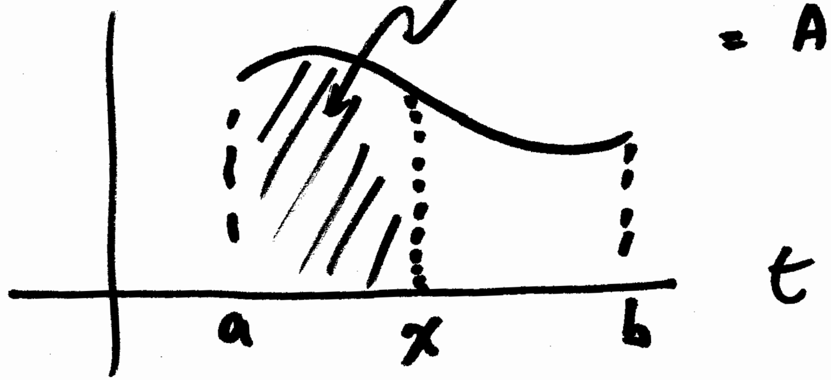
12A3

FTC (Part #1) Fundamental Theorem of Calculus

If $f(x)$ cont. on $[a, b]$ & $g(x) = \int_a^x f(t)\,dt$

then $\frac{d}{dx} g(x) = f(x)$



$g(x) = \int_a^x f(t)\,dt$

= Area, as a fcn. of x

if $f(x) \geq 0$

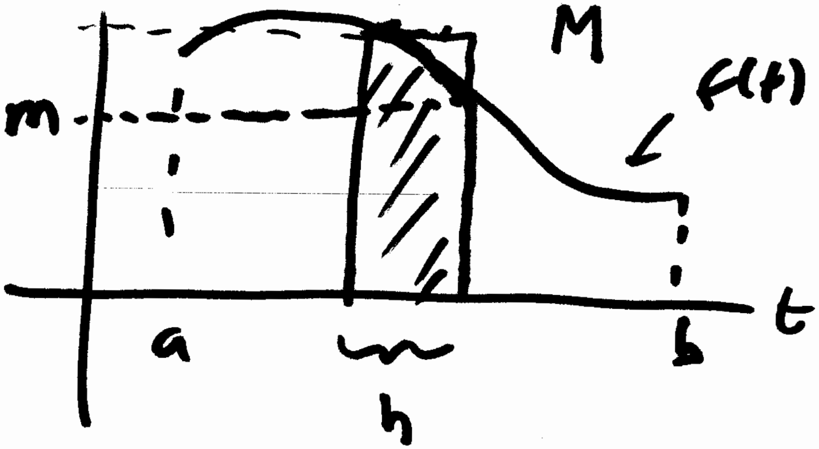
"proof"

$$g(x) = \int_a^x f(t)\,dt$$

$$\frac{d}{dx} g(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$= \lim_{h \to 0} \frac{\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt}{h}$$

$$= \lim_{h \to 0} \int_x^{x+h} f(t)\,dt \Big/ h$$



$$\frac{mh}{h} \le \frac{\int_x^{x+h} f(t)\,dt}{h} \le \frac{Mh}{h}$$

$$\lim_{h \to 0} \cancel{m} \le \lim_{h \to 0} \int_x^{x+h} \cancel{f}(t)\,dt \Big/ h \le \lim_{h \to 0} \cancel{M}$$

$$f(x) \leq g'(x) \leq f(x)$$

Sol $g'(x) = f(x)$

eg. $\dfrac{d}{dx}\left(\displaystyle\int_{-\pi^{48}}^{x} \underbrace{\tanh(t + \arctan(t^3 - 5))}_{f(t)}\, dt\right)$ "$g(x)$"

$= \tanh(x + \arctan(x^3 - 5))$

$g'(x) = f(x)$

Remember! if $g(x) = \int_a^x f(t)\, dt$

$\Rightarrow \underline{g'(x) = f(x)}$

eg: $\frac{d}{dx} \int_{1/5}^{x^2} \sin^{-1}(t)\,dt$ ← oh oh! not $\underline{\underline{x}}$

Consider $g(x) = \int_{1/5}^{x} \sin^{-1}(t)\,dt = \int_a^x f(t)\,dt$

by FTC $g'(x) = f(x) = \underline{\underline{\sin^{-1}(x)}}$.

$= \frac{d}{dx} g(x^2) = g'(x^2) \cdot 2x = f(x^2) \cdot 2x$

$= \underline{\underline{2x \sin^{-1}(x^2)}}$.

In general: If I have $\frac{d}{dx} \int_a^{b(x)} f(t)\,dt$

↑ const.

$\underline{\underline{\text{thcm}}}$ $g(x) = \int_a^x f(t)\,dt \Rightarrow g'(x) = f(x)$

by FTC.

$$\Rightarrow \boxed{\frac{d}{dx} \int_a^{b(x)} f(t)\,dt = f(b(x))\,b'(x)}$$

$$= g'(b(x))\,b'(x)$$

$\underline{\underline{\text{eg.}}}$

$$\frac{d}{dx} \int_{\sin x}^{57} \sqrt{1+t^3}\,dt$$

$$= \frac{d}{dx} (-1) \int_{57}^{\sin x} \sqrt{1+t^3}\,dt$$

$$= -\frac{d}{dx}\, g(\sin x) = -g'(\sin x) \cdot \cos x$$

$$= -f(\sin x)\cos x$$

$$= -\left(\sqrt{1+\sin^3 x}\right)\cos x$$

So In general

$$\frac{d}{dx}\int_{a(x)}^{b} f(t)\,dt \quad \Rightarrow \quad \text{let } g(x) = \int_{b}^{x} f(t)\,dt$$

$b \leftarrow b$ const.

$$\Rightarrow g'(x) = f(x)$$

$$= \frac{d}{dx}\left(-g(a(x))\right) = -g'(a(x))\,a'(x)$$

$$= -\underline{f(a(x))\,a'(x)}$$

So

$$\boxed{\frac{d}{dx}\int_{a(x)}^{b} f(t)\,dt = -f(a(x))\,a'(x)}$$

eg $\displaystyle \frac{d}{dx} \int_{x^2}^{x^3} \tanh(t)\,dt$

$$= \frac{d}{dx} \int_0^{x^3} \tanh(t)\,dt + \frac{d}{dx} \int_{x^2}^{0} \tanh(t)\,dt$$

$$= \tanh(x^3) \cdot 3x^2 + (-1) \tanh(x^2) \cdot 2x$$

$$= \tan(x^3) \cdot 3x^2 - \tanh(x^2) \cdot 2x.$$

In general

$$\boxed{\frac{d}{dx} \int_{a(x)}^{b(x)} f(t)\,dt = f(b(x))\,b'(x) - f(a(x))\,a'(x)}$$

FTC pt 2

$$g(x) = \int_a^x f(t)\,dt$$

$$\int_a^b f(t)\,dt = g(b)$$

$$g(a) = \int_a^a f(t)\,dt = 0$$

$$= g(b) - g(a)$$

Notice

Say $F(x)$ is any antideriv. of $f(x)$ $\Rightarrow$ $\frac{d}{dx} F(x) = f(x)$

but $g'(x) = f(x)$ too! $\Rightarrow$ $g(x) = F(x) + c$

for some c constant!

$$\int_a^b f(t)\,dt = g(b) - g(a) = F(b) + \cancel{c} - (F(a) + \cancel{c})$$

$$= F(b) - F(a)$$

So FTC #2 is:

If $f(x)$ is cont. on $[a,b]$

$$\Rightarrow \int_a^b f(x)\,dx = F(b) - F(a)$$

$$= F(x)\Big|_a^b \quad = \text{net change!}$$

$$= \int f(x)\,dx \Big|_a^b$$

eg. $\int_0^6 x^2\,dx = \frac{1}{3}x^3\Big|_0^6 = \frac{1}{3}6^3 - \frac{1}{3}(0^3) = \boxed{\frac{6^3}{3}}$

eg. $$\int_1^{\sqrt{3}} \frac{1}{1+x^2}\,dx = \arctan x \Big|_1^{\sqrt{3}}$$

$$= \tan^{-1}(\sqrt{3}) - \tan^{-1}(1)$$

$$= \pi/3 - \pi/4 = \pi/12$$

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eg. $$\int_1^e x^2 - 17x + \frac{1}{x}\,dx$$

$$= \frac{1}{3}x^3 - \frac{17}{2}x^2 + \ln x \Big|_1^e$$

$$= \left(\frac{1}{3}e^3 - \frac{17}{2}e^2 + 1\right) - \left(\frac{1}{3} - \frac{17}{2} + 0\right)$$

or

$$= \frac{1}{3}x^3 \Big|_1^e - \frac{17}{2}x^2 \Big|_1^e + \ln x \Big|_1^e$$

$$= \frac{1}{3}(e^3-1) - \frac{17}{2}(e^2-1) + 1 - 0 \checkmark$$

eg. $$\int_1^e \frac{(\ln x)^2}{x}\,dx$$

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

$$\frac{1}{3}\frac{d}{dx}(\ln x)^3 = \cancel{\frac{3}{3}}(\ln x)^2/x$$

$$= \frac{1}{3}(\ln x)^3 \Big|_1^e$$

$$= \frac{1}{3}(1^3 - 0^3) = \frac{1}{3}$$

Need an easier trick to reverse chain rule!

Let's look at a chain rule integral!

$$\frac{d}{dx} f(g(x)) = f'(g(x))\, g'(x)$$

$$\int f'(g(x))\, g'(x)\, dx = f(g(x)) + C$$

In Leibnitz, let $g(x) = u \Rightarrow g'(x) = du/dx$

$$\int f'(u) \frac{du}{\cancel{dx}} \cancel{dx} = f(u) + C$$

$$= \int f'(u)\, du$$