Announcements

Topics:

- Analysis of Autonomous DEs – Population Models (8.1)
- Equilibria, Phase-Line Diagrams, Stability (8.2 and 8.3)
- Solving Separable DEs (8.4)

To Do:

- Read sections 8.1, 8.2, 8.3, and 8.4 in the textbook
- Work on Assignments and Suggested Practice Problems assigned on the webpage under the SCHEDULE + HOMEWORK link

Equilibria

Definition:
A value, m^*, of the state variable is called an **equilibrium** of the autonomous differential equation

$$\frac{dm}{dt} = f(m)$$

if $f(m^*) = 0$.

Equilibria

Example:

Find the equilibria of the following autonomous DEs.

(a) $\dfrac{dx}{dt} = 1 - e^{x}$

(b) Selection Model

$$\frac{dp}{dt} = (\mu - \lambda)p(1 - p)$$

Phase-Line Diagrams

- A **phase-line diagram** is a graphical display of the qualitative behaviour of solutions to autonomous DEs.

- A phase-line diagram summarizes where the state variable is increasing, where it is decreasing, and where it is unchanged.

Phase-Line Diagrams

<u>Construction</u>:

The horizontal line represents the state variable.

For values of the state variable where the DE=0, large dots are drawn to indicate equilibrium solutions.

For values of the state variable where the DE is positive, right-pointing arrows are drawn to indicate that the solution is increasing.

For values of the state variable where the DE is negative, left-pointing arrows are drawn to indicate that the solution is decreasing.

The size of the arrows corresponds to the magnitude of the rate of change.

Phase-Line Diagrams

Example:

Newton's Law of Cooling

$$\frac{dT}{dt} = \alpha(A - T)$$

Phase-Line Diagrams

Example:

For the autonomous DE:

$$\frac{dx}{dt} = 1 - e^{x}$$

(i) Graph *dx/dt* as a function of *x*.

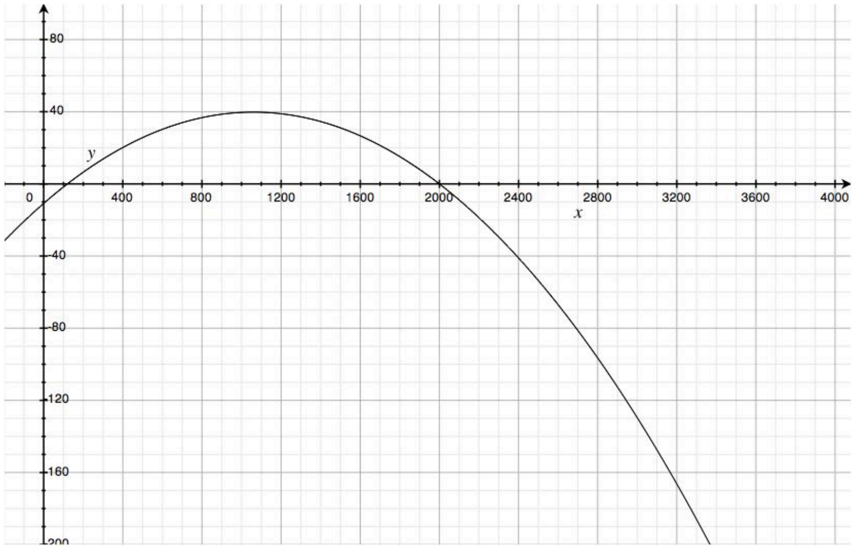
(ii) Create a phase-line diagram.

(iii) Draw some solution curves.

Phase-Line Diagrams

Example:
Draw a phase-line diagram for the modified logistic model

$$\frac{dP}{dt} = 0.09P\left(1 - \frac{P}{2000}\right)\left(1 - \frac{120}{P}\right)$$



Stability

Definition:

An equilibrium of an autonomous DE is **stable** if solutions that begin near the equilibrium approach the equilibrium.

An equilibrium of an autonomous DE is **unstable** if solutions that begin near the equilibrium move away from the equilibrium.

Stability Theorem for Autonomous DEs

Suppose $$\frac{dm}{dt} = f(m)$$

is an autonomous DE with an equilibrium at m^*.

The equilibrium at m^* is **stable** if $f'(m^*) < 0$

and **unstable** if $f'(m^*) > 0$.

Whenever $f'(m^*) = 0$ the stability theorem does not apply and we must analyze it another way.

Stability Theorem for Autonomous DEs

Examples:

Use the stability theorem to determine whether the equilibria in the following models are stable or unstable. Compare with your phase-line diagrams.

(a) $\dfrac{dP}{dt} = 0.08P\left(1 - \dfrac{P}{1000}\right)$

(b) $\dfrac{dx}{dt} = 1 - e^{x}$

A Model for a Disease (starts on p. 614 in your textbook)

Suppose a disease is circulating in a population. Individuals recover from this disease unharmed but are susceptible to reinfection.

Let I denote the fraction of infected individuals in a population.

Then, the rate at which the fraction of infected individuals is changing is given by

$$\frac{dI}{dt} = \alpha I(1 - I) - \mu I$$

where α and μ are positive constants.

A Model for a Disease
(starts on p. 614 in your textbook)

Suppose we start with a few infected individuals, i.e., a small value of I, will this disease ever die out?

Separable Differential Equations

A **separable differential equation** is a differential equation in which we can separate the unknown function (also called the state variable) from the independent variable.

The general form of a separable equation is

$$\frac{dy}{dx} = g(x) \cdot h(y)$$

where *h(y)* is a function of *y* only, and *g(x)* is a function of *x* only.

Separable Differential Equations

It is possible to find an algebraic solution for these types of DEs.

To solve this equation, first **separate the equation** so that all *y*'s are on one side of the equation and all *x*'s are on the other side:

$$f(y)dy = g(x)dx \quad \text{where} \quad f(y) = \frac{1}{h(y)}$$

Separable Differential Equations

Next, **integrate both sides of the equation**:

$$\int f(y)dy = \int g(x)dx$$

This equation defines *y* implicitly as a function of *x*.

Finally, **solve this equation for *y* in terms of *x***, if possible.

Separation of Variables

Example:

Solve the pure-time differential equation

$$\frac{dP}{dt} = \frac{5}{1+2t}, \quad P(0) = 0$$

Separation of Variables

Example:
Solve the following autonomous differential equations.

(a) $\dfrac{dP}{dt} = 1 + P^2, \quad P(0) = 1$

(b) <u>Exponential Growth Model</u>

$$\frac{dP}{dt} = kP, \quad P(0) = P_0$$

Separation of Variables

Example:

Solve the following "mixed" differential equations.

(a) $\frac{dy}{dt} = -8ty^2, \quad y(0) = 3$

(b) $P' - 2P\sqrt{t} = 0, \quad P(0) = 250$

Solution of Logistic Model

Read example 8.4.5 in your textbook (starting on p. 623) which solves the logistic differential equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right)$$

Solution of Logistic Model

Result:

The solution of the logistic equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right)$$

with initial condition $P(0)=P_0$ is

$$P(t) = \frac{L}{1 + \frac{L-P_0}{P_0} e^{-kt}}$$

Solution of Logistic Model

Recall Previous Example:

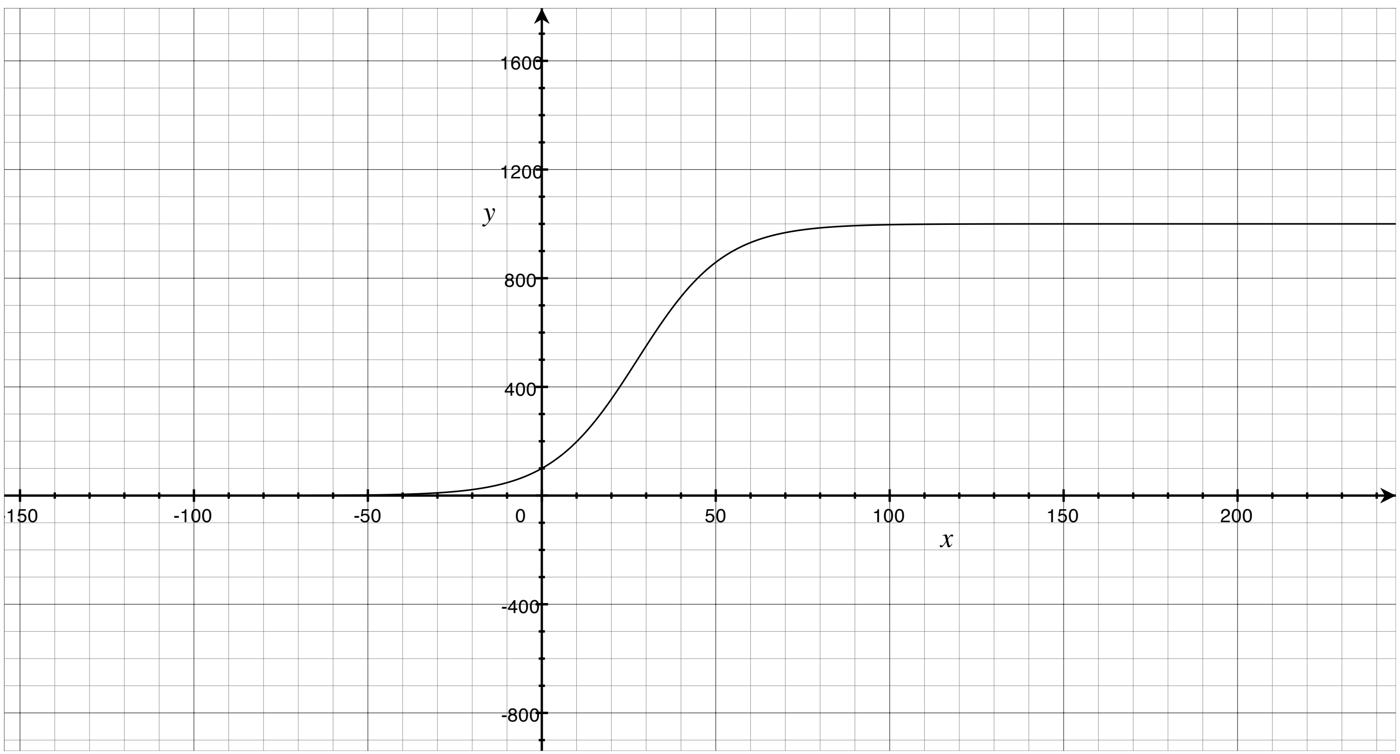
A population grows according to the logistic model

$$\frac{dP}{dt} = 0.08P\left(1 - \frac{P}{1000}\right)$$

with initial population $P(0) = 100.$

Solution of Logistic Model

The solution to this IVP is $P(t) = \dfrac{1000}{1 + \frac{1000-100}{100} e^{-0.08t}}$



Solution of Selection Model

Example p. 628, #24.
The rate of spread of a virus is proportional to the product of the fraction, $I(t)$, of the population who are infected and the fraction of those who are not yet infected, where t is the time in days.

(a) Write an autonomous DE for $I(t)$.
(b) Suppose that initially 15 people of the total population of 500 were infected and that 3 days later, 125 people were infected. Solve the equation.