Summary of Logarithm Properties

Limits :

$$0 < a < 1 \Rightarrow \lim_{x \to \infty} \log_a x = -\infty \qquad \lim_{x \to 0^+} \log_a x = \infty$$

$$a > 1 \Rightarrow \lim_{x \to \infty} \log_a x = \infty \qquad \lim_{x \to 0^+} \log_a x = -\infty$$

$$\lim_{x \to \infty} \ln x = \infty \qquad \lim_{x \to 0^+} \ln x = -\infty$$

Core Properties :

$$\log_a bc = \log_a b + \log_a c \qquad \ln bc = \ln b + \ln c$$

$$\log_a b^c = c \log_a b \qquad \ln b^c = c \ln b$$

$$\log_a b = \frac{\ln b}{\ln a}$$

$$b > 0 \Rightarrow b = e^{\ln b} = a^{\log_a b}$$

Values :

$$\log_a a = 1, \quad \log_a 1 = 0, \quad \log_a \left(1/a \right) = -1$$

Derivatives :

$$\frac{d}{dx} \log_a x = \frac{1}{x \ln a} \qquad \frac{d}{dx} \ln x = \frac{1}{x}$$

$$\frac{d}{dx} \ln f(x) = \frac{f'(x)}{f(x)}$$

Other Derivatives :

$$\frac{d}{dx} a^x = a^x \ln a \qquad \frac{d}{dx} \ln (ax) = \frac{1}{x} \qquad \frac{d}{dx} \ln \left(x^p \right) = \frac{p}{x}$$

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$$

Logarithm Practice Problems:

1. Use logarithms to find the value of t in the following equations:

a) $e^{3t} = 8^2$ **b)** $3e^{2t+1} = 27$ **c)** $3^{2t} = 2^t$ **d)** $2e^{3t} - 1 = 4$ **e)** $\dfrac{200}{1-e^{2t}} = 100$

2. Given $\log a = 0.34$ and $\log b = 0.17$, evaluate the expressions:

a) $\log ab$ **b)** $\log\left(\dfrac{a}{b}\right)$ **c)** $\log(a^3\sqrt{b})$ **d)** $\log_b a$

3. Use the laws of logarithms to rewrite the following as single logarithmic expressions:

a) $2\ln x - \ln(x-1)$ **b)** $3\log_2 x + \log_2 y$ **c)** $\ln(x+1) - \ln(x-1)$

d) $2+3\ln x + \ln y - \ln z$ **e)** $\dfrac{1}{2}\ln a - \ln b + \dfrac{3}{2}\ln c$ **f)** $\dfrac{1}{3}\ln\left(x^2\right) + \dfrac{1}{4}\ln(x+1)$

g) $4\ln(\sin x) - 5\ln(\cos x) + 1$ **h)** $\ln(x+1) + c\ln(y^2-1) + d$ **i)** $\log_a(\sin^2 x + 2) - 3\log_a(y\tan y)$

4. Expand and simplify the following logarithms:

a) $\ln(2x)$ **b)** $\ln\left(\sqrt{x}\right)$ **c)** $\ln(3x\sqrt{x})$ **d)** $\ln(10^x)$ **e)** $\ln(5\sin x)$

f) $\ln(x(x-1)(x-2))$ **g)** $\ln\left(\dfrac{x+5}{\pi x}\right)$ **h)** $\ln(x^2-4)$ **i)** $\ln(2^{\sin x})$ **j)** $\ln\left(\dfrac{x}{2^x}\right)$

k) $\ln\left(e^2 x\right)$ **l)** $\ln(3^{3x}(x-1))$ **m)** $\ln\left(xe^{-x^2}\right)$ **n)** $\ln\left(e^{\pi^2}/x\right)$ **o)** $\ln\left(\dfrac{e^x}{1-e^x}\right)$

p) $\log_2\left(\dfrac{\sqrt{x-1}}{x+2}\right)$ **q)** $\ln\left(e^{x^2\sin(x)\sqrt{x+1}}\right)$ **r)** $\ln\left(\dfrac{\sqrt{x-1}}{\sqrt[3]{x^2+1}}\right)$

5. Use logarithms & logarithm properties to find the value of x in the following equations:

a) $\ln(2x) = 1$ **b)** $\ln\left(\dfrac{5\pi}{x^2}\right) + 2 = 0$ **c)** $\ln\left(\dfrac{x}{x+1}\right) = 0$ **d)** $\ln 3 + \ln(x+1) = 0$

e) $2\ln x = 0$ **f)** $\dfrac{1}{2}\ln(2x) - \dfrac{1}{2}\ln(x^2+1) = 0$ **g)** $\ln(x+1) + \ln(-4x) = 0$

h) $2e^{3x} = 1$ **i)** $\dfrac{e^x - 1}{x} = 0$ **j)** $\dfrac{4e^{2x} - 9}{\sqrt{\sin x + x^2 + \ln x}} = 0$ **k)** $\dfrac{1+e^x}{1-e^x} = 4$

l) $2e^{x^2} = 8$ **m)** $2^x = 5^{3x}$ **n)** $xe^x = 0$ **o)** $\left(x^2-1\right)e^{2x-3} = 0$ **p)** $\sqrt{x+2}\ln x = 0$

Logarithm Practice Solutions:

1. **a)** $t = 2\ln 2$ **b)** $t = \ln 3 - \frac{1}{2}$ **c)** $t = 0$ **d)** $t = \frac{1}{3}\ln\left(\frac{5}{2}\right)$ **e)** No Solution

2. **a)** 0.51 **b)** 0.17 **c)** 1.105 **d)** 2

3. **a)** $\ln\left(\frac{x^2}{x-1}\right)$ **b)** $\log_2 x^3 y$ **c)** $\ln\left(\frac{x+1}{x-1}\right)$ **d)** $\ln\left(\frac{x^3 y e^2}{z}\right)$

e) $\ln\left(\frac{\sqrt{ac^3}}{b}\right)$ **f)** $\ln\left(x^{2/3}(x+1)^{1/4}\right)$ **g)** $\ln\left(\frac{e\cdot\sin^4 x}{\cos^5 x}\right)$ **h)** $\ln\left((x+1)(y^2-1)^c e^d\right)$

i) $\log_a\left(\frac{\sin^2 x + 2}{y^3 \tan^3 y}\right)$

4. **a)** $\ln 2 + \ln x$ **b)** $\frac{1}{2}\ln x$ **c)** $\ln 3 + \frac{3}{2}\ln(x)$ **d)** $x \ln 10$

e) $\ln(5) + \ln(\sin x)$ **f)** $\ln(x) + \ln(x-1) + \ln(x-2)$ **g)** $\ln(x+5) - \ln\pi - \ln x$

h) $\ln(x-2) + \ln(x+2)$ **i)** $\ln 2 \cdot \sin x$ **j)** $\ln x - x\ln 2$ **k)** $2 + \ln x$

l) $3x\ln 3 + \ln(x-1)$ **m)** $\ln(x) - x^2$ **n)** $\pi^2 - \ln x$ **o)** $x - \ln\left(1 - e^x\right)$

p) $\frac{1}{2}\log_2(x-1) - \log_2(x+2)$ **q)** $x^2 \sin x\sqrt{x+1}$ **r)** $\frac{1}{2}\ln(x-1) - \frac{1}{3}\ln\left(x^2+1\right)$

5. **a)** $x = \frac{e}{2}$ **b)** $x = \pm\sqrt{5\pi e}$ **c)** *No Solution* **d)** $x = -\frac{2}{3}$

e) $x = 1$ (*only*!) **f)** $x = 1$ **g)** $x = -\frac{1}{2}$

h) $x = -\frac{1}{3}\ln 2$ **i)** $x = 0$ **j)** $x = \ln\left(\frac{3}{2}\right)$ **k)** $x = \ln\left(\frac{3}{5}\right)$

l) $x = \pm\sqrt{2\ln 2}$ **m)** $x = 0$ **n)** $x = 0$ **o)** $x = \pm 1$ **p)** $x = -2$ *or* 1

For additional logarithm property problems, and derivative problems, see the course text, Stewart's Calculus: Early Transcendentals 7th ed.

Basic Properties: 1.6 #35-41, 51-54
Derivatives: **3.6** #1-16, 23, 24, 25, 27-34

(And for additional reading and examples, read the parts of sections 1.6 on ln(x), and 3.6 on derivatives of ln/log. Leave logarithmic differentiation for later lectures!)