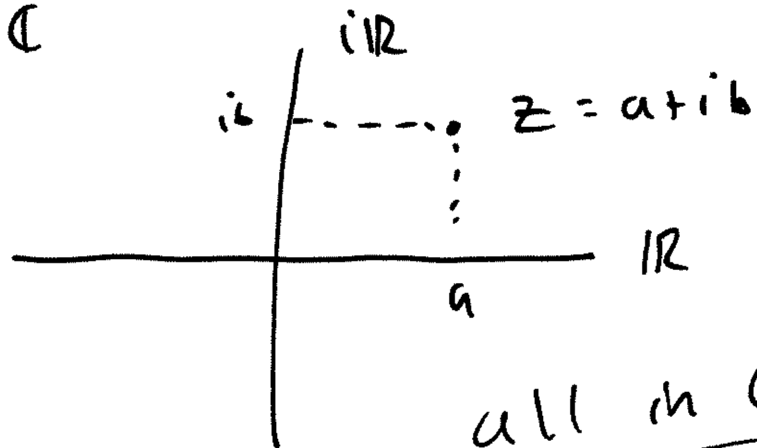
1ZC3 Last Day Complex Plane



all in $\mathbb{C}$

eg. $z = 5 - 7i$

eg $w = 6 - 2i$

eg $z = 5i$ } purely imaginary

eg $w = 4$ } purely real

If $z = a+ib \Rightarrow \mathrm{Re}(z) = a,\ \mathrm{Im}(z) = b$

modulus of $z = |z| = \sqrt{a^2+b^2}$

$\theta = \arg(z) =$ argument of z, ccw angle to $\mathbb{R}^+$

$= \arctan\left(\frac{b}{a}\right)$ if $\mathrm{Re}(z) > 0$

$\arctan\left(\frac{b}{a}\right) \pm \pi$ if $\mathrm{Re}(z) < 0$

Generally use "principal argument", $\theta \in (-\pi, \pi]$

(in other texts, $\theta \in [0, 2\pi)$)

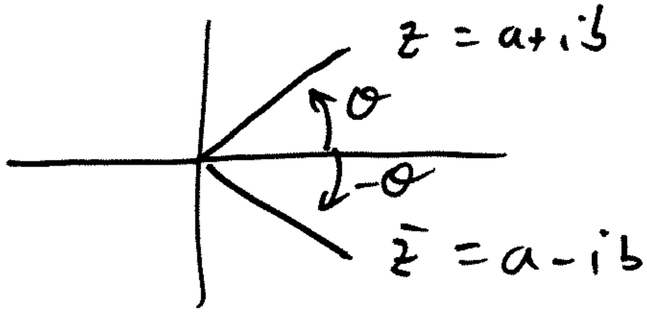
$\Rightarrow$ polar form $\quad z = \overset{r=|z|}{r}(\cos \overset{\arg(z)}{\theta} + i \sin\theta) = r \operatorname{cis}\theta$

Remember if $z = a + ib \Rightarrow \bar{z} = a - ib$

complex conjugate

if $z = r \operatorname{cis}\theta \Rightarrow$

$$\bar{z} = r\,\overline{\operatorname{cis}\theta} = r\,(\overline{\cos\theta + i\sin\theta})$$
$$= r(\cos\theta - i\sin\theta)$$
$$= r(\cos(-\theta) + i\sin(-\theta))$$
$$\bar{z} = r \operatorname{cis}(-\theta)$$



Trig is a Lie!!

$$\text{cis}\,\theta = e^{i\theta}$$

$$\left.e^{i\theta} = \cos\theta + i\sin\theta\right\}$$ **Euler's Formula!**

Why? notice $\frac{d}{d\theta}\,\text{cis}\,\theta = \frac{d}{d\theta}(\cos\theta + i\sin\theta)$

$$= -\sin\theta + i\cos\theta$$

$$= i\,(\cos\theta + i\sin\theta) = i\cdot\text{cis}\,\theta$$

$\Rightarrow (\text{cis}\,\theta)' = i\cdot\text{cis}\,\theta$.

$\Rightarrow$ $\text{cis}\,\theta$ is a particular solution to

$$\boxed{y' = i\cdot y}$$

General solution to $y' = i \cdot y$ is $y = (\text{const}) \cdot e^{i\theta}$

but $\operatorname{cis}\theta \big|_{\theta=0} = \cos(0) + i\sin(0) = 1$

$$y = (\text{const})\, e^{i\theta}\Big|_{\theta=0} = 1$$

$\text{const} = 1 \Rightarrow \boxed{y = e^{i\theta}}$

$$\Rightarrow \boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

or

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{i^{2n} x^{2n}}{(2n)!}$$

$$i\sin x = \sum_{n=0}^{\infty} i(-1)^n \frac{x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{i^{2n+1} x^{2n+1}}{(2n+1)!}$$

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!}$$

So $e^{i\theta} = \cos\theta + i\sin\theta$ always!

1) $z = r\,\mathrm{cis}\,\theta \Rightarrow z = re^{i\theta}$

2)
$$e^{i\theta} - e^{-i\theta} = \cancel{\cos\theta} + i\sin\theta - \cancel{\cos(-\theta)} - i\sin(-\theta)$$
$$= 2i\sin\theta$$

$$\boxed{i\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2} = \sinh(i\theta)}$$

$$e^{i\theta} + e^{-i\theta} = \cos\theta + i\cancel{\sin\theta}$$
$$+ \cos(-\theta) + i\cancel{\sin(-\theta)}$$
$$= 2\cos\theta$$

$$\boxed{\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \cosh(i\theta)}$$

Fun Identity, eg.

$e^{-i\theta}$ $\Rightarrow$ {if $z = re^{i\theta}$, $\bar{z} = re^{-i\theta}$}

$$|e^{i\theta}| = \sqrt{e^{i\theta} \cdot \overline{e^{i\theta}}} = \sqrt{\operatorname{cis}\theta \cdot \overline{\operatorname{cis}\theta}}$$
$$= \sqrt{\operatorname{cis}\theta \cdot \operatorname{cis}(-\theta)}$$
$$= \sqrt{e^{i\theta} \cdot e^{-i\theta}} = 1$$

$\Rightarrow$ {$z = e^{i\theta}$ unit circle}

$$1 = e^{i\theta} \cdot e^{-i\theta} = (\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta)$$

$$= \cos^2\theta - \cancel{i\cos\theta\sin\theta} + \cancel{i\cos\theta\sin\theta} - i^2\sin^2\theta$$

$$1 = \cos^2\theta + \sin^2\theta$$

$$e^{i(2\theta)} = \underline{\cos(2\theta) + i\sin(2\theta)}$$

$$\shortparallel$$

$$e^{i\theta} \cdot e^{i\theta} = (\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta)$$

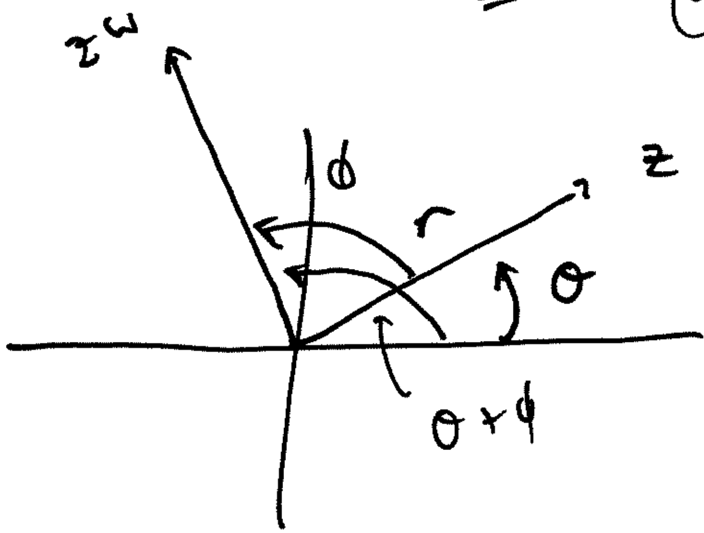
$$= \cos^2\theta + e^{i2}\sin^2\theta + i\sin\theta\cos\theta + i\cos\theta\sin\theta$$

$$= (\cos^2\theta - \sin^2\theta) + i(2\sin\theta\cos\theta)$$

$$= \cos(2\theta) + i\sin(2\theta)$$

Notice if $z = re^{i\theta}$, $w = \rho e^{i\phi}$

$$zw = r\rho\, e^{i(\theta+\phi)}$$



& $z^n = (re^{i\theta})^n$

$= r^n e^{in\theta}$

$\frac{1}{z} = z^{-1} = \frac{1}{r} e^{-i\theta}$, $r \neq 0$

e.g. if $z = 1-\sqrt{3}\,i$ find z^{50}

Solution $r = |z| = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{1+3} = 2$

$$\arg(z) = \tan^{-1}(b/a) = \tan^{-1}\left(\frac{-\sqrt{3}}{1}\right)$$

$$= -\tan^{-1}(\sqrt{3}) = -\pi/3$$

$Re(z) = 1 > 0$ ✓ ok!

$$z^{50} = (1-\sqrt{3}i)^{50} = (re^{i\theta})^{50} = 2^{50} e^{i\,50\cdot(-\pi/3)}$$

$$= 2^{50} e^{-i(50\pi/3)}$$ ← drop multiples of $2\pi = \frac{6\pi}{3}$

$+ 60\pi/3 = \underline{10 \text{ rotatn}}$

$$= 2^{50} e^{i(10\pi/3)}$$

$-\underline{1 \text{ rotn}} = -6\pi/3$

$\in (-\pi, \pi]$

$$= 2^{50} e^{i(4\pi/3)} = 2^{50} e^{-i(2\pi/3)}.$$

$\in [0, 2\pi)$

$-1 \text{ rotn} = -\frac{6\pi}{3}$

$\Rightarrow$ z^{50} in $a+ib$ form!

$$= 2^{50} \text{cis}\left(-2\pi/3\right)$$

$$= 2^{50}\cos\left(2\pi/3\right) - i2^{50}\sin\left(2\pi/3\right)$$

$$= 2^{50}\left(-\frac{1}{2} - \frac{\sqrt{3}i}{2}\right) = \underline{\underline{-2^{49}\left(1+\sqrt{3}i\right)}}$$