MATHEMATICS 1LT3 TEST 2

Day Class
Duration of Test: 60 minutes
McMaster University, 10 February 2011

E. Clements

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FIRST NAME (please print) : ____________

FAMILY NAME (please print) : ____________

Student No.: ____________

THIS TEST HAS 8 PAGES AND 6 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE.

Total number of points is 50. Marks are indicated next to the problem number in square brackets. Any Casio fx991 (or lower, non-graphing) calculator is allowed.

USE PEN TO WRITE YOUR TEST. IF YOU USE A PENCIL, YOUR TEST WILL NOT BE ACCEPTED FOR REMARKING (IF NEEDED).

You need to show work to receive full credit.

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1. Write a differential equation and an initial condition to describe the following situations.

[2] (a) A population of amoeba starts with 120 amoeba and grows with a constant per capita reproduction rate of 1.3 per member per day.

$$\frac{P'}{P} = 1.3 \Rightarrow \frac{dP}{dt} = 1.3P, \qquad P(0) = 120$$

[2] (b) The population of an isolated island in the Pacic Ocean is 7500. Initially, 13 people are infected with a virus. The rate of change of the number of infected people is proportional to the product of the number of people who are infected and the number of people who are not yet infected.

if

$$\frac{dI}{dt} = kI(7500 - I), \qquad I(0) = 13$$

2. Consider the initial value problem

$$\frac{dy}{dt} = 2t - 4, \quad y(0) = 3$$

[3] (a) Determine a formula for the exact solution and use it to calculate $y(2)$.

$$y(t) = \int (2t-4)\,dt = t^2 - 4t + C$$

$$y(0) = 3 \Rightarrow \quad 0^2 - 4(0) + C = 3 \Rightarrow C = 3$$

$$\therefore\ y(t) = t^2 - 4t + 3$$

$$y(2) = 2^2 - 4(2) + 3 = -1$$

[3] (b) Use Euler's method with step size $\Delta t = 1$ to approximate $y(2)$.

$t_0 = 0$

$y_0 = 3$

$t_1 = t_0 + \Delta t$
$= 0 + 1$
$= 1$

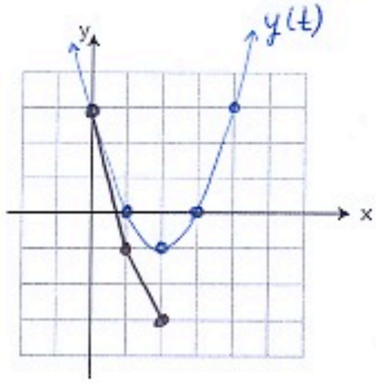
$y_1 = y_0 + y'(t_0)\Delta t$
$= 3 + (2(0) - 4)(1)$
$= -1$

$t_2 = t_1 + \Delta t$
$= 1 + 1$
$= 2$

$y_2 = y_1 + y'(t_1)\Delta t$
$= -1 + (2(1) - 4)(1)$
$= -3$

$\therefore\ y(2) \approx -3$

[3] (c) Graph your solution curve from part (a). On the same axis, plot the points from part (b) and connect them with straight lines. How could we improve our estimate in part (b)?



$$y(t) = t^2 - 4t + 3 = (t-3)(t-1)$$

We could improve our estimate by using a smaller step size.

3. A population of birds is observed to grow according to the logistic equation

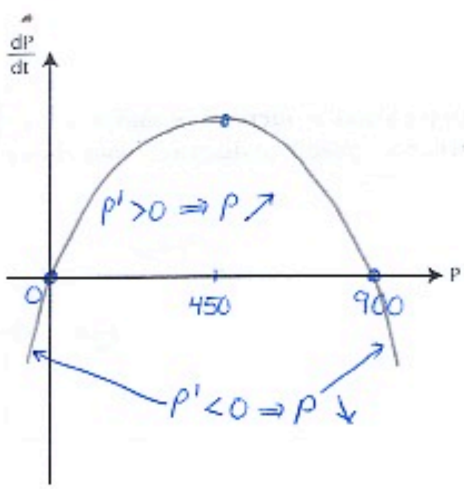
$$\frac{dP}{dt} = 0.08P(1 - \frac{P}{900})$$

[2] (a) Determine the equilibrium solutions. What does the number 900 represent?

$\frac{dP}{dt} = 0$ when $P = 0$ or $P = 900$

900 is the carrying capacity of the popn (ie the max. # of individuals the popn can sustain)

[3] (b) Graph $\frac{dP}{dt}$ as a function of P. Draw a phase-line diagram.



[3] (c) Determine the stability of your equilibrium solutions from part (a) using the stability theorem. Does your diagram in part (b) support these results?

$P' = 0.08P - \frac{0.08}{900}P^2$

$P'' = 0.08 - \frac{0.16}{900}P$

$P''(0) = 0.08 > 0 \Rightarrow P = 0$ is an unstable eqn

$P''(900) = -0.08 < 0 \Rightarrow P = 900$ is a stable eqn.

This supports our results in (b) since the unstable eqn $P=0$ repels solns whereas the stable eqn $P=900$ attracts them.

4. Suppose that we know the same population of birds in question 3 will die out if the population size drops below 50 birds.

[1] (a) Write a new model (just modify the model in the previous question) to reflect this observation.

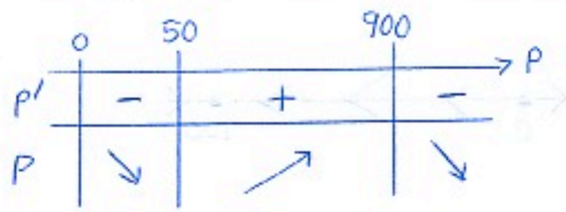
"existential threshold"

$$\frac{dP}{dt} = 0.08P\left(1 - \frac{P}{900}\right)\left(1 - \frac{50}{P}\right)$$

[1] (b) Determine the equilibrium solutions of this new model.

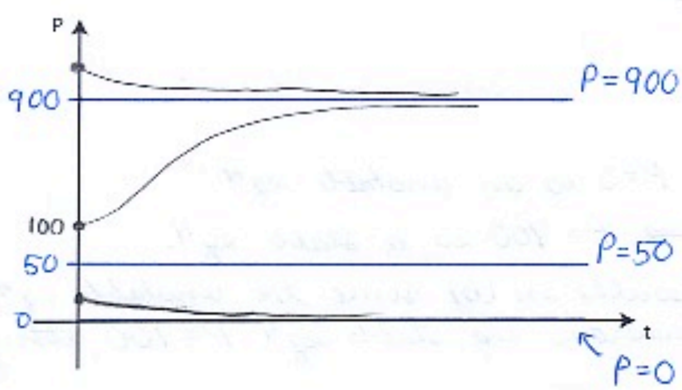
$$\frac{dP}{dt} = 0 \text{ when } P = 0,\ P = 900,\ P = 50$$

[3] (c) Analyze the model to determine when the population is increasing and when is it decreasing (You may summarize your results in a chart or a phase-line diagram, your choice).



OR

[4] (c) Sketch the equilibrium solutions and sketch typical solution curves corresponding to the initial conditions $P(0) = 20$, $P(0) = 100$, and $P(0) = 1000$.



5. Solve the following separable differential equatons.

[4] (a) $\frac{dy}{dt} = 0.4e^{-y}$, $y(0) = 1$

$\Rightarrow \int e^{y}\,dy = \int 0.4\,dt$

$e^{y} = 0.4t + C$

$y = \ln(0.4t + C)$

$y(0) = 1 \Rightarrow 1 = \ln C \Rightarrow C = e$

$\therefore\ y(t) = \ln(0.4t + e)$

[4] (b) $\frac{dP}{dt} = 2t^3 e^{P}$, $P(0) = 1$

$\Rightarrow \int e^{-P}\,dP = \int 2t^3\,dt$

$-e^{-P} = \frac{1}{2}t^4 + C \quad |(\times -1)$

$e^{-P} = -\frac{1}{2}t^4 - C \quad |\ln$

$-P = \ln(-\frac{1}{2}t^4 - C)$

$P = -\ln(-\frac{t^4}{2} - C)$

$P(0) = 1 \Rightarrow 1 = -\ln(-C) \Rightarrow -1 = \ln(-C) \Rightarrow e^{-1} = -C$

$\Rightarrow C = -\frac{1}{e}$

$\therefore\ P(t) = -\ln\left(\frac{1}{e} - \frac{t^4}{2}\right)$

6. Newton's Law of Cooling can be used in forensic science to determine an approximate time of death. After death, the rate of change in the temperature of a body is given by the differential equation

$$\frac{dT}{dt} = \alpha(A - T) \quad (*)$$

where $T(t)$ is the temperature in $°C$ at time t in hours, A is the ambient temperature and α is a positive constant.

[2] (a) Determine the equilibrium solution and describe the behaviour of solutions (i.e. when will solutions increase and when will they decrease).

$$\frac{dT}{dt} = 0 \text{ when } \alpha(A-T) = 0 \Rightarrow \alpha = 0 \text{ or } \underline{\underline{T=A}}$$

eqⁿ solⁿ

If the temperature of the object is $< A$, then the temp. will increase towards A. If the temp. T is $> A$, then object will cool off and the temp. will decrease towards A.

[4] (b) Solve this separable differential equation to verify that the temperature of a dead body at time t is given by $T(t) = A - Ke^{-\alpha t}$, where K is a constant.

$$(*) \rightarrow \int \frac{1}{A-T}\, dT = \int \alpha\, dt$$

$$-\ln|A-T| = \alpha t + C$$

$$\ln|A-T| = -\alpha t - C \ | e$$

$$|A-T| = e^{-\alpha t - C} = e^{-C} \cdot e^{-\alpha t}$$

$$A - T = \pm e^{-C} \cdot e^{-\alpha t}$$

$$\Rightarrow \quad T = A - Ke^{-\alpha t} \quad \text{where } K = \pm e^{-C}$$

[4] (c) At 4 a.m. (call this time $t = 0$) a body is found in a hotel room. The police measure the body's temperature to be $28°C$. At 5 a.m. the medical examiner arrives and determines the temperature to be $27°C$. Assuming the hotel room had a constant temperature of $20°C$, solve for the constants in part (b).

$T(0) = 28°C$

$T(1) = 27°C$

$A = 20°C$

$$T(t) = 20 - Ke^{-\alpha t}$$

① $\begin{cases} T(0) = 28 \Rightarrow 28 = 20 - K\underbrace{e^{0}}_{=1} \Rightarrow K = -8 \\ T(t) = 20 + 8e^{-\alpha t} \end{cases}$

② $T(1) = 27 \Rightarrow 27 = 20 + 8e^{-\alpha(1)}$

$\Rightarrow \frac{7}{8} = e^{-\alpha} \quad | \ln$

$\Rightarrow \ln\frac{7}{8} = -\alpha$

$\Rightarrow \alpha = -\ln\frac{7}{8} \approx 0.1335$

$\therefore \quad T(t) = 20 + 8e^{\ln\frac{7}{8}t}$

[2] (d) Assuming that the person had a normal body temperature of $37°C$ when he or she died, determine the approximate time of death for this person.

Solve $T(t) = 37$:

$20 + 8e^{\ln\frac{7}{8}t} = 37$

$e^{\ln\frac{7}{8}t} = \frac{17}{8} \quad | \ln$

$\ln\frac{7}{8}t = \ln\frac{17}{8}$

$t = \ln\frac{17}{8} / \ln\frac{7}{8} \approx -5.64$ hours

$\therefore$ This person died approximately -5.64 hours before 4am so the approx. time of death is 10:22pm.

$0.64h \times \frac{60 \text{ min}}{h} = 38.4 \text{ min}$

4am $- 5h, 38$ min $= 10:22$ pm