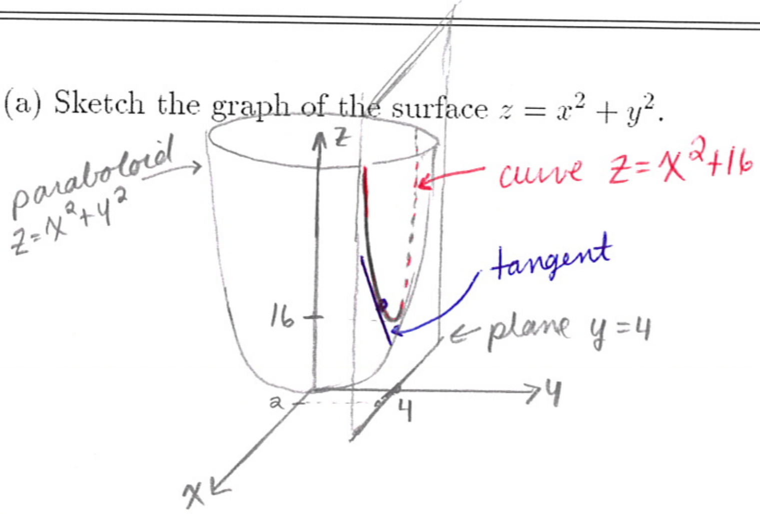
ASSIGNMENT 5

Sections 4, 5, and 6 in the Red Module

1. (a) Sketch the graph of the surface $z = x^2 + y^2$.



(b) Explain how to obtain the curve with the property that the slope of its tangent at (2,4) is equal to the partial derivative $f_x(2,4)$. Add the curve and its tangent to the graph of the surface in part (a).

Fix $y = 4$.
Then $z = x^2+16$ is the curve of intersection between the paraboloid and the plane $y=4$.

2. Assume that the function $T(x, y, t)$ models the temperature (in degrees Celsius) at time t in a city located at a longitude of x degrees and a latitude of y degrees. The time t is measured in hours. What is the meaning of the partial derivative $T_t(x, y, t)$? What are its units? What is most likely going to be the sign of $T_y(x, y, t)$ for Winnipeg, Manitoba in January?

$T_t(x,y,t)$ is the rate of change of temperature with respect to time. Its units are $°C/h$ (degrees Celsius per hour).

$T_y(x,y,t)$ is the rate of change of temperature with respect to latitude. As latitude increases from Winnipeg, we would expect temperature to decrease, hence $T_y(x,y,t)$ will most likely be negative.

3. Below is an excerpt from a table of values of I, the temperature-humidity index, which is the perceived air temperature when the actual temperature is T (degrees fahrenheit), and the relative humidity is h (percent).

$T \downarrow$ $h \rightarrow$	20	30	40	50	60	70
80	74	76	78	82	83	86
85	81	82	84	86	90	94
90	86	90	93	96	101	106
95	94	94	98	107	111	125
100	99	101	109	122	129	138

(a) Write the definition (equation) of the partial derivative of $I(T, h)$ with respect to h.

$$I_h(T,h) = \lim_{\Delta h \to 0} \frac{I(T,h+\Delta h) - I(T,h)}{\Delta h}$$

(b) Approximate $I_h(95, 40)$ and interpret your answer, i.e., write a statement to explain what this number represents, including units.

$\Delta h = 10$: $I_h(95,40) \approx \dfrac{I(95,50) - I(95,40)}{10} = \dfrac{107-98}{10} = 0.9 \ \dfrac{\text{humidex points}}{\text{\% humidity}}$

$\Delta h = -10$: $I_h(95,40) \approx \dfrac{I(95,30) - I(95,40)}{-10} = \dfrac{94-98}{-10} = 0.4 \ \dfrac{\text{humidex points}}{\text{\% humidity}}$

average of estimates $= \dfrac{0.9 + 0.4}{2} = 0.65 \ \dfrac{\text{humidex points}}{\text{\% humidity}}$

$\therefore$ When temperature is 95°F and humidity is at 40%, the humidex is 98°F and is increasing at a rate of 0.65 humidex points per percent increase in humidity.

4. Compute the indicated partial derivatives.

(a) $f(x,y) = \dfrac{4x - xy}{x^2+y^2}$; $f_x(x,y)$

$$f(x,y) = \frac{x(4-y)}{x^2+y^2}$$

$$f_x(x,y) = \frac{(4-y)(x^2+y^2) - x(4-y)\cdot 2x}{(x^2+y^2)^2}$$

$$= \frac{(4-y)(y^2-x^2)}{(x^2+y^2)^2}$$

(b) $h(x,t) = te^{\sqrt{x-4t^2}}$; $h_t(5,1)$

$$h_t(x,t) = 1\cdot e^{\sqrt{x-4t^2}} + t\cdot e^{\sqrt{x-4t^2}}\cdot \frac{1}{2\sqrt{x-4t^2}}\cdot(-8t)$$

$$= e^{\sqrt{x-4t^2}}\left(1 - \frac{4t^2}{\sqrt{x-4t^2}}\right)$$

$$h_t(5,1) = e^{\sqrt{5-4(1)^2}}\left(1 - \frac{4(1)^2}{\sqrt{5-4(1)^2}}\right) = -3e$$

5. Let $f(x, y) = \ln(3x - y + 1)$. >0

(a) Compute the partial derivatives of f.

$$f_x(x,y) = \frac{3}{3x - y + 1}$$

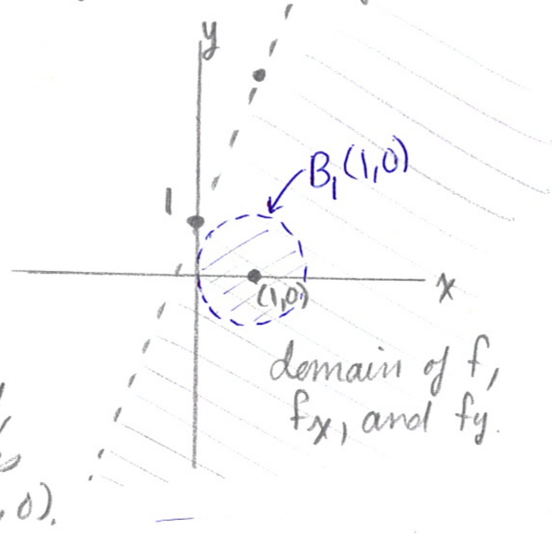
$$f_y(x,y) = \frac{-1}{3x - y + 1}$$

$3x-y+1 \neq 0$ (and also $3x-y+1>0$ since domain$(f_y) \subseteq$ domain(f))

(b) Find and sketch the domains of f_x and f_y. (*Recall*: The domain of a derivative of a function is always a *subset* of the domain of the function).

domain$(f) = \{(x,y) \in \mathbb{R}^2 \mid y < 3x+1\}$

domain$(f_x) = \{(x,y) \in \mathbb{R}^2 \mid y < 3x+1\} =$ domain(f_y)



(c) Is f differentiable at $(1, 0)$? Explain.

YES! The partial derivatives are rational functions and are ∴ continuous on their domain. The point $(1,0)$ is well inside the domain. In fact, take $B_1(1,0)$. f_x and f_y are continuous on $B_1(1,0) \Rightarrow f$ is differentiable at $(1,0)$.

(d) Find the equation of the tangent plane to the surface $f(x, y) = \ln(3x - y + 1)$ at the point $(1, 0)$. Is this tangent plane a good approximation of the surface near the point of tangency? Explain.

$$z = f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y-0)$$
$$= \ln 4 + \frac{3}{4}(x-1) - \frac{1}{4}y$$

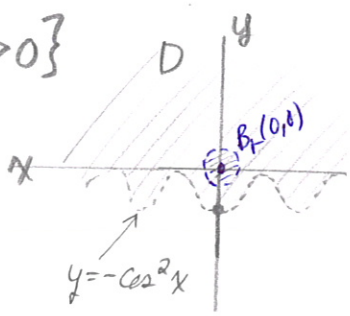
This tangent plane is a good approximation of the surface near $(1,0)$ since the f^N is differentiable at $(1,0)$.

6. Consider the function $f(x, y) = \sqrt{y + \cos^2 x}$.

(a) Using Theorem 6, show that the function is differentiable at $(0, 0)$.

$$f_x = \frac{2\cos x(-\sin x)}{2\sqrt{y+\cos^2 x}} = \frac{-\sin 2x}{2\sqrt{y+\cos^2 x}} \qquad f_y = \frac{1}{2\sqrt{y+\cos^2 x}}$$

$D = \text{domain}(f_x) = \text{domain}(f_y) = \{(x,y) \in \mathbb{R}^2 \mid y + \cos^2 x > 0\}$



Note that $B_{0.1}(0,0) \subset D$. $\because$ f_x and f_y are continuous on their domains $\therefore$ they are continuous on $B_{0.1}(0,0) \Longrightarrow f$ is differentiable at $(0,0)$

choose something small enough

(b) Verify the linear approximation $\sqrt{y + \cos^2 x} \approx 1 + \frac{1}{2}y$ at (0,0).

$\because$ f is diff. at $(0,0)$ $\therefore$ $f(x,y) \approx L_{(0,0)}(x,y)$.

verify: $L_{(0,0)}(x,y) = 1 + \frac{1}{2}y$.

$$L_{(0,0)}(x,y) = f(0,0) + f_x(0,0)x + f_y(0,0)y$$

$$= \underbrace{\sqrt{0+\cos^2 0}}_{=1} - \frac{\overbrace{\sin 0}^{=0}}{2\underbrace{\sqrt{0+\cos^2 0}}_{=1}}x + \frac{1}{2\underbrace{\sqrt{0+\cos^2 0}}_{=1}}y$$

$$= 1 + \frac{1}{2}y \checkmark$$

$\therefore$ The linear approximation is correct.

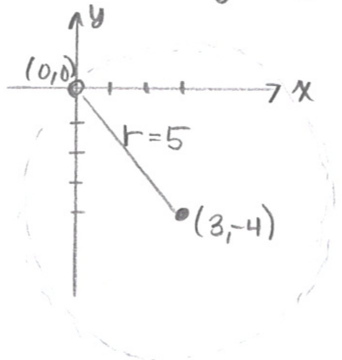
7. Using Theorem 6, show that the function $f(x, y) = xy(x^2 + y^2)^{-1}$ is differentiable at the point $(3, -4)$. What is the largest open disk centred at $(3, -4)$ that you can use?

$$f_x = y \cdot (x^2+y^2)^{-1} + xy(-1)(x^2+y^2)^{-2}(2x)$$

$$= (x^2+y^2)^{-2}\left[y(x^2+y^2) - 2x^2y\right] = (x^2+y^2)^{-2}(y^3 - x^2y)$$

$$f_y = (x^2+y^2)^{-2}(x^3 - y^2x)$$

domain of f_x and f_y : $\mathbb{R}^2 \setminus \{(0,0)\}$



f_x and f_y are continuous on their domains and $B_5(3,-4) \subset$ domain of f_x and f_y.

$\Rightarrow$ f_x and f_y are continuous on $B_5(3,-4)$

$\Rightarrow$ f is differentiable at $(3,-4)$.

8. Suppose that $z = x^2 y \sin x$, where $x = 6t$ and $y = e^t$. Use the Chain Rule to find $z'(t)$.

$$\frac{\partial z}{\partial x} = 2xy\sin x + x^2 y\cos x \qquad \frac{dx}{dt} = 6$$

$$\frac{\partial z}{\partial y} = x^2 \sin x \qquad \frac{dy}{dt} = e^t$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

$$= (2xy\sin x + x^2 y\cos x)\cdot 6 + x^2\sin x\cdot e^t$$

$$= 36te^t\left[2\sin 6t + 6t\cos 6t + t\sin 6t\right]$$

9. Suppose that $z = \dfrac{ab-1}{b^2+1}$, where $a = 3s$ and $b = st$. Use the Chain Rule to find $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$ when $s = 1$ and $t = 1$.

$$\frac{\partial z}{\partial a} = \frac{b}{b^2+1}\ ; \quad \frac{\partial z}{\partial b} = \frac{a(b^2+1)-(ab-1)2b}{(b^2+1)^2} = \frac{a-ab^2+2b}{(b^2+1)^2}$$

$$\frac{\partial a}{\partial s} = 3 \qquad \frac{\partial a}{\partial t} = 0 \qquad \frac{\partial b}{\partial s} = t \qquad \frac{\partial b}{\partial t} = s$$

when $s=1$ & $t=1$, $a=3$ and $b=1$.

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial a}\cdot\frac{\partial a}{\partial s} + \frac{\partial z}{\partial b}\cdot\frac{\partial b}{\partial s}$$

$$\left.\frac{\partial z}{\partial s}\right|_{\substack{s=1\\t=1}} = \frac{1}{2}\cdot 3 + \frac{1}{2}\cdot 1 = 2$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial a}\cdot\frac{\partial a}{\partial t} + \frac{\partial z}{\partial b}\cdot\frac{\partial b}{\partial t}$$

$$\left.\frac{\partial z}{\partial t}\right|_{\substack{s=1\\t=1}} = \frac{1}{2}\cdot 0 + \frac{1}{2}\cdot 1 = \frac{1}{2}$$

10. Wheat production W in a given year depends on the average temperature T and the annual rainfall R. Scientists estimate that the average temperature is rising at a rate of 0.15^oC/year and rainfall is decreasing at a rate of 0.1 cm/year. They also estimate that, at current production levels, $\frac{\partial W}{\partial T} = -2$ and $\frac{\partial W}{\partial R} = 8$.

(a) What is the significance of the signs of these partial derivatives?

$\frac{\partial W}{\partial T} = -2 \Rightarrow$ wheat production decreases as average temperature increases

$\frac{\partial W}{\partial R} = 8 \Rightarrow$ wheat production increases as annual rainfall increases

(b) Estimate the current rate of change of wheat production, $\frac{dW}{dt}$.

$$\frac{dW}{dt} = \frac{\partial W}{\partial T}\cdot\frac{dT}{dt} + \frac{\partial W}{\partial R}\cdot\frac{dR}{dt}$$
$$= (-2)(+0.15) + (8)(-0.1)$$
$$= -1.1 \text{ units of wheat / year.}$$

So wheat production is decreasing at about 1.1 units per year.

11. Suppose f is a differentiable function of x and y, and $g(r,s) = f(\underbrace{2r-s}_{x}, \underbrace{s^2-4r}_{y})$. Use the table of values below to calculate $g_r(1,2)$ and $g_s(1,2)$.

r s

	f	g	f_x	f_y
$(0,0)$	3	6	4	8
$(1,2)$	6	3	2	5

$r=1, s=2 \Rightarrow x=0, y=0$

$$g_r = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial r} = f_x\cdot(2) + f_y\cdot(-4)$$
$$g_r(1,2) = f_x(0,0)\cdot 2 + f_y(0,0)\cdot(-4) = (4)(2) + (8)(-4) = -24$$
$$g_s = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s} = f_x\cdot(-1) + f_y\cdot(2s)$$
$$g_s(1,2) = f_x(0,0)\cdot(-1) + f_y(0,0)\cdot(4) = (4)(-1) + (8)(4) = 28$$

THE END