MATH 1B03/1ZC3 Winter 2019

# Lecture 8: Determinants I

Instructor: Dr Rushworth January 29th

## Determinants via cofactor expansion

(from Chapter 2.1 of Anton-Rorres)

Matrices encode information. Often we don't need all of the information contained in a matrix, and wish to extract a certain part of it.

An example of this is trace of a matrix: given a square matrix $A$, the trace $tr(A)$ is a number containing some information about $A$.

A more important number we can extract from a matrix is the determinant. We have actually seen the determinant of a $2 \times 2$ matrix already: if

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then the determinant of $A$ is

$$det(A) = ad - bc$$

Recall that $A$ is invertible if and only if $ad - bc \neq 0$. This result extends to square matrices of any size: a matrix is invertible if and only if it has non-zero determinant. For this and other reasons the determinant is an important quantity in linear algebra.

We already know how to compute the determinant of $2 \times 2$ matrices, and we will use this to compute the determinant of larger matrices. Given an $n \times n$ matrix, we compute its determinant in the following way:

1. break the matrix down into a collection of $(n-1) \times (n-1)$ matrices
2. break those down further into $(n-2) \times (n-2)$ matrices
3. keep breaking down until we produce a collection of $2 \times 2$ matrices

4. reassemble of the determinant of the $n \times n$ matrix from the collection of $2 \times 2$ determinants

(This is an example of mathematical induction.)

**Definition 8.1: Determinant**

Let $A$ be a square matrix. The determinant of $A$ is written $det(A)$ or $|A|$. It is a number.

**Definition 8.2: Minors and Cofactors**

Let $A$ be an $n \times n$ matrix. Denote by $A[i, j]$ the matrix formed from $A$ by deleting the $i$-th row and the $j$-th column.
The $ij$-th minor of $A$ is the number

$$M_{i,j} := det(A[i, j]).$$

The $ij$-th cofactor of $A$ is the number

$$C_{i,j} := (-1)^{i+j} M_{i,j}.$$

Warning: do not confuse the minor $M_{i,j}$ with the notation $(M)_{ij}$ for entries of a matrix.

To find $A[i, j]$, cover the $i$-th row and $j$-th column, then write down the remaining matrix. For example, if

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

then $A[2, 3]$ is found by considering

$$A = \begin{bmatrix} 1 & 2 & \text{[redacted]} \\ \text{[redacted]} & & \\ 7 & 8 & \end{bmatrix}$$

and

$$A[2, 3] = \begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix}$$

**Example 8.3**

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \quad A[1,1] = \begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix}, \quad A[2,3] = \begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix}$$

$$\begin{aligned} M_{1,1} &= det(A[1,1]) \\ &= det\left(\begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix}\right) \\ &= 45 - 48 \\ &= -3 \end{aligned} \qquad \begin{aligned} M_{2,3} &= det(A[2,3]) \\ &= det\left(\begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix}\right) \\ &= 8 - 14 \\ &= -6 \end{aligned}$$

$$\begin{aligned} C_{1,1} &= (-1)^{1+1} M_{1,1} \\ &= (-1)^2(-3) \\ &= -3 \end{aligned} \qquad \begin{aligned} C_{2,3} &= (-1)^{2+3} M_{2,3} \\ &= (-1)^5(-6) \\ &= 6 \end{aligned}$$

Note that there are more minors and cofactors of $A$ to compute.

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$$B = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \quad B[4,1] = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

but we don't know how to compute $M_{4,1}$ yet.

**Question 8.4**

Compute the remaining minors and cofactors of $A$ in the example above.

Using minors and cofactors we can compute the determinant of matrices larger than $3 \times 3$. We are going to compute the determinant of larger matrices by computing lots of $2 \times 2$ determinants.

## Fact 8.5: Cofactor expansion

Let $A = [a_{ij}]$ be an $n \times n$ matrix, and $C_{i,j}$ its cofactors. Then $det(A)$ can be found via cofactor expansion along the $i$-th row

$$\begin{aligned} det(A) &= \sum_{k=1}^{n} a_{ik} C_{i,k} \\ &= a_{i1} C_{i,1} + a_{i2} C_{i,2} + \cdots + a_{in} C_{i,n} \end{aligned}$$

or via cofactor expansion along the $j$-th column

$$\begin{aligned} det(A) &= \sum_{k=1}^{n} a_{kj} C_{k,j} \\ &= a_{1j} C_{1,j} + a_{2j} C_{2,j} + \cdots + a_{nj} C_{n,j} \end{aligned}$$

The idea: as $A$ is $n \times n$, the cofactors $C_{i,j}$ are determinants of $(n-1) \times (n-1)$ matrices. To compute these determinants, we apply cofactor expansion again, and obtain determinants of $(n-2) \times (n-2)$ matrices. We keep applying cofactor expansion until we hit $2 \times 2$ determinants, which we know how to compute!

Its important to note that it doesn't matter which row or column we expand along: we will always arrive at the same answer.

## Example 8.6

In the $3 \times 3$ case the formula for expansion along the $i$-th row is

$$det(A) = a_{i1} C_{i,1} + a_{i2} C_{i,2} + a_{i3} C_{i,3}$$

If we expand along the first row (so that $i = 1$) this becomes

$$det(A) = a_{11} C_{1,1} + a_{12} C_{1,2} + a_{13} C_{1,3}$$

Lets use this to compute $det(A)$ for

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & 0 & 5 \end{bmatrix}$$

The formula becomes

$$
\begin{aligned}
det(A) &= 1(-1)^{1+1}\begin{vmatrix}1 & 2\\ 0 & 5\end{vmatrix} + 2(-1)^{1+2}\begin{vmatrix}0 & 2\\ 1 & 5\end{vmatrix} + 3(-1)^{1+3}\begin{vmatrix}0 & 1\\ 1 & 0\end{vmatrix}\\
&= (-1)^2 5 + 2(-1)^3(-2) + 3(-1)^4(-1)\\
&= 5 + 4 - 3\\
&= 6
\end{aligned}
$$

Lets compute the determinant again, expanding along the $1$-st column:

$$
\begin{aligned}
det(A) &= 1(-1)^{1+1}\begin{vmatrix}1 & 2\\ 0 & 5\end{vmatrix} + 0(-1)^{1+2}\begin{vmatrix}2 & 3\\ 0 & 5\end{vmatrix} + 1(-1)^{1+3}\begin{vmatrix}2 & 3\\ 1 & 2\end{vmatrix}\\
&= 5 + 1\\
&= 6
\end{aligned}
$$

What about a $4 \times 4$? We have to keep expanding. Expanding along the first row:

$$
\begin{aligned}
det\left(\begin{bmatrix}1 & 0 & 1 & 0\\ 0 & 2 & 1 & -1\\ 1 & 0 & 2 & -1\\ 0 & 1 & -1 & 0\end{bmatrix}\right) =& 1(-1)^{1+1}\begin{vmatrix}2 & 1 & -1\\ 0 & 2 & -1\\ 1 & -1 & 0\end{vmatrix} + 0(-1)^{1+2}\begin{vmatrix}0 & 1 & -1\\ 1 & 2 & -1\\ 0 & -1 & 0\end{vmatrix}\\
&+ 1(-1)^{1+3}\begin{vmatrix}0 & 2 & -1\\ 1 & 0 & -1\\ 0 & 1 & 0\end{vmatrix} + 0(-1)^{1+4}\begin{vmatrix}0 & 2 & 1\\ 1 & 0 & 2\\ 0 & 1 & -1\end{vmatrix}
\end{aligned}
$$

to complete this determinant we need to repeat the process on the remaining $3 \times 3$ matrices.

The difficulty of computing matrix determinants grows very fast with the size of the matrix. In fact, the computation of the determinant of an $n \times n$ matrix requires $n! = n(n-1)(n-2)\cdots(2)(1)$ individual computations.

For example, a $5{\times}5$ determinant requires $120$ calculations, and a $6{\times}6$ determinant requires $720$ calculations!

This is an example of a task in linear algebra very well suited to computers, but not so well suited to humans.

## Chequerboard pattern of signs

There is a nice way of determining the signs $(-1)^{i+j}$ in the cofactor expansion. The following chequerboard pattern of $+$ and $-$ signs can be used to speed up calculations.

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

This pattern extends to matrices of any size.

## Spotting nice rows and columns

Another way to speed up the calculation of determinants is by spotting particularly nice rows and columns to expand along. In earlier examples we encountered terms like

$$0(-1)^{2+3}\begin{vmatrix} 1 & 0 \\ 3 & 2 \end{vmatrix}$$

which we could ignore (due to the $0$). In general, we should always pick the row or column with the most $0$'s as entries.

### Example 8.7

Let

$$A = \begin{bmatrix} 2 & 0 & 0 & -1 \\ 1 & -1 & 0 & -1 \\ 4 & 1 & 2 & 5 \\ 1 & 0 & 0 & 2 \end{bmatrix}$$

Expanding along the $3$-rd column:

$$\begin{aligned} det(A) &= 2det\left(\begin{bmatrix} 2 & 0 & -1 \\ 1 & -1 & -1 \\ 1 & 0 & 2 \end{bmatrix}\right), \text{ then expand along } 2\text{-nd column} \\ &= 2\left(-1det\left(\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\right)\right) \\ &= 2(-1(4+1)) \\ &= -10 \end{aligned}$$

(Note that we have used the chequerboard pattern of signs to speed up this calculation.)

This trick can be used to prove the following fact.

## Fact 8.8: Determinant of a triangular matrix

Let $A = [a_{ij}]$ be a triangular square matrix (either upper or lower). Then

$$det(A) = a_{11}a_{22}\cdots a_{nn}$$

i.e. it is the product of the diagonal entries.

**Proof:** Assume that $A$ is lower triangular (the proof for the upper triangular case is very similar). Then

$$A = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & 0 & \cdots & a_{nn} \end{bmatrix}$$

Expanding about the $1$-st row we obtain

$$det(A) = a_{11} det\left(\begin{bmatrix} a_{22} & 0 & \cdots & 0 \\ a_{32} & a_{33} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}\right), \text{ and again}$$

$$= a_{11} a_{22} det\left(\begin{bmatrix} a_{33} & 0 & \cdots & 0 \\ a_{43} & a_{44} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ a_{n3} & a_{n4} & \cdots & a_{nn} \end{bmatrix}\right)$$

Repeatedly expanding about the first row, we obtain

$$det(A) = a_{11} a_{22} \cdots a_{nn}.$$

■

Recall that a diagonal matrix is upper triangular, so that this fact allows us to quickly find the determinant of diagonal matrices, too. In particular we can use it to show that

$$det(I_n) = 1.$$

**Example 8.9**

**Question**: Compute the determinant of

$$A = \begin{bmatrix} 1 & 8 & 12 & 3 & 17 \\ 0 & 7 & 41 & -2 & 0 \\ 0 & 0 & 4 & -5 & 3 \\ 0 & 0 & 0 & 2 & 7 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

**Answer**: $det(A) = 1 \cdot 7 \cdot 4 \cdot 2 \cdot 1 = 56$, as $A$ is triangular.

Taking the transpose of a matrix does not change the determinant.

**Fact 8.10: Determinant of the transpose**

Let $A$ be a square matrix. Then

$$det(A) = det(A^T)$$

**Proof:** The columns of $A$ are the rows of $A^T$ (and vice versa). We have seen that $det(A)$ can be computed by expanding along any row or column, from which the result follows. ■

## Determinants via row reduction

(from Chapter 2.2 of Anton-Rorres)

Determinants are extremely important in the study of matrices, and it is therefore no surprise that multiple ways to compute them have been developed. In addition to cofactor expansion, we can compute the determinant of a matrix using elementary row operations.

**Fact 8.11**

Let $A$ be a square matrix. If $A$ has a row or column of $0$'s, then $det(A) = 0$.

**Proof:** Simply use cofactor expansion along the row or column of $0$'s. ■

In addition to this fact, we also saw that the determinant of an upper triangular matrix was very easy to find (recall Theorem 8.8).

In previous lectures we saw that we can convert any matrix into RREF using Gauss-Jordan elimination, and that a matrix in RREF is automatically upper triangular.

Therefore, if we understand the affect of elementary row operations on the determinant, we will have another way to compute the determinant of a matrix.

**Fact 8.12: Determinants via row reduction**

Let $A$ and $B$ be square matrices of the same size, related by exactly one elementary row operation.

1. If $B$ is obtained from $A$ by swapping two rows, then

$$det(B) = -det(A)$$

E.g.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = - \begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

2. If $B$ is obtained from $A$ by multiplying a row by the scalar $\lambda$, then

$$det(B) = \lambda det(A)$$

E.g.

$$\begin{vmatrix} \lambda a_{11} & \lambda a_{12} & \lambda a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \lambda \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

3. If $B$ is obtained from $A$ by adding a multiple of one row to another row, then

$$det(B) = det(A)$$

E.g.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} + a_{21} & a_{12} + a_{22} & a_{13} + a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

**Question 8.13**

Prove these facts using cofactor expansion.

The idea is to use elementary row operations to speed up calculation of the determinant, either by finding a row or column of $0$'s, or by converting the matrix into an upper triangular matrix.

As was the case when we were using elementary row operations to solve SLEs, Gauss-Jordan elimination will always yield a solution: follow the recipe to convert the matrix into RREF, keeping track of how you are changing the determinant, and you will eventually arrive at an answer.

However, if you can spot a faster way to produce a row or column of $0$'s, or to produce an upper triangular matrix, do it.

### Example 8.14

**Question**: Compute the determinant of the following matrix using row reduction

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 0 & 5 \end{bmatrix}$$

**Answer**:

$$\begin{aligned}
\begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 0 & 5 \end{vmatrix} &\overset{R1\leftrightarrow R2}{=} -\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & 0 & 5 \end{vmatrix} \\
&\overset{-R1+R3}{=} -\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -2 & 2 \end{vmatrix} \\
&\overset{2R2+R3}{=} -\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 6 \end{vmatrix} \\
&= -6, \quad \text{as the matrix is upper triangular}
\end{aligned}$$

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**Question**: Compute the determinant of the following matrix using row reduction

$$B = \begin{bmatrix} 0 & 4 & -3 \\ 2 & -2 & 0 \\ -1 & -3 & 3 \end{bmatrix}$$

**Answer**:

$$\begin{aligned}
\begin{vmatrix} 0 & 4 & -3 \\ 2 & -2 & 0 \\ -1 & -3 & 3 \end{vmatrix} &\overset{R1+R3}{=} \begin{vmatrix} 0 & 4 & -3 \\ 2 & -2 & 0 \\ -1 & 1 & 0 \end{vmatrix} \\
&= 2\begin{vmatrix} 0 & 4 & -3 \\ 1 & -1 & 0 \\ -1 & 1 & 0 \end{vmatrix} \\
&\overset{R2+R3}{=} 2\begin{vmatrix} 0 & 4 & -3 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{vmatrix} \\
&= 0, \text{ as there is a row of 0's}
\end{aligned}$$

Recall that

$$det(A) = det(A^T)$$

(we saw this in Theorem 8.10). We can use this fact to speed up computing the determinant via row reduction.

## Example 8.15

**Question**: Compute the determinant of the following matrix using row reduction

$$A = \begin{bmatrix} 3 & 6 & 5 & -9 \\ -1 & -2 & 8 & 3 \\ 4 & 5 & 1 & -12 \\ -5 & 2 & 4 & 15 \end{bmatrix}$$

**Answer**:

$$\begin{vmatrix} 3 & 6 & 5 & -9 \\ -1 & -2 & 8 & 3 \\ 4 & 5 & 1 & -12 \\ -5 & 2 & 4 & 15 \end{vmatrix} \overset{\text{take transpose}}{=} \begin{vmatrix} 3 & -1 & 4 & -5 \\ 6 & -2 & 5 & 2 \\ 5 & 8 & 1 & 4 \\ -9 & 3 & -12 & 15 \end{vmatrix}$$

$$\overset{3R1+R4}{=} \begin{vmatrix} 3 & -1 & 4 & -5 \\ 6 & -2 & 5 & 2 \\ 5 & 8 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

$$= \quad 0, \text{ as there is a row of } 0\text{'s}$$

The following fact is also very useful to speed up determinant calculations.

**Fact 8.16**

Let $A$ be a square matrix. If $A$ has a column which is a scalar multiple of another column, or a row which is a scalar multiple of another row. Then $det(A) = 0$.

**Question 8.17**

Prove this fact.

## Suggested Problems

Practice the material covered in this lecture by attempting the following questions from Chapter 2.1 of Anton-Rorres, starting on page $111$

- Questions $3, 17, 21, 25, 31, 36$

and from Chapter 2.2, starting on page $117$

- Questions $9, 13, 17, 21, 26$

- True/False questions (a), (b), (c), (e)