COMPSCI/SFWRENG 2FA3
Discrete Mathematics with Applications II
Winter 2020

Extra Credit Assignment 3

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Extra Credit Assignment 3 consists of three problems concerning Kleene algebas. You must write your solution to the problem using LaTeX.

Please submit Extra Credit Assignment 3 as two files, `EC_Assignment_3_`*`YourMacID`*`.tex` and `EC_Assignment_3_`*`YourMacID`*`.pdf`, to the Extra Credit Assignment 3 folder on Avenue under Assessments/Assignments. *`YourMacID`* must be your personal MacID (written without capitalization). The `EC_Assignment_3_`*`YourMacID`*`.tex` file is a copy of the LaTeX source file for this assignment (`EC_Assignment_3.tex` found on Avenue under Contents/Assignments) with your solution entered after the problem. The `EC_Assignment_3_`*`YourMacID`*`.pdf` is the PDF output produced by executing

```
pdflatex EC_Assignment_3_YourMacID
```

This assignment is due **Sunday, March 8, 2020 before midnight.** You are allow to submit the assignment multiple times, but only the last submission will be marked. **Late submissions and files that are not named exactly as specified above will not be accepted!** It is suggested that you submit your preliminary `EC_Assignment_3_`*`YourMacID`*`.tex` and `EC_Assignment_3_`*`YourMacID`*`.pdf` files well before the deadline so that your mark is not zero if, e.g., your computer fails at 11:50 PM on March 8.

Although you are allowed to receive help from the instructional staff and other students, your submission must be your own work. Copying will be treated as academic dishonesty! If any of the ideas used in your submission were obtained from other students or sources outside of the lectures and tutorials, you must acknowledge where or from whom these ideas were obtained.

Background

Let $\Sigma_{\text{ka}} = (\{K\}, \{0, 1\}, \{+, \cdot, ^*\}, \emptyset, \tau)$ where:

1. $\tau(0) = \tau(1) = K$.
2. $\tau(+) = \tau(\cdot) = K \times K \to K$.
3. $\tau(^*) = K \to K$.

We will write $a \cdot b$ as simply ab.

Let $T_{\mathsf{ka}} = (\Sigma_{\text{ka}}, \Gamma_{\text{ka}})$ be and MSFOL theory of Kleene algebras where Γ_{ka} contain the following axioms:

1. $\forall\, x, y, z : K \;.\; x + (y + z) = (x + y) + z$.
2. $\forall\, x, y : K \;.\; x + y = y + x$.
3. $\forall\, x : K \;.\; x + 0 = x$.
4. $\forall\, x : K \;.\; x + x = x$.
5. $\forall\, x, y, z : K \;.\; x(yz) = (xy)z$.
6. $\forall\, x : K \;.\; x1 = x$.
7. $\forall\, x : K \;.\; 1x = x$.
8. $\forall\, x : K \;.\; x0 = 0$.
9. $\forall\, x : K \;.\; 0x = 0$.
10. $\forall\, x, y, z : K \;.\; x(y + z) = xy + xz$.
11. $\forall\, x, y, z : K \;.\; (x + y)z = xz + yz$.
12. $\forall\, x : K \;.\; 1 + xx^* = x^*$.
13. $\forall\, x : K \;.\; 1 + x^*x = x^*$.
14. $\forall\, x, y, z : K \;.\; y + xz \leq z \Rightarrow x^*y \leq z$.
15. $\forall\, x, y, z : K \;.\; y + zx \leq z \Rightarrow yx^* \leq z$.

Note: $x \leq y$ stands for $x + y = y$.

Extra Credit Problem [2 bonus points]

1. Prove that the following Σ_{ka}-structures are models of T_{ka}:

   a. $\mathcal{M}_1 = (\mathcal{L}, \emptyset, \{\epsilon\}, \cup, \mathsf{concatenation}, {}^*)$ where $\mathcal{L} = \mathcal{P}(\Sigma^*)$ is the set of all languages over some alphabet Σ. Note that technically $\mathcal{M}_1 = (\{D_K\}, I)$ where $D_K = \mathcal{L}$, $I(0) = \emptyset$, $I(1) = \{\epsilon\}$, $I(+) = \cup$, $I(\cdot) = \mathsf{concatenation}$, and $I({}^*) = {}^*$. Notice that Σ_{ka} and Σ are very different: Σ_{ka} is a signature and Σ is an alphabet.

   b. $\mathcal{M}_2 = (\mathcal{L}_{\mathrm{reg}}, \emptyset, \{\epsilon\}, \cup, \mathsf{concatenation}, {}^*)$ where $\mathcal{L}_{\mathrm{reg}}$ is the set of regular languages over some alphabet Σ.

   c. $\mathcal{M}_3 = (\mathcal{E}_{\mathrm{reg}}, \emptyset, \epsilon, +, \cdot, {}^*)$ where $\mathcal{E}_{\mathrm{reg}}$ is the set of regular expressions over some alphabet Σ in which equivalent regular expressions are consider equal.

   d. $\mathcal{M}_4 = (\{F, T\}, F, T, \vee, \wedge, f_T)$ where F and T are the standard truth values and $f_T(F) = f_T(T) = T$.

2. Let $(K, 0, 1, +, \cdot, {}^*)$ be an arbitrary model of T_{ka}. Prove the following:

   a. $(K, 0, +)$ is an idempotent commutative monoid.

   b. $(K, 0, 1, \cdot)$ is a monoid with an annihilator.

   c. $(K, 0, 1, +, \cdot)$ is a semiring.

   d. $(K, \leq)$ is a weak partial order with least upper bounds (what is called a *join-semilattice*).

3. Assuming Axioms 1–13, prove that Axiom 14 and the sentence

   $$\forall\, x, y : K \,.\, xy \leq y \Rightarrow x^* y \leq y$$

   are logically equivalent.

Put your name, MacID, and date here.

Put your solution here.