MATHEMATICS 1LS3 TEST 4

Day Class
Duration of Examination: 60 minutes
McMaster University, 28 November 2016

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First name (PLEASE PRINT): SOLUTIONS

Family name (PLEASE PRINT): ______________

Student No.: ______________

THIS TEST HAS 8 PAGES AND 6 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE.

Total number of points is 40. Marks are indicated next to the problem number. Any non-graphing calculator is allowed.

EXCEPT ON QUESTIONS 1 AND 2, YOU MUST SHOW CORRECT WORK TO EARN CREDIT.

USE PEN TO WRITE YOUR TEST. IF YOU USE A PENCIL YOUR TEST WILL NOT BE ACCEPTED FOR REMARKING (IF NEEDED).

You must show work to receive full credit.

Problem	Points	Mark
1	4	
2	6	
3	8	
4	6	
5	6	
6	10	
TOTAL	40	

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1. **Multiple choice questions: circle ONE answer. No justification is needed.**

(a)[2] Which of the following improper integrals are convergent?

(I) $\int_1^\infty x^{-1.5}\,dx$ CONV. $p=1.5$ (II) $\int_1^\infty x^{-1}\,dx$ DIV. $p=1$ (III) $\int_1^\infty x^{-0.5}\,dx$ DIV. $p=0.5$

(A) none (B) I only (C) II only (D) III only

(E) I and II (F) I and III (G) II and III (H) all three

$\int_1^\infty \frac{1}{x^p}\,dx$ is convergent if $p>1$

divergent if $p \le 1$

(b)[2] Identify all correct statements about the dynamical system $m_{t+1} = 1.4m_t$, $m_0 = 1$.

(I) The updating function is $f(m_t) = \cancel{1.4}$ $1.4m_t$

(II) The corresponding backward dynamical system is $m_t = \cancel{\frac{1.4}{m_{t+1}}}$ $\frac{m_{t+1}}{1.4}$

(III) $m_t = 1.4^t$ for all $t \ge 1$. ✓

(A) none (B) I only (C) II only (D) III only

(E) I and II (F) I and III (G) II and III (H) all three

$\left.\begin{array}{l} m_0 = 1 \\ m_{t+1} = 1.4m_t \end{array}\right\} m_t = 1 \cdot 1.4^t$

from memory, or:

$m_1 = 1.4m_0 = 1.4$

$m_2 = 1.4m_1 = (1.4)(1.4) = 1.4^2$

etc.

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2. Identify each statement as true or false (circle your choice). You do not need to justify your answer.

(a)[2] A population of bacteria triples every hour. Every hour, before reproduction, 700 bacteria are removed. The population starts with 1,000 bacteria. Let p_t denote the population size (i.e., number of bacteria) at time t. The dynamical system which describes this population is given by $p_{t+1} = 3(p_t - 700)$, $p_0 = 1000$.

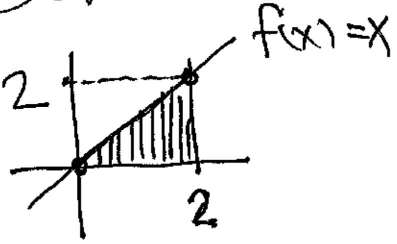
removed first

then triples/reproduces

TRUE FALSE

(b)[2] The solid of revolution whose volume is given by $\pi \int_0^2 x^2\, dx$ is a cone of base radius ~~4~~ and height 2. OK

2



TRUE FALSE

(c)[2] The dynamical system $h_{t+1} = 1.5h_t + 0.45$ describes the height of a tree in metres, where t is time in years. Converted so that the height is in centimetres, this dynamical system reads $H_{t+1} = 150H_t + 45$.

TRUE FALSE

$$H_{t+1} = 100\, h_{t+1}$$

in cm

$$= 100(1.5 h_t + 0.45)$$
$$= 1.5 \cdot 100 h_t + 45$$
$$= 1.5 H_t + 45$$

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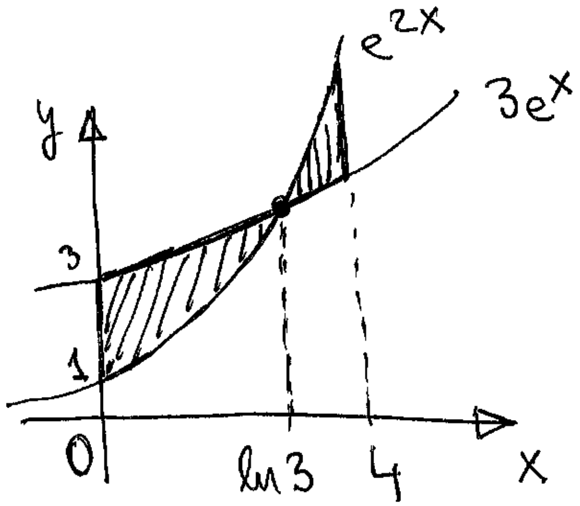
Questions 3-6: You must show CORRECT work to receive full credit.

3. (a)[4] Sketch (shade) the region bounded by the graphs of $y = 3e^x$ and $y = e^{2x}$ on $[0, 4]$. Write a formula for its area. **Do not evaluate the integral(s) involved.**



$$e^{2x} = 3e^x \quad | \div e^x$$
$$e^x = 3$$
$$x = \ln 3$$

$$A = \int_0^{\ln 3} (3e^x - e^{2x})\,dx + \int_{\ln 3}^{4} (e^{2x} - 3e^x)\,dx$$

note: $\quad A = \int_0^4 |3e^x - e^{2x}|\,dx$

is correct $\rightarrow$ but then we need to expand it to get the above answer

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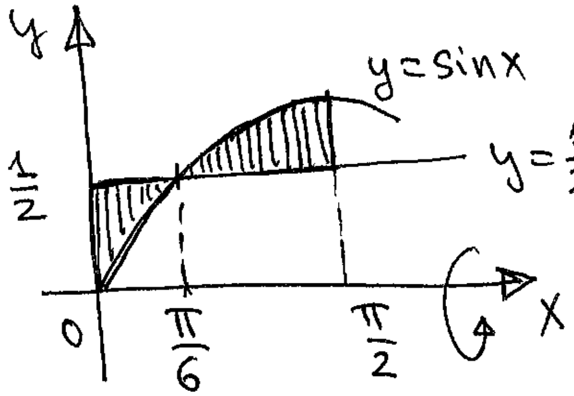
(b)[4] Consider the region bounded by the graphs of $y = \sin x$, $y = 1/2$, $x = 0$ and $x = \pi/2$. Write a formula for the volume of the solid obtained by revolving this region about the x-axis. **Do not evaluate the integral(s) involved.**



$\sin x = \frac{1}{2}$

$\rightarrow x = \pi/6$

$$V = \pi \int_0^{\pi/6} \left[\left(\frac{1}{2}\right)^2 - (\sin x)^2 \right] dx + \pi \int_{\pi/6}^{\pi/2} \left[(\sin x)^2 - \left(\frac{1}{2}\right)^2 \right] dx$$

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4. (a)[1] Write the Taylor polynomial $T_2(x)$ for the function $f(x) = e^x$ at $x = 0$.

$$T_2(x) = 1 + x + \frac{x^2}{2}$$

(b)[2] Use (a) to show that the function $f(x) = xe^{-x^2}$ can be approximated by the polynomial $T(x) = x - x^3 + \frac{x^5}{2}$ near $x = 0$.

$$xe^{-x^2} = x\left(1 + (-x^2) + \frac{(-x^2)^2}{2}\right)$$

$$= x - x^3 + \frac{x^5}{2}$$

(b)[3] Use your answer to (b) to find an estimate for $\int_0^1 xe^{-x^2}\,dx$.

$$\approx \int_0^1 \left(x - x^3 + \frac{x^5}{2}\right) dx$$

$$= \frac{1}{2}x^2 - \frac{1}{4}x^4 + \frac{1}{12}x^6 \Big|_0^1$$

$$= \frac{1}{2} - \frac{1}{4} + \frac{1}{12} = \frac{1}{3}$$

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5. (a)[3] Determine whether the improper integral $\int_5^6 \frac{1}{\sqrt{x-5}}\,dx$ is convergent or divergent. If convergent, find its value.

$$= \lim_{T\to 5^+} \int_T^6 \frac{1}{\sqrt{x-5}}\,dx$$

5 6

$$= \lim_{T\to 5^+} \left.\frac{(x-5)^{1/2}}{1/2}\right|_T^6$$

$$= \lim_{T\to 5^+} \left[2 - 2(T-5)^{1/2}\right] = \underline{\underline{2}}$$

thus convergent

(b)[3] Determine whether the improper integral $\int_1^\infty \frac{3}{(1+x)^{4/3}}\,dx$ is convergent or divergent. If convergent, find its value.

$$= \lim_{T\to\infty} \int_1^T \frac{3}{(1+x)^{4/3}}\,dx$$

$$= \lim_{T\to\infty} 3\cdot \left.\frac{(1+x)^{-1/3}}{-\frac{1}{3}}\right|_1^T$$

$$= \lim_{T\to\infty} \left.\left(-9\cdot\frac{1}{\sqrt[3]{1+x}}\right)\right|_1^T$$

$$= \lim_{T\to\infty} \left(-9\cdot \underbrace{\cancel{\frac{1}{\sqrt[3]{1+T}}}}_{0}\right) - \left(-9\cdot\frac{1}{\sqrt[3]{2}}\right) = \frac{9}{\sqrt[3]{2}}$$

≈ 7.14

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6. (a)[3] Find $\int x^2 \ln x \, dx$. $= \left\{ \begin{matrix} u = \ln x \rightarrow u' = \frac{1}{x} \\ v' = x^2 \rightarrow v = \frac{x^3}{3} \end{matrix} \right\}$

$$= uv - \int vu' dx = \frac{x^3}{3} \ln x - \int \frac{x^2}{3} dx$$

$$= \underline{\frac{x^3}{3} \ln x - \frac{x^3}{9}} + C$$

(b)[3] Find $\int \frac{(1+\sqrt{x})^3}{\sqrt{x}} \, dx$. $= \left\{ \begin{matrix} u = 1 + \sqrt{x} \\ \frac{du}{dx} = \frac{1}{2\sqrt{x}} \rightarrow \frac{dx}{\sqrt{x}} = 2du \end{matrix} \right\}$

$$= \int u^3 \cdot 2du = 2\frac{u^4}{4} = \underline{\frac{1}{2}(1+\sqrt{x})^4} + C$$

(c)[4] Find the most general antiderivative of the function $f(x) = \frac{3-2x}{1+x^2}$.

$$\int f(x) dx = \int \frac{3-2x}{1+x^2} dx$$

$$= \int \frac{3}{1+x^2} dx - \int \underbrace{\frac{2x}{1+x^2}}_{} dx = \underline{3 \arctan x - \ln(1+x^2) + C}$$

substitute $u = 1 + x^2$
or guess

THE END