MATH 1LT3E

A#3 SOL^{N}S

#1. (a) $\frac{dy}{dx} = \frac{y^2 \cos x}{1+y^2}$

eq^m sol^{n}s: $\frac{dy}{dx} = 0$ when $\frac{y^2 \cos x}{1+y^2} = 0 \Rightarrow y^2 \cos x = 0 \Rightarrow \boxed{y=0}$

all other sol^{n}s:

$$\int \frac{1+y^2}{y^2}\,dy = \int \cos x\,dx$$

$$\Rightarrow \int (y^{-2}+1)\,dy = \int \cos x\,dx$$

$$\Rightarrow \frac{y^{-1}}{-1} + y = \sin x + C$$

$$\Rightarrow \boxed{y - \frac{1}{y} = \sin x + C} \quad \text{implicit sol}^n$$

(b) $(x^2+1)\frac{dy}{dx} = xy \Rightarrow \frac{dy}{dx} = \frac{xy}{x^2+1}$

eq^m sol^n: $\frac{dy}{dx} = 0$ when $\frac{xy}{x^2+1} = 0 \Rightarrow \boxed{y=0}$

all other sol^{n}s:

$$\int \frac{1}{y}\,dy = \int \frac{x}{x^2+1}\,dx$$

aside: Let $u = x^2+1$. Then $du = 2x\,dx \Rightarrow dx = \frac{du}{2x}$

$$\ln|y| = \int \frac{x}{u}\cdot\frac{du}{2x}$$

$$= \frac{1}{2}\ln|u| + C$$

$$= \frac{1}{2}\ln(x^2+1) + C$$

aside: $a\ln b = \ln b^a$

$$= \ln\sqrt{x^2+1} + C$$

$$\Rightarrow |y| = \sqrt{x^2+1}\cdot e^C$$

$$\Rightarrow y = \pm e^C\cdot\sqrt{x^2+1} = k\sqrt{x^2+1} \text{ where } k = \pm e^C.$$

#2. (a) $\frac{dP}{dt} = P(1+t) + 1(t+1) = (t+1)(P+1)$

$$\int \frac{1}{P+1}\,dP = \int (t+1)\,dt$$

$$\ln|P+1| = \frac{t^2}{2} + t + C$$

$$|P+1| = e^{\frac{t^2}{2}+t+C}$$

$$P+1 = \pm e^C \cdot e^{\frac{t^2}{2}+t}$$

$$P = k \cdot e^{\frac{t^2}{2}+t} - 1 \quad \text{where } k = \pm e^C$$

$$P(0) = 50 \Rightarrow 50 = k \cdot e^0 - 1 \Rightarrow k = 51$$

$$\therefore P(t) = 51 e^{\frac{t^2}{2}+t} - 1$$

(b) $\int (2y + e^{3y})\,dy = \int x \cos x\,dx$

aside:
$u = x \quad dv = \cos x\,dx$
$du = dx \quad v = \sin x$

$$y^2 + \frac{1}{3}e^{3y} = x\sin x - \int \sin x\,dx$$

$$= x\sin x + \cos x + C$$

$$y(0) = 0 \Rightarrow 0^2 + \frac{1}{3}e^0 = 0 \cdot \sin 0 + \cos 0 + C \Rightarrow C = -\frac{2}{3}$$

$$\therefore y^2 + \frac{1}{3}e^{3y} = x\sin x + \cos x - \frac{2}{3}$$

$$\Rightarrow 3y^2 + e^{3y} = 3x\sin x + 3\cos x - 2 \quad \left(\text{implicit sol}^{n}\right)$$

#3 (a) $\frac{dp}{dt} = p(1-p)$

$$\int \frac{1}{p(1-p)}\,dp = \int 1\,dt$$

$$\Rightarrow \int \left(\frac{1}{p} + \frac{1}{1-p}\right)dp = \int 1\,dt$$

$$\ln|p| - \ln|1-p| = t + C$$

$$\ln\left|\frac{p}{1-p}\right| = t + C$$

$$\left|\frac{p}{1-p}\right| = e^{t+C}$$

$$\frac{p}{1-p} = \pm e^{c} \cdot e^{t}$$

$$\frac{1}{\pm e^{c} \cdot e^{t}} = \frac{1-p}{p}$$

call this "K" $\rightarrow$ $\frac{1}{\pm e^{c}} \cdot e^{-t} = \frac{1}{p} - 1$

$$K \cdot e^{-t} + 1 = \frac{1}{p}$$

$$\Rightarrow p = \frac{1}{Ke^{-t}+1} \quad \text{where } K = \pm\frac{1}{e^{c}}$$

<u>aside</u>:

$$\frac{1}{p(1-p)} = \frac{A}{p} + \frac{B}{1-p}$$

$$= \frac{A(1-p) + Bp}{p(1-p)}$$

$$\Rightarrow 1 = A(1-p) + Bp$$

when $p = 0$, $\boxed{1 = A}$

when $p = 1$, $\boxed{1 = B}$

$$\therefore \frac{1}{p(1-p)} = \frac{1}{p} + \frac{1}{1-p}$$

(b) $p(0) = 0.01 \Rightarrow 0.01 = \frac{1}{Ke^{0}+1} \Rightarrow K+1 = \frac{1}{0.01} = 100$

so, $K = 99$

$$\therefore p(t) = \frac{1}{99e^{-t}+1}$$

$$\lim_{t\to\infty} p(t) = \frac{1}{99\underbrace{e^{-\infty}}_{0}+1} = 1$$

(f) $t_0 = 0$ $\quad h = 6$

$C_0 = 15$

$M_0 = 40$

$t_1 = t_0 + h = 6$

$C_1 = C_0 + \left.\frac{dC}{dt}\right|_{\substack{C=C_0 \\ M=M_0}} \cdot h = 15 + \left[-0.05(15) + 0.0001(15)(40)\right]6 \cong 11$

$M_1 = M_0 + \left.\frac{dM}{dt}\right|_{\substack{C=C_0 \\ M=M_0}} \cdot h = 40 + \left[0.1(40) - 0.005(15)(40)\right]6 \cong 46$

$t_2 = t_1 + h = 12$

$C_2 = 11 + \left[-0.05(11) + 0.0001(11)(46)\right]6 \approx 8$

$M_2 = 46 + \left[0.1(46) - 0.005(11)(46)\right]6 \approx 59$

$t_3 = 18$ $\quad t_4 = 24$

$C_3 = 6$ $\quad C_4 = 4$

$M_3 = 80$ $\quad M_4 = 114$

$\therefore$ After two years, there will be approximately 4 cats and 114 mice.

(g) In 2 years, (24 months), the pop^n of mice will be approximately 139 and the pop^n of cats will be approximately 5 cats.

In 7 years (84 months), there will be approximately 54 cats and 61 mice.

#3. (c) $p(0) = 0.5 \Rightarrow 0.5 = \dfrac{1}{Ke^{0}+1} \Rightarrow K+1 = \dfrac{1}{0.5} = 2 \Rightarrow K = 1$

$\therefore \; P(t) = \dfrac{1}{1e^{-t}+1}$

$\lim_{t\to\infty} p(t) = \dfrac{1}{1e^{-\infty}+1} = 1$

Note: The selection eq^{n} $\frac{dp}{dt} = p(1-p)$ has two equilibrium sol^{n}s: $p = 0$ and $p = 1$.

By the stability thm,

$f'(p) = 1 - 2p$

$f'(0) = 1 > 0 \Rightarrow p^* = 0$ is unstable

$f'(1) = -1 < 0 \Rightarrow p^* = 1$ is stable.

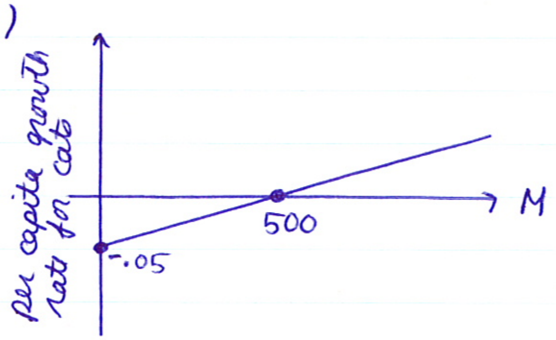
So, for any $p(t) > 0$ ($p(t) = \frac{1}{Ke^{-t}+1} > 0$ for any K)

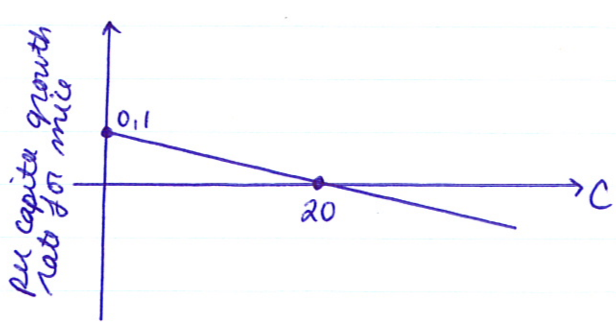
as $t \to \infty$, $p(t) \to 1$ since this is the stable eq^{m}.

#4. (a) $\dfrac{dC}{dt} = -0.05C + 0.0001MC$

$\dfrac{dM}{dt} = 0.1M - 0.005MC$

(b)



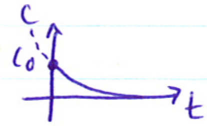


$\dfrac{\frac{dC}{dt}}{C} = 0.0001M - 0.05$

$\dfrac{\frac{dM}{dt}}{M} = -0.005C + 0.1$

(c) (i) $C>0,\ M=0$:

$\frac{dC}{dt} = -0.05C$ ... cat pop^N will die out

(sol^N: $C(t) = C_0 e^{-0.05t}$ )

$\frac{dM}{dt} = 0$ ... mouse pop^N remains at 0

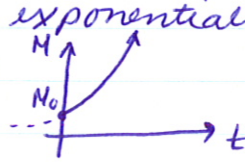
(sol^N: $M(t) = 0$ for all $t \geq 0$)

(ii) $M>0,\ C=0$:

$\frac{dC}{dt} = 0$ ... cat pop^N remains at 0

(sol^N: $C(t) = 0$ for all $t \geq 0$)

$\frac{dM}{dt} = 0.1M$ ... mouse pop^N grows exponentially

(sol^N: $M(t) = M_0 e^{0.1t}$ )

(d) $\frac{dC}{dt} = 0$ and $\frac{dN}{dt} = 0$ when either $C=0$ and $M=0$ (both pop^{N}s starting at 0 will remain at 0 forever) or $C=20$ and $M=500$. The second pair represents a ecological equilibrium of these two pop^{N}s.

(e) $C_0 = 15,\ M_0 = 40$:

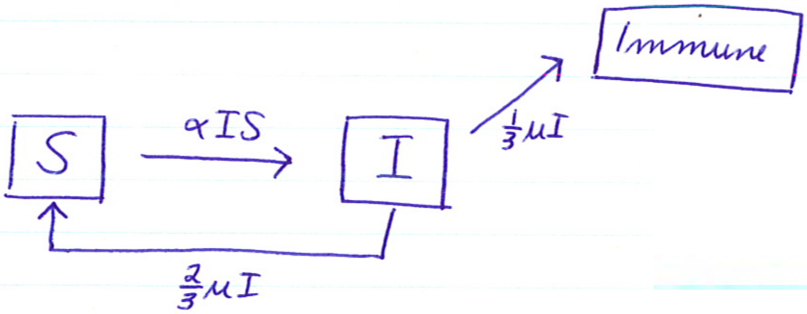
Since the cat-to-mouse ratio is higher than it would be if these pop^{N}s were in a state of eq^N, I would expect both pop^{N}s to decrease in the immediate future.

Once the cat pop^N falls low enough, the mice pop^N will have a chance to grow and eventually, the cat pop^N will increase too. Long-term, these pop^{N}s will continue to oscillate over time.

#5. (a) x and y represent two species that cooperate for mutual benefit. The growth rate of both species is increased by interact^{n}s between x and y since the coefficients of the "xy" term is positive in both eq^{n}s. (Note: The growth rate of pop^n x is decreased by interactions w/ members of its own species as indicated by the negative coefficient of the "$x \cdot x$" term)

(b) x and y represent two species that compete for the same resources. The presence of y ($y>0$) will have a negative effect on the growth rate of x ("$-0.006xy$") and vice versa. Also, within each species there is competition for resources indicated by the "$-0.0002x^2$" and "$-0.00008y^2$" terms which decrease $\frac{dx}{dt}$ and $\frac{dy}{dt}$ respectively.

#6.



$$\frac{dI}{dt} = \alpha IS - \mu I$$

$$\frac{dS}{dt} = -\alpha IS + \frac{2}{3}\mu I$$