COMPSCI/SFWRENG 2FA3
Discrete Mathematics with Applications II
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Week 06 Exercises with Solutions

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Background Definitions

Consider the following definitions:

1. $\Sigma_{\text{stack}} = (\mathcal{B}, \mathcal{C}, \mathcal{F}, \mathcal{P}, \tau)$ where:

   a. $\mathcal{B} = \{\mathsf{Element}, \mathsf{Stack}\}$.
   b. $\mathcal{C} = \{\mathsf{error}, \mathsf{bottom}\}$.
   c. $\mathcal{F} = \{\mathsf{push}, \mathsf{pop}, \mathsf{top}\}$.
   d. $\mathcal{P} = \emptyset$.
   e. $\tau(\mathsf{error}) = \mathsf{Element}$.
   f. $\tau(\mathsf{bottom}) = \mathsf{Stack}$.
   g. $\tau(\mathsf{push}) = \mathsf{Element} \times \mathsf{Stack} \to \mathsf{Stack}$.
   h. $\tau(\mathsf{pop}) = \mathsf{Stack} \to \mathsf{Stack}$.
   i. $\tau(\mathsf{top}) = \mathsf{Stack} \to \mathsf{Element}$.

2. $\Sigma_{\text{grp}} = (\{G\}, \{e\}, \{*, \mathsf{inv}\}, \emptyset, \tau)$ where $\tau(e) = G$, $\tau(*) = G \times G \to G$, and $\tau(\mathsf{inv}) = G \to G$.

3. Let Γ_{grp} be the following set of Σ-sentences:

Assoc	$\forall\, x, y, z : G \,.\, (x * y) * z = x * (y * z)$.
IdLeft	$\forall\, x : G \,.\, e * x = x$.
IdRight	$\forall\, x : G \,.\, x * e = x$.
InvLeft	$\forall\, x : G \,.\, \mathsf{inv}(x) * x = e$.
InvRight	$\forall\, x : G \,.\, x * \mathsf{inv}(x) = e$.

4. A *partition* of a set S is a nonempty set U of subsets of S such that, for all $x \in S$, x is a member of exactly one member of U. Hence (1) the members of U are disjoint and (2) their union equals S.

5. A *lattice* is a weak partial order $(U, \leq)$ such that each pair of elements of U has both a least upper bound and a greatest lower bound.

6. Let $M_1 = (D_1, e_1, *_1)$ and $M_2 = (D_2, e_2, *_2)$ be two monoids. A *monoid homomorphism* from M_1 to M_2 is a function $h : D_1 \to D_2$ such that:

   a. $h(x *_1 y) = h(x) *_2 h(y)$ for all $x, y \in D_1$.
   b. $h(e_1) = e_2$.

Exercises

1. Construct in MSFOL a theory $T = (\Sigma_{\mathsf{stack}}, \Gamma_{\mathsf{stack}})$ of stacks. Γ_{stack} should contain axioms that say:

   a. The top of the bottom stack is the error element.
   b. Let s be a stack obtained by pushing an element e onto a stack s'. The top of s is e.
   c. Pop of the bottom stack is the bottom stack.
   d. Let s be a stack obtained by pushing an element e onto a stack s'. The pop of s is s'.

   Solution:
   Σ_{stack} is defined in Background Definitions.
   Γ_{stack} contains the following:

   a. $\mathsf{top}(\mathsf{bottom}) = \mathsf{error}$
   b. $\forall\, e : \mathsf{Element},\ s' : \mathsf{Stack}\ .\ \mathsf{top}(\mathsf{push}(e,\ s')) = e$
   c. $\mathsf{pop}(\mathsf{bottom}) = \mathsf{bottom}$
   d. $\forall\, e : \mathsf{Element},\ s' : \mathsf{Stack}\ .\ \mathsf{pop}(\mathsf{push}(e,\ s')) = s'$

2. A *group* is a monoid with an inverse operation. $T_{\mathrm{grp}} = (\Sigma_{\mathrm{grp}}, \Gamma_{\mathrm{grp}})$ is a theory of groups. Show that models of T_{grp} can be directly derived from $(\mathbb{Z}, 0, +)$ and $(\mathbb{Q}, 1, *)$ but not from $(\mathbb{N}, 0, +)$ and $(\mathbb{Z}, 1, *)$.

   Solution:

   In each case, we must show/argue that the function provided for $*$ is associative, that the constant provided for e is a left and right identity for that operator, and provide a function for inv which is a right and left inverse of the function for $*$.

   $(\mathbb{Z}, 0, +)$: $+$ is associative, 0 is the identity for $+$, and for inv we may take the unary $-$, since $x + -x = 0$ and $-x + x = 0$.

   $(\mathbb{Q}, 1, *)$: Take $\mathbb{Q}\backslash\{0\}$ as G, rather than $\mathbb{Q}$, because there is no multiplicative inverse for 0. Then, $*$ is associative, 1 is the identity for $*$, and for inv we may take the unary "recipricol", since $x * \frac{1}{x} = 1$ and $\frac{1}{x} * x = 1$ (since we have removed 0).

$(\mathbb{N}, 0, +)$: while $+$ is associative and 0 is the identity for $+$, we cannot provide an inverse for plus; this can be seen because e.g. $1 + y \neq 0$ for any y.

$(\mathbb{Z}, 1, *)$: while $*$ is associative and 1 is the identity for $*$, we cannot provide an inverse for $*$; this can be seen because e.g. $1 + y \neq 0$ for any y..

3. Let $\Sigma = (\alpha, p : \alpha \to \mathbb{B}, q : \alpha \to \mathbb{B})$ be a signature of MSFOL. What should Γ be so that each model for the theory $T = (\Sigma, \Gamma)$ is a set of values partitioned into two components defined by p and q.

   Solution:

   Let Γ contain the following Σ-sentences:

   a. $\forall x : \alpha \, . \, \neg(p\, x \wedge q\, x)$.
   b. $\forall x : \alpha \, . \, p\, x \vee q\, x$.

   Let $\mathcal{M} = (\{D_\alpha\}, I)$ be a model for T. Then clearly the two sets

   $$\{d \in D_\alpha \mid V^{\mathrm{M}}_{\phi[x:\alpha \mapsto d]} p\,(x : \alpha) = T\}$$

   and

   $$\{d \in D_\alpha \mid V^{M}_{\phi[x:\alpha \mapsto d]} q\,(x : \alpha) = T\}$$

   are a partition of D_α.

4. Let Σ_{pairs} be the signature $(\mathcal{B}, \emptyset, \mathcal{F}, \emptyset, \tau)$ where:

   a. $\mathcal{B} = \{\alpha, \beta, \gamma\}$.
   b. $\mathcal{F} = \{\mathsf{mkPair}, \mathsf{left}, \mathsf{right}\}$ where $\tau(\mathsf{mkPair}) = \alpha \times \beta \to \gamma$, $\tau(\mathsf{left}) = \gamma \to \alpha$, and $\tau(\mathsf{right}) = \gamma \to \beta$.

   What should Γ_{pairs} be so that $T_{\text{pairs}} = (\Sigma_{\text{pairs}}, \Gamma_{\text{pairs}})$ is a theory of mathematical structures that contain (1) sets A, B, and C where C is a set of values that have the same structure as ordered pairs of members of A and B and (2) functions to construct and destruct the pairs in C?

   Solution:
   Γ_{pairs} should consist of the following sentences:

   a. OnlyPairs $\forall z : \gamma \, . \, \exists\, x : \alpha, y : \beta \, . \, \mathsf{mkPair}\ x\ y = z$
   b. mkPairInj $\forall x, x' : \alpha, y, y' : \beta \, . \, \mathsf{mkPair}\ x\ y = \mathsf{mkPair}\ x'\ y'$
      $\Rightarrow (x = x' \wedge y = y')$
   c. IsLeftProj $\forall x : \alpha, y : \beta \, . \, \mathsf{left}\ (\mathsf{mkPair}\ x\ y) = x$
   d. IsRightProj $\forall x : \alpha, y : \beta \, . \, \mathsf{right}\ (\mathsf{mkPair}\ x\ y) = y$

5. Let $\Sigma_{\mathsf{lattice}} = (\{U\}, \emptyset, \emptyset, \{\leq\}, \tau)$ where $\tau(\leq) = U \times U \to \mathbb{B}$. Construct in MSFOL a theory $T = (\Sigma_{\mathsf{lattice}}, \Gamma_{\mathsf{lattice}})$ of lattices.

   Solution.

   $\Sigma_{\mathsf{lattice}}$ is defined in Background Definitions.

   $\Gamma_{\mathsf{lattice}}$ contain the following:

   a. $\forall\, x \in U \,.\, x \leq x$
   b. $\forall\, x, y \in U \,.\, (x \leq y \wedge y \leq x) \Rightarrow x = y$
   c. $\forall\, x, y, z \in U \,.\, (x \leq y \wedge y \leq z) \Rightarrow x \leq z$
   d. $\forall\, x, y \in U \,.\, \exists\, z \in U \,.$
      $x \leq z \wedge y \leq z \wedge (\forall\, w \in U \,.\, x \leq w \wedge y \leq w \Rightarrow z \leq w).$
   e. $\forall\, x, y \in U \,.\, \exists\, z \in U \,.$
      $z \leq x \wedge z \leq y \wedge (\forall\, w \in U \,.\, w \leq x \wedge w \leq y \Rightarrow w \leq z).$

6. Explain why it is not possible to construct a theory of well-founded relations in MSFOL.

   Solution:

   Recall the definition of a well founded relation: a binary relation R on a set U is well founded if every non-empty subset of U has an R-minimal element.

   Since it is not possible to quantify over subsets of a sort in MSFOL, it is not possible to construct a theory of well-founded relations in MSFOL.

7. Construct in MSFOL a theory of vector spaces.

   Solution: Let $T_{vs} = (\Sigma_{vs}, \Gamma_{vs})$ where Σ_{vs} contains the following:

   - $\mathcal{B} = \{F, V\}$
   - $\mathcal{C} = \{1, \vec{\mathbf{0}}\}$
   - $\mathcal{F} = \{*, \cdot, +, +_F, -\}$
   - $\mathcal{P} = \{F, V\}$
   - τ has:
     - $\tau(1) = F$
     - $\tau(\vec{\mathbf{0}}) = V$
     - $\tau(*) = F \times F \to F$
     - $\tau(\cdot) = F \times V \to V$
     - $\tau(+) = V \times V \to V$
     - $\tau(+_F) = F \times F \to F$
     - $\tau(-) = V \to V$

   and Γ_{vs} contains the following axioms:

- $\forall \vec{x}, \vec{y} : V \,.\, \vec{x} + \vec{y} = \vec{y} + \vec{x}$
- $\forall \vec{x}, \vec{y}, \vec{z} : V \,.\, (\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z})$
- $\forall \vec{x} : V \,.\, \vec{\mathbf{0}} + \vec{x} = \vec{x}$
- $\forall \vec{x} : V \,.\, \vec{x} + \vec{\mathbf{0}} = \vec{x}$
- $\forall \vec{x} : V \,.\, \vec{x} + (-\vec{x}) = \vec{\mathbf{0}}$
- $\forall r, s : F,\, \vec{x} : V \,.\, r \,.\, (s \cdot \vec{x}) = (r * s) \cdot \vec{x}$
- $\forall r, s : F,\, \vec{x} : V \,.\, (r +_F s) \cdot \vec{x} = r \cdot \vec{x} + s \cdot \vec{x}$
- $\forall r : F,\, \vec{x}, \vec{y} : V \,.\, r \cdot (\vec{x} + \vec{y}) = r \cdot \vec{x} + r \cdot \vec{y}$
- $\forall \vec{x} : V \,.\, 1 \cdot \vec{x} = \vec{x}$

8. Construct in MSFOL a theory T of monoid homomorphisms where each model for T contains a monoid homomorphism.

   Solution:

   Let $\Sigma = (\{M_1, M_2\}, \{e_1, e_2\}, \{\circ_1, \circ_2, h\}, \emptyset, \tau)$ where τ is defined as follows:

   a. $\tau\, e_1 = M_1$.
   b. $\tau\, e_2 = M_2$.
   c. $\tau \circ_1 = M_1 \times M_1 \to M_1$.
   d. $\tau \circ_2 = M_2 \times M_2 \to M_2$.
   e. $\tau\, h = M_1 \to M_2$.

   Let $T = (\Sigma, \Gamma)$ where Γ contains the following Σ-sentences:

   a. $\forall x, y, z : M_1 \,.\, (x \circ_1 y) \circ_1 z = x \circ_1 (y \circ_1 z)$.
   b. $\forall x : M_1 \,.\, e \circ_1 x = x$.
   c. $\forall x : M_1 \,.\, x \circ_1 e = x$.
   d. $\forall x, y, z : M_2 \,.\, (x \circ_2 y) \circ_2 z = x \circ_2 (y \circ_2 z)$.
   e. $\forall x : M_2 \,.\, e \circ_2 x = x$.
   f. $\forall x : M_2 \,.\, x \circ_2 e = x$.
   g. $\forall x, y : M_1 \,.\, h\,(x \circ_1 y) = (h\, x) \circ_2 (h\, y)$.
   h. $h\, e_1 = e_2$.

   If $\mathcal{H} = (\{D_{M_1}, D_{M_2}\}, I)$ is a model for T, then clearly $I\, h$ is a monoid homomorphism. If g is a monoid homomorphism in some mathematical model, then clearly there is a Σ-interpretation $\mathcal{H} = (\{D_{M_1}, D_{M_2}\}, I)$ such that $g = I\, h$. Therefore, T is a theory of monoid homomorphisms.