MATH 1B03/1ZC3 Winter 2019

Lecture 1: Systems of linear equations

Instructor: Dr Rushworth January 8th

Course information

Welcome to the combined nightclass for MATH 1B03/1ZC3.

Course Instructor: Dr Rushworth

Email: rushworw@mcmaster.ca

Email me with all questions about mathematics, and simple questions about the course (i.e. “when is the exam?", or “which topics are testable?").

If you have a more detailed question about the course administration (including MSAF and SAS), or if you have a request for relief/special consideration, **email the course coordinator**, Dr Childs via childsa@mcmaster.ca.

The best way to ask questions is to come to my office hours:

- Mondays 12:00 to 13:00
- Wednesdays 14:00 to 15:00
- Fridays 10:00 to 11:00

in Hamilton Hall 319. If you are not able to come to any of the above times, send me an email to schedule an appointment.

Textbook: *Elementary Linear Algebra - Applications Version*, 11th Edition, Anton and Rorres.

Link to notes: The course notes can be accessed via tinyurl.com/LAnotes2019.

Tutorials/Labs/Assignments/Etc: All other course information can be found on the course page in childsmath.

Course page

All of the important course information is available on the course page in childsmath (we are not using Avenue): log in to childsmath, click on 1B03/1ZC3, then click on "Course Information". Answers to most administrative questions can be found on the course page; try to find the answer to your question there before emailing Dr Childs.

Notes

The course notes will be uploaded to my webpage. The notes **are not a substitute for coming to the lectures**. If you miss the lectures you will be missing out on much of the content, you will miss a chance to ask questions, and you will miss out on hearing questions asked by other students which might help you.

Evaluation

The course is evaluated as follows

- 6 assignments - 2% each
- 5 labs - 2% each
- 2 midterms - 19% each
- 1 final - 40%

Good habits

If you are in your first year, you have the opportunity to develop good studying habits which will benefit you for your entire university career. If you are retaking this course, the best way to improve your grade is to form new, better, habits of studying.

Dr Childs has a brilliant section on the course page entitled 'How to Study', which contains a wealth of information on how to develop and maintain good studying habits. I highly recommend that you read it and follow its advice. Here are some of the key points

1. Never miss a lecture
2. Ask questions: during and after a lecture, at office hours, via email
3. Complete all of the assignments and practice problems

One of the worst habits a student can fall in to is neglecting a course until close to the final, perhaps even using their MSAF to avoid sitting a midterm which they could otherwise have sat. Here is some convincing evidence that this is a bad idea. In the course I taught last semester

- the average grade of students who sat **both** midterms: **B**
- the average grade of students who sat **only one** midterm: **D**
- the average grade of students who **did not sit either** midterm: **F**

In fact, the average course mark of students who did not sit either midterm was less than 10 %.

The point of this evidence is not that if you need to MSAF a midterm for medical or personal reasons then you are doomed. The point is that the best way to maximise your grade is to keep up with the course, and prepare very well for both midterms.

Other important information

- There are no tutorials in the first week
- No calculators are permitted on any of the exams (either the midterms or the final)
- If you are registered with SAS, book your tests as soon as possible.

Systems of linear equations

(corresponding to Chapter 1.1 of Anton-Rorres)

Linear algebra is the study of vector spaces and transformations between them. These transformations take the form of matrices. If you are a mathematician, then you should be excited already. If you are not a mathematician, I'll give you a short explanation why this course will be useful to you.

Mathematics is the study of the structure of information. Any time you attempt to codify, classify, or characterize information, you are doing mathematics. We are not interested in single pieces of information, or even in collections of information: we want to understand every possible way in which information can be given to us.

For example, a trained sheepdog may understand that two sheep add one sheep makes three sheep. But its unlikely that this knowledge will help the sheepdog to count horses, or days, or any other thing. It has learned how to add sheep, but it can't transfer this to other contexts[1].

A mathematician understands that there is something important about the fact that two add one equals three, regardless of whether we are counting sheep, horses, or days, or anything else: we look at the abstract equation $2 + 1 = 3$. By doing so, we can prove facts about numbers themselves, and apply these facts to every context that numbers arise in.

In this course we are not interested in the numbers, $1, 2, 3, \ldots$ - we are interested in systems of linear equations. These systems appear across biology, chemistry, physics, business, econmics, engineering, and just about any other subject you can think of. Understanding these systems abstractly allows us to better apply them to the subjects they appear in.

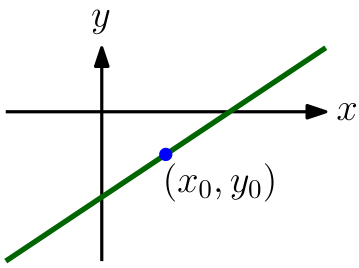
What is a linear equation?

We need to know what a linear equation is before we can talk about systems of them. Recall that the equation

$$2x + 3y = 5$$

describes the following line in the (x, y)-plane:

[1] this example is taken from *The Skeleton Key of Mathematics* by D.E. Littlewood

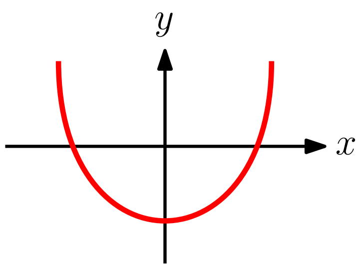


Because of this, we say that it is a linear equation in 2-variables (x and y are the variables). Any point (x_0, y_0) on the line is a solution to the linear equation: that is because $2x_0 + 3y_0 = 5$.

The equation

$$x^2 - y = 4$$

is not a linear equation, because it describes the following curve, not a straight line:



In general, if an equation contains a variable with a power higher than 1, it is not a linear equation.

We do not want to be constrained to the (x, y)-plane, however, so we need to look at equations which have more than 2 variables.

Definition 1.1

Let $a_1, a_2, \ldots, a_n, b$ be constants, and $x_1, x_2, \ldots, x_n$ be variables. A linear equation in n variables is an equation of the form

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

We refer to the constant a_i as the coefficient of the variable x_i. If $b = 0$ we say that the linear equation is homogeneous.

A linear equation in n variables still describes a line, but now the line sits in n-dimensional space, rather than the 2-dimensional (x, y)-plane. Therefore, a solution is still a point on this line.

Definition 1.2

Let

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

be a linear equation in n variables. A solution is a set of constants $s_1, s_2, \ldots, s_n$ such that

$$a_1s_1 + a_2s_2 + \ldots + a_ns_n = b.$$

In other words, a solution is simply a collection of numbers which we can plug into the left hand side of the equation, and obtain the right hand side.

Example 1.3

When there are only two or three variables present we often use x, y and z to denote them, as in the following examples.

The equation

$$17x + 6y - \frac{3}{2}z = 0$$

is a linear equation in the variables x, y, z.

The equation

$$2x - y^2x^2 - y^3 = 4$$

is not a linear equation (it contains powers higher than 1).

Question 1.4

Is

$$\pi x - \sqrt{6}y = 5^{\frac{1}{3}}$$

a linear equation?

Systems of equations

A system of linear equations is a collection of linear equations a.k.a. a system of simultaneous equations. For example

$$\begin{aligned} x + y + z &= 2 \\ x - y - z &= 1. \end{aligned} \tag{$\star$}$$

is a system of 2 equations in 3 variables. A solution to this system is a triple of numbers $(x_0, y_0, z_0,)$ which is a solution to both the equations in the system: the triple $\left(\frac{3}{2}, 0, \frac{1}{2}\right)$ is a solution, for example.

Remember that an individual linear equation describes a line, and a solution to the equation is a point on that line. Therefore, a solution to a system of equations is a point which lies on all of the lines described by the equations in the system. Our first task is to learn how to solve a systems of linear equations i.e. find every possible solution to the system.

How many different solutions can a system of linear equations possess? There are exactly three possibilities.

Fact 1.5

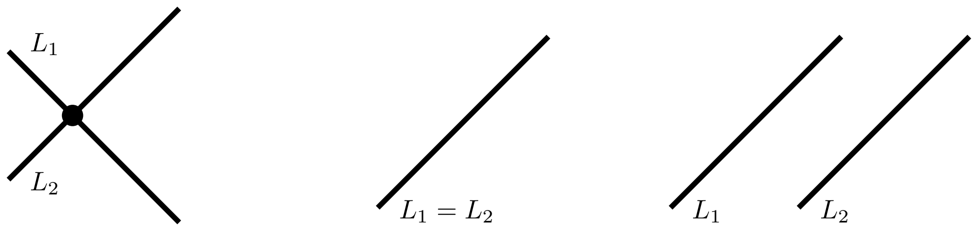
A system of m linear equations in n variables has

1. a unique solution, or
2. infinitely many solutions, or
3. no solutions.

There are no other possibilities.

If a system does not possess any solutions it is often referred to as an inconsistent system.

Let's see what these possiblities look like in the case of a system of 2 equations in 2 variables. If L_1 and L_2 are the lines described by the equations of the system, the possibilities are:



lines intersection once
unique solution

lines overlap
infinitely many solutions

lines parallel
no solutions

These pictures can be generalized to more equations and more variables, but the three possibilities remain the same.

Example 1.6

Here are some examples of systems and their solutions. You could find these solutions by plotting the lines described by the equations - later on we are going to see a much faster way of doing it.

The system

$$\begin{aligned} x + y &= 1 \\ x - y &= 0 \end{aligned}$$

has the unique solution $\left(\frac{1}{2}, \frac{1}{2}\right)$.

The system

$$\begin{aligned} x + y &= 1 \\ 2x + 2y &= 2 \end{aligned}$$

has infinitely many solutions.

The system

$$\begin{aligned} x + 2y &= 1 \\ x + 2y &= 2 \end{aligned}$$

has no solutions (it is an inconsistent system).

Augmented matrices

When the number of variables and equations is small, we can write out the entire system easily (like we did in Equation (★) and Example 1.6). If the number of variables and equations is large, writing the system out like this will become tedious and unwieldy. Instead, we package up the information in a nicer way using a matrix[2].

A matrix is a rectangular array of numbers, like this

$$\begin{bmatrix} 2 & 3 & 6 \\ -1 & 0 & 5 \end{bmatrix}$$

This matrix has two rows (the horizontal sections) and three columns (the vertical sections). When specifying the rows and columns, we always start from the top left corner of the matrix. For example, $\begin{matrix} 6 \\ 5 \end{matrix}$ is the third column, and $-1 \quad 0 \quad 5$ is the second row of this matrix. The numbers are referred to as the entries of the matrix.

We can write a system of linear equations as a matrix. Consider the system of 2 equations in 3 variables

$$\begin{aligned} x_1 - 3x_2 + 3x_3 &= 1 \\ 2x_1 + x_2 - x_3 &= 5 \end{aligned}$$

The augmented matrix of this system is

$$\begin{bmatrix} 1 & -3 & 3 & 1 \\ 2 & 1 & -1 & 5 \end{bmatrix}$$

To produce the augmented matrix we have placed the coefficients of the variable x_1 in the first column, the coefficients ofx_2 in the second column, and the coefficients ofx_3 in the third column. Finally, we have placed the right hand sides of the equations in the fourth column.

This matrix is called an augmented matrix to help us remember that the right-most column contains the right hand sides of the equations, while the other columns contain the coefficients. We will solve systems of linear equations by manipulating their augmented matrices.

[2]singular: matrix, plural: matrices

It is important to notice that we can change how we write the system without changing its solutions. For example, given the system

$$\begin{aligned} x + y &= 1 \\ x - y &= 0 \end{aligned}$$

it is clear that if we swap the order of the equations we will not change the solutions of the system (after all, we are not changing the lines the system describes). That is, we can write the system as

$$\begin{aligned} x - y &= 0 \\ x + y &= 1 \end{aligned}$$

instead. What will this do to the augmented matrix of the system? It will swap two of its rows!

In addition to swapping two equations, there are two other operations we can apply to a system that will not change its solutions. These three operations are the elementary operations.

Definition 1.7

The elementary operations on a system of linear equations are

1. Swap two equations
2. Multiply one equation by a non-zero number
3. Add a multiple of one equation to another equation

Fact 1.8

The elementary operations do not change the solutions of a system of linear equations.

Each of the elementary operations on a system of linear equations has an affect on the augmented matrix of the system.

Definition 1.9

The elementary row operations on a matrix are

1. Swap two rows
2. Multiply one row by a non-zero number
3. Add a multiple of one row to another row

Applying an elementary operation to a system of linear equations is exactly the same as applying an elementary row operation to its augmented matrix. Therefore, we can use the augmented matrix and elementary row operations to solve a system of linear equations. Here is an example.

Example 1.10

Consider the system of 2 equations in 2 variables

$$\begin{aligned} x + y &= 1 \\ x - y &= 0 \end{aligned}$$

Its augmented matrix is

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

Let's use elementary row operations to simplify this matrix. Denote the first row by $R1$, and the second row by $R2$. Then

$$\begin{bmatrix}1 & 1 & 1\\1 & -1 & 0\end{bmatrix}$$
$$\rightsquigarrow \quad \text{add } R2 \text{ to } R1$$
$$\begin{bmatrix}2 & 0 & 1\\1 & -1 & 0\end{bmatrix}$$
$$\rightsquigarrow \quad \text{multiply } R1 \text{ by } \tfrac{1}{2}$$
$$\begin{bmatrix}1 & 0 & \frac{1}{2}\\1 & -1 & 0\end{bmatrix}$$
$$\rightsquigarrow \quad \text{add } -1 \times R1 \text{ to } R2$$
$$\begin{bmatrix}1 & 0 & \frac{1}{2}\\0 & -1 & -\frac{1}{2}\end{bmatrix}$$
$$\rightsquigarrow \quad \text{multiply } R2 \text{ by } -1$$
$$\begin{bmatrix}1 & 0 & \frac{1}{2}\\0 & 1 & \frac{1}{2}\end{bmatrix}$$

What system does this final augmented matrix describe? It is

$$\begin{aligned}1x + 0y &= \frac{1}{2}\\0x + 1y &= \frac{1}{2}\end{aligned}$$

and we see immediately that the unique solution is $\left(\frac{1}{2}, \frac{1}{2}\right)$ (just as we said it was in Example 1.6).

This is an example of the method we are going to use to solve systems of linear equations: form the augmented matrix, and apply elementary row operations to obtain a more simple matrix, from which we can “read off” the solution.

Suggested problems

Practice the material in this lecture by attempting the following problems in **Chapter 1.1** of Anton-Rorres, starting on page 8

- Questions 5, 7, 9, 13, 21
- True/False questions (a), (c), (e), (h)