McMaster University
Math 1LT3

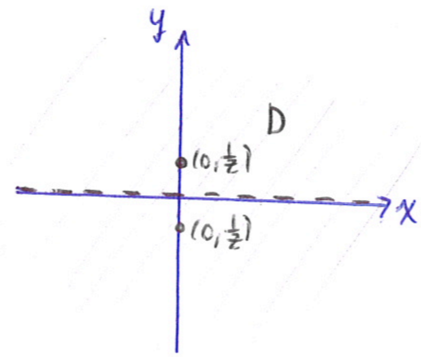
ASSIGNMENT 4

1. Consider the function $f(x,y) = \dfrac{e^x}{y}$.

(a) Find and sketch the domain of f.

$y \neq 0$

$D = \{(x,y) \in \mathbb{R}^2 \mid y \neq 0\}$



(b) Determine the range of f.

$\oplus$ / $\oplus$ or $\ominus$ $(y \neq 0)$: $\dfrac{e^x}{y} = z \Rightarrow y = \dfrac{e^x}{z} \quad (z \neq 0)$

Choose $x = 0$ and $y = \frac{1}{z}$ for any $z \in \mathbb{R} \setminus \{0\}$.

<u>Note</u>: $(0, \frac{1}{z}) \in D$.

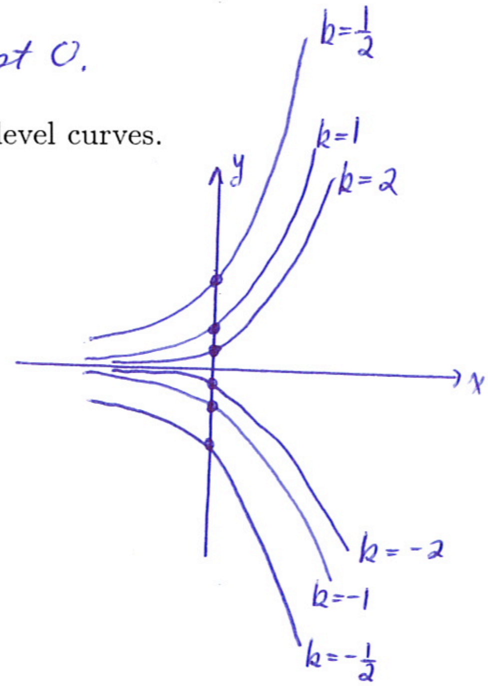
Then $f(0, \frac{1}{z}) = \dfrac{e^0}{\frac{1}{z}} = z$.

$\therefore$ The range of f is all of $\mathbb{R}$ except 0.

(c) Sketch a contour map of f. Include at least 5 level curves.

$\dfrac{e^x}{y} = k \quad (k \in \mathbb{R} \setminus \{0\})$.

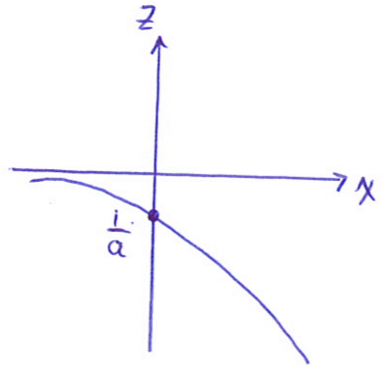
$\Rightarrow y = \dfrac{1}{k} e^x$



(d) Treat y as a parameter and sketch a graph in two-dimensions to illustrate how f depends on x. (Consider the case when $y < 0$ and then when $y > 0$.)

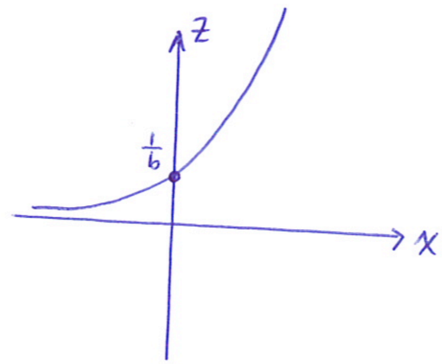
Let $y = a$ where $a < 0$.

$$f(x, a) = \frac{e^x}{a}$$



Let $y = b$ where $b > 0$.

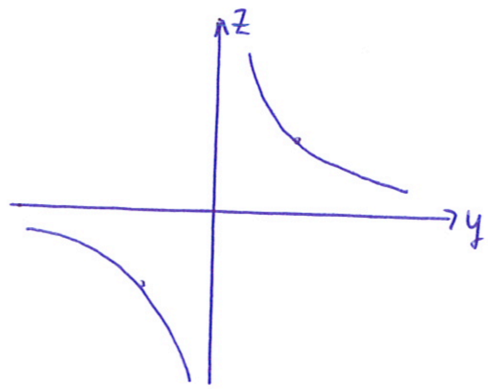
$$f(x, b) = \frac{e^x}{b}$$



(e) Treat x as a parameter and sketch a graph in two-dimensions to illustrate how f depends on y.

Let $x = c$ where $c \in \mathbb{R}$.

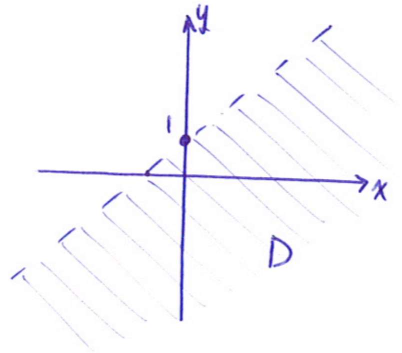
$$f(c, y) = \frac{e^c}{y} \Big\} \oplus$$



2. Find and sketch the domain of the following functions.

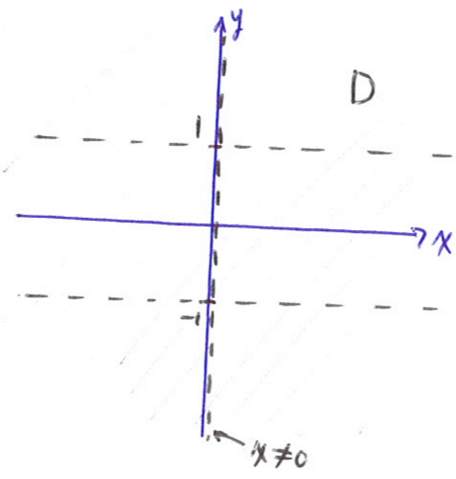
(a) $f(x, y) = \ln(1 + x - y)$

$1 + x - y > 0 \Rightarrow y < x + 1$



(b) $g(x, y) = \dfrac{3x + 1}{xy^2 - x}$

$xy^2 - x \neq 0 \Rightarrow x(y^2 - 1) \neq 0 \Rightarrow x \neq 0, y \neq \pm 1$

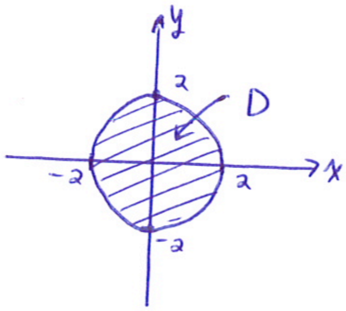


3. Let $f(x, y) = \sqrt{4 - x^2 - y^2}$.

(a) Find and sketch the domain.

$4 - x^2 - y^2 \geq 0 \Rightarrow x^2 + y^2 \leq 4$

all points on the edge and inside of a circle centred at $(0,0)$ with radius 2



(b) Determine the range.

$\underbrace{\sqrt{4 - x^2 - y^2}}_{+} = z$ where $z \geq 0$.

$4 - x^2 - y^2 = z^2 \rightarrow \underbrace{x^2 + y^2}_{+} = 4 - z^2 \Rightarrow 4 - z^2 \geq 0$

$\Rightarrow z^2 \leq 4$

$\Rightarrow -2 \leq z \leq 2$

Since $z \geq 0$ and $-2 \leq z \leq 2$, we have that $0 \leq z \leq 2$

(c) Create a contour map for the function.

$\sqrt{4-x^2-y^2} = k$ where $k \in [0,2]$

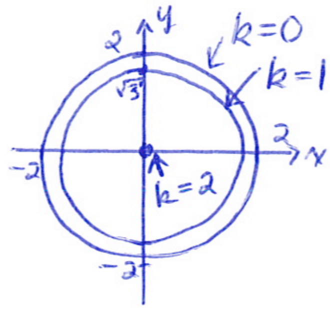
$4-x^2-y^2 = k^2 \Rightarrow x^2+y^2 = 4-k^2$

So, level curves are circles w/ centre $(0,0)$ and radius $\sqrt{4-k^2}$

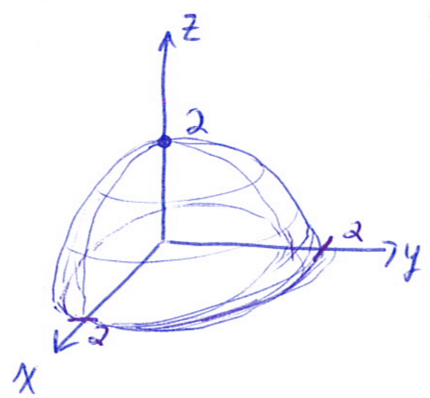
$k=0 \Rightarrow x^2+y^2=4$

$k=1 \Rightarrow x^2+y^2=3$

$k=2 \Rightarrow x^2+y^2=0$



(d) Sketch the graph of the function.

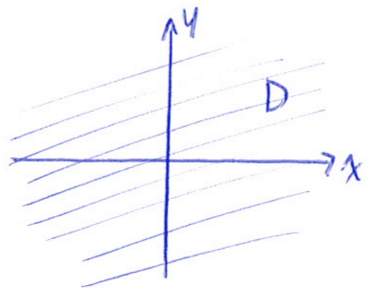


top half of a sphere w/ centre $(0,0,0)$ and radius 2

4. Let $g(x, y) = 8 + x^2 + y^2$.

(a) Find and sketch the domain.

domain : $\mathbb{R}^2$



(b) Determine the range.

$$z = \underbrace{8 + \underbrace{x^2 + y^2}_{\geq 0}}_{\geq 8}$$

$\therefore\ z \geq 8$

(c) Create a contour map for the function.

$8 + x^2 + y^2 = k$ where $k \geq 8$

$x^2 + y^2 = k - 8$

level curves are circles centred at $(0,0)$ w/ radius $\sqrt{k-8}$

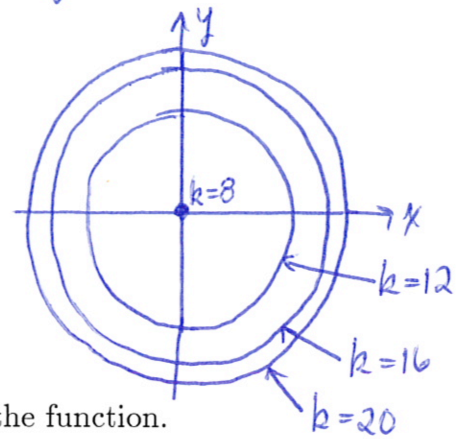
$k = 8 \Rightarrow x^2 + y^2 = 0 \quad (r=0)$

$k = 12 \Rightarrow x^2 + y^2 = 4 \quad (r=2)$

$k = 16 \Rightarrow x^2 + y^2 = 8 \quad (r \approx 2.8)$

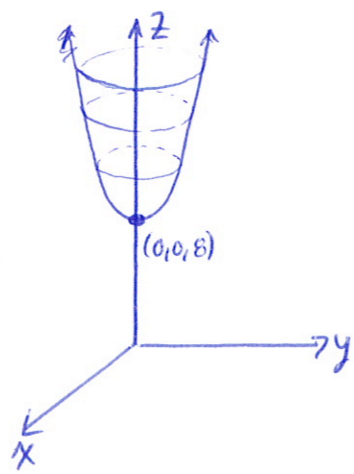
$k = 20 \Rightarrow x^2 + y^2 = 12 \quad (r \approx 3.5)$

contour map:



(d) Sketch the graph of the function.

paraboloid



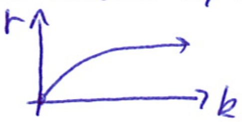
5. Question 4 on p. V1-28.

paraboloid

$z = x^2 + y^2$

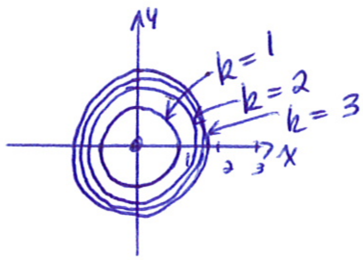
level curves: $x^2 + y^2 = k$, $k \geq 0$

concentric circles w/ radius $r = \sqrt{k}$



as $k \to \infty$, r increases but at a decreasing rate

contour map:

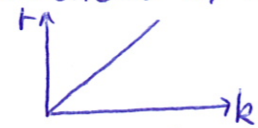


cone

$z = \sqrt{x^2 + y^2}$

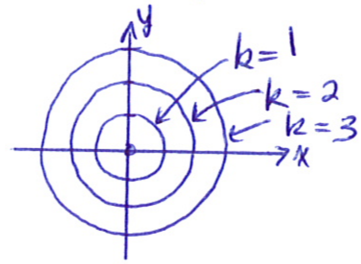
level curves: $x^2 + y^2 = k^2$, $k \geq 0$

concentric circles w/ radius $r = k$.



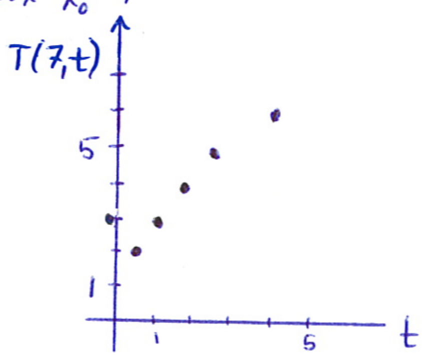
as $k \to \infty$, r increases at a constant rate

contour map:

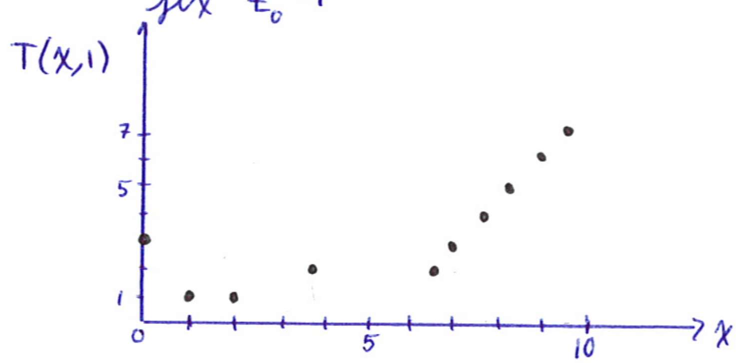


6. Question 32 on p. V1-30.

fix $x_0 = 7$



fix $t_0 = 1$



7. (a) In your own words, explain what is meant by $\lim_{(x,y)\to(a,b)} f(x,y) = L$.

The z-values approach L more and more closely as (x, y) approaches (a, b) more and more closely along every path in the domain of f.

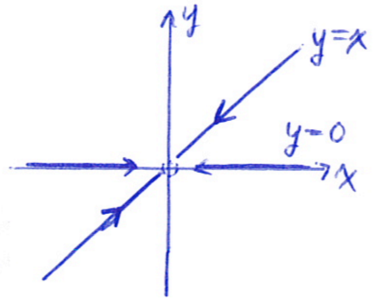
(b) Explain how you would show that $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist.

You must find two paths P_1 and P_2 in the domain of f such that

$$\lim_{(x,y)\to(a,b)} f(x,y) \text{ along } P_1 \neq \lim_{(x,y)\to(a,b)} f(x,y) \text{ along } P_2$$

8. Show that the following limits do not exist. Sketch the domains and paths involved.

(a) $\lim_{(x,y)\to(0,0)} \overbrace{\frac{(x-y)^2}{x^2+y^2}}^{f}$



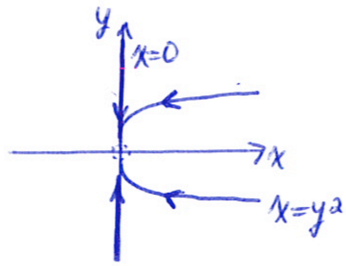
domain : $\mathbb{R}^2 \setminus \{(0,0)\}$

Along $y=0$:
$f(x,0) = \frac{x^2}{x^2} = 1 \Rightarrow f(x,y) \to 1$ as $(x,y) \to (0,0)$ along $y=0$

Along $y=x$:
$f(x,x) = \frac{0^2}{2x^2} = 0 \Rightarrow f(x,y) \to 0$ as $(x,y) \to (0,0)$ along $y=x$

$\therefore \lim_{(x,y)\to(0,0)} f(x,y)$ D.N.E.

(b) $\lim_{(x,y)\to(0,0)} \overbrace{\frac{2xy^2}{x^2+y^4}}^{f}$



domain : $\mathbb{R}^2 \setminus \{(0,0)\}$

Along $x=0$:
$f(0,y) = \frac{0}{y^4} = 0 \Rightarrow f(x,y) \to 0$ as $(x,y) \to (0,0)$ along $x=0$

Along $x=y^2$:
$f(y^2,y) = \frac{2y^2 \cdot y^2}{(y^2)^2+y^4} = \frac{2y^4}{2y^4} = 1 \Rightarrow f(x,y) \to 1$ as $(x,y) \to (0,0)$ along $x=y^2$

$\therefore \lim_{(x,y)\to(0,0)} f(x,y)$ D.N.E.

9. (a) Explain what you would have to show in order to prove that a function $f(x, y)$ is continuous at (a, b).

you would have to show that the limit of the function f as (x,y) approaches (a,b) exists and is equal to the value of the function at (a,b), ie,

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

(b) Find a function g such that $\lim_{(x,y)\to(5,4)} g(x, y)$ exists but g is not continuous at $(5, 4)$.

Many possible answers... here's one:

$$g(x,y) = \begin{cases} x+y & \text{if } (x,y) \neq (5,4) \\ 10 & \text{if } (x,y) = (5,4) \end{cases}$$

so, $\lim_{(x,y)\to(5,4)} g(x,y) = \lim_{(x,y)\to(5,4)} (x+y) = 9$ (limit exists)

but $g(5,4) = 10 \Rightarrow$ by the def^{n} of continuity, g is not continuous at $(5,4)$.

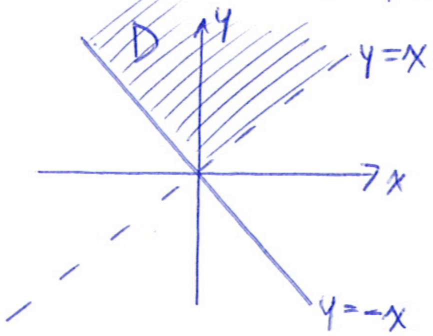
(c) Find and sketch the largest domain on which $z = \ln(y - x) + \sqrt{y + x}$ is continuous.

$y - x > 0$ AND $y + x \geq 0$

$\Rightarrow$ $y > x$ AND $y \geq -x$

Since z is a combination of continuous functions, it is continuous on its domain, ie, continuous on

$$D = \{(x,y) \in \mathbb{R}^2 \mid y > x \text{ and } y \geq -x\}$$



10. Use the definition of continuity to show that

$$h(x,y) = \begin{cases} 4 - e^{-x-y+2} & \text{if} \quad (x,y) \neq (1,1) \\ 3 & \text{if} \quad (x,y) = (1,1) \end{cases}$$

is continuous at $(1,1)$.

① $\lim_{(x,y)\to(1,1)} h(x,y) = \lim_{(x,y)\to(1,1)} (4 - e^{-x-y+2}) = 4 - e^0 = 3$

② $h(1,1) = 3$

③ Since $\lim_{(x,y)\to(1,1)} h(x,y) = 3 = h(1,1)$, by the def^n of continuity, h is continuous at $(1,1)$.