MATHEMATICS 1LT3 TEST 3

Evening Class
Duration of Test: 60 minutes
McMaster University

E. Clements

26 March 2015

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FIRST NAME (please print): Sol^NS

FAMILY NAME (please print): ____________

Student No.: ____________

THIS TEST HAS 8 PAGES AND 9 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE.

Total number of points is 40. Marks are indicated next to the problem number in square brackets. You may use the McMaster standard calculator, Casio fx991 MS+, on this test.

USE PEN TO WRITE YOUR TEST. IF YOU USE A PENCIL, YOUR TEST WILL NOT BE ACCEPTED FOR REMARKING (IF NEEDED).

You need to show work to receive full credit, except for Multiple Choice.

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1. State whether each statement is true or false. Explain your reasoning.

(a) [2] It is possible to assign probabilities to sets A and B in the following way:

$$P(A) = 0.5,\ P(B) = 0.2,\ \text{and}\ P(A \cap B) = 0.4$$

FALSE. $A \cap B \subseteq B$ which implies that $P(A \cap B) \leq P(B)$ (visualize this by interpreting probability as area).
Here, $\underset{0.4}{\underset{\|}{P(A\cap B)}} > \underset{0.2}{\underset{\|}{P(B)}}$

(b) [2] Suppose that X is a random variable such that $E(X) = 30$ and $E(X^2) = 1000$. Then the standard deviation of X is 10.

$\sigma = \sqrt{Var(X)}$

$Var(X) = E(X^2) - [E(X)]^2$
$= 1000 - 30^2$
$= 100$

so, $\sigma = \sqrt{100} = 10$ $\therefore$ TRUE

Name:______________

Student No.: ______________

2. Multiple Choice. Clearly circle the one correct answer.

(a) [3] Which of the following functions has a local maximum or minimum at $(0,0)$?

(I) $f(x,y) = x^4 + y^4$ min (II) $g(x,y) = -x^4 - y^4$ max (III) $h(x,y) = x^4 - y^4$ saddle

(A) none (B) I only (C) II only (D) III only

(E) I and II (F) I and III (G) II and III (H) all three

(b) [3] Consider the random experiment of rolling a fair, 6-sided die. Let A be the event of rolling an even number and let B be the event of rolling a number greater than 3. Which of the following are correct?

(I) $P(A \cup B) = 2/3$ (II) $P(A \cap B) = 1/3$ (III) $P(B^C) = 1/2$

(A) none (B) I only (C) II only (D) III only

(E) I and II (F) I and III (G) II and III (H) all three

$A = \{2,4,6\}$, $B = \{4,5,6\}$, $A \cup B = \{2,4,5,6\}$, $A \cap B = \{4,6\}$, $B^c = \{1,2,3\}$

$P(A \cup B) = \frac{4}{6} = \frac{2}{3}$ $P(A \cap B) = \frac{2}{6} = \frac{1}{3}$ $P(B^c) = \frac{3}{6} = \frac{1}{2}$

(c) [3] A quiz has 5 multiple choice questions, each with 3 choices. Without reading them, a student answers all questions. Which of the following are true?

(I) The probability that the student answers at least one correctly is about 0.87. ✓
(II) The probability that the student answers more than three correctly is about 0.67. ✗
(III) The probability that the student answers all questions correctly is about 0.0041. ✓

(A) none (B) I only (C) II only (D) III only

(E) I and II (F) I and III (G) II and III (H) all three

C = "correct", W = "wrong" $P(C) = \frac{1}{3}$, $P(W) = \frac{2}{3}$

(I) $1 - P(WWWWW) = 1 - (\frac{2}{3})^5 \approx 0.87$

(II) $P(CCCCC) + P(WCCCC) + P(CWCCC) + P(CCWCC) + P(CCCWC) + P(CCCCW)$
$= (\frac{1}{3})^5 + 5(\frac{2}{3})(\frac{1}{3})^4 \approx 0.045$

(III) $P(CCCCC) = (\frac{1}{3})^5 \approx 0.0041$

Name:__________________
Student No.: __________________

3. Consider the function $f(x, y) = x^3 - 2y^2 + 3xy + 4$.

(a) [2] Find all critical points of f.

$f_x = 3x^2 + 3y \qquad f_y = -4y + 3x$

$f_x = 0 \Rightarrow \quad 3(x^2+y) = 0 \Rightarrow y = -x^2$ ①

$f_y = 0 \Rightarrow \quad -4y + 3x = 0 \Rightarrow y = \frac{3}{4}x$ ②

sub ② into ①: $\frac{3}{4}x = -x^2 \Rightarrow x^2 + \frac{3}{4}x = 0 \Rightarrow x(x + \frac{3}{4}) = 0$

$\Rightarrow x = 0$ or $x = -\frac{3}{4}$

sub $x = 0$ into ①: $y = 0$

sub $x = -\frac{3}{4}$ into ①: $y = -(-\frac{3}{4})^2 = -\frac{9}{16}$

$\therefore$ critical points are $(0,0)$ and $(-\frac{3}{4}, -\frac{9}{16})$.

(b) [3] Using the second derivatives test, classify the critical points from part (a).

$f_{xx} = 6x, \quad f_{xy} = 3, \quad f_{yy} = -4$

$\therefore D = (6x)(-4) - 3^2 = -24x - 9$

$D(0,0) = -9 \Rightarrow f$ has a saddle point at $(0,0)$.

$$\left.\begin{aligned} D(-\tfrac{3}{4}, \tfrac{-9}{16}) &= -24(-\tfrac{3}{4}) - 9 = 9 \\ f_{xx}(-\tfrac{3}{4}, \tfrac{-9}{16}) &= 6(-\tfrac{3}{4}) = -\tfrac{9}{2} \end{aligned}\right\} \Rightarrow f \text{ has a local max. at } (-\tfrac{3}{4}, \tfrac{-9}{16})$$

4. A fair, 6-sided die is rolled.

(a) [2] What is the probability that the sum of the two numbers that come up is 8?

$A = \{(2,6),(3,5),(4,4),(5,3),(6,2)\}$ $|S| = 36.$

$P(A) = \frac{|A|}{|S|} = \frac{5}{36}$

(b) [2] Using conditional probability, determine the probability that ~~exactly~~ one die is a 4 given that the sum is 7.

$C =$ "sum is 7" $= \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$

$A =$ "one is a 4"

$A \cap C = \{(3,4),(4,3)\}$

$P(A|C) = \frac{P(A \cap C)}{P(C)} = \frac{\frac{2}{36}}{\frac{6}{36}} = \frac{1}{3}$

Name:____________________

Student No.: __________________

5. [2] A certain population consists of 20% children (C), 30% adolescents (T), and 50% adults (D). The probabilities that a certain member of this population catches the flu are 0.45 for a child, 0.2 for an adolescent, and 0.15 for an adult. What is the probability that a randomly selected member of this population has the flu?

A = "has flu"

$$\begin{aligned} P(A) &= P(A|C)P(C) + P(A|T)P(T) + P(A|D)P(D) \\ &= 0.45(0.2) + 0.2(0.3) + 0.15(0.5) \\ &= 0.225 \end{aligned}$$

∴ The probability that a randomly selected member has the flu is 0.225.

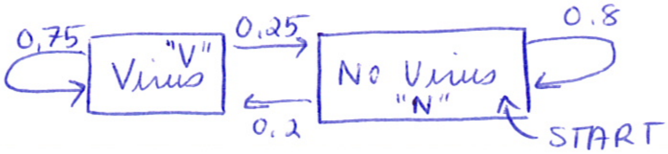
6. [3] A surveyed population consists of 60% females ($♀$) and 40% males ($♂$). Of all males, 35% are smokers, and of all females, 20% are smokers. What is the probability that a randomly selected smoker (S) is female?

S = "smoker"

$$\begin{aligned} P(♀|S) &= \frac{P(S|♀)P(♀)}{P(S|♀)P(♀) + P(S|♂)P(♂)} \\ &= \frac{(0.2)(0.6)}{(0.2)(0.6) + (0.35)(0.4)} \\ &\approx 0.461 \end{aligned}$$

∴ The probability that a randomly selected smoker is a female is about 0.461.

7. [3] Suppose that if a certain virus is present in a population, it will be present in the following month with a chance of 75%. If it is absent from the population, the virus will be absent the following month with a chance of 80%. Suppose that the virus is currently absent from the population. What is the probability that over the next three months the virus will be present for exactly one month?



$P(VNN) + P(NVN) + P(NNV)$

$= (0.2)(0.25)(0.8) + (0.8)(0.2)(0.25) + (0.8)(0.8)(0.2)$

$= 0.208$

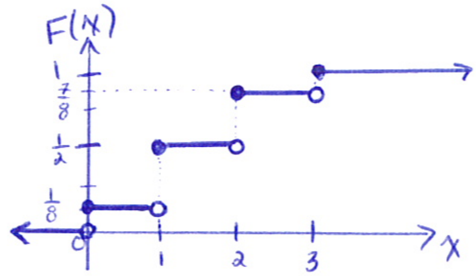
∴ The probability that the virus will be present one month out of the next three months is 0.208.

8. [4] Flip a fair coin three times. Let X count the number of heads that occur. Determine the cumulative distribution function, $F(x)$. Sketch the graph of $F(x)$.

range of $X = \{0, 1, 2, 3\}$ $\quad F(x) = P(X \leq x)$

x	$p(x)$
0	$\frac{1}{8}$
1	$\frac{3}{8}$
2	$\frac{3}{8}$
3	$\frac{1}{8}$

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{8} & 0 \leq x < 1 \\ \frac{1}{2} & 1 \leq x < 2 \\ \frac{7}{8} & 2 \leq x < 3 \\ 1 & x \geq 3 \end{cases}$$



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Student No.: ________________

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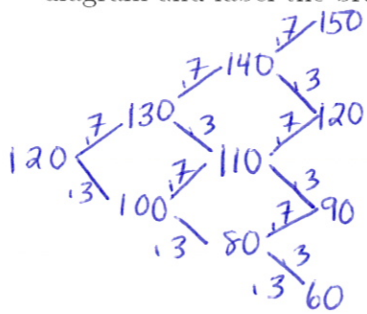
9. A population of tigers T_t, where $t = 0, 1, 2...$, is modelled by $T_{t+1} = T_t + I_t$. The immigration term, I_t, is equal to 10 with a 70% chance and -20 with a 30% chance. Suppose that $T_0 = 120$.

(a) [2] What is the sample space for the population of tigers after 3 years? Draw a tree diagram and label the branches with probabilities.



$S = \{60, 90, 120, 150\}$

(b) [2] Let X be the number of tigers in the population after 3 years. Determine the probability mass function for X.

x	$p(x)$
60	$(0.3)^3 = 0.027$
90	$3(0.3)^2(0.7) = 0.189$
120	$3(0.3)(0.7)^2 = 0.441$
150	$(0.7)^3 = 0.343$

(c) [2] What is the expected value of the population after 3 years?

$$E(X) = 60 \cdot p(60) + 90 \cdot p(90) + 120 \cdot p(120) + 150 \cdot p(150)$$
$$= 123$$

$\therefore$ The expected value of the population in three years is 123 tigers.