Announcements

Topics:

- Solutions in the Phase Plane (8.7)

In the Functions of Several Variables module:

- **Section 1:** Introduction to Functions of Several Variables (Basic Definitions and Notation)
- **Section 2:** Graphs, Level Curves + Contour Maps
- **Section 3:** Limits and Continuity

To Do:

- Read section 8.7 in the textbook and sections 1, 2, and 3 in the "Functions of Several Variables" module
- Work on Assignments and Suggested Practice Problems assigned on the webpage under the SCHEDULE + HOMEWORK link

Single Variable Calculus

Definition:

A real-valued function *f* of one variable is a rule that assigns to each real number *x* in a set D called the domain a unique real number *y* in a set R called the range.

We denote this by $y = f(x)$.

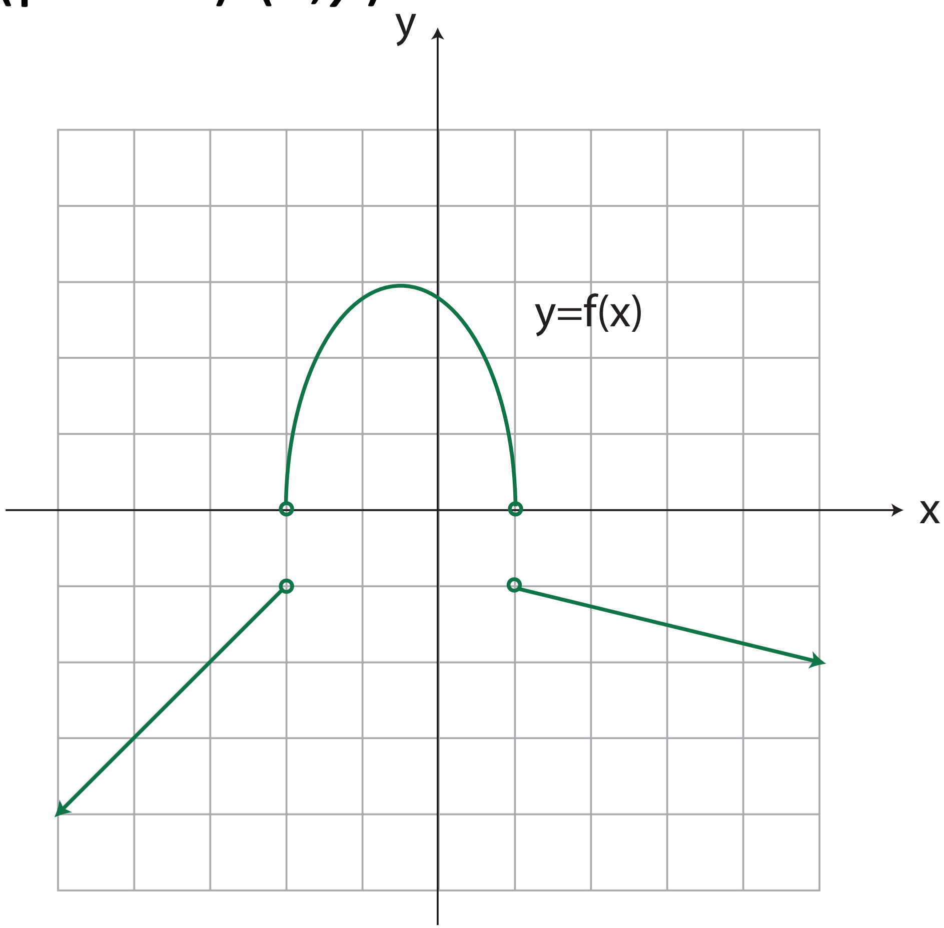
Single Variable Calculus

Domain of *f(x)*:
The set of all *x*-values for which *f(x)* is defined as a real number. (All possible *x*-values the equation will accept as input).

Range of *f(x)*:
The set of all *y*-values that *f* can attain.
(All possible output values).

Single Variable Calculus

The **graph** of a function *f* is a set of all ordered pairs (points) (*x*,*y*) where *x* is in the domain of *f* and *y=f(x)*.



Domain?

Range?

Functions of Two Variables

Definition:

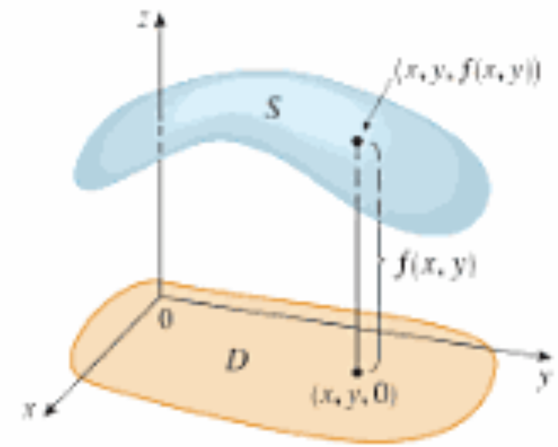
A real-valued function f of two variables is a rule that assigns to each ordered pair of real numbers (x,y) in a set D called the domain a unique real number z in a set R called the range.

We denote this by

$$z = f(x,y).$$

Functions of Two Variables

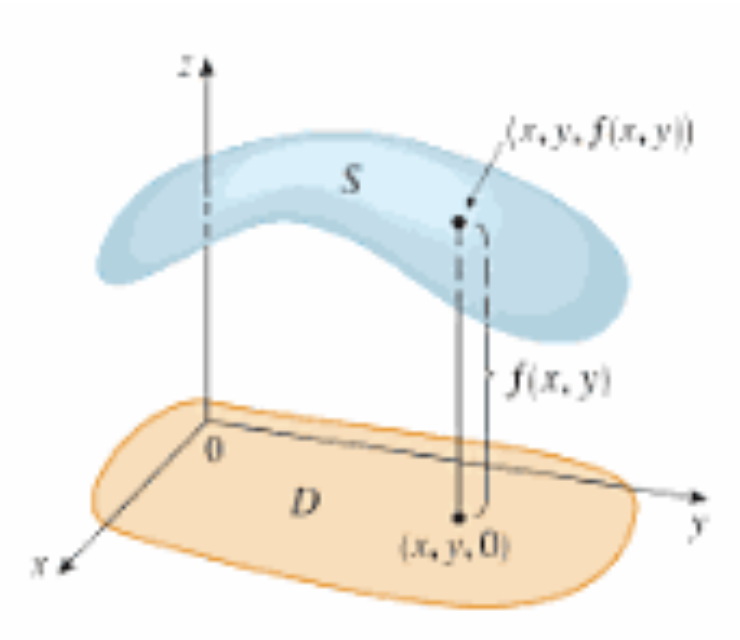
Domain of *f(x,y)*:
The set of all ordered pairs (x,y) for which $f(x,y)$ is a real number. (A subset of the xy-plane, R^2).



Range of *f(x,y)*:
The set of all z-values that f can attain. (A subset of the real number line, R).

Functions of Two Variables

The **graph** of a function $z=f(x,y)$ of two variables is the set of points (x,y,z) in the space R^3 such that $z=f(x,y)$ for some (x,y) in the domain of f.



Functions of Two Variables

Example: Body Mass Index

$$BMI(w,h) = \frac{w}{h^2}$$

where w is a person's weight in kilograms and h their height in metres.

BMI is the **dependent variable;**
w and h are the two **independent variables**.

Functions of Two Variables

Example: Body Mass Index

(a) Determine the Body Mass Index of a person weighing 56 kg with a height of 174 cm.

(b) Compute $BMI(56,h)$ and $BMI(w,1.74)$ and analyze the resulting functions.

Functions of Two Variables

Example: Body Mass Index

WEIGHT lbs	100	105	110	115	120	125	130	135	140	145	150	155	160	165	170	175	180	185	190	195	200	205	210	215
kgs	45.5	47.7	50.0	52.3	54.5	56.8	59.1	61.4	63.6	65.9	68.2	70.5	72.7	75.0	77.3	79.5	81.8	84.1	86.4	88.6	90.9	93.2	95.5	97.7
HEIGHT in/cm	Underweight					Healthy					Overweight					Obese				Extremely obese				
5'0" - 152.4	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42
5'1" - 154.9	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	36	37	38	39	40
5'2" - 157.4	18	19	20	21	22	22	23	24	25	26	27	28	29	30	31	32	33	33	34	35	36	37	38	39
5'3" - 160.0	17	18	19	20	21	22	23	24	24	25	26	27	28	29	30	31	32	32	33	34	35	36	37	38
5'4" - 162.5	17	18	18	19	20	21	22	23	24	24	25	26	27	28	29	30	31	31	32	33	34	35	36	37
5'5" - 165.1	16	17	18	19	20	20	21	22	23	24	25	25	26	27	28	29	30	30	31	32	33	34	35	35
5'6" - 167.6	16	17	17	18	19	20	21	21	22	23	24	25	25	26	27	28	29	29	30	31	32	33	34	34
5'7" - 170.1	15	16	17	18	18	19	20	21	22	22	23	24	25	25	26	27	28	29	29	30	31	32	33	33
5'8" - 172.7	15	16	16	17	18	19	19	20	21	22	22	23	24	25	25	26	27	28	28	29	30	31	32	32
5'9" - 175.2	14	15	16	17	17	18	19	20	20	21	22	22	23	24	25	25	26	27	28	28	29	30	31	31
5'10" - 177.8	14	15	15	16	17	18	18	19	20	20	21	22	23	23	24	25	25	26	27	28	28	29	30	30
5'11" - 180.3	14	14	15	16	16	17	18	18	19	20	21	21	22	23	23	24	25	25	26	27	28	28	29	30
6'0" - 182.8	13	14	14	15	16	17	17	18	19	19	20	21	21	22	23	23	24	25	25	26	27	27	28	29
6'1" - 185.4	13	13	14	15	15	16	17	17	18	19	19	20	21	21	22	23	23	24	25	25	26	27	27	28
6'2" - 187.9	12	13	14	14	15	16	16	17	18	18	19	19	20	21	21	22	23	23	24	25	25	26	27	27
6'3" - 190.5	12	13	13	14	15	15	16	16	17	18	18	19	20	20	21	21	22	23	23	24	25	25	26	26
6'4" - 193.0	12	12	13	14	14	15	15	16	17	17	18	18	19	20	20	21	22	22	23	23	24	25	25	26

BMI Chart

Functions of Two Variables

Example:

Find and sketch the domain of each function.

(a) $f(x,y) = \ln(x + y - 1)$

(b) $f(x,y) = \sqrt{xy}$

(c) $BMI(w,h) = \dfrac{w}{h^2}$

Functions of Two Variables

Example:

Determine the range of each function.

(a) $f(x,y) = \ln(x + y - 1)$

(b) $f(x,y) = e^{1-x^2-y^2}$

(c) $f(x,y) = \dfrac{1}{x^2 - y^2}$

Functions of Two Variables

Linear Functions:
Linear functions in two variables are of the form

$$f(x,y) = ax + by + c$$

'linear' because the exponent of both *x* and *y* is 1

where *a*, *b*, and *c* are real numbers.

<u>Domain</u>: all of R^2

<u>Graph</u>: plane

<u>Example</u>: $f(x,y) = 6 - 3x - 2y$

*Note: A linear functions is just a special case of a polynomial function (next)

Functions of Two Variables

Polynomial Functions:
A polynomial functions in two variables is a sum of terms of the form

$$cx^k y^l$$

where *c* is a real number and *k* and *l* are non-negative integers.

<u>Domain</u>: all of R^2

<u>Examples</u>:

$$f(x,y) = 4 - x^2 - y^2 \qquad g(x,y) = 3xy + x^4y^3 - 1$$

Functions of Two Variables

Rational Functions:
A rational function in two variables is a quotient of two polynomials in two variables.

<u>Domain</u>: all of R^2 except points at which the denominator = 0

<u>Examples</u>:

$$f(x,y) = \frac{x-y}{1+x^2+y^2} \qquad g(x,y) = \frac{3xy + x^4y^3 - 1}{x^2 - y^2}$$

Functions of Two Variables

Example:

Sketch the graph of each function.

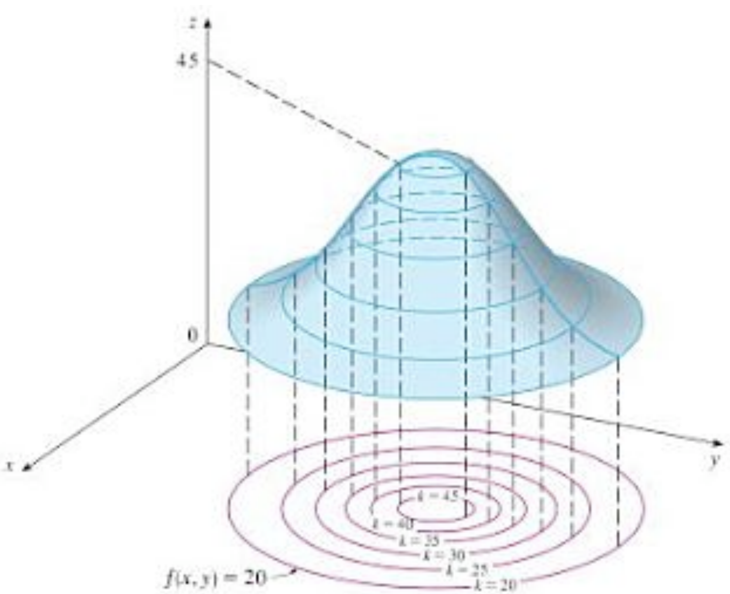
Describe domain and range.

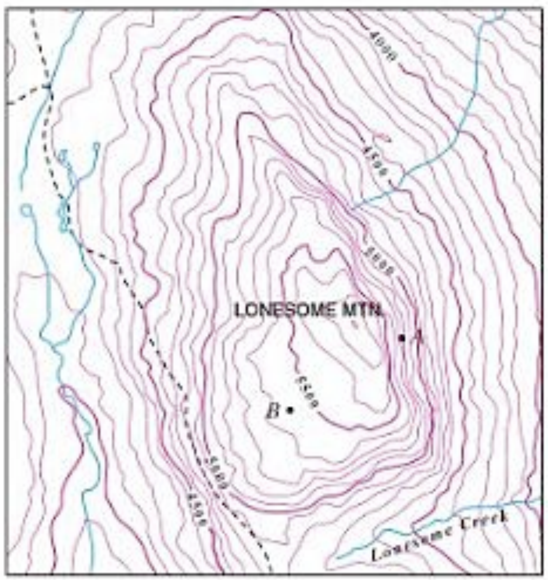
(a) $f(x,y) = 6 - 3x - 2y$

(b) $f(x,y) = 4 - x^2 - y^2$

Contour Maps and Level Curves

In general, sketching the graphs of functions of two variables (surfaces) is difficult so instead we sketch 2-dimensional representations of these surfaces in R^2 called <u>*contour maps*</u>.





Contour Maps and Level Curves

Level Curves:
The level curves of a function f of two variables are the curves with equations

$$f(x,y) = k$$

where k is a constant in the <u>**RANGE**</u> of the function.

A level curve $f(x,y) = k$ is a curve in the domain of f along which the graph of f has height k.

Contour Maps and Level Curves

Contour Maps:
A contour map is a collection of level curves.

To visualize the graph of f from the contour map, imagine raising each level curve to the indicated height.

The surface is steep where the level curves are close together and it is flatter where they are farther apart.

Contour Maps and Level Curves

Examples:

Draw a contour map for the following functions showing several level curves. Compare them to the surfaces we drew previously.

(a) $f(x,y) = 6 - 3x - 2y$

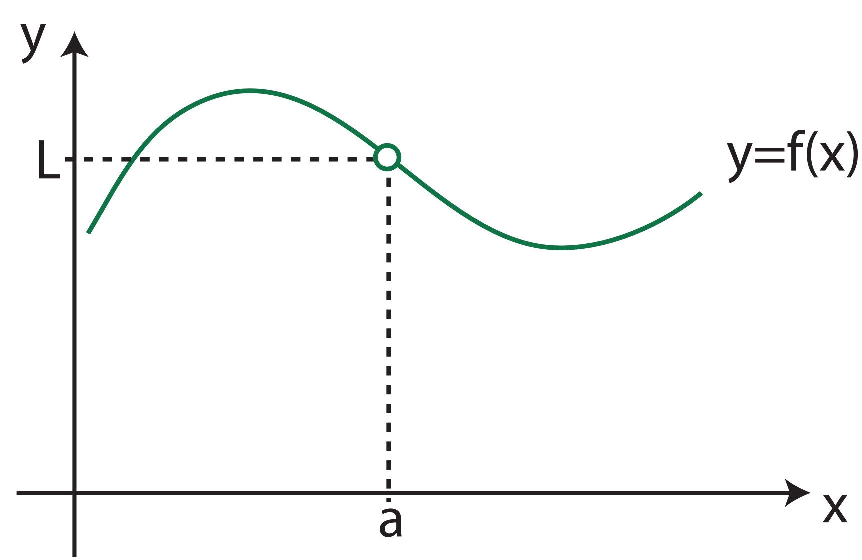
(b) $f(x,y) = 4 - x^2 - y^2$

Limit of a Function in R^2

Definition:

$$\lim_{x \to a} f(x) = L$$

means that the *y*-values can be made arbitrarily close (as close as we'd like) to *L* by taking the *x*-values sufficiently close to *a*, from either side of *a*, but not equal to *a*.



Existence of a Limit in R^2

The limit exists if and only if the left and right limits both exist (equal a real number) and are the same value.

Existence of a Limit in R^2

Example:

Evaluate the following limits or show that they do not exist.

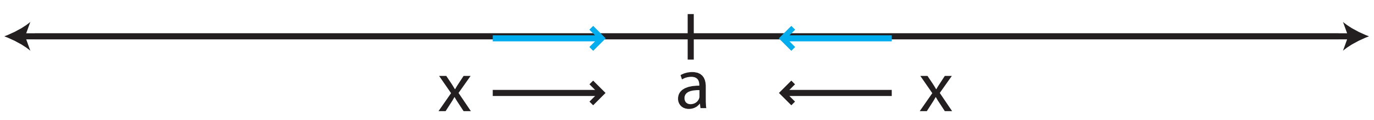
(a) $\lim_{x \to 1} f(x)$ where $f(x) = \begin{cases} x & \text{when } x < 1 \\ \dfrac{1}{x^2} & \text{when x} \geq 1 \end{cases}$

(b) $\lim_{x \to 0} \dfrac{|x|}{x}$

(c) $\lim_{x \to 0} \dfrac{1}{x^2}$

Existence of a Limit in R^2

It is relatively easy to show that this type of limit exists since there are only <u>two</u> ways to approach the number a along the real number line: *either from the left or from the right*

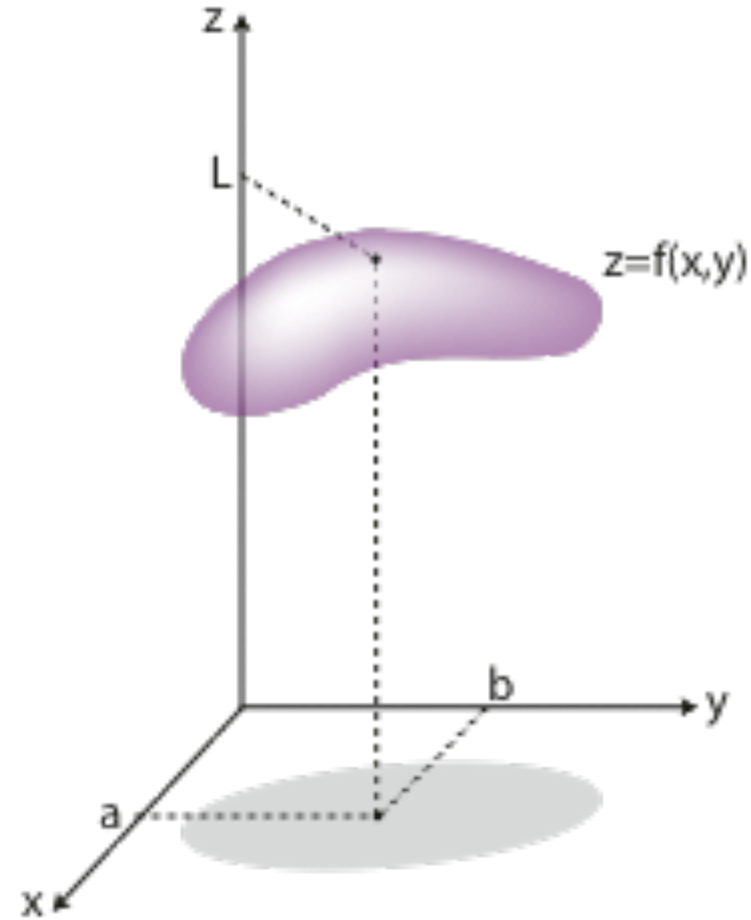


Limit of a Function in R^3

Definition:

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

means that the z-values approach L as (x,y) approaches (a,b) along *every path* in the domain of f.



Existence of a Limit in R^3

In general, it is difficult to show that such a limit exists because we have to consider the limit along all possible paths to (a,b).

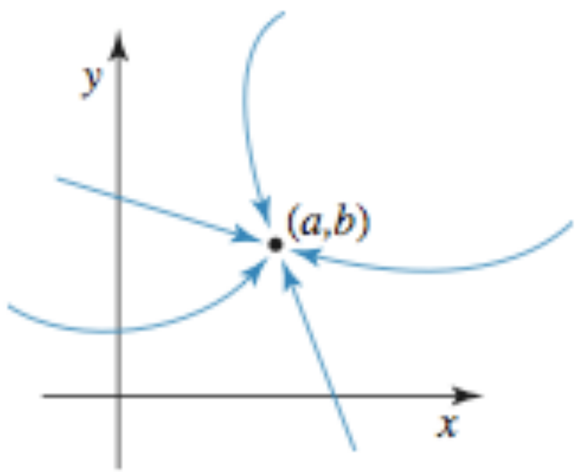


FIGURE 3.2 Paths leading to (a, b)

Existence of a Limit in R^3

However, to show that a limit **doesn't** exist, all we have to do is to find two *different* paths leading to (a,b) such that the limit of the function along each path is different (or does not exist).

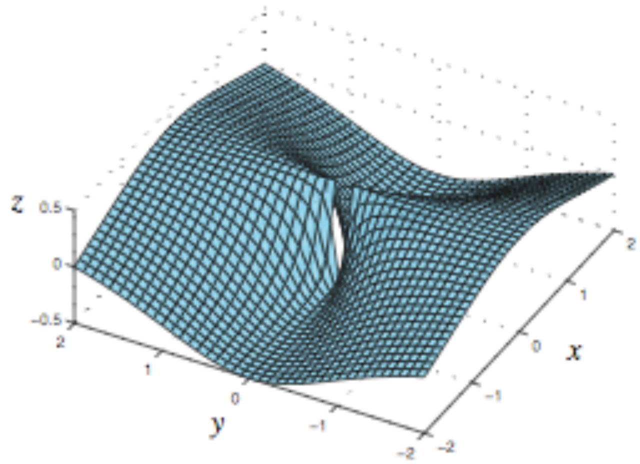


FIGURE 3.4 The graph of $f(x, y) = \frac{y^2 - x^2}{2x^2 + 3y^2}$

Existence of a Limit in R³

Example:

Show that the following limits **do not** exist.

(a) $\lim\limits_{(x,y)\to(0,0)} \dfrac{6x^3y}{2x^4+y^4}$

(b) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2+\sin^2 y}{2x^2+y^2}$

Limit Laws

Theorem:
Assume that $\lim_{(x,y)\to(a,b)} f(x,y)$ and $\lim_{(x,y)\to(a,b)} g(x,y)$ exist (i.e. are real numbers). Then

(a) $\lim_{(x,y)\to(a,b)} \left(f(x,y) \pm g(x,y)\right) = \lim_{(x,y)\to(a,b)} f(x,y) \pm \lim_{(x,y)\to(a,b)} g(x,y)$

(b) $\lim_{(x,y)\to(a,b)} \left(c\,f(x,y)\right) = c \lim_{(x,y)\to(a,b)} f(x,y)$, where c is any constant.

Limit Laws

Theorem (continued):

(c) $$\lim_{(x,y)\to(a,b)}\big(f(x,y)\times g(x,y)\big) = \lim_{(x,y)\to(a,b)} f(x,y)\times \lim_{(x,y)\to(a,b)} g(x,y)$$

(d) $$\lim_{(x,y)\to(a,b)}\frac{f(x,y)}{g(x,y)} = \frac{\displaystyle\lim_{(x,y)\to(a,b)} f(x,y)}{\displaystyle\lim_{(x,y)\to(a,b)} g(x,y)}, \quad \text{provided} \lim_{(x,y)\to(a,b)} g(x,y)\neq 0.$$

Some Basic Rules

For the function $f(x,y) = x$, $\lim_{(x,y)\to(a,b)} f(x,y) = \lim_{(x,y)\to(a,b)} x = a$

For the function $f(x,y) = y$, $\lim_{(x,y)\to(a,b)} f(x,y) = \lim_{(x,y)\to(a,b)} y = b$

For the function $f(x,y) = c$, $\lim_{(x,y)\to(a,b)} f(x,y) = \lim_{(x,y)\to(a,b)} c = c$

Evaluating Limits

Example #10:

Using the properties of limits, evaluate $\displaystyle\lim_{(x,y)\to(2,-2)} \frac{1}{xy-4}$.

Direct Substitution

Theorem:

If $f(x,y)$ is a polynomial or rational function (in which case (a,b) must be in the domain of f), then

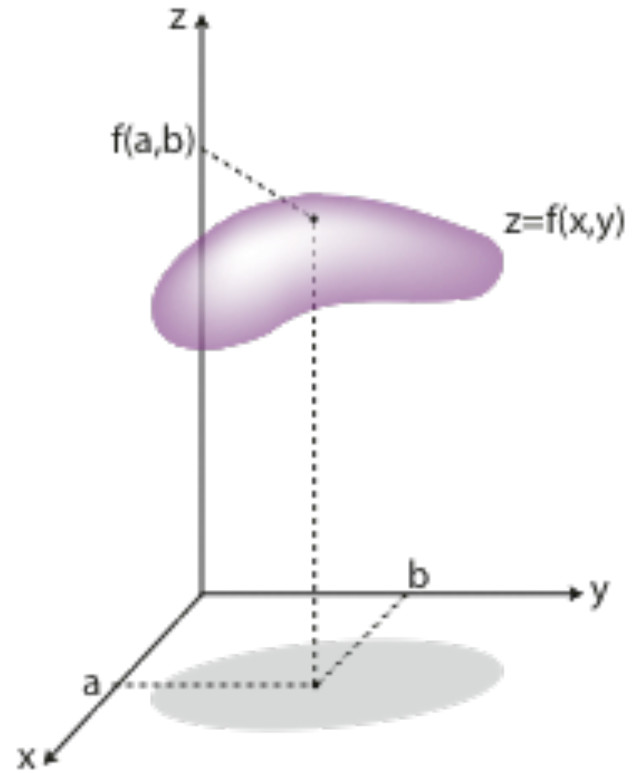
$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

Continuity of a Function in R^3

Intuitive idea:

A function is continuous if its graph has no holes, gaps, jumps, or tears.

A continuous function has the property that a small change in the input produces a small change in the output.

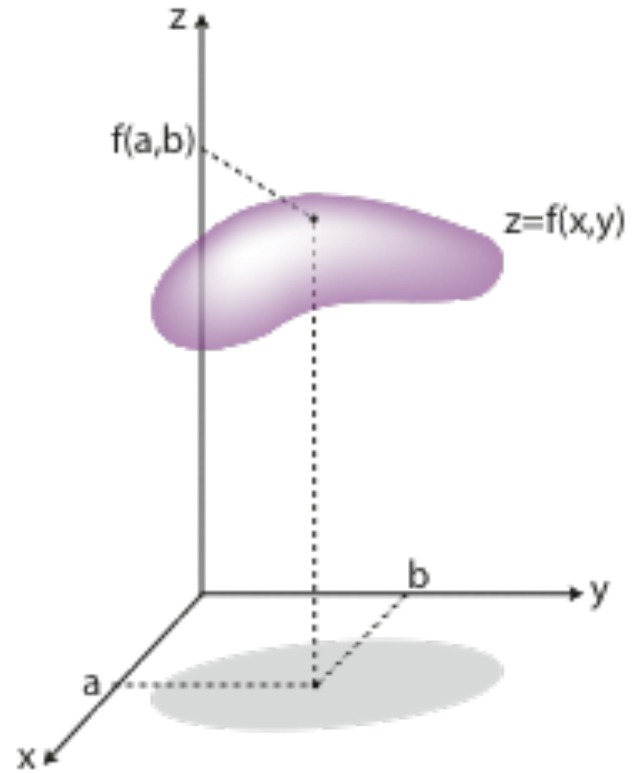


Continuity of a Function in R^3

Definition:

A function f is continuous at the point (a,b) if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$



Continuity of a Function in R^3

Example:

Determine whether or not the function

$$f(x,y)=\begin{cases} x^2+y^2+4 & \text{if } (x,y)\neq(0,0) \\ 1 & \text{if } (x,y)=(0,0) \end{cases}$$

is continuous at (0,0).

Which Functions Are Continuous?

A function is **continuous** if it is continuous at every point in its domain.

Basic Continuous Functions:

- ✓ polynomials
- ✓ rational functions
- ✓ exponential functions
- ✓ logarithmic functions
- ✓ trigonometric functions
- ✓ root functions

Which Functions Are Continuous?

Combining Continuous Functions:
The sum, difference, product, quotient, and composition of continuous functions is continuous where defined.

Example:
Find the largest domain on which each function is continuous.

(a) $f(x,y) = 3x^2y^4 - 2y^3$

(b) $f(x,y) = e^{x^2y} + \sqrt{x + y^2}$

Limits of Continuous Functions

By the definition of continuity, if a function is continuous at a point, then we can evaluate the limit simply by **direct substitution**.

Example: Evaluate each limit.

(a) $\lim_{(x,y)\to(0,-1)} \left(3x^2y^4 - 2y^3\right)$

(b) $\lim_{(x,y)\to(0,-1)} \left(e^{x^2y} + \sqrt{x + y^2}\right)$