Discrete Mathematics with Applications I

COMPSCI&SFWRENG 2DM3

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Plan for Today

- CALCCHECK$_{\text{Web}}$ (interacting with Exercises 1.1, 1.2)
- Meaning of Boolean Operators
- Conjunctional Operators

Truth Values

Boolean constants/values: *false*, *true*

The type of Boolean values: $\mathbb{B}$

— This is the type of propositions, for example: $(x = 1) : \mathbb{B}$

— For any type t, equality $\_{=}\_$ can be used on expressions of that type: $\_{=}\_ : t \to t \to \mathbb{B}$

Boolean operators:

- $\neg\_ : \mathbb{B} \to \mathbb{B}$ — negation, complement, "logical not"
- $\_\wedge\_ : \mathbb{B} \to \mathbb{B} \to \mathbb{B}$ — conjunction, "logical and"
- $\_\vee\_ : \mathbb{B} \to \mathbb{B} \to \mathbb{B}$ — disjunction, "logical or"
- $\_\Rightarrow\_ : \mathbb{B} \to \mathbb{B} \to \mathbb{B}$ — implication, "implies", "if … then …"
- $\_\equiv\_ : \mathbb{B} \to \mathbb{B} \to \mathbb{B}$ — equivalence, "if and only if", "iff"
- $\_\not\equiv\_ : \mathbb{B} \to \mathbb{B} \to \mathbb{B}$ — inequivalence, "exclusive or"

Binary Boolean Operators: Conjunction

Args.		$\wedge$	
F	F	F	The moon is green, and $2+2=7$.
F	T	F	The moon is green, and $1+1=2$.
T	F	F	$1+1=2$, and the moon is green.
T	T	T	$1+1=2$, and the sun is a star.

Binary Boolean Operators: Disjunction

Args.		$\vee$	
F	F	F	The moon is green, or $2+2=7$.
F	T	T	The moon is green, or $1+1=2$.
T	F	T	$1+1=2$, or the moon is green.
T	T	T	$1+1=2$, or the sun is a star.

This is known as "inclusive or" — see textbook p.34.

Some Laws for the Boolean Operators

(3.12) **Double negation**: $\neg\neg p \equiv p$
(3.36) **Symmetry of $\wedge$**: $p \wedge q \equiv q \wedge p$
(3.37) **Associativity of $\wedge$**: $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
(3.38) **Idempotency of $\wedge$**: $p \wedge p \equiv p$
(3.39) **Identity of $\wedge$**: $p \wedge true \equiv p$
(3.40) **Zero of $\wedge$**: $p \wedge false \equiv false$
(3.42) **Contradiction**: $p \wedge \neg p \equiv false$

(3.24) **Symmetry of $\vee$**: $p \vee q \equiv q \vee p$
(3.25) **Associativity of $\vee$**: $(p \vee q) \vee r \equiv p \vee (q \vee r)$
(3.26) **Idempotency of $\vee$**: $p \vee p \equiv p$
(3.29) **Zero of $\vee$**: $p \vee true \equiv true$
(3.30) **Identity of $\vee$**: $p \vee false \equiv p$
(3.28) **Excluded Middle**: $p \vee \neg p$

(3.45) **Distributivity of $\vee$ over $\wedge$**: $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
(3.46) **Distributivity of $\wedge$ over $\vee$**: $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
(3.47) **De Morgan**: $\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\quad$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$

Truth Values and Equivalence

Boolean constants/values: *false*, *true*

The set/type of Boolean values: $\mathbb{B}$

Equality of Boolean values is also called **equivalence** and written $\equiv$

$p \equiv q$ can be read as: p is equivalent to q
or: p exactly when q
or: p if-and-only-if q
or: p iff q

p	q	$p \equiv q$	
false	*false*	*true*	The moon is green **iff** $2 + 2 = 7$.
false	*true*	*false*	The moon is green **iff** $1 + 1 = 2$.
true	*false*	*false*	$1 + 1 = 2$ **iff** the moon is green.
true	*true*	*true*	$1 + 1 = 2$ **iff** the sun is a star.

Table of Precedences

- $[x := e]$ (textual substitution) **(highest precedence)**
- $.$ (function application)
- unary prefix operators $+$, $-$, $\neg$, $\#$, $\sim$, $\mathcal{P}$
- $**$
- $\cdot$ $/$ $\div$ **mod** **gcd**
- $+$ $-$ $\cup$ $\cap$ $\times$ $\circ$ $\bullet$
- $\downarrow$ $\uparrow$
- $\#$
- $\triangleleft$ $\triangleright$ $\hat{}$
- $=$ $\neq$ $<$ $>$ $\in$ $\subset$ $\subseteq$ $\supset$ $\supseteq$ $|$ (conjunctional)
- $\vee$ $\wedge$
- $\Rightarrow$ $\not\Rightarrow$ $\Leftarrow$ $\not\Leftarrow$
- $\equiv$ $\not\equiv$ **(lowest precedence)**

All non-associative binary infix operators associate to the left, except $**$, $\triangleleft$, $\Rightarrow$, $\rightarrow$, which associate to the right.

Conjunctional Operators

Chains can involve different conjunctional operators:

$$
\begin{aligned}
& 1 < i \leq j < 5 = k \\
\equiv\ & \langle \text{ conjunctional operators } \rangle \\
& 1 < i \quad \wedge \quad i \leq j \quad \wedge \quad j < 5 \quad \wedge \quad 5 = k \\
\equiv\ & \langle \quad \wedge \quad \text{has lower precedence } \rangle \\
& (1 < i) \quad \wedge \quad (i \leq j) \quad \wedge \quad (j < 5) \quad \wedge \quad (5 = k)
\end{aligned}
$$

$$
\begin{aligned}
& x < 5 \in S \subseteq T \\
\equiv\ & \langle \text{ conjunctional operators } \rangle \\
& x < 5 \quad \wedge \quad 5 \in S \quad \wedge \quad S \subseteq T \\
\equiv\ & \langle \quad \wedge \quad \text{has lower precedence } \rangle \\
& (x < 5) \quad \wedge \quad (5 \in S) \quad \wedge \quad (S \subseteq T)
\end{aligned}
$$

Equality versus Equivalence

The operators $=$ (as Boolean operator) and $\equiv$

- have the **same meaning** (represent the same function),
- but **are used with different notational conventions:**
  - different precedences ($\equiv$ has lowest)
  - different **chaining behaviour**:
    - $\equiv$ is associative:

$$\boxed{(p \equiv q \equiv r) \quad = \quad ((p \equiv q) \equiv r) \quad = \quad (p \equiv (q \equiv r))}$$

    - $=$ is **conjunctional**:

$$\boxed{(p = q = r) \qquad = \qquad ((p = q) \quad \wedge \quad (q = r))}$$

Binary Boolean Operators: Equivalence

Args.		$\equiv$	
F	F	T	The moon is green **iff** $2 + 2 = 7$.
F	T	F	The moon is green **iff** $1 + 1 = 2$.
T	F	F	$1 + 1 = 2$ **iff** the moon is green.
T	T	T	$1 + 1 = 2$ **iff** the sun is a star.

Binary Boolean Op.: Inequivalence ("exclusive or")

Args.		$\not\equiv$	
F	F	F	Either the moon is green, or $2 + 2 = 7$.
F	T	T	Either the moon is green, or $1 + 1 = 2$.
T	F	T	Either $1 + 1 = 2$, or the moon is green.
T	T	F	Either $1 + 1 = 2$, or the sun is a star.

Binary Boolean Operators: Implication

Args.		$\Rightarrow$	
F	F	T	If the moon is green, then $2 + 2 = 7$.
F	T	T	If the moon is green, then $1 + 1 = 2$.
T	F	F	If $1 + 1 = 2$, then the moon is green.
T	T	T	If $1 + 1 = 2$, then the sun is a star.

$$p \Rightarrow q \qquad \equiv \qquad \neg p \vee q$$

If you don't eat your spinach, I'll spank you.	$\equiv$	You eat your spinach, or I'll spank you.

Binary Boolean Operators: Consequence

Args.		$\Leftarrow$	
F	F	T	The moon is green **if** $2 + 2 = 7$.
F	T	F	The moon is green **if** $1 + 1 = 2$.
T	F	T	$1 + 1 = 2$ **if** the moon is green.
T	T	T	$1 + 1 = 2$ **if** the sun is a star.

$$p \Leftarrow q \qquad \equiv \qquad p \vee \neg q$$