**COMPSCI/SFWRENG 2FA3**
**Discrete Mathematics with Applications II**
**Winter 2020**

# Week 02 Exercises

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## Background Definitions

1. The notation $\sum_{i=m}^{n} f(i)$ is defined by:

$$\sum_{i=m}^{n} f(i) = \begin{cases} 0 & \text{if } m > n \\ f(n) + \sum_{i=m}^{n-1} f(i) & \text{if } m \leq n \end{cases}$$

2. The Fibonacci sequence $\mathsf{fib} : \mathbb{N} \to \mathbb{N}$ is defined by:

$$\mathsf{fib}(n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ \mathsf{fib}(n-1) + \mathsf{fib}(n-2) & \text{if } n \geq 2 \end{cases}$$

3. Let $a, b \in \mathbb{Z}$. $a$ *divides* $b$, written $a \mid b$, if $b = ac$ for some $c \in \mathbb{Z}$.

## Exercises

1. Prove the following statements:

   a. The sum of two odd integers is an even integer.
   b. If $x$ is an even integer, then $x^2$ is also even.
   c. Let $a, b, c, d \in \mathbb{Z}$. If $a \mid b$ and $c \mid d$, then $ac \mid bd$.
   d. The square root of 2 is an irrational number.

2. Prove the following statements by weak induction:

   a. $\sum_{i=0}^{n} 2i = n(n+1)$ for all $n \in \mathbb{N}$.
   b. $\sum_{i=1}^{n} (2i-1) = n^2$ for all $n \in \mathbb{N}$.
   c. $\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ for all $n \in \mathbb{N}$.
   d. $\sum_{i=0}^{n-1} 2^i = 2^n - 1$ for all $n \in \mathbb{N}$.

e. $\sum_{i=0}^{n} \mathsf{fib}(i) = \mathsf{fib}(n+2) - 1$ for $n \in \mathbb{N}$.

f. $\sum_{i=0}^{n} (\mathsf{fib}(i))^2 = \mathsf{fib}(n) * \mathsf{fib}(n+1)$ for all $n \in \mathbb{N}$.

3. Prove the following statements by strong induction:

   a. If $n \in \mathbb{N}$ with $n \geq 2$, then $n$ is a product of prime numbers.

   b. $\mathsf{fib}(n) < 2^n$ for all $n \in \mathbb{N}$.

   c. It takes $n - 1$ divisions to break up a rectangular chocolate bar containing $n$ squares into individual squares.

4. Let $t_n, s_n, o_n$ be the $n$th triangle, square, and oblong numbers, respectively, where $n \in \mathbb{N}$.

   a. Define $t_n, s_n, o_n$ by recursion.

   b. Prove by induction that every triangle number is exactly half of an oblong number.

   c. Prove by induction that the sum of every two consecutive triangle numbers is a square number.