Math 1LS3 Week 5: Limits and Continuity

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This week, we will finish 3.3 and cover 3.4. Next week is 3.5–4.5.
Reminder: please carefully read Table 3.3.4 on p.191 for the precise meaning of "limit does not exist".

1. Overview
2. Limits at Infinity
3. Algebra Tricks
4. Horizontal Asymptotes
5. Long Term Comparisons
6. Continuity

Overview

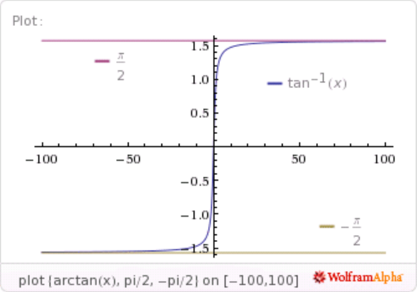
Last week we defined $\lim_{x \to a} f(x)$, and evaluated some limits. Recall:

- For most limits involving *decent functions f*: use "direct substitution"
- If direct substitution yields e.g. $\frac{17}{\pm\infty}$, the limit is 0
- If direct substitution yields e.g. $\frac{17}{0}$:
  - You probably have a **vertical asymptote** of f
  - Evaluate $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ separately
  - See if you get 17/(small positive)$\to +\infty$ or 17/(small negative)$\to -\infty$
- If direct substitution yields ∞/∞, $0/0$, $\infty - \infty$ (or other "indeterminate forms"):
  - You might have a **hole** or **asympote** of f
  - Try dividing numerator and denominator by "largest" of all terms
  - Try finding and cancelling common factors

This week, we'll focus on these "decent" (**continuous**) functions.

Limits at Infinity

Here is a graph of *arctan*(x) (a reflection of *tan*(x) restricted):

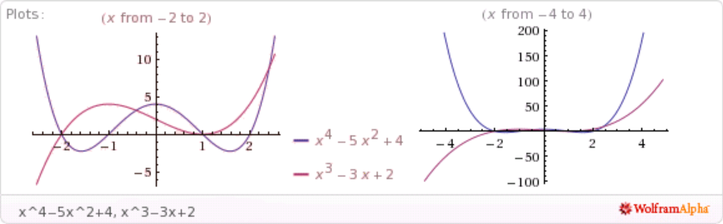


Limits "at infinity" describe long-term behavior.

$$\lim_{x \to +\infty} arctan(x) = \frac{\pi}{2} \qquad \lim_{x \to -\infty} f(x) = -\frac{\pi}{2}$$

Finite limits at $\pm\infty$ correspond to *horizontal* asymptotes.

Polynomials at Infinity



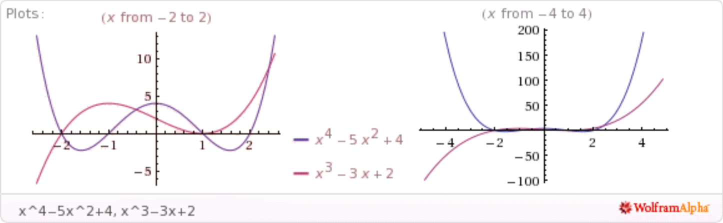
Theorem

Let $P(x)$ be a nonconstant polynomial with positive leading coefficient.

$$\lim_{x \to +\infty} P(x) = +\infty.$$

Theorem

Let $P(x)$ be a nonconstant polynomial with positive leading coefficient.
If P has even degree, $\lim_{x \to -\infty} P(x) = \infty$.
If P has odd degree, $\lim_{x \to -\infty} P(x) = -\infty$.



Example

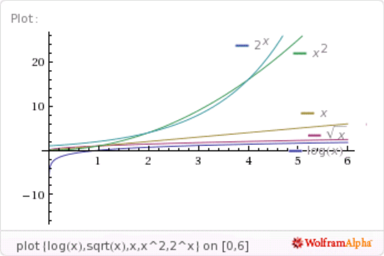
- $\displaystyle\lim_{t\to\infty} t^5 - t^4 + 3 = +\infty,$
- $\displaystyle\lim_{t\to-\infty} t^5 - t^4 + 3 = -\infty,$
- $\displaystyle\lim_{t\to\infty} t^6 - t^3 + 3 = +\infty,$
- $\displaystyle\lim_{t\to-\infty} -2t^5 + t^4 = - * -\infty = +\infty.$

Functions that Approach Infinity at Infinity

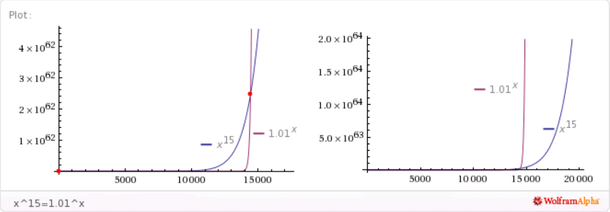
Here are many functions $f(x)$ with

$$\lim_{x \to \infty} f(x) = \infty.$$

$$2^x \gg x^2 \gg x \gg \sqrt{x} \gg \ln(x)$$



Any exponential eventually beats *any* polynomial:

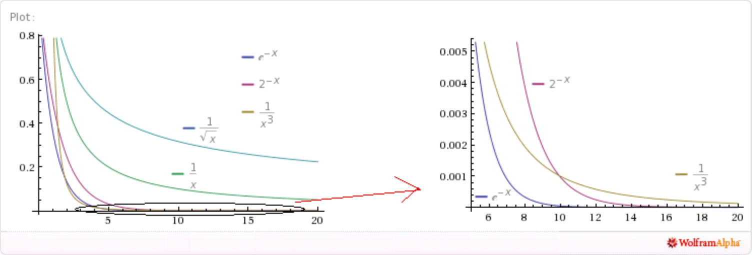


This tells us long term behavior of their ratios:

$$\lim_{x \to \infty} \frac{e^x}{15x^4 + 29x^3} = \lim \frac{\textit{realbig}}{\textit{big}} = \infty \qquad \lim_{x \to \infty} \frac{500x^{999}}{1.000001^x} = \lim \frac{\textit{big}}{\textit{realbig}} = 0$$

Functions that Approach Zero at Infinity

Here are many functions $g(x)$ with $\lim_{x\to\infty} g(x) = 0$.



$$e^{-x} \ll 2^{-x} \ll x^{-3} \ll \frac{1}{x} \ll \frac{1}{\sqrt{x}}$$

This tells us long term behavior of their ratios:

$$\lim_{x\to\infty} \frac{e^{-x}}{x^{-2}} = \lim \frac{\mathit{crazysmall}}{\mathit{small}} = 0 \qquad \lim_{t\to\infty} \frac{\frac{1}{t}}{\frac{1}{t^2}} = \lim \frac{\mathit{small}}{\mathit{crazysmall}} = +\infty$$

Algebra Tricks 2: Dividing by Large Powers of x

Problem

Compute

$$\lim_{t \to -\infty} \frac{t^2 + t - 2}{t^2 - t}$$

Solution

If we plug in, we get $\frac{\infty}{\infty}$. Not defined.
Trick: divide numerator and denominator by leading power of t:

$$\lim_{t \to -\infty} \frac{t^2 + t - 2}{t^2 - t} = \lim_{t \to -\infty} \frac{\frac{t^2+t-2}{t^2}}{\frac{t^2-t}{t^2}} = \lim_{t \to -\infty} \frac{1 - \frac{1}{t} - \frac{2}{t^2}}{1 - \frac{1}{t}} = \frac{1 - 0 - 0}{1 - 0} = 1$$

Limits Problems (i)

To find horizontal asymptotes, compute $\lim_{x \to \pm\infty} f(x)$.

Problem

Find the horizontal asymptotes of:

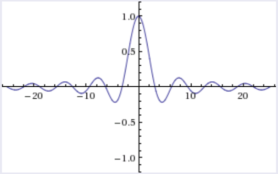
$$(i)\ \frac{\sin(x)}{x}, \qquad (ii)\ e^{-2x^3+3x-6}, \qquad (iii)\ \coth(x) := \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

Solution

$$\lim_{x\to\infty} \frac{\sin(x)}{x} = \lim \frac{\textit{bounded}}{\textit{big}} = 0.$$

(i)

$$\lim_{x\to-\infty} \frac{\sin(x)}{x} = \lim \frac{\textit{bounded}}{-\textit{big}} = 0.$$



Horizontal asymptote: $y = 0$ *(as* $x \to \pm\infty$*).*

Limits Problems (ii)

To find horizontal asymptotes, compute $\lim_{x\to\pm\infty} f(x)$.

Problem

Find the horizontal asymptotes of:

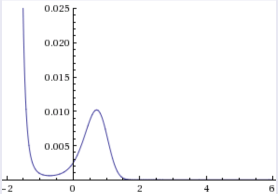
$$(i)\ \frac{\sin(x)}{x}, \qquad (ii)\ e^{-2x^3+3x-6}, \qquad (iii)\ \coth(x) := \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

Solution

(ii)

$$\lim_{x\to-\infty} e^{-2x^3+3x-6} = e^{\left(\lim_{x\to-\infty} -2x^3+3x-6\right)} = e^{+\infty} = +\infty$$

$$\lim_{x\to\infty} e^{-2x^3+3x-6} = e^{\left(\lim_{x\to\infty} -2x^3+3x-6\right)} = e^{-\infty} = 0.$$



Horizontal asymptote: $y = 0$ (as $x \to +\infty$ only).

Limits Problems (iii)

To find horizontal asymptotes, compute $\lim_{x\to\pm\infty} f(x)$.

Problem

Find the horizontal asymptotes of:

$$(i)\ \frac{\sin(x)}{x}, \qquad (ii)\ e^{-2x^3+3x-6}, \qquad (iii)\ \coth(x) := \frac{e^x+e^{-x}}{e^x-e^{-x}}$$

Solution

$$\lim_{x\to\infty}\frac{e^x+e^{-x}}{e^x-e^{-x}} = \lim_{x\to\infty}\frac{e^x+e^{-x}}{e^x-e^{-x}} \div \frac{e^x}{e^x} = \lim_{x\to\infty}\frac{1+e^{-2x}}{1-e^{-2x}} = \frac{1+0}{1-0} = 1$$

$$\lim_{x\to-\infty}\frac{e^x+e^{-x}}{e^x-e^{-x}} = \lim_{x\to-\infty}\frac{e^x+e^{-x}}{e^x-e^{-x}} \div \frac{e^{-x}}{e^{-x}} = \lim_{x\to\infty}\frac{e^{2x}+1}{e^{2x}-1} = \frac{0+1}{0-1} = -1$$

Horizontal asymptotes: $y = 1$ *(as* $x \to \infty$*)*, $y = -1$ *(as* $x \to -\infty$*)*.

Using Limits to Graph

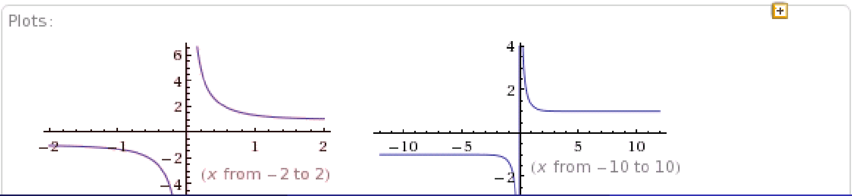
Problem

Graph $\coth(x) := \frac{e^x+e^{-x}}{e^x-e^{-x}}$. *[Recall:* $\lim_{x\to\infty} \coth(x) = 1, \ \lim_{x\to-\infty} \coth(x) = -1$*]*

Solution

The horizontal asymptotes are $y = \pm 1$*.*
To find the vertical asymptotes, see where denominator is zero and check one-sided limits.

$$\lim_{x\to 0^+} \frac{e^x + e^{-x}}{e^x - e^{-x}} = +\infty, \ \lim_{x\to 0^-} \frac{e^x + e^{-x}}{e^x - e^{-x}} = -\infty$$



Long Term Comparisons

Suppose $f, g \to \infty$ as $x \to \infty$. **Which grows faster in the long term?**

$$\boxed{\text{Key idea: } \frac{\text{smaller}}{\text{bigger}} = \text{small and } \frac{\text{bigger}}{\text{smaller}} = \text{big.}}$$

So to compare two amounts: divide either by the other. See what you get.

- If $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = \infty$ then $f \gg g$ and $g \ll f$.
  - f grows much faster than g.
  - g grows much slower than f.
- If $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = 0$ then $f \ll g$ and $g \gg f$.
  - f grows much slower than g.
  - g grows much faster than f.
- If $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = 17$ (say), then $f \& g$ grow at roughly the same rate.

Long Term Comparisons

Problem

- *Which approaches infinity more quickly:* $100x^2$ *or* $x^2 \ln(x)$*?*
- *Which approaches zero more quickly:* 2^{-x} *or* 3^{-x}*?*
- *Which approaches zero more quickly:* e^{-x} *or* $e^{-(x+1)}$*?*

To compare f,g, compute $\lim_{x \to \infty} \frac{f(x)}{g(x)}$.

Solution

- $\lim_{x \to \infty} \frac{100x^2}{x^2 \ln(x)} = \lim_{x \to \infty} \frac{100}{\ln(x)} = 0$. *So* $x^2 \ln(x)$ *approaches infinity more quickly.*
- $\lim_{x \to \infty} \frac{2^{-x}}{3^{-x}} = \lim_{x \to \infty} 1.5^x = \infty$. *So* 3^{-x} *approaches zero more quickly.*
- $\lim_{x \to \infty} \frac{e^{-x}}{e^{-(x+1)}} = \lim_{x \to \infty} e = e$. *So they approach 0 at the same rate.*

Long Term Comparisons: A warning

Caution: These few slides are about **long term growth**, i.e. $(x \to \infty)$.

- When $f \gg g$ or $f \ll g$ is written, there is always an (implicit or explicit) *context*.
- e.g. $e^x \gg e^{-x}$ as $x \to \infty$, but $e^{-x} \gg e^x$ as $x \to -\infty$.

Good exercise: list all the basic functions you know $(1/x, \sin(x), \arctan(x), \ln(x))$ and compare them as $x \to \infty$, as $x \to -\infty$, as $x \to 0^+$, as $x \to 0^-$.

Idea of Continuity

Definition

A *continuous* function is one with a "connected" graph.
Fine print: the technical term is **arc–connected**. Assumes the domain is an interval.

We'll see a more formal definition soon.

Problem

Draw a continuous and a discontinuous function.

Continuous at a Point

Before formally defining continuous, a more basic notion:

Definition

The function f is continuous at the number b if:

1. $\lim_{x \to b} f(x)$ exists;
2. $f(b)$ exists (i.e., b is in the domain of f); and
3. $\lim_{x \to b} f(x) = f(b)$

Problem

Find examples of functions that satisfy:

- *1 but not 2*
- *2 but not 1*
- *1 and 2 but not 3*

The signum function

The **signum function** tells you if a number has a plus or minus sign.

Problem

Where is the signum function continuous?

$$sgn(x) := \begin{cases} 1 & \textit{if } x > 0 \\ 0 & \textit{if } x = 0 \\ -1 & \textit{if } x < 0 \end{cases}$$

Solution

Everywhere except at $x = 0$.

Note: $|x| = \mathrm{sgn}(x) \cdot x$. In particular, $|x| = -x$ if $x < 0$. If this is counterintuitive: HW– convince yourself it's true.

Continuous on an Open Interval

Definition

f is <u>continuous on (a,b)</u> if f is continuous at each point of (a,b).

Note: a can be $-\infty$ and/or b can be $+\infty$.

Problem

- *Is* $f(x) = 1/x$ *continuous on* $(-\infty,\infty)$*?*
- *Is* $g(x) = \tan(x)$ *continuous on* $(-\pi/2, \pi, 2)$*?*

Solution

- *No.* f *is not continous at 0 because* $f(0)$ *is not defined.*
- *Yes, by the direct substitution rule. (Since domain contains* $(-\pi/2, \pi/2)$*.)*

Continuous on a Closed Interval

When checking continuity on $[c, d]$, only require *endpoint continuity* at endpoints:

Definition

The function f is continuous at the left endpoint c if:

1. $\lim\limits_{x \to c^+} f(x)$ exists;
2. $f(c)$ exists (i.e., c is in the domain of f); and
3. $\lim\limits_{x \to c^+} f(x) = f(c)$

Definition

The function f is continuous at the right endpoint d if:

1. $\lim\limits_{x \to d^-} f(x)$ exists;
2. $f(d)$ exists (i.e., d is in the domain of f); and
3. $\lim\limits_{x \to d^-} f(x) = f(d)$

Lots of Functions are Continuous

Many "basic" functions are continuous at every point of their domains:

- $e^x, \ln(x), \sin(x), \cos(x), |x|, x$, constants

Adding, subtracting, multiplying continuous functions yields continuous functions.

Dividing continuous functions yields a function continuous *on its domain*.

If f is continuous at a and g is continuous at $f(a)$ then $g \circ f$ is continuous at a.

Moral: most functions you write down by a single rule are continuous *on their domains*.

Building up Continuous Functions

Problem

Where is $\tan(x)$ *continuous?* $\cot(x)$*?* $\tan(x)\cot(x)$*?* $\tan(x^2)$*?*

Answer: on their domains. (Why?)

Solution

- $\tan(x)$ *is continuous everywhere* ***except*** $\ldots, -\frac{5\pi}{2}, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$.
- $\cot(x)$ *is continuous everywhere* ***except*** $\ldots, -2\pi, -\pi, 0, \pi, 2\pi, \ldots$.
- $\tan(x)\cot(x)$ *is continuous everywhere* ***except*** $\ldots, -\frac{5\pi}{2}, -2\pi, -\frac{3\pi}{2}, -\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi, \frac{5\pi}{2}, \ldots$.
- $\tan(x^2)$ *is continuous everywhere* ***except*** $-\sqrt{5\pi/2}, -\sqrt{3\pi/2}, -\sqrt{\pi/2}, \sqrt{\pi/2}, \sqrt{3\pi/2}, \sqrt{5\pi/2}, \ldots$.

Piecewise-Defined Functions

Problem

Is there a number c such that f is continuous everywhere? What is it? Same question for g.

$$f(x) = \begin{cases} x^2 + 2 & x < 0 \\ x - c & x \geq 0 \end{cases}$$

$$g(x) = \begin{cases} x^2 + 2 & x < 0 \\ c & x = 0 \\ x + 3 & x > 0 \end{cases}$$

Solution

For f, YES. Take $c = -2$.

For g, $\lim_{x \to 0^-} g(x) = 2 \neq 3 = \lim_{x \to 0^+} g(x)$, so NO.

Input and Output Precision

Consider the updating function $b_{t+1} = 2b_t$, t in days.

Problem

We need to know tomorrow's population to within 100,000 bacteria. We measure the current value and find 2 million bacteria. What do we predict for tomorrow? How accurate must our measurement be?

- 100,000 is called the output tolerance.
- 50,000 is called the input tolerance.
- Fancy definition of continuous function: for every input value and every desired output tolerance, there is a corresponding input tolerance that guarantees the desired output tolerance.
- What happens at a jump discontinuity?

Tangential remark: scientists sometimes use **the derivative** to approximate the (output tolerance : input tolerance) ratio.