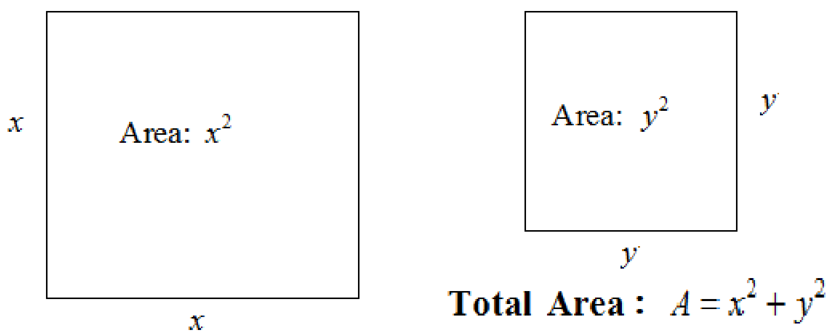
<u>Optimization Example</u>: ***When things go badly***

Question:

The entire length of one hundred meter length of fence is used to enclose two square plots of land. What size are the two plots of land when we have maximized the area?

Solution:

Often it is a good idea to draw a diagram to get an idea what's happening.



From the diagram, you can see that we've labeled the sides of our square regions as length x, and length y, giving us areas x^2 and y^2 respectively.

Also, the total amount of fencing is 100m, so we have the equations:

Total Area: $A = x^2 + y^2$ Total Perimiter: $P = 4x + 4y = 100$

We wish to maximize area, so let's write A in terms of only one variable.

From the perimeter:

$$4x + 4y = 100 \Rightarrow x + y = 25 \Rightarrow y = 25 - x$$

Substituting this into the area formula:

$$A = x^2 + y^2 \Rightarrow A(x) = x^2 + (25 - x)^2$$

$$\Rightarrow A(x) = 2x^2 - 50x + 625$$

Remember, that we have less than 100m total wire and we *must* enclose two regions, so x cannot properly be zero. This means $x \in (0, 25)$

So let's proceed by checking to see what are critical points are:

$$A'(x) = \frac{d}{dx}\left(2x^2 - 50x + 625\right) = 4x - 50$$

Critical Point Check :

Case #1

$$A'(x) = 4x - 50 = 0 \ \Rightarrow \ \ x = \frac{25}{2} = 12.5$$

Case #2

Clearly $A'(x)$ is defined everywhere. No c.p. from this case.

So $x = 12.5$ is our only critical point.

Notice from our perimeter equation that $x = 12.5,$ which means $y = 12.5,$ and we have two squares, each area $\frac{625}{4}m^2,$ with a combined area $\frac{625}{2}m^2.$

The naive assuption would be that this represents the maximum area we are looking for.

But what what happens if we have all, or very nearly all of the wire surrounding the one box, and next to nothing in the other?

Then $4x \approx 100 \ \Rightarrow \ \ x \approx 25 \Rightarrow \ A \approx 25^2 = 625$

This is TWICE the area at our critical point values! Why? What has happened?

What has gone wrong, is that we have failed to notice that the lone c.p. we found is NOT a local max, but a local minimum!

$$A'(x) = 4x - 50 \Rightarrow \begin{cases} A'(x) = 4x - 50 > 0 \text{ for } x \in (12.5, \ 25) \\ A'(x) = 4x - 50 = 0 \quad \text{for } x = 12.5 \\ A'(x) = 4x - 50 < 0 \text{ for } x \in (0, 12.5) \end{cases}$$

So by the first derivative test, $x = 12.5$ is a local min, and by our optimization logic, since this is an isolated c.p., $x = 12.5$ is the location of the absolute minimum area.

We've found exactly what we DIDN'T want!

Instead, since this critical point represents the configuration for the absolute maximum area, the maximum must occur on the boundary. And this absolute maximum area is only valid if we slightly extend our problem to allow all the fencing to surround one region, and none around the other.

If we don't do that, then we can approach, but never attain any absolute maximum area.

The moral of this story:

IF you use an isolated critical point method to find your absolute max or min, you MUST check what the type of critical point it is!

If it is the wrong type, then the absolute maximum only occurs at the endpoints of the region. (*And then is attained, ie. is a valid max or min, if that endpoint is closed.*)