12A3

Trig Substitution

Given an integral containing $\sqrt{a^2 - x^2}$, a +ve const.

Let $x = a \sin t$, $dx = a \cos t \, dt$ } don't forget!

$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2 \sin^2 t} = \sqrt{a^2 \cos^2 t} = a \cos t$

$t \in [-\pi/2, \pi/2]$ ← range of $\sin^{-1}(t)$

eg. $\int \frac{x^2}{\sqrt{4 - x^2}} \, dx$ Integrate using trig. sub.

Solution

Let $x = a\sin t = 2\sin t$, $dx = 2\cos t\, dt$

$$\sqrt{4-x^2} = \sqrt{4-4\sin^2 t} = \sqrt{4\cos^2 t} = 2\cos t.$$

$$\int \frac{x^2}{\sqrt{4-x^2}}\,dx = \int \frac{4\sin^2 t}{\cancel{2\cos t}} \cdot \cancel{2\cos t}\,dt$$

$$= 4\int \sin^2 t\,dt$$ don't forgot:

$\sin^2 t = \frac{1}{2}(1-\cos(2t))$

$$= 2\int 1 - \cos(2t)\,dt$$

$$= 2t - \frac{\cancel{2}}{\cancel{2}}\sin(2t) + c$$ convert back to x functions!

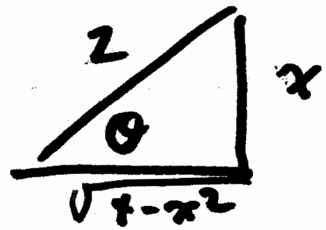
$\sin(2t) = 2\sin t\cos t$

$$= 2\sin^{-1}\left(\frac{x}{2}\right) - 2\cdot\left(\frac{x}{2}\right)\left(\cancel{?}\right) + C$$

$$\frac{\sqrt{4-x^2}}{2}$$

$x = 2\sin t$

$t = \sin^{-1}\left(\frac{x}{2}\right)$



$$\sin t = \frac{\text{opp}}{\text{hyp}} = \frac{x}{2}$$

$$\cos(t) = \frac{\text{adj}}{\text{hyp}} = \frac{\sqrt{4-x^2}}{2}$$

$$= 2\sin^{-1}\left(\frac{x}{2}\right) - \frac{1}{2}x\sqrt{4-x^2} + C$$

eg. $\displaystyle\int \frac{dx}{x^2\sqrt{1+9x^2}}$

let $x = \frac{1}{3}\tan t \Rightarrow \sqrt{1+9x^2} = \sqrt{1+\tan^2 t}$

$= \sqrt{\sec^2 t}$

$\& \; dx = \frac{1}{3}\sec^2 t \, dt$ $\qquad = \sec t;\quad t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$= \int \frac{1}{\left(\frac{1}{9}\tan^2 t\right) \cancel{\sec t}} \cdot \frac{1}{3}\sec^{\cancel{2}} t \, dt$$

$$= 3 \int \frac{\sec t}{\tan^2 t} \, dt = 3 \int \frac{\cos^{\cancel{2}} t}{\sin^2 t} \cdot \frac{1}{\cancel{\cos t}} \, dt$$

$$= 3 \int \frac{\cos t}{\sin^2 t} \, dt$$

let $u = \sin t$

$du = \cos t \, dt$

$$= 3 \int \frac{1}{u^2} du = 3 \int u^{-2} du = -3u^{-1} + c$$

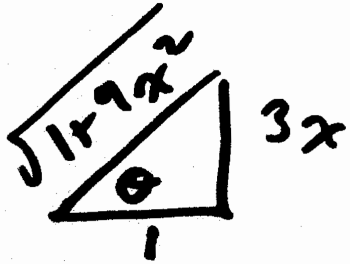
$$= -\frac{3}{u} + c = -\frac{3}{\sin t} + c$$

$x = \frac{1}{3}\tan t$

$t = \arctan(3x)$

$$= -\frac{3}{\sin t} + C = \frac{-3}{\left(\frac{3x}{\sqrt{1+9x^2}}\right)} + C$$

$$= \boxed{-\frac{\sqrt{1+9x^2}}{x} + C}$$



$$\tan\theta = 3x = \frac{\text{opp}}{\text{adj}}$$

$$\sin(\theta) = \frac{\text{opp}}{\text{hyp}} = \frac{3x}{\sqrt{1+9x^2}}$$

In general if my integrand has $\sqrt{a^2+x^2}$, $a>0$

then let $x = a\tan t$, $dx = a\sec^2 t\, dt$

$$\sqrt{a^2+x^2} = \sqrt{a^2+a^2\tan^2 t} = \sqrt{a^2(1+\tan^2 t)}$$

$$= \sqrt{a^2\sec^2 t} = a\sec t$$

if $t \in (-\frac{\pi}{2}, \pi/2)$.

If my integral has $\sqrt{x^2-a^2}$

Then let $x = a\sec t$

$dx = a\sec t\tan t\,dt$

$$\sqrt{x^2-a^2} = \sqrt{a^2\sec^2 t - a^2} = \sqrt{a^2\tan^2 t}$$
$$= a\tan t$$

eg. Solve by trig. substitution:

$$\int \frac{x}{\sqrt{x^2-25}}\,dx$$

$x = 5\sec t$

$\sqrt{x^2-25} = \sqrt{25\sec^2 t - 25}$

$= \sqrt{5^2\tan^2 t} = 5\tan t$

$$= \int \frac{5\sec t}{\cancel{5}\,\cancel{\tan t}} \cdot \cancel{5}\sec t\,\cancel{\tan t}\,dt$$

$$= 5 \int \sec^2 t \, dt$$

$$= 5 \tan t + c = \underline{\underline{\sqrt{x^2 - 25} + c}}$$

or by subs $\int \frac{x}{\sqrt{x^2-25}} dx$, let $u = x^2 - 25$

$du = 2x\,dx$

$\frac{1}{2} du = x\,dx$

$$= \int \frac{1}{2} \cdot u^{-\frac{1}{2}} \, du$$

$$= \cancel{\frac{1/2}{1/2}} u^{1/2} = \sqrt{u} + c = \underline{\underline{\sqrt{x^2 - 25} + c}}$$

The moral? Don't use trig. sub. if you don't have to!