12C3

Last Day: Matrix Multiplication

eg
$$\begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2(-1) + 3(2) \\ 1(-1) + 5(2) \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \end{bmatrix}$$

square → 2×2, 2×1 ← col. vector

match

⇒ result is outer dimensions: 2×1

or
$$\begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}(-1) + \begin{bmatrix} 3 \\ 5 \end{bmatrix}(2)$$

look at as a combination of columns!

$$= \begin{bmatrix} 2(-1) + 3(2) \\ 1(-1) + 5(2) \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \end{bmatrix}.$$

Eg

Let's make an order matrix for parts!

$$\begin{array}{c c} & \text{Order \#1} \\ & \begin{bmatrix} 3 \\ 5 \end{bmatrix} \begin{array}{l} \text{Part 1} \\ \text{Part 2} \end{array} \end{array}$$

Let's make a list of cost of parts:

$$\begin{array}{r c} & \begin{array}{cc} \text{Part 1} & \text{Part 2} \end{array} \\ \begin{array}{r} \text{Amazon} \\ \text{Ebay} \end{array} & \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \end{array}$$

Now: Multiply

$$\begin{array}{r c} & \begin{array}{cc} P_1 & P_2 \end{array} \\ \begin{array}{r} \text{Am.} \\ \text{Eb.} \end{array} & \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \end{array} \begin{array}{c} \text{Order \#1} \\ \begin{bmatrix} 3 \\ 5 \end{bmatrix} \begin{array}{l} P_1 \\ P_2 \end{array} \end{array} = \begin{bmatrix} 2(3) + 3(5) \\ 1(3) + 4(5) \end{bmatrix} = \begin{array}{c} \text{Order 1} \\ \begin{bmatrix} 21 \\ 23 \end{bmatrix} \begin{array}{l} \text{Am} \\ \text{Eb.} \end{array} \end{array}$$

$$\underbrace{}_{C} \quad \underbrace{}_{N} = \underbrace{}_{P}$$

(cost) (# ordered) (amount paid)

ie $\boxed{CN = P}$

<u>What if</u> Order size unknown & Cost & Paid amount are known? Can we find N?

$$\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 21 \\ 23 \end{bmatrix}$$

$$\hookrightarrow \left.\begin{aligned} 2x + 3y &= 21 \\ x + 4y &= 23 \end{aligned}\right\} \text{Linear System!}$$

This works for any linear system!

eg

$$\begin{aligned} 2x + 5y - z &= 2 \\ x + 7y + z &= 3 \\ -3x + 2y &= 5 \end{aligned}$$

$$\underbrace{\begin{bmatrix} 2 & 5 & -1 \\ 1 & 7 & 1 \\ -3 & 2 & 0 \end{bmatrix}}_{\text{coefficient matrix}} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\text{Variables}} = \underbrace{\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}}_{\text{constants}}$$

$$A \quad X = B$$

Goal: Invent division! $\Rightarrow$ First, invent exponentials

$$\left\{\begin{aligned} A^1 &= A \\ A^2 &= AA \\ A^3 &= AAA \\ A^p &= \underbrace{AAA\cdots A}_{p \text{ times!}} \end{aligned}\right\} \iff$$

Say A is $m \times n$

$\underbrace{A}_{m\times n}\underbrace{A}_{m\times n}$ is defined iff $m = n$

$\Rightarrow A^p$ only defined if A is square!

$A^0 = ?$ $\rightsquigarrow$ For all A, $\underset{n\times n}{A}\,\underset{n\times n}{A^0} = A^1 = \underset{n\times n}{A}$

$A^0 = I = I_n$ ← nxn sized

"Identity matrix" (acts like 1 but wider!)

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$I_n = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}$$

1 on "principal diagonal"

<u>0 elsewhere!</u>

$AI = A$, $IA = A$ for <u>all</u> A (square or not)

(as long as dimensions match!)

So what is A^{-1}?

<u>We'll define</u> A^{-1} is the matrix such that

$$AA^{-1} = A^{-1}A = I \qquad (A \text{ is } \underline{n \times n}).$$

Is A^{-1} unique for a given A?

Say $BA = AB = I \Rightarrow B$ is an inverse of A

& $CA = AC = I \Rightarrow C$ is another inverse of A

then $B = BI = BAC = IC = C \Rightarrow B = C$

$\Rightarrow$ inverse is unique (if exists!)

note "A^{-1} = inverse of A", defined as $A^{-1}A = AA^{-1} = I$

If it exists $\Rightarrow$ "A is invertible" or non-singular

If it does not exist $\Rightarrow$ "A is not invertible

or singular

Properties of Inverse

1) $(A^{-1})^{-1} = A$

2) If $\underline{k \neq 0}$ $\quad (kA)^{-1} = \frac{1}{k} A^{-1}$

Proof $\quad (kA)^{-1} \cdot kA = I = kA \cdot (kA)^{-1}$

does $\frac{1}{k} A^{-1}$ fit?

Check $\quad \left(\frac{1}{k} A^{-1}\right)(kA) = \frac{k}{k} A^{-1}A = 1 I = I$

$(kA)\left(\frac{1}{k} A^{-1}\right) = \frac{k}{k} AA^{-1} = I$ ✓

3) $(AB)^{-1} = B^{-1}A^{-1}$

Proof

$AB B^{-1} A^{-1} = A I A^{-1} = AA^{-1} = I$

$B^{-1}A^{-1}AB = B^{-1} I B = B^{-1}B = I$

4) $(A^T)^{-1} = (A^{-1})^T$

5) $(A + B)^{-1} =$ <u>Horror!</u>

<u>Sample Problem</u> If A is an $n \times n$ matrix find an expression for A^{-1} <u>if</u>

$$3A^3 + 2A^2 + 7A + 2I = 0$$

<u>Solution</u>

$$3A^3 + 2A^2 + 7A = -2I$$

$$-\frac{3}{2}A^3 - A^2 - \frac{7}{2}A = I$$

$$\underbrace{\left(-\frac{3}{2}A^2 - A - \frac{7}{2}I\right)}_{A^{-1}} A = I = A \underbrace{\left(-\frac{3}{2}A^2 - A - \frac{7}{2}I\right)}_{A^{-1}}$$

by defn!

Why the "I" above?

Consider: Say $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Then $A^2 + 2A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + 2\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$= \left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$= (A + 2I)A$

If we don't use the I here, this object is not a matrix!

"$\underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}_{\text{matrix}} + \underbrace{2}_{\text{scalar}}$" is undefined