MATHEMATICS 1LT3 TEST 3

Day Class
Duration of Test: 60 minutes
McMaster University

E. Clements

21 March 2013

FIRST NAME (please print): Sol^{N}s.

FAMILY NAME (please print): ____________

Student No.: ____________

THIS TEST HAS 8 PAGES AND 10 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE.

Total number of points is 40. Marks are indicated next to the problem number in square brackets. Any Casio fx991 (or lower, non-graphing) calculator is allowed.

USE PEN TO WRITE YOUR TEST. IF YOU USE A PENCIL, YOUR TEST WILL NOT BE ACCEPTED FOR REMARKING (IF NEEDED).

You need to show work to receive full credit, except for question 1 (Multiple Choice).

Page	Points	Mark
2	6	
3	6	
4	7	
5	5	
6	7	
7	4	
8	5	
TOTAL	40	

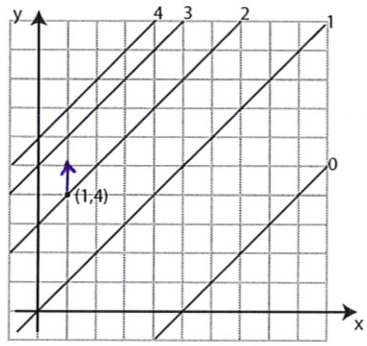
MATH 1LT3 * Test 3 * 21 March 2013 Name:

Student No.:

1. **Multiple Choice.** Clearly **circle** the one correct answer.

(a) [3] For the contour diagram of $f(x, y)$ below, which of the following is/are true?

(I) $f_{yy}(1, 4) > 0$ ✓
(II) $\nabla f(1, 4)$ points in the same direction as $\mathbf{v_1} = -\mathbf{i} + \mathbf{j}$ ✓
(III) $D_{\mathbf{v_2}} f(1, 4) = 0$ when $\mathbf{v_2} = \mathbf{i} + \mathbf{j}$ ✓



(A) none (B) I only (C) II only (D) III only
(E) I and II (F) I and III (G) II and III (H) all three

(b) [3] Consider the random experiment of rolling a fair, six-sided die. Define A to be the event that you roll a number less than 3 and B to be the event that you roll an even number. Which of the following statements is/are true?

(I) $P(B^C) = \frac{1}{2}$ ✓ (II) $P(A \cup B) = \frac{2}{3}$ ✓ (III) $P(A \cap B) = \frac{1}{6}$ ✓

(A) none (B) I only (C) II only (D) III only
(E) I and II (F) I and III (G) II and III (H) all three

Name:________________

Student No.: ________________

2. (a) [4] Compute the directional derivative of the function $f(x,y) = \arctan\left(\frac{3y}{x}\right)$ at the point $(1,1)$ in the direction of the vector $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j}$.

$\|\vec{v}\| = \sqrt{3^2+4^2} = 5$

$\hat{u} = \frac{1}{\|\vec{v}\|}\vec{v} = \frac{1}{5}(3\hat{\imath}+4\hat{\jmath}) = \underbrace{\frac{3}{5}}_{u_1}\hat{\imath} + \underbrace{\frac{4}{5}}_{u_2}\hat{\jmath}$

$f_x = \frac{1}{1+\left(\frac{3y}{x}\right)^2}\cdot\left(-\frac{3y}{x^2}\right) \quad \ldots \quad f_x(1,1) = \frac{-3}{10}$

$f_y = \frac{1}{1+\left(\frac{3y}{x}\right)^2}\cdot\left(\frac{3}{x}\right) \quad \ldots \quad f_y(1,1) = \frac{3}{10}$

$$D_{\vec{v}}f(1,1) = f_x(1,1)\cdot u_1 + f_y(1,1)u_2$$
$$= \frac{-3}{10}\left(\frac{3}{5}\right) + \frac{3}{10}\left(\frac{4}{5}\right)$$
$$= \frac{3}{50} \quad (= 0.06)$$

(b) [2] At the point $(1,1)$, is the function f increasing or decreasing in the direction of $\mathbf{v}$? What is the maximum rate of increase of $f(x,y)$ at $(1,1)$?

Since $D_{\vec{v}}f(1,1) > 0$, $f(x,y)$ is increasing at $(1,1)$ in the direction $\vec{v}$.

max. rate of increase at $(1,1) = \|\nabla f(1,1)\|$

$$\nabla f(1,1) = -\frac{3}{10}\hat{\imath} + \frac{3}{10}\hat{\jmath}$$

$$\|\nabla f(1,1)\| = \sqrt{\left(\frac{-3}{10}\right)^2 + \left(\frac{3}{10}\right)^2} = \sqrt{\frac{18}{100}} = \frac{3}{5\sqrt{2}} \quad (\simeq 0.42)$$

$\therefore$ the maximum rate of increase of $f(x,y)$ at $(1,1)$ is $\frac{3}{5\sqrt{2}}$.

Name:________________
Student No.: ________________

---

3. (a) [1] Write the formula for the degree-2 Taylor polynomial of the function $f(x, y)$ at the point (a, b).

$$T_2(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + \dots$$
$$\dots + \frac{f_{xx}(a,b)}{2}(x-a)^2 + f_{xy}(a,b)(x-a)(y-b) + \frac{f_{yy}(a,b)}{2}(y-b)^2$$

(b) [4] Choose an appropriate point (a, b) and use a degree-2 Taylor polynomial to approximate the value of $\sqrt{3.9}\ln 1.01$.

Let $f(x,y) = \sqrt{x}\ln y$. Choose $(a,b) = (4,1)$. $(\because (4,1) \approx (3.9, 1.01))$

$f(4,1) = \sqrt{4}\ln 1 = 0$

$f_x = \frac{1}{2\sqrt{x}}\cdot \ln y \; \dots \; f_x(4,1) = 0$

$f_{xx} = \frac{-1}{4x^{3/2}}\cdot \ln y \; \dots \; f_{xx}(4,1) = 0$

$f_y = \frac{\sqrt{x}}{y} \; \dots \; f_y(4,1) = 2$

$f_{yy} = -\frac{\sqrt{x}}{y^2} \; \dots \; f_{yy}(4,1) = -2$

$f_{xy} = f_{yx} = \frac{1}{2\sqrt{x}y} \; \dots \; f_{xy}(4,1) = \frac{1}{4}$

$$T_2(x,y) = 2(y-1) + \frac{1}{4}(x-4)(y-1) - (y-1)^2$$

So, $\sqrt{3.9}\ln 1.01 \approx 2(1.01-1) + \frac{1}{4}(3.9-4)(1.01-1) - (1.01-1)^2$

$\approx 2(0.01) + 0.25(-0.1)(0.01) - (0.01)^2$

≈ 0.01965

4. [2] Using a geometric argument, determine whether the function $f(x, y) = 2x^4 - y^4$ has a local minimum, a local maximum, or a saddle point at the critical point $(0, 0)$.

$f(0,0) = 0$

Could 0 be a local max, ie $f(x,y) \le 0 \; \forall (x,y)$ near $(0,0)$?

when $y=0$, $f(x,y) = 2x^4 \geq 0$ in any $B_r(0,0)$, $r>0$

So, near $(0,0)$ there will always be z-values greater than $0 \Rightarrow f(x,y)$ cannot have a local max at $(0,0)$.

Could 0 be a local min?

when $x=0$, $f(x,y) = -y^4 \le 0$ for any $B_r(0,0)$, $r>0$.

So, near $(0,0)$ there will always be z-values less than $0 \Rightarrow f(x,y)$ cannot have a local min at $(0,0)$.

$\therefore f(x,y)$ has a saddle point at $(0,0)$.

Name:____________________

Student No.: ____________________

5. Consider the function $f(x,y) = (x^3 - x)e^{-y^2}$ and its partial derivatives given below.

$f_x = (3x^2 - 1)e^{-y^2}$ $\qquad$ $f_y = -2y(x^3 - x)e^{-y^2}$

$f_{xx} = 6xe^{-y^2}$ $\qquad$ $f_{yy} = -2(x^3 - x)(1 - 2y^2)e^{-y^2}$

$f_{xy} = -2y(3x^2 - 1)e^{-y^2}$

(a) [2] Find all critical points of $f(x,y)$.

$$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \Rightarrow \begin{cases} (3x^2-1)e^{-y^2} = 0 \\ -2y(x^3-x)e^{-y^2} = 0 \end{cases} \Rightarrow \begin{cases} 3x^2 - 1 = 0 \\ -2y\cdot x(x^2-1) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x = \pm\frac{1}{\sqrt{3}} \\ y = 0,\ x = 0, \pm 1 \end{cases}$$

critical points: $\left(-\frac{1}{\sqrt{3}}, 0\right)$ and $\left(\frac{1}{\sqrt{3}}, 0\right)$

(b) [3] Use the second derivatives test to classify each critical point in (a) as either a local maximum, local minimum, or a saddle point of $f(x,y)$.

$$D = 6xe^{-y^2}\cdot(-2)(x^3-x)(1-2y^2)\cdot e^{-y^2} - \left[-2y(3x^2-1)e^{-y^2}\right]^2$$

$$D\left(-\tfrac{1}{\sqrt{3}}, 0\right) = -\frac{6}{\sqrt{3}}\cdot(-2)\left(-\frac{1}{\sqrt{3}}\right)\left(\left(\frac{-1}{\sqrt{3}}\right)^2 - 1\right) = \frac{8}{3} > 0$$

$$f_{xx}\left(\tfrac{-1}{\sqrt{3}}, 0\right) = \frac{-6}{\sqrt{3}} < 0$$

$\Rightarrow$ f has a local max. at $\left(-\frac{1}{\sqrt{3}}, 0\right)$.

$$D\left(\tfrac{1}{\sqrt{3}}, 0\right) = \frac{6}{\sqrt{3}}(-2)\left(\frac{1}{\sqrt{3}}\right)\left(\left(\frac{1}{\sqrt{3}}\right)^2 - 1\right) = \frac{8}{3} > 0$$

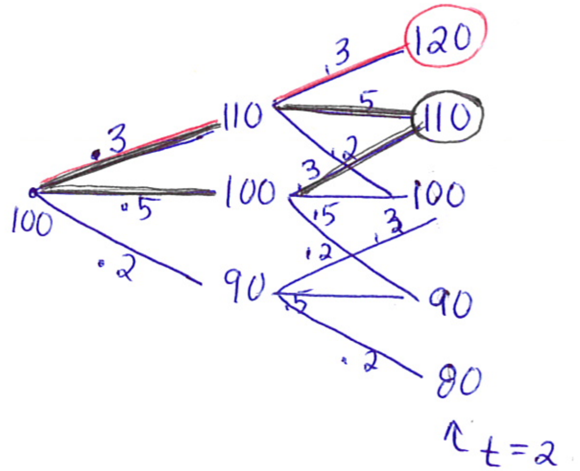
$$f_{xx}\left(\tfrac{1}{\sqrt{3}}, 0\right) = \frac{6}{\sqrt{3}} > 0$$

$\Rightarrow$ f has a local min at $\left(\frac{1}{\sqrt{3}}, 0\right)$.

Name:________________
Student No.: ________________

6. Consider a population of 100 elephants. Suppose that within any given year, there is a 30% chance the population will increase by 10, a 50% chance it will stay the same, and a 20% chance it will decrease by 10. Assume that immigration from year to year is independent.

(a) [2] Write the sample space for the population of elephants after 2 years. (Hint: Draw a tree diagram and label the branches with probabilities).



$S = \{80, 90, 100, 110, 120\}$.

(b) [3] What is the probability that the population will have increased after 2 years?

$= P(110) + P(120)$

$= (.3)(.5) + (.5)(.3) + (.3)^2$

$= 2(.3)(.5) + (.3)^2$

$= 0.39$

There is a 39% chance that the pop^n will have increased after 2 years.

(c) [2] Suppose that conditions changed and now there is a 60% chance that the population will increase by 8 and a 40% it will decrease by 10 in a given year. What is more likely to happen to the number of elephants over time? An increase or a decrease? Explain.

Consider what is likely to happen over 10 years.

6 years, pop^n will increase by 8

4 years, pop^n will decrease by 10

Net elephants in 10 years:

$6 \times 8 + 4 \times (-10) = \underline{8}$.

$\therefore$ The pop^n is likely to increase over time.

Name:

Student No.:

7. [2] The incidence of asthma in young adults is 6.4% for females and 4.5% for males. Suppose that you survey a population consisting of 55% females and 45% males. What is the probability that a randomly chosen young adult has asthma?

Let A = event of having asthma.

$$P(A) = P(A|♀)\cdot P(♀) + P(A|♂)P(♂)$$
$$= (0.064)(0.55) + (0.045)(0.45)$$
$$= 0.05545$$

∴ A randomly chosen young adult from this pop^N has about a 5.5% chance of having asthma.

8. [2] Suppose that in 1999, the incidence of bacterial meningitis was 3.4 per 100,000. A test for meningitis shows a positive result in 85% of people who have it and in 7% of people who do not have it. If a person tests positive for bacterial meningitis, what is the probability that they have it?

Let B = event of having bacterial meningitis.

$$P(B|\oplus) = \frac{P(\oplus|B)\cdot P(B)}{P(\oplus|B)\cdot P(B) + P(\oplus|B^c)\cdot P(B^c)}$$
$$= \frac{(0.85)\left(\frac{3.4}{100\,000}\right)}{(.85)\left(\frac{3.4}{100\,000}\right) + (0.07)\left(1 - \frac{3.4}{100\,000}\right)}$$
$$\simeq 0.000\,41270$$

∴ If a person tests positive for meningitis, there is only a 0.04% chance they have it.

Name:________________

Student No.: ________________

9. [3] The average efficacy of a birth control pill is about 97.5% per year and the average efficacy of a condom is 86% per year. If a sexually active woman uses both the pill and condoms (and assuming that the two preventative measures are independent), what is the probability that she will get pregnant at least once in 4 years?

P(pregnant in year i using both condoms $\overset{\cap}{\text{and}}$ the pill)

= P(pregnant in year i using condoms) $\cdot$ P(pregnant in year i using pills)

$= (0.025)(0.14)$

$= 0.0035$

Let A be the event of getting pregnant at least once in 4 years

$P(A) = 1 - P(A^c)$

$= 1 - P(\bigcap_{i=1}^{4} A_i^c)$ where A_i^c = event of NOT pregnant in year i

$= 1 - \prod_{i=1}^{4} P(A_i^c)$

$= 1 - (1 - 0.0035)^4$

$= 0.0139$

$\therefore$ There is about a 1.4% chance she will get pregnant.

10. [2] If A and B are both subsets of S, then $P(A \cap B) = P(A)P(B)$. True or false? Explain.

NO. This is only true when A and B are independent!

THE END