MATHEMATICS 1LS3 TEST 2

Day Class
Duration of Examination: 60 minutes
McMaster University, 14 February 2012

E. Clements

FIRST NAME (please print): Sol^{N}s

FAMILY NAME (please print): ______

Student No.: ______

THIS TEST HAS 8 PAGES AND 6 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE.

Total number of points is 40. Marks are indicated next to the problem number. Any non-graphing calculator is allowed.

USE PEN TO WRITE YOUR TEST. IF YOU USE A PENCIL YOUR TEST WILL NOT BE ACCEPTED FOR REMARKING (IF NEEDED).

You need to show work to receive full credit.

Problem	Points	Mark
1	6	
2	7	
3	6	
4	7	
5	6	
6	8	
TOTAL	40	

MATH 1LS3 * Test 2 * 14 February 2012

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1. The dynamics of caffeine absorption and replacement can be described by the discrete time dynamical system

$$c_{t+1} = .0.87c_t + d$$

where c_t denotes the amount of caffeine (in mg) present in your body at time t (in hours) and d is the amount of caffeine taken every hour. Suppose that the initial amount of caffeine is $c_0 = 50$ mg and every hour you consume a small coffee (60 mg of caffeine).

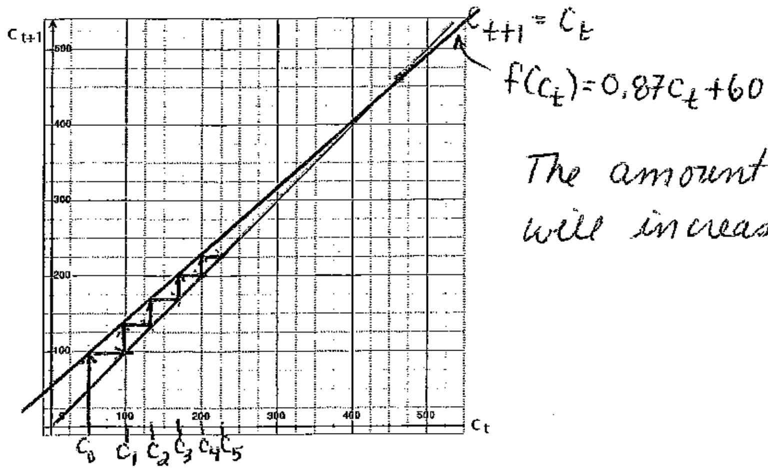
(a) [1] Determine the equilibrium amount of caffeine.

$$d = 60 \Rightarrow c_{t+1} = 0.87c_t + 60$$

$$c^* = 0.87c^* + 60$$

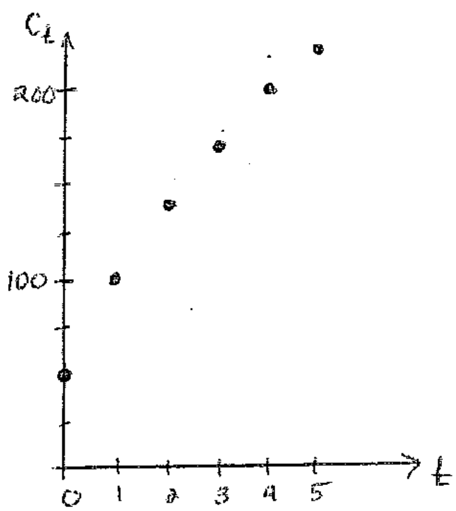
$$\Rightarrow c^* = \frac{60}{0.13} \doteq 461.5 \text{ mg}$$

(b) [3] Graph the updating function and the diagonal. Cobweb for 5 steps starting from $c_0 = 50$ mg. Describe what happens to the amount of caffeine in your body over the next 5 hours.



The amount of caffeine in your body will increase over the next 5 hours.

(c) [2] Roughly (no calculations required!) plot the solution points you found in part (b). Label the axes.



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2. Consider the model for bacterial population growth given by $b_{t+1} = \dfrac{12}{1+0.01b_t} b_t$, where b_t represents the number of bacteria in a culture at time t, in hours.

(a) [2] How do the dynamics of this system differ from the dynamics of the system $b_{t+1} = rb_t$, where r is a constant?

In this model, the per capita production rate is inversely proportional to population size (as b_t increases, $r(b_t)$ decreases). In the model $b_{t+1} = rb_t$, the per capita production rate is constant (ie does not depend on population size, b_t).

(b) [3] If a culture is found to contain 500 bacteria, what was the size of the population one hour ago?

inverse:

$$b_{t+1} = \frac{12b_t}{1+0.01b_t}$$

$$(1+0.01b_t)b_{t+1} = 12b_t$$

$$b_{t+1} + 0.01b_t(b_{t+1}) = 12b_t$$

$$0.01b_t(b_{t+1}) - 12b_t = -b_{t+1}$$

$$b_t(0.01b_{t+1} - 12) = -b_{t+1}$$

$$\Rightarrow b_t = \frac{-b_{t+1}}{0.01b_{t+1} - 12}$$

so, $b_{t-1} = \dfrac{-b_t}{0.01b_t - 12}$

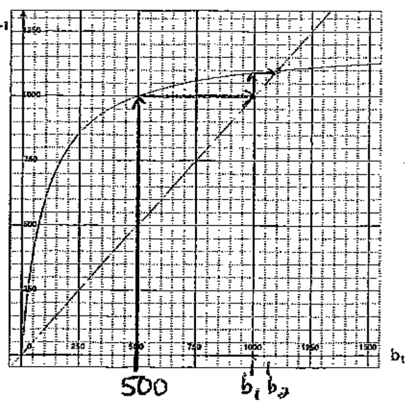
when $b_0 = 500$,

$$b_{-1} = \frac{-500}{0.01(500) - 12}$$

$\cong 71$ bacteria

(c) [2] Below is the graph of the updating function and the diagonal for $b_{t+1} = \dfrac{12}{1+0.01b_t} b_t$.

$b^* = 1100$



Starting from $b_0 = 500$, cobweb several steps. Describe what happens to this population over time.

The population will increase rapidly at first but then slow down as it approaches the equilibrium.

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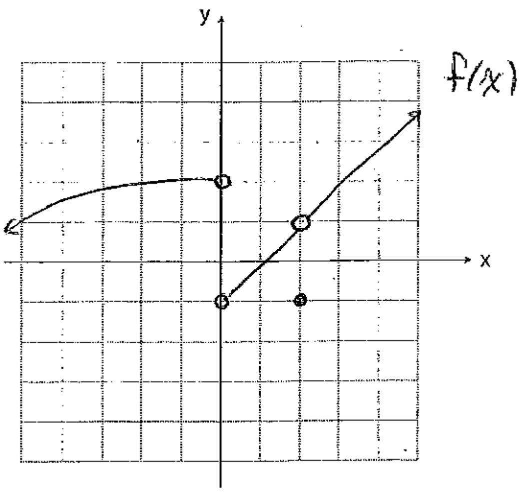
3. (a) [2] In your own words, explain what is meant by the expression $\lim_{x \to a} f(x) = L$. If $f(x)$ is continuous at $x = a$, what must the value of L be?

$\lim_{x \to a} f(x) = L$, means that the values of the function approach L as x approaches a from either side of a (but $x \neq a$). We can make $f(x)$ as close as we'd like to L by choosing x sufficiently close to a.

If $f(x)$ is continuous at $x=a$, then $\lim_{x \to a} f(x) = f(a)$ and so L must equal $f(a)$.

(b) [3] Sketch a possible graph of a function $f(x)$ that satisfies all of the following conditions:

$\lim_{x \to 0^-} f(x) = 2$, $\lim_{x \to 0^+} f(x) = -1$, $f(0)$ is not defined, $f(2) = -1$, and $\lim_{x \to 2} f(x) = 1$.



(c) [1] At what x-values is the function in part (b) discontinuous? What types of discontinuities does it have there?

f has a jump discontinuity at $x=0$ and a removable discontinuity at $x=2$.

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4. Evaluate each limit, or explain why it does not exist.

(a) [2] $\lim_{x \to 4} \frac{\sqrt{x} - 2}{4 - x}$

$$= \lim_{x \to 4} \frac{\cancel{x-4}}{-(\cancel{x-4})(\sqrt{x}+2)}$$

$$= \lim_{x \to 4} \frac{-1}{\sqrt{x}+2}$$

$$= \frac{-1}{\sqrt{4}+2}$$

$$= -\frac{1}{4}$$

(b) [3] $\lim_{x \to \infty} (\ln x - \ln(x-1))$

$$= \lim_{x \to \infty} \ln\left(\frac{x}{x-1}\right)$$

$$= \lim_{x \to \infty} \ln\left(\frac{1}{1-\frac{1}{x}}\right)$$

$$= \ln\left(\frac{1}{1-\cancel{\frac{1}{\infty}}0}\right)$$

$$= \ln 1$$

$$= 0$$

(c) [2] $\lim_{x \to \infty} \frac{0.2x^{-5}}{e^{-x}}$ $\quad \left(= \frac{0}{0}\right)$

$e^{-x} \to 0$ faster than $0.2x^{-5} \to 0$

$\Rightarrow \frac{0.2x^{-5}}{e^{-x}} \to \frac{\#}{0} \to +\infty$

so, $\lim_{x \to \infty} \frac{0.2x^{-5}}{e^{-x}} = \infty$ (D.N.E.)

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5. Suppose that the absorption of a toxic chemical is modelled by

$$\alpha(c) = \frac{2c}{1+10c}$$

where α represents the amount absorbed (in mg) and c represents the concentration of the chemical (in mmol/L).

(a) [1] If the concentration of the chemical changes from 10 mmol/L to 20 mmol/L, what is the average rate of change in the amount absorbed. *Remember units!*

$$= \frac{\alpha(20) - \alpha(10)}{20-10}$$

$$= \frac{\frac{2(20)}{1+10(20)} - \frac{2(10)}{1+10(10)}}{10}$$

$$= \frac{2}{20\,301} \frac{mg}{mmol/L}$$

(b) [3] Using the limit definition of the derivative, find $\alpha'(c)$. At what rate is the substance absorbed when the concentration is 10mmol/L? *Remember units!*

$$\alpha'(c) = \lim_{h\to 0} \frac{\frac{2(c+h)}{1+10(c+h)} - \frac{2c}{1+10c}}{h}$$

$$= \lim_{h\to 0} \frac{2(c+h)(1+10c) - 2c(1+10(c+h))}{(1+10(c+h))(1+10c)\,h}$$

$$= \lim_{h\to 0} \frac{2\cancel{h}}{(1+10(c+h))(1+10c)\cancel{h}}$$

$$= \frac{2}{(1+10(c+0))(1+10c)}$$

$$= \frac{2}{(1+10c)^2}$$

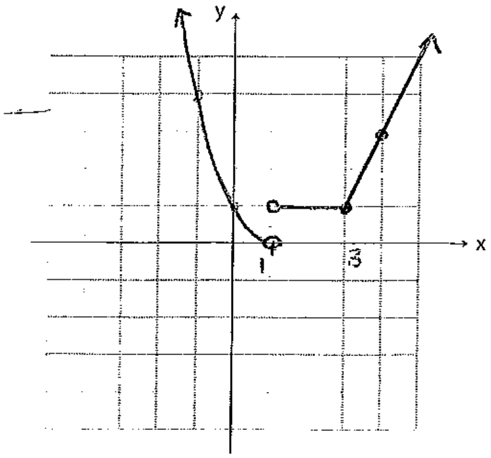
(c) [2] What will happen long-term to the amount your body absorbs if the concentration of this chemical continues to increase steadily?

$$\lim_{c\to\infty} \alpha(c) = \lim_{c\to\infty} \frac{2}{\frac{1}{c}+10} = \frac{2}{\frac{1}{\infty}+10} = \frac{1}{5}$$

Long-term, you body will $\frac{1}{5}$ mg of this chemical.

6. Consider the function $f(x) = \begin{cases} (x-1)^2 & \text{if } x < 1 \\ 1 & \text{if } 1 < x \leq 3 \\ 2x-5 & \text{if } x > 3 \end{cases}$.

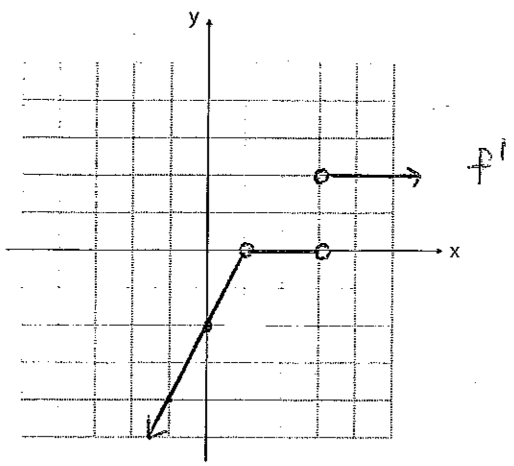
(a) [3] Sketch the graph of f.



(b) [2] Where is f not differentiable? Why is f not differentiable there?

f is not differentiable at $x=1$ because it is discontinuous here and it is not differentiable at $x=3$ because the graph has a "corner" here.

(c) [3] Sketch the graph of the derivative, f'.



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ROUGH WORK

THE END