**COMPSCI/SFWRENG 2FA3**

**Discrete Mathematics with Applications II**

**Winter 2020**

# Assignment 3 with Solutions

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Assignment 3 consists of four problems. You must write your solutions to the problems using LaTeX.

Please submit Assignment 3 as two files, `Assignment_3_`*`YourMacID`*`.tex` and `Assignment_3_`*`YourMacID`*`.pdf`, to the Assignment 3 folder on Avenue under Assessments/Assignments. *`YourMacID`* must be your personal MacID (written without capitalization). The `Assignment_3_`*`YourMacID`*`.tex` file is a copy of the LaTeX source file for this assignment (`Assignment_3.tex` found on Avenue under Contents/Assignments) with your solution entered after each problem. The `Assignment_3_`*`YourMacID`*`.pdf` is the PDF output produced by executing

```
pdflatex Assignment_3_YourMacID
```

This assignment is due **Sunday, February 9, 2020 before midnight.** You are allow to submit the assignment multiple times, but only the last submission will be marked. **Late submissions and files that are not named exactly as specified above will not be accepted!** It is suggested that you submit your preliminary `Assignment_3_`*`YourMacID`*`.tex` and `Assignment_3_`*`YourMacID`*`.pdf` files well before the deadline so that your mark is not zero if, e.g., your computer fails at 11:50 PM on February 9.

**Although you are allowed to receive help from the instructional staff and other students, your submission must be your own work. Copying will be treated as academic dishonesty! If any of the ideas used in your submission were obtained from other students or sources outside of the lectures and tutorials, you must acknowledge where or from whom these ideas were obtained.**

## Background

Let $\Sigma = (\mathcal{B}, \mathcal{C}, \mathcal{F}, \mathcal{P}, \tau)$ be a finite signature of MSFOL, $F_\Sigma$ be the set of $\Sigma$-formulas, and $A \in F_\Sigma$. Recall that the members of $F_\Sigma$ are certain strings of symbols. A *subformula* of $A$ is a $B \in F_\Sigma$ such that $B$ is a substring of $A$. For example, let $A$ be the formula $((0 = 2) \wedge (3 \mid 4))$, i.e., $A$ is the string "$((0 = 2) \wedge (3 \mid 4))$". Then "$(0 = 2)$", "$(3 \mid 4)$", and "$((0 = 2) \wedge (3 \mid 4))$" are the subformulas of $A$, and "$(0 =$ " and "$\wedge$" are two substrings of $A$ that are not subformulas of $A$.

A function $f : A \rightarrow B$ is *total* if it is defined on *all* members of $A$. A function $f : A \rightarrow B$ is a *partial* if it is be undefined on *some* members of $A$. For example, the square root function $\sqrt{\cdot} : \mathbb{R} \rightarrow \mathbb{R}$ is a partial function since $\sqrt{r}$ is undefined for all $r \in \mathbb{R}$ with $r < 0$. If $f, g : A \rightarrow B$ are partial or total functions, then $f$ is a *subfunction* of $g$, written $f \sqsubseteq g$, if the domain $D_f$ of $f$ is a subset of the domain of $g$ and, for all $x \in D_f$, $f(x) = g(x)$. In other words, $f$ is a subfunction of $g$ if $g(a)$ is defined and $f(a) = g(a)$ whenever $f(a)$ is defined.

## Problems

1. **[10 points]** Let $\mathsf{subformulas} : F_\Sigma \rightarrow \mathcal{P}(F_\Sigma)$ be the function that maps a formula $A \in F_\Sigma$ to the set of subformulas of $A$. Define $\mathsf{subformulas}$ by structural recursion using pattern matching.

   **Put your name, MacID, and date here.**

   **Solution**:

   We define $\mathsf{subformulas}$ by structural recursion using pattern matching as follows:

   a. $\mathsf{subformulas}((s = t)) = \{(s = t)\}$.
   b. $\mathsf{subformulas}(p(t_1, \ldots, t_n)) = \{p(t_1, \ldots, t_n)\}$.
   c. $\mathsf{subformulas}(\neg A) = \{\neg A\} \cup \mathsf{subformulas}(A)$.
   d. $\mathsf{subformulas}((A \Rightarrow B)) =$

   $$\{(A \Rightarrow B)\} \cup \mathsf{subformulas}(A) \cup \mathsf{subformulas}(B).$$

   e. $\mathsf{subformulas}((\forall\, x : \alpha \,.\, A)) = \{(\forall\, x : \alpha \,.\, A)\} \cup \mathsf{subformulas}(A)$.

2. **[10 points]** Suppose $F$ is the set of partial and total functions $f : \mathbb{N} \rightarrow \mathbb{N}$.

   a. Show that $(F, \sqsubseteq)$ is a weak partial order but not a weak total order.
   b. Describe the set of minimal elements of $(F, \sqsubseteq)$.
   c. Describe the set of maximal elements of $(F, \sqsubseteq)$.
   d. Does $(F, \sqsubseteq)$ have a minimum element? If so, what is it?

e. Does $(F, \sqsubseteq)$ have a maximum element? If so, what is it?

**Put your name, MacID, and date here.**

**Solution**:

a. *Proof.* We must show that $(F, \sqsubseteq)$ is reflexive, antisymmetric, and transitive.

   *Reflexivity.* Obviously, $f \sqsubseteq f$ for any $f \in F$, so $\sqsubseteq$ is reflexive.

   *Antisymmetry.* Let $f, g \in F$ such that $f \sqsubseteq g$ and $g \sqsubseteq f$. Then (1) $D_f \subseteq D_g$ and $D_g \subseteq D_f$ and so $D_f = D_g$ and (2) for all $x \in D_f$, $f(x) = g(x)$. Therefore, $f = g$ and so $\sqsubseteq$ is antisymmetric.

   *Transitivity.* Let $f, g, h \in F$ such that $f \sqsubseteq g$ and $g \sqsubseteq h$. Then (1) $D_f \subseteq D_g$ and $D_g \subseteq D_h$ and so $D_f \subseteq D_h$ and (2) for all $x \in D_f$, $f(x) = g(x)$ and for all $x \in D_g$, $g(x) = h(x)$ and so for all $x \in D_f$, $f(x) = h(x)$. Therefore, $f \sqsubseteq h$ and so $\sqsubseteq$ is transitive.

   Therefore, $(F, \sqsubseteq)$ is a weak partial order. $\square$

b. Let $e$ be the empty function. Then $D_e = \emptyset$. Obviously, $e \sqsubseteq f$ for all $f \in F$. Therefore, the set of minimal elements is $\{e\}$.

c. Let $t$ be a total function in $F$. Then $D_t = \mathbb{N}$. Obviously, there is no $f \in F$ except $t$ itself such that $t \sqsubseteq f$. Therefore, the set of maximal elements is the infinite set of total functions in $F$.

d. Since the empty function is the only minimal element, it is the minimum element in $F$.

e. Since there is more than one total function in $F$, there is no maximum element in $F$.