Announcements

Topics:

In the Probability and Statistics module:

- **Sections 1 + 2:** Introduction to Stochastic Models
- **Section 3:** Basics of Probability Theory
- **Section 4:** Conditional Probability; Law of Total Probability
- **Section 5:** Independence

To Do:

- Work on Assignments and Suggested Practice Problems assigned on the webpage under the SCHEDULE + HOMEWORK link

Deterministic Models

So far, we have been studying idealized, *deterministic* models where the outcome is always *certain*.

Stochastic Models

Now, more realistically, we study models which account for *random* or *chance* factors. These factors are unpredictable, unknown events which have an effect on the process we are studying.

Stochastic Model

Definition:

A *stochastic model* is a mathematical model that describes processes (such as biological processes) that are driven by chance (random) events.

Stochastic Population Models

Example: Lion Population with Immigration

Suppose that a population p_t of lions at time t is described by the stochastic dynamical system

$$p_{t+1} = p_t + I_t$$

where the term I_t represents *possible* immigration of lions and where time t=0,1,2,... is measured in years.

Stochastic Population Models

Example: Lion Population with Immigration

Suppose that initially there are 160 lions and that the immigration term is defined as

$$I_t = \begin{cases} 12 & \text{with 50\% chance} \\ 0 & \text{with 50\% chance} \end{cases}$$

Analyze *possible* dynamics of this population over the next three years.

Stochastic Population Models

Example: Lion Population with Immigration

Trials/simulations:

t	I_t	p_t
0		160
1		
2		
3		

t	I_t	p_t
0		160
1		
2		
3		

Random Experiment and Sample Space

Definition:
A *random experiment* is an experiment that is repeatable but has an uncertain outcome.

The set of all possible outcomes of a random experiment is called the *sample space* of that experiment.

Stochastic Population Models

Example: Lion Population with Immigration

Sample Space:
At the end of 3 years, what are the possible sizes of the lion population? Do all occur with equal likelihood?

Expectations:
What is your prediction for the population of lions in the next 3 years?

Sample Spaces and Events

Definition: Simple Event and Event

A single outcome of a random experiment is called a *simple event*. An *event* is a collection (or set) of simple events.

Statistic

Definition:

A *statistic* is a set of numerical values that can summarize the outcomes of a random experiment.

Stochastic Models

Example #6.
A population of leopards p_t is modelled by

$$p_{t+1} = p_t + I_t \quad \text{where} \quad I_t = \begin{cases} 10 & \text{with a 90\% chance} \\ -100 & \text{with a 10\% chance} \end{cases}$$

What is more likely to happen to the number of leopards over time – a net increase or decrease? Or will the population remain at about the same size? Explain.

Elements of Set Theory

An **empty set** (denoted by $\varnothing$) is a set that contains no elements.

A set A is a **subset** of a set S, denoted by $A \subseteq S$, if A is empty or contains some or all of the elements of S.

For any set A, $\varnothing \subseteq A$, and $A \subseteq A$.

Elements of Set Theory

Definition: Intersection of Sets

The intersection $A \cap B$ of sets A and B is the set of elements that belong to both A *and* B.

Note that $A \cap B = B \cap A$.

Definition: Mutually Exclusive Events

Two events A and B are said to be mutually exclusive if they are *disjoint*, i.e., if $A \cap B = \emptyset$.

Elements of Set Theory

Definition: Union of Sets

The union $A \cup B$ of sets A and B is the set of elements that belong to either A *<u>or</u>* B.

Note that $A \cup B = B \cup A$.

Definition: Compliment of a Set

The compliment A^C of a subset $A \subseteq S$ is the set of all elements in S that are not in A.

Note that

$$S^C = \varnothing, \quad \varnothing^C = S, \quad \left(A^C\right)^C = A, \quad A \cup A^C = S, \quad A \cap A^C = \varnothing.$$

Elements of Set Theory

Exercise:
Consider the random experiment of rolling a fair, 6-sided die.

(a) Write out the sample space, S.

(b) Define an event and a simple event.

(c) Define two mutually exclusive events.

Elements of Set Theory

Exercise:

Consider the random experiment of rolling a fair, 6-sided die. Let A be the event of rolling an even number and let B be the event of rolling a number greater than 2.

Determine the following:

(a) $A \cap B$ (b) $A \cup B$ (c) A^C

Probability

Definition: Probability

Let S denote a sample space. A probability is a function P that assigns, to each event A in S, a unique real number P(A), called the probability of A.

The function P satisfies the following properties:

(i) $0 \le P(A) \le 1$ for any event $A \subseteq S$.

(ii) $P(\varnothing) = 0$ and $P(S) = 1$.

(iii) If A and B are mutually exclusive events, then $P(A \cup B) = P(A) + P(B)$.

Probability

Probability as Area:

Let the area of the sample space S be 1 and the area of the empty set to be 0. Any other event (subset of S) has area between 0 and 1, i.e.,

$$\text{for } A \subseteq S, \quad P(A) = \text{area of } A.$$

Probability

Exercise:
Consider the random experiment of rolling a fair, 6-sided die. Let A be the event of rolling an even number and let B be the event of rolling a number greater than 2.

Determine the following:

(a) $P(A)$ (b) $P(B)$ (c) $P(A \cap B)$

Probability

Theorem: Probability of a Complimentary Event

If A is an event, then $P(A^C) = 1 - P(A)$.

Theorem: Probability of the Union of Two Events

If A and B are two events, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Probability

Exercise:
Consider the random experiment of rolling a fair, 6-sided die. Let A be the event of rolling an even number and let B be the event of rolling a number greater than 2.

Determine the following:

(a) $P(B^C)$ (b) $P(A \cup B)$

Assigning Probability to Events

Assume that the sample space of an experiment is finite, i.e., that it contains n distinct, simple events $E_1, E_2, \cdots, E_n$.

Then, $E_1 \cup E_2 \cup \cdots \cup E_n = S$

and $P(S) = P(E_1) + P(E_2) + \cdots + P(E_n) = 1$.

Assigning Probability to Events

Theorem: Assigning Probabilities to Equally Likely Simple Events

Assume that S is a finite sample space in which all outcomes (simple sets) are *equally likely*. The probability of an event $A \subseteq S$ is

$$P(A) = \frac{|A|}{|S|}$$

Assigning Probability to Events

Recall: Lion Population with Immigration

Consider a population of lions described by the stochastic dynamical system

$$p_{t+1} = p_t + I_t \quad \text{where} \quad I_t = \begin{cases} 12 & \text{with 50\% chance} \\ 0 & \text{with 50\% chance} \end{cases}$$

and time t=0,1,2,... is measured in years. If initially there are 160 lions, determine the possible population sizes 3 years later and the probability of each population size occurring.

Conditional Probability

So far, we have defined *unconditional probability*, i.e., the probability that an event has occurred, disregarding any factors that might affect it.

Conditional probability is the probability of an event occurring when we already know that another event has taken place.

Conditional Probability

Example #12.
A coin is tossed three times.

(a) Find the probability that at least two heads occurred.

(b) Find the probability that at least two heads occurred, given that at least one toss resulted in heads.

Conditional Probability

Definition:

The probability of an event A *conditional* on an event C is given by

$$P(A \mid C) = \frac{P(A \cap C)}{P(C)}$$

provided that $P(C) \neq 0.$

We read this as "the probability of A, given C".

Conditional Probability

Notes:

(1) Conditional probability does not "commute"

(2) $P(A \cap C)$ can be calculated in two different ways

(3) $P(A \mid C)$ in two extreme cases

Application

Example 4.4: Incidence of Heart Attacks in Canada

Based on a survey that examined the 1950-1999 data on cardiovascular diseases in Canada, 1.53% of adult Canadians who suffered a heart attack were male and 0.54% of adult Canadians who suffered a heart attack were female.

Application

Example 4.4: Incidence of Heart Attacks in Canada

Assuming an equal sex ratio, answer the following:

(a) What is the probability that an adult male has suffered a heart attack?

(b) What is the probability that a randomly chosen adult Canadian has suffered a heart attack?

(c) What is the probability that a person who has suffered a heart attack is male?

Partitions

Definition:

Let S be a sample space and $E_1, E_2, ..., E_n \ (n \geq 1)$ be events in S. If

(i) $E_1, E_2, ..., E_n$ are *mutually exclusive*, i.e.,

$$E_i \cap E_j = \varnothing \quad \text{for all } \ i \neq j, \ 1 \leq i, j \leq n$$

(ii) $E_1, E_2, ..., E_n$ are *collectively exhaustive*, i.e.,

$$S = E_1 \cup E_2 \cup \cdots \cup E_n$$

then we say that $E_1, E_2, ..., E_n$ form a **partition** of the sample space S.

The Law of Total Probability

Assume that events $E_1, E_2, ..., E_n$ $(n \geq 1)$ form a partition of a sample space S. For any event A in S,

$$P(A) = P(A \mid E_1)P(E_1) + P(A \mid E_2)P(E_2) + \cdots + P(A \mid E_n)P(E_n)$$

The Law of Total Probability

Tree Diagrams

Start with the sample space S and branch it out into the events that form the partition and assign probabilities to each branch.

Create further branches and assign (this time, conditional) probabilities to them.

To find the probability of some event "A" occurring, multiply the probabilities along each path leading to A and add up these products.

Application

Example #26.
A certain medical condition (could be high blood pressure) comes in three forms, X, Y, and Z, with prevalences of 45%, 35%, and 20%, respectively. The probability that a person will need emergency medical attention is 10% if he has the X form, 5% if he has the Y form, and 45% if he has the Z form. What is the probability that a person who has the condition will require emergency medical attention?

Bayes' Theorem

Assume that events $E_1, E_2, \ldots, E_n \ (n \geq 1)$ form a partition of a sample space S. Let A be an event. Then

$$P(E_i \mid A) = \frac{P(A \mid E_i)P(E_i)}{P(A \mid E_1)P(E_1) + P(A \mid E_2)P(E_2) + \cdots + P(A \mid E_n)P(E_n)}$$

Application

Example #30.
The average incidence of autism spectrum disorder is 45 cases per 10, 000. A test for the disorder shows a positive result in 96% of people who have the disorder, and in 1% of people who do not have it (*false positive*).

(a) What is the probability that a randomly selected person tests positive for the disorder?

(b) If a person tests positive for the disorder, what is the probability that they have it?

Independent Events

If our knowledge about an event does not tell us anything about the probability of another event occurring, then the two events are independent.

Definition:

Two events A and B are *independent* if $P(A \mid B) = P(A)$.

Equivalently, A and B are *independent* if

$$P(A \cap B) = P(A)P(B).$$

Application

Example #20.
The average efficacy of an oral contraceptive (birth control pill) is about 97.5% per year. This means that, within a year, 2.5% of sexually active women who are taking the pill will get pregnant. What is the probability that a sexually active woman who takes birth control pills will get pregnant **at least** once in a 5-year period?