**COMPSCI/SFWRENG 2FA3**
**Discrete Mathematics with Applications II**
**Winter 2020**

# Assignment 4 with Solutions


**Dr. William M. Farmer**
**McMaster University**


Revised: February 16, 2020

Assignment 4 consists of two problems. You must write your solutions to the problems using LaTeX.

Please submit Assignment 4 as two files, `Assignment_4_`*`YourMacID`*`.tex` and `Assignment_4_`*`YourMacID`*`.pdf`, to the Assignment 4 folder on Avenue under Assessments/Assignments. *`YourMacID`* must be your personal MacID (written without capitalization). The `Assignment_4_`*`YourMacID`*`.tex` file is a copy of the LaTeX source file for this assignment (`Assignment_4.tex` found on Avenue under Contents/Assignments) with your solution entered after each problem. The `Assignment_4_`*`YourMacID`*`.pdf` is the PDF output produced by executing

```
pdflatex Assignment_4_YourMacID
```

This assignment is due **Sunday, February 16, 2019 before midnight.** You are allow to submit the assignment multiple times, but only the last submission will be marked. **Late submissions and files that are not named exactly as specified above will not be accepted!** It is suggested that you submit your preliminary `Assignment_4_`*`YourMacID`*`.tex` and `Assignment_4_`*`YourMacID`*`.pdf` files well before the deadline so that your mark is not zero if, e.g., your computer fails at 11:50 PM on February 16.

**Although you are allowed to receive help from the instructional staff and other students, your submission must be your own work. Copying will be treated as academic dishonesty! If any of the ideas used in your submission were obtained from other students or sources outside of the lectures and tutorials, you must acknowledge where or from whom these ideas were obtained.**

## Background

1. Let $(S, <)$ be a strict partial order. $(S, <)$ is *dense* if, for all $x, y \in S$ with $x < y$, there is some $z \in S$ such that $x < z < y$.

2. A *queue* is a finite sequence of elements for which elements are added ("enqueued") to the back of the sequence and removed ("dequeued") from the front of the sequence. An *empty queue* is a queue with no members. A *singleton queue* is a queue with a single element that is obtain by enqueuing an element to an empty queue.

## Problems

1. **[10 points]** Construct in MSFOL a theory $T$ of strict total orders that are dense and have minimum and maximum elements. Give two models for $T$.

   **Put your name, MacID, and date here.**

   **Solution:**

   Let $\Sigma = (\{\alpha\}, \{\mathsf{min}, \mathsf{max}\}, \emptyset, \{<\}, \tau)$ where $\tau(\mathsf{min}) = \tau(\mathsf{max}) = \alpha$ and $\tau(<) = \alpha \times \alpha \to \mathbb{B}$ and $T = (\Sigma, \Gamma)$ where $\Gamma$ contains the following $\Sigma$-sentences:

   a. $\forall x : \alpha \mathrel{.} \neg(x < x)$.
   b. $\forall x, y : \alpha \mathrel{.} x < y \Rightarrow \neg(y < x)$.
   c. $\forall x, y, z : \alpha \mathrel{.} (x < y \wedge y < z) \Rightarrow x < z$.
   d. $\forall x, y : \alpha \mathrel{.} x < y \vee y < x \vee x = y$.
   e. $\forall x, y : \alpha \mathrel{.} x < y \Rightarrow (\exists z : \alpha \mathrel{.} x < z \wedge z < y)$.
   f. $\forall x : \alpha \mathrel{.} x \neq \mathsf{min} \Rightarrow \mathsf{min} < x$.
   g. $\forall x : \alpha \mathrel{.} x \neq \mathsf{max} \Rightarrow x < \mathsf{max}$.

   Let $\mathcal{M}$ be a model for $T$. The first four axioms say $\mathcal{M}$ is a strict total order; the fifth says $\mathcal{M}$ is dense; and the last two say $\mathcal{M}$ has minimum and maximum elements.

   Note: The constant symbols $\mathsf{min}$ and $\mathsf{max}$ are convenient to have in the signature, but they are not strictly needed.

   Let $\mathcal{M}_1 = (\{D_\alpha\}, I)$ where $D_\alpha = \mathbb{Q} \cup \{-\infty, +\infty\}$, $I(\mathsf{min}) = -\infty$, $I(\mathsf{max}) = +\infty$, and $I(<)$ is the usual strict total order on $\mathbb{Q} \cup \{-\infty, +\infty\}$. It follows from Exercise 4 of the Week 04 exercises that $\mathcal{M}_1$ is a model for $T$.

   Let $\mathcal{M}_2 = (\{D_\alpha\}, I)$ where $D_\alpha = \{r \in \mathbb{R} \mid 0 \leq r \leq 1\}$, $I(\mathsf{min}) = 0$, $I(\mathsf{max}) = 1$, and $I(<)$ is the usual strict total order on $\mathbb{R}$. Clearly, $\mathcal{M}_2$ is a strict total order that is dense and has minimum and maximum elements. Hence $\mathcal{M}_2$ is a model for $T$.

2. **[10 points]** Let $\Sigma_{\mathsf{queue}} = (\mathcal{B}, \mathcal{C}, \mathcal{F}, \mathcal{P}, \tau)$ where:

   a. $\mathcal{B} = \{\mathsf{Element}, \mathsf{Queue}\}$.
   b. $\mathcal{C} = \{\mathsf{error}, \mathsf{empty}\}$.
   c. $\mathcal{F} = \{\mathsf{front}, \mathsf{back}, \mathsf{enqueue}, \mathsf{dequeue}\}$.
   d. $\mathcal{P} = \emptyset$.
   e. $\tau(\mathsf{error}) = \mathsf{Element}$.
   f. $\tau(\mathsf{empty}) = \mathsf{Queue}$.
   g. $\tau(\mathsf{front}) = \mathsf{Queue} \rightarrow \mathsf{Element}$.
   h. $\tau(\mathsf{back}) = \mathsf{Queue} \rightarrow \mathsf{Element}$.
   i. $\tau(\mathsf{enqueue}) = \mathsf{Element} \times \mathsf{Queue} \rightarrow \mathsf{Queue}$.
   j. $\tau(\mathsf{dequeue}) = \mathsf{Queue} \rightarrow \mathsf{Queue}$.

   Construct in MSFOL a theory $T = (\Sigma_{\mathsf{queue}}, \Gamma_{\mathsf{queue}})$ of queues. $\Gamma_{\mathsf{queue}}$ should contain axioms that say:

   a. The front of an empty queue is the error element.
   b. The front of a singleton queue is the single element in the queue.
   c. Let $q$ be a queue obtain by enqueuing $e$ to a nonempty queue $q'$. The front of $q$ is the front of $q'$.
   d. The back of an empty queue is the error element.
   e. Let $q$ be a queue obtain by enqueuing $e$ to a queue $q'$. The back of $q$ is $e$.
   f. The dequeue of an empty queue is the empty queue.
   g. The dequeue of a singleton queue is the empty queue.
   h. Let $q$ be a queue obtain by enqueuing $e$ to a nonempty queue $q'$. The dequeue of $q$ is the enqueue of $e$ to the dequeue of $q'$.

   **Put your name, MacID, and date here.**

   **Solution**:

   $\Sigma_{\mathsf{queue}}$ is already defined, so we need only define $\Gamma_{\mathsf{queue}}$. $\Gamma_{\mathsf{queue}}$ contains the following eight axioms corresponding the eight informal statements given above: That is, it suffices that $\Gamma$ contains the following axioms:

   a. $\mathsf{front}(\mathsf{empty}) = \mathsf{error}$.
   b. $\forall\, e : \mathsf{Element}\ .\ \mathsf{front}(\mathsf{enqueue}(e, \mathsf{empty})) = e$.
   c. $\forall\, e : \mathsf{Element}, q : \mathsf{Queue}\ .$
      $q \neq \mathsf{empty} \Rightarrow \mathsf{front}(\mathsf{enqueue}(e, q)) = \mathsf{front}(q)$.
   d. $\mathsf{back}(\mathsf{empty}) = \mathsf{error}$.
   e. $\forall\, e : \mathsf{Element}, q : \mathsf{Queue}\ .\ \mathsf{back}(\mathsf{enqueue}(e, q)) = e$.

f. $\mathsf{dequeue}(\mathsf{empty}) = \mathsf{empty}$.

g. $\forall\, e : \mathsf{Element}\ .\ \mathsf{dequeue}(\mathsf{enqueue}(e, \mathsf{empty})) = \mathsf{empty}$.

h. $\forall\, e : \mathsf{Element}, q : \mathsf{Queue}\ .$
$q \neq \mathsf{empty} \Rightarrow \mathsf{dequeue}(\mathsf{enqueue}(e, q)) = \mathsf{enqueue}(e, \mathsf{dequeue}(q))$.