MATH 1B03/1ZC3 Winter 2019

Lecture 24: Row, column, and null space

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The row, column, and null space of a matrix

(from Chapter 4.7 of Anton-Rorres)

Recall that a given $n \times m$ matrix M defines a transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$. Given a vector $\mathbf{x} \in \mathbb{R}^m$, it is transformed to a vector $\mathbf{y} \in \mathbb{R}^n$ using the rule

$$M\mathbf{x} = \mathbf{y}$$

The matrix M defines important subspaces of $\mathbb{R}^n$ and $\mathbb{R}^m$.

Definition 24.1: Row and column vectors

Let A be an $n \times m$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

The row vectors of A are the $1 \times m$ vectors

$$\mathbf{r}_i = \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{im} \end{bmatrix}$$

The column vectors of A are the $n \times 1$ vectors

$$\mathbf{c}_i = \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{bmatrix}$$

Example 24.2

Given

$$M = \begin{bmatrix} 2 & 0 \\ -3 & 5 \\ 2 & 4 \end{bmatrix}$$

the row vectors are

$$\begin{aligned} \mathbf{r}_1 &= [2 \quad 0] \\ \mathbf{r}_2 &= [-3 \quad 5] \\ \mathbf{r}_3 &= [2 \quad 4] \end{aligned}$$

and the column vectors are

$$\mathbf{c}_1 = \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} 0 \\ 5 \\ 4 \end{bmatrix}$$

Definition 24.3: Row and column space

Let A be an $n \times m$ matrix. The row space of A, denoted row(A), is defined

$$\text{row}(A) = \text{span}\left(\{\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n\}\right)$$

It is the span of the row vectors of A, and is a subspace of $\mathbb{R}^m$.
The column space of A, denoted col(A), is defined

$$\text{col}(A) = \text{span}\left(\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_m\}\right)$$

It is the span of the column vectors of A, and is a subspace of $\mathbb{R}^n$.

Notice that the row and column vectors of a matrix are not necessarily linearly independent, so that they do not automatically form a basis for the row or the column space.

Example 24.4

Given

$$M = \begin{bmatrix} 2 & 0 \\ -3 & 5 \\ 2 & 4 \end{bmatrix}$$

we have

$$\text{row}(M) = \text{span}\left(\{[2 \quad 0], [-3 \quad 5], [2 \quad 4]\}\right) = \mathbb{R}^2$$

and

$$\text{col}(M) = \text{span}\left(\left\{\begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 4 \end{bmatrix}\right\}\right)$$

Notice that the column vectors of A are linearly independent, so $\text{col}(M)$ is a plane in $\mathbb{R}^3$.

Definition 24.5: Null space

Let A be an $n \times m$ matrix. The null space of A, denoted $\text{null}(A)$, is the subspace of vectors $\mathbf{x} \in \mathbb{R}^m$ such that

$$A\mathbf{x} = \mathbf{0}$$

The null space of a matrix is also known as the kernel of A. Notice that, for any matrix A we always have $\mathbf{0} \in \text{null}(A)$ as

$$A\mathbf{0} = \mathbf{0}.$$

We have been secretly finding bases for null spaces of matrices in earlier sections of this course, whenever we found all of the solutions to homogeneous matrix equations i.e. those of the form

$$M\mathbf{x} = \mathbf{0}$$

Example 24.6

Question: Let

$$M = \begin{bmatrix} 5 & 2 & 2 \\ 4 & 0 & 5 \\ 1 & 2 & -3 \end{bmatrix}$$

Find a basis for null(M).
Answer: Consider the equation

$$\begin{bmatrix} 5 & 2 & 2 \\ 4 & 0 & 5 \\ 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying Gauss-Jordon elimination to M we obtain

$$\begin{bmatrix} 1 & 0 & 5/4 \\ 0 & 1 & -17/8 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore x_3 is free, setting $x_3 = s$ we obtain

$$x_1 = -\frac{5}{4}s$$
$$x_2 = \frac{17}{8}s$$

Therefore $\mathbf{x} \in$ null(M) if and only if it is of the form

$$\mathbf{x} = s \begin{bmatrix} -5/4 \\ 17/8 \\ 1 \end{bmatrix}$$

and

$$\left\{ \begin{bmatrix} -5/4 \\ 17/8 \\ 1 \end{bmatrix} \right\}$$

is a basis for null(M).

Knowledge of the row and column space of a matrix A can help us understand solutions to equations of the form

$$A\mathbf{x} = \mathbf{b}$$

Recall that we say the equation $A\mathbf{x} = \mathbf{b}$ is <u>consistent</u> if it possesses at least one solution.

Fact 24.7

Let A be an $n \times m$ matrix. The equation

$$A\mathbf{x} = \mathbf{b}$$

has a unique solution if and only if $\mathbf{b} \in \text{col}(A)$.

Proof: Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

Notice that

$$A \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} = \mathbf{c}_1$$

that is, the first column vector of A.

Similarly

$$A \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{c}_i$$

i–th position

so that given

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$$

we have

$$\begin{aligned} A\mathbf{x} &= A\left(x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + x_m \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right) \\ &= \underbrace{x_1\mathbf{c}_1 + x_2\mathbf{c}_1 + \cdots + x_m\mathbf{c}_m}_{\text{a linear combination of the column vectors}} \end{aligned}$$

Thus we conclude that $A\mathbf{x} = \mathbf{b}$ if and only if $\mathbf{b} \in \text{col}(A)$. ■

Finding bases for the row, column, and null space

For the reason described above, and others, it is often desirable to have a basis for the row, column, and or null space of a matrix. In light of the following facts we can use Gauss-Jordon elimination to quickly find bases for the row and null space (the column space is slightly harder).

Fact 24.8

Let A and B be matrices related by a sequence of elementary row operations. Then $\text{row}(A) = \text{row}(B)$ and $\text{null}(A) = \text{null}(B)$.

Recipe 24.9: Finding a basis for row and null space

Given a matrix M use this recipe to find bases for row(M) and null(M).

Step 1: Perform Gauss-Jordon elimination on M to obtain the row echelon form R.

Step 2: The row vectors of R which contain leading 1's form a basis for row(M).
A basis for null(M) can be found by solving the matrix equation

$$R\mathbf{x} = \mathbf{0}$$

Notice that solving the equation $R\mathbf{x} = \mathbf{0}$ is very simple: R is in row echelon form, so we can simply read off the solution.

If a matrix is given to us already in row echelon form, then we do not need to do any work at all, and can simply read off the basis vectors.

Example 24.10

Question: Find bases for the row and null space of the matrix

$$A = \begin{bmatrix} -4 & 6 & -3 & 2 \\ -4 & 6 & 5 & -2 \\ 6 & -3 & 4 & 0 \end{bmatrix}$$

Answer: Apply Gauss-Jordan elimination to A to obtain

$$R = \begin{bmatrix} 1 & 0 & 0 & 9/16 \\ 0 & 1 & 0 & 11/24 \\ 0 & 0 & 1 & -1/2 \end{bmatrix}$$

Every row vector contains a leading 1 so

$$\left\{ \begin{bmatrix} 1 & 0 & 0 & \frac{9}{16} \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & \frac{1}{24} \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & -\frac{1}{2} \end{bmatrix} \right\}$$

forms a basis for row(M).

To find a basis for null(M), solve the equation $R\mathbf{x} = \mathbf{0}$. Consider

$$\begin{bmatrix} 1 & 0 & 0 & 9/16 \\ 0 & 1 & 0 & 11/24 \\ 0 & 0 & 1 & -1/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This yields

$$\begin{aligned} x_1 &= -\frac{9}{16}x_4 \\ x_2 &= -\frac{11}{24}x_4 \\ x_3 &= \frac{1}{2}x_4 \end{aligned}$$

with x_4 free. Setting $x_4 = s$ we obtain

$$\mathbf{x} = s \begin{bmatrix} -9/16 \\ -11/24 \\ 1/2 \\ 1 \end{bmatrix}$$

so that

$$\left\{ \begin{bmatrix} -9/16 \\ -11/24 \\ 1/2 \\ 1 \end{bmatrix} \right\}$$

is a basis for null(M).

This method allows us to easily find bases for the row and null space of a matrix. Notice that the basis vectors we obtain are not necessarily rows of the original matrix.

Sometimes it is useful to have a basis for the row space given in terms of the rows of the original matrix. Also, we need a systematic method to find a basis for the column space of a matrix. We can do both of these things as follows.

Recipe 24.11: Finding bases for row(A), col(A) in terms of the row and column vectors of A

Let A be an $n \times m$ matrix with column vectors

$$\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_m$$

Use this recipe to find bases for row(A) and col(A) in terms of the row and column vectors of A.
Once we have found this basis, we can write the remaining columns in terms of the basis vectors easily.

Step 1: Apply Gauss-Jordan elimination to A to obtain the row echelon form R.

Step 2: Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ be the column vectors of R.
The vector $\mathbf{c}_i$ is a basis vector of col(A) if and only if $\mathbf{v}_i$ contains a leading 1.

Step 3: Let $\mathbf{c}_j$ be a column vector of A which is not a basis vector of col(A).
To write $\mathbf{c}_j$ as a linear combination of the basis vectors, consider $\mathbf{v}_j$ the associated column vector of R.
The vector $\mathbf{v}_j$ may be written as a linear combination of the column vectors to the left which contain leading 1's. This linear combination also holds for $\mathbf{c}_j$.
For example, if

$$\mathbf{v}_4 = 2\mathbf{v}_1 - 3\mathbf{v}_2$$

then

$$\mathbf{c}_4 = 2\mathbf{c}_1 - 3\mathbf{c}_2$$

Step 4: To find a basis of row(A) in terms of the row vectors of A, apply Steps 1. and 2. to A^T to obtain a basis for col(A^T).
To obtain a basis for row(A), take the transpose of the basis vectors of col(A^T).

We will not go into the details of the facts underpinning this recipe. For the curious, it can be justified by proving that elementary row operations do not break linear independence of a subset of column vectors.

Example 24.12

Question: Find a basis for col(M) and row(M) in terms of the row and column vectors of M, where

$$M = \begin{bmatrix} -10 & -5 & -4 \\ 6 & 3 & 9 \\ 2 & 1 & 3 \end{bmatrix}$$

Write the row and column vectors which are not basis vectors in terms of the basis.
Answer: Apply Gauss-Jordan elimination to obtain

$$R = \begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The column vectors containing leading 1's are

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

so the column vectors of M

$$\mathbf{c}_1 = \begin{bmatrix} -10 \\ 6 \\ 2 \end{bmatrix}, \mathbf{c}_3 = \begin{bmatrix} -4 \\ 9 \\ 3 \end{bmatrix}$$

form a basis for col(A). The column vector $\mathbf{v}_2$ of R may be written

$$\mathbf{v}_2 = \frac{1}{2}\mathbf{v}_1$$

so

$$\mathbf{c}_2 = \frac{1}{2}\mathbf{c}_1$$
$$\begin{bmatrix} -5 \\ 3 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -10 \\ 6 \\ 2 \end{bmatrix}$$

as required.

To find a basis for row(A), repeat the process on M^T

$$M^T = \begin{bmatrix} -10 & 6 & 2 \\ -5 & 3 & 1 \\ -4 & 9 & 3 \end{bmatrix}$$

Gauss-Jordan elimination yields

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 \end{bmatrix}$$

The column vectors containing leading 1's are

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

and

$$\mathbf{v}_3 = \frac{1}{3}\mathbf{v}_2$$

Therefore

$$\left\{ \begin{bmatrix} -10 \\ -5 \\ -4 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \\ 9 \end{bmatrix} \right\}$$

forms a basis for col(M^T) and

$$\{[-10 \quad -5 \quad -4], [6 \quad 3 \quad 9]\}$$

forms a basis for row(M) with

$$\mathbf{r}_3 = \frac{1}{3}\mathbf{r}_2$$

We can use this recipe to find bases for subspaces, by constructing a matrix and applying the method of Recipe 24.11.

Example 24.13

Question: Find a subset of the given vectors which forms a basis for their span, and express the other vectors in terms of the basis vectors

$$\begin{aligned}
\mathbf{u}_1 &= (-5, 4, -2, 0) \\
\mathbf{u}_2 &= (-3, 2, 3, -4) \\
\mathbf{u}_3 &= (-1, -4, -1, 2) \\
\mathbf{u}_4 &= (-1, 3, -2, -3) \\
\mathbf{u}_5 &= (-5, -4, 3, 2)
\end{aligned}$$

Answer: Form a matrix whose columns are the vectors in question

$$M = \begin{bmatrix} -5 & -3 & -1 & -1 & -5 \\ 4 & 2 & -4 & 3 & -4 \\ -2 & 3 & -1 & -2 & 3 \\ 0 & -4 & 2 & -3 & 2 \end{bmatrix}$$

and apply the method given in Recipe 24.11. Gauss-Jordan elimination yields

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 & 12/25 \\ 0 & 1 & 0 & 0 & 9/10 \\ 0 & 0 & 1 & 0 & 53/50 \\ 0 & 0 & 0 & 1 & -29/25 \end{bmatrix}$$

We conclude that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ forms a basis for the span, and

$$\mathbf{u}_5 = \frac{12}{25}\mathbf{u}_1 + \frac{9}{10}\mathbf{u}_2 + \frac{53}{50}\mathbf{u}_3 - \frac{29}{25}\mathbf{u}_4$$

Suggested Problems

Practice the material covered in this lecture by attempting the following questions from Chapter 4.4 of Anton-Rorres, starting on page 246

- Questions 3, 9, 15, 17, 19
- True/False (a), (c), (d), (i)