Math 1LS3 Week 8: Advanced Derivative Applications

Owen Baker

McMaster University

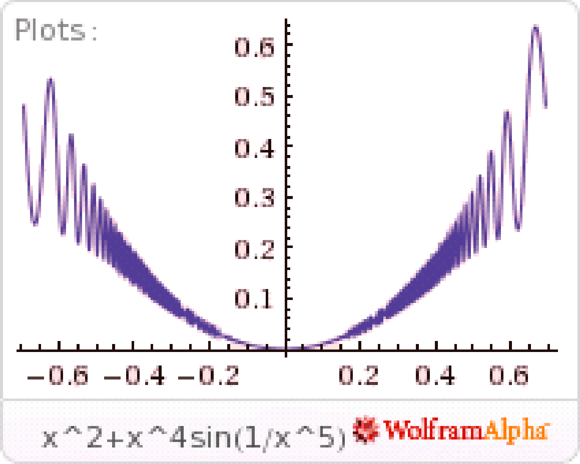
Oct. 29–Nov. 2, 2012

Outline

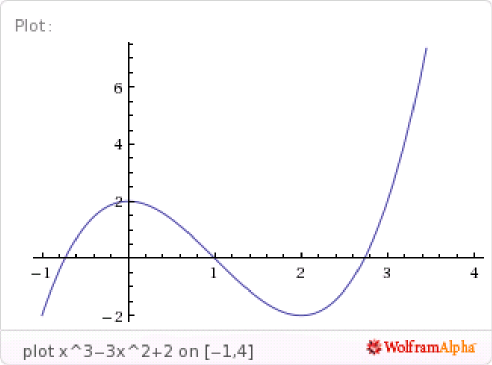
This week, we will cover 5.1 (absolute extreme values) and 5.3 (leading behaviour and l'Hôpital's rule).
Next week is 5.5,5.6, start of ch. 6.
Work through all solved examples in section 5.3 in order to practice leading behaviour.

An Example Where First Derivative Test Fails

$$y = x^2 + x^4 \sin\left(\frac{1}{x^5}\right)$$



Global vs Local Extrema



$f(x) = x^3 - 3x^2 + 2$ with domain [-1,4].

- f has an absolute (global) max at $x = a$ if $f(a) \geq f(b)$ for all b in f's domain.
- f has a relative (local) max at $x = a$ if $f(a) \geq f(b)$ for all b in f's domain *near* a.

Caution: by fiat, endpoints are not allowed to be relative max/min.

Extreme Value Theorem

Theorem (Extreme Value Theorem)

Every continuous function with domain a closed interval $[a, b]$ attains a global maximum and a global minimum value.

Every global max (or min) is either:

- a local max (or min), and hence a critical point.
- an endpoint

To find the absolute extreme values, just list the critical points and endpoints and see which has the biggest/smallest function value.

Example

Does e^x attain a minimum value? Which value? A maximum?

Absolute Extrema: Example

Problem

Find the global max/min for $f(x) = x^3 - 3x^2 + 2$ *with domain [-1,4].*

Solution

$$f'(x) = 3x^2 - 6x$$

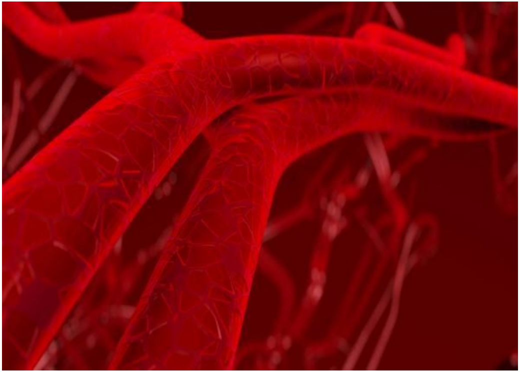
Find the critical points and endpoints.

- *Where does* $f'(x)$ *does not exist? (just technically endpoints)*
- *Where does* $f'(x) = 0$*?* $x = 0$ *and* $x = 2$*.*

The critical points are: $x = 0, x = 2$*, the endpoints:* $x = -1, x = 4$*.*
Function values: $f(0) = 2$*,* $f(2) = -2$*,* $f(-1) = -2$*,* $f(4) = 18$*.*
Absolute max value: 18 (at $x = 4$*).*
Absolute min value: -2 (at $x = -1$ *AND* $x = 2$*).*

Why didn't we need to use the first/second derivative test?

vascular branching

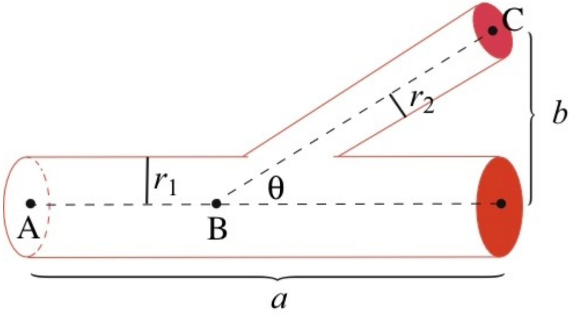


Poiseuille's Law:

resistance R is proportional to length of the tube L and inversely proportional to the fourth power of its radius r

$$R = C\frac{L}{r^4}$$

vascular branching:
objective: minimize resistance, thus minimizing energy (i.e. minimizing work the heart needs to produce)



Optimal Vascular Branching

$$R(\theta) = C\left(\frac{a - b\cot\theta}{r_1^4} + \frac{b\csc\theta}{r_2^4} \right)$$

critical point:

$$\cos q = \frac{r_2^4}{r_1^4}$$

Optimal Vascular Branching

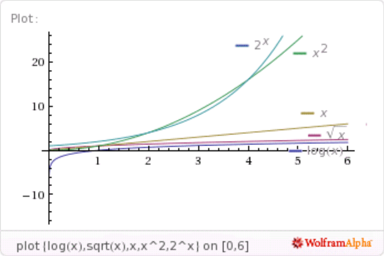
ratio of sizes of vessels (ratio of their radii)	optimal branching angle $\theta = \arccos \frac{r_2^4}{r_1^4}$
0.9	0.85 rad = 49.0 degrees
0.8	1.15 rad = 65.8 degrees
0.67=2/3	1.37 rad = 78.6 degrees
0.5	1.51 rad = 86.4 degrees

Leading Behaviour at Infinity

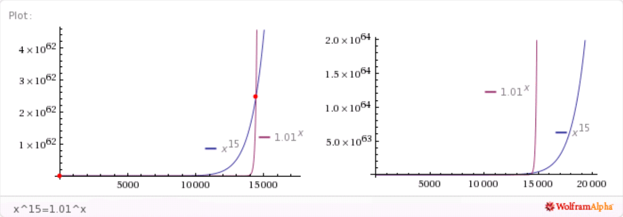
Here are many functions $f(x)$ with

$$\lim_{x\to\infty} f(x) = \infty.$$

$$2^x \gg x^2 \gg x \gg \sqrt{x} \gg \ln(x)$$



Any exponential eventually beats *any* polynomial:



This tells us long term behaviour of their ratios:

$$\lim_{x\to\infty} \frac{e^x}{15x^4+29x^3} = \lim \frac{realbig}{big} = \infty \qquad \lim_{x\to\infty} \frac{500x^{999}}{1.000001^x} = \lim \frac{big}{realbig} = 0$$

Leading Behaviour at Infinity

- Leading behaviour at ∞=long-term behaviour=asymptotic behaviour
- Only the dominant term(s) matters
- Write $f_\infty(x)$ for leading behaviour of $f(x)$ as $x \to \infty$.

Example

If $f(x) = 500 * 2^x - \frac{1}{2} * 2.1^x + x^{100} - x^{55} + 3$, then $f_\infty(x) = -\frac{1}{2} * 2.1^x$.

Example

If $g(x) = \frac{2+3x+5x^2}{1-x^3}$, then $g_\infty(x) = \frac{5x^2}{-x^3} = -\frac{5}{x}$.

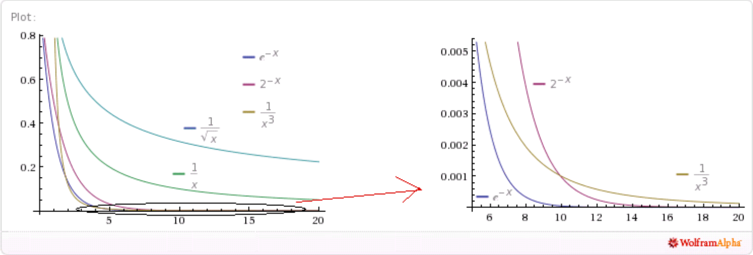
Notation: $\gg$ means *dominates*

$$2.1^x \gg 2^x \gg x^{100} \gg x^{55} \gg x^0 = 1$$

In the next few slides, we'll review dominant terms.

Functions that Approach Zero at Infinity

Here are many functions $g(x)$ with $\lim_{x\to\infty} g(x) = 0$.



$$e^{-x} \ll 2^{-x} \ll x^{-3} \ll \frac{1}{x} \ll \frac{1}{\sqrt{x}}$$

This tells us long term behaviour of their ratios:

$$\lim_{x\to\infty} \frac{e^{-x}}{x^{-2}} = \lim \frac{crazysmall}{small} = 0 \qquad \lim_{t\to\infty} \frac{\frac{1}{t}}{\frac{1}{t^2}} = \lim \frac{small}{crazysmall} = +\infty$$

Summary: Dominant Terms at Infinity

Dominant term goes to infinity faster (or to zero slower)

- $a^x \gg x^b$ ($a > 1$, any b) (at ∞) exp. dominates poly.
- $a^x \gg b^x$ if $a > b$ (at ∞) exp. with bigger base dominates
- $x^a \gg x^b$ if $a > b$ (at ∞) power with bigger degree dominates
- $x^a \gg \ln(x)$ if $a > 0$ (at ∞) power dominates log
- $a^x, x^b, \ln(x) \gg 1$ ($a > 1, b > 0$) (at ∞) exp.,pow.,log dominates constant
- $1 \gg x^{-b}, a^x$ ($a < 1, b > 0$) (at ∞) constant dominates exp./power decay
- $x^b \gg a^x$ ($a < 1$, any b) (at ∞) power decay dominates exp. decay

Work through ALL solved examples in section 5.3.

Leading Behaviour near Zero

- Different terms dominate near zero than near ∞
- Again, only the dominant term(s) matters
- Write $f_0(x)$ for leading behaviour near 0.

Example

If $f(x) = x^{100} - x^{55} + 3$, then $f_0(x) = 3$.

Example

If $g(x) = \frac{2x+3x^2+5x^3}{3-x^3}$, then $g_0(x) = \frac{2x}{3}$.

Notation: $\gg$ means *dominates* (*location* depends on context)

$$1 = x^0 \gg x^{55} \gg x^{100} \qquad \text{(near zero)}$$

Dominant Terms Near Zero

Dominant term goes to infinity faster (or to zero slower)

- Exponential Functions: 3^x is near 1 for x near 0
- Powers: $1 \gg \sqrt{x} \gg x \gg x^2 \gg x^3$, etc. (reverse of near ∞)

Near zero or infinity, only the *absolute value (size)* of the term matters in determining what dominates.

Example

- If $f(x) = 3x^2 - 4x^5 + x^7$, then $f_0(x) = 3x^2$ and $f_\infty(x) = x^7$.
- If $g(x) = 2e^x + 5x^2$, then $g_0(x) = 2$ and $g_\infty(x) = 2e^x$.

Example of Leading Behavior: Absorption Functions

How much of a chemical in the air does the lung absorb?

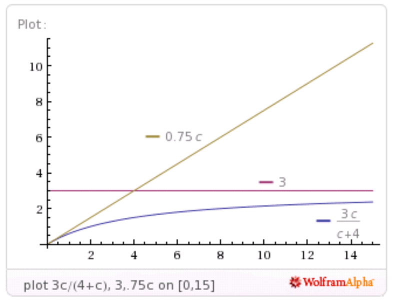
- Depends on the concentration of chemical
- But how? Model: absorption function $\alpha(c)$
- Next few slides: behaviour at ∞ and near 0 for some absorption fns

Example of Leading Behavior: Absorption Functions

Example (Absorption Function with Saturation (5.3.11))

$$\alpha(c) = \frac{Ac}{k+c}$$

Leading behaviour at infinity: $\alpha_\infty(c) = \frac{Ac}{c} = A$
Leading behaviour at zero: $\alpha_0(c) = \frac{Ac}{k} = \frac{A}{k}c$.



Behave like a constant near $\infty \leftrightarrow$ horizontal asymptote

Behaviour near zero approximated by tangent line to α at $(0,0)$

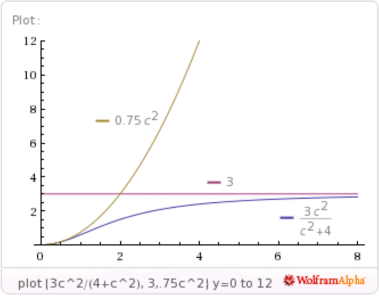
Example of Leading Behavior: Absorption Functions

Example (Saturation with Threshold (5.3.12))

$$\alpha(c) = \frac{Ac^2}{k + c^2}$$

Leading behaviour at infinity: $\alpha_\infty(c) = \frac{Ac^2}{c^2} = A$

Leading behaviour at zero: $\alpha_0(c) = \frac{Ac^2}{k} = \frac{A}{k}c^2$.



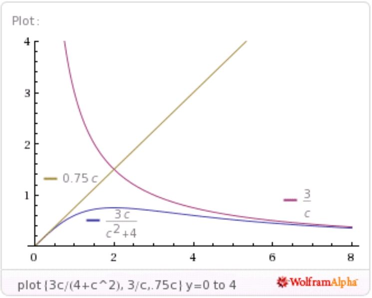
Behave like a constant near ∞ $\leftrightarrow$ horizontal asymptote

Approximated by parabola near $(0, 0)$

Example (Saturation with Overcompensation II (5.3.13))

$$\alpha(c) = \frac{Ac}{k + c^2}$$

Leading behaviour at infinity: $\alpha_\infty(c) = \frac{Ac}{c^2} = \frac{A}{c}$
Leading behaviour at zero: $\alpha_0(c) = \frac{Ac}{k} = \frac{A}{k}c.$



Behaves like hyperbola near infinity

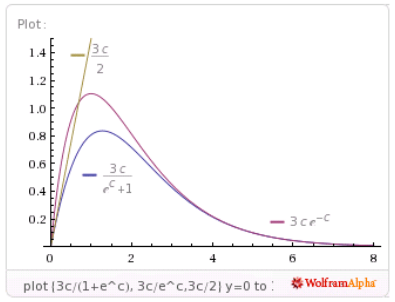
Behaves like tangent line to $(0, 0)$ near $c = 0$

Example of Leading Behavior: Absorption Functions

Example (Saturation with Overcompensation (5.3.14))

$$\alpha(c) = \frac{Ac}{1 + e^c}$$

Leading behaviour at infinity: $\alpha_\infty(c) = \frac{Ac}{e^c}$
Leading behaviour at zero: $\alpha_0(c) = \frac{Ac}{1+1} = \frac{A}{2}c.$



Rapid decay near infinity

Behaves like tangent line to $(0, 0)$ near $c = 0$.

l'Hôpital's Rule: Intro

Use derivatives to help compute limits.

Theorem (l'Hôpital)

If

$$\lim_{x\to a}\frac{f(x)}{g(x)}$$

is of the form $\frac{0}{0}$ *or* $\frac{\infty}{\infty}$ *then:*

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Why does it work? Taylor polynomials + Leading Behavior

l'Hôpital Example

Problem

Find $\lim_{x \to 0} \frac{\sin(x)}{x}$.

Solution

Try directly subbing: 0/0. *Apply l'Hôpital:*

$$\lim_{x \to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} \frac{\sin'(x)}{x'} = \lim_{x \to 0} \frac{\cos(x)}{1} = \cos(0) = \boxed{1}$$

l'Hôpital's Rule only works for *indeterminate forms*.
Fortunately, we only need it for indeterminate forms.
Always try direct substitution first!

List of indeterminate forms:

- $\frac{0}{0}$
- $\frac{\pm\infty}{\pm\infty}$
- $0 \cdot \pm\infty$
- $\infty - \infty$
- 1^{∞}
- 0^{0}
- ∞^{0}

Not indeterminate: $0/\pm\infty$, $17/0$, $\pm\infty/0$, $0^{\pm\infty}$.

More l'Hôpital examples

Problem

Evaluate:

$$\lim_{x \to 0} \frac{e^{2x} - 2x - 1}{x^2}$$

Solution

$$\lim_{x \to 0} \frac{e^{2x} - 2x - 1}{x^2} = \lim_{x \to 0} \frac{2e^{2x} - 2}{2x} = \lim_{x \to 0} \frac{4e^{2x}}{2} = \frac{4 \cdot e^0}{2} = \boxed{2}$$

More l'Hôpital examples

Problem

Evaluate:

$$\lim_{x \to \frac{\pi}{2}} \frac{\cot(x)}{\cot(3x)}$$

Solution

$$\lim_{x \to \frac{\pi}{2}} \frac{\cot(x)}{\cot(3x)} = \lim_{x \to \frac{\pi}{2}} \frac{-\csc^2(x)}{-3\csc^2(3x)} = \frac{-1}{-3} = \boxed{\frac{1}{3}}$$

Indeterminate Product

For indeterminate forms other than $\frac{0}{0}$ or $\frac{\infty}{\infty}$, use algebra to express the limit in one of these two forms.

Problem

Find $\lim_{x \to 0^+} x^2 \ln(x)$.

Solution

$$\lim_{x \to 0^+} x^2 \ln(x) = \lim_{x \to 0^+} \frac{\ln(x)}{x^{-2}} = \lim_{x \to 0^+} \frac{\ln'(x)}{(x^{-2})'} = \lim_{x \to 0^+} \frac{\frac{1}{x}}{-2x^{-3}}$$

$$= \lim_{x \to 0^+} -\frac{1}{2}x^2 = \boxed{0}$$

For indeterminate forms other than $\frac{0}{0}$ or $\frac{\infty}{\infty}$, use algebra to express the limit in one of these two forms.

Problem

Evaluate $\lim\limits_{x\to\infty}\left[\ln(2x^2+1)-\ln(x^2-17x)\right]$

Solution

$$\lim_{x\to\infty}\left[\ln(2x^2+1)-\ln(x^2-17x)\right]=\lim_{x\to\infty}\ln\left(\frac{2x^2+1}{x^2-17x}\right)$$

$$=\ln\left(\lim_{x\to\infty}\frac{2x^2+1}{x^2-17x}\right)=\ln\left(\lim_{x\to\infty}\frac{4x}{2x-17}\right)=\ln\left(\lim_{x\to\infty}\frac{4}{2}\right)=\boxed{\ln(2)}$$

Indeterminate Exponential: Compound Interest

Problem

Evaluate $\lim_{n\to\infty} \left(1+\frac{r}{n}\right)^{nt}$.

Solution

$$L = \lim_{n\to\infty} \left(1+\frac{r}{n}\right)^{nt}$$

$$\ln(L) = \lim_{n\to\infty} \ln\left(\left(1+\frac{r}{n}\right)^{nt}\right) = \lim_{n\to\infty} nt\ln\left(1+\frac{r}{n}\right) = \lim_{n\to\infty} \frac{t\log\left(1+\frac{r}{n}\right)}{n^{-1}}$$

$$= t\lim_{n\to\infty} \frac{\left(1+\frac{r}{n}\right)^{-1}\cdot -rn^{-2}}{-n^{-2}} = t\lim_{n\to\infty} r\left(1+\frac{r}{n}\right)^{-1} = tr\left(1+\frac{r}{\infty}\right)^{-1} = rt$$

$$L = \lim_{n\to\infty} \left(1+\frac{r}{n}\right)^{nt} = e^{rt}$$ *Don't forget this step!*

Hence formula for compound interest: $P = P_0e^{rt}$.