LEC 3

Today **Row, Null & Column Space**

Let's look back at our linear systems $A\vec{x} = \vec{b}$

$A \in M_{mn}$, i.e. $m \times n$ matrix, our coefficient matrix.

$\vec{x} \in \mathbb{R}^n$ "input" vector: $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ = variables

$\vec{b} \in \mathbb{R}^m$ "output vector" $\vec{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$ = constants

row(A) i.e. the "row space of A"

the span of row vectors of A

eg $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ \vdots & & \\ a_{31} & \cdots & a_{33} \end{bmatrix} \begin{matrix} \leftarrow \vec{r}_1 \\ \\ \leftarrow \vec{r}_3 \end{matrix}$

In general

$A = \begin{bmatrix} \vec{r}_1 \\ \vdots \\ \vec{r}_m \end{bmatrix}$

$\text{row}(A) = \text{Span}(\{\vec{r}_1 \dots \vec{r}_m\}) \subseteq \mathbb{R}^{\underline{n}}$

$n = \#$ of <u>variables</u> in <u>system</u>

<u>**Notice**</u> <u>Row Ops</u> on A change $\underline{\vec{r}}$'s but <u>not</u> $\text{Span}(\{\vec{r}_1 \dots \vec{r}_m\})$

ie $\text{row}(A)$ <u>does</u> <u>not</u> change under <u>row ops</u>

<u>why?</u> Let $A = \begin{bmatrix} \vec{r}_1 \\ \vdots \\ \vec{r}_m \end{bmatrix}$ & $\text{row}(A) = \text{Span}(\{\vec{r}_1 \dots \vec{r}_m\})$

If I mult. a row by $\underline{k \neq 0}$

$\text{row}(\text{new } A)$

$= \text{Span}(\{k\vec{r}_1, \vec{r}_2, \vec{r}_3 \dots \vec{r}_m\})$

$=$ Same <u>span</u>

If I swap two rows, eg. $\vec{r}_1$ & $\vec{r}_2$

$$\text{Span}(\{\vec{r}_1, \vec{r}_2, \dots \vec{r}_m\}) = \text{Span}(\{\vec{r}_2, \vec{r}_1, \vec{r}_3 \dots \vec{r}_m\})?$$

Say I add $k\vec{r}_2$ to $\vec{r}_1$

$$\text{Span}(\{\vec{r}_1 + k\vec{r}_2, \vec{r}_2 \dots \vec{r}_m\}) = \text{Span}(\{\vec{r}_1 \dots \vec{r}_m\})?$$

Yes! Again both are in each other's span!

$\Rightarrow$ row ops do not change row(A)

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eg. Let $A = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 5 & 6 & 5 \end{bmatrix}$, let's put it into RREF.

$R_2 - 2R_1$ & $R_3 - 3R_1$.

$$\begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & -1 & 0 & -7 \\ 0 & -1 & 0 & -7 \end{bmatrix}$$

$R_3 - R_2$, then $R_2 \cdot (-1)$

$$\begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 - 2R_2$

RREF

$$\begin{bmatrix} 1 & 0 & 2 & -10 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

In RREF any col. with a leading 1 is only non-zero at that 1.

⇒ Leading 1 rows not L.C. of rest!

All non-zero rows in RREF are L.I

Basis for row(A)

In genrl   If write vectors as rows:

- RREF non-zero rows $\Rightarrow$ pretty basis of Span!
- # leading 1's = dimension

$= \text{rank}(A) = \dim(\text{row}(A))$
$= \#$ leading 1's

If rank < # rows = m $\Rightarrow$ rows L. D.

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null space of A = null(A) = $\{ \vec{x} \mid A\vec{x} = \vec{0} \} \subseteq \mathbb{R}^n$

vectors same length as rows.

notice null(A) = zero eigenspace (if A is $n \times n$) i.e. $A\vec{x} = \lambda \vec{x}$ with $\lambda = 0$.

$$E\vec{c}_1 = E(a\vec{c}_2 + b\vec{c}_3) = a E\vec{c}_2 + b E\vec{c}_3$$

And if $E\vec{c}_1 = a E\vec{c}_2 + b E\vec{c}_3$, multi. by E^{-1}

$$E^{-1}E\vec{c}_1 = a E^{-1}E\vec{c}_2 + b E^{-1}E\vec{c}_3$$

$$\vec{c}_1 = a\vec{c}_2 + b\vec{c}_3$$

In general any $\vec{c}_i$ is a L.C. of other $\vec{c}_j$'s iff $E\vec{c}_i$ is the same L.C. of $E\vec{c}_j$'s

Column Space of A = col(A) = span of col. of A $\subseteq \mathbb{R}^m$

(if A is $m \times n$, m rows, n col).

Unfortunately row ops mess up col. vectors!

but don't mess up relation of col. vectors.

Remember row op. on A $\Leftrightarrow$ EA, E elementary

(& thus invertible!)

Say for example I have 3 col vectors

& $\vec{c}_1 = a\vec{c}_2 + b\vec{c}_3 \quad a, b \in \mathbb{R}$.

$\Rightarrow$ dim(null(A)) = nullity of A = # parameters in solution

& if A is $m \times n$, n = # variables = # leading 1's + # parameters

= rank + nullity

= dim(row(A)) + dim(null(A)).

(notice this also means if we solve $(A - \lambda I)\vec{x} = \vec{0}$ to get eigenvectors $\Rightarrow$ Finding basis of null$(A - \lambda I)$

$\Rightarrow$ basis eigenvectors are bases of each eigenspace (i.e. subspaces!))

geometric multiplicity = eigenspace dimension

Let's find a <u>basis</u> for our <u>nullspace</u> of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 6 & 9 \\ 2 & 4 & 6 \end{bmatrix}$$

<u>Solution</u> $A\vec{x} = \vec{0} \Rightarrow$ solve $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 3 & 6 & 9 & 0 \\ 2 & 4 & 6 & 0 \end{array}\right]$ $R_2 - 3R_1$ & $R_3 - 2R_1$

$$\hookrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ & 0 & & 0 \end{array}\right] \Rightarrow \begin{cases} x = 0 - 2s - 3t \\ y = s \\ z = t \end{cases}$$

$$\hookrightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2s - 3t \\ s \\ t \end{bmatrix} = t\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} + s\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

<u>when solving</u> each parameter has a vector with <u>1</u> in a position where <u>all</u> <u>others</u> are <u>0</u>

$\Rightarrow$ L.I $\Rightarrow$ <u>basis</u> of null(A).