Optimization

How do we optimize?

Step 1 Read the Question: Watch for variables & what we are trying to optimize!

Step #2: Picture! (If possible) to visualize!

Step #3: Name those variables: Set names to values you're working with!

Step #4: Write equations / relations known.

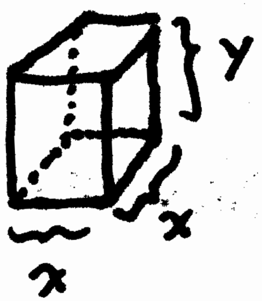
Step #5: Reduce to put optimized quanty in terms of one variable!

Step #6: Take derivative & get max (or min)

Step #7: Re-read the question! Make sure your answer is what you are looking for

eg. Say I want a $1000\,cm^3$ volume closed cardboard box. The bottom of this rectangular box must be square. What is the **height** of the box when the surface area is minimized?

Solution



V = volume

A = surface area

$$1000 = x^2 y \qquad A = 2x^2 + 4xy$$

$$y = \frac{1000}{x^2} \Rightarrow A = 2x^2 + 4x\left(\frac{1000}{x^2}\right)$$

$$= 2x^2 + \frac{4000}{x}$$

$$\frac{dA}{dx} = 4x - \frac{4000}{x^2} = \frac{4}{x^2}(x^3 - 1000)$$

$$\Rightarrow \frac{dA}{dx} = 0 \text{ at } \underline{x = 10} \qquad \text{DNE } \underline{\text{at } x = 0}$$

To minimize Surface Area (A) $\Rightarrow$ usually get c.n.

& test c.n. & endpoints

$$A = 2x^2 + \frac{4000}{x}, \quad x \in (0, \infty)$$

oh no! not closed bounded interval!

$\Uparrow$

no endpoints to test! Evil

Evil

But

If $f(x)$ is cont. on (a, b) ← an open interval!

and if there exists $\boxed{\text{exactly one}}$ critical number.

then if it is a local min $\Rightarrow$ It's y-value is abs. min

& if it is a local max $\Rightarrow$ It's y-value is abs max

eg

$\leftarrow$ local min

$\leftarrow$ local max

So back to box

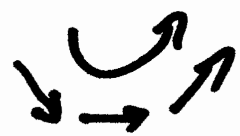
We know $A(x) = 2x^2 + \frac{4000}{x}$ & $A'(x) = 4x - 4000/x^2$

$$= \frac{4}{x^2}(x^3 - 1000)$$

$\Rightarrow$ c.n. at $x = 10$ $\Rightarrow$ check for abs. min

eg first deriv. test $x > 10 \Rightarrow \frac{dA}{dx} > 0$

$x < 10 \Rightarrow dA/dx < 0$



$\Rightarrow$ local min at $\underline{x=10}$ $\Rightarrow$ location of Abs. Min. of A.

Question wanted height i.e. y

$$y x^2 = V = 10000 \Rightarrow y \cdot 10^2 = 1000$$

$$y = 10$$

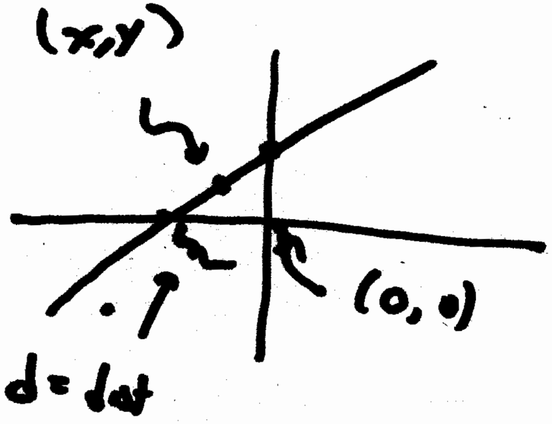
TAH DAH

Stupid Optimization Tricks

e.g. Say I want to find point (x, y) on line $y = 2x+1$ closest to $(0,0)$

Solution

Goal: Minimise dist



$$d = \sqrt{x^2 + y^2}$$

$\uparrow$ but

$y = 2x+1$

$$d = \sqrt{x^2 + (2x+1)^2}$$

Instead define $G = x^2 + (2x+1)^2$

G should have a min. at same x as d.

So use G instead! Cheat!

$$\frac{d}{dx} G = 2x + 2(2x+1)\cdot 2$$

$$= 10x + 4 = 0 \Rightarrow x = \frac{-4}{10} = -\frac{2}{5}$$

Check have 1 c.n. ($x = -0.4$):

Is this a local min? $\frac{d^2}{dx^2} G = 10 > 0$

$\Rightarrow$ local min by 2nd deriv. test!

$\Rightarrow$ yes! $x = -\frac{4}{10} = -\frac{2}{5}$ is min G point

$\Rightarrow$ minimizes dist!

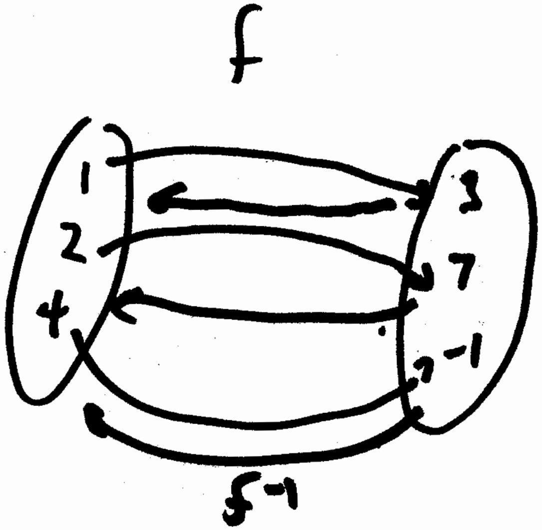
we wanted (x,y), but $y = 2x+1 = -\frac{4}{5}+1 = \frac{1}{5}$

$\cancel{= -\frac{4}{5}+1 = 3/5}$

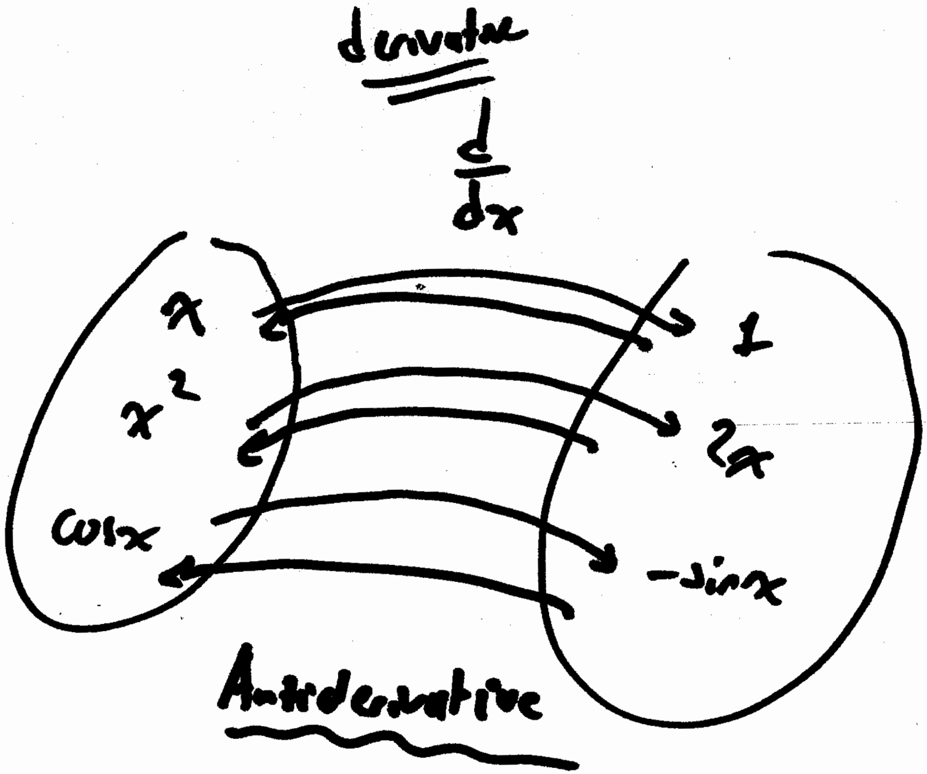
$$(x,y) = \left(-\frac{2}{5}, \frac{1}{5}\right)$$

minimizes our distance!

Antidifferentiation



$\Rightarrow$



But

$$\frac{d}{dx} x^2 = 2x$$

$$\frac{d}{dx}(x^2 + 4) = 2x$$

$$\cdots \frac{d}{dx}(x^2 + 10^{100}) = 2x$$

" $\frac{d}{dx}$ " is not 1-1 mapping!

antiderivative $f(x)$ is not unique

Its clear If $F'(x) = f(x)$ & $G'(x) = f(x)$ $\Big\} \Rightarrow F(x) - G(x) = C = \text{const}$

Unique up to constant

" General Antiderivative " " Integral "

$$\int 2x \, dx = x^2 + C$$