Discrete Mathematics with Applications I

COMPSCI&SFWRENG 2DM3

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Reachability

Recall:

- **Transitive closure** R^+ is reachability via at least one R-step
- **Reflexive transitive closure** R^* is reachability via any number of R-steps

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Let a directed graph $G = (V, E)$ with vertex/node set V and edge relation $E : V \leftrightarrow V$ be given.

Formalise:

- No edge ends at node s
- No edge starts at node s
- Node n_2 is reachable from node n_1 via a three-edge path
- Every node is reachable from node r
- From every node, each node is reachable
- Each node in the set S : **set** V is reachable from every node in S
- A node n is said to "lie on a cycle" if there is a non-empty path from n to n
- No node lies on a cycle
- Each node has at most one predecessor
- Every node is reachable from node r

Plan for Today

- **Sets of Functions with Selected Properties:** $\nrightarrow$, $\rightarrowtail$, $\twoheadrightarrow$, . . .
- **Inverses, Categories**
- **Bags (Multisets)**
- **Equivalence Classes, Partitions**
- Graph Concepts via Relations: Closures, Reachability

Recall: Properties of Heterogeneous Relations

A relation $R : B \leftrightarrow C$ is called:

univalent determinate	$R^{\smile} \mathbin{;} R \subseteq \mathrm{Id}$	$\forall\, b, c_1, c_2 \bullet b\,(R)\,c_1 \land b\,(R)\,c_2 \Rightarrow c_1 = c_2$
total	$\mathit{Dom}\ R = \llcorner B \lrcorner$ $\mathrm{Id} \subseteq R \mathbin{;} R^{\smile}$	$\forall\, b : B \bullet (\exists\, c : C \bullet b\,(R)\,c)$
injective	$R \mathbin{;} R^{\smile} \subseteq \mathrm{Id}$	$\forall\, b_1, b_2, c \bullet b_1\,(R)\,c \land b_2\,(R)\,c \Rightarrow b_1 = b_2$
surjective	$\mathit{Ran}\ R = \llcorner C \lrcorner$ $\mathrm{Id} \subseteq R^{\smile} \mathbin{;} R$	$\forall\, c : C \bullet (\exists\, b : B \bullet b\,(R)\,c)$
a **mapping**	iff it is univalent and total	
bijective	iff it is injective and surjective	

Univalent relations are also called **(partial) functions**.

Mappings are also called **total functions**.

Function Sets — Z Definition and Description [Spivey 1992]

In Z, $X \leftrightarrow Y = \mathbb{P}(X \times Y)$, and $x \mapsto y = (x, y)$ is an abbreviation for pairs.

$\pfun$	–	Partial functions
$\fun$	–	Total functions
$\pinj$	–	Partial injections
$\inj$	–	Total injections
$\psurj$	–	Partial surjections
$\surj$	–	Total surjections
$\bij$	–	Bijections

$$X \pfun Y == \{ f : X \leftrightarrow Y \mid (\forall x : X;\ y_1, y_2 : Y \bullet (x \mapsto y_1) \in f \land (x \mapsto y_2) \in f \Rightarrow y_1 = y_2) \}$$

$$X \fun Y == \{ f : X \pfun Y \mid \operatorname{dom} f = X \}$$

$$X \pinj Y == \{ f : X \pfun Y \mid (\forall x_1, x_2 : \operatorname{dom} f \bullet f(x_1) = f(x_2) \Rightarrow x_1 = x_2) \}$$

$$X \inj Y == (X \pinj Y) \cap (X \fun Y)$$

$$X \psurj Y == \{ f : X \pfun Y \mid \operatorname{ran} f = Y \}$$

$$X \surj Y == (X \psurj Y) \cap (X \fun Y)$$

$$X \bij Y == (X \surj Y) \cap (X \inj Y)$$

If X and Y are sets, $X \pfun Y$ is the set of partial functions from X to Y. These are relations which relate each member x of X to at most one member of Y. This member of Y, if it exists, is written $f(x)$. The set $X \to Y$ is the set of total functions from X to Y. These are partial functions whose domain is the whole of X; they relate each member of X to exactly one member of Y.

Function Sets — Z Definition and Laws [Spivey 1992]

In Z, $X \leftrightarrow Y = \mathbb{P}(X \times Y)$, and $x \mapsto y = (x, y)$ is an abbreviation for pairs, and $S \circ R = R \mathbin{;} S$.

$$X \pfun Y == \{ f : X \leftrightarrow Y \mid (\forall x : X;\ y_1, y_2 : Y \bullet (x \mapsto y_1) \in f \land (x \mapsto y_2) \in f \Rightarrow y_1 = y_2) \}$$

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$$X \surj Y == (X \psurj Y) \cap (X \fun Y)$$

$$X \bij Y == (X \surj Y) \cap (X \inj Y)$$

Laws:

$$f \in X \pfun Y \Leftrightarrow f \circ f^{\sim} = \mathrm{id}(\operatorname{ran} f)$$

$$f \in X \pinj Y \Leftrightarrow f \in X \pfun Y \land f^{\sim} \in Y \pfun X$$
$$f \in X \inj Y \Leftrightarrow f \in X \fun Y \land f^{\sim} \in Y \pfun X$$

$$f \in X \pinj Y \Rightarrow f(\!| S \cap T |\!) = f(\!| S |\!) \cap f(\!| T |\!)$$

$$f \in X \bij Y \Leftrightarrow f \in X \fun Y \land f^{\sim} \in Y \fun X$$
$$f \in X \psurj Y \Rightarrow f \circ f^{\sim} = \mathrm{id}\ Y$$

Function Sets

For two sets $A : \mathbf{set}\ t_1$ and $B : \mathbf{set}\ t_2$, we adopt the following **function set** definitions from Z:

Z	CalcCheck			
$f \in A \to B$	$f \in A \mathrel{\ooalign{$\leftrightarrow$}} B$	`\tfun`	total function	$Dom\, f = A \ \wedge\ f^{\smile} \mathbin{;} f \subseteq \mathbb{I}\, B$
$f \in A \nrightarrow B$		`\pfun`	partial function	$Dom\, f \subseteq A \ \wedge\ f^{\smile} \mathbin{;} f \subseteq \mathbb{I}\, B$
$f \in A \rightarrowtail B$		`\tinj`	total injection	$f \mathbin{;} f^{\smile} = \mathbb{I}\, A \ \wedge\ f^{\smile} \mathbin{;} f \subseteq \mathbb{I}\, B$
$f \in A \mathrel{\rightarrowtail\!\!\!\!\!+} B$		`\pinj`	partial injection	$f \mathbin{;} f^{\smile} \subseteq \mathbb{I}\, A \ \wedge\ f^{\smile} \mathbin{;} f \subseteq \mathbb{I}\, B$
$f \in A \twoheadrightarrow B$		`\tsurj`	total surjection	$Dom\, f = A \ \wedge\ f^{\smile} \mathbin{;} f = \mathbb{I}\, B$
$f \in A \mathrel{\twoheadrightarrow\!\!\!\!\!+} B$		`\psurj`	partial surjection	$Dom\, f \subseteq A \ \wedge\ f^{\smile} \mathbin{;} f = \mathbb{I}\, B$
$f \in A \rightarrowtail\!\!\!\!\twoheadrightarrow B$		`\tbij`	total bijection	$f \mathbin{;} f^{\smile} = \mathbb{I}\, A \ \wedge\ f^{\smile} \mathbin{;} f = \mathbb{I}\, B$
	$f \in A \mathrel{\rightarrowtail\!\!\!\!\twoheadrightarrow\!\!\!\!\!+} B$	`\pbij`	partial bijection	$f \mathbin{;} f^{\smile} \subseteq \mathbb{I}\, A \ \wedge\ f^{\smile} \mathbin{;} f = \mathbb{I}\, B$

Properties of Heterogeneous Relations — Notes

univalent	$R^{\smile} \mathbin{;} R$	$\subseteq$	Id	$\forall\, b, c_1, c_2 \bullet b \,(R)\, c_1 \wedge b \,(R)\, c_2 \Rightarrow c_1 = c_2$
surjective	Id	$\subseteq$	$R^{\smile} \mathbin{;} R$	$\forall\, c : C \bullet (\exists\, b : B \bullet b \,(R)\, c)$
total	Id	$\subseteq$	$R \mathbin{;} R^{\smile}$	$\forall\, b : B \bullet (\exists\, c : C \bullet b \,(R)\, c)$
injective	$R \mathbin{;} R^{\smile}$	$\subseteq$	Id	$\forall\, b_1, b_2, c \bullet b_1 \,(R)\, c \wedge b_2 \,(R)\, c \Rightarrow b_1 = b_2$

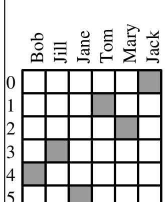
All these properties are defined for arbitrary relations! (Not only for functions!)

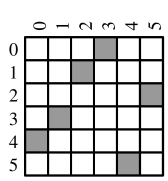
- R is univalent and surjective
  iff $R^{\smile} \mathbin{;} R = Id$
  iff $R^{\smile}$ is a left-inverse of R
- R is total and injective
  iff $R \mathbin{;} R^{\smile} = Id$
  iff $R^{\smile}$ is a right-inverse of R

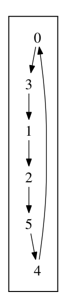
Inverses of Total Functions

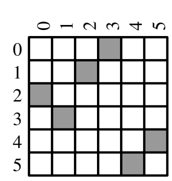
(14.43) **Definition:** Let $f : B \leftrightarrow C$ be a **mapping**.
An **inverse of** f is a mapping $g : C \leftrightarrow B$ such that $f \mathbin{;} g = \mathbb{I}\,\llcorner B \lrcorner$ and $g \mathbin{;} f = \mathbb{I}\,\llcorner C \lrcorner$.

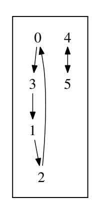
- f has an inverse iff f is a bijective mapping.
- The inverse of a bijective mapping f is its converse $f^{\smile}$.
- A homogeneous bijective mapping is also called a **permutation**.











Inverses of Total Functions (ctd.)

(14.43) **Definition:** Let $f : B \leftrightarrow C$ be a **mapping**.
An **inverse of** f is a mapping $g : C \leftrightarrow B$ such that $f \fatsemi g = \mathbb{I}\llcorner B \lrcorner$ and $g \fatsemi f = \mathbb{I}\llcorner C \lrcorner$.

Theorem: If g is an inverse of $f : B \to C$, then $g = f^{\smile}$.
Proving $f^{\smile} \subseteq g \subseteq f^{\smile}$, for using antisymmetry of $\subseteq$ **:**

$$
\begin{array}{ll}
 & f^{\smile} \\
= & \langle \text{ Identity of } \fatsemi \ \rangle \\
 & f^{\smile} \fatsemi \mathrm{Id} \\
= & \langle\ g \text{ is an inverse of } f \ \rangle \\
 & f^{\smile} \fatsemi f \fatsemi g \\
\subseteq & \langle \text{ “Mon. } \fatsemi \text{” w. } f \text{ is univalent, that is, } f^{\smile} \fatsemi f \subseteq \mathrm{Id} \ \rangle \\
 & \mathrm{Id} \fatsemi g \\
= & \langle \text{ Identity of } \fatsemi \ \rangle \\
 & g \\
\subseteq & \langle \text{ Identity of } \fatsemi; \quad \text{“Mon. } \fatsemi \text{” w. } f \text{ is total, that is, } \mathrm{Id} \subseteq f \fatsemi f^{\smile} \ \rangle \\
 & g \fatsemi f \fatsemi f^{\smile} \\
= & \langle\ g \text{ is an inverse of } f\text{; Identity of } \fatsemi \ \rangle \\
 & f^{\smile}
\end{array}
$$

General Inverses

A general setting that has an associative composition $\fatsemi$ with identities Id is called a **category**.

In **any** category: **Definition of Inverse:** g **is an inverse of** $f : B \to C$ **iff** $f \fatsemi g = \mathrm{Id}$ and $g \fatsemi f = \mathrm{Id}$.

In the category of sets and **total functions**, we have (see previous slide):

Theorem: Characterisation of inverse total functions:
$f : B \longrightarrow C$ has an inverse iff f is bijective.

In the category of sets and **arbitrary relations**, we have:

Theorem: Characterisation of inverse relations:
$R : B \leftrightarrow C$ has an inverse iff R is a bijective mapping.

Proof: Exercise!

In the category of **monotone mappings between partial orders**,
a bijective monotone mapping **may not** have an inverse,
e.g. $f : \{a, b\} \to \{\sigma, \tau\}$ with orders as indicated:

$$
\begin{tikzcd}
b \arrow[r, "f"] & \tau \\
a \arrow[r, "f"] & \sigma \arrow[u, "\leq"']
\end{tikzcd}
$$

(The inverse of f as just a mapping is not monotone.)

“Multisets” or “Bags”

A **bag** (or **multiset**) is “like a set, but each element can occur any (finite) number of times”.

Bag comprehension and enumeration: Written as for sets, but with delimiters $\Lbag$ and $\Rbag$.

Sets versus bags example:

$$
\begin{array}{lclclcl}
\{\, x : \mathbb{Z} \mid -2 \leq x \leq 2 \bullet x \cdot x \,\} & = & \{4, 1, 0\} & = & \{0, 1, 4\} & = & \{0, 0, 0, 1, 1, 4\} \\
\Lbag\, x : \mathbb{Z} \mid -2 \leq x \leq 2 \bullet x \cdot x \,\Rbag & = & \Lbag 4, 1, 0, 1, 4 \Rbag & = & \Lbag 0, 1, 1, 4, 4 \Rbag & \neq & \Lbag 0, 1, 4 \Rbag
\end{array}
$$

The operator $\_\#\_ : t \to \mathit{Bag}\ t \to \mathbb{N}$ counts the number of occurrences of an element in a bag:

$$1 \# \Lbag 0, 0, 0, 1, 1, 4 \Rbag = 2$$

Bag extensionality and **bag inclusion** are defined via all occurrence counts:

$$B = C \quad \equiv \quad (\forall\, x \bullet x \# B = x \# C) \qquad\qquad B \subseteq C \quad \equiv \quad (\forall\, x \bullet x \# B \leq x \# C)$$

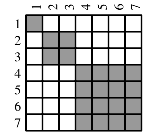
Bag union, intersection, difference:

$$
\begin{array}{lcl}
x \# (B \cup C) & = & (x \# B) + (x \# C) \\
x \# (B \cap C) & = & (x \# B) \downarrow (x \# C) \\
x \# (B - C) & = & (x \# B) \mathbin{\dot{-}} (x \# C)
\end{array}
$$

In Z, a bag **is** the function producing the counts of its elements: $\mathit{Bag}\ A = A \nrightarrow (\mathbb{N} - \{0\})$
— This view is useful for **implementation via “dictionary” datatypes**.

Equivalence Relations, Equivalence Classes, Partitions

Recall: A (homogeneous) relation $R : B \leftrightarrow B$ is called:



reflexive	Id	$\subseteq$	R	$(\forall\, b : B \bullet b \mathrel{(} R \mathrel{)} b)$
symmetric	$R^{\smile}$	$=$	R	$(\forall\, b, c : B \bullet b \mathrel{(} R \mathrel{)} c \equiv c \mathrel{(} R \mathrel{)} b)$
transitive	$R \mathbin{⨾} R$	$\subseteq$	R	$(\forall b, c, d \bullet b \mathrel{(} R \mathrel{)} c \mathrel{(} R \mathrel{)} d \Rightarrow b \mathrel{(} R \mathrel{)} d)$
idempotent	$R \mathbin{⨾} R$	$=$	R	
equivalence	$\mathrm{Id} \subseteq R = R \mathbin{⨾} R$	$=$	$R^{\smile}$	reflexive, transitive, symmetric

Definition (14.34): Let Ξ be an equivalence relation on B. Then $[b]_{\Xi}$. the **equivalence class of** b, is the subset of elements of B that are equivalent (under Ξ) to b:

$$x \in [b]_{\Xi} \quad \equiv \quad x \mathrel{(} \Xi \mathrel{)} b \qquad\qquad \text{Equivalently:} \qquad [b]_{\Xi} \;=\; \Xi\,(\!|\, \{b\} \,|\!)$$

Theorem: For an equivalence relation Ξ on B, the set $\{\, b : B \bullet \Xi\,(\!|\, \{b\} \,|\!) \,\}$ of equivalence classes of Ξ is a partition of $\llcorner B \lrcorner$.

Definition (11.76): If $T : \mathbf{set}\ t$ and $S : \mathbf{set}\ (\mathbf{set}\ t)$, then:

S is a **partition of** T $\equiv$ $(\forall u, v \mid u \in S \wedge v \in S \wedge u \neq v \bullet u \cap v = \{\}) \wedge (\bigcup u \mid u \in S \bullet u) = T$

The partition view can be useful for **implementing** equivalence relations.