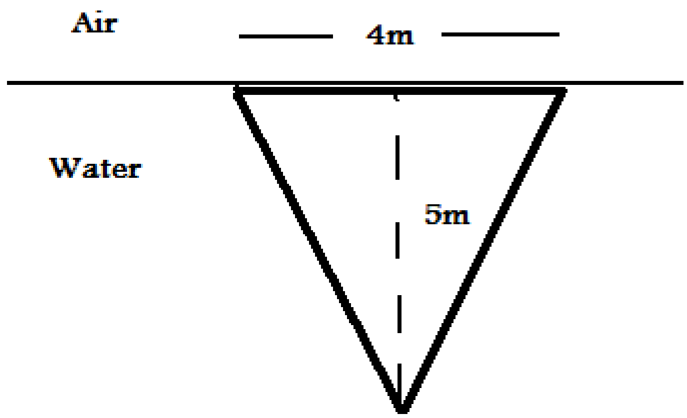
Extra Example of Hydrostatic Pressure

A Simple Triangular Plate

Question:

A triangular plate with base 4 meters across and 5 meters tall (see diagram), is submerged just under water, tip pointing downwards. What is the total water pressure on one face of the plate?



Solution:

We know at any given depth the water pressure is given by the equation:

$$P = \rho g \cdot (\text{depth})$$

So the total force on any thin horizontal strip, of vertical thickness, Δx is:

$$F_{slice} = P \cdot \text{Area} = \rho g \cdot (\text{depth}) \cdot (\text{width}) \cdot \Delta x$$

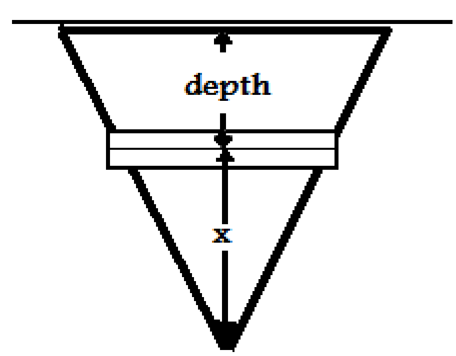
Now let's stop here a second, and think about what we have left to do:

Clearly, our *x*-variable, then, is somehow a measure of the depth, but how exactly?

Really what we have to choose is where $x = 0$ (ie the point we are measuring *from*) and whether x increases or decreases with depth. Once we have the x well defined, we would proceed to construct an equation for the width as a function of x. Then we are free to rewrite our F's into a sum, and thus into an integral.

Now, for illustration purposes, I'll finish up the calculation two different ways, using two different typical choices of x.

Version #1: *Choose x as height up from tip of triangle*



If our x is chosen to measure the distance up from the tip of the triangle, upwards, the actual depth into the water, then, is height of the triangle subtract x. That is, depth = $5 - x$.

Our equation for the force on one horizontal slice then simplifies to:

$$F_{slice} = P \cdot \text{Area} = \rho g \cdot (5 - x) \cdot (\text{width}) \cdot \Delta x$$

Now, how does width change with x?

If we slice the triangle at height x, we are creating the base of a new, similar triangle to the original. So the ratio of height to with is the same as that of the entire triangle. This means:

$$\text{width}/x = \text{width}/\text{height} = 4/5$$

So then we have that:

$$\text{width} = 4x/5$$

So our force on any given slice equation becomes:

$$F_{slice} = P \cdot \text{Area} = \rho g \cdot (5 - x) \cdot (4x/5) \cdot \Delta x$$

If we add these slices up we then find that the total force on our plate is approximately:

$$\text{Net Force} \simeq \sum_{Slices} F_{slice} = \sum_i \rho\, g(5 - x_i)\left(\frac{4x_i}{5}\right)\Delta x = \sum_i \rho\, g \cdot 4x_i \cdot \left(1 - \frac{x_i}{5}\right) \cdot \Delta x$$

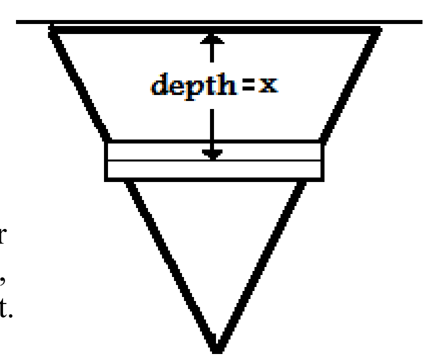
And we can convert this, in the limit as our slices become thin, into the integral:

$$\text{Net Force} = \int_{tip}^{top} \rho\, g \cdot 4x \cdot \left(1 - \frac{x}{5}\right) \cdot dx = 4\rho\, g \int_0^5 x - \frac{1}{5}x^2\, dx$$

Which we can solve:

$$\text{Net Force} = 2\rho\, gx^2 - \frac{4}{15}\rho\, gx^3 \Big|_0^5 = \rho\, g\left(50 - \frac{100}{3}\right) = \frac{50}{3}\rho\, g$$

Version #2: *Choose x as depth from surface*



If x measures the actual depth into the water, then our equation for the force on one horizontal slice simplifies:

$$F_{slice} = P \cdot \text{Area} = \rho g x \cdot (\text{width}) \cdot \Delta x$$

Now we just have to consider the width. Unlike above, the equation for width isn't quite as simple. We are still working with a triangle, though, so we know the width should change linearly with our change in height. That is:

$$\text{width} = \text{m} \cdot x + b$$

First let's find the "m".

At the surface (depth 0), we have $x = 0$ and the full width of 4m. At depth 5m, we have $x = 5$ and we're at the tip, so width = 0. This gives:

$$\text{m} = \frac{\Delta\, width}{\Delta\, x} = \frac{4 - 0}{0 - 5} = -\frac{4}{5}$$

Furthermore, since at $x = 0$, the width is 4m, $b = 4$, and we have:

$$\text{width} = -\frac{4}{5}x + 4 = 4\left(1 - \frac{x}{5}\right)$$

So now we can rewrite the force on one slice as:

$$\text{F}_{slice} = \text{P} \cdot \text{Area} = \rho\, g x \cdot (\text{width})\, \Delta x = \rho\, g x \cdot 4\left(1 - \frac{x}{5}\right) \Delta x$$

If we add these slices up we then find that the total force on our plate is approximately:

$$\text{Net Force} \simeq \sum_{Slices} \text{F}_{slice} = \sum_{i} \rho\, g x_i \cdot 4\left(1 - \frac{x_i}{5}\right) \Delta x = \sum_{i} \rho\, g \cdot 4 x_i \cdot \left(1 - \frac{x_i}{5}\right) \cdot \Delta x$$

Look familiar?

This is the same equation as in the other method, just x has a different physical interpretation!

As before, we can convert this, in the limit as our slices become thin, into the integral:

$$\text{Net Force} = \int_{surface}^{tip} \rho\, g \cdot 4x \cdot \left(1 - \frac{x}{5}\right) \cdot dx = 4\rho\, g \int_0^5 x - \frac{1}{5}x^2\, dx = \frac{50}{3}\rho\, g$$

As before.

In general there are infinite possible choices of how to assign x to describe our height, but each variant, in the end, must produce an integral which, even if it looks different superficially, only differs from any other possible choice of integral by choice of variable.

By simply doing an appropriate substitution, any such integral for the same physical situation, would turn into any other such integral. And of course, different form or not, as definite integrals, all will ultimately end up with the same numerical result.