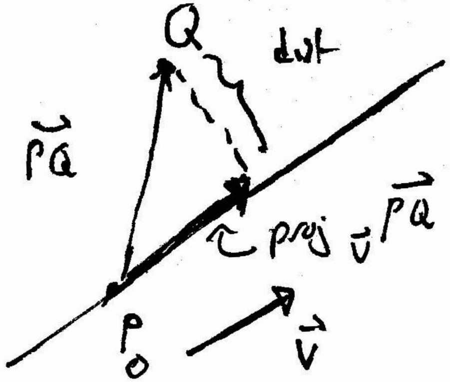
1ZC3

Last Day

Point-Line Distance

$\vec{u} = \vec{PQ}$



$$
\begin{aligned}
\text{dist} &= \| \text{orth}_{\vec{v}} \vec{PQ} \| \\
&= \| \vec{u}_{\perp} \| \\
&= \| \vec{u} - \text{proj}_{\vec{v}} \vec{u} \| \\
&= \| \vec{u} - \frac{\langle u, v \rangle}{\|\vec{v}\|^2} \vec{v} \|
\end{aligned}
$$

eg. Find the distance from $Q = (0,0,1)$ & the line.

$\vec{P}(t) = (1,2,3) + t(1,0,0)$

Solution $\vec{v} = (1,0,0)$, $\vec{u} = \vec{P_0 Q} = \vec{Q} - \vec{P_0}$

$$
\begin{aligned}
&= (0,0,1) - (1,2,3) \\
&= (-1, -2, -2)
\end{aligned}
$$

$$\text{dist} = \left\| \vec{u} - \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{v}\|^2} \vec{v} \right\| = \left\| (-1, -2, -2) - \frac{(-1+0+0)}{\sqrt{1^2+0^2+0^2}} (1,0,0) \right\|$$

$$= \|(-1, -2, -2) + (1, 0, 0)\|$$

$$= \|(0, -2, -4)\| = \sqrt{0^2 + 2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$$

Side note

$$\|\vec{u}_\perp\| = \|\vec{u} - \text{proj}_{\vec{v}} \vec{u}\| = \left\| \vec{u} - \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{v}\|^2} \vec{v} \right\| \geq 0$$

$$= \sqrt{\| \quad " \quad \|^2}$$

$$= \langle \vec{u}, \vec{u} \rangle + \frac{\langle \vec{u}, \vec{v} \rangle^2}{\|\vec{v}\|^4} \cancel{\langle \vec{v}, \vec{v} \rangle}_{\|\vec{v}\|^2} - 2 \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{v}\|^2} \langle \vec{u}, \vec{v} \rangle$$

$$= \|\vec{u}\|^2 + \frac{(\langle \vec{u}, \vec{v} \rangle)^2}{\|\vec{v}\|^2} - 2 \frac{|\langle \vec{u}, \vec{v} \rangle|^2}{\|\vec{v}\|^2}$$

$$= \|\vec{u}\|^2 - \cancel{\ } \langle \vec{u}, \vec{v} \rangle^2 / \|\vec{v}\|^2 \geq 0$$

$$\Rightarrow \quad \|\vec{u}\|^2 \|\vec{v}\|^2 \geq \cancel{\ } |\langle u, v \rangle|^2$$

C-S Inequality!

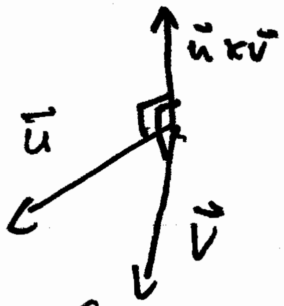
Cross Product (i.e. Vector product)

$$\vec{u} \times \vec{v} = \vec{w} \quad \Rightarrow \quad \vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$$

Properties

1) $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$
2) $(k\vec{u}) \times \vec{v} = k(\vec{u} \times \vec{v})$
3) $(\vec{u} + \vec{w}) \times \vec{v} = (\vec{u} \times \vec{v}) + (\vec{w} \times \vec{v})$
4) $\vec{u} \times \vec{u} = \vec{0}$
5) $\vec{u} \times \vec{0} = \vec{0}$

In $\mathbb{R}^3$ (only ...)



$\vec u \times \vec v \perp$ to $\vec u$ & $\vec v$

$$\|\vec u \times \vec v\| = \|\vec u\| \, \|\vec v\| \sin\theta$$

$$\vec i \times \vec j = \vec k, \quad \vec j \times \vec k = \vec i, \quad \vec k \times \vec i = \vec j$$

$$\vec u \times \vec v = (u_2 v_3 - u_3 v_2,\ u_3 v_1 - v_3 u_1,\ u_1 v_2 - u_2 v_1)$$

$$= \begin{vmatrix} \vec i & \vec j & \vec k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

expand our det. along row #1.

$$= M_{11} \vec i - M_{12} \vec j + M_{13} \vec k$$

$$= (M_{11}, -M_{12}, M_{13}) = \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, -\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right)$$

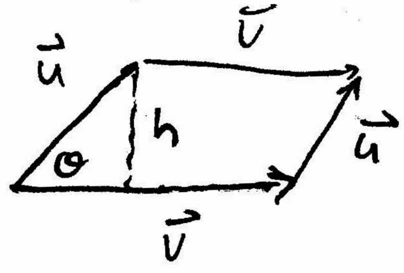
eg. If $\vec{u} = (1, 0, 5)$, $\vec{v} = (-1, 2, 1)$ find $\vec{u} \times \vec{v}$

Solution

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 5 \\ -1 & 2 & 1 \end{vmatrix} = \underbrace{\begin{vmatrix} 0 & 5 \\ 2 & 1 \end{vmatrix}}_{-10} \vec{i} - \underbrace{\begin{vmatrix} 1 & 5 \\ -1 & 1 \end{vmatrix}}_{1+5=6} \vec{j} + \underbrace{\begin{vmatrix} 1 & 0 \\ -1 & 2 \end{vmatrix}}_{2} \vec{k}$$

$$= (-10, -6, 2)$$

eg. Find the area of the $\parallel$grm generated by $\vec{u} = (1, 2)$, $\vec{v} = (3, 4)$



Area = base · height

$$= \|\vec{v}\| \cdot h = \|\vec{v}\| \frac{\|\vec{u}\times\vec{v}\|}{\|\vec{v}\|}$$

$$= \|\vec{u}\times\vec{v}\|$$

$$h = \|\vec{u}\| \sin\theta = \|\vec{u}\|\|\vec{v}\| \sin\theta / \|\vec{v}\| = \frac{\|\vec{u}\times\vec{v}\|}{\|\vec{v}\|}$$

Area of ∥gm is $\|\vec{u}\times\vec{v}\|$

Fake $\mathbb{R}^3$

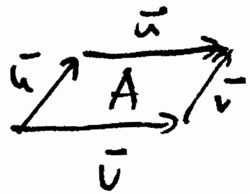
e.g. Let $\vec{u} = (1,2)$, $\vec{v} = (3,4)$ $\rightarrow$ $\vec{u} = (1,2,0)$, $\vec{v} = (3,4,0)$

$$\vec{u}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 0 \\ 3 & 4 & 0 \end{vmatrix} = \left(\begin{vmatrix} 2 & 0 \\ 4 & 0 \end{vmatrix}, -\begin{vmatrix} 1 & 0 \\ 3 & 0 \end{vmatrix}, \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} \right)$$

$$= (0, 0, \overset{-2}{\cancel{4-6}})$$

$$\|\vec{u} \times \vec{v}\| = \|(0, 0, -2)\| = 2$$

Notice In general if $\vec{u} = (u_1, u_2)$ & $\vec{v} = (v_1, v_2)$



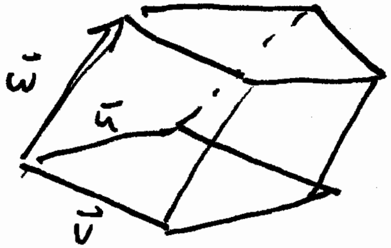
$$A = \|\vec{u} \times \vec{v}\|$$

$$= \| (0, 0, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \|$$

$$= \left| \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right|$$

$$= |\det[\vec{u} | \vec{v}]|$$

Triple Product & Parallelepipeds



$$\text{Vol} = (\text{Area of base}) \cdot \text{height}$$

$$= \|\vec{u} \times \vec{v}\| \cdot \|\operatorname{proj}_{\vec{n}} \vec{w}\|$$

$$= \cancel{\|\vec{u} \times \vec{v}\|} \left\| \frac{(\vec{w} \cdot (\vec{u} \times \vec{v}))}{\|\vec{u} \times \vec{v}\|^2} (\vec{u} \times \vec{v}) \right\|$$

$$= \frac{|\vec{w} \cdot (\vec{u} \times \vec{v})|}{\cancel{\|\vec{u} \times \vec{v}\|^2}} \cancel{\|\vec{u} \times \vec{v}\|}$$

$$\text{Vol} = |\vec{w} \cdot (\vec{u} \times \vec{v})|$$

$\vec{w} \cdot (\vec{u} \times \vec{v})$ = triple product notice $\vec{w} \cdot (\vec{u} \times \vec{v})$ is a scalar

$$= \vec{w} \cdot \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

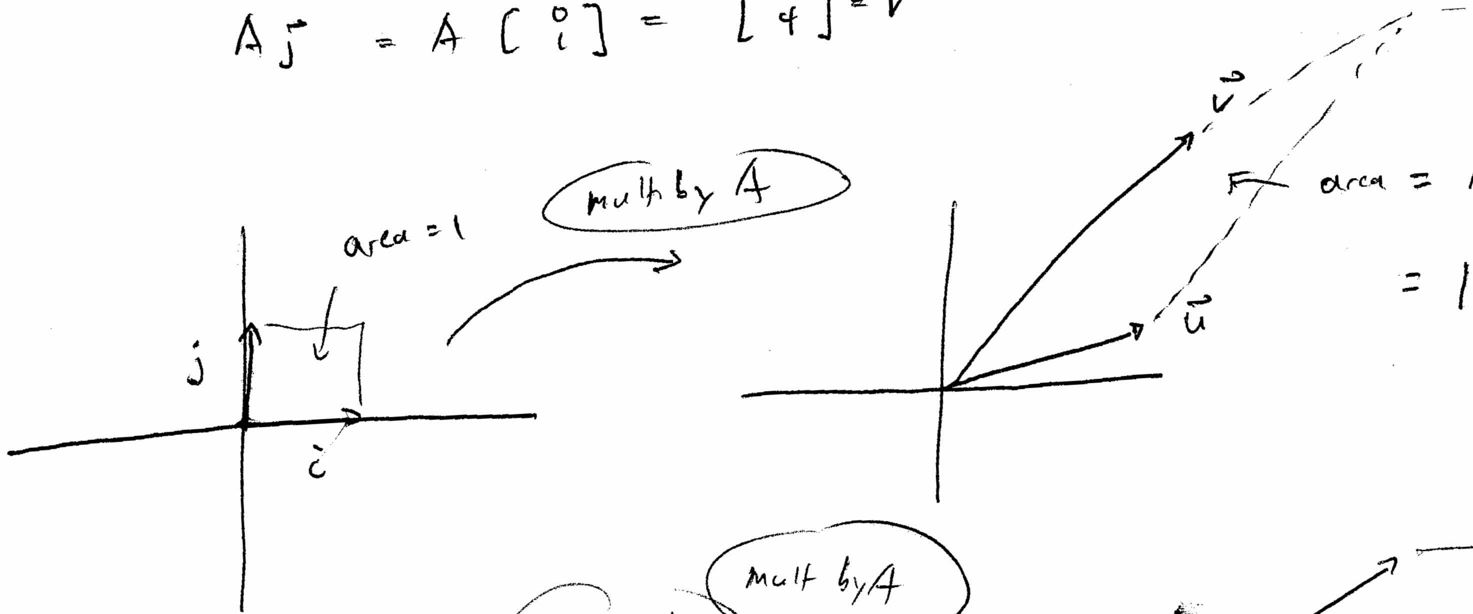
$$= (w_1, w_2, w_3) \cdot (M_{11}, -M_{12}, M_{13})$$

$$= w_1 M_{11} - w_2 M_{12} + w_3 M_{13}$$

$$= \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \vec{w} \cdot (\vec{u} \times \vec{v})$$

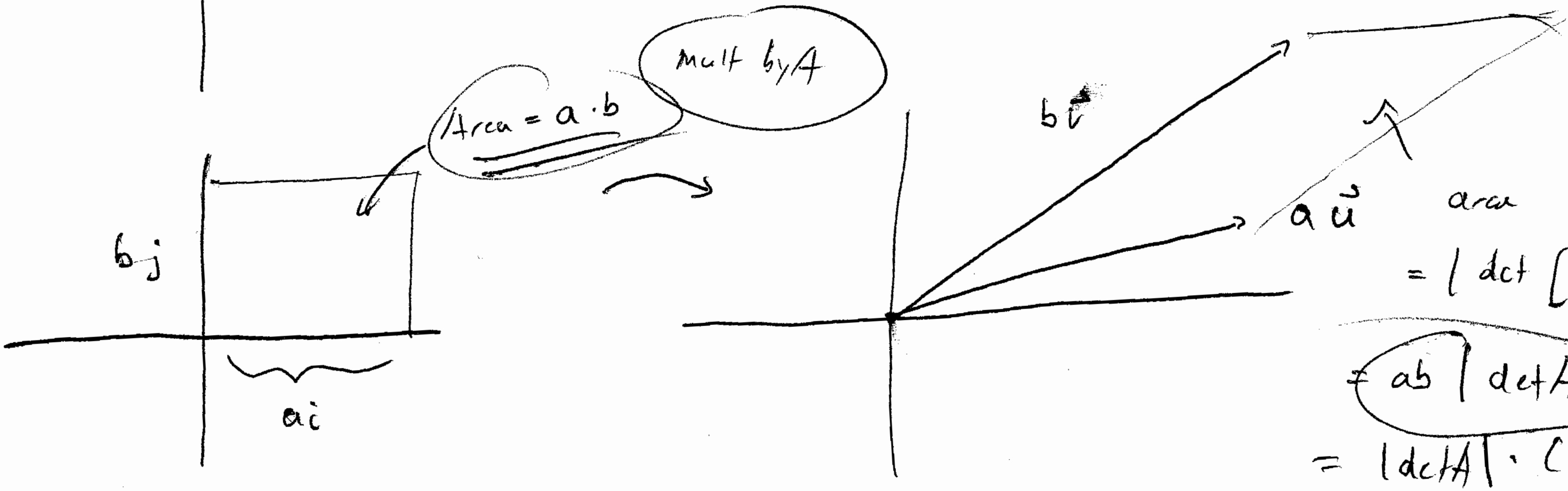
Say $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ & $\vec{u} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

$$\left.\begin{aligned} \Rightarrow\ A\vec{i} &= A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \vec{u} \\ A\vec{j} &= A\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \vec{V} \end{aligned}\right\} \Rightarrow A = [\vec{u} | \vec{V}]$$



area = Area of llgrm

$= |\det A|$

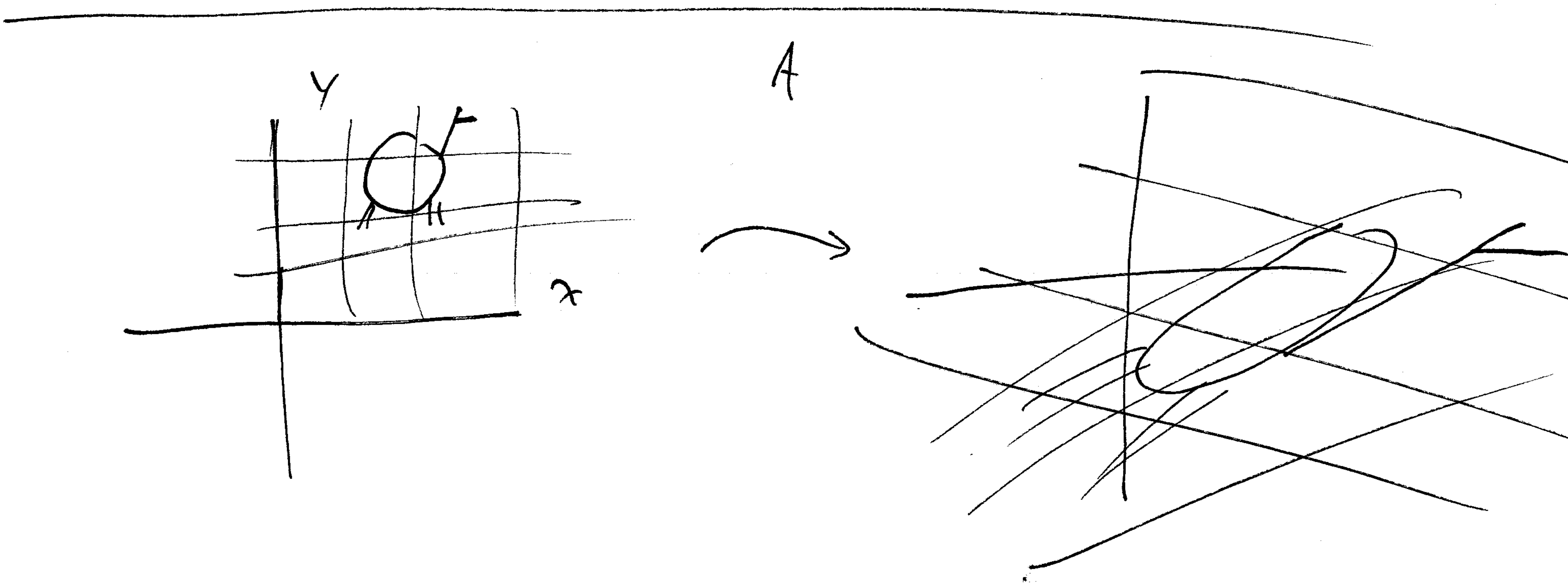


area

$= |\det [a\vec{u} | b\vec{v}]|$

$= ab\,|\det A|$

$= |\det A| \cdot$ (original area)


y
x
A