Constructive Proofs aren't Square; Deriving the formula for the Sum of Squares

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Abstract

The story of how a precocious Gauss derived the formula for the sum of integers has firmly entered mathematical lore.

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \tag{1}$$

Its older cousin, the formula for the sum of squares is often taught, but it is rarely derived.

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \tag{2}$$

This is likely because the proof that Archemedes devised, though brilliant, is rather laborious. In this paper, we would like to present a simplified, and hopefully, a more intuitive method to derive the latter formula.

1 Proof

1.1 Step 1

Let us begin by observing figure (a). The top layer has 1 cube. The second layer has 4, the third 9, and so on. When we put them all together, we get a pyramid (b) that has as many cubes as the sum of 1 to n^2.

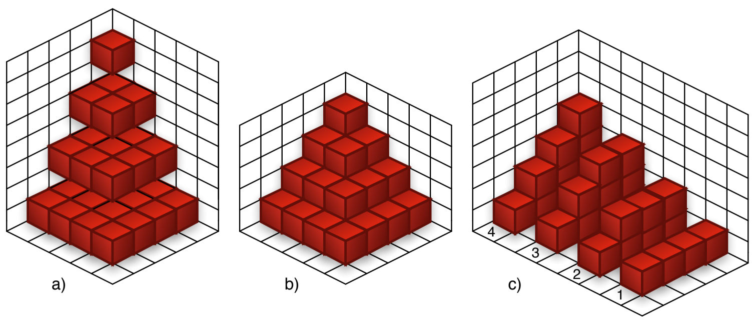

Figure 1: **Sum of squares by slices.**

Of course, there is more than one way to slice a pyramid. Let us take a look at figure (c), where our pyramid is sliced vertically. The first row of this pyramid has 4 elements, the second, $4+3$, the third, $4+3+2$ and so on. When we generalize this for an aribtrary slice, we see that the first row is always n. Since we're counting from the base of the pyramid instead of the top, the last row will be $n-i+1$. That is, for i = 2 being the second slice, the top row is $4-2+1=3$. Thus, we need the following sum.

$$n+(n-1)+(n-2)+...+(n-i+1)=\sum_{n-i+1}^{n} i \tag{3}$$

Since,

$$\sum_{1}^{n} i=\sum_{1}^{n-i} i+\sum_{n-i+1}^{n} i \tag{4}$$

each slice will be

$$\sum_{n-i+1}^{n} i=\frac{n(n+1)}{2}-\frac{(n-i)(n-i+1)}{2} \tag{5}$$

$$\sum_{n-i+1}^{n} i=\frac{2ni-i^2+i}{2} \tag{6}$$

and the sum of these slices should equal the sum of our squares.

$$\sum_{i=1}^{n} i^2=\sum_{i=1}^{n}\frac{2ni-i^2+i}{2} \tag{7}$$

From this point, deriving the formula is a matter of some rearranging.

$$2\sum_{i=1}^{n} i^2=\sum_{i=1}^{n} 2ni-i^2+i \tag{8}$$

$$3\sum_{i=1}^{n} i^2=\sum_{i=1}^{n} 2ni+i \tag{9}$$

$$3\sum_{i=1}^{n} i^2=(2n+1)\sum_{i=1}^{n} i \tag{10}$$

$$3\sum_{i=1}^{n} i^2=\frac{(2n+1)n(n+1)}{2} \tag{11}$$

$$\sum_{i=1}^{n} i^2=\frac{n(n+1)(2n+1)}{6} \tag{12}$$

1.2 Conclusion

The sum of squares is frequently encountered and easy to prove by induction. As this provides little insight, this proof by construction aims to shed some light on the issue.