MATHEMATICS 1LT3 TEST 2

Evening Class
Duration of Test: 60 minutes
McMaster University

E. Clements

6 March 2017

FIRST NAME (please print): Sol^{N}s

FAMILY NAME (please print): ____________________

Student No.: ____________________

THIS TEST HAS 8 PAGES AND 10 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE.

Total number of points is 40. Marks are indicated next to the problem number in square brackets. You may use the McMaster standard calculator, Casio fx991 MS+, on this test.

USE PEN TO WRITE YOUR TEST. IF YOU USE A PENCIL, YOUR TEST WILL NOT BE ACCEPTED FOR REMARKING (IF NEEDED).

You need to show work to receive full credit, except for Question 1.

Problem	Points	Mark
1	6	
2	6	
3	3	
4	2	
5	3	
6	3	
7	4	
8	5	
9	4	
10	4	
TOTAL	40	

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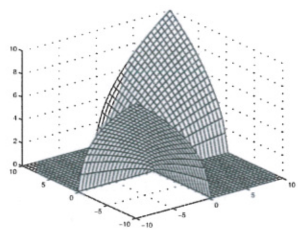
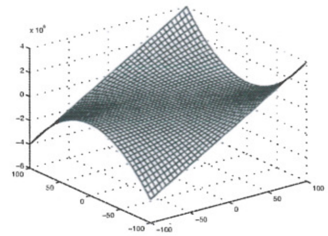
1. For part (a), write the letter corresponding to the graph of the function next to the equation in the space provided. For part (b), clearly circle the one correct answer.

(a) [3] Match the equation of each function with its graph below.

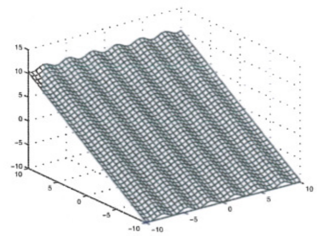
$f(x,y) = \sin^2 x + y$ C $\qquad g(x,y) = \sqrt{xy}$ A $\qquad h(x,y) = x - y^2 + 4xy^2$ B



(A)



(B)



(C)

(b) [3] The humidex $H(T,h)$ is a measure used by meteorologists to describe the combined effects of heat and humidity on an average person's feeling of hotness. The table below provides values of humidex based on measurements of temperature T (in degrees Celsius) and relative humidity h (given as a percent).

	$T = 22$	$T = 26$	$T = 30$	$T = 34$
$h = 70$	27	33	41	49
$h = 60$	25	32	38	46
$h = 50$	24	30	36	43

Which of the following statements is/are true?

(I) $H_T(30,60)$ is negative $\qquad$ (II) $H_h(26,60)$ is positive $\qquad$ (III) $H_T(26,60)$ is negative

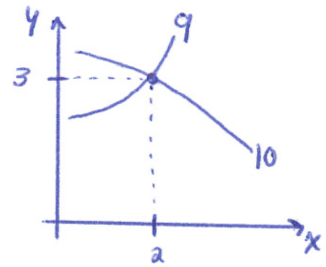
(A) none $\qquad$ (B) I only $\qquad$ (C) II only $\qquad$ (D) III only

(E) I and II $\qquad$ (F) I and III $\qquad$ (G) II and III $\qquad$ (H) all three

2. State whether each statement is **true or false**. **Explain** your reasoning.

(a) [2] In creating a contour map for a function $f(x, y)$, it is possible for the level curves of two different values to intersect each other.

FALSE.



Suppose the contour map for $f(x,y)$ contains two level curves with different values that intersect at $(2,3)$.
Then $f(2,3) = 9$ and $f(2,3) = 10$ which violates the definition of a function.

(b) [2] The function $f(x, y) = \begin{cases} \dfrac{2e^{x+y}}{xy - 1} & \text{if} \quad (x, y) \neq (1, -1) \\ -1 & \text{if} \quad (x, y) = (1, -1) \end{cases}$ is continuous at $(1, -1)$.

$$\lim_{(x,y)\to(1,-1)} f(x,y) = \lim_{(x,y)\to(1,-1)} \frac{2e^{x+y}}{xy-1} = \frac{2e^{1-1}}{(1)(-1)-1} = \frac{2}{-2} = -1$$

$$f(1,-1) = -1$$

TRUE. Since $\lim_{(x,y)\to(1,-1)} f(x,y) = f(1,-1)$, we conclude that f is continuous at $(1,-1)$ by the definition of continuity.

(c) [2] If $z = x^y$, then $z_{yx} = x^{y-1}(y \ln x + 1)$.

$$z_y = x^y \cdot \ln x$$

$$z_{yx} = y x^{y-1} \cdot \ln x + x^y \cdot \frac{1}{x}$$

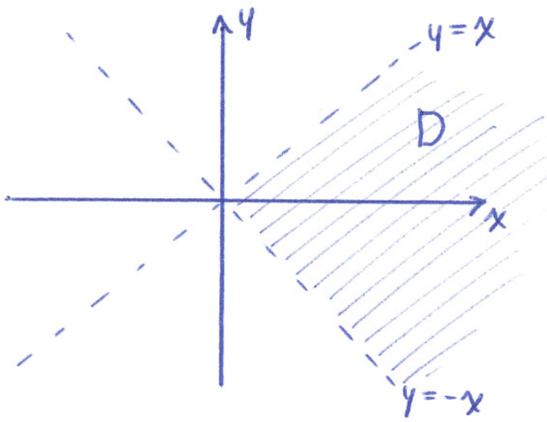
$$= x^{y-1}(y \ln x + 1)$$

$\therefore$ TRUE.

3. [2] Find and sketch the domain of $f(x,y) = \ln(x+y) + \ln(x-y)$.

$x+y>0$ and $x-y>0$

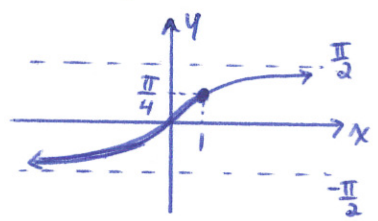
$\Rightarrow$ $y>-x$ and $y<x$



4. [2] Determine the range of the function $g(x,y) = \arctan(1-x^2-y^2)$. You do not need to provide a formal proof as we did in class, but you have to correctly explain your reasoning.

$x^2+y^2 \geq 0 \Rightarrow -x^2-y^2 \leq 0 \Rightarrow 1-x^2-y^2 \leq 1$

Consider $y = \arctan x$



when $x \in \mathbb{R}$, $y \in (-\frac{\pi}{2}, \frac{\pi}{2})$

when $x \leq 1$, $y \in (-\frac{\pi}{2}, \frac{\pi}{4}]$

$\therefore$ range of $g(x,y)$ is $(-\frac{\pi}{2}, \frac{\pi}{4}]$

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5. [3] Sketch a contour diagram of the function $f(x,y) = e^{x^{-2}y}$ by drawing level curves which correspond to $k = 0.5$, $k = 1$, and $k = 1.5$. Label all curves in your diagram.

$x \neq 0$

level curves: $e^{x^{-2}y} = k$, where $k > 0$

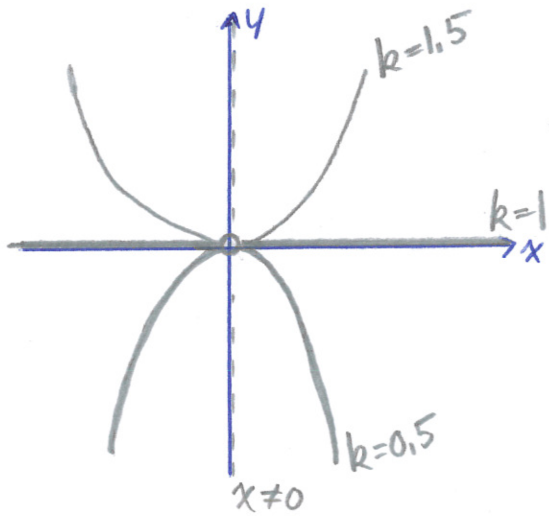
$\Rightarrow x^{-2}y = \ln k$

$\Rightarrow y = \ln k \, x^2$

$k = 0.5 \Rightarrow y = \ln 0.5 \, x^2 \approx -0.693 x^2$

$k = 1 \Rightarrow y = \ln 1 \, x^2 = 0$

$k = 1.5 \Rightarrow y = \ln 1.5 \, x^2 \approx 0.405 x^2$



6. [3] Show that $\lim_{(x,y)\to(0,0)} \underbrace{\frac{y \sin x}{x^2 + y^2}}_{f}$ does not exist.

① $f(0,y) = \frac{y \sin 0}{0^2 + y^2} = \frac{0}{y^2} = 0$

② $f(x,x) = \frac{x \sin x}{x^2 + x^2} = \frac{\sin x}{2x}$

$\lim_{x\to 0} \frac{\sin x}{2x} \overset{LH}{=} \lim_{x \to 0} \frac{\cos x}{2} = \frac{\cos 0}{2} = \frac{1}{2}$

Since $f(x,y) \to 0$ as $(x,y) \to (0,0)$ along $x = 0$ and $f(x,y) \to \frac{1}{2}$ as $(x,y) \to (0,0)$ along $y = x$, $\lim_{(x,y)\to(0,0)} f(x,y)$ D.N.E.

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7. [4] Approximate the value of $\sin 0.75 \cos 3$ using the linearization of an appropriate function $f(x, y)$ at a suitable base point (a, b). Round your answer to two decimal places.

$$\left.\begin{array}{l} 0.75 = \frac{3}{4} \approx \frac{\pi}{4} \\ 3 \approx \pi \end{array}\right\} \Rightarrow \text{use } (a,b) = \left(\tfrac{\pi}{4}, \pi\right) \text{ and } f(x,y) = \sin x \cdot \cos y$$

$$f\left(\tfrac{\pi}{4}, \pi\right) = \sin\tfrac{\pi}{4}\cos\pi = \frac{-1}{\sqrt{2}}$$

$$f_x = \cos x \cdot \cos y \; \ldots \; f_x\left(\tfrac{\pi}{4}, \pi\right) = \cos\tfrac{\pi}{4} \cdot \cos\pi = \frac{-1}{\sqrt{2}}$$

$$f_y = \sin x \cdot (-\sin y) \; \ldots \; f_y\left(\tfrac{\pi}{4}, \pi\right) = \sin\tfrac{\pi}{4} \cdot (-\sin\pi) = 0$$

$$L_{(\frac{\pi}{4}, \pi)}(x,y) = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\left(x - \tfrac{\pi}{4}\right) + 0(y - \pi)$$

$$\therefore \sin 0.75 \cos 3 \approx -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\left(0.75 - \tfrac{\pi}{4}\right) \approx -0.68$$

8. Let $f(x, y) = \sqrt{xy}$.

(a) [3] Compute f_x and f_y. Shade the region on which f_x and f_y are continuous.

$$f_x = \frac{1}{2\sqrt{xy}}(y)$$

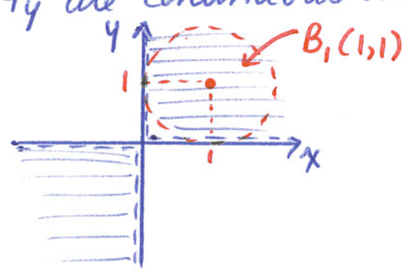
$$f_y = \frac{1}{2\sqrt{xy}}(x)$$

domain: $xy > 0$

$\Rightarrow x > 0$ and $y > 0$

OR $x < 0$ and $y < 0$

f_x and f_y are continuous on their domain



(b) [2] Explain why f is differentiable at $(1, 1)$.

Since f_x and f_y are continuous on any $B_r(1,1)$ where $r \leq 1$, by Theorem 6, f is differentiable at $(1,1)$.

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9. [4] Compute the directional derivative of the function $f(x, y) = x^3 y - \frac{4}{xy}$ at the point $(1, 1)$ in the direction specified by $\mathbf{v} = 4\mathbf{i} - 3\mathbf{j}$.

$\|\vec{v}\| = \sqrt{4^2 + (-3)^2} = 5 \Rightarrow \hat{u} = \underbrace{\frac{4}{5}}_{u_1}\hat{i} \underbrace{- \frac{3}{5}}_{u_2}\hat{j}$

$f_x = 3x^2y - \frac{4}{y}\left(-\frac{1}{x^2}\right) \ldots f_x(1,1) = 7$

$f_y = x^3 - \frac{4}{x}\left(-\frac{1}{y^2}\right) \ldots f_y(1,1) = 5$

$D_{\vec{v}} f(1,1) = 7\left(\frac{4}{5}\right) + 5\left(\frac{-3}{5}\right) = \frac{13}{5}$

10. [4] Find the maximum rate of change of $f(x, y) = 2ye^x + e^{-x}$ at the point $(0, 0)$ and the direction in which it occurs.

max. rate of change of f at $(0,0) = \|\nabla f(0,0)\|$

$f_x = 2ye^x - e^{-x} \ldots f_x(0,0) = -1$

$f_y = 2e^x \ldots\ldots\ldots f_y(0,0) = 2$

$\nabla f(0,0) = -1\hat{i} + 2\hat{j}$ ← direction in which the max. rate of change occurs

$\|\nabla f(0,0)\| = \sqrt{(-1)^2 + 2^2} = \sqrt{5}$

↑ max. rate of change of f at $(0,0)$