Last Day   Vector Spaces

If V is a set, $\vec{u}, \vec{v}, \vec{w} \in V$ & $k, \ell \in \mathbb{R}$

then V is a real vector space if:

1) $\vec{u} + \vec{v} \in V$

2) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$

3) $\vec{u} + \vec{v} + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
$= (\vec{u} + \vec{v}) + \vec{w}$

4) There exists a unique $\vec{0}$
$\vec{u} + \vec{0} = \vec{u}$

5) Each $\vec{u}$ has a "$-\vec{u}$"
$\vec{u} + (-\vec{u}) = \vec{0}$

6) $k\vec{u} \in V$

7) $k(\vec{u} + \vec{v}) = k\vec{u} + k\vec{v}$

8) $(k + \ell)\vec{u} = k\vec{u} + \ell\vec{u}$

9) $(k\ell)(\vec{u}) = k(\ell(\vec{u}))$

10) $1\vec{u} = \vec{u}$

eg. Say $V = \mathbb{R}^3$ new addn. $\vec{u} + \vec{v} = \vec{u} \times \vec{v}$

& the standard scalar multiplication.

Is this a vector space? ie a V space

Axiom # 2

$\vec{u} + \vec{v} = \vec{u} \times \vec{v}$

$$\left.\begin{aligned} \vec{v} + \vec{u} &= \vec{v} \times \vec{u} \\ &= -\vec{u} \times \vec{v} \\ &\neq \vec{u} \times \vec{v} \end{aligned}\right\} \text{FAIL}$$

$LS \neq RS$

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eg. Let $V = \mathbb{R}^2$ with the usual addition but

$k(u_1, u_2) = (ku_1, 0)$

Under the new rules, is this still a vspace?

Solution

All axiom 1-5 work automatically (usual addn! no change!)

need to test 6-10.

eg 8) $(k+l)\vec{u} = k\vec{u} + l\vec{u}$

Check

$LS = (k+l)\vec{u}$

$= (k+l)(u_1, u_2)$

new rule!

$= ((k+l)u_1, 0)$

$RS = k\vec{u} + l\vec{u}$

$= k(u_1, u_2) + l(u_1, u_2)$

$= (ku_1, 0) + (lu_1, 0)$

$= (ku_1 + lu_1, 0)$

$= ((k+l)u_1, 0)$

$LS = RS$

$\Rightarrow$ 8 holds!

But (U) Failed $1(u_1, u_2) = (u_1, 0) \neq (u_1, u_2)$

$\Rightarrow$ not a vspace

As an exercise try to check 10 axioms for:

$V = \mathbb{R}$, $\vec{u} + \vec{v} = u + v + 1$

$\vec{u} = u$

$\vec{v} = v$

$k\vec{u} = ku - k + 1$

} new operations

Vector Subspace

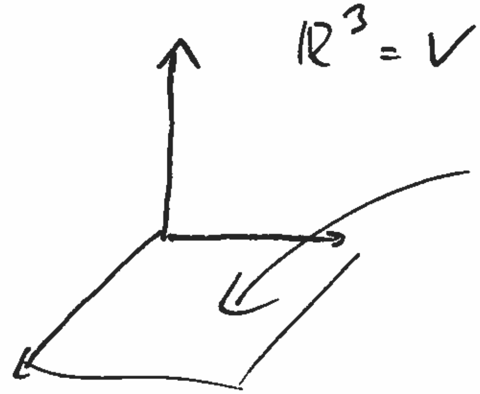
If V is a vector space (ie a set with defined addn. & mult. that follow 10 axioms)

A subset of V, ie $S \subseteq V$ is a subspace of V

Subset symbol

if S is a vspace using addn. & mult. rules of V.

eg



x-y plane is a copy of $\mathbb{R}^2$ inside $\mathbb{R}^3$

Subspace!

$S \subseteq V$ (a vector space) is a subspace if

1) $\vec{u}+\vec{v} \in S$ if $u, v \in S$ ← closed under addn.

2) $k\vec{u} \in S$ if $k \in \mathbb{R}$, $\vec{u} \in S$ ← closed under scalar multn.

(3) S is non empty)

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eg. Is $S = \{ (a, 0, 0) \mid a \in \mathbb{R} \}$ a subspace of $\mathbb{R}^3$? (under the usual rules)

Solution Is it closed under addn?

Let $\left.\begin{array}{l} \vec{u} = (u_1, 0, 0) \\ \vec{v} = (v_1, 0, 0) \end{array}\right\} \in S$

$\vec{u}+\vec{v}$

$= (\underbrace{u_1+v_1}, 0, 0) \in S$ ✓

$(\#, 0, 0)$

So closed under addition!

Is it closed under scalar multn?

$$\vec{u} \in S \Rightarrow \vec{u} = (u_1, 0, 0)$$

$$k\vec{u} = (\underbrace{ku_1}_{= \#}, \underbrace{0}_{=0}, \underbrace{0}_{=0}) \in S \quad \checkmark$$

closed under multn!

Closed under both addn & multn

& not empty eg $(0,0,0) \in S$

$\Rightarrow$ S is a subspace

Compare if $S = \{ (a, 2, 0), \ a \in \mathbb{R} \} \subseteq \mathbb{R}^3$

not always 2

multn $k\vec{u} =$ $k(u_1, 2, 0) = (ku_1, 2k, 0) \notin S$

Fails both closures

Addn $\vec{u}+\vec{v} =$ $(u_1, 2, 0) + (v_1, 2, 0) = (u_1+v_1, 4, 0)$

$\Rightarrow$ not a subspace

Is set of all $\{(a, b) \mid ab = 0\}$ a subspace of standard $\mathbb{R}^2$?

"S

Check Addn $\vec{u} = (u_1, u_2) \in S \Rightarrow u_1 u_2 = 0$

$\vec{v} = (v_1, v_2) \in S \Rightarrow v_1 v_2 = 0$

$\vec{u} + \vec{v} = (u_1 + v_1, u_2 + v_2)$ Is it always in S?

ie does "$ab = 0$"?

$$(u_1 + v_1)(u_2 + v_2)$$

$$= \underbrace{\cancel{u_1 u_2}}_{0} + v_1 u_2 + v_2 u_1 + \underbrace{\cancel{v_1 v_2}}_{0}$$

$$= v_1 u_2 + u_1 v_2 \leftarrow \text{\underline{not} \underline{always} } \underline{\underline{0}}$$

eg $\vec{u} = (1,0) \in S$
$\vec{v} = (0,1) \in S$

$\vec{u} + \vec{v} = (1,1) \notin S$

$1 \cdot 1 = 1 \neq 0$