12C3

Last Day: Vegas!

A set, S, which is a subset of a vspace is also a vspace (ie a subspace) if

Vegas! $\left\{\begin{array}{l} \text{1) } S \text{ not empty (or, equivalently, } \vec{0} \in S) \\ \text{2) } S \text{ is closed under addition} \\ \text{3) } S \text{ is closed under scalar multiplication} \end{array}\right.$

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eg. Consider $S \subseteq M_{33}$ such that $S = \{$all symmetric matrices$\}$

Is this a subspace (using usual arithmetic?)

Solution 1) Is it empty? No $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in S$

2) Check addition Let $A \in S \Rightarrow A^T = A$

$B \in S \Rightarrow B^T = B$

Closed under addn

Is $A + B \in S$? $\quad (A+B)^T = A^T + B^T = A + B$ ✓

$\Rightarrow A + B \in S$ ✓

3) <u>Check scalar multn</u>

Closed under multn

$A \in S$, $A^T = A$ & let $k \in \mathbb{R}$

is $kA \in S$? $\quad (kA)^T = kA^T = kA$

$\Rightarrow kA \in S$ ✓

So yes! It's a subspace!

Why do we do vspace / subspace & axioms?

Remember the vspace $\quad V = \mathbb{R}^+ = \{ x \in \mathbb{R} \mid x > 0 \}$

& $\vec{x} + \vec{y} = xy \,, \quad k\vec{x} = x^k$

We determined $\underline{\underline{\vec{0} = 1}}$ since $\vec{x} + \vec{0} = x \cdot \# = x = \vec{x}$

where $\underline{\underline{\# = \vec{0}}}$

Let's now arbitrarily choose our "1" ie $\vec{\imath} = \underline{\underline{2}}$

We can write __any__ $\vec{v} \in V$ as some $p \cdot \vec{\imath}$ where $\underline{\underline{p \in \mathbb{R}}}$.

eg. Let $\vec{a} = 8, \ \vec{b} = 4$ $\qquad \vec{a} + \vec{b} = 8 \cdot 4 = 32$

& $\vec{a} = 3\vec{\imath} \quad \vec{b} = 2\vec{\imath}$ $\qquad = 3\vec{\imath} + 2\vec{\imath} = 5\vec{\imath}$

Note here: If $\vec{x} = p\vec{\imath} \iff$ $\boxed{\begin{aligned} x &= 2^p \\ p &= \log_2 x \end{aligned}}$

Similarly Remember $\mathbb{P}_2 = \{ ax^2 + by^2 + c \mid a, b, c \in \mathbb{R} \}$

$$(x^2 + 2x - 1) + (3x^2 - 7) = 4x^2 + 2x - 8$$

Let $\vec{i} = x^2$, $\vec{j} = x$, $\vec{k} = 1$

$$(1, 2, -1) + (3, 0, -7) = (4, 2, -8)$$

Span & Linear Independence: The Road to Dimension!

First Define a Linear Combination (LC) of vectors $\vec{v}_1 \dots \vec{v}_n$

~~we~~ have the form $a_1 \vec{v}_1 + a_2 \vec{v}_2 + \dots + a_n \vec{v}_n$

& a_i are real scalars.

eg $3\vec{i} + 2\vec{j}$, $7\vec{u} + 6\vec{v} + 10\vec{w}$,

$-2\vec{p} + 7\vec{q} - 2\vec{r} + 5\vec{u} + 6\vec{v}$ etc.

<u>Define</u> a <u>Span</u> of a <u>set</u> <u>of</u> <u>vectors</u> is all possible linear combinations of those vectors

eg. $\{a\vec{i} + b\vec{j} \mid a, b \in \mathbb{R}\} = \text{Span}\{\vec{i}, \vec{j}\}$

<u>Notice</u> All <u>Spans</u> are <u>subspaces</u> of their v-spaces

(we'll also see <u>any</u> <u>finite</u> dim. subspace is the <u>Span</u> of some vectors!).

Typical Question eg Is $\begin{bmatrix}1\\1\\3\end{bmatrix} \in \text{Span}\left\{\begin{bmatrix}1\\3\\-1\end{bmatrix}, \begin{bmatrix}0\\1\\2\end{bmatrix}\right\}$

(equivalently: Is $\begin{bmatrix}1\\1\\3\end{bmatrix}$ a L.C. of $\begin{bmatrix}1\\3\\-1\end{bmatrix}, \begin{bmatrix}0\\1\\2\end{bmatrix}$?)

Solution $\begin{bmatrix}1\\1\\3\end{bmatrix}$ is a L.C. of others <u>if</u>

$$a\begin{bmatrix}1\\3\\-1\end{bmatrix} + b\begin{bmatrix}0\\1\\2\end{bmatrix} = \begin{bmatrix}1\\1\\3\end{bmatrix}$$ <u>for some a, b</u>.

$$\begin{bmatrix}1 & 0\\3 & 1\\-1 & 2\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}1\\1\\3\end{bmatrix}$$

<u>Try to solve!</u>

$$\left[\begin{array}{cc|c}1 & 0 & 1\\3 & 1 & 1\\-1 & 2 & 3\end{array}\right]$$ $R_2 - 3R_1$ & $R_3 + R_1$

$$\hookrightarrow \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 2 & 2 \end{array}\right] \quad R_3 - 2R_2$$

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 6 \end{array}\right] \leftarrow \text{no soln!}$$

not in Span!

Not a L.C.

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eg. Does $S = \{ x^2, x-1, x+1 \}$ span $\mathbb{P}_2$?

ie. does $\text{Span}(S) = \mathbb{P}_2$?

ie are all elements of $\mathbb{P}_2$ in $\text{Span}(S)$?

ie. Is any arbitrary $ax^2 + bx + c \in \text{Span}(S)$?

Solution

Is there a $k_1, k_2, k_3 \in \mathbb{R}$ such that:

for any a, b, c, such that

$$k_1 x^2 + k_2(x-1) + k_3(x+1) = ax^2 + bx + c$$

i.e. $k_1 = a$, $k_2 + k_3 = b$, $-k_2 + k_3 = c$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Does a soln. exist for any a, b, c?

A has $\det A = 1 \cdot \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 1 \cdot (1+1)$

$= 2 \neq 0$

$\Rightarrow$ A inv. exists $\Rightarrow$ always has a solution.

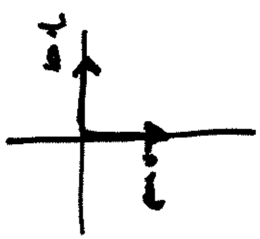
(by our <u>Mega</u> equivalence theorem!).

$\Rightarrow \{x^2, x-1, x+1\}$ $\underline{\underline{\text{Span's } \mathbb{P}_2}}$.

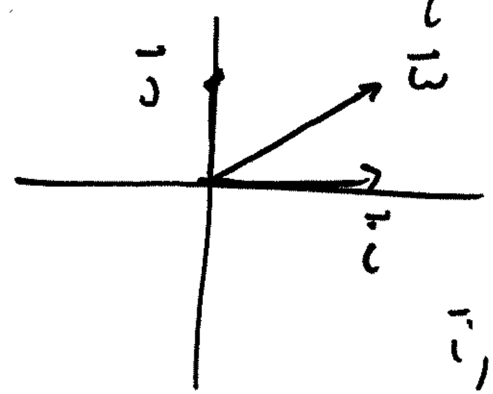
<u>Spanning is Crap</u> (at least on it's own!)

$\underline{\underline{\{\vec{i}, \vec{j}\} \text{ span } \mathbb{R}^2}}$

eg



$\vec{w} = (1,1)$



$\vec{i}, \vec{j}, \vec{w}$ $\underline{\underline{\text{span } \mathbb{R}^2}}$.

eg $(1, 3) = \vec{v}$

$\vec{v} = 1\vec{i} + 3\vec{j}$

$\vec{v} = (1, 3)$

$= 1\vec{w} + 2\vec{j}$

$$= 3\vec{w} - 2\vec{i}$$

$$= \frac{1}{2}\vec{w} + \frac{1}{2}\vec{i} + \frac{5}{2}\vec{j}$$

ok.