MATHEMATICS 1LS3 TEST 3

Day Class
Duration of Examination: 60 minutes
McMaster University, 5 November 2012

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FIRST NAME (please print): SOLUTIONS

FAMILY NAME (please print): ______

Student No.: ______

THIS TEST HAS 8 PAGES AND 7 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE.

Total number of points is 40. Marks are indicated next to the problem number. Any non-graphing calculator is allowed.

USE PEN TO WRITE YOUR TEST. IF YOU USE A PENCIL YOUR TEST WILL NOT BE ACCEPTED FOR REMARKING (IF NEEDED).

You need to show work to receive full credit.

Problem	Points	Mark
1	6	
2	6	
3	5	
4	5	
5	7	
6	5	
7	6	
TOTAL	40	

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1. **Multiple choice questions: circle ONE answer. No justification is needed.**

(a)[3] It is known that $f'(a) = 0$ and $f''(a) < 0$. Which statements is/are true?

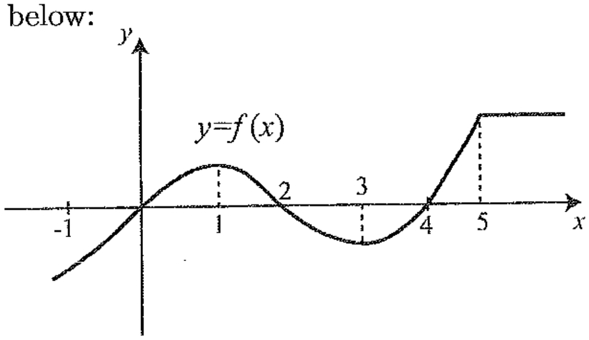
(I) $f(x)$ is concave down at a ✓

(II) The linear approximation of $f(x)$ at $x = a$ is a horizontal line ✓

(III) $f(x)$ has a relative minimum at a ✗

(A) none	(B) I only	(C) II only	(D) III only
(E) I and II	(F) I and III	(G) II and III	(H) all three

(b)[3] Determine which of the statements is/are true for the function $f(x)$ whose graph is given below:



(I) $x = 2$ is a critical point (critical number) of $f(x)$ ✗

(II) $x = 3$ is a critical point (critical number) of $f(x)$ ✓

(III) $x = 5$ is a critical point (critical number) of $f(x)$ ✓

(A) none	(B) I only	(C) II only	(D) III only
(E) I and II	(F) I and III	(G) II and III	(H) all three

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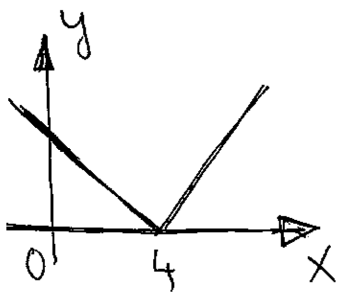
2. **Identify each statement as true or false (circle your choice). No justification is needed.**

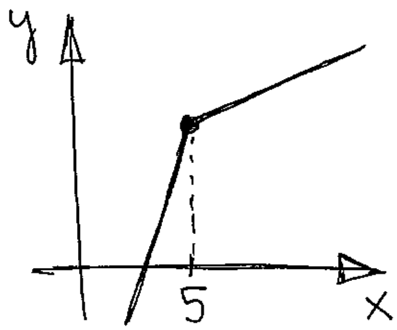
(a)[2] The function $f(x) = |x - 4|$ is differentiable at $x = 0$.

TRUE FALSE



(b)[2] If $x = 5$ is in the domain of $f(x)$ and $f'(5)$ does not exist, then $f(x)$ must have an extreme value (i.e., either minimum or maximum) at $x = 5$.

TRUE FALSE



(c)[2] $x = \pi/4$ is a critical point (critical number) of the function $f(x) = \sin x + \cos x$

TRUE FALSE

$f'(x) = \cos x - \sin x$

$f'(\pi/4) = \cos(\pi/4) - \sin(\pi/4) = 0$

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3. (a)[1] Write the formula for the Taylor polynomial $T_2(x)$ of a function $f(x)$ based at $a = 1$.

$$T_2(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2$$

(b)[3] Find the Taylor polynomial $T_2(x)$ for the function $f(x) = \sqrt[3]{x}$ near $x = 1$.

$$f(x) = x^{1/3} \to f(1) = 1$$

$$f'(x) = \frac{1}{3}x^{-2/3} \to f'(1) = \frac{1}{3}$$

$$f''(x) = \frac{1}{3}\left(-\frac{2}{3}\right)x^{-5/3} \to f''(1) = -\frac{2}{9}$$

$$\text{so } T_2(x) = 1 + \frac{1}{3}(x-1) - \frac{1}{9}(x-1)^2$$

(c)[1] Use your answer in (b) to find an approximation of $f(x) = \sqrt[3]{1.2}$ $\approx T_2(1.2)$

$$T_2(1.2) = 1 + \frac{1}{3}(0.2) - \frac{1}{9}(0.2)^2 \approx 1.062$$

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4. (a) [2] Give the statement of the Extreme Value Theorem. Clearly separate assumption(s) from conclusion(s).

assume: $f(x)$ is continuous on a closed interval $[a,b]$

then: $f(x)$ has an absolute max. and an absolute min. in $[a,b]$

(b) [3] Find the absolute maximum and the absolute minimum of the function $f(x) = \dfrac{\ln x}{x^2}$ on the interval $[1, 4]$.

$$f'(x) = \frac{\frac{1}{x}\cdot x^2 - \ln x\,(2x)}{x^4} = \frac{x - 2x\ln x}{x^4} = \frac{1-2\ln x}{x^3}$$

$$f'(x) = 0 \rightarrow 1 - 2\ln x = 0, \quad \ln x = \frac{1}{2}, \quad x = e^{1/2}$$

no other critical points

x	$f(x)$
1	$\ln 1/1^2 = 0$
4	$\ln 4/4^2 \approx 0.0866$
$e^{1/2} \approx 1.6487$	$\dfrac{\ln(e^{1/2})}{(e^{1/2})^2} = \dfrac{1}{2e} \approx 0.1839$

abs. min. $= 0$ at $x = 1$

abs. max $= \dfrac{1}{2e} \approx 0.1839$ at $x = e^{1/2}$

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5. (a)[2] Find $f'(0)$ if $f(x) = \arctan(x^2) - e^{x^2+x}$.

$$f'(x) = \frac{1}{1+x^4} \cdot 2x - e^{x^2+x} \cdot (2x+1)$$

$$f'(0) = 0 - 1 = -1$$

(b)[3] Find all critical points (critical numbers) of the function $f(x) = x^2 e^{-x}$.

$$f'(x) = 2x e^{-x} + x^2 e^{-x}(-1) = x e^{-x}(2-x)$$

$f'(x) = 0 \rightarrow x = 0, 2$ (two critical points)

$f'(x)$ dne $\rightarrow$ no such x

(c)[2] Identify interval(s) where the function $f(x)$ from (b) is decreasing.

x		0		2	
f'	$-$		$+$		$-$
f	$\searrow$		$\nearrow$		$\searrow$

$(-\infty, 0)$ and $(2, \infty)$

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6. Consider $f(x) = x^{2/3}(x-1)^2$.

(a)[1] Show that $f'(x) = \dfrac{(x-1)\left(\frac{8x}{3} - \frac{2}{3}\right)}{x^{1/3}}$.

$$f'(x) = \frac{2}{3}x^{-1/3}(x-1)^2 + x^{2/3} \cdot 2(x-1)$$

$$= \frac{1}{x^{1/3}}(x-1)\left[\frac{2}{3}(x-1) + 2x\right]$$

$$= \frac{(x-1)\left(\frac{8x}{3} - \frac{2}{3}\right)}{x^{1/3}}$$

$$\frac{8x-2}{3} = 0$$

$$8x = 2$$

$$x = 1/4$$

(b)[4] Find all relative extreme values of $f(x)$.

from (a): cp's are $x = 0, 1, 1/4$

x		0		1/4		1	
f'	⊖		⊕		⊖		⊕
f	↘		↗		↘		↗

rel. min. at $x = 0, 1$

rel. max. at $x = 1/4$

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7. (a)[3] Using implicit differentiation, find y' if $x^2y^3 = x - e^y$.

$$2xy^3 + x^2 \cdot 3y^2 y' = 1 - e^y y'$$

$$y'(3x^2y^2 + e^y) = 1 - 2xy^3$$

$$y' = \frac{1 - 2xy^3}{3x^2y^2 + e^y}$$

(b)[3] Let $f(x) = \dfrac{3}{x^2} + 7$. **Using the definition** of the derivative, find $f'(x)$. (No credit is given if differentiation rules are used.)

$$f'(x) = \lim_{h \to 0} \frac{\frac{3}{(x+h)^2} + \cancel{7} - \left(\frac{3}{x^2} + \cancel{7}\right)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \cdot \frac{3x^2 - 3(x+h)^2}{(x+h)^2 \cdot x^2} \to \cancel{3x^2} - \cancel{3x^2} - 6xh - h^2$$

$$= \lim_{h \to 0} \frac{-6x - h}{(x+h)^2 \cdot x^2} = \frac{-6x}{x^4} = -\frac{6}{x^3}$$

THE END