MATH 1B03/1ZC3 **Winter 2019**

Lecture 9: Determinants II

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Determinants of elementary matrices
(from Chapter 2.2 of Anton-Rorres)

Recall that an elementary matrix is a matrix which can be obtained from the identity matrix by applying exactly one elementary row operation. Using Fact 8.12 it is very easy to compute the determinant of elementary matrices.

First, recall that $det(I_n) = 1$. Combining this with Fact 8.12 we obtain the following.

Fact 9.1: Determinants of elementary matrices

Let E be an elementary $n \times n$ matrix. Then

1. If E is obtained from I_n by swapping a row, then $det(E) = -1$.
2. If E is obtained from I_n by multiplying a row by a scalar λ, then $det(E) = \lambda$.
3. If E is obtained from I_n by adding a multiple of one row to another, then $det(E) = 1$.

Mixing and matching

Often the quickest way to compute the determinant of a matrix is to combine both the method of cofactor expansion and row reduction. The idea is to use row reduction to produce 'nice' rows or columns, and then expand along them. Remember that when using row operations we must keep track of how the determinant is changing.

Example 9.2

Question: Compute the determinant of the matrix

$$A = \begin{bmatrix} 2 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

Answer:

$$\begin{aligned}
\begin{vmatrix} 2 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{vmatrix} & \overset{-2R2+R1}{=} \begin{vmatrix} 0 & 1 & 1 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{vmatrix} \\
& \overset{-R3+R4}{=} \begin{vmatrix} 0 & 1 & 1 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 3 & 3 & 3 \end{vmatrix} \\
& \overset{-3R1+R4}{=} \begin{vmatrix} 0 & 1 & 1 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 6 \end{vmatrix}
\end{aligned}$$

Now expand along first column:

$$\begin{aligned}
\begin{vmatrix} 0 & 1 & 1 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 6 \end{vmatrix} &= (-1) \begin{vmatrix} 1 & 1 & -1 \\ 2 & 1 & 0 \\ 0 & 0 & 6 \end{vmatrix}, \quad \text{now expand along third row} \\
&= (-1)(6) \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} \\
&= -6(1-2) \\
&= 6
\end{aligned}$$

Properties of determinants

(from Chapter 2.3 of Anton-Rorres)

Now that we have two reliable ways of computing the determinant, we will look at some nice properties of the determinant which can make our lives easier.

The determinant and scalar multiplication

The determinant interacts nicely with scalar multiplication.

Fact 9.3: Determinant and scalar multiplication

Let A be an $n \times n$ matrix, and λ a scalar. Then $det(\lambda A) = \lambda^n \det(A)$.

Proof:

$$det(\lambda A) = \begin{vmatrix} \lambda a_{11} & \lambda a_{12} & \cdots & \lambda a_{1n} \\ \lambda a_{21} & \lambda a_{22} & \cdots & \lambda a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ \lambda a_{n1} & \lambda a_{n2} & \cdots & \lambda a_{nn} \end{vmatrix}$$

$$= \lambda \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \lambda a_{21} & \lambda a_{22} & \cdots & \lambda a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ \lambda a_{n1} & \lambda a_{n2} & \cdots & \lambda a_{nn} \end{vmatrix}$$

$$= \lambda^2 \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ \lambda a_{n1} & \lambda a_{n2} & \cdots & \lambda a_{nn} \end{vmatrix}$$

$\vdots$ repeat this for the remaining rows

$$= \lambda^n \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$= \lambda^n det(A)$$

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The determinant of a sum

In the case of matrix addition, the determinant does not interact nicely, in general. That is,

$$det(A + B) \neq det(A) + det(B).$$

For example, consider the matrices

$$A = \begin{bmatrix} 1 & 7 & -4 \\ 0 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix}$$

so that $det(A) = 8$, and

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

so that $det(B) = 4$. Then

$$A + B = \begin{bmatrix} 2 & 7 & -4 \\ 0 & 4 & 3 \\ 0 & 0 & 6 \end{bmatrix}$$

and $det(A + B) = 24$. Notice that $det(A) + det(B) = 12$, so that $det(A + B) \neq det(A) + det(B)$.

However, there is a special case in which the determinant of the sum is the sum of the determinants.

Fact 9.4

Let A and B be square matrices of the same size. Assume that A is equal to B except in the k-th row.
Let C be the matrix with rows equal to those of A (and B), except in the k-th row: let the k-th row of C be equal to the sum of the k-th row of A and the k-th row of B.
Then $det(C) = det(A) + det(B)$.

Example 9.5

Let

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 0 & 4 & -2 \\ 8 & 2 & -5 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 1 & 3 & -2 \\ 2 & -3 & 0 \\ 8 & 2 & -5 \end{bmatrix}$$

Notice that A is equal to B except in the second row. We have $det(A) = 0$, and $det(B) = -11$.

The matrix

$$C = \begin{bmatrix} 1 & 3 & -2 \\ 2 & 1 & -2 \\ 8 & 2 & -5 \end{bmatrix}$$

is equal to A and B, except in the second row. The second row of C is the sum of the second row of A and the second row of B, so that we have $det(C) = det(A) + det(B) = -11$.

The determinant of a product

Unlike the awkward fact about matrix addition, matrix multiplication interacts very well with the determinant.

Fact 9.6: Determinant of a product

Let A and B be square matrices of the same size. Then

$$det(AB) = det(A)det(B).$$

We can express this fact in words as 'the determinant of the product is the product of the determinants'.

The proof of this fact is more complicated than those we have seen so far, and we will not cover it.

The determinant and invertibility

One of the main reasons for the importance of the determinant is that it allows us to check if a matrix is invertible or not.

Fact 9.7

Let A be a square matrix. Then A is invertible if and only if $det(A) \neq 0$.

As this fact contains an 'if and only if', it follows that a square matrix A is singular if and only if $det(A) = 0$. Againg, the proof of this fact is a bit too complicated for this course.

We know have a very simple way of checking if a matrix is invertible or not, just by computing its determinant.

As a consequence of Theorem 9.7 we obtain the following fact.

Fact 9.8

Let A be an invertible matrix. Then

$$det(A^{-1}) = \frac{1}{det(A)}.$$

Proof: Let A be invertible. Then A^{-1} exists and $A^{-1}A = I$. Recall that $det(I) = 1$. Then

$$\begin{aligned} 1 &= det(I) \\ &= det(A^{-1}A) \\ &= det(A^{-1})det(A) \end{aligned}$$

and we can conclude that

$$det(A^{-1}) = \frac{1}{det(A)}.$$

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The adjoint of a matrix

Using the determinant and cofactors of a matrix, we can produce a method to always find the inverse of a matrix, if it exists. This method yields a formula for the inverse, which, after enough computation, gives us the inverse. This is in contrast to the inversion algorithm, which required us to pick a sequence of elementary row operations.

In fact, this method is how we obtained the formula for a 2×2 matrix.

The downside of this method is that it is incredibly computationally intensive i.e. it requires a very large amount of computation. Moreover, the number of computations we need to do grows exponentially with the size of the matrix.

To emphasise how slow this it is, 1048576 individual computations are required to compute the determinant of a 20×20 matrix using this method.

Using this method for anything other than a $2{\times}2$ is slower than the other methods we have seen, so it should not be used for larger matrices.

Definition 9.9: Matrix adjoint

Let A be an $n \times n$ matrix. Recall that $C_{i,j}$ is the ij-th cofactor of A, and is defined

$$C_{i,j} = (-1)^{i+j} det(A[i,j])$$

where $A[i,j]$ is the matrix formed from A by deleting the i-th row and the j-th column.
Let C be the matrix of cofactors i.e.

$$(C)_{ij} = C_{i,j}$$

The adjoint of A is denoted $adj(A)$ and is defined

$$adj(A) = C^T.$$

That is, it is the transpose of the matrix of cofactors of A.

The adjoint is also known as the adjugate. (The term 'adjoint' is also used to refer to a different concept in linear algebra, which we will not see in this course).

Example 9.10

Question: Compute the adjoint of the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & -2 \\ 1 & 2 & 4 \end{bmatrix}$$

Answer: We can use the chequerboard pattern of $+$ and $-$ signs we saw earlier to speed up this calculation: recall

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

Now compute all the minors $M_{i,j} = det(A[i, j])$.

$$det(A[1,1]) = \begin{vmatrix} 3 & -2 \\ 2 & 4 \end{vmatrix} = 16$$

$$det(A[2,3]) = \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = 0$$

$$det(A[1,2]) = \begin{vmatrix} 0 & -2 \\ 1 & 4 \end{vmatrix} = 2$$

$$det(A[3,1]) = \begin{vmatrix} 2 & 0 \\ 3 & -2 \end{vmatrix} = -4$$

$$det(A[1,3]) = \begin{vmatrix} 0 & 3 \\ 1 & 2 \end{vmatrix} = -3$$

$$det(A[3,2]) = \begin{vmatrix} 1 & 0 \\ 0 & -2 \end{vmatrix} = -2$$

$$det(A[2,1]) = \begin{vmatrix} 2 & 0 \\ 2 & 4 \end{vmatrix} = 8$$

$$det(A[3,3]) = \begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} = 3$$

$$det(A[2,2]) = \begin{vmatrix} 1 & 0 \\ 1 & 4 \end{vmatrix} = 4$$

Therefore

$$C = \begin{bmatrix} 16 & -2 & -3 \\ -8 & 4 & 0 \\ -4 & 2 & 3 \end{bmatrix}$$

and

$$adj(A) = \begin{bmatrix} 16 & -8 & -4 \\ -2 & 4 & 2 \\ -3 & 0 & 3 \end{bmatrix}$$

Why would we ever to go so much trouble? Because of the following fact.

Fact 9.11: The inverse via the adjoint

Let A be a square matrix. If $det(A) \neq 0$, then

$$A^{-1} = \frac{1}{det(A)} adj(A).$$

Therefore, we can obtain the inverse of a matrix from its adjoint.

Example 9.12

Question: Use the adjoint to find the inverse of

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & -2 \\ 1 & 2 & 4 \end{bmatrix}$$

Answer: We have that

$$adj(A) = \begin{bmatrix} 16 & -8 & -4 \\ -2 & 4 & 2 \\ -3 & 0 & 3 \end{bmatrix}$$

so we need to compute $det(A)$. Using cofactor expansion, and expanding along the first column we obtain

$$\begin{aligned} \begin{vmatrix} 1 & 2 & 0 \\ 0 & 3 & -2 \\ 1 & 2 & 4 \end{vmatrix} &= det(A[1,1]) + det(A[3,1]) \quad = \begin{vmatrix} 3 & -2 \\ 2 & 4 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 31 & -2 \end{vmatrix} \\ &= 16 - 4 \\ &= 12 \end{aligned}$$

so that

$$A^{-1} = \frac{1}{12} \begin{bmatrix} 16 & -8 & -4 \\ -2 & 4 & 2 \\ -3 & 0 & 3 \end{bmatrix}$$

Question 9.13

Verify that this method returns the same answer as the inversion algorithm.

Example 9.14

Question: Consider the matrix

$$B = \begin{bmatrix} 0 & c & 1 \\ -c & 3 & 9 \\ 0 & 5 & 1 \end{bmatrix}$$

For which values of c is B invertible?
Answer: Compute $det(B)$, as it tells us exactly when B is invertible. By cofactor expansion along the first column we obtain

$$\begin{aligned} det(B) &= (-c)(-1)\begin{vmatrix} c & 1 \\ 5 & 1 \end{vmatrix} \\ &= c(c-5) \end{aligned}$$

The matrix B is singular if and only if $det(B) = 0$, so we consider

$$c(c-5) = 0$$

which has solutions at $c = 0$ and $c = 5$.
Therefore B is invertible for all values of c except $c = 0$ and $c = 5$.

Suggested Problems

Practice the material covered in this lecture by attempting the following questions from Chapter 2.3 of Anton-Rorres, starting on page 127

- Questions $17, 19, 33, 35$
- True/False questions (a), (b), (c), (f), (g)