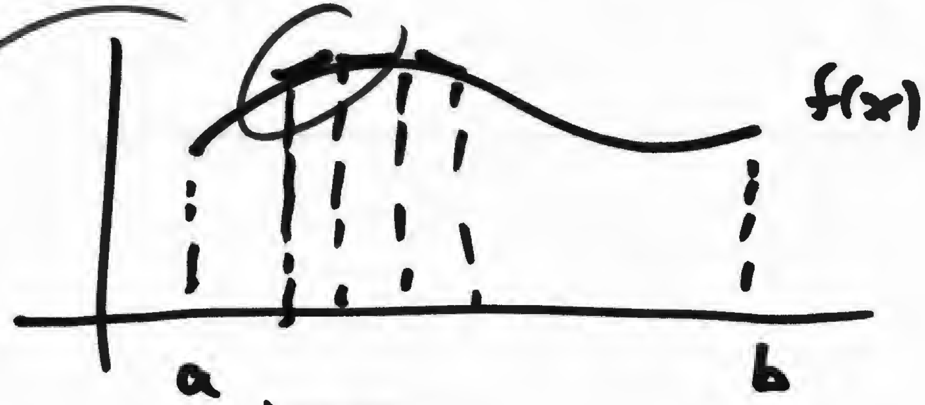
1ZA3

Arc Length - how long is my curve?



zoom in!

decompose our curve into approx. straight subsections

$$(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2$$

$$\Delta s = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

so $\Delta s = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\ \Delta x$

by MVT $\quad \frac{\Delta y}{\Delta x} = f'(c)$

for some c on interval!

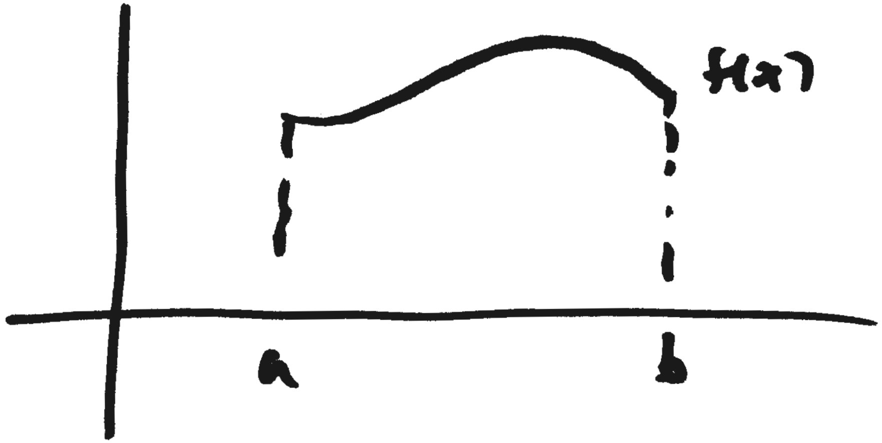
$\approx \sqrt{1 + (f'(x_i))^2}\ \Delta x$

so net arc length $\approx \sum_{i=1}^{n} \Delta s = \sum_{i=1}^{n} \sqrt{1 + (f'(x_i))^2}\ \Delta x$

& as $n \to \infty$

Arc Length $= \int_a^b \sqrt{1 + (f'(x))^2}\ dx$

Alternate interpretation.



Say as I draw $f(x)$

I move <u>right</u> at 1 unit x per unit t.

Pen velocity $= \vec{v} = \left(\frac{dx}{dt}, \frac{dy}{dt}\right)$

speed $= \|\vec{v}\| = \sqrt{\underbrace{\left(\frac{dx}{dt}\right)^2}_{1} + \underbrace{\left(\frac{dy}{dt}\right)^2}_{dy/dx}} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$

$$\boxed{\text{dist} = \int_{\text{start}}^{\text{finish}} \text{speed}\, dt = \int_a^b \sqrt{1+(dy/dx)^2}\, dx}$$

$x = t + c$

$dx = dt$

eg Find the arclength on $[1, 8]$ of $y = x^{2/3} = f(x)$,

$$f'(x) = \frac{2}{3} x^{-\frac{1}{3}}$$

Solution

$$\text{Arc Length} = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

$$= \int_1^8 \sqrt{1 + \frac{4}{9} x^{-2/3}}\, dx$$

$$= \int_1^8 \sqrt{\frac{9x^{2/3} + 4}{9x^{2/3}}}\, dx$$

$$= \int_1^8 \frac{1}{3} x^{-\frac{1}{3}} \sqrt{9x^{2/3} + 4}\, dx$$

let $u = 9x^{2/3} + 4$, $\quad x = 1 \Rightarrow u = 13$

$du = 6x^{-1/3}\,dx \quad\quad x = 8 \Rightarrow u = 40$

$\frac{1}{6}\,du = x^{-\frac{1}{3}}\,dx$

$$\frac{1}{6}\cdot\frac{1}{3}\int_{18}^{40} u^{\frac{1}{2}}\,du = \frac{1}{18}\cdot\frac{2}{3}\,u^{3/2}\Big|_{13}^{40}$$

$$= \frac{1}{27}\left(40^{3/2} - 18^{3/2}\right)$$

Note We don't have to work in x.

equivalently $\quad$ Arc Length $= \displaystyle\int_{y_0}^{y_1} \sqrt{1 + (g'(y))^2}\,dy$

eg. Integrating in "$\underline{\underline{y}}$", find the arclength of

$y = x^{2/3}$ for $x \in [1, 8]$

Solution

$$y = x^{2/3} \rightsquigarrow x = y^{3/2} = g(y) \rightsquigarrow g'(y) = \frac{3}{2} y^{\frac{1}{2}}$$

$$x \in [1, 8] \rightsquigarrow y \in [1, 4]$$

$$\text{Arc Length} = \int_{y_0}^{y_1} \sqrt{1 + (g'(y))^2}\, dy$$

$$= \int_1^4 \sqrt{1 + \frac{9}{4} y}\, dy$$

Let $u = 1 + \frac{9}{4}y$ $\quad du = \frac{9}{4}dy$

$dy = \frac{4}{9}du$

$y = 1 \Rightarrow u = \frac{13}{4}$ $\quad y = 4 \Rightarrow u = 10$

$$\text{Arclength} = \int_{13/4}^{10} u^{\frac{1}{2}} \cdot \frac{4}{9}\,du$$

$$= \frac{4}{9} \cdot \frac{2}{3} u^{3/2} \Big|_{13/4}^{10}$$

$$= \frac{8}{27} \cdot \left(10^{3/2} - \left(\frac{13}{4}\right)^{3/2}\right)$$

$$= \frac{1}{27}\left(40^{3/2} - 13^{3/2}\right)$$

$8 = 4^{3/2}$

$8 \cdot 10^{3/2}$

$= 4^{3/2} \cdot 10^{3/2}$

$= 40^{3/2}$ etc.

eg. Let $f(x) = \frac{1}{4}x^4 + \frac{1}{8x^2}$, $x \in [1, 3]$

Find the arclength!

Solution

Arclength $= \int_1^3 \sqrt{1 + (f'(x))^2}\, dx$

$f(x) = \frac{1}{4}x^4 + \frac{1}{8x^2}$

"the magic half"

$(f'(x))^2 = \left(x^3 - \frac{1}{4x^3}\right)^2 = x^6 - \frac{1}{2} + \frac{1}{16x^6}$

$$1 + (f'(x))^2 = x^6 \underline{\underline{- \frac{1}{2}}} + \frac{1}{16x^6} \underline{\underline{+1}}$$

$$= x^6 + \frac{1}{2} + \frac{1}{16x^6}$$

$$= \left(x^3 + \frac{1}{4x^3}\right)^2$$

<u>Arc Length</u> $= \displaystyle\int_1^3 \sqrt{1 + (f'(x))^2}\, dx$

$$= \int_1^3 \sqrt{\left(x^3 + \frac{1}{4x^3}\right)^2}\, dx$$

$$= \int_1^3 x^3 + \frac{1}{4x^3}\, dx$$

$$= \frac{1}{4}x^4 - \frac{1}{8x^2} \Big|_1^3 = \underline{\underline{\#}}$$

Alternate advanced notation

Arclength from $x=a$, as a **function**

$$S(x) = \int_a^x \sqrt{1+(f'(t))^2}\, dt$$

t plays role of x in integral!

More **generally**.

$$S_C = \int_C ds = \int \sqrt{(1+f'(x))^2}\, dx$$

$$= \int \sqrt{1+(g'(y))^2}\, dy$$

$$\int dx = x + c$$