Discrete Mathematics with Applications I

COMPSCI&SFWRENG 2DM3

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All Propositional Axioms of the Equational Logic E — Fill in the Blanks!

(3.1) Axiom, **Associativity of** $\equiv$:

(3.2) Axiom, **Symmetry of** $\equiv$:

(3.3) Axiom, **Identity of** $\equiv$:

(3.8) Axiom, **Definition of** *false*:

(3.9) Axiom, **Commutativity of** $\neg$ **with** $\equiv$:

(3.10) Axiom, **Definition of** $\not\equiv$:

(3.24) Axiom, **Symmetry of** $\vee$:

(3.25) Axiom, **Associativity of** $\vee$:

(3.26) Axiom, **Idempotency of** $\vee$:

(3.27) Axiom, **Distributivity of** $\vee$ **over** $\equiv$:

(3.28) Axiom, **Excluded middle**:

(3.35) Axiom, **Golden rule**:

(3.57) Axiom, **Definition of Implication**:

(3.58) Axiom, **Definition of** $\Leftarrow$**, Consequence**:

Read Textbook Chapter 3!

Read LADM Chapter 3 (pp. 41–68):

- 3.1: Propositional Calculus Preliminaries
- 3.2: Equivalence and true
- 3.3: Negation, Inequivalence, and false
- 3.4: Disjunction
- 3.5: Conjunction
- 3.6: Implication

Plan for Today

- **Textbook Chapter 3: Propositional Calculus**
  - Revisiting conjunction
  - Implication
  - Knights and Knaves

Theorems Relating $\wedge$ and $\vee$

(3.43) **Absorption**: $p \wedge (p \vee q) \equiv p$
$p \vee (p \wedge q) \equiv p$

(3.44) **Absorption**: $p \wedge (\neg p \vee q) \equiv p \wedge q$
$p \vee (\neg p \wedge q) \equiv p \vee q$

(3.45) **Distributivity of $\vee$ over $\wedge$**: $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

(3.46) **Distributivity of $\wedge$ over $\vee$**: $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

(3.47) **De Morgan**: $\neg(p \wedge q) \equiv \neg p \vee \neg q$
$\neg(p \vee q) \equiv \neg p \wedge \neg q$

De Morgan's Laws

Prove:

(3.47) **De Morgan**: $\neg(p \wedge q) \equiv \neg p \vee \neg q$
$\neg(p \vee q) \equiv \neg p \wedge \neg q$

Use, in particular:

(3.32) $t \vee u \equiv t \vee \neg u \equiv t$

(3.35) **Axiom, Golden rule**: $t \wedge u \equiv t \equiv u \equiv t \vee u$

Theorems Relating $\wedge$ and $\equiv$

(3.48) **(3.48)** $p \wedge q \quad \equiv \quad p \wedge \neg q \quad \equiv \quad \neg p$

(3.49) **(3.49)** $p \wedge (q \equiv r) \quad \equiv \quad p \wedge q \quad \equiv \quad p \wedge r \quad \equiv \quad p$

(3.50) **(3.50)** $p \wedge (q \equiv p) \quad \equiv \quad p \wedge q$

(3.51) **Replacement**: $(p \equiv q) \wedge (r \equiv p) \quad \equiv \quad (p \equiv q) \wedge (r \equiv q)$

Alternative Definitions of $\equiv$ and $\not\equiv$

(3.52) **Definition of $\equiv$**: $p \equiv q \quad \equiv \quad (p \wedge q) \vee (\neg p \wedge \neg q)$

(3.53) **Definition of $\not\equiv$**: $p \not\equiv q \quad \equiv \quad (\neg p \wedge q) \vee (p \wedge \neg q)$

Implication

(3.57) **Axiom, Definition of Implication**: $\boxed{p \Rightarrow q \;\equiv\; p \vee q \;\equiv\; q}$

(3.58) **Axiom, Definition of Consequence**: $\boxed{p \Leftarrow q \quad \equiv \quad q \Rightarrow p}$

Rewriting Implication:

(3.59) (Alternative) **Definition of Implication**: $p \Rightarrow q \;\equiv\; \neg p \vee q$

(3.60) (Dual) **Definition of Implication**: $p \Rightarrow q \;\equiv\; p \wedge q \;\equiv\; p$

(3.61) **Contrapositive**: $p \Rightarrow q \quad \equiv \quad \neg q \Rightarrow \neg p$

The "Golden Rule" and Implication

(3.35) **Axiom, Golden rule**: $\boxed{p \wedge q \quad \equiv \quad p \equiv q \quad \equiv \quad p \vee q}$

Can be used as:

- $p \wedge q \quad = \quad (p \equiv q \quad \equiv \quad p \vee q)$
- $(p \equiv q) \quad = \quad (p \wedge q \quad \equiv \quad p \vee q)$
- ...
- $(p \wedge q \quad \equiv \quad p) \quad \equiv \quad (q \quad \equiv \quad p \vee q)$

(3.57) **Axiom, Definition of Implication**: $p \Rightarrow q \;\equiv\; p \vee q \;\equiv\; q$

(3.60) (Dual) **Definition of Implication**: $p \Rightarrow q \;\equiv\; p \wedge q \;\equiv\; p$

Weakening/Strengthening Theorems

"$p \Rightarrow q$" can be read "p is stronger-than-or-equivalent-to q"

"$p \Rightarrow q$" can be read "p is at least as strong as q"

(3.76a) $p \quad \Rightarrow p \vee q$

(3.76b) $p \wedge q \quad \Rightarrow p$

(3.76c) $p \wedge q \quad \Rightarrow p \vee q$

(3.76d) $p \vee (q \wedge r) \quad \Rightarrow p \vee q$

(3.76e) $p \wedge q \quad \Rightarrow p \wedge (q \vee r)$

Implication Theorems 2

(3.62) $p \Rightarrow (q \equiv r) \quad \equiv \quad p \wedge q \quad \equiv \quad p \wedge r$

(3.63) **Distributivity of $\Rightarrow$ over $\equiv$:**

$$p \Rightarrow (q \equiv r) \quad \equiv \quad p \Rightarrow q \quad \equiv \quad p \Rightarrow r$$

(3.64) **Self-distributivity of $\Rightarrow$:**

$$p \Rightarrow (q \Rightarrow r) \quad \equiv \quad (p \Rightarrow q) \Rightarrow (p \Rightarrow r)$$

(3.65) **Shunting:** $p \wedge q \Rightarrow r \quad \equiv \quad p \Rightarrow (q \Rightarrow r)$

Some Property Names

Let $\odot$ and $\oplus$ be binary operators and $\square$ be a constant.

($\odot$ and $\oplus$ and $\square$ are ***metavariables*** *for operators.)*

- "**$\odot$ is symmetric**": $x \odot y \;=\; y \odot x$
- "**$\odot$ is associative**": $(x \odot y) \odot z \;=\; x \odot (y \odot z)$
- "**$\odot$ is mutually associative with $\oplus$** (from the left)":
  $$(x \odot y) \oplus z \;=\; x \odot (y \oplus z)$$

For example:

- $+$ **is** mutually associative with $-$:
  $$(x + y) - z \;=\; x + (y - z)$$
- $-$ **is not** mutually associative with $+$:
  $$(5 - 2) + 3 \neq 5 - (2 + 3)$$

Some Property Names (ctd.)

Let $\odot$ and $\oplus$ be binary operators and $\square$ be a constant.

($\odot$ and $\oplus$ and $\square$ are ***metavariables*** *for operators.)*

- "**$\odot$ is symmetric**": $x \odot y \;=\; y \odot x$
- "**$\odot$ is associative**": $(x \odot y) \odot z \;=\; x \odot (y \odot z)$
- "**$\odot$ is mutually associative with $\oplus$** (from the left)":
  $$(x \odot y) \oplus z \;=\; x \odot (y \oplus z)$$
- "**$\odot$ is idempotent**": $x \odot x \;=\; x$
- "**$\square$ is a unit/identity of $\odot$**": $\square \odot x \;=\; x$ and $x \odot \square \;=\; x$
- "**$\square$ is a zero of $\odot$**": $\square \odot x \;=\; \square$ and $x \odot \square \;=\; \square$
- "**$\odot$ distributes over $\oplus$ from the left**":
  $$x \odot (y \oplus z) \;=\; (x \odot y) \oplus (x \odot z)$$
- "**$\odot$ distributes over $\oplus$ from the right**":
  $$(y \oplus z) \odot x \;=\; (y \odot x) \oplus (z \odot x)$$
- "**$\odot$ distributes over $\oplus$**":
  $\odot$ distributes over $\oplus$ from the left **and**
  $\odot$ distributes over $\oplus$ from the right

Implication Theorems 3

(3.66)	$p \wedge (p \Rightarrow q)$	$\equiv$	$p \wedge q$	$\langle \ldots \quad p \wedge q \equiv p \rangle$
(3.67)	$p \wedge (q \Rightarrow p)$	$\equiv$	p	$\langle \ldots \quad p \wedge q \equiv p \rangle$
(3.68)	$p \vee (p \Rightarrow q)$	$\equiv$	$true$	$\langle \ldots \quad \neg p \vee q \rangle$
(3.69)	$p \vee (q \Rightarrow p)$	$\equiv$	$q \Rightarrow p$	$\langle \ldots \quad p \vee q \equiv q \rangle$
(3.70)	$p \vee q \Rightarrow p \wedge q$	$\equiv$	$p \equiv q$	$\langle \ldots$ Golden Rule $\ldots \rangle$

Implication Theorems 4

(3.71) **Reflexivity of $\Rightarrow$:** $p \Rightarrow p \quad \equiv \quad true$

(3.72) **Right-zero of $\Rightarrow$:** $p \Rightarrow true \quad \equiv \quad true$

(3.73) **Left-identity of $\Rightarrow$:** $true \Rightarrow p \quad \equiv \quad p$

(3.74) $p \Rightarrow false \quad \equiv \quad \neg p$ — sometimes this is: Definition of $\neg$

(3.75) ***ex falso quodlibet:*** $false \Rightarrow p \quad \equiv \quad true$

Implication Theorems 5

(3.77) **Modus ponens:** $p \wedge (p \Rightarrow q) \Rightarrow q$

(3.78) **Case analysis:** $(p \Rightarrow r) \wedge (q \Rightarrow r) \;\equiv\; (p \vee q \Rightarrow r)$

(3.79) **Case analysis:** $(p \Rightarrow r) \wedge (\neg p \Rightarrow r) \;\equiv\; r$

Implication Theorems 6

(3.80) **Mutual implication:** $(p \Rightarrow q) \wedge (q \Rightarrow p) \;\equiv\; p \equiv q$

(3.80b) **Reflexivity wrt. Equivalence:** $(p \equiv q) \Rightarrow (p \Rightarrow q)$

(3.81) **Antisymmetry:** $(p \Rightarrow q) \wedge (q \Rightarrow p) \;\Rightarrow\; (p \equiv q)$

(3.82a) **Transitivity:** $(p \Rightarrow q) \wedge (q \Rightarrow r) \;\Rightarrow\; (p \Rightarrow r)$

(3.82b) **Transitivity:** $(p \equiv q) \wedge (q \Rightarrow r) \;\Rightarrow\; (p \Rightarrow r)$

(3.82c) **Transitivity:** $(p \Rightarrow q) \wedge (q \equiv r) \;\Rightarrow\; (p \Rightarrow r)$