MATH 1B03/1ZC3 Winter 2019

Lecture 7: SLEs continued

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We have seen that matrix inverses are important in the study of solutions to SLEs. The following facts are useful when finding and using matrix inverses.

Fact 7.1

Let A and B be square matrices of the same size. Then

1. if $AB = I$, then $B = A^{-1}$
2. if $BA = I$, then $B = A^{-1}$
3. if $B = A^{-1}$, then $A = B^{-1}$ also
4. if AB is invertible then A and B are invertible also

In particular 3. allows us to 'take the inverse of both sides', as in the following example.

Example 7.2

Question: Consider the equation

$$(AB + C)^{-1} = D$$

for A, B, C, and D matrices. Solve for C.

Answer:

$$\begin{aligned} (AB + C)^{-1} &= D \\ AB + C &= D^{-1}, \text{ take the inverse of both sides} \\ C &= D^{-1} - AB \end{aligned}$$

We can now add to our list of statements equivalent to A being invertible.

Fact 7.3

Let A be an $n \times n$ matrix. The following are equivalent (that is, either all of them are true, or all of them are false):

1. A is invertible.
2. The matrix equation $A\mathbf{x} = 0$ has the unique solution $\mathbf{x} = 0$ (where 0 is the $n \times 1$ zero matrix).
3. The RREF of A is I_n.
4. A may be represented as a product of elementary matrices. That is $A = E_1 E_2 \cdots E_n$, where E_i is an elementary matrix.
5. The matrix equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$ (and is therefore consistent).

(Note that 5. implies 2. by picking $\mathbf{b} = 0$.)

Later on in the course we will see a geometric way to interpret these facts. If A is square 5. allows us to quickly solve SLEs which A appears in by determining if it is singular.

Non-square and singular matrices

Fact 6.7 only applies to square invertible matrices. We can use similar techniques to solve SLEs involving non-square or singular matrices, however. In particular, we can determine if a SLE has solutions by studying the inverse of the associated matrix.

Recipe 7.4: Determining if a SLE is consistent

Step 1: Given a SLE, write down the associated matrix equation $A\mathbf{x} = \mathbf{b}$, and determine if A is square. If A is not square, go to Step 2.

If A is square, use the inversion algorithm to determine if A is invertible. If A is singular, go to Step 2.
If A is invertible, the SLE is consistent and has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

Step 2: Form the augmented matrix of the SLE. Use Gauss-Jordan elimination to put the augmented matrix into RREF (note we are not using A).
The RREF will yield conditions which must be met for the SLE to be consistent (that is, to have solutions).

Example 7.5

Question: Find conditions on the constants b_1 and b_2 such that the following SLE is consistent

$$\begin{aligned} 6x_1 - 4x_2 &= b_1 \\ 3x_1 - 2x_2 &= b_2 \end{aligned}$$

Answer: The associated matrix equation is

$$\begin{bmatrix} 6 & -4 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

so that

$$A = \begin{bmatrix} 6 & -4 \\ 3 & -2 \end{bmatrix}$$

The matrix A is square, so check if A is singular.

$$\left[\begin{array}{cc|cc} 6 & -4 & 1 & 0 \\ 3 & -2 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|cc} 6 & -4 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} & 1 \end{array}\right] \qquad -\tfrac{1}{2}R1 + R2$$

as we encounter a row of 0's, A is singular.
The augmented matrix is

$$\begin{bmatrix} 6 & -4 & b_1 \\ 3 & -2 & b_2 \end{bmatrix}$$

Applying Gauss-Jordan elimination to the augmented matrix, we encounter

$$\left[\begin{array}{cc|c} 6 & -4 & b_1 \\ 0 & 0 & -\frac{1}{2}b_1 + b_2 \end{array}\right]$$

The SLE will only be consistent if $-\frac{1}{2}b_1 + b_2 = 0$.

Diagonal, triangular, and symmetric matrices

(from Chapter 1.7 of Anton-Rorres)

Carrying on our study of matrices, we will introduce some special types of matrices which have nice properties.

Diagonal matrices (defined only for square matrices)

Definition 7.6: Diagonal matrix

A square matrix is diagonal if all entries not on the main diagonal are zero.

Examples of diagonal matrices are

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 19 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

An $n \times n$ diagonal matrix is often written

$$\begin{bmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{bmatrix}$$

It is very easy to tell if a diagonal matrix is invertible, and to find the inverse if it is.

Fact 7.7

Let

$$D = \begin{bmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{bmatrix}$$

be a diagonal matrix. Then D is invertible if and only if every entry $d_i \neq 0$. If D is invertible then

$$D^{-1} = \begin{bmatrix} \frac{1}{d_1} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{d_2} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{d_n} \end{bmatrix}$$

Powers of diagonal matrices are also easy to find.

Fact 7.8

Let

$$D = \begin{bmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{bmatrix}$$

be a diagonal matrix. Then

$$D^k = \begin{bmatrix} d_1^k & 0 & 0 & \cdots & 0 \\ 0 & d_2^k & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & d_n^k \end{bmatrix}$$

for any integer k.

Multiplying matrices by diagonal matrices is faster than usual matrix multiplication.

The following equations generalize to matrices of any size. Multiplying a matrix from the left by a diagonal matrix multiplies the rows by the diagonal entries:

$$\begin{bmatrix} \boxed{d_1} & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \begin{bmatrix} \boxed{a_{11} \quad a_{12} \quad a_{13} \quad a_{14}} \\ a_{21} \quad a_{22} \quad a_{23} \quad a_{24} \\ a_{31} \quad a_{32} \quad a_{33} \quad a_{34} \end{bmatrix} = \begin{bmatrix} \boxed{d_1a_{11} \quad d_1a_{12} \quad d_1a_{13} \quad d_1a_{14}} \\ d_2a_{21} \quad d_2a_{22} \quad d_2a_{23} \quad d_2a_{24} \\ d_3a_{31} \quad d_3a_{32} \quad d_3a_{33} \quad d_3a_{34} \end{bmatrix}$$

Multiplying a matrix from the right by a diagonal matrix multiplies the columns by the diagonal entries:

$$\begin{bmatrix} a_{11} & \boxed{\begin{matrix} a_{12} \\ a_{22} \\ a_{32} \end{matrix}} & a_{13} & a_{14} \\ a_{21} & & a_{23} & a_{24} \\ a_{31} & & a_{33} & a_{34} \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0 & 0 \\ 0 & \boxed{d_2} & 0 & 0 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & d_4 \end{bmatrix} = \begin{bmatrix} d_1a_{11} & \boxed{\begin{matrix} d_2a_{12} \\ d_2a_{22} \\ d_2a_{32} \end{matrix}} & d_3a_{13} & d_4a_{14} \\ d_1a_{21} & & d_3a_{23} & d_4a_{24} \\ d_1a_{31} & & d_3a_{33} & d_4a_{34} \end{bmatrix}$$

Question 7.9

Verify these equations via matrix multiplication.

Example 7.10

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 2a_{11} & 2a_{12} & 2a_{13} \\ a_{21} & a_{22} & a_{23} \\ 3a_{31} & 3a_{32} & 3a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 2a_{11} & a_{12} & 3a_{13} \\ 2a_{21} & a_{22} & 3a_{23} \\ 2a_{31} & a_{32} & 3a_{33} \end{bmatrix}$$

Triangular matrices (defined only for square matrices)

The next best thing to a diagonal matrix is a triangular matrix.

Definition 7.11: Triangular matrices

A square matrix is upper triangular if the only nonzero entries are above or on the main diagonal.
A square matrix is lower triangular if the only nonzero entries are below or on the main diagonal.
A square matrix is triangular if it is either upper or lower triangular.

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}}_{\text{upper}} \quad \underbrace{\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}_{\text{lower}}$$

Note that a diagonal matrix is both upper and lower triangular. A matrix in REF or RREF is upper triangular.

Fact 7.12: Properties of triangular matrices

1. The transpose of an upper triangular matrix is lower triangular, and vice versa.
2. The product of two upper triangular matrices is upper triangular. The product of two lower triangular matrices is lower triangular.
3. A triangular matrix is invertible if and only if its diagonal entries are non-zero.
4. The inverse of an upper triangular matrix is upper triangular. The inverse of a lower triangular matrix is lower triangular.

3. is very important and gives us a quick way to check if a triangular matrix is invertible. We will see deep reasons for these facts as we progress.

Example 7.13

$$\begin{bmatrix} 2 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 7 \end{bmatrix} \quad \begin{bmatrix} 4 & 0 & 0 \\ 5 & 4 & 0 \\ 1 & 12 & 1 \end{bmatrix}$$

$\underline{\text{singular}}$ $\quad$ $\underline{\text{invertible}}$

Products of triangular matrices

If A and B are both upper triangular (or both lower triangular) matrices, we can easily determine the diagonal entries of the products AB and BA.

Fact 7.14: Products of triangular matrices

Let A and B be upper triangular matrices (or both lower triangular matrices). Then

$$(AB)_{ii} = (BA)_{ii} = (A)_{ii}\,(B)_{ii}$$

That is, the diagonal entries of AB and BA can be found by multiplying the diagonal entries of A and B. Remember that $AB \neq BA$, in general, so that the non-diagonal entries cannot be found this way.

Example 7.15

Let

$$A = \begin{bmatrix} 1 & 0 \\ 8 & 4 \end{bmatrix}, \; B = \begin{bmatrix} 2 & 0 \\ 5 & 3 \end{bmatrix}$$

They are both lower triangular, so the diagonal entries of AB and BA are

$$(AB)_{11} = (BA)_{11} = 1 \cdot 2 = 2$$
$$(BA)_{22} = (BA)_{22} = 4 \cdot 3 = 12$$

But

$$AB = \begin{bmatrix} 1 & 0 \\ 8 & 4 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 36 & 12 \end{bmatrix}$$

and

$$BA = \begin{bmatrix} 2 & 0 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 8 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 29 & 12 \end{bmatrix}$$

Notice that the nondiagonal elements are not equal.

Symmetric matrices (defined only for square matrices)

Definition 7.16

A matrix A is symmetric if $A^T = A$.

Notice that only square matrices can be symmetric: if A is an $m \times n$ matrix, then A^T is an $n \times m$ matrix. If $A^T = A$ then the matrices must be of the same size, so that $m = n$ and A is square.

A diagonal matrix is automatically symmetric.

Example 7.17

The following matrices are symmetric

$$\begin{bmatrix} 4 & -2 & -2 \\ -2 & 3 & 0 \\ -2 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We can write the symmetric condition $A^T = A$ as

$$\begin{aligned} (A)_{ij} &= \left(A^T\right)_{ji}, \quad \text{by definition of the transpose} \\ &= (A)_{ji}, \quad \text{as } A^T = A \end{aligned}$$

Therefore A is symmetric if $(A)_{ij} = (A)_{ji}$.

Fact 7.18: Properties of symmetric matrices

Let A and B be symmetric matrices of the same size, and λ a scalar. Then

1. A^T is symmetric
2. $A + B$ and $A - B$ are symmetric
3. λA is symmetric

2. and 3. allow us to add, subtract, and multiply symmetric matrices by scalars to produce another symmetric matrix.

However, the product of two symmetric matrices is not necessarily symmetric. Consider the following example:

$$\begin{bmatrix} 3 & 4 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 8 & -3 \\ -3 & 0 \end{bmatrix} = \begin{bmatrix} 12 & \boxed{-9} \\ \boxed{29} & -12 \end{bmatrix}$$

as $29 \neq -9$, the resulting matrix is not symmetric.

Fact 7.19

If A is invertible and symmetric, then A^{-1} is symmetric also.

Proof: Let A be invertible and symmetric. Therefore A^{-1} exists and $A^T = A$. Recall that

$$(A^{-1})^T = (A^T)^{-1}.$$

Then

$$\begin{aligned} (A^{-1})^T &= (A^T)^{-1} \\ &= A^{-1} \end{aligned}$$

so A^{-1} is symmetric. ■

Let M be a matrix of any size. The products MM^T and M^TM are common in applications of matrix theory. These matrix products are always square.

Question 7.20

Check that MM^T and M^TM are square, where M is a matrix of any size.

Fact 7.21

Let M be a matrix of any size.

1. MM^T and M^TM are symmetric
2. If M is invertible, then MM^T and M^TM are invertible also

Example 7.22

Question: Find all constants a, b and c such that the matrix

$$A = \begin{bmatrix} 2 & a-2b+2c & 2a+b+c \\ 3 & 5 & a+c \\ 0 & -2 & 7 \end{bmatrix}$$

is symmetric.
Answer:

$$A^T = \begin{bmatrix} 2 & 3 & 0 \\ a-2b+2c & 5 & -2 \\ 2a+b+c & a+c & 7 \end{bmatrix}$$

If $A^T = A$ then

$$\begin{aligned} a-2b+2c &= 3 \\ 2a+b+c &= 0 \\ a+c &= -2 \end{aligned}$$

Solving this system yields

$$\begin{aligned} a &= -11 \\ b &= -9 \\ c &= -13 \end{aligned}$$

Question: Find a diagonal matrix D such that

$$D^5 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Answer: If

$$D = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$$

then

$$D^5 = \begin{bmatrix} d_1^5 & 0 & 0 \\ 0 & d_2^5 & 0 \\ 0 & 0 & d_3^5 \end{bmatrix}$$

If

$$D^5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

then

$$\begin{aligned} d_1^5 &= -1, \ d_2^5 &= 3, \ d_3^5 = 5 \\ d_1 &= -1, \ d_2 &= 3^{\frac{1}{5}}, \ d_3^5 = 5^{\frac{1}{5}} \end{aligned}$$

so that

$$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3^{\frac{1}{5}} & 0 \\ 0 & 0 & 5^{\frac{1}{5}} \end{bmatrix}$$

Suggested problems

Practice the material in this lecture by attempting the following problems in **Chapter 1.7** of Anton-Rorres, starting on page 72

- Questions 19, 21, 26, 34, 35, 41, 43, 44, 45
- True/False questions (b), (c), (f), (i)