Math 1LS3 Week 7: Derivative Applications

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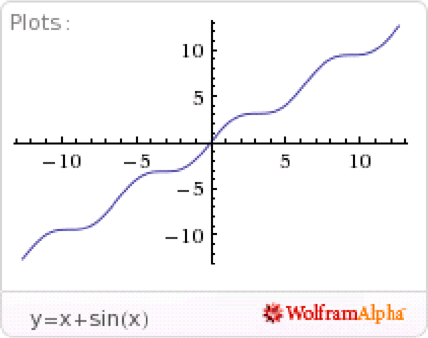
This week, we will cover 4.6,4.7, and the beginning of 5.1. Not many sections, but the problems will be multi-step. At this point, it is essential that you can quickly compute derivatives.

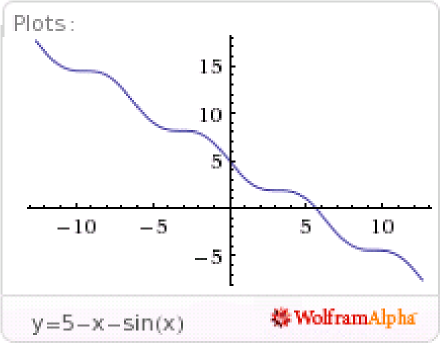
1. Concavity and Curve-Sketching
2. Approximating Functions by Taylor Polynomials
3. Extreme Values (Optimization)

Increasing vs. Decreasing

Increasing Function



Decreasing Function



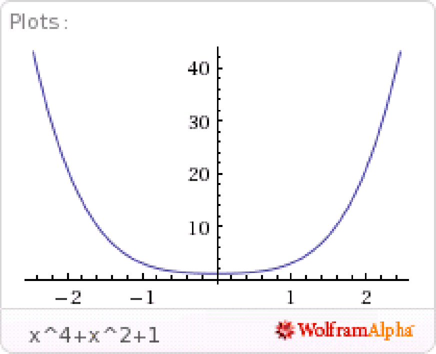
Positive Slope ($y' \geq 0$)

Draw some tangent lines.

Negative Slope ($y' \leq 0$)

Concave Up vs. Concave Down

Concave Up

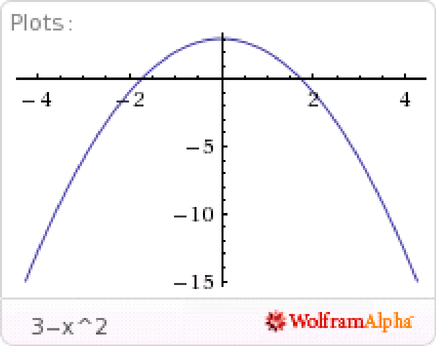


Slope increases

y' increases

$y'' \geq 0$

Concave Down



Slope decreases

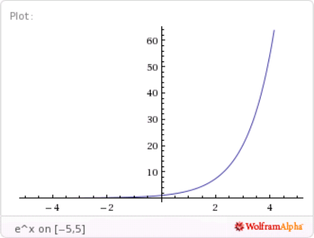
y' decreases

$y'' \leq 0$

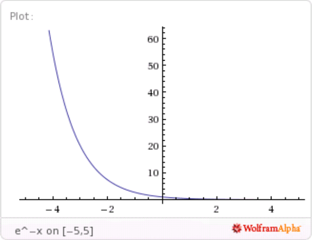
The derivative of the derivative (y'') is called the *second derivative*.

Concave Up vs. Concave Down

Exponential growth is concave (up or down)? up!



Exponential decay is concave (up or down)? up!



Steps for Curve Sketching

To graph $y = f(x)$:

1. Compute $f'(x)$ and $f''(x)$.
2. Draw three numberlines: one for $f(x)$, one for $f'(x)$, one for $f''(x)$.
3. Indicate on the first numberline where $f(x)$ is zero, where not defined.
4. For each region on first numberline, determine if $f(x)$ is $+$ or $-$.
5. Do the same for the other two numberlines.
6. Sketch graph: f tells above/below x-axis, f' tells increasing/decreasing; f'' tells concave up/down
7. Show where f changes sign (cross x-axis), where f' changes sign (peak or valley), where f'' changes sign (*inflection point*)
8. Don't forget asymptotes, jumps, corners, vertical tangents, cusps, holes

Example

Sketch $g(x) = x^2 e^{-x}$, the *gamma distribution with* $n = 2$.

The gamma distribution with $n = 2$: $g(x) = x^2e^{-x}$

$$g(x) = x^2 e^{-x}$$

Find g' and g'':

$$g'(x) = 2xe^{-x} - x^2e^{-x} = (2x - x^2)e^{-x}$$

$$g''(x) = (x^2 - 4x + 2)e^{-x}$$

Where are g', g'' zero or undefined?

$$g(x) = 0 \Rightarrow x^2 = 0$$

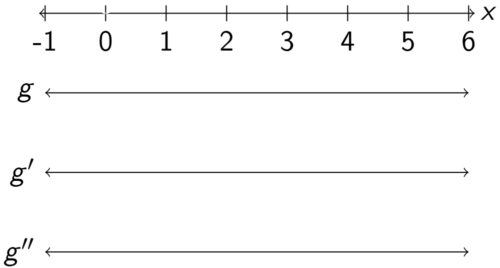
$$\Rightarrow x = 0$$

$$g'(x) = 0 \Rightarrow 2x - x^2 = 0$$

$$\Rightarrow x = 0 \text{ or } x = 2$$

$$g''(x) = 0 \Rightarrow x^2 - 4x + 2 = 0$$

$$\Rightarrow x = 0.59 \text{ or } x = 3.4$$



$$g(x) = x^2 e^{-x}$$
$$g'(x) = (2x - x^2)e^{-x}$$
$$g''(x) = (x^2 - 4x + 2)e^{-x}$$

Where are g, g', g'' positive?

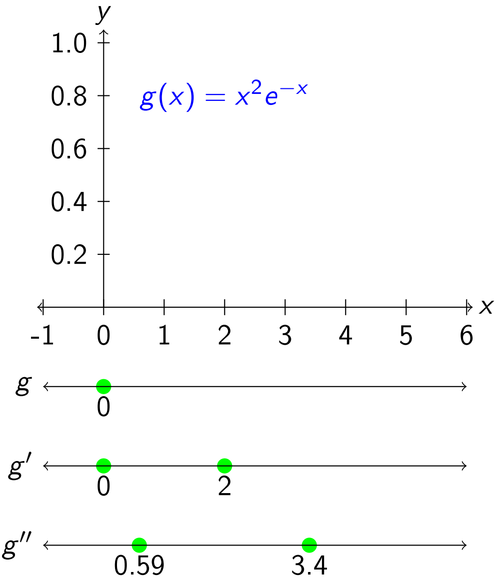
x	$g(x)$
-1	$+$
1	$+$

x	$g'(x)$
-1	$-$
1	$+$
3	$-$

x	$g''(x)$
0	$+$
1	$-$
4	$+$

As $x \to \infty, g(x) \to 0.$

As $x \to -\infty, g(x) \to \infty.$

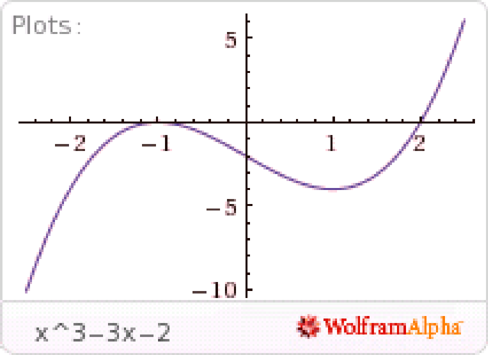


Example: Graph $y = x^3 - 3x - 2$

If we have time, let's do this example. (Or do it at home, or in tutorial.)

Problem

Graph $y = x^3 - 3x - 2$.



Linear Approximations

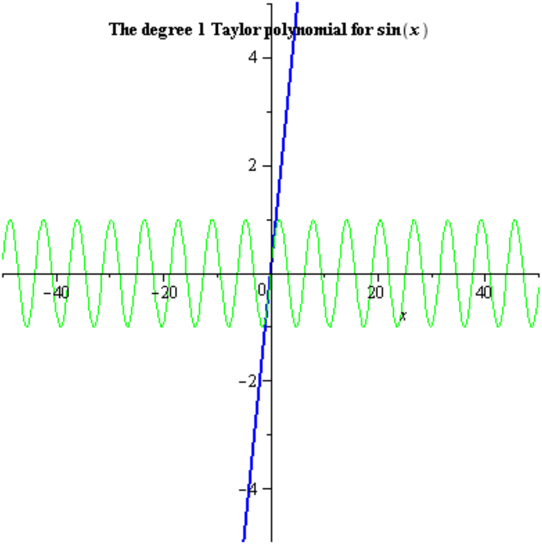
- We've seen how to find secant and tangent lines to $y = f(x)$.
- Tangent lines give local approximations for the graph.
- Only good near point of tangency.
- Can we find even better approximations?

Crucial observation: *near 0,*

$$x \gg x^2 \gg x^3 \gg x^4 \gg \cdots$$

$$\boxed{\begin{array}{c}\text{Linear Approximation}\\ \text{(Tangent Line)}\end{array}} \xrightarrow[\text{(fine–tuning)}]{\text{add more terms}} \boxed{\begin{array}{c}\text{Taylor Polynomial}\\ \text{(Better Approx.)}\end{array}}$$

Taylor Polynomials for $\sin(x)$



Approximations for $\sin(x)$ near $x = 0$.

As n grows,

- Accurate on wider interval
- Greater precision

Quadratic Approximations (Degree 2 Taylor Polys)

If $y = L(x)$ is the line tangent to $y = f(x)$ at $x = a$ then:

- $L(a) = f(a)$
- $L'(a) = f'(a)$

How should we define "best" quadratic approximation $y = Q(x)$ to $y = f(x)$ at $x = a$?

- $Q(a) = f(a)$
- $Q'(a) = f'(a)$
- $Q''(a) = f''(a)$

Note: tangent lines have a **point of tangency**; Taylor polys have a **centre**.

Quadratic Approximation to $y = e^x$ at $x = 0$

Problem

Find the parabola $Q(x)$ that best approximates $y = f(x) = e^x$ near $x = 0$.

Solution

Write $Q(x) = a + bx + cx^2$. *We want to find* a, b, c.

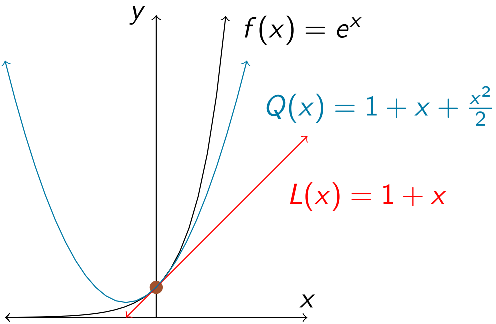
$$Q(0) = f(0) \quad \Longrightarrow \quad a = f(0) = 1.$$

$$Q'(0) = f'(0) \quad \Longrightarrow \quad b = f'(0) = 1.$$

$$Q''(0) = f''(0) \quad \Longrightarrow \quad c = f''(0)/2 = 1/2.$$

So $Q(x) = 1 + x + \frac{1}{2}x^2$.

Quadratic Approximation to $y = e^x$ at $x = 0$



- Curve $f(x) = e^x$.
- Linear approximation (tangent line) $L(x) = 1 + x$.
- Quadratic approx (deg 2 Taylor) $Q(x) = 1 + x + \frac{x^2}{2}$.
- These approximations good near **centre**: $x = 0$.

Quadratic Approximation: General Formula

Theorem

The best quadratic approximation $Q(x)$ to a curve $f(x)$ at $x = a$ is:

$$Q(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

[Distance Dropped]=[initial speed]$*$[time]+4.9$*$[time]^2 is a special case.

Theorem (Taylor's Theorem)

The best degree n approximation polynomial $T_n(x)$ to a curve $f(x)$ at $x = a$ is the nth Taylor Polynomial:

$$T_n(x) = f(a)+f'(a)(x-a)+\frac{f''(a)}{2}(x-a)^2+\frac{f'''(a)}{6}(x-a)^3+\cdots+\frac{f^{(n)}}{n!}(x-a)^n$$

Note: $n!$ is n factorial. $f^{(n)}$ is the n^{th} derivative of f.

Taylor polynomial for $\sin(x)$

Problem

Find the degree 4 Taylor polynomial T_4 for $f(x) = \sin(x)$.

Solution

$$
\begin{aligned}
& & &\text{so } f(0) = \sin(0) = 0 \\
f'(x) &= \cos(x) & &\text{so } f'(0) = 1 \\
f''(x) &= -\sin(x) & &\text{so } f''(0) = 0 \\
f'''(x) &= -\cos(x) & &\text{so } f'''(0) = -1 \\
f''''(x) &= \sin(x) & &\text{so } f''''(0) = 0
\end{aligned}
$$

Thus $f(x) \approx f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 + \frac{f''''(0)}{24}x^4 = x - \frac{1}{6}x^3.$

In fact, $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \cdots.$

Taylor Polynomial for $1/x$

Does $f(x) = 1/x$ have a Taylor Polynomial at $x = 0$?

Problem

Find the degree 3 Taylor Polynomial for $f(x) = 1/x$ about $x = 1$.

Solution

$f(x) = x^{-1}$ so $f(1) = 1$.
$f'(x) = -x^{-2}$ so $f'(1) = -1$.
$f''(x) = 2x^{-3}$ so $f''(1) = 2$.
$f'''(x) = -6x^{-4}$ so $f'''(1) = -6$.
The degree 3 Taylor polynomial is:

$$T_3(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 + \frac{f'''(1)}{6}(x-1)^3$$
$$= 1 - (x-1) + (x-1)^2 - (x-1)^3.$$

Estimating Using Taylor Series

Problem

Estimate $1/1.01$ *to 5 decimal places without using long division.*

Solution

Use Taylor series near $x = 1$. *(Why?)*

We saw $T_3(x) = 1 - (x-1) + (x-1)^2 - (x-1)^3$. *So*

$T_3(1.01) = 1 - (.01) + (.01)^2 - (.01)^3 = 1 - .01 + .0001 - .000001 = .990099$

The real answer is just slightly more (you'll add .00000001 next).

$$1/1.01 \approx .99010$$

Calculators, mental math make use of Taylor series.

Common Taylor Series

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \cdots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \cdots$$

Awesome consequences:

$$e^{ix} = \cos(x) + i\sin(x)$$

$$e^{i\pi} = -1$$

More awesomeness

$$\frac{1}{1-r} = 1 + r + r^2 + r^3 + r^4 + r^5 + \cdots$$

Writing $r = -x^2$:

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \cdots$$

The left side is the derivative of $\arctan(x)$. What is the right side the derivative of?

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} + \cdots$$

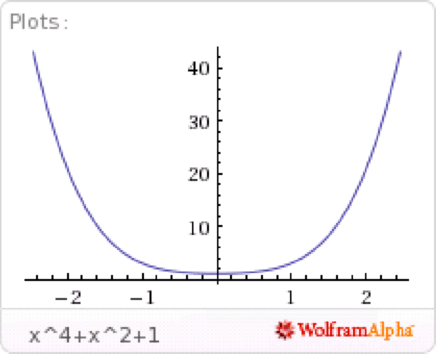
Now plug in $x = 1$:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots$$

Maximizing, Minimizing

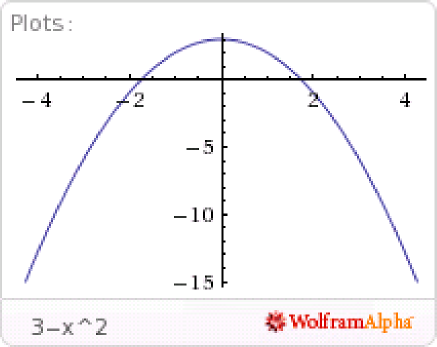
Natural question: for which number(s) x is $f(x)$ maximized/minimized?
Smooth functions have horizontal tangent at (internal) max/min.

Minimum



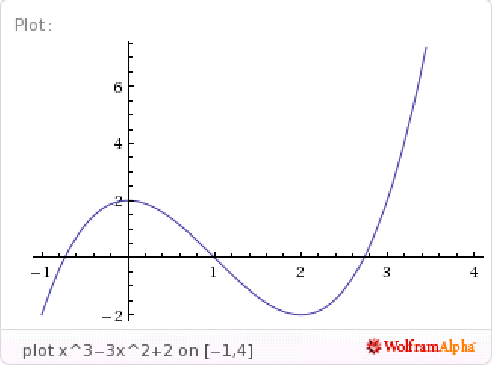
y decreases then increases
$y' = 0$ at minimum

Maximum



y increases then decreases
$y' = 0$ at maximum

Global vs Local Extrema



$f(x) = x^3 - 3x^2 + 2$ with domain [-1,4].

- f has an absolute (global) max at $x = a$ if $f(a) \geq f(b)$ for all b *in* f*'s domain*. [Note: we study global extremes next week.]
- f has a relative (local) max at $x = a$ if $f(a) \geq f(b)$ for all b *near* a.

Caution: by fiat, endpoints are not allowed to be relative max/min.

Critical Points

Definition

$x = c$ is a critical point for $f(x)$ if either of the following:

- $f'(c) = 0$
- $f'(c)$ does not exist

Theorem

Local maxima/minima can only occur at critical points.

Caution: critical points need not be local maxima or minima.

Finding Critical Points: Example

Problem

Find the critical points for $f(x) = x^3 - 3x^2 + 2$.

Solution

$$f'(x) = 3x^2 - 6x$$

- *Where does* $f'(x)$ *fail to exist?* *f' exists everywhere.*
- *Where does* $f'(x) = 0$*?* *At $x = 0$ and at $x = 2$.*

The critical points are: $x = 0, x = 2$.

Testing Critical Points

Critical points need not be local maxima/minima. Test them:

Theorem (First Derivative Test)

- *If f increases up to $x = c$ and decreases after, then $x = c$ is a local max.*
- *If f decreases up to $x = c$ and increases after, then $x = c$ is a local min.*
- *If f continues to increase or decrease at $x = c$, then $x = c$ is not a local extremum.*

Theorem (Second Derivative Test)

If f'' is continuous near c and $f'(c) = 0$ then:

- *If $f''(c) < 0$ (so f is concave down), then $x = c$ is a local max.*
- *If $f''(c) > 0$ (so f is concave up), then $x = c$ is a local min.*
- *If $f''(c) = 0$ test fails (try first derivative test).*

First Derivative Test: Deja Vu?

Problem

Use the first derivative test to test the critical points $x = 0$ and $x = 2$ for $f(x) = x^3 - 3x^2 + 2$.

Solution

$$f'(x) = 3x^2 - 6x$$

Where is f' positive, negative, zero? Test points!

$f' > 0$ for $x < 0$. $f' < 0$ for $x \in (0, 2)$. $f' > 0$ for $x > 2$.

So $x = 0$ is a relative maximum and $x = 2$ is a relative minimum.

Note: the first derivative test is essentially the same process you do in graphing.

Second Derivative Test: an example

Problem

Use the second derivative test to test the critical points $x = 0$ *and* $x = 2$ *for* $f(x) = x^3 - 3x^2 + 2$*.*

Solution

$$f'(x) = 3x^2 - 6x$$

$$f''(x) = 6x - 6$$

$f''(0) = -6 < 0 \quad \Rightarrow \quad$ *At* $x = 0$*,* f *has a relative maximum.*
$f''(2) = 6(2) - 6 = 6 > 0 \quad \Rightarrow \quad$ *At* $x = 2$*,* f *has a relative minimum.*

Example

Problem

Find the relative extrema of $y = \sqrt[3]{x}(x-2)^2$.

Solution

Step 1: Find critical points. Step 2: Test them.

$$y' = \frac{1}{3}x^{-2/3}(x-2)^2 + x^{1/3} * 2(x-2)$$

$$y'' = \frac{4(7x^2 - 4x - 2)}{9x^{5/3}}$$

- *Where is the derivative undefined? At* $x = 0$
- *Where is* $y' = 0$*? At* $x = \frac{2}{7}$ *and* $x = 2$

Use derivative tests: local min at $x = 2$*, local max at* $x = \frac{2}{7}$.
($x = 0$ *is neither.)*