1ZA3

Last Day: Integration by Parts

e.g. $\int_0^{\pi} x\cos x\,dx$

$u = x \Rightarrow \frac{du}{dx} = 1$

so $du = dx$

$= \int u\,dv = uv - \int v\,du$ & $\int dv = \int \cos x\,dx$

$= \sin x = v$

$= x\sin x\Big|_0^{\pi} - \int_0^{\pi} \sin x\,dx$

$= x\sin x + \cos x\Big|_0^{\pi}$

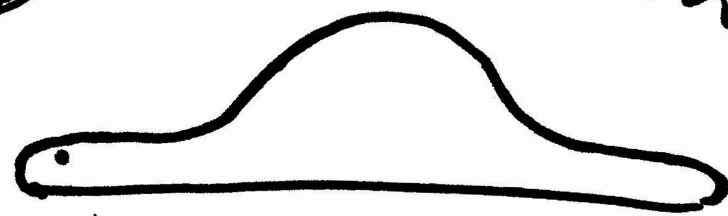
$= (0-0) + (-1) - 1 = \underline{\underline{-2}}.$

Remember General Guideline

Best u: $\ln x$, $\tan^{-1} x$, x^2, x^3 etc.

$\downarrow$

worst u: e^x, $\sin x$, $\cosh x$,



Important!

eg. $\int 1 \cdot \ln x \, dx$

$= \int u \, dv$

$= uv - \int v \, du$

$$\begin{cases} u = \ln x \Rightarrow \dfrac{du}{dx} = \dfrac{1}{x}, \; du = \dfrac{1}{x} dx \\ \int dv = \int 1 \, dx \Rightarrow v = x \end{cases}$$

$= x \ln x - \int x \cdot \frac{1}{x} dx$

$= x \ln x - x + C$

eg. $\int e^x \cos x \, dx$

$= \int u \, dv$

$\begin{cases} u = e^x \Rightarrow du = e^x dx \\ \int dv = \int \cos x \, dx \Rightarrow v = \sin x \end{cases}$

$= uv - \int v \, du \quad = e^x \sin x - \int e^x \sin x \, dx$

$\int e^x \sin x \, dx$

$= \int u \, dv$

$\begin{cases} u = e^x \Rightarrow du = e^x dx \\ v = \int dv = \int \sin x \, dx = -\cos x \end{cases}$

$= uv - \int v \, du$

$= -e^x \cos x + \int e^x \cos x \, dx$

$\underline{\text{so}}$ $\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx$

$= e^x \sin x + e^x \cos x - \int e^x \cos x \, dx$

$\underline{\text{so}}$ $2 \int e^x \cos x \, dx = e^x \sin x + e^x \cos x + C$

$\int e^x \cos x \, dx = \frac{1}{2} \left(e^x \sin x + e^x \cos x \right) + C$

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eg. $\int x^3 e^{x^2} \, dx$

Let's substitute!

Let $\underline{u = x^2}$ $\Rightarrow$ $du = 2x \, dx$

$dx = \frac{1}{2x} \, du$

$= \int x^3 e^u \frac{1}{2x} \, du$

$= \frac{1}{2} \int x^2 e^u \, du = \frac{1}{2} \int u e^u \, du$

$$= \frac{1}{2}\int w\,dv = \left(wv - \int v\,dw\right)/2$$

$$w = u \qquad du = dw$$

$$\int dv = \int e^u\,du \Rightarrow v = e^u$$

$$= \left(u e^u - \int e^u\,du\right)/2$$

$$= \frac{1}{2}\left(u e^u - e^u\right) + C$$

$$= \frac{1}{2}\left(x^2 e^{x^2} - e^{x^2}\right) + C$$

$$= \frac{e^{x^2}(x^2-1)}{2} + C$$

Trig Integration

$$\int \cos x\,dx = \sin x + C, \qquad \int \sin x\,dx = -\cos x + C$$

$$\int \sin x \cdot \cos^5 x \, dx = \int u^5 (-1)\, du$$

$u = \cos x, \quad du = -\sin x \, dx$

$$= -1 \cdot \frac{1}{6} u^6 = -\frac{1}{6}\cos^6 x + C$$

eg $\int \sin^3 x \cos^4 x \, dx$

let $u = \cos x$

$-du = \sin x \, dx$

$$= \int \sin^2 x \cos^4 x \, \sin x \, dx$$

$$= -\int \sin^2 x \cdot u^4 \, du$$

$\sin^2 x = 1 - \cos^2 x$

$= 1 - u^2$

$$= \int (u^2 - 1)\, u^4 \, du$$

$$= \int u^6 - u^4 \, du = \frac{1}{7}u^7 - \frac{1}{5}u^5 + C$$

$$= \frac{1}{7}\cos^7 x - \frac{1}{5}\cos^5 x + C$$

In general Given $\int \cos^m x \sin^n x \, dx$

If **n odd**: let $u = \cos x$, $(-du) = \sin x \, dx$

& convert to cosines: $\sin^2 x = 1 - \cos^2 x$
$= 1 - u^2$

If **m odd**: let $u = \sin x$, $du = \cos x \, dx$

& convert to **sines**! $\cos^2 x = 1 - \sin^2 x$
$= 1 - u^2$

If **both** odd, **do either!**

(but note: answers may **look** different! One in $\cos x$ **only**
One in $\sin x$ **only**! But equivalent! Since $\cos^2 x = 1 - \sin^2 x$

What if m, n both even?

eg. $$\int \cos^2 x \sin^2 x \, dx$$

$$\cos^2 x = \frac{1}{2}(1+\cos(2x))$$

$$\sin^2 x = \frac{1}{2}(1-\cos(2x))$$

$$= \int \frac{1}{2}(1+\cos(2x))\frac{1}{2}(1-\cos(2x))\,dx$$

$$= \frac{1}{4}\int 1 - \boxed{\cos^2(2x)}\,dx$$

$$= \frac{1}{4}\int 1 - \boxed{\left(\frac{1}{2}(1+\cos(4x))\right)}\,dx$$

$$= \frac{1}{8}\int 2 - 1 - \cos(4x)\,dx$$

$$= \frac{1}{8}\int 1 - \cos(4x)\,dx$$

$$= \frac{1}{8}x - \frac{1}{8}\cdot\frac{1}{4}\sin(4x) + C$$

$$= \frac{1}{8}x - \frac{1}{32}\sin(4x) + C$$