ASSIGNMENT 14

LS

1(a) $f''(x) > 0$ ... f is concave up

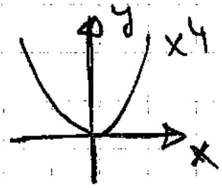
$f''(x) < 0$ ... f is concave down

inflection point is a point on the graph of f where f changes concavity

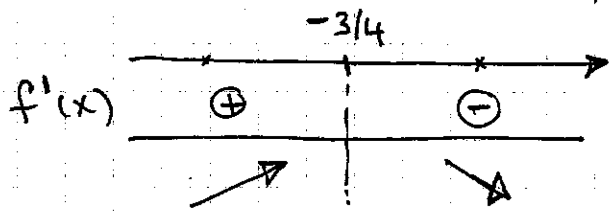
(b) No. for instance, $f(x) = x^4$:

$f'(x) = 4x^3$, $f''(x) = 12x^2 \rightarrow$ so $f''(0) = 0$

however 0 is <u>not</u> an inflection point of f



(c) $f'(x) = -4x - 3 = 0 \rightarrow 4x = -3,\ x = -3/4 \leftarrow$ critical point

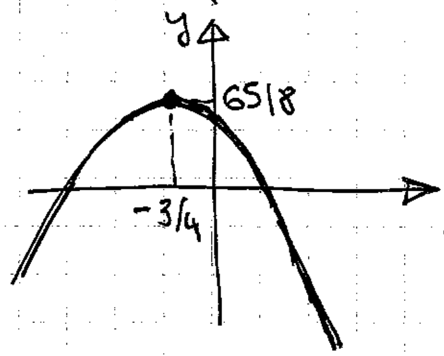


f increasing on $(-\infty, -3/4)$ f decreasing on $(-3/4, \infty)$

$f(-3/4) = -2\left(-\frac{3}{4}\right)^2 - 3\left(-\frac{3}{4}\right) + 7 = -\frac{9}{8} + \frac{9}{4} + 7 = \frac{65}{8}$

is the highest point

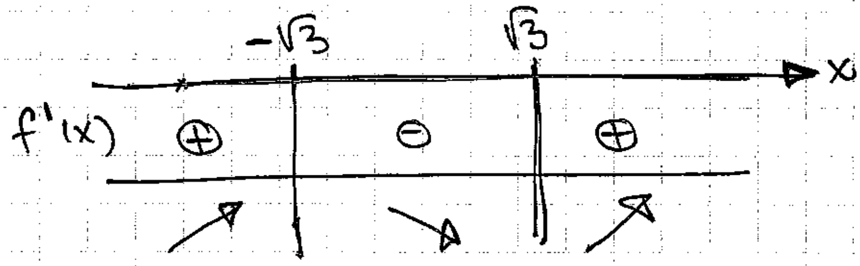
$f''(x) = -2 < 0$ for all $x \rightarrow f$ is concave down



(graph was not needed)

2. $f'(x) = 3x^2 - 9 = 0 \rightarrow 3(x^2-3)=0$

so $x = \pm\sqrt{3}$ are critical points

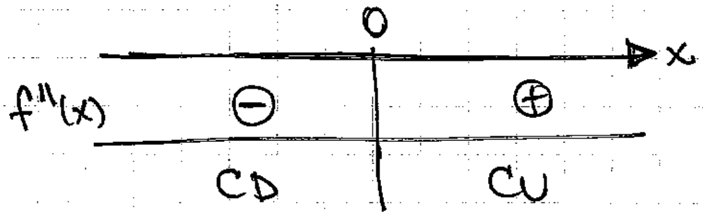


so f is increasing on $(-\infty, -\sqrt{3})$, $(\sqrt{3}, \infty)$

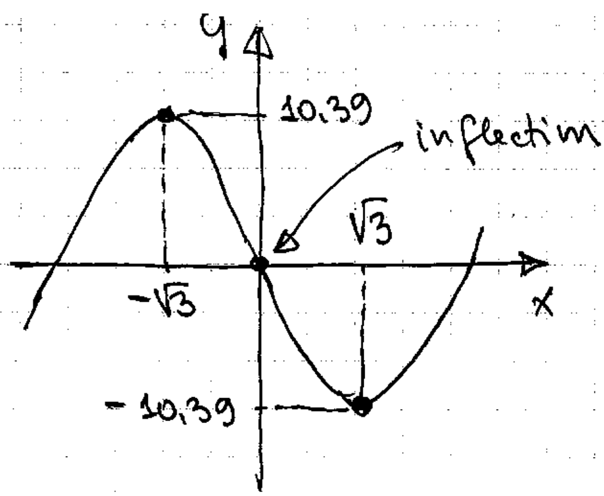
decreasing on $(-\sqrt{3}, \sqrt{3})$

$f(\sqrt{3}) = -10.39$ $\quad f(-\sqrt{3}) = 10.39$

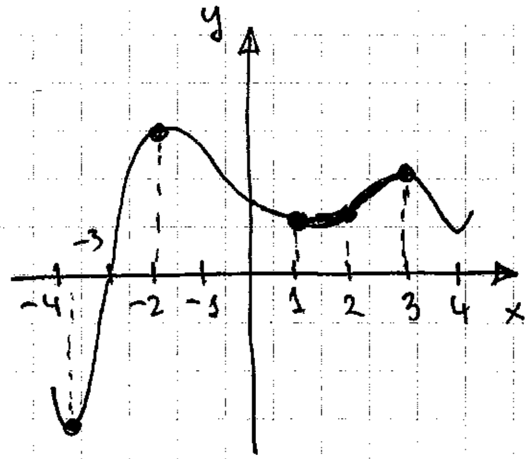
$f''(x) = 9x - 0 = 0 \rightarrow x = 0$



f is concave up on $(0, \infty)$, down on $(-\infty, 0)$



3.



ⓐ critical point → look for places where tangent is horizontal (minimum, maximum)

$x = -2, 1, 3$, between -4 and -3

ⓑ increasing → any point in $(1,3)$

ⓒ decreasing → any point in $(-2,1)$

ⓓ positive second der. (CU) – $x = 1$; $x = 0$, and so on

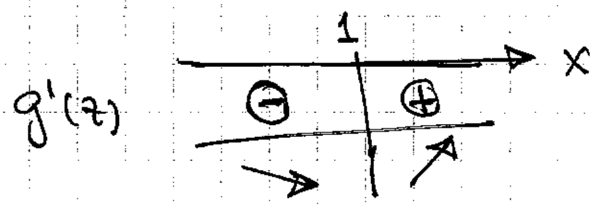
ⓔ CD → $x = -2$; $x = 3$, and so on

ⓕ not clear from the graph → could be $x = 2$ (or near 2)

$x = -3$ (or near -3)

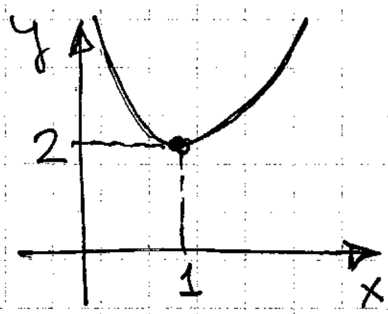
4. $g(z) = z + \frac{1}{z}$, $z > 0$

$g'(z) = 1 - \frac{1}{z^2} = 0 \to \frac{1}{z^2} = 1$, $z^2 = 1$, $z = 1$ (because $z > 0$)



$z = 1$ is a critical point; $g(1) = 2$

$g''(z) = -(-2)z^{-3} = \frac{2}{z^3} > 0$ for $z > 0 \Longrightarrow$ concave up



$\lim_{z \to 0^+} g(z) = \infty$

$\lim_{z \to \infty} g(z) = \infty$

5. (a) $f(x) = \cos x$

$f'(x) = -\sin x$

$f''(x) = -\cos x$

$f^{(3)}(x) = \sin x$

$f^{(4)}(x) = \cos x \to f^{(8)}(x) = \cos x \to f^{(12)}(x) = \cos x$

every derivative divisible by 4 is equal to $\cos x$

$\Rightarrow f^{(100)}(x) = \cos x$

(b) $f = e^{-2x}$

$f' = -2e^{-2x}$

$f'' = (-2)e^{-2x}(-2) = 2^2 e^{-2x}$

$f''' = 2^2 e^{-2x}(-2) = -2^3 e^{-2x}$

$f^{(4)} = -2^3 e^{-2x}(-2) = 2^4 e^{-2x}$

$\Rightarrow f^{(66)}(x) = \underbrace{(-1)^{66}} 2^{66} e^{-2x}$

$= 2^{66} e^{-2x}$

6. (a) Linear approximation of $f(x)$ at $x=a$ is the equation of the tangent line to $f(x)$ at $x=a$.

(b) $f(x) = e^{-3x} \to f(0) = e^0 = 1$

$f'(x) = -3e^{-3x} \to f'(0) = -3e^0 = -3$

$$L(x) = f(0) + f'(0)(x-0) = 1 - 3x$$

(c) Pick two points on the graph of a function and join them with a straight line (= secant line)

(d) $x=0 \rightarrow f(0)=e^0=1$; $x=1 \rightarrow f(1)=e^{-3}$ — secant connects $(0,1)$ and $(1, e^{-3})$

point-slope equation: $y-1=\dfrac{e^{-3}-1}{1}(x-0)$

$$y=(e^{-3}-1)x+1$$

or, can write $\underset{\uparrow \text{ secant line}}{S(x)} = (e^{-3}-1)x+1$

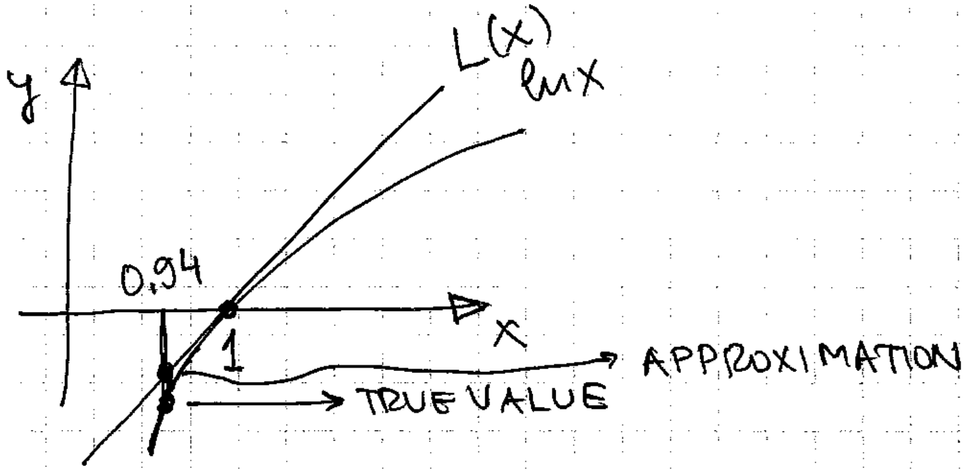
(e) Approximation of a function $f(x)$ near $x=a$ using a quadratic function

$$T_2(x)=f(0)+f'(0)x+\frac{f''(0)}{2}x^2$$

(f)

$$\left.\begin{aligned} f &= e^{-3x} \rightarrow f(0)=1 \\ f' &= -3e^{-3x} \rightarrow f'(0)=-3 \\ f'' &= 9e^{-3x} \rightarrow f''(0)=9 \end{aligned}\right\} \quad T_2(x)=1-3x+\frac{9}{2}x^2$$

7. (a) The value 0.94 is close to 1. We will construct the tangent line to $y=\ln x$ at $x=1$ and use it to approximate $\ln 0.94$



tangent at $x=1$:

$$\left.\begin{aligned} f = \ln x &\rightarrow f(1)=0 \\ f' = \frac{1}{x} &\rightarrow f(1)=1 \end{aligned}\right\} \quad \begin{aligned} L(x) &= 0+1(x-1) \\ L(x) &= x-1 \end{aligned}$$

so $\ln 0{,}94 = f(0{,}94) \approx L(0{,}94) = 0{,}94-1 = \underline{\underline{-0{,}06}}$

(b) Use quadratic approximation (continue (a)):

$f'' = -\frac{1}{x^2} \rightarrow f''(1) = -1$

$$\begin{aligned} T_2(x) &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 \\ &= 0 + 1(x-1) - \tfrac{1}{2}(x-1)^2 \\ &= (x-1) - \tfrac{1}{2}(x-1)^2 \end{aligned}$$

$$\begin{aligned} \ln 0{,}94 &= f(0.94) \approx T_2(0{,}94) \\ &= (0{,}94-1) - \tfrac{1}{2}(0{,}94-1)^2 = -0{,}06180 \end{aligned}$$

[TRUE VALUE: $\ln 0{,}94 \approx -0{,}0618754\ldots$]

8 (a) $f(x) = \sqrt{x}$, $a=4$

tangent line: $f = \sqrt{x} \longrightarrow f(4) = 2$

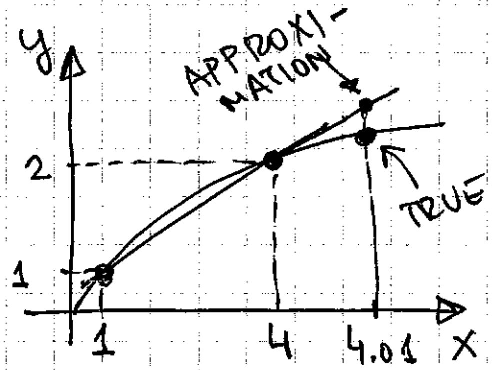
$f' = \frac{1}{2\sqrt{x}} \longrightarrow f'(4) = \frac{1}{4}$

$$L(x) = 2 + \tfrac{1}{4}(x-4) = \tfrac{1}{4}x + 1$$

$$\sqrt{4{,}01} = f(4{,}01) \approx L(4{,}01) = \frac{4{,}01}{4} + 1 = \underline{\underline{2{,}0025}}$$

for secant line, pick another point where we know f(x); for instance $x=1 \to \sqrt{x}=1$

or $x=9 \to \sqrt{x}=3$



in this case points defining secant are (1,1) and (4,2)

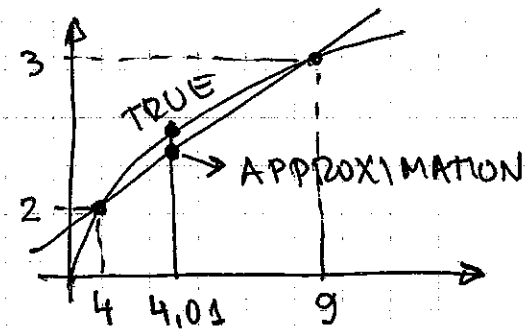
$$y-1=\frac{2-1}{4-1}(x-1)$$

$$y-1=\frac{1}{3}(x-1)$$

$$S(x)=y=\frac{1}{3}x+\frac{2}{3}$$

$$\sqrt{4.01}=f(4.01)\approx S(4.01)=\frac{1}{3}(4.01)+\frac{2}{3}=\underline{\underline{2.00333...}}$$

OR;



connect (4,2) and (9,3) with a secant:

$$y-2=\frac{3-2}{9-4}(x-4)$$

$$y-2=\frac{1}{5}(x-4)$$

$$S(x)=y=\frac{1}{5}x+\frac{6}{5}$$

$$\sqrt{4.01}=f(4.01)\approx S(4.01)=\frac{1}{5}(4.01)+\frac{6}{5}=\underline{\underline{2.002}}$$

(b) $f=\sqrt{x} \longrightarrow f(4)=2$

$f'=\frac{1}{2\sqrt{x}} \longrightarrow f'(4)=\frac{1}{4}$

$f''=\frac{1}{2}\left(-\frac{1}{2}\right)x^{-3/2}=-\frac{1}{4}\frac{1}{\sqrt{x^3}}=-\frac{1}{32}$

$$T_2(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2$$

$$\sqrt{4{,}01} = f(4{,}01) \approx T_2(4{,}01)$$

$$= 2 + \frac{1}{4}(4{,}01-4) - \frac{1}{64}(4{,}01-4)^2$$

$$= \underline{2{,}002498}4375$$

TRUE VALUE $\sqrt{4{,}01} \approx \underline{2{,}002498}3945$