MATHEMATICS 1LT3 TEST 2

Evening Class
Duration of Test: 60 minutes
McMaster University

E. Clements

26 February 2015

40

FIRST NAME (please print): SolNs

FAMILY NAME (please print): ____________

Student No.: ____________

THIS TEST HAS 8 PAGES AND 11 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE.

Total number of points is 40. Marks are indicated next to the problem number in square brackets. You may use the McMaster standard calculator, Casio fx991 MS+, on this test.

USE PEN TO WRITE YOUR TEST. IF YOU USE A PENCIL, YOUR TEST WILL NOT BE ACCEPTED FOR REMARKING (IF NEEDED).

You need to show work to receive full credit, except for Multiple Choice.

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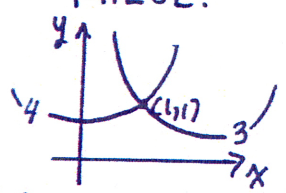
1. State whether each statement is true or false. Explain your reasoning.

(a) [2] In a contour map for a function of two variables, $z = f(x, y)$, it is possible for level curves of two different values to intersect each other.

FALSE.



Suppose that two level curves of different values did intersect at a point (a,b) in the domain of f. Then $f(a,b)=k_1$ and $f(a,b)=k_2$ where $k_1 \neq k_2$ and so f is not a function.

$f(1,1)=3$ and $f(1,1)=4 \Rightarrow f$ is not a function

(b) [2] If $f(x, y)$ approaches 2 as (x, y) approaches $(0, 0)$ along all lines $y = mx$, where m is a real number, then $\lim_{(x,y)\to(0,0)} f(x, y) = 2$.

FALSE. In order for $\lim_{(x,y)\to(0,0)} f(x,y) = 2$, f must approach 2 as $(x,y)\to(0,0)$ along ALL paths in the domain of f.

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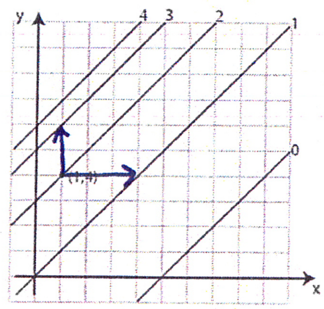
2. Multiple Choice. Clearly circle the one correct answer.

(a) [3] Consider the function $f(x, y)$ whose contour map is given below. Which of the following is/are positive?

(I) $f(1, 4) = 2 > 0$ (II) $f_x(1, 4) < 0$ (III) $f_y(1, 4) > 0$



(A) none (B) I only (C) II only (D) III only

(E) I and II (F) I and III (G) II and III (H) all three

(b) [3] Which of the following functions is/are differentiable at $(0, 0)$?

(I) $f(x, y) = \sqrt{x^2 + y^2}$ X (II) $g(x, y) = 1 - x^2 - y^2$ ✓ (III) $h(x, y) = \sqrt{1 - x^2 - y^2}$ ✓

(A) none (B) I only (C) II only (D) III only

(E) I and II (F) I and III (G) II and III (H) all three

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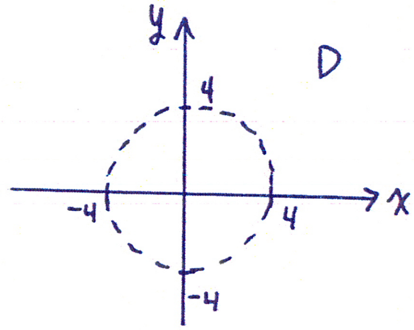
3. Find and sketch the domains of the following functions.

(a) [2] $f(x, y) = \dfrac{x}{16 - x^2 - y^2}$

$16 - x^2 - y^2 \neq 0$

$x^2 + y^2 \neq 16$

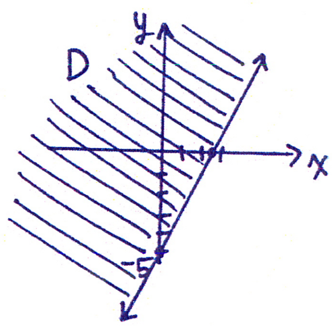
all points in $\mathbb{R}^2$ except those along the circle with centre $(0,0)$ and radius 4



(b) [2] $g(x, y) = \sqrt{5 - 2x + y}$

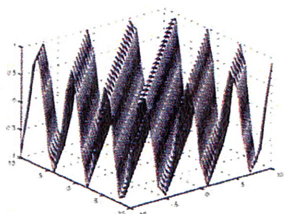
$5 - 2x + y \geq 0$

$y \geq 2x - 5$

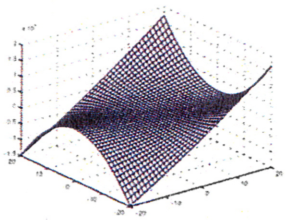


4. [3] Match the equation of each function with its graph below. Write the letter corresponding to the graph of the function next to the equation in the space provided.

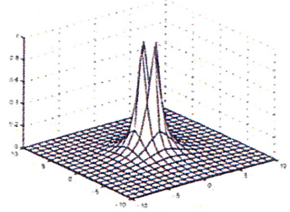
$f(x, y) = x^4 - 2xy^2$ B $g(x, y) = \dfrac{1}{x^4 + y^2}$ C $h(x, y) = \sin(x - y)$ A



(A)



(B)



(C)

5. Consider the function $f(x, y) = 2\arctan(xy)$.

(a) [1] Determine the range of f. [You do not need to prove your result formally as we did in class.]

$$-\frac{\pi}{2} < \arctan(xy) < \frac{\pi}{2} \Rightarrow -\pi < 2\arctan(xy) < \pi$$

$$\therefore z \in (-\pi, \pi)$$

(b) [3] Create a contour map for f. Include level curves corresponding to $k = -2$, $k = 0$, and $k = 2$.

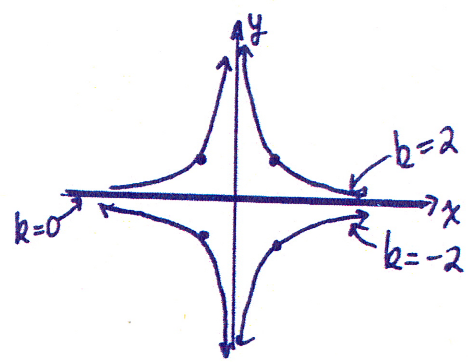
Let $2\arctan(xy) = k$ where $k \in (-\pi, \pi)$.

Then $y = \dfrac{\tan(\frac{k}{2})}{x}$.

$k = -2 \Rightarrow y = \dfrac{\tan(-1)}{x}$

$k = 0 \Rightarrow y = \dfrac{\tan(0)}{x} = 0$

$k = 2 \Rightarrow y = \dfrac{\tan(1)}{x}$



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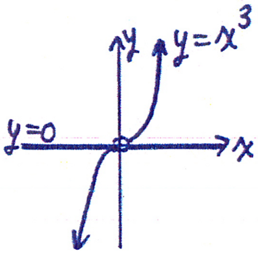
6. [3] Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^6+y^2}$ does not exist. Sketch the domain of the function and the paths involved.



$f(x,0) = \frac{0}{x^6} = 0$ ... $f \to 0$ as $(x,y) \to (0,0)$ along $y=0$

$f(x,x^3) = \frac{x^3 \cdot x^3}{x^6 + (x^3)^2} = \frac{x^6}{2x^6} = \frac{1}{2}$ ... $f \to \frac{1}{2}$ as $(x,y) \to (0,0)$ along $y = x^3$.

$\therefore \lim\limits_{(x,y)\to(0,0)} f(x,y)$ D.N.E.

7. [2] Use the definition of continuity to show that

$$f(x,y) = \begin{cases} \dfrac{\cos(xy)}{x^2+y^2+1} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

is not continuous at $(0,0)$.

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{\cos(xy)}{x^2+y^2+1} = \frac{\cos 0}{1} = 1$$

$(x,y) \neq (0,0)$

$f(0,0) = 0$

Since $\lim\limits_{(x,y)\to(0,0)} f(x,y) \neq f(0,0)$, f is not continuous at $(0,0)$.

8. The table below shows the values of the wind chill index $W(T, v)$, with temperatures measured in degrees Celsius and wind speed in kilometres per hour.

	$T = -20$	$T = -15$	$T = -10$	$T = -5$
$v = 40$	-34.1	-27.4	-20.8	-14.1
$v = 30$	-32.6	-26.0	-19.5	-13.0
$v = 20$	-30.5	-24.2	-17.9	-11.6

$\Delta T = \pm 5$

(a) [2] Using the table, estimate $W_T(-15, 30)$. fixed

$$W_T(-15,30) \approx \frac{W(-10,30)-W(-15,30)}{-10-(-15)} \approx \frac{-19.5-(-26.0)}{5} \approx 1.3$$

$$W_T(-15,30) \approx \frac{W(-20,30)-W(-15,30)}{-20-(-15)} \approx \frac{-32.6-(-26.0)}{-5} \approx 1.32$$

$$\text{average} = \frac{1.3+1.32}{2} = 1.31$$

$$\therefore W_T(-15,30) \approx 1.31$$

(b) [2] Using the formula for the wind chill

$$W(T, v) = 13.12 + 0.6215T - 11.37v^{0.16} + 0.3965Tv^{0.16}$$

calculate the value $W_T(-15, 30)$ and ~~compare with your estimate in part (a)~~. interpret your result.

$$W_T = 0.6215 + 0.3965v^{0.16}$$

$$W_T(-15,30) = 0.6215 + 0.3965(30)^{0.16} \approx 1.30$$

When the temperature is $-15°C$ and the wind is 30 km/h, the wind chill index is increasing at a rate of 1.3 wind chill index units per °C.

9. Let $f(x, y) = \sqrt{xy}$.

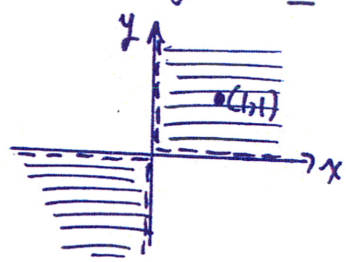
(a) [2] Compute f_x and f_y. Shade the region on which f_x and f_y are continuous.

$$f_x = \frac{1}{2\sqrt{xy}} \cdot y \qquad f_y = \frac{1}{2\sqrt{xy}} \cdot x$$

both functions have the same domain and are continuous on this domain (because they are algebraic functions)

Domain of f_x and f_y:

$xy > 0 \Rightarrow x > 0$ and $y > 0$ OR $x < 0$ and $y < 0$



(b) [2] Explain why f is differentiable at $(1, 1)$.

By the theorem used in class, since f_x and f_y are continuous on $B_1(1,1)$, f is differentiable at $(1,1)$.

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10. [3] Compute the directional derivative of the function $f(x,y) = x^3y + 2xy^2 - y^3$ at the point $(1,3)$ in the direction specified by $\mathbf{v} = 2\mathbf{i} + \mathbf{j}$.

$\|\vec{v}\| = \sqrt{2^2+1^2} = \sqrt{5} \Rightarrow \hat{u} = \underbrace{\frac{2}{\sqrt{5}}}_{u_1}\hat{i} + \underbrace{\frac{1}{\sqrt{5}}}_{u_2}\hat{j}$

$f_x = 3x^2y + 2y^2 \; \ldots \; f_x(1,3) = 3(1)^2(3) + 2(3)^2 = 27$

$f_y = x^3 + 4xy - 3y^2 \ldots \; f_y(1,3) = 1^3 + 4(1)(3) - 3(3)^2 = -14$

$$D_u f(1,3) = f_x(1,3)u_1 + f_y(1,3)u_2$$
$$= 27\left(\frac{2}{\sqrt{5}}\right) - 14\left(\frac{1}{\sqrt{5}}\right)$$
$$= \frac{40}{\sqrt{5}}$$

11. [3] Find the maximum rate of change of $g(x,y) = \arctan\left(\frac{3y}{x}\right)$ at the point $(1,1)$ and the direction in which it occurs.

$g_x = \frac{1}{1+\left(\frac{3y}{x}\right)^2}\left(\frac{-3y}{x^2}\right) = \frac{-3y}{x^2+9y^2} \; \cdots \; g_x(1,1) = -\frac{3}{10}$

$g_y = \frac{1}{1+\left(\frac{3y}{x}\right)^2}\left(\frac{3}{x}\right)\left(\frac{x}{x}\right) = \frac{3x}{x^2+9y^2} \; \cdots \; g_y(1,1) = \frac{3}{10}$

$\nabla g(1,1) = -\frac{3}{10}\hat{i} + \frac{3}{10}\hat{j}$ $\qquad \|\nabla g(1,1)\| = \sqrt{\left(-\frac{3}{10}\right)^2 + \left(\frac{3}{10}\right)^2} = \sqrt{\frac{18}{100}} = \frac{3\sqrt{2}}{10}$

$\therefore$ The maximum rate of change at $(1,1)$ is $\frac{3\sqrt{2}}{10}$ and occurs in the direction $\vec{v} = -\hat{i} + \hat{j}$ (same direction as $\nabla g(1,1)$).

THE END