Example: Which is approximately the upper bound for the difference between $\sum_{n=2}^{\infty} \frac{\ln(n)}{n^3}$ and $\sum_{n=2}^{9} \frac{\ln(n)}{n^3}$?

4.3 The Comparison Tests (Chapter 11.4)

We focus on series now with non-negative terms, i.e., $S = \sum_{n=1}^{\infty} a_n$ with $a_n \geq 0$.

$\Rightarrow \qquad S_{n+1} \qquad\qquad S_n$

Thus, if S_n is ________________, then $\{S_n\}$ converges, i.e., $\sum_{n=1}^{\infty} a_n$ ________________.

Comparison Test:

Consider series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ with $0 \leq a_n \leq b_n$ for all $n = 1, 2, 3, \ldots$.

$$\sum_{n=1}^{\infty} a_n \qquad\qquad \Longrightarrow \qquad \sum_{n=1}^{\infty} b_n$$

and

$$\sum_{n=1}^{\infty} b_n \qquad\qquad \Longrightarrow \qquad \sum_{n=1}^{\infty} a_n$$

Relaxing of conditions are possible:

Example:

$A)$ $$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^5 + n + 1}}$$

$B)$ $$\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$$

Example:

$C)$ $$\sum_{n=1}^{\infty} \frac{1+\cos(n)}{n^2+1}$$

$D)$ $$\sum_{n=9}^{\infty} \frac{2}{\sqrt{n}-1}$$

Can you apply the Comparison test to discuss the convergence of $\sum_{n=1}^{\infty} \frac{\cos(n)}{n^3}$?

Limit Comparison Test:

Consider $\sum a_n$ and $\sum b_n$ with $a_n, b_n > 0$.

If $$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$,

then ______________________________________.

Explanation:

If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then for sufficiently large n, we have

$$\left|\frac{a_n}{b_n} - c\right| \qquad\qquad \Rightarrow$$

What can we say if the limit does not exist?