12A3

l'Hôpital's Rule

If I have $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \underbrace{\text{"}\dfrac{0}{0}\text{" or } \pm\text{"}\dfrac{\infty}{\infty}\text{"}}_{\text{Indeterminate forms}}$

then $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x \to a} \dfrac{f'(x)}{g'(x)}$ (if it exists)

e.g. $\lim\limits_{x \to 0} \sin x / x = \dfrac{\sin(0)}{0} = \text{"}\dfrac{0}{0}\text{"}$ } indeterminate!

$$\overset{(H)}{=} \lim_{x \to 0} \frac{\cos x}{1} = \cos(0) = \underline{\underline{1}}$$

eg $$\lim_{x \to 4} \frac{x - 4}{x^2 - 16} = \text{"}\frac{0}{0}\text{"}$$

$$\overset{(H)}{=} \lim_{x \to 4} \frac{1}{2x - 0} = \frac{1}{2(4)} = \frac{1}{8}$$

<u>Please</u> do not use (H) on limits that <u>do not</u> have $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form! It will <u>FAIL</u>.

<u>notice</u> limit does <u>not</u> need to be $x \to a$

can be $x \to a^+$, a^-, $+\infty$, $-\infty$ etc.

eg $\lim_{x\to\infty} \frac{e^x + x}{2e^x + 17} = \frac{\infty + \infty}{\infty + 17} = \text{"}\frac{\infty}{\infty}\text{"}$ Indet

$\sim e^x/2e^x = \frac{1}{2}$

$\overset{H}{=} \lim_{x\to\infty} \frac{e^x + 1}{2e^x + 0} = \lim \frac{e^\infty + 1}{2e^\infty + 0} = \text{"}\frac{\infty}{\infty}\text{"}$

way #1

$\overset{H}{=} \lim_{x\to\infty} \frac{\cancel{e^x}}{2\cancel{e^x}} = \boxed{\frac{1}{2}}$

eg $\lim_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^\infty - e^{-\infty}}{e^\infty + e^{-\infty}} = \frac{\infty - 0}{\infty + 0} = \text{"}\frac{\infty}{\infty}\text{"}$

$\overset{H}{=} \lim_{x\to\infty} \frac{e^x + e^{-x}}{e^x - e^{-x}} = \text{"}\frac{\infty}{\infty}\text{"}$

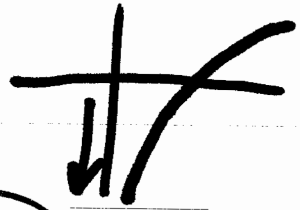
$\sim \frac{e^x}{e^x} = 1$ ✓

or $$\lim_{x\to\infty} \frac{\overset{\div e^x}{e^x - e^{-x}}}{\underset{\div e^x}{e^x + e^{-x}}} = \lim_{x\to\infty} \frac{1 - \cancel{e^{-2x}}^{\,0}}{1 + \cancel{e^{-2x}}_{\,0}} = \frac{1}{1} = 1$$

(H) doesn't always do the job! Watch out!

---

l'Hôpital's: 2nd Type



$y = \ln x$

$$\lim_{x\to 0^+} x\ln x = 0\cdot \ln(0^+) = 0\cdot(-\infty)$$

Indet. product!

rewrite as a fraction to get "$\frac{0}{0}$" or "$\frac{\infty}{\infty}$"

$$\lim_{x\to 0^+} \frac{\overset{\leftarrow f}{\ln x}}{\underset{\leftarrow g}{(1/x)}} = \frac{-\infty}{(1/0^+)} = -\text{"}\frac{\infty}{\infty}\text{"}$$ so (H)

$$= \lim_{x \to 0^+} \frac{1/x \leftarrow f'}{(-1)x^{-2} \leftarrow g'} = \lim_{x \to 0^+} (-x) = \underline{\underline{0}}$$

Always Simplify!

---

eg. $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = \left(1 + \frac{1}{\infty}\right)^\infty = (1^+)^\infty = \text{"} 1^\infty \text{"}$ form!

(not exactly 1)$^\infty$ is indeterminate!

Let's turn this into a product form using ln

Let $y = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$

so $\ln y = \ln \left(\lim_{x \to \infty} \text{"} \right) = \lim_{x \to \infty} \ln \left[\left(1 + \frac{1}{x}\right)^x \right]$

$$= \lim_{x\to\infty} x \ln\left(1+\frac{1}{x}\right) = \infty \ln\left(1+\frac{1}{\infty}\right)$$

$$= \infty \ln(1^+) = \underline{\underline{\infty \cdot 0}}$$

<u>2nd</u> H <u>Type!</u>

$$= \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\left(1/x\right)} = \frac{\ln(1)}{\left(1/\infty\right)} = \text{"}\frac{0}{0}\text{"}$$

Indet. fraction! $\Rightarrow$ H

$$\overset{H}{=} \lim_{x\to\infty} \frac{\frac{1}{(1+1/x)}\cdot \left(0-\cancel{\frac{1}{x^2}}\right)}{\left(\cancel{-1/x^2}\right)}$$

$$= \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}} = \frac{1}{1+\frac{1}{\infty}} = \frac{1}{1+0} = \underline{\underline{1}} = \underline{\underline{\ln y}}$$

$$y = e^1 = \boxed{\underline{\underline{e}} = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x}$$

Summary **l'Hôpital Cases**

Case #1 $\lim_{x \to a} \frac{f(x)}{g(x)} = "\frac{0}{0}"$ or $"\frac{\pm\infty}{\infty}"$

$\Rightarrow \lim_{x \to a} \frac{f(x)}{g(x)} \stackrel{H}{=} \lim_{x \to a} \frac{f'(x)}{g'(x)}$ if exists.

Case #2 $\lim_{x \to a} f(x) \cdot g(x) = "0 \cdot \infty"$

$\Rightarrow$ rewrite as $= \lim_{x \to a} \frac{f(x)}{\left(1/g(x)\right)} = "\frac{0}{0}"$ or $"\frac{\infty}{\infty}"$

Goto case #1.

Case #3 $\lim_{x \to a} (f(x))^{g(x)} =$ "1^{∞}", "∞^0", "0^0"

let $\ln y = \ln\left(\lim_{x \to a} f(x)^{g(x)}\right)$

$= \lim_{x \to a} g(x) \ln(f(x)) =$ "$0 \cdot \infty$"

a case #2 problem! Goto case #2

$y = e^{(\text{final result})}$ of case #2

i.e. Don't forget to exp. the final answer!

eg. $\lim_{x\to\infty}(x^3 - x^2) = \infty - \infty = ?!$

$= \lim_{x\to\infty} x^2(x-1) = \infty \cdot \infty = \underline{\underline{\infty}}$ } Just factor!

(or $x^3 >> x^2$ for x large!

so $\lim_{x\to\infty} \frac{x^3 - x^2}{1} \sim \lim_{x\to\infty} \frac{x^3}{1} = \infty$)

eg $\lim_{x\to 0^+} x^{1/x^2} = 0^{(1/0^2)} = 0^{1/0^+} = 0^{\infty} = \underline{\underline{0}}$

not indeterminate! It's 0

Note

$$\infty^{\infty} = \infty$$

$$\infty^{-\infty} = \frac{1}{\infty^{\infty}} = \frac{1}{\infty} = 0$$