MATHEMATICS 1LT3 TEST 3

Day Class
Duration of Test: 60 minutes
McMaster University, 8 March 2011

E. Clements

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FIRST NAME (please print) : Sol^{N}s
FAMILY NAME (please print) :
Student No.:

THIS TEST HAS 8 PAGES AND 9 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE.

Total number of points is 50. Marks are indicated next to the problem number in square brackets. Any Casio fx991 (or lower, non-graphing) calculator is allowed.

USE PEN TO WRITE YOUR TEST. IF YOU USE A PENCIL, YOUR TEST WILL NOT BE ACCEPTED FOR REMARKING (IF NEEDED).

You need to show work to receive full credit.

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1. [5] Solve the linear differential equation

$$x^2y' + xy = 1, \text{ where } x > 0, \text{ and } y(1) = 2.$$

$$\xrightarrow{(\div x^2)} \quad y' + \frac{1}{x}y = \frac{1}{x^2}$$

$$I(x) = e^{\int \frac{1}{x}dx} = e^{\ln|x|} = |x| = x \quad \because x > 0$$

$$\xrightarrow{\times I(x)} \quad xy' + y = \frac{1}{x}$$

$$(xy)' = \frac{1}{x}$$

$$xy = \int \frac{1}{x}\,dx$$

$$xy = \ln|x| + C = \ln x + C \quad \because x > 0$$

$$\Rightarrow y = \frac{\ln x + C}{x}$$

$$y(1) = 2 \rightarrow 2 = \frac{\ln 1 + C}{1} \Rightarrow C = 2$$

$$\text{So, } y = \frac{\ln x + 2}{x}$$

2. The following pair of equations represent the population growth of two different species where one is the predator and the other is the prey.

$$\frac{dx}{dt} = -0.05x + 0.0001xy$$
$$\frac{dy}{dt} = 0.1y - 0.005xy$$

[2] (a) Which of the variables, x or y, represents the predator population and which represents the prey population? Explain.

x represents the # of predators ∵ the coefficient of the interaction term xy is ⊕ so interact^{n}s increase pop^{n} size.
y represents the # of prey ∵ the coefficient of the interact^{n} term xy is ⊖ so interact^{n}s decrease pop^{n} size.

[2] (b) Determine the equilibrium solutions.

$$x' = x(-0.05 + 0.0001y)$$
$$y' = y(0.1 - 0.005x)$$

$$\left.\begin{matrix} x'=0 \\ y'=0 \end{matrix}\right\} \Rightarrow \left.\begin{matrix} x=0 \\ y=0 \end{matrix}\right\} \text{ or } \left.\begin{matrix} y=500 \\ x=20 \end{matrix}\right\}$$

[3] (c) If $x_0 = 25$ and $y_0 = 40$, approximate the size of both populations after one year using Euler's method and a step size of 6 months. Here, t is measured in months.

$\Delta t = 6$
$x_0 = 25$
$y_0 = 40$

$x_1 = 25 + (-.05(25) + .0001(25)(40))(6) \approxeq 18$
$y_1 = 40 + (0.1(40) - 0.005(40)(25))(6) = 34$

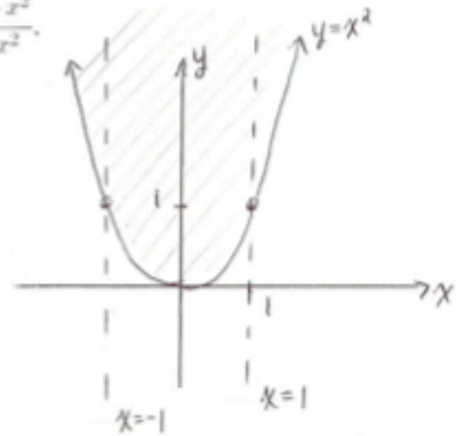
$x_2 = 18 + (-.05(18) + .0001(18)(34))(6) \approx 13$
$y_2 = 34 + (0.1(34) - .005(34)(18))(6) \approx 36$

∴ After 1 year, there are approximately 13 predators and 36 prey.

3. [3] Find and sketch the domain of $f(x,y) = \dfrac{\sqrt{y-x^2}}{1-x^2}$.

① $1-x^2 \neq 0 \Rightarrow x \neq \pm 1$

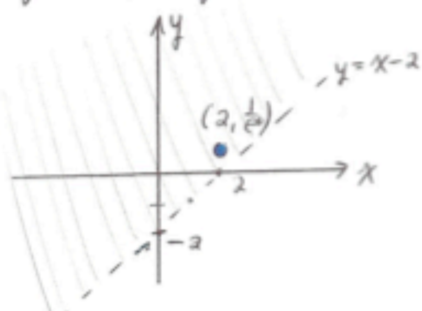
② $y - x^2 \geqslant 0 \Rightarrow y \geqslant x^2$



4. Consider the function $f(x,y) = \ln(2-x+y)$.

[2] (a) Find and sketch the domain of f.

$2 - x + y > 0 \Rightarrow y > x - 2$



[3] (b) Show that -6 is in the range of f (i.e. find a point (x,y) in the domain of f such that $f(x,y) = -6$).

set $f(x,y) = -6$: $\ln(2-x+y) = -6$

$2-x+y = e^{-6}$

$y = e^{-6} - 2 + x$

choose $x = 2$. Then $y = e^{-6} = \frac{1}{e^6}$.

$(2, \frac{1}{e^6}) \in \text{domain}(f)$ and $f(2, e^{-6}) = \ln(2-2+e^{-6})$

$= \ln e^{-6}$

$= -6$ ✓

$\therefore -6$ is in the range of f.

5. [12] Match the function (a) with its graph (labeled A-F) and (b) with its contour map (labeled I-VI).

$z = \sin\sqrt{x^2+y^2}$ (a) B (b) III $z = x^2y^2e^{-x^2-y^2}$ (a) C (b) II

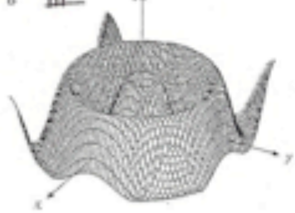
$z = \dfrac{1}{x^2+4y^2}$ (a) F (b) V $z = x^3 - 3xy^2$ (a) A (b) VI

$z = \sin x \sin y$ (a) D (b) IV $z = \sin^2 x + \frac{1}{4}y^2$ (a) E (b) I

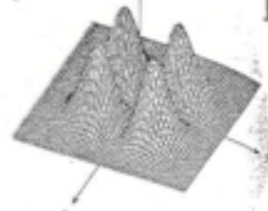
$x = 0, \pi, 2\pi, \ldots \quad z = 0$
$y = 0, \pi, 2\pi, \ldots \quad z = 0$

A VI

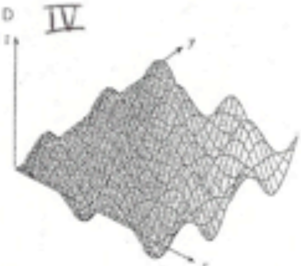
B III



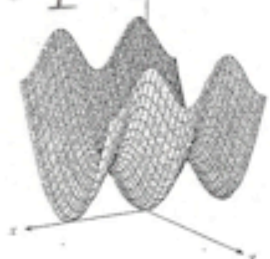
C II



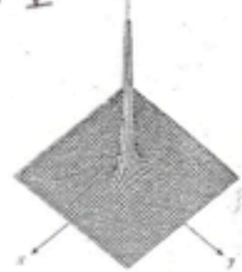
D IV



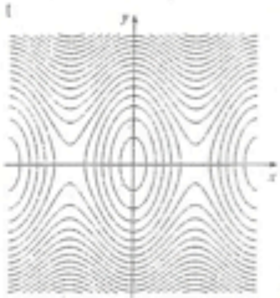
E I



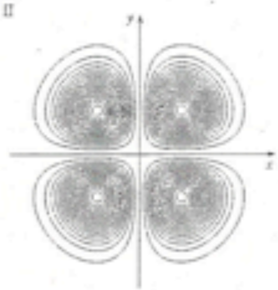
F V



I

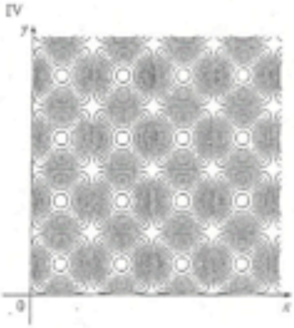


II

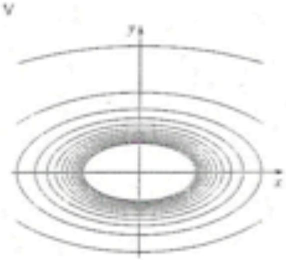


III

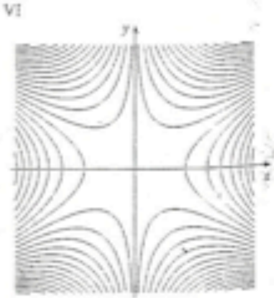
IV



V



VI



6. Show that the following limits do not exist.

[4] (a) $\underbrace{\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{3x^2+y^2}}_{f}$

$f(x,0) = \frac{x^2}{3x^2} = \frac{1}{3}$

$f(x,y) \to \frac{1}{3}$ as $(x,y) \to (0,0)$ along $y=0$

$f(0,y) = \frac{y^2}{y^2} = 1$

$f(x,y) \to 1$ as $(x,y) \to (0,0)$ along $x=0$.

$\Rightarrow \lim_{(x,y)\to(0,0)} f(x,y)$ D.N.E.

[4] (b) $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$ (Hint: Compute limits along $x=0$ and $x=y^2$)

$f(0,y) = 0$

$f(x,y) \to 0$ as $(x,y) \to (0,0)$ along $x=0$.

$f(y^2,y) = \frac{y^4}{y^4+y^4} = \frac{1}{2}$

$f(x,y) \to \frac{1}{2}$ as $(x,y) \to (0,0)$ along $x=y^2$.

$\Rightarrow \lim_{(x,y)\to(0,0)} f(x,y)$ D.N.E.

7. [3] If $\lim_{(x,y)\to(1,2)} f(x,y) = 5$ along all lines $y = mx$ where m is any real number, can we conclude that this limit exists? Why or why not?

No. We have only shown that the limit along all straight lines through the origin is 5. What about limit along a curve through the origin? $y=x^2$ for ex.

8. Let $f(x,y) = \ln(x^2 + y^2 - 4)$.

[3] (a) Determine the set of points at which f is continuous. Sketch this set.

f is a composit^n of cts f^{n}s $\Rightarrow$ f is cts on its domai

$$x^2+y^2-4>0$$

$$x^2+y^2>4$$

[1] (b) Evaluate $\lim\limits_{(x,y)\to(2,1)} \ln(x^2 + y^2 - 4)$. direct subs. $\because$ cts at $(2,1)$.

$$\lim_{(x,y)\to(2,1)} \ln(x^2+y^2-4) = \ln(2^2+1^2-4) = \ln 1 = 0$$

9. [3] Using the definition, determine whether or not this piece is continuous on R^2

$$f(x,y) = \begin{cases} \dfrac{\cos y}{x^2+y^2+1} & \text{if } (x,y) \neq (0,0) \\ 1 & \text{if } (x,y) = (0,0) \end{cases}$$

is continuous at $(0,0)$.

$$\lim_{(x,y)\to(0,0)} f(x,y) = \frac{\overbrace{\cos 0}^{=1}}{0^2+0^2+1} = 1$$

$$f(0,0) = 1$$

$$\therefore \lim_{(x,y)\to(0,0)} f(x,y) = 1 = f(0,0) \quad \therefore f \text{ is continuous at } (0,0).$$