McMaster University
Math 1LT3

ASSIGNMENT 3

Section 8.5, 8.6, 8.7

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1. Suppose that the size of a room is five times that of an object in the room, but the specific heat of the room is twice that of the object (meaning that a large amount of heat produces only a small change in the temperature of the room).

(a) Write a system of autonomous differential equations describing the temperature of the object and the temperature of the room (see examples 8.5.3 and 8.5.4).

large room lowers α_2
specific heat of room > specific heat of object lowers α_2

$$\Rightarrow \alpha_2 = \frac{1}{5} \cdot \frac{1}{2}\alpha = \frac{1}{10}\alpha$$

$$\therefore \frac{dT}{dt} = \alpha(A - T) \quad \text{and} \quad \frac{dA}{dt} = \frac{\alpha}{10}(T - A)$$

(b) Suppose that $T(0) = 60$ and $A(0) = 20$. Approximate the values of $T(1)$ and $A(1)$ using Euler's method, with a step size of $\Delta t = 0.5$. Assume that $\alpha = 0.4$.

$T_0 = 60$ $\quad$ $\Delta t = 0.5$ $\quad$ $T' = 0.4(A - T)$
$A_0 = 20$ $\quad$ $A' = 0.04(T - A)$

$$T_1 = T_0 + 0.4(A_0 - T_0)\cdot\Delta t = 60 + 0.4(20-60)(0.5) = 52$$

$$A_1 = A_0 + 0.04(T_0 - A_0)\Delta t = 20 + 0.04(60-20)(0.5) = 20.8$$

$$T_2 = T_1 + 0.4(A_1 - T_1)\Delta t = 52 + 0.4(20.8 - 52)(0.5) = 45.8$$

$$A_2 = A_1 + 0.04(T_1 - A_1)\Delta t = 20.8 + 0.04(52 - 20.8)(0.5) = 21.4$$

$\therefore T(1) \approx 45.8$ and $A(1) \approx 21.4$
(so, object cools quickly, room warms up slowly)

2. Write systems of differential equations describing the following situations. You may make up parameter values as needed.

(a) Two predators that must eat each other to survive, but with the per capita growth rate of each reduced by competition with its own species.

2 predators that must eat each other to survive:

$$\frac{dp_1}{dt} = -p_1 + 0.2p_1p_2 = (-1 + 0.2p_2)p_1$$

$$\frac{dp_2}{dt} = -p_2 + 0.3p_1p_2 = (\underline{-1 + 0.3p_1})p_2$$

per capita growth rate

now with per capita growth rates reduced by competition:

$$\frac{dp_1}{dt} = (-1 + 0.2p_2 - 0.4p_1)p_1 = -p_1 + 0.2p_1p_2 - 0.4p_1p_1$$

$$\frac{dp_2}{dt} = (-1 + 0.3p_1 - 0.1p_2)p_2 = -p_2 + 0.3p_1p_2 - 0.1p_2p_2$$

(b) Two competitors where the per capita growth rate of each type is affected only by the population size of the other type.

$$\frac{da}{dt} = \mu\left(1 - \frac{b}{K_a}\right)a$$

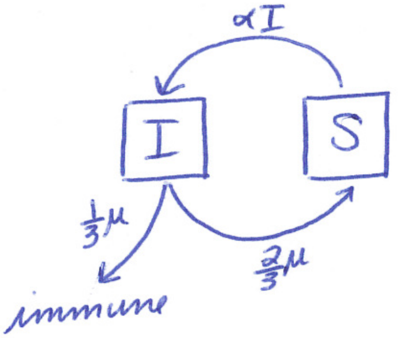
$$\frac{db}{dt} = \lambda\left(1 - \frac{a}{K_b}\right)b$$

3. Consider the two-dimensional extension of the basic disease model,

$$\frac{dI}{dt} = \alpha IS - \mu I, \qquad \frac{dS}{dt} = -\alpha IS + \mu I$$

Write systems of differential equations describing the following situations.

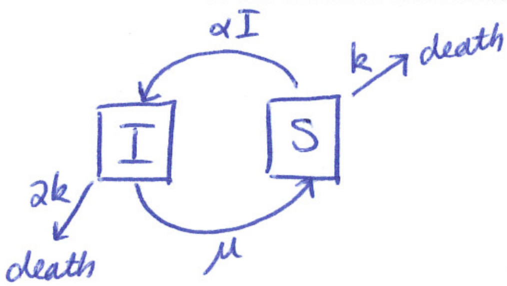
(a) Suppose that one-third of the individuals who leave the infected class through recovery become permanently immune and that the other two-thirds become susceptible again.



$$\frac{dI}{dt} = \alpha IS - \mu I \text{ (same)}$$

$$\frac{dS}{dt} = -\alpha IS + \frac{2}{3}\mu I$$

(b) Suppose that all individuals become susceptible upon recovery (as in the basic model) but that there is a source of mortality, whereby susceptible individuals die at a per capita rate k but infected individuals die at a per capita rate that is twice as large.



$$\frac{dI}{dt} = \alpha IS - \mu I - 2kI$$

$$\frac{dS}{dt} = -\alpha IS + \mu I - kS$$

4. For each system of differential equations, (i) find and graph the nullclines in the phase plane, (ii) identify the equilibria, (iii) add direction arrows to your phase-plane diagram, and (iv) sketch a phase-plane trajectory starting from the given initial conditions.

(a) Predator-prey model $\frac{db}{dt} = (1 - 0.05p)b$, $\frac{dp}{dt} = (-3 + 0.02b)p$, with $b(0) = 180$, $p(0) = 30$.

$\frac{db}{dt} = 0$ when $1 - 0.05p = 0$ or $\boxed{b = 0}$

$\Rightarrow \boxed{p = 20}$

two components of the b-nullcline

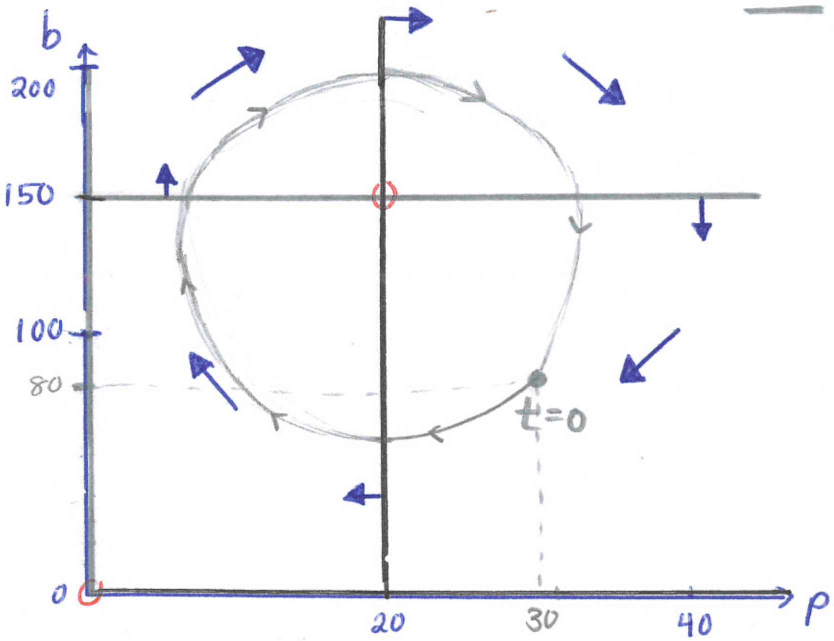
— black pen

$\frac{dp}{dt} = 0$ when $-3 + 0.02b = 0$ or $p = 0$

$\Rightarrow b = 150$

two components of the p-nullcline

— pencil



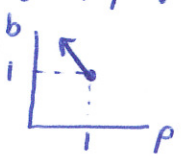
equilibria : $(0,0)$ and $(20,150)$

test

$(p,b) = (1,1)$

$b' > 0 \Rightarrow b \nearrow$

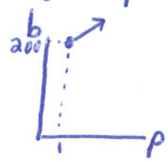
$p' < 0 \Rightarrow p \downarrow$



$(p,b) = (1,200)$

$b' > 0 \Rightarrow b \nearrow$

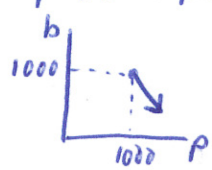
$p' > 0 \Rightarrow p \nearrow$



$(p,b) = (1000,1000)$

$b' < 0 \Rightarrow b \downarrow$

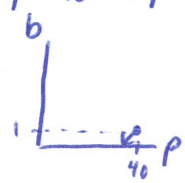
$p' > 0 \Rightarrow p \nearrow$



$(p,b) = (40,1)$

$b' < 0 \Rightarrow b \downarrow$

$p' < 0 \Rightarrow p \downarrow$



(b) Competition model $\frac{da}{dt} = 2(1 - \frac{a+b}{100})a$, $\frac{db}{dt} = (1 - \frac{a+b}{200})b$, with $a(0) = 20$, $b(0) = 50$.

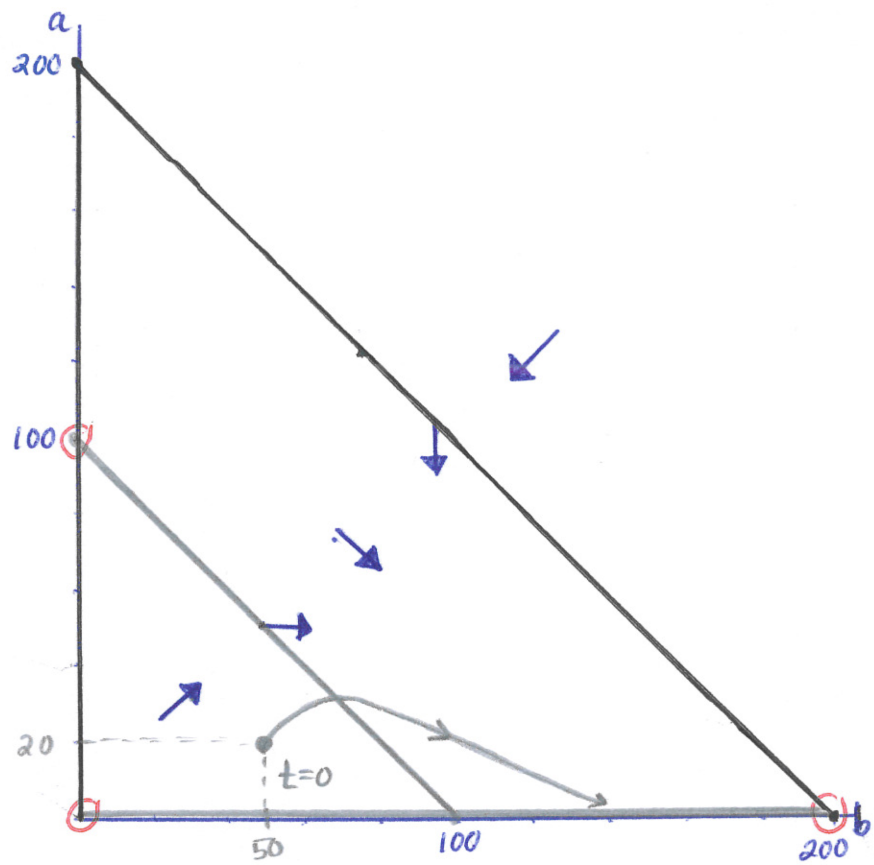
$\frac{da}{dt} = 0$ when $a = 0$ or $1 - \frac{a+b}{100} = 0$

$a + b = 100$

$a = -b + 100$ pencil

$\frac{db}{dt} = 0$ when $b = 0$ or $1 - \frac{a+b}{200} = 0$

$a = -b + 200$ black pen



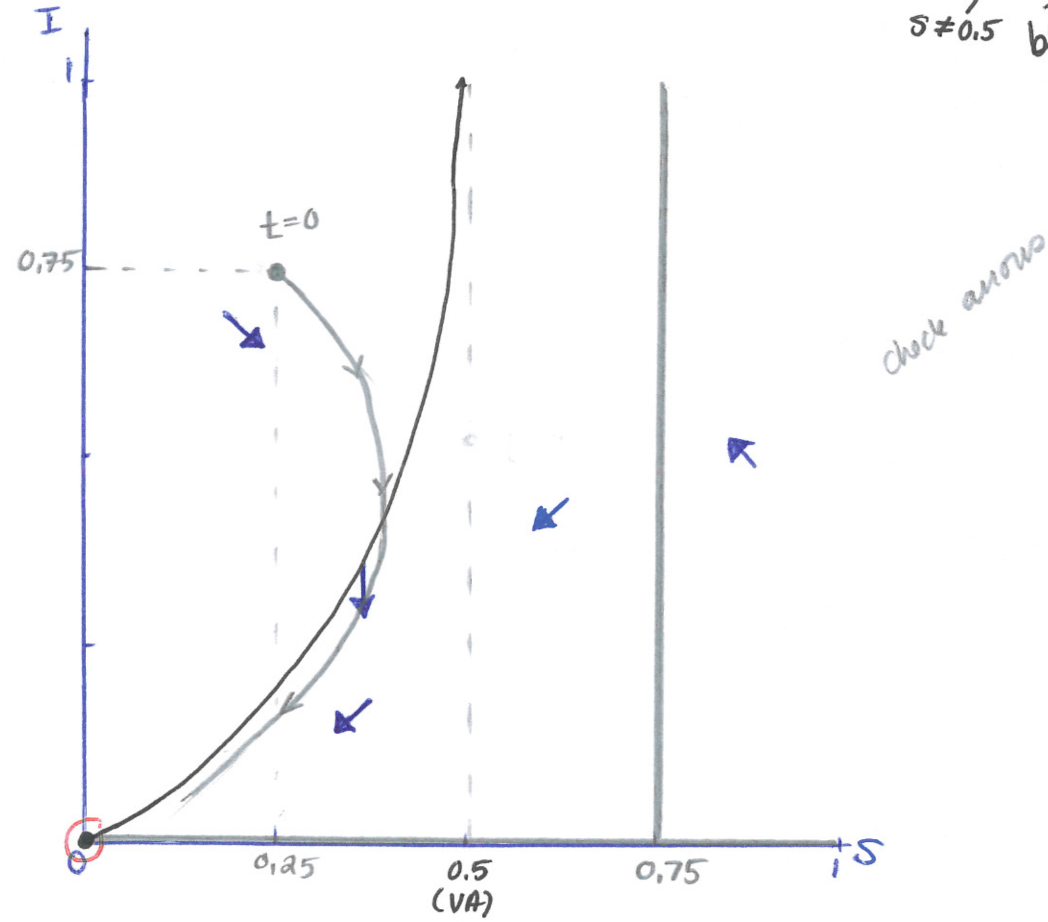
equilibria: $(0,0)$, $(200,0)$, $(0,100)$

(c) Modified disease model $\frac{dI}{dt} = 2IS \overbrace{- I - 0.5I}^{-1,5I}$, $\frac{dS}{dt} = -2IS + I - 0.5S$, with $I(0) = 0{,}75$ $S(0) = 0.25$.

$\frac{dI}{dt} = 0$ when $I(2S - 1.5) = 0 \Rightarrow \boxed{I = 0}$ or $\boxed{S = 0{,}75}$ pencil

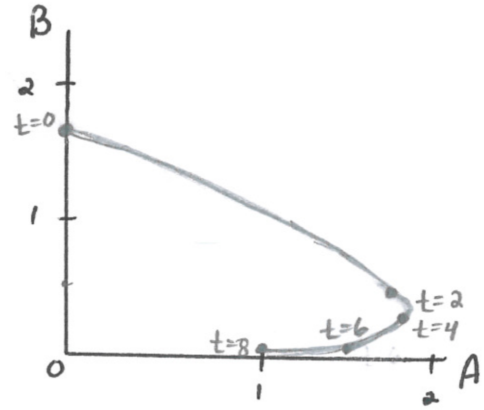
$\frac{dS}{dt} = 0$ when $I(-2S+1) - 0{,}5S = 0 \Rightarrow I = \frac{0{,}5S}{1-2S} = \frac{S}{2-4S}$

$S \neq 0{,}5$ black pen

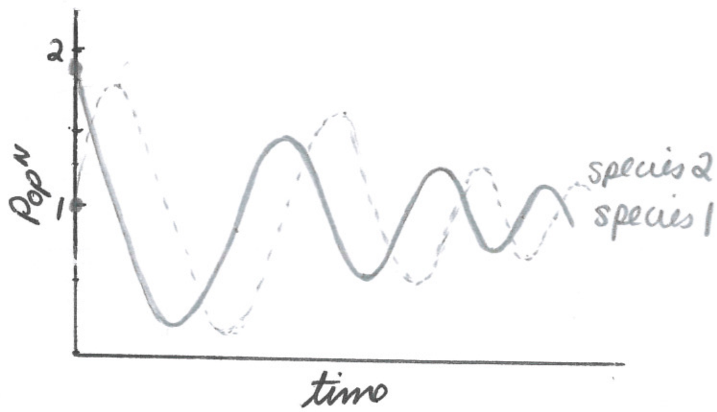


equilibrium : $(0,0)$

5. p.657, question 6



6. p.657, question 8



7. p.658, question 10