Math 1LS3 Week 9: Dynamical Systems – Discrete and Continuous

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Week 9: In class, we cover 5.5, 5.6 (stability of dynamical systems); start of integration (ch 6) section 6.1 (not including Euler's Method). On your own:

- Work through all solved examples in section 5.3 (if you didn't do so last week)
- Study cobwebbing and Qualitative Dynamical Systems (p.380–386)
- Work through all examples 5.5.1 – 5.5.8 and 5.6.1 – 5.6.5

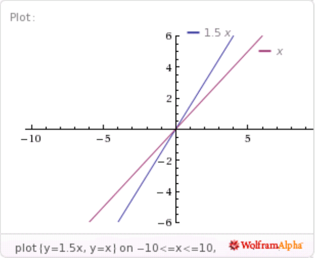
A DTDS consists of:

- Initial value m_0
- Updating function f: $m_1 = f(m_0)$, $m_2 = f(m_1)$, etc. $m_{t+1} = f(m_t)$.

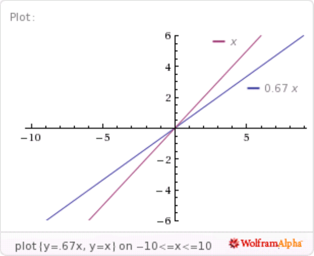
Recall:

- **Solution**: m_t as a function of t alone
- **Cobwebbing**: graphical technique for seeing how system evolves.
- **Equilibrium point**: m^* such that $f(m^*) = m^*$. Graphically, it's where $y = f(x)$ intersects $y = x$. If a DTDS starts at equilibrium, it stays there forever.
- **Stable equilibrium**: start near $m^* \implies$ stay near m^* for all time.
- **Unstable equilibrium**: start near m^* (but not at m^*) $\implies$ eventually go away from m^*.

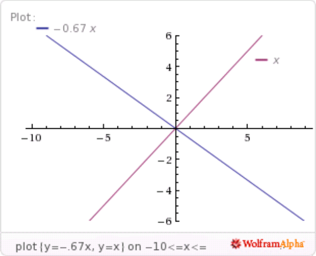
Linear Dynamics



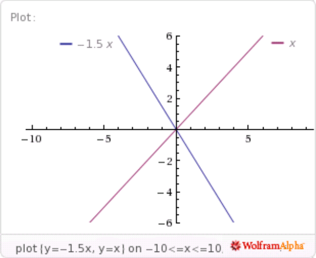
Slope > 1: Unstable, Monotone



$0 <$ Slope < 1: Stable, Monotone



$-1 <$ Slope < 0: Stable, Oscillating

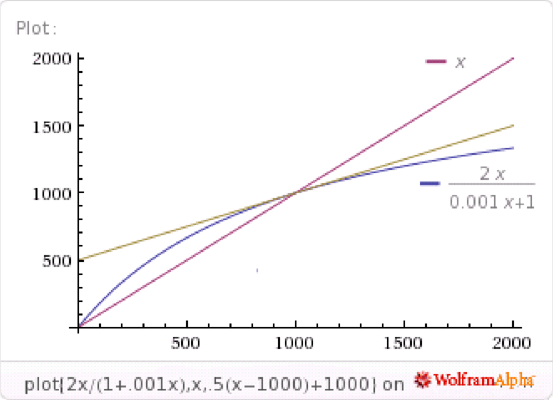


Slope < -1: Unstable, Oscillating

Derivatives and Stability

Key points:

- For linear dynamics, only slope matters
- For nonlinear dynamics, study stability using tangent line



Derivatives and Stability: Summary

Theorem (p.385)

Suppose x^* is an equilibrium point for a DTDS with updating function f.

If ...	*then x^* is ...*	*and behavior is ...*
$f'(x^*) > 1$	*unstable*	*monotonic*
$0 < f'(x^*) < 1$	*stable*	*monotonic*
$-1 < f'(x^*) < 0$	*stable*	*oscillating*
$f'(x^*) < -1$	*unstable*	*oscillating*

Example

Problem (5.5.3)

Consider the DTDS: $c_{t+1} = .75c_t + 1.25$. *Find and analyze the equilibria.*

Solution

To find equilibria, solve:

$$c^* = .75c^* + 1.25$$

Algebra $\implies$ *unique equilibrium:* $c^* = 5.00$.
To classify, compute $f'(5.00)$.

$$f'(c) = .75 \implies f'(5.00) = .75$$

This number is between 0 and 1, so equilibrium is stable and monotone.

Harder Example

Problem (5.5.7)

Given: per capita production is $\frac{2}{1+.001x}$. *Find and analyze the equilibria.*

Solution

Updating function: $f(x)$*=[per capita production]*x* $= \frac{2x}{1+.001x}$.
To find equilibria, solve:

$$x^* = \frac{2x^*}{1 + .001x^*}$$

One solution: $x^* = 0$. *Else:* $1 = \frac{2}{1+.001x^*}$.

$$\implies 1 + .001x^* = 2 \implies .001x^* = 1 \implies x^* = 1000$$

Equilibria are $\boxed{x^* = 0}$ *and* $\boxed{x^* = 1000}$ *Now to analyze them...*

Harder Example: Continued

Problem (5.5.7)

Given: per capita production is $\frac{2}{1+.001x}$. *Find and analyze the equilibria.*

Solution

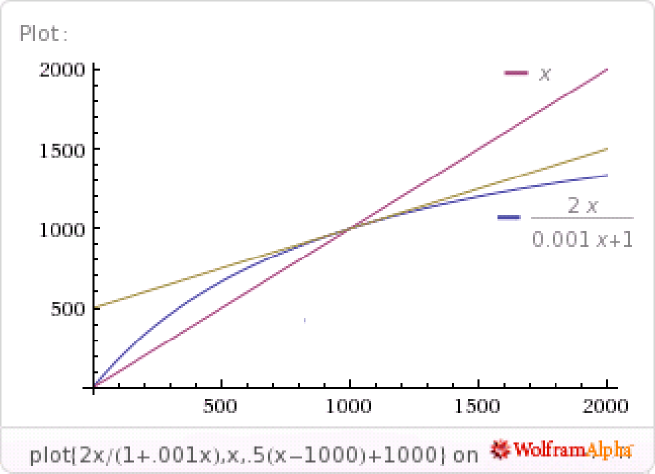
We found: equilibria are $x^* = 0$ *and* $x^* = 1000$ *for* $f(x) = \frac{2x}{1+.001x}$.
To classify, compute $f'(0)$ *and* $f'(1000)$.

$$f'(x) = \frac{2(1+.001x) - 2x(.001)}{(1+.001x)^2} = \frac{2}{(1+.001x)^2}$$

$f'(0) = 2$, *so* 0 *is an unstable, monotone equilibrium.*
$f'(1000) = \frac{2}{2^2} = 0.5$, *so* 1000 *is a stable, monotone equilibrium.*

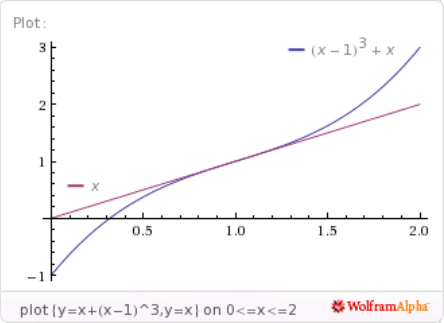
Note: this example was graphed a few slides back (and next slide).

Harder Example: Stable Equilibrium at $x^* = 1000$



Suppose $f(x)$ is **increasing** near an equilibrium point x^*.

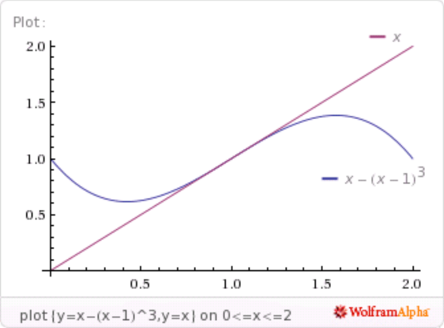
The following technique works even if $f'(x^*) = 1$:



From below $y = x$ to above $y = x$

Acts like slope > 1

Unstable

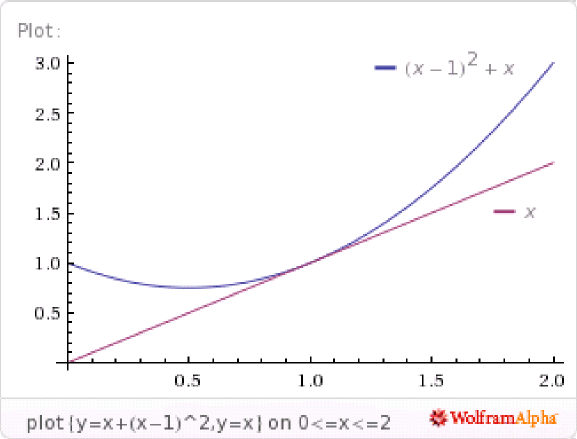


From above $y = x$ to below $y = x$

Acts like slope < 1

Stable

A Half-Stable Equilibrium



From above to above (pictured): stable on left, unstable on right.
From below to below (not shown): unstable on left, stable on right.

Logistic DTDS

$$x_{t+1} = rx_t(1 - x_t)$$

Read cobwebbing discussion on p. 380 and study Examples 5.6.1–5.6.4 on different r values at home. (Also study example 5.6.5).

Problem

What are the equilibrium values?

Solution

$$x^* = rx^*(1 - x^*)$$

One solution: $\boxed{x^* = 0}$. *Else:* $1 = r(1 - x^*)$

$$\implies \frac{1}{r} = 1 - x^* \implies \boxed{x^* = 1 - \frac{1}{r}.}$$

Logistic DTDS: Oscillating, Stable Equilibria

Problem

For which values of r does logistic DTDS have oscillating, stable equilibrium at $1 - \frac{1}{r}$?

Solution

Must solve $-1 < f'(x) < 0$.
$f(x) = rx(1-x) = rx - rx^2 \implies f'(x) = r - 2rx$. *So solve:*

$$-1 < r - 2r\left(1 - \frac{1}{r}\right) < 0$$

$$\iff -1 < r - 2(r-1) < 0 \iff -1 < 2 - r < 0 \iff 2 < r < 3$$

Good exercise: check that $r = 3$ has a half-stable equilibrium.

Optional:Ricker Model

Problem

Analyze the equilibria for the Ricker Model $f(x) = rxe^{-x}$.

Model says per capita production decays exponentially.

DiffEq Overview: Thought Experiment

You're in a car that travels along x-axis in the positive direction. You have:

- Speedometer
- Stopwatch
- Notebook, Pencil, Coffee

Can you accurately determine:

1. Distance traveled in 1 hour?
2. x-coord at time 1 hour?

You want to *integrate* speed.
You want to solve the differential equation "$\frac{dx}{dt}$ =speed" for $x(t)$.
For question 2, you need to know $x(0)$ ("initial condition").

Differentiation and Integration

Integration is the **inverse operation** of differentiation.

Examples:

Position x	Speed dx/dt
Mass m	(Mass) Growth Rate dm/dt
Population Size P	(Population) Growth Rate dP/dt
Amount of Sodium in Cell n	Rate of Sodium Entry to Cell dn/dt
Area of Circle A	Perimeter of Circle dA/dr

$$\xrightarrow{\text{Differentiate (Take Derivative)}}$$

$$\xleftarrow{\text{Integrate (Solve Diff. Eq.)}}$$

Differential Equations: Terminology

- **Differential Equation**: an equation involving derivative(s) of a function
  - Example: $f''(t) = tf'(t) + f(t)$ or $\frac{dy}{dx} = y^2$.
- **Pure-Time DiffEq**: When t is independent variable, expresses $f'(t)$ in terms of t alone
  - Example: $f'(t) = t^2$ or $\frac{dx}{dt} = 1 - e^t$.
- **Autonomous DiffEq**: When t is independent variable, expresses $f'(t)$ in terms of $f(t)$ alone
  - Example: $f'(t) = f(t)^2$ or $\frac{dy}{dt} = y + 1$.
- **Solution of a DiffEq**: A function making the two sides of a DiffEq equal *as functions*.
  - Example: $y = \sin(x) + 3$ is a solution to $y' = \cos(x)$.
- **Initial Condition**: Extra requirement a solution to a diffeq must satisfy.
  - Example: $f(0) = 3$

Pure-Time or Autonomous

The thought-experiment with the car is which kind of DiffEq? Pure-Time

Discrete-Time Dynamical Systems, given by an updating function, most resemble which kind of DiffEq? Autonomous

Verifying a Solution

Problem

Check that the diff. eq. $y'' = -y$ *with initial conditions* $y(0) = 3, y'(0) = 1$ *has* $y = 3\cos(x) + \sin(x)$ *as a solution.*

Solution

Left Hand Side:
$y'' = (3\cos x + \sin x)'' = (-3\sin x + \cos x)' = -3\cos x - \sin x.$
Right Hand Side: $-y = -3\cos x - \sin x.$
LHS=RHS, so $y = 3\cos(x) + \sin(x)$ *satisfies diff. eq.*✓

$y(0) = 3\cos(0) + \sin(0) = 3.$ $y'(0) = -3\sin(0) + \cos(0) = 1.$
$y = 3\cos(x) + \sin(x)$ *satisfies initial conditions.* ✓

Solving an Autonomous DiffEq

Looking at a diffeq, you can often tell some things about solutions right away.

Problem (6.1.4)

If $P'(t) = 0.0092P(t)$ models biological population growth, does the population increase or decrease? What's the concavity?

Solution

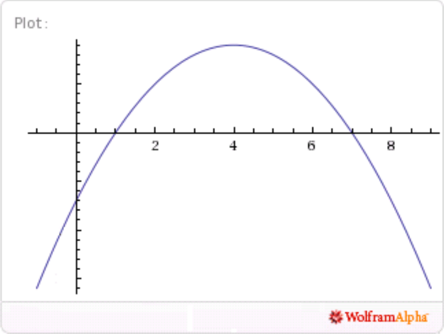
Since $P(t)$ is a biological population, $P(t) \geq 0$.
So $P'(t) = 0.0092P(t) \geq 0$. So $P(t)$ (increases/decreases)? increases

For concavity, compute $P''(t) = 0.0092P'(t) \geq 0$. So P is concave up.

Can you think of a kind of increasing, concave up function that's proportional to its rate of change? Exp. growth: $P = Ce^{rt}$.
You could sub $P(t) := Ce^{rt}$ into the diff. eq. and solve for r.

Problem (6.1.6)

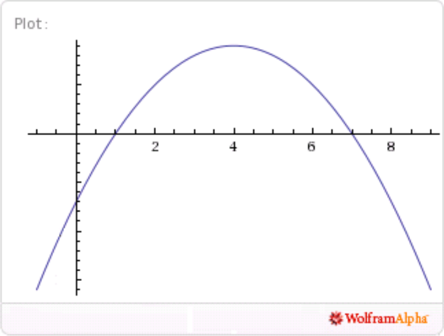
Given the graph of $M'(t)$ on the left, and the initial condition $M(5) = 10$, graph $M(t)$.

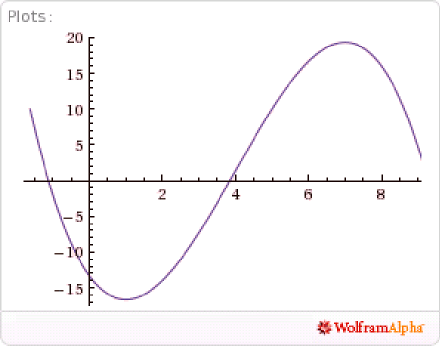


Graphing a Pure-Time Diff Eq solution

Problem (6.1.6)

Given the graph of $M'(t)$ on the left, and the initial condition $M(5) = 10$, graph $M(t)$.





Constructing an Autonomous DiffEq Model

Autonomous DiffEqs modelling real-world problems often come from thinking about what the derivative should be proportional to.

Problem (Assignment 18.6(b))

At time $t = 0$, Dr. Baker starts spreading a rumour on the McMaster campus. Assume 15,000 students on campus. $S(t)$ is the number who have heard the rumour by time t. The rate at which S grows should be proportional to:

- *The number of people who have heard it, AND*
- *The number of people who have not heard it.*

Solution

$S'(t) = k \cdot S(t) \cdot (15000 - S(t))$.
Initial condition: $S(0) = 1$.

Why should $S'(t)$ be proportional to these two things?

DiffEqs and Stability

Problem (p.417 #36)

Caribou population diff. eq.: $P'(t) = 2P(t)\left(1 - \frac{P(t)}{2500}\right)$, $P(t) > 0$.
Find a constant solution. What does this solution tell us?

Solution

If $P(t) = c$ *is constant, then* $P'(t) = 0$. *So:*

$$0 = 2P(t)\left(1 - \frac{P(t)}{2500}\right)$$

$P(t) \neq 0$, *so* $1 - \frac{P(t)}{2500} = 0$.
Thus $P(t) = 2500$. *If you check, this actually is a solution.*

- *2500 is an equilibrium for the population.*

DiffEqs and Stability

Problem (p.417 #36)

Caribou population diff. eq.: $P'(t) = 2P(t)\left(1 - \frac{P(t)}{2500}\right)$, $P(t) > 0$. *We saw 2500 is an equilibrium. Is* $P(t)$ *increasing/decreasing for* $P(t) > 2500$*? For* $P(t) < 2500$*? What does this say about the equilibrium?*

Solution

If $P(t) > 2500$*, then*

$P'(t) = 2P(t)\left(1 - \frac{P(t)}{2500}\right)$ *=(pos)(pos)(neg)=negative. So:*

- *P decreases if it's more than 2500.*

If $0 \leq P(t) < 2500$*, then* $P'(t) =$*(pos)(pos)(pos)*> 0*. So:*

- *P increases if it's less than 2500.*

In summary, 2500 is a stable equilibrium.