1ZA3

Average Value

Say I have a continuous $f(x)$ on $[a,b]$

then Ave. $f(x)$ on $[a,b]$ $= \dfrac{\int_a^b f(x)\,dx}{b-a}$

eg. Find ave. value of $f(x) = x^2 - 2x$ on $[-1, 2]$

Solution

$$\text{Ave} = \frac{\int_a^b f(x)\,dx}{b-a} = \frac{1}{2-(-1)} \int_{-1}^{2} x^2 - 2x\,dx$$

$1/3$

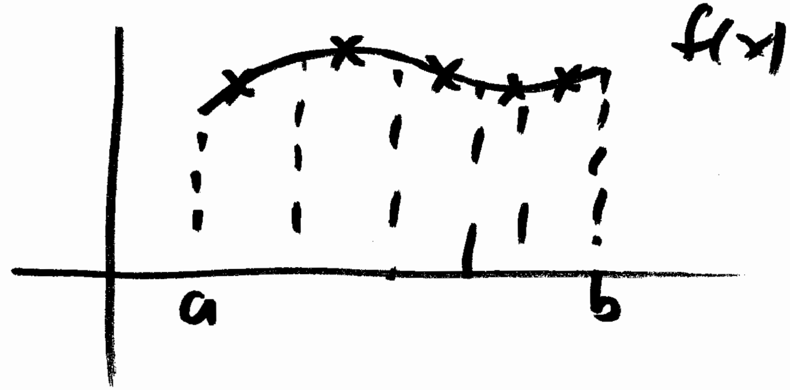
$$= \frac{1}{3}\left(\frac{1}{3}x^3 - x^2 \Big|_{-1}^{2} \right)$$

$$= \frac{1}{9}\left(\cancelto{9}{2^3+1}\right) - \frac{1}{3}\left(\cancelto{3}{4-1}\right)$$

$$= 1 - 1 = \underline{\underline{0}}$$

Why does it work?

Formally:



$$\text{Ave of } f(x) \simeq \text{Ave. sample} = \sum_{i=1}^{n} f(x_i) / n$$

$$= \sum_{i=1}^{n} f(x) \left(\frac{1}{n} \cdot \frac{b-a}{b-a} \right), \quad \frac{1}{n} \cdot (b-a) = \Delta x$$

$$= \frac{\sum_{i=1}^{n} f(x_i)\,\Delta x}{b-a}$$

so exact Ave. $= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x \Big/ (b-a)$

$$= \left(\int_a^b f(x)\,dx\right)\frac{1}{b-a}$$

Informally

$A = \int_a^b f(x)\,dx \longleftrightarrow A = (\text{Ave})(b-a)$

$$\Rightarrow \int_a^b f(x)\,dx = \text{Ave}\,(b-a)$$

$$\Rightarrow \text{Ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

e.g. $\int x e^x \, dx$ ← Nice looking product
need a "product rule"

$\Rightarrow$ let's try to reverse product rule!

Say $u = f(x)$, $v = g(x)$

$$(uv)' = u'v + v'u.$$

Integrate!

$$uv + c = \int v u' dx + \int u v' dx$$

$$\int u v' dx = uv - \int v u' dx$$

or

$$\int u \frac{dv}{\cancel{dx}} \cancel{dx} = uv - \int v \frac{du}{\cancel{dx}} \cancel{dx}$$

$$\int u\, dv = uv - \int v\, du$$

Same

Integration by Parts

d/dx

$$\int u v' dx = uv - \int v u' dx$$

$\int dx$

This is an Integral Identity

eg. $\int x \underbrace{e^x dx}$

$= \int u \, dv$

let $u = x \Rightarrow \frac{du}{dx} = 1$ $\quad du = 1dx$

$v' = e^x \Rightarrow v = \int e^x dx$

$= e^x \cancel{+ C}$

drop it!

(it'll go away)

$= uv - \int v \, du$

$= xe^x - \int e^x dx$

$= xe^x - e^x + c = (x-1)e^x + c$

What if I made the "wrong" choice?

$$\int x e^x \, dx \qquad u = e^x , \quad u' = e^x, \quad du = e^x \, dx$$

$$= \int u \, dv \qquad v = \int dv = \int x \, dx = \frac{1}{2} x^2$$

$$= uv - \int v \, du$$

$$= e^x \cdot \frac{1}{2} x^2 - \frac{1}{2} \int x^2 e^x \, dx$$

"Not Even Wrong"

General Guideline for "u" choice in I. by P.

best u $\ln x$, $\tan^{-1} x$

"mid" u x^2, x^3, etc.

worst u e^x, $\cos x$, $\sin x$, $\cosh(x)$ etc.

eg. $\int_1^e x^2 \ln x \, dx$

$u = \ln x \Rightarrow du = \frac{1}{x} dx$

$v' = x^2 \Rightarrow v = \frac{1}{3} x^3$

$= \int u \, dv$

$= uv - \int v \, du = \left((\ln x) \frac{1}{3} x^3 - \int \frac{1}{3} x^3 \cdot \frac{1}{x} dx \right) \Big|_1^e$

$= \frac{1}{3} x^3 \ln x \Big|_1^e - \frac{1}{3} \int_1^e x^2 dx$

don't forget

$= \frac{1}{3} x^3 \ln x \Big|_1^e - \frac{1}{9} x^3 \Big|_1^e$

$= \frac{1}{3}(e^3 - 0) - \frac{1}{9}(e^3 - 1)$

$= \frac{1}{9}(2e^3 + 1)$