Complex numbers info sheet

This sheet is a revision aid and is not a list of testable topics

Definition 14.1: The imaginary unit

Denote by i the imaginary unit, defined $i = \sqrt{-1}$.

$$\frac{1}{i} = \frac{i}{ii} = \frac{i}{-1} = -i$$

Definition 14.2: Complex number

A complex number z is a number of the form

$$z = a + ib$$

for a, b real numbers.
The real part of z is

$$\text{Re}(z) = a.$$

The imaginary part of z is

$$\text{Im}(z) = b.$$

Example 14.3

Let $z = -2 + i$, $w = 2 + 3i$. Then

$$\begin{aligned} z + w &= -2 + 2 + i(1 + 3) \\ &= 4i \\ z - w &= -4 - 2i \end{aligned}$$

Let $z = 12 - 4i$, then $\frac{1}{4}z = 3 - i$.

Let $z = -2 + i$, $w = 2 + 3i$. Then

$$\begin{aligned} zw &= (-2 + i)(2 + 3i) \\ &= -4 - 6i + 2i - 6i^2 \\ &= 6 - 4 + i(2 - 6) \\ &= 2 - 4i \end{aligned}$$

Let $z = 2 - i$, then

$$\begin{aligned} z^3 &= (2 - i)(2 - i)(2 - i) \\ &= (2 - i)(4 - 4i + i^2) \\ &= (2 - i)(3 - 4i) \\ &= 6 - 8i - 3i + 4i^2 \\ &= 2 - 11i \end{aligned}$$

Definition 14.4: Complex conjugate

Let $z = a + ib$ be a complex number. The complex conjugate of z is denote $\overline{z}$, and is defined

$$\overline{z} = a - ib$$

Example 14.5

If $z = 18 - 7i$ then $\overline{z} = 18 + 7i$.

Fact 14.6

Let z be a complex number and $\overline{z}$ its conjugate. Then

$$z\overline{z} = \overline{z}z$$

is a real number.

Definition 14.7: Modulus

Let $z = a + ib$ be a complex number. The modulus of z is denoted $|z|$ and is defined

$$|z| = \sqrt{a^2 + b^2}$$

Fact 14.8

Let z be a complex number. Then

$$|z|^2 = z\overline{z}$$

Definition 14.9: Reciprocal of a complex number

Let $z = a + ib$ be a non-zero complex number. We define

$$z^{-1} = \frac{1}{z} = \frac{1}{|z|^2}\overline{z}$$

Example 14.10

Question: Express the complex number

$$\frac{7 - 3i}{-2 + 5i}$$

in the form $a + ib$.
Answer: Let $z = 7 - 3i$ and $w = -2 + 5i$.
Then

$$|w|^2 = (-2)^2 + 5^2 = 29$$

and

$$\overline{w} = -2 - 5i$$

Then

$$\begin{aligned} \frac{7 - 3i}{-2 + 5i} &= \frac{z}{w} \\ &= \frac{1}{|w|^2}z\overline{w} \\ &= \frac{1}{29}(7 - 3i)(-2 - 5i) \\ &= \frac{1}{29}(-14 - 35i + 6i + 15i^2) \\ &= \frac{1}{29}(-29 - 29i) \\ &= -1 - i \end{aligned}$$

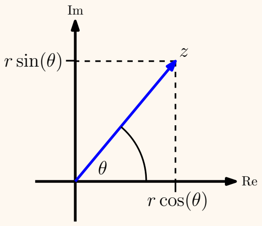
Fact 14.11: Properties of the complex conjugate

Let z and w be complex numbers. Then

- $\overline{z+w} = \overline{z} + \overline{w}$
- $\overline{z-w} = \overline{z} - \overline{w}$
- $\overline{zw} = (\overline{z})(\overline{w})$
- $\overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}}$
- $\overline{\overline{z}} = z$

Definition 15.12: Polar form

Let $z = a + ib$ be a complex number with $|z| = r$, then



where θ is the angle z makes to the positive real axis. Then

$$z = r\left(\cos(\theta) + i\sin(\theta)\right)$$

is the polar form of the complex number z.
The angle θ is the argument of z (always in radians).
The principle argument of z is denoted $Arg(z)$ and is the argument θ such that $-\pi < \theta \leqq \pi$.

Example 15.13

Let $z = 2\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right)$.
Then

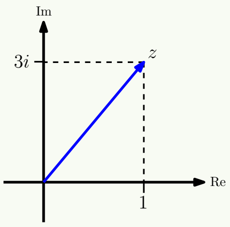
$$\theta = \frac{\pi}{2} + 2\pi = \frac{5\pi}{2}$$
$$\theta = \frac{\pi}{2} - 6\pi = -\frac{11\pi}{2}$$

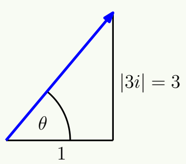
are both possible arguments of z, but $Arg(z) = \frac{\pi}{2}$, as $-\pi < \frac{\pi}{2} \leqq \pi$.

Question: Express the complex number

$$z = 1 + 3i$$

in polar form.
Answer: Sketch the vector form of z:





and form a triangle with side lengths equal to $Re(z)$ and $Im(z)$ (given on the right). Using the trigonometric identity

$$\tan = \frac{opposite}{adjacent}$$

we see that

$$\tan(\theta) = \frac{3}{1}$$

and

$$\theta = \arctan(3) \approx 1.25\ldots$$

Next find the modulus of z:

$$|z| = \sqrt{1^2 + 3^2} = \sqrt{10}$$

Therefore the polar form of z is

$$z = \sqrt{10}(\cos(1.25\ldots) + i\sin(1.25\ldots))$$

Fact 15.14: Operations in polar form

Let $z = r_1\left(\cos(\theta_1) + i\sin(\theta_1)\right)$, and $w = r_2\left(\cos(\theta_2) + i\sin(\theta_2)\right)$ be complex numbers. Then

1. $zw = r_1 r_2\left(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right)$
2. $\frac{z}{w} = \frac{r_1}{r_2}\left(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right)$
3. $|zw| = |z||w|$
4. $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$
5. $Arg\left(\frac{z}{w}\right) = Arg(z) - Arg(w)$
6. $Arg(zw) = Arg(z) + Arg(w)$

Fact 15.15: De Moivre's Formula

Let z be a complex number with $z = r\left(\cos(\theta) + i\sin(\theta)\right)$. For any positive integer n, we have

$$z^n = r^n\left(\cos(n\theta) + i\sin(n\theta)\right)$$

Fact 15.16: Complex roots

Let $z = r\left(\cos(\theta) + i\sin(\theta)\right)$ be a complex number, and n a positive integer.
There are exactly n n-th roots of z, and they are given by

$$z^{\frac{1}{n}} = r^{\frac{1}{n}}\left(\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right)$$

for $k = 0, 1, 2, \ldots, n-1$.

Fact 15.17

Let $z = r(\cos(\theta) + i\sin(\theta))$ be a complex number. Then

$$z = re^{i\theta}$$