Interpretation of the Partial Derivatives of z=f(x,y)

Example:

Determine the signs of $f_x(1,2)$ and $f_y(1,2)$ on the graph below.



Partial Derivatives of z=f(x,y)

Example:

If $f(x,y)=\sqrt{4-x^2-y^2}$, find $f_x(1,0)$ and $f_y(1,0)$ and interpret geometrically.

Partial Derivatives of z=f(x,y)

Example: *Humidex*
The humidex $H(T,h)$ is a measure used by meteorologists to describe the combined effects of heat and humidity on an average person's feeling of hotness. The table below contains values of humidex based on measurements of temperature T (in degrees Celsius) and relative humidity h (given as a percent).

	T=22	T=26	T=30	T=34
h=70	27	33	41	49
h=60	25	32	38	46
h=50	24	30	36	43

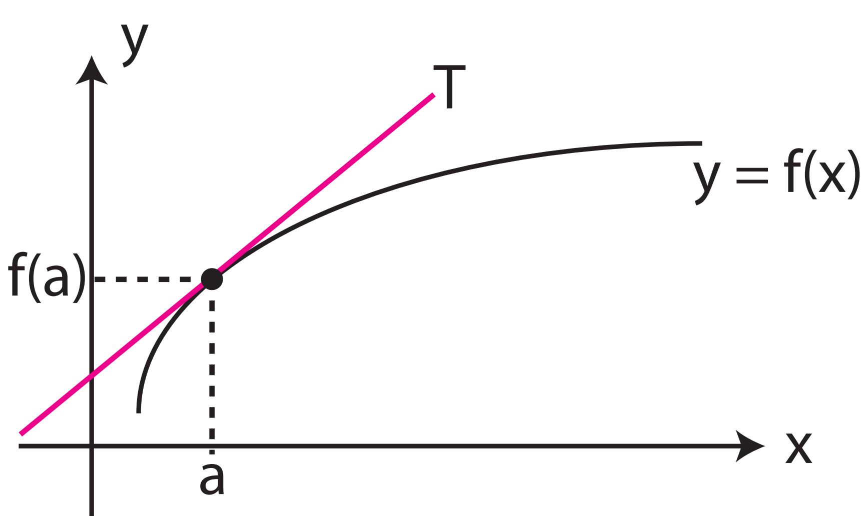
Estimate $H_T(30, 60)$ and interpret your answer.

Tangent Lines

Let $y=f(x)$ be a differentiable function in R^2.

Equation of the tangent line to the graph of f at $(a,f(a))$:

$$y - f(a) = f'(a)(x-a)$$



Linearization of *f* at *x=a*:

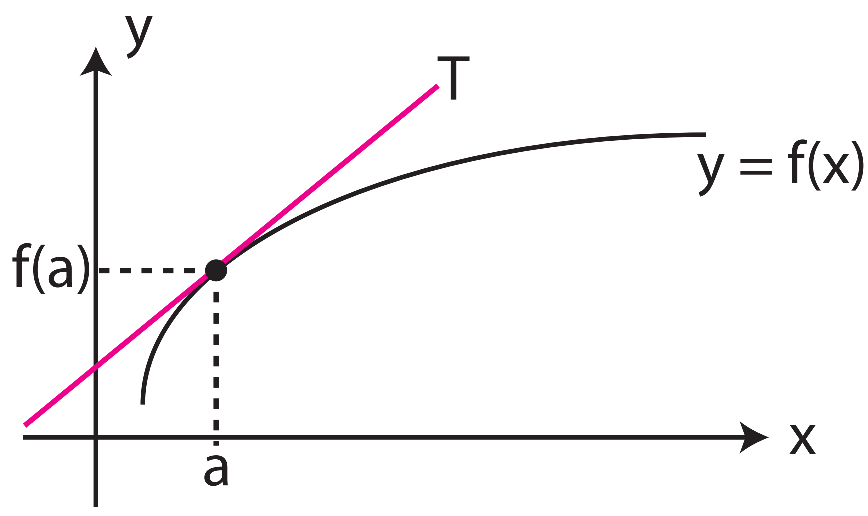
$$L_a(x) = f(a) + f'(a)(x-a)$$

L because this is a Linear function

Tangent Lines

The function *f* is approximately equal to its linearization at $(a, f(a))$ when the value of *x* is close to *a*.

as you zoom in around $(a, f(a))$, the line T more and more closely resembles the curve *f*



Linear approximation of *f* at *x*=*a*:

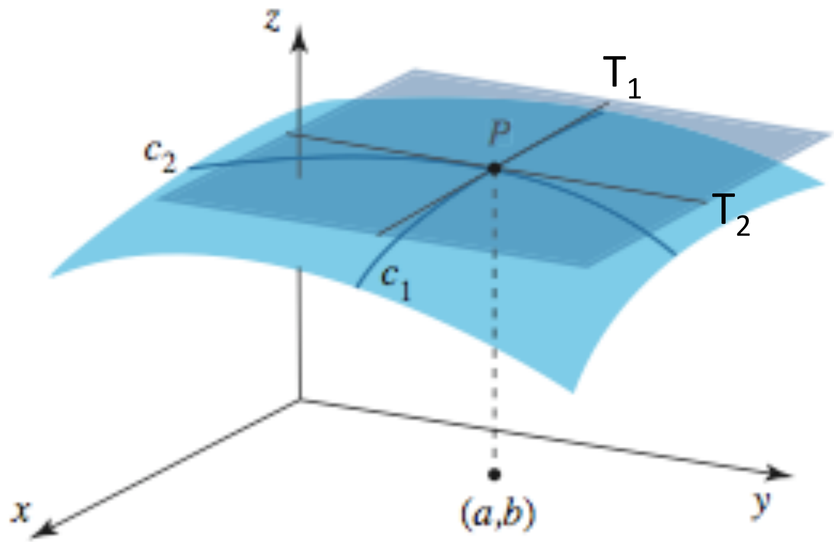
$$f(x) \approx f(a) + f'(a)(x - a)$$

Tangent Planes

Let $z=f(x,y)$ be a function in R^3 with continuous partial derivatives f_x and f_y.

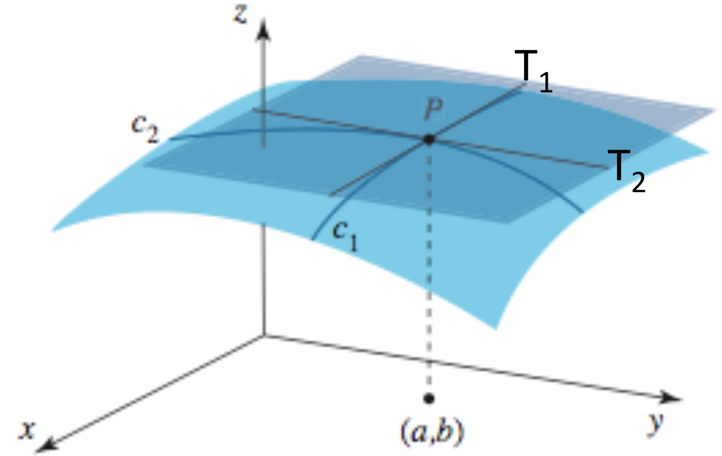
<u>Definition</u>: **Tangent Plane**

The plane that contains the point P and the tangent lines T_1 and T_2 at P is called the *tangent plane to the surface z=f(x,y) at P.*



Tangent Planes

Equation of the tangent plane to the surface $z=f(x,y)$ at $(a, b, f(a,b))$:

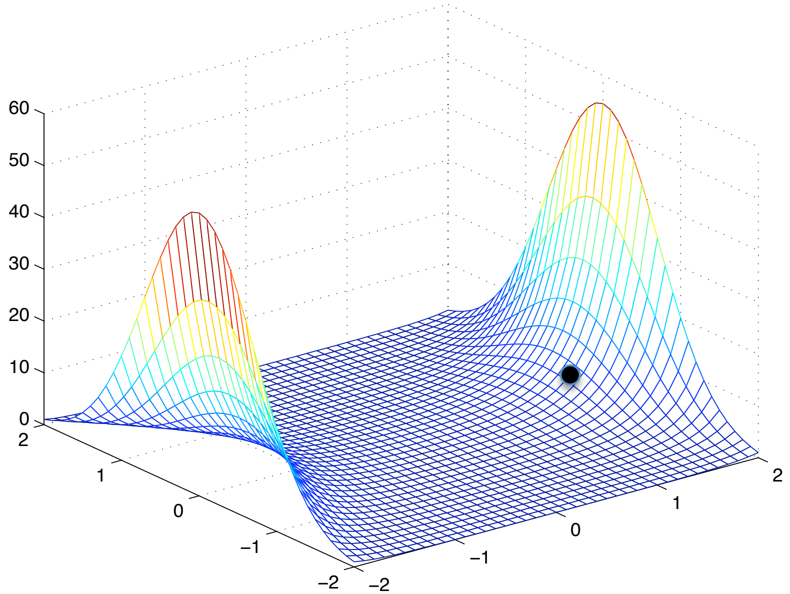


as you zoom in around $(a,b, f(a,b))$, the tangent plane more and more closely resembles the surface f

Tangent Planes

Example:

Find an equation of the tangent plane to the surface $z = e^{x^2 - y^2}$ at the point (1,-1, 1).



Linearization and Linear Approximation

<u>Definition</u>:
Assume that $z = f(x,y)$ has continuous partial derivatives at (a,b).

Linearization of *f* at (*a*,*b*):

$$L_{(a,b)}(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

Linear approximation (or tangent plane approximation) of *f* at (*a*,*b*):

$$f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

Linearization and Linear Approximation

Example:

Find the linearization of $f(x,y) = \ln(x - 3y)$ at $(7,2)$ and use it to approximate $f(6.9, 2.06)$.

Differentiability in R^2

- A function $f(x)$ is differentiable at a point $x=a$ if $f'(a)$ exists,

i.e. if the limit

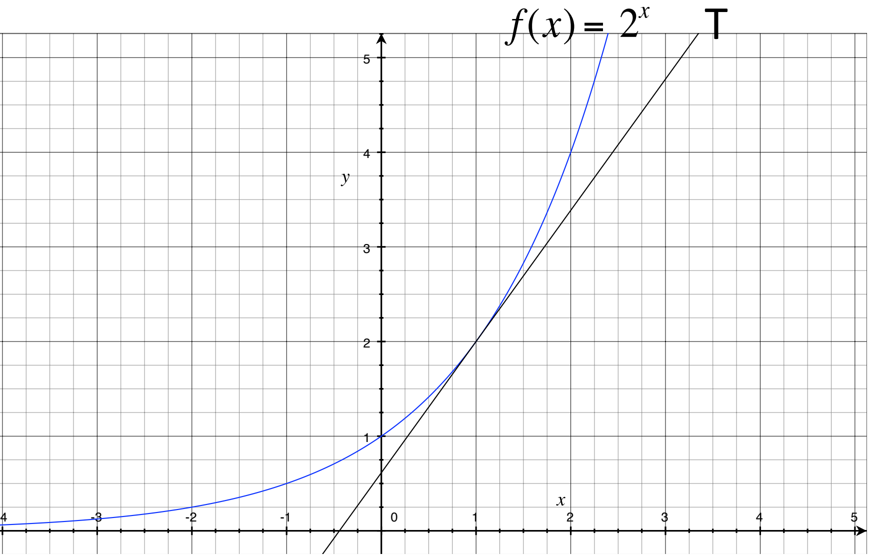
$$f'(a) = \left.\frac{df}{dx}\right|_{x=a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

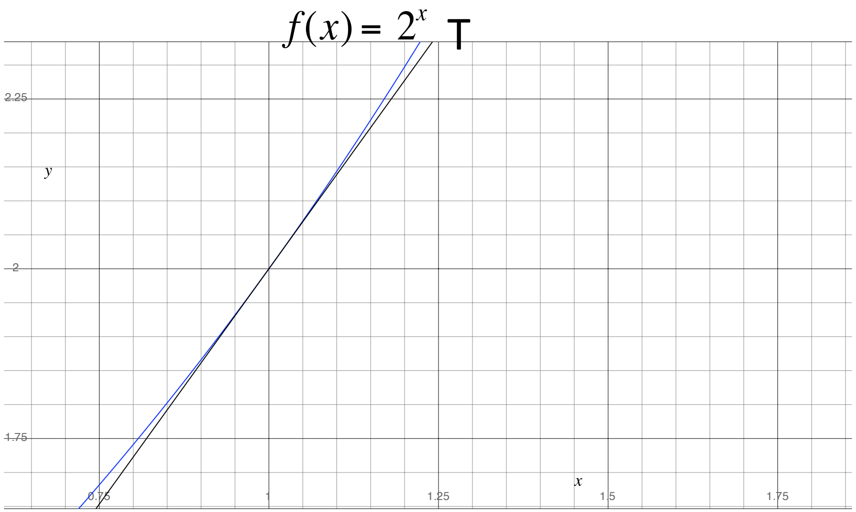
equals a real number.

Differentiability in R^2

- Geometrically, a function is differentiable at $x=a$ if its tangent line is *well-defined* at $(a,f(a))$.
- A well-defined tangent line has the property that it closely resembles the graph of the function on <u>both sides</u> of $x=a$ as we move closer and closer to the point (i.e., as we zoom in around the point, the curve and its tangent line become indistinguishable).

Differentiability in R^2

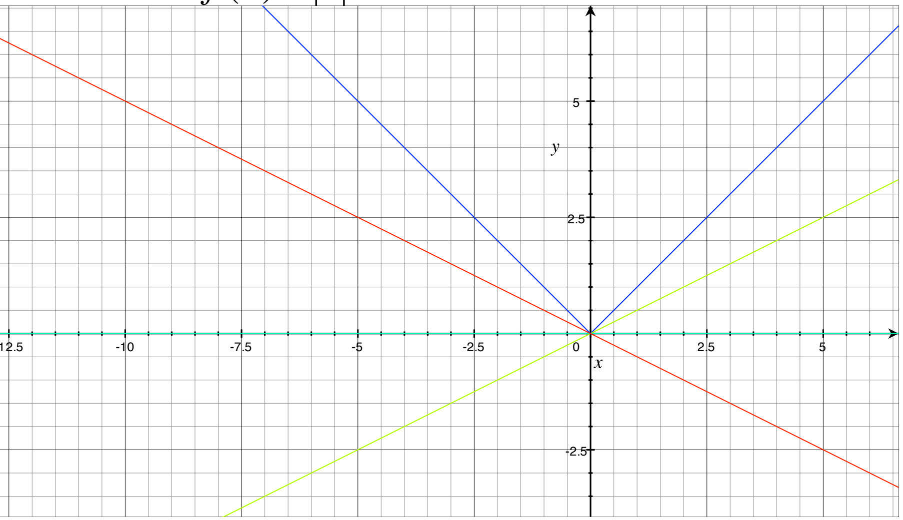




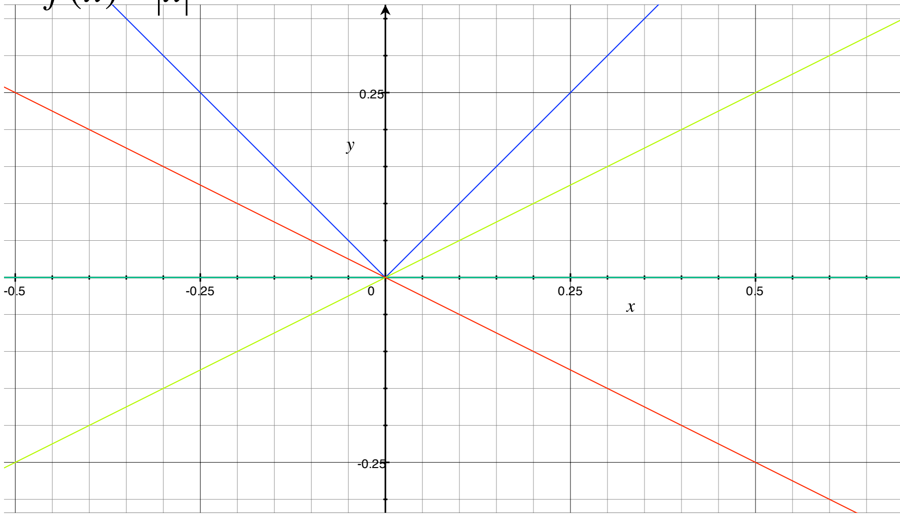
* $f(x)$ is differentiable at x=1

Differentiability in R^2

$f(x)=|x|$



$f(x)=|x|$



* $f(x)$ is NOT differentiable at x=0

Differentiability in R^3

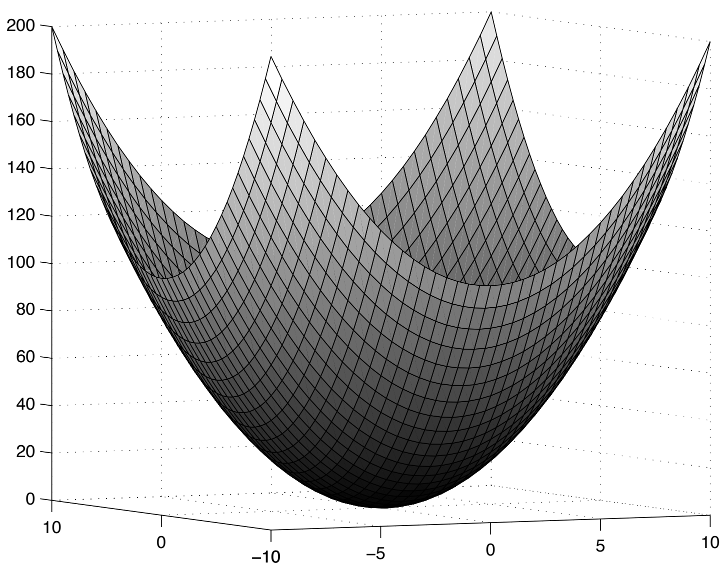
- Theoretically, a function $f(x,y)$ is differentiable at $(x,y)=(a,b)$ if the *directional* derivative of f exists in *EVERY* direction at (a,b). (This is impossible to check directly using the algebraic definition of the derivative.)

Differentiability in R^3

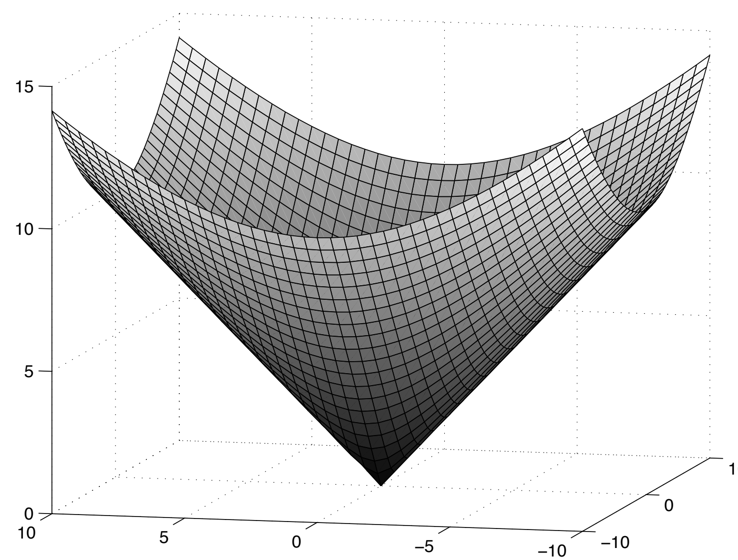
- Geometrically, a function $f(x,y)$ is differentiable at a point $(x,y)=(a,b)$ if its tangent **plane** is *well-defined* at (a,b).
- A well-defined tangent plane has the property that it closely resembles the graph of the function ***all around*** the point (a,b) as we move closer and closer to the point (i.e., as we zoom in around the point, the surface and its tangent plane become indistinguishable).

Differentiability in R^3

$f(x) = x^2 + y^2$



$f(x) = \sqrt{x^2 + y^2}$



Differentiable at (0,0)

<u>NOT</u> differentiable at (0,0)

Differentiability for a Function of Two Variables

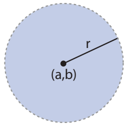
When a function $f(x,y)$ is differentiable at a point (a,b), we say that its linearization $L_{(a,b)}(x,y)$ is a good approximation to f near (a,b) and so the linear approximation

$$f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

is valid for (x,y) near (a,b).

Theorems

Sufficient Condition for Differentiability



Assume that f is defined on an open disk $B_r(a,b)$ centred at (a,b), and that the partial derivatives f_x and f_y are continuous on $B_r(a,b)$. Then f is differentiable at (a,b).

Differentiability Implies Continuity

Assume that a function f is differentiable at (a,b). Then it is continuous at (a,b).

Differentiability for a Function of Two Variables

Example:

Verify that the linear approximation

$$\frac{2x+3}{4y+1} \approx 3+2x-12y$$

is valid for (x,y) near $(0,0)$.

Differentiability for a Function of Two Variables

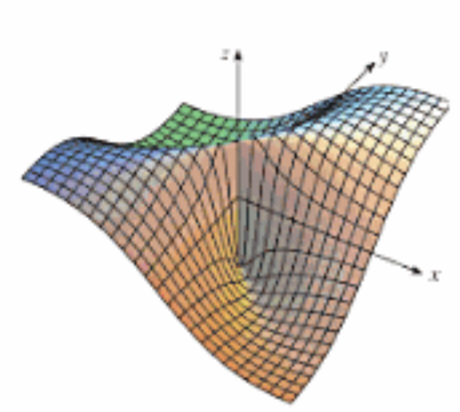
Example #16.

Show that the function $f(x,y) = x\tan y$ is differentiable at (0,0). What is the largest open disk centred at (0,0) on which *f* is differentiable?

Differentiability for a Function of Two Variables

Example in your text:

$$f(x,y)=\begin{cases} \dfrac{xy}{x^2+y^2} & if \quad (x,y)\neq(0,0) \\ 0 & if \quad (x,y)=(0,0) \end{cases}$$



Using the formula, and ignoring the fact that the partial derivatives are ***not*** continuous at (0,0), we find the linearization (tangent plane approximation) to be

$$L_{(0,0)}(x,y)=0$$

Differentiability for a Function of Two Variables

Example in your text:

However this is *not* a good approximation since the error between this linearization and the function does not approach 0 as (x,y) approaches (0,0).

For instance, along y=x, f(x,x)= ½ and the difference between the tangent plane and and the surface will remain constant at ½ (i.e. will not go to zero):

$$\text{error} = \left|f(x,y) - L_{(0,0)}(x,y)\right| = \left|\frac{1}{2} - 0\right| = \frac{1}{2}$$

Second-Order Partial Derivatives

Let *f* be a differentiable function of two variables, *x* and *y*.

Then *f* has two partial derivatives:

$$\frac{\partial f}{\partial x} \qquad\qquad \frac{\partial f}{\partial y}$$

Differentiating these expressions again (i.e. finding partial derivatives of partial derivatives), we obtain four second-order partial derivatives.

Second-Order Partial Derivatives

Differentiating $\dfrac{\partial f}{\partial x}$ with respect to *x*, we obtain:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} \quad \text{or} \quad (f_x)_x = f_{xx}$$

Differentiating $\dfrac{\partial f}{\partial y}$ with respect to *y*, we obtain:

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} \quad \text{or} \quad (f_y)_y = f_{yy}$$

Second-Order Partial Derivatives

Differentiating $\frac{\partial f}{\partial x}$ with respect to *y*, we obtain:

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} \quad \text{or} \quad (f_x)_y = f_{xy}$$

Differentiating $\frac{\partial f}{\partial y}$ with respect to x, we obtain:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} \quad \text{or} \quad (f_y)_x = f_{yx}$$

Second-Order Partial Derivatives

Some ways in which these derivatives are useful:

1. Second-order derivatives can help us to determine behaviour of first-order derivatives

2. We can use second-order derivatives to create quadratic approximations to functions (an improvement over linear approximations)

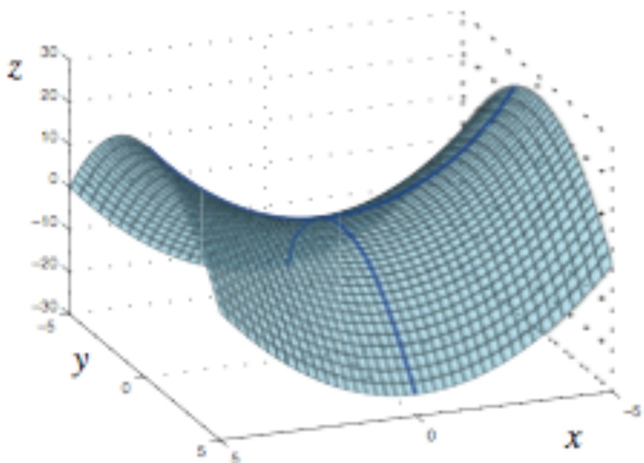
3. We can build partial differential equations which are used to model many real-life phenomena

Second-Order Partial Derivatives

Example:

Consider the function $f(x,y) = x^2 - y^2$.

Compute $f_{xx}(0,0)$ and $f_{yy}(0,0)$. What does this tell you about the shape of the graph of f at $(0,0)$?



$f(x,y) = x^2 - y^2$

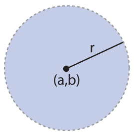
Second-Order Partial Derivatives

Example:

#18. Compute all second-order derivatives for the function $f(x,y) = \dfrac{xy}{x^2+1}$.

Equality of Mixed Partial Derivatives

Theorem:
Assume that a function f is defined in an open disk $B_r(a,b)$ and that the partial derivatives f_{xy} and f_{yx} are continuous on $B_r(a,b)$.



Then $f_{xy}(a,b)=f_{yx}(a,b)$.

Second-Order Partial Derivatives

Example:

Using the table of values below, determine whether the following partial derivatives are positive, negative, or zero:

$$f_x(4,1), \quad f_{xx}(4,1), \quad f_y(4,1), \quad f_{yy}(4,1)$$

	x=3	x=4	x=5	x=6
y=0	5.9	6.1	6.8	6.7
y=1	5.6	6	6.2	6.3
y=2	5.4	5.7	6.1	6.5