1ZC3 Last Day Basis & Dimension

A basis of a Vspace is a Linearly Independent & Spanning Set of the space.

$\Rightarrow$ each basis of a Vspace gives a co-ordinate system in which vectors can be expressed uniquely

All bases of a given Vspace are same size
for a space V this is $\dim(V)$ the dimension
(can be a #, can be ∞)

$(\text{size of LI sets}) \leq \dim V \leq (\text{Size of Spans})$

& if U is a subspace of V

$\dim(U) \leq \dim(V)$ & $U = V$ iff $\dim(U) = \dim(V)$

Also last day showed if any two of

① Spans, ② LI ③ # vectors $= \dim(V)$

are true $\Rightarrow$ all 3 are true $\Rightarrow$ we have a basis eg. $\left\{ \overset{\vec{u}_1}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}, \overset{\vec{u}_2}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}, \overset{\vec{u}_3}{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}, \overset{\vec{u}_4}{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}} \right\} = S$

Is this a <u>basis</u> of M_{22} (2×2 matrices).

<u>Solution</u> (1) $\dim(M_{22}) = 4 =$ <u># of vectors</u> ✓

$= n(S)$

<u>Check L.I.</u>

L.I. iff $a\vec{u}_1 + b\vec{u}_2 + c\vec{u}_3 + d\vec{u}_4 = 0$

iff $a = b = c = d = 0$

that mean $a\begin{bmatrix}1&0\\0&1\end{bmatrix}+b\begin{bmatrix}0&1\\1&0\end{bmatrix}+c\begin{bmatrix}1&0\\0&-1\end{bmatrix}+d\begin{bmatrix}0&1\\-1&0\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$

only for $a=b=c=d=0$

$$\begin{cases} a+c=0 & b+d=0 \\ b-d=0 & a-c=0 \end{cases}$$

could solve! if $a=b=c=d=0$ only $\Rightarrow$ L.I.

or as Lin. Systm

$$\left[\begin{array}{cccc|c} 1&0&1&0&0\\ 0&1&0&1&0\\ 0&1&0&-1&0\\ 1&0&-1&0&0 \end{array}\right]$$

does this have a unique soln?

If det of this is $\neq 0 \Rightarrow$ invertible!

$\Rightarrow$ only 0 solution!

$$\begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{vmatrix} \underset{R_3 - R_2}{\overset{R_4 - R_1}{=}} \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & -2 & 0 \end{vmatrix}$$

Flip R_3 & R_4.

$$= - \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{vmatrix} \neq 0$$

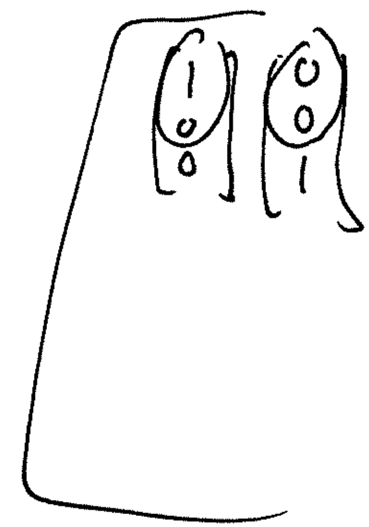
$\Rightarrow$ L. I. set!

The Wronskian Detects L. I. functions

Wronskian ~~of~~ n function is an $n \times n$ matrix determinant

$$\det\left(\begin{bmatrix} f_1(x) & f_2(x) & f_3(x) & \cdots & f_n(x)\\ f_1'(x) & f_2'(x) & \cdots & & f_n'(x)\\ \vdots & & & & \\ f_1^{(n-1)}(x) & \cdots & & & f_n^{(n-1)}(x)\end{bmatrix}\right).$$

If $\underline{\text{Wron} \neq 0} \Rightarrow \{f_1(x) \dots f_n(x)\}$ is L.I.



eg. Are $\sin(x)$ & $\cos(x)$ L.I.?

Solution

$$\text{Wron} = \det\begin{bmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{bmatrix}$$

$$= -\sin^2 x - \cos^2 x = -1 \neq 0 \quad \text{L.I}$$

Orthogonality & Basis

We need to working in an "inner product space"

i.e. we need a real vspace & an inner product $\langle \vec{u}, \vec{v} \rangle$.

Remember $\langle \vec{u}, \vec{v} \rangle$ input: 2 vectors in V

output: $\# \in \mathbb{R}$

Properties:

1) $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$
2) $\langle k\vec{u}, \vec{v} \rangle = k \langle \vec{u}, \vec{v} \rangle$
3) $\langle \vec{u} + \vec{w}, \vec{v} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{w}, \vec{v} \rangle$
4) $\langle \vec{u}, \vec{u} \rangle \geq 0$ ($\|\vec{u}\|^2 = \langle \vec{u}, \vec{u} \rangle$
5) $\langle \vec{u}, \vec{u} \rangle = 0$ iff $\vec{u} = \vec{0}$

If $\vec{u}, \vec{v} \neq 0$ but $\langle \vec{u}, \vec{v} \rangle = 0 \iff$ orthogonal vectors

An orthogonal set is a set of non-zero vectors that are pairwise orthogonal

i.e. $\{\vec{v}_1 \ldots \vec{v}_n\}$ such that $\vec{v}_i \cdot \vec{v}_j = 0$ iff $i \neq j$

An orthonormal set is a orthogonal set where all vectors have length 1 ie. $\{\vec{v}_1, \dots \vec{v}_n\}$ such that $\vec{v}_i \cdot \vec{v}_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

Note any orthogonal set is automatically L.I

Proof Say $a_1 \vec{v}_1 + \dots a_n \vec{v}_n = \vec{0}$

dot with any $\vec{v}_i$, say $\vec{v}_1$

$$\underline{a_1 \vec{v}_1 \cdot \vec{v}_1} + a_2 \cancelto{0}{\vec{v}_2 \cdot \vec{v}_1} + \dots a_n \cancelto{0}{\vec{v}_n \cdot \vec{v}_1} = \cancelto{0}{\vec{0} \cdot \vec{v}_1}$$

$$a_1 \|\vec{v}_1\|^2 = 0$$

$\|\vec{v}_1\| \neq 0 \Rightarrow a_1 = 0$

by same logic, any $a_i = 0 \Rightarrow$ V's L.I
(only!)

If I have an orthogonal basis, ie n(set) = dim V

$\Rightarrow$ very pretty co-ordinate calculation!

If $\vec{u} \in \text{Span}(\{\vec{v}_1, \dots \vec{v}_n\})$

$$\Rightarrow \vec{u} = a_1 \vec{v}_1 + \dots a_n \vec{v}_n$$

$$\langle \vec{u}, \vec{v}_i \rangle = a_1 \cancel{\langle \vec{v}_1, \vec{v}_i \rangle}^{0} + \dots a_n \cancel{\langle \vec{v}_n, \vec{v}_i \rangle}^{0}$$

$$= a_i \cdot \|\vec{v}_i\|^2$$

$$a_i = \langle \vec{u}, \vec{v}_i \rangle / \|\vec{v}_i\|^2$$

& $\vec{v}$'s are orthonormal

$$a_i = \langle \vec{u}, \vec{v}_i \rangle$$