12C3

Last Day

Span: A set $\{\vec{u}_1, \dots \vec{u}_n\} \subseteq V$ spans V if all $\vec{v} \in V$ is a L.C. of $\{\vec{u}_1, \dots \vec{u}_n\}$

L.I. A set $\{\vec{u}_1, \dots \vec{u}_n\} \subseteq V$ is Linearly Independent if

$$a_1\vec{u}_1 + \dots + a_n\vec{u}_n = \vec{0} \quad \text{iff} \quad a_1 = a_2 = \dots = a_n = \vec{0}.$$

A basis is a L.I. Set which Spans V

Last Day we showed

1) any L.I. set in V has fewer or equal vectors than any basis

2) All bases of a vspace are the same size

that size is the dimension of V (called $\dim(V)$)

ie minimum # of vectors to span V = $\dim(V)$

note any basis for V has a unique L.C. of basis vectors for any $\vec{v} \in V$ as a L.C. of $\{\vec{u}_1 \dots \vec{u}_n\}$

these coeffs of $\vec{u}_1 \dots \vec{u}_n$ ie

$$\vec{v} = a_1 \vec{u}_1 + \dots a_n \vec{u}_n$$

a_i unique $n = \dim V$

a_i's are co-ordinates in this basis

eg Previously $\mathbb{R}^2$ obvious basis $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

$\hat{i}$ $\hat{j}$

Any 2 non parallel vectors also work!

eg $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right\}$

$\mathbb{R}^2$ is 2-dimensional!

$\dim(\mathbb{R}^2) = 2$

$\dim(\mathbb{R}^n) = n$

$\dim(\mathbb{C}) = 2$

as a real Vspace

$\dim(M_{nm}) = nm$

$n \times m$ matrices!

$\dim(\mathbb{P}_2) = 3$

order 2 or less polynomials!

Check $\mathbb{P}_2$ $\qquad \mathbb{P}_2 = \{ax^2+bx+c \mid a,b,c \in \mathbb{R}\}$

$\{x^2, x, 1\}$ span $\mathbb{P}_2$ (by definition!)

is this L.I?

$$ax^2 + bx + c\cdot 1 = 0 = 0x^2 + 0x + 0$$

equal iff coeff. equal!

$a=0$ $\quad c=0$ $\quad$ only!

$b=0$

Yes! L.I!

L.I. & Span $\Rightarrow$ basis!

$\{x^2, x, 1\}$ 3 elements! $\Rightarrow$ $\dim(\mathbb{P}_2) = 3$

In general $\dim(\mathbb{P}_n) = n+1$.

$\dim(C_0[0,1]) = \infty$ $\quad$ $C_0[0,1]$ is cont. functions on $[0,1]$

$\dim(\text{Maclaurin series}) = \infty$ $\quad$ etc! No finite spans!

Reminder: A basis of a vspace is a L.I. spanning set

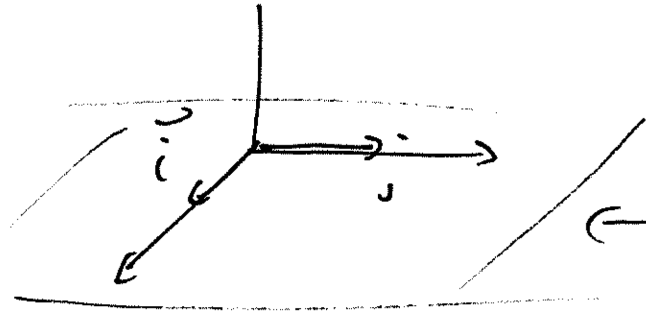
$\Rightarrow$ All L.I. sets are bases of their spans

ie if $\{\vec{v}_1 \ldots \vec{v}_n\}$ is L.I.

$\Rightarrow$ $\{\vec{v}_1, \ldots \vec{v}_n\}$ is a basis of $\text{Span}(\{\vec{v}_1 \ldots \vec{v}_n\})$

eg in $\mathbb{R}^3$ $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ span x-y plane! basis of xy plane!

In $\mathbb{R}^3$



$\leftarrow$ copy of $\mathbb{R}^2$

Say I have a LI set in V a vspace! $\{\vec{v}_1, \ldots \vec{v}_m\}$

Say # of vectors in LI set, $m <$ dim$(V) = n$

$\Rightarrow$ not a basis! (of V)

$\Rightarrow$ does not span!

$\Rightarrow$ vectors exist not. L.C. of our set, $\vec{u}$

$\Rightarrow \{\vec{v}_1 \dots \vec{v}_n, \vec{u}\}$ also L.I

does # elemts in set $= n$?

no

Yes!

Can't add any more! If any vector not span

$\Leftrightarrow$ not a L.C. of $\{\vec{v}_1 \dots \vec{v}_n\}$

I could add it!

$\Rightarrow$ new L.I set has $n+1$ elemts

Not Possible!

$\Rightarrow$ Must be a basis

Must Span!

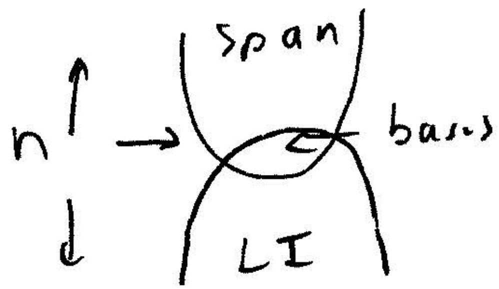
In Summary

1) **Plus / Minus Theorem**

If $\dim(V) = n$, any spanning set has $\geq n$ elements

any L.I set has $\leq n$ elements

any basis has <u>n</u> elements.



Any (finite) span can <u>reduce</u> to a basis (drop L.D. vectors)

Any LI set in a finite dim. space can be <u>grown</u> to a <u>basis</u> by adding vectors not in L.C..

<u>or</u>

2) If U subspace of V $\Rightarrow$ $\dim(U) \leq \dim(V)$

& $\dim(U) = \dim(V)$ <u>iff</u> $U = V$

$\Rightarrow$ To check if a subset of V is a <u>basis</u>

Check 1) LI 2) Span 3) $\# = \dim V$

<u>any 2 of 3 will do!</u>

Example! Which of the following is a <u>basis</u> of the <u>given V space</u>?

1) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$ in $\mathbb{R}^3$ <u>no</u> $n = 2$

$\dim(\mathbb{R}^2) = 3$

$2 \neq 3$.

2) $\{x+1, x-1, 5\}$ in $\mathbb{P}_1$ no $n=3$

$\dim(\mathbb{P}_1) = 2 \neq 3$

3) $\{x^2, x^2-x, x^2-x+1\}$ in $\mathbb{P}_2$ $n = 3 = \dim(\mathbb{P}_2)$

but need L.I.!

Check LI

To show $a=b=c=0$ only!

$$ax^2 + b(x^2-x) + c(x^2-x+1) = 0$$

$$\Downarrow$$

$$(a+b+c)x^2 + (-b-c)x + c(1) = 0$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

det $\neq 0 \Rightarrow$ invertible $\Rightarrow$ trivial soln. only!

$\Rightarrow$ L.I ✓ $\Rightarrow$ basis

(b n = 3) ✓

Express x^2-2x+5 in above basis.

Solution $a x^2 + b(x^2 - x) + c(x^2 - x + 1) = x^2 - 2x + 5$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & -2 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

$c = 5$

$-b - 5 = -2$

$b = -3$

$a + b + c = 1$

$a - 3 + 5 = 1 \qquad a = -1.$

$\Rightarrow \quad (x^2 - 2x + 5) = (-1)\ x^2$

$\qquad - 3 \cdot (x^2 - x).$

$\qquad + 5\,(x^2 - x + 1)$

or in basis $(-1, -3, 5)$