MATHEMATICS 1LS3E TEST 3

Evening Class
Duration of Examination: 75 minutes
McMaster University, 4 June 2012

E. Clements

FIRST NAME (please print): SOL^{NS}

FAMILY NAME (please print): ______

Student No.: ______

THIS TEST HAS 8 PAGES AND 12 QUESTIONS. QUESTIONS BEGIN ON PAGE 2. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE.

Total number of points is 40. Marks are indicated next to the problem number. Any non-graphing calculator is allowed.

USE PEN TO WRITE YOUR TEST. IF YOU USE A PENCIL YOUR TEST WILL NOT BE ACCEPTED FOR REMARKING (IF NEEDED).

You need to show work to receive full credit.

Page	Points	Mark
2	7	
3	8	
4	6	
5	5	
6	4	
7	4	
8	6	
TOTAL	40	

Continued on next page

1. [3] Using the limit definition, find the derivative of $f(x) = \dfrac{x-1}{x+3}$.

$$f'(x) = \lim_{h\to 0} \frac{\dfrac{(x+h)-1}{(x+h)+3} - \dfrac{x-1}{x+3}}{h}$$

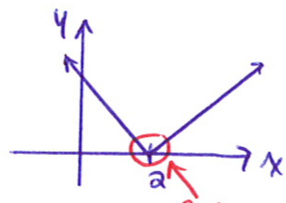
$$= \lim_{h\to 0} \frac{(x+h-1)(x+3) - (x-1)(x+h+3)}{(x+h+3)(x+3)} \cdot \frac{1}{h}$$

$$= \lim_{h\to 0} \frac{4}{(x+h+3)(x+3)}$$

$$= \frac{4}{(x+3)^2}$$

2. [2] True or false: If a function $f(x)$ is continuous at $x = 2$, then $f(x)$ is differentiable ($f'(2)$ exists) at $x = 2$. Explain.

FALSE! For example, $f(x) = |x-2|$ is continuous everywhere but not differentiable at $x = 2$ (graph has a corner).



Corner $\Rightarrow f'(2)$ D.N.E.

3. [2] Find $f'(1)$ if $f(x) = 2^{\ln x} + (\ln x)^2$.

$$f'(x) = 2^{\ln x} \cdot \ln 2 \cdot \frac{1}{x} + 2(\ln x) \cdot \frac{1}{x}$$

$$f'(1) = \underbrace{2^{\cancel{\ln 1} \to 0}}_{=1} \cdot \ln 2 \cdot \frac{1}{1} + 2\cancel{\ln 1}^{\,0} \cdot \frac{1}{1}$$

$$= \ln 2$$

Name:____________________
Student No.: ____________________

4. [3] Determine an equation of the tangent to the graph of $y = (\sin x + \cos x)^3$ at $x = 0$.

$y' = 3(\sin x + \cos x)^2 \cdot (\cos x - \sin x)$ $y(0) = (0+1)^3 = 1$

$y'(0) = 3(0+1)^2 \cdot (1-0) = 3$

$\therefore\ y - 1 = 3(x-0) \Rightarrow y = 3x+1$

5. [3] Determine all critical numbers of the function $g(x) = x^{\frac{1}{3}}(x-1)$.

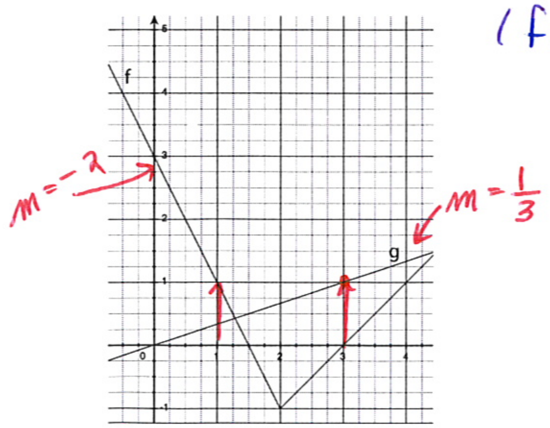
$g' = \frac{1}{3}x^{-2/3} \cdot (x-1) + x^{1/3} \cdot (1) = \frac{1}{3}x^{-2/3}((x-1) + 3x) = \dfrac{4x-1}{3x^{2/3}}$

$g' = 0$ when $4x - 1 = 0 \Rightarrow x = \frac{1}{4}$

g' dne when $3x^{2/3} = 0 \Rightarrow x = 0$

Since the domain of $g(x)$ is $\mathbb{R}$,
both 0 and $\frac{1}{4}$ are the critical #s of $g(x)$.

6. [2] By referring to the graphs of two functions f and g below, compute $(f \circ g)'(3)$.



$(f \circ g)'(3) = f'(g(3)) \cdot g'(3)$

$= f'(1) \cdot g'(3)$

$= -2 \cdot \frac{1}{3}$

$= -\frac{2}{3}$

7. (a) [3] Determine the intervals of increase and decrease for $f(x) = 4x^2 + \frac{1}{x}$.

$f' = 8x - \frac{1}{x^2} = \frac{8x^3 - 1}{x^2}$

$f' = 0 \Rightarrow 8x^3 - 1 = 0 \Rightarrow x^3 = \frac{1}{8} \Rightarrow x = \frac{1}{2}$

		0		$\frac{1}{2}$	
f'	$-$		$-$		$+$
f	↘		↘		↗

(b) [1] State any local extreme values of $f(x)$ and the x-values at which they occur.

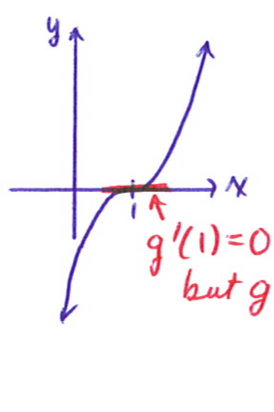
f has a local min. when $x = \frac{1}{2}$

local min. value:

$f\left(\frac{1}{2}\right) = 4\left(\frac{1}{2}\right)^2 + \frac{1}{\frac{1}{2}} = 3$

(c) [2] Give an example (sketch a graph, or write down a formula) of a continuous function $g(x)$ such that $g'(1) = 0$, but g does not have an extreme value at $x = 1$.

$g(x) = (x-1)^3$



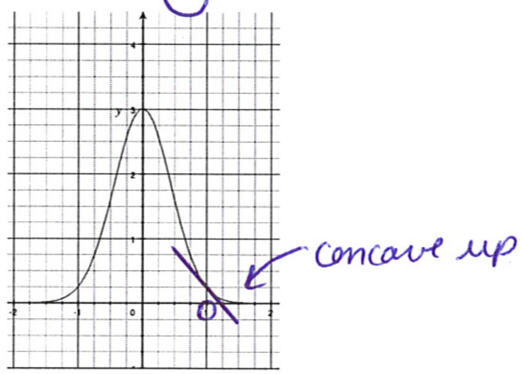
$g'(1) = 0$
but g has no extreme values

Name:__________________

Student No.: ________________

8. [2] In the graph of $f(x)$ below, which of the following are negative?

(I) $f(1)$ $+$ (II) $f'(1)$ $-$ (III) $f''(1)$ $+$



(A) none (B) I only (C) II only (D) III only

(E) I and II (F) I and III (G) II and III (H) all three

9. [3] Find the absolute extreme values of the function $f(x) = \dfrac{\ln x}{x}$ on the interval $[1, 3]$.

$$f' = \frac{\frac{1}{x}\cdot x - \ln x \cdot 1}{x^2} = \frac{1-\ln x}{x^2}$$

$f' = 0$ when $1 - \ln x = 0 \Rightarrow \ln x = 1 \Rightarrow x = e$

x	$f(x)$
1	$\frac{\ln 1}{1} = 0$ ← ABS. MIN.
e	$\frac{\ln e}{e} = \frac{1}{e} \approx 0.368$ ← ABS. MAX.
3	$\frac{\ln 3}{3} \approx 0.366$

10. (a) [3] Find the Taylor polynomial $T_2(x)$ for the function $f(x) = \arctan x$ near $x = 1$.

$f(x) = \arctan x$ $\qquad f(1) = \arctan 1 = \frac{\pi}{4}$

$f'(x) = \frac{1}{1+x^2} = (1+x^2)^{-1}$ $\qquad f'(1) = \frac{1}{2}$

$f''(x) = -1(1+x^2)^{-2} \cdot 2x$ $\qquad f''(1) = -\frac{1}{2}$

$= \frac{-2x}{(1+x^2)^2}$

$$T_2(x) = \frac{\pi}{4} + \frac{1}{2}(x-1) - \frac{\frac{1}{2}}{2}(x-1)^2$$

(b) [1] Use your answer to (a) to find an approximation of arctan 1.1.

$$\arctan 1.1 \approx \frac{\pi}{4} + \frac{1}{2}(1.1-1) - \frac{1}{4}(1.1-1)^2$$

$$\approx \frac{3.14}{4} + \frac{1}{2}(0.1) - \frac{1}{4}(0.1)^2$$

$$\approx 0.83$$

11. (a) [2] Let $f(x) = xe^{4x}$. Show that $f''(x) = e^{4x}(16x + 8)$.

$f' = 1 \cdot e^{4x} + x \cdot e^{4x}(4) = e^{4x}(1 + 4x)$

$f'' = e^{4x} \cdot 4(1 + 4x) + e^{4x}(4)$

$= e^{4x}(16x + 8)$

(b) [2] Find the interval where the graph of $f(x) = xe^{4x}$ is concave up.

$f'' = 0$ when $16x + 8 = 0 \Rightarrow x = -\frac{1}{2}$

	-1	$-\frac{1}{2}$
f''	$-$	$+$
f	$\cap$	$\cup$

$\therefore$ the graph of f is concave up on $(-\frac{1}{2}, \infty)$.

Name:________________
Student No.: ________________

12. Evaluate the following limits using L'Hopital's Rule.

(a) [3] $\lim_{x\to 1} \frac{\ln x}{x^2-1}$ $\left(= \frac{\ln 1}{1^2-1} = \frac{0}{0}\right)$

$\overset{LH}{=} \lim_{x\to 1} \frac{\frac{1}{x}}{2x}$

$= \lim_{x\to 1} \frac{1}{2x^2}$

$= \frac{1}{2}$

(b) [3] $\lim_{x\to\infty} x^2 e^{-x}$ $(= \infty \cdot 0)$

$= \lim_{x\to\infty} \frac{x^2}{e^x}$ $\left(= \frac{\infty}{\infty}\right)$

$\overset{LH}{=} \lim_{x\to\infty} \frac{2x}{e^x}$ $\left(= \frac{\infty}{\infty}\right)$

$\overset{LH}{=} \lim_{x\to\infty} \frac{2}{e^x}$

$= \frac{2}{\infty}$

$= 0$

THE END