MATH 1B03/1ZC3 Winter 2019

Lecture 20: General vector spaces

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Subspaces and linear independence

(from Chapter 4.2 of Anton-Rorres)

Let's continue our study of subspaces of vector spaces.

Let V be a vector spaces and $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ a set of vectors in V. If span$(S) = V$ we say that the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ generate V.

We saw that the set

$$S = \{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}$$

generates $\mathbb{R}^4$ i.e. that span$(S) = \mathbb{R}^4$. We can check if a set spans $\mathbb{R}^n$ in the following way.

Recipe 20.1: Does S span $\mathbb{R}^n$?

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a set of n vectors in $\mathbb{R}^n$. Use this recipe to check if span$(S) = \mathbb{R}^n$.

Step 1: Construct the matrix

$$A = \begin{bmatrix} \leftarrow & \mathbf{v}_1 & \rightarrow \\ \leftarrow & \mathbf{v}_2 & \rightarrow \\ & \vdots & \\ \leftarrow & \mathbf{v}_n & \rightarrow \end{bmatrix}$$

Step 2: Compute $det(A)$.
If $det(A) \neq 0$, then span$(S) = \mathbb{R}^n$
If $det(A) = 0$, then span$(S) \neq \mathbb{R}^n$.

We shall see later that if $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ and $k < n$, then span$(S) \neq \mathbb{R}^n$. If $k > n$ we can still answer the question, but we need some more terminology.

Example 20.2

Question: Let $\mathbf{v}_1 = (2, 0, 1)$, $\mathbf{v}_2 = (-3, 4, 7)$ and $\mathbf{v}_3 = (-1, 4, 8)$. Does span $(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}) = \mathbb{R}^3$?
Answer: We have

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 4 & 7 \\ -1 & 4 & 8 \end{bmatrix}$$

Compute $det(A)$:

$$\begin{vmatrix} 2 & 0 & 1 \\ 3 & 4 & 7 \\ -1 & 4 & 8 \end{vmatrix} = 2(32 - 28) + (-12 + 4) = 8 - 8 = 0$$

therefore span $(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}) \neq \mathbb{R}^3$.
Also notice that

$$(-1, 4, 8) = (-3, 4, 7) + (2, 0, 1)$$

so that $\mathbf{v}_3$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. We shall see why this is important later.

An important subspace of $\mathbb{R}^n$ is formed by the solutions to homogeneous equations.

Fact 20.3

Let A be an $m \times n$ matrix. The set of solutions to the equation

$$A\mathbf{x} = \mathbf{0}$$

is a subspace of $\mathbb{R}^n$.

Proof: Fist, we are required to prove that if $\mathbf{x}_1$ and $\mathbf{x}_1$ are both solutions to $A\mathbf{x} = \mathbf{0}$, then $\mathbf{x}_1 + \mathbf{x}_2$ is a solution also.

We have

$$
\begin{aligned}
A(\mathbf{x}_1 + \mathbf{x}_2) &= A\mathbf{x}_1 + A\mathbf{x}_2 \\
&= 0
\end{aligned}
$$

as required, as $\mathbf{x}_1$ and $\mathbf{x}_1$ are both solutions.

Second, we are required to prove that if $\mathbf{x}_1$ is a solution to $A\mathbf{x} = \mathbf{0}$, then $k\mathbf{x}_1$ is a solution also, for any scalar k.

We have

$$
\begin{aligned}
A(k\mathbf{x}_1) &= kA\mathbf{x}_1 \\
&= 0
\end{aligned}
$$

as required, as $\mathbf{x}_1$ is a solution. ■

Given two subspaces we can produce a new subspace by taking their intersection.

Definition 20.4: Intersection

Let X and Y be sets. The intersection of X and Y is denoted $X \cap Y$ and is defined to the the set of elements in **both** X and Y.

Fact 20.5

Let V be a vector space and W_1, W_2 subspaces. Then $W_1 \cap W_2$ is a subspace.

Question 20.6

Prove this fact.
Hint: you only need to check $W_1 \cap W_2$ is closed under addition and scalar multiplication.

Linear independence

(from Chapter 4.3 of Anton-Rorres)

As is usual when it comes to studying abstract mathematical objects, we want to describe vector spaces in the simplest possible way. We do this by using the notion of span, together with linear independence.

Definition 20.7: Linear Independence

Let V be a vector space and $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ a set of vectors in V. The set S is linearly independent if the equation

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n = 0$$

has the unique solution $k_1 = k_2 = \cdots k_n = 0$.
A set which is not linearly independent is linearly dependent.

What does this mean? A set S is linearly independent if and only if no vector in S can be written as a linear combination of the others.

Therefore, all we have to do to prove a set is linearly dependent is write one of its vectors as a linear combination of the others.

Proving a set is linearly independent is slightly harder, but we will see methods of doing so.

Example 20.8

We saw earlier that

$$(-1, 4, 8) = (-3, 4, 7) + (2, 0, 1)$$

Therefore the set $\{(2, 0, 1), (-3, 4, 7), (-1, 4, 8)\}$ is **linearly dependent**.
To this is explicitly, notice that the equation

$$k_1 (2, 0, 1) + k_2 (-3, 4, 7) + k_3 (-1, 4, 8) = 0$$

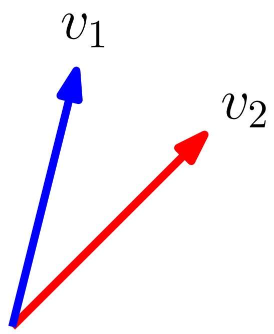
has the solution $k_1 = -1$, $k_2 = -1$ and $k_3 = 1$.

We can understand linear independence geometrically as follows.

Consider the set of $S = \{\mathbf{v}_1, \mathbf{v}_2\}$ of two vectors in $\mathbb{R}^n$. If S is linearly independent then

$$\mathbf{v}_1 \neq k\mathbf{v}_2$$

for all scalars k. Geometrically, this means that $\mathbf{v}_1$ does not lie on the line defined by $\mathbf{v}_2$:

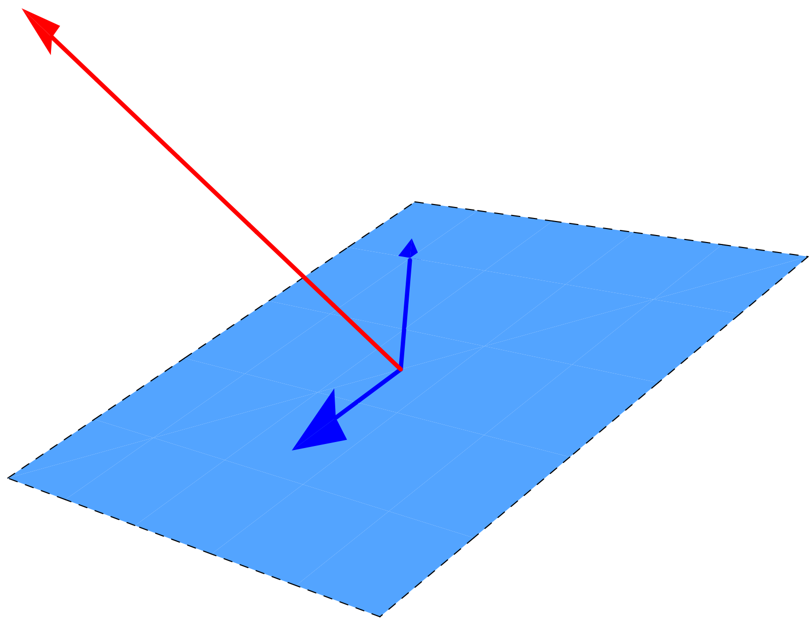


Notice that $\mathbf{v}_1$ is not necessarily orthogonal to $\mathbf{v}_2$.

Further, if $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set of three vectors in $\mathbb{R}^n$ then

$$\mathbf{v}_3 \neq k\mathbf{v}_1 + m\mathbf{v}_2$$

for all scalars k, and m. Therefore $\mathbf{v}_3 \notin \text{span}(\{\mathbf{v}_1, \mathbf{v}_2\})$, and $\mathbf{v}_3$ does not lie in the plane described by $\mathbf{v}_1$, $\mathbf{v}_2$:



Notice again that $\mathbf{v}_3$ is not necessarily orthogonal to $\mathbf{v}_1$ and $\mathbf{v}_2$, it is only required not to lie in the plane defined by them.

When the number of vectors is small it is often easy to check if a set is linearly independent by inspection. When the numbers get large, however, we need to check more systematically.

Fact 20.9

If S is a set of n linearly independent vectors in $\mathbb{R}^n$ then span$(S) = \mathbb{R}^n$.

Recipe 20.10: Checking linear independence in $\mathbb{R}^n$

Use this method to determine if a set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_1, \ldots, \mathbf{v}_m$ is linearly independent in $\mathbb{R}^n$.

Step 1: If $m > n$ then S is linearly dependent.

Step 2: If $m = n$, then form the matrix

$$A = \begin{bmatrix} \leftarrow & \mathbf{v}_1 & \rightarrow \\ \leftarrow & \mathbf{v}_2 & \rightarrow \\ & \vdots & \\ \leftarrow & \mathbf{v}_n & \rightarrow \end{bmatrix}$$

Compute $det(A)$. If $det(A) = 0$ then S is linearly dependent.
If $det(A)$. If $det(A) \neq 0$ then S is linearly independent.

Step 2: If $m < n$, consider the equation

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots k_m\mathbf{v}_m = 0$$

Let $\mathbf{v}_i = \left(v_1^i, v_2^i, \ldots, v_n^i, \right)$. Then equation yields the system of linear equations

$$\begin{aligned} k_1v_1^1 + k_2v_1^2 + \cdots + k_mv_1^m &= 0 \\ k_1v_2^1 + k_2v_2^2 + \cdots + k_mv_2^m &= 0 \\ &\vdots \\ k_1v_n^1 + k_2v_n^2 + \cdots + k_mv_n^m &= 0 \end{aligned}$$

Solve this system (via Gaussian elimination, for example). If $k_1 = k_2 = \cdots = k_m = 0$ is the unique solution, then S is linearly independent.

We will understand the methods behind this recipe later on; for now, we can use it to determine linear independence.

Example 20.11

Question: Is the set

$$\{(2, 3), (-8, 5), (3, -7)\}$$

linearly independent in $\mathbb{R}^2$?
Answer: The set has 3 elements but we are in $\mathbb{R}^2$, so it must be linearly dependent.

Question: Is the set

$$\{(7, 3), (0, 4)\}$$

linearly independent in $\mathbb{R}^2$?
Answer: We can determine by inspection that the set is linearly dependent: clearly $(0, 4)$ cannot be written as a scalar multiple of $(7, 3)$.
We can double check this as follows. We have 2 vectors in $\mathbb{R}^2$, so form the matrix

$$A = \begin{bmatrix} 7 & 3 \\ 0 & 4 \end{bmatrix}$$

and compute the determinant

$$\begin{vmatrix} 7 & 3 \\ 0 & 4 \end{vmatrix} = 28 \neq 0$$

Therefore the set is linearly independent.

Question: Is the set

$$\{(2, 2, 0, -2), (8, -3, -1, 0), (-12, 10, 2, -4)\}$$

linearly independent in $\mathbb{R}^4$?
Answer: Consider the equation

$$k_1 (2, 2, 0, -2) + k_2 (8, -3, -1, 0) + k_3 (-12, 10, 2, -4) = 0$$

This yields the SLE

$$\begin{aligned} 2k_1 + 8k_2 - 12k_3 &= 0 \\ 2k_1 - 3k_2 + 10k3 &= 0 \\ -k_2 + 2k_3 &= 0 \\ -2k_1 - 4k_3 &= 0 \end{aligned}$$

To solve this system, apply Gauss-Jordon elimination to the augmented matrix

$$\begin{bmatrix} 2 & 8 & -12 & 0 \\ 2 & -3 & 10 & 0 \\ 0 & -1 & 2 & 0 \\ -2 & 0 & -4 & 0 \end{bmatrix}$$

The reduced row echelon form is:

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

from which we obtain

$$\begin{aligned} k_1 + 2k_3 &= 0 \\ k_2 - 2k_3 &= 0 \end{aligned}$$

which has $k_1 = -2$, $k_2 = 2$, $k_3 = 1$ as a solution. Therefore the set is linearly dependent.
Notice that

$$(-12, 10, 2, -4) = 2\,(2, 2, 0, -2) - 2\,(8, -3, -1, 0)\,.$$

In vector space which are not $\mathbb{R}^n$ we must rely on the definition of linear independence.

Example 20.12

Recall that P_2 is the vector space of polynomials of degree at most 2. Is the set

$$\{7x^2 + 3,\ 11x - 4,\ 2x^2 - x + 6\}$$

linearly independent in P_2?

Answer: Consider the equation

$$k_1\left(7x^2+3\right)+k_2(11x-4)+k_3\left(2x^2-x+6\right)=0$$
$$(7k_1+2k_3)\,x^2+(11k_2-k_3)\,x+3k_1-4k_2+6k_3=0$$

where we have expanded and collected like terms.
This yields the simultaneous equations

$$\begin{aligned}7k_1+2k_3&=0\\11k_2-k_3&=0\\3k_1-4k_2+6k_3&=0\end{aligned}$$

The associated augmented matrix is

$$\begin{bmatrix}7&2&0&0\\0&11&-1&0\\3&-4&6&0\end{bmatrix}$$

Applying Gauss-Jordon elimination we obtain

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\end{bmatrix}$$

and we can conclude that $k_1=k_2=k_3=0$ is the unique solution. Therefore the set $\{7x^2+3,\ 11x-4,\ 2x^2-x+6\}$ is linearly independent in P_2.

Suggested Problems

Practice the material covered in this lecture by attempting the following questions from Chapter 4.2 of Anton-Rorres, starting on page 200

- Questions $1, 2, 3, 7, 10, 11, 13$
- True/False (e), (g), (h), (j)

and from Chapter 4.3 of Anton-Rorres, starting on page 210

- Questions $1, 3, 9, 22, 23(b)$
- True/False (b), (d), (f), (g)