12A3

Last Day: Breaking Up Rationals into "Partial Fractions"

A Rational Function is a fcn. of form: $\dfrac{\text{polynomial}}{\text{polynomial}}$

eg $$\frac{x^2 - 2x + 3}{7x^5 + 6x^2}$$

We can break these up by following rules..

1) If (order of numerator) $\geq$ (order of denominator)

$\Rightarrow$ perform "synthetic" division so

order on top < order on bottom;

2) $\dfrac{1}{(x-a)(x-b)} = \dfrac{A}{x-a} + \dfrac{B}{x-b}$, A, B const.

3) $\dfrac{1}{(x-a)(x-b)(x-c)\cdots} = \dfrac{A}{x-a} + \dfrac{B}{x-b} + \dfrac{C}{x-c} + \cdots\cdots$

4) Repeated factors ?

eg. $\dfrac{1}{(x-a)^3(x-b)^7} = \dfrac{A_1}{x-a} + \dfrac{A_2}{(x-a)^2} + \dfrac{A_3}{(x-a)^3}$

$+ \dfrac{B_1}{(x-b)} + \dfrac{B_2}{(x-b)^2} + \cdots \dfrac{B_7}{(x-b)^7}$.

One term (with its own constant) for each power of any repeated factor, up to max.

Today: Irreducible Quadratics

eg $$\frac{1}{x^3+4x} = \frac{1}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

$(x-2i)(x+2i)$? **No!** Keeping it real

not real!

In general each irreducible quadratic factor gets a term with "$Bx+C$" up top!

eg $$\int \frac{1}{(x-a)(x-b)(x^2+c^2)^3} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{Cx+D}{x^2+c^2}$$

Again, one term for each power up to max. for a repeated irreducible quadratic factor! $\longrightarrow$ $+ \dfrac{Ex+F}{(x^2+c)^2} + \dfrac{Gx+H}{(x^2+c)^3}$

Let's see how this works in an integral problem

eg. $\displaystyle\int \frac{1}{x(x^2+4)}\,dx \longrightarrow \frac{1}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$

so $1 = Ax^2 + 4A + Bx^2 + Cx$

match powers

$$\left.\begin{aligned} x^2:&\quad 0 = A + B \\ x:&\quad 0 = C \\ \text{const:}&\quad 1 = 4A \end{aligned}\right\} \rightarrow \quad \begin{aligned} A &= \tfrac{1}{4} \\ B &= -A = -\tfrac{1}{4} \\ C &= 0. \end{aligned}$$

So $\int \frac{1}{x(x^2+4)}\,dx = \int \frac{(1/4)}{x}\,dx + \int \frac{(-\frac{1}{4})x + 0}{x^2+4}\,dx$

$= \frac{1}{4}\ln|x| - \frac{1}{4}\int \underbrace{\frac{x}{x^2+4}}\,dx$

Let $u = x^2+4$

$du = 2x\,dx$

$= \frac{1}{4}\ln|x| - \frac{1}{8}\int \frac{1}{u}\,du$ $\quad \ln|u| + c$

$= \frac{1}{4}\ln|x| - \frac{1}{8}\ln|x^2+4| + C$

What complications can occur?

1) What if "C" in $\dfrac{Bx+C}{x^2+4}$ was not 0

$\Rightarrow$ need to integrate $\displaystyle\int \frac{C}{x^2+4}\,dx = C\int \frac{1}{x^2+4}\,dx$

$$\int \frac{1}{x^2+4}\,dx = \frac{1}{4}\int \frac{1}{\frac{x^2}{4}+1}\,dx = \frac{1}{4}\int \frac{1}{\left(\frac{x}{2}\right)^2+1}\,dx$$

Let $u = x/2 \Rightarrow du = \frac{1}{2}\,dx$ or $dx = 2\,du$

$$= \frac{1}{4}\int \frac{1}{u^2+1}\cdot 2\,du = \frac{1}{2}\tan^{-1}(u) + C$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$$

In general $$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

Complication #2 "Weirder" Irreducible Quadratics

e.g. $$\int \frac{1}{x^2+x+1}\,dx \rightarrow ax^2+bx+c = x^2+x+1$$

↑ check its descriminant: $b^2-4ac<0$

$\Rightarrow$ irred. quadratic!

no real roots / factors!

here $b = c = 1 \Rightarrow b^2 - 4ac = -3 < 0$ ✓

irreducible!

Let's complete the square

$$x^2 + bx + c = \left(x + \frac{b}{2}\right)^2 + c - \left(\frac{b}{2}\right)^2 \quad ✓$$

$$= \left(x^2 + \cancel{2}x\frac{b}{\cancel{2}} + \cancel{\frac{b^2}{4}}\right) + c - \cancel{\frac{b^2}{4}}$$

here $\underbrace{x^2 + x + 1}_{b=c=1} = \left(x + \frac{1}{2}\right)^2 + \cancel{c}^{\,1} - \left(\frac{1}{2}\right)^2$

$$= \left(x + \frac{1}{2}\right)^2 + 3/4$$

So

$$\int \frac{1}{x^2+x+1}\,dx = \int \frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}}\,dx$$

let $u = x+\frac{1}{2}$, $dx = du$

$$= \int \frac{1}{u^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,du$$

"$\frac{1}{u^2+a^2}$" form!

$a = \sqrt{3}/2$

$$= \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right)+C$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x+\frac{1}{2}}{\sqrt{3}/2}\right)+C$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+C$$

So for practice!

$$\int \frac{x^4 + 7}{x^2(x^2 + 2x + 5)} \, dx$$

Note: Very Long!