(The) Last Day Integral Recap!

Today: First Half Recap!

Inverse function

eg. $f(x) = \dfrac{x^3}{1-x^3}$ find $f^{-1}(x)$

Solution.

$$y = \frac{x^3}{1-x^3}$$

$$y - yx^3 = x^3$$

$$y = x^3(1+y)$$

$$x^3 = \frac{y}{1+y}$$

$$x = \left(\frac{y}{1+y}\right)^{1/3}$$

& reverse names! $\quad y = \left(\frac{x}{1+x}\right)^{\frac{1}{3}} = \underline{\underline{f^{-1}(x)}}$

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eg. $\quad f(x) = 1 + x + e^{x^3+x}$

Find $f^{-1}(f(10)) = \underline{\underline{10}}$

Don't forget formula for $\frac{d}{dx} f^{-1}(x)$

$$\frac{d}{dx}\left(f^{-1}(x)\right) = \frac{1}{f'(f^{-1}(x))}$$

$$f(f^{-1}(x)) = x$$

$$\Big\downarrow \frac{d}{dx}$$

$$f'(f^{-1}(x))(f^{-1})' = 1$$

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$$

so

$f(3) = 5 \qquad f(5) = -1$

$f'(3) = 2 \qquad f'(-1) = 9$

Find $\frac{d}{dx} f^{-1}(x)$ at $x = 5$.

Solution

$$= \frac{1}{f'(f^{-1}(x))} \quad \text{at } x = 5.$$

$$= \frac{1}{f'(f^{-1}(5))}$$

$$= \frac{1}{f'(3)} = \boxed{\frac{1}{2}}$$

$f^{-1}(5) = ?$

$5 = f(?)$

$? = \underline{\underline{3}}$

Hypabolics

$$\cosh(x) = \frac{e^{x}+e^{-x}}{2}, \quad \sinh(x) = \frac{e^{x}-e^{-x}}{2}$$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$$

$$\operatorname{sech}(x) = \frac{1}{\cosh(x)} = \frac{2}{e^{x}+e^{-x}}$$

eg. Evaluate $\operatorname{csch}(\ln 4)$

Solution $\quad \operatorname{csch}(x) = \dfrac{1}{\sinh(x)} = \dfrac{2}{e^{x} - e^{-x}}$

$$\operatorname{csch}(\ln 4) = \frac{1}{\sinh(\ln 4)} = \frac{2}{e^{\ln 4} - e^{-\ln 4}}$$

$$= \frac{2}{4 - \frac{1}{4}}$$

$$= \frac{8}{16 - 1} = \boxed{\frac{8}{15}}.$$

$$e^{-\ln 4} = \frac{1}{e^{\ln 4}} = \frac{1}{4}$$

or

$$e^{-\ln 4} = e^{\ln(4^{-1})} = 4^{-1} = \frac{1}{4}$$

eg $\cosh(x) + \sinh(x) = \dfrac{e^x + \cancel{e^{-x}}}{2} + \dfrac{e^x - \cancel{e^{-x}}}{2}$

$= \dfrac{2e^x}{2} = e^x$.

$\cosh(x) - \sinh(x) = \dfrac{\cancel{e^x} + e^{-x}}{2} - \left(\dfrac{\cancel{e^x} - e^{-x}}{2}\right)$

$= \dfrac{2e^{-x}}{2} = e^{-x}$

eg $\cosh^2(x) - \sinh^2(x) = (\cosh(x) + \sinh(x))(\cosh(x) - \sinh(x))$

$= e^x \cdot e^{-x} = 1$

$f(x)$ cont. at $x=a$ <u>if</u> $\lim_{x\to a} f(x) = f(a)$

$f(x)$ is <u>left</u> <u>cont.</u> at $x=a$ if $\lim_{x\to a^-} f(x) = f(a)$

<u>right</u> <u>cont</u> at $x=a$ if $\lim_{x\to a^+} f(x) = f(a)$!

$f(x)$ cont. at $x=a$ <u>iff</u> $\lim_{x\to a^+} f(x) = f(a) = \lim_{x\to a^-} f(x)$.

eg.

$$f(x) = \begin{cases} x^3 + 2ax\,, & x > 1 \\ \frac{1}{x} + 5\,, & x \le 1. \end{cases}$$

find "a" such that $f(x)$ is cont. at $x=1$.

Solution we note $x^3 + 2ax$ & $\frac{1}{x}+5$

both cont. on domain

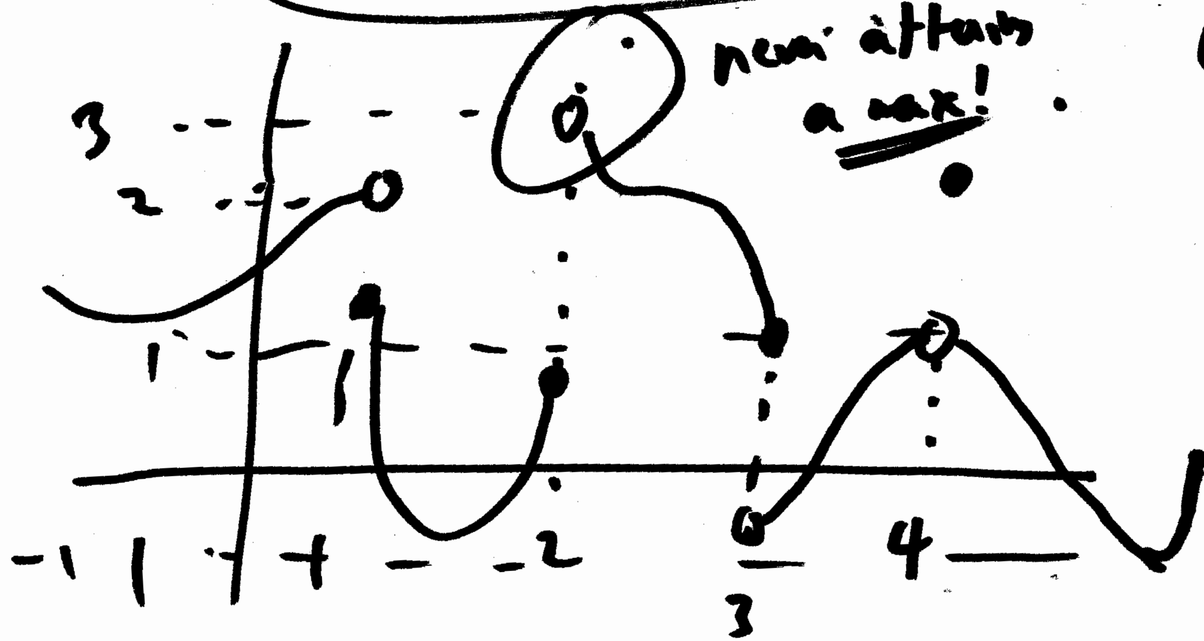
so

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x^3 + 2ax = 1 + 2a$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{1}{x}+5 = 1+5 = 6$$
$$= \underline{\underline{f(1)}}.$$

$\underline{\underline{\text{So}}}$ $1+2a = 6 \Rightarrow 2a = 5$ $\quad a = \frac{5}{2}$

eg



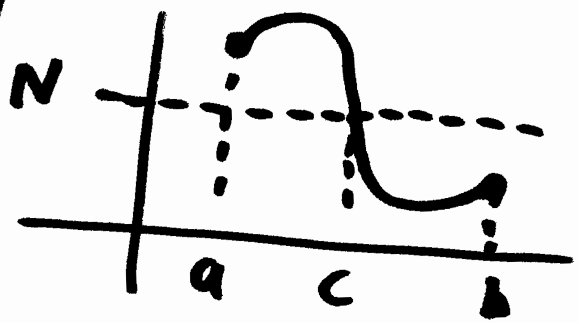
never attains a $\underline{\underline{\text{max!}}}$.

Left. cont only at $x = 2, 3$

right cont only at $x = 1$

For above graph: Abs. min = -1, Abs. Max = none!

IVT Intermediate Value Theorem



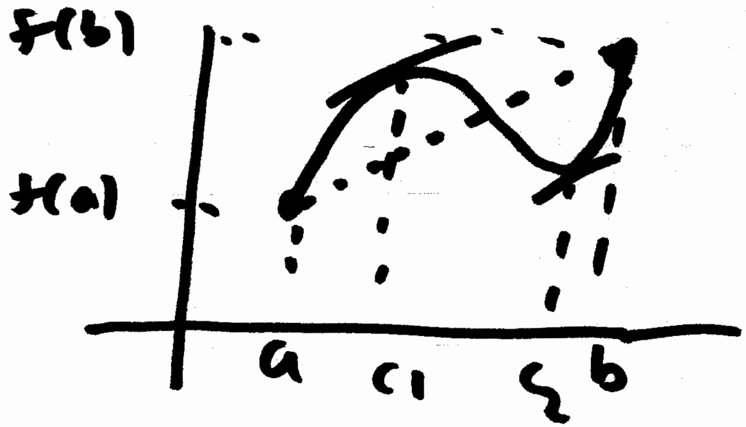
If $f(x)$ cont. on $[a,b]$ closed!

& $N \in [f(a), f(b)]$

(or $[f(b), f(a)]$

then there exists at least one $c \in [a,b]$ such that $f(c) = N$

MVT Mean Value Theorem



If $f(x)$ cont. on $[a,b]$ & diff. on (a,b) then there exits at least one c such that $f'(c) = \frac{f(b)-f(a)}{b-a}$

The "secant slope"

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eg Find the derivative of $y = (\sin x)^x$, $0 \le x \le \pi$.

(Logarithmic differentiation!)

Solution

1) $\ln y = \ln\left((\sin x)^x\right)$ ← "log" both sides

2) $\ln y = x \ln(\sin x)$ ← simplify using "ln" properties.

3) $\dfrac{y'}{y} = 1 \ln(\sin x) + x \cdot \dfrac{\cos x}{\sin x}$ ← diff it

$= \ln(\sin x) + x\cot(x)$

solve for y'; sub in y

4) $y' = y\left(\quad " \quad \right)$

$= (\sin x)^x \left(\ln(\sin x) + x\cot x \right)$

l'Hopitals   Indeterminate Form

Quotient Form (base)   "$\frac{0}{0}$"   "$\pm\frac{\infty}{\infty}$"

Product Form : $0 \cdot (\pm\infty)$ $\Rightarrow$ re-write as quotient!

Power Form : ∞^0, 0^0, 1^∞

approach 1 not **exactly** 1.

Not L'Hopitals: $\infty^{\infty} = \infty$

$0^{\infty} = 0$