MATH 1B03/1ZC3 Winter 2019

Lecture 12: Diagonalization continued

Instructor: Dr Rushworth February 26th

THIS MATERIAL IS NOT TESTABLE ON MIDTERM 1 ON FEBRUARY 25TH.

Diagonalization continued

(from Chapter 5.2 of Anton-Rorres)

In the previous lecture we that the matrix

$$B = \begin{bmatrix} 4 & 1 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & -2 \end{bmatrix}$$

has eigenvalues $\lambda_1 = -2$, $\lambda_2 = 2$, and $\lambda_3 = 3$. As B is 3×3 and has 3 distinct eigenvalues it is diagonalizable. The eigenvectors can be computed using the method we saw in Lecture 10. They are

$$\text{eigenvalue}: \quad \lambda_1 = -2 \quad \lambda_2 = 2 \quad \lambda_3 = 3$$

$$\text{basis:} \qquad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 4 \\ -8 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 5 \\ -5 \\ 1 \end{bmatrix}$$

Therefore the matrix

$$P = \begin{bmatrix} 0 & 4 & 5 \\ 0 & -8 & -5 \\ 1 & 1 & 1 \end{bmatrix}$$

diagonalizes B: that is

$$D = P^{-1}BP$$

is a diagonal matrix. What are the elements on the diagonal of D?

Using the inversion algorithm we find that P^{-1} is given by

$$P^{-1} = \begin{bmatrix} -\frac{3}{20} & \frac{1}{4} & 1 \\ -\frac{1}{4} & \frac{1}{4} & 1 \\ \frac{2}{5} & 0 & 0 \end{bmatrix}$$

Lets compute the product $P^{-1}BP$:

$$\begin{aligned} D &= P^{-1}BP \\ &= \begin{bmatrix} -\frac{3}{20} & \frac{1}{4} & 1 \\ -\frac{1}{4} & \frac{1}{4} & 1 \\ \frac{2}{5} & 0 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} 0 & 4 & 5 \\ 0 & -8 & -5 \\ 1 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{3}{20} & \frac{1}{4} & 1 \\ -\frac{1}{4} & \frac{1}{4} & 1 \\ \frac{2}{5} & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 8 & 15 \\ 0 & -16 & -15 \\ -2 & 2 & 3 \end{bmatrix} \\ &= \begin{bmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \end{aligned}$$

The eigenvalues of B have appeared on the diagonal of D! This always happens, as described in the following fact.

Fact 12.1

Let A be a diagonalizable matrix, with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Let $\mathbf{x}_i$ be the eigenvector associated to the eigenvalue λ_i. Let

$$P = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

be the matrix of eigenvectors.
Then P diagonalizes A so that $A = PDP^{-1}$, where the diagonal matrix D

has the form

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

That is, it is the matrix with the eigenvalues of A on the diagonal, and 0's elsewhere.

Notice that the order in which the eigenvectors appear in P will be the order the eigenvalues appear in D.

We can use this fact to compute powers of matrices quickly.

Computing matrix powers via diagonalization

Let A be a square matrix. To compute A^k we need to compute $k - 1$ individual matrix products. For example,

$$\begin{aligned} A^2 &= AA, \text{ one product} \\ A^4 &= AAAA, \text{ three products} \\ A^{10} &= AAAAAAAAAA, \text{ nine products} \end{aligned}$$

If A is diagonalizable, we can compute any power A^k using only three matrix products.

To see this, let A be diagonalizable with

$$A = PDP^{-1}.$$

Using Fact 12.1 we see that D has the form

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A. Using Fact 7.8 from Lecture 7, we see that

$$D^k = \begin{bmatrix} \lambda_1^k & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2^k & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n^k \end{bmatrix}$$

Now consider A^k

$$\begin{aligned} A^k &= \left(PDP^{-1}\right)^k \\ &= \underbrace{PDP^{-1}PDP^{-1}\cdots PDP^{-1}}_{k \text{ times}} \\ &= PD^kP^{-1}, \text{ as } P^{-1}P = I \end{aligned}$$

The matrix D^k is very easy to compute (as we have seen above). Therefore we can compute A^k via

$$A^k = PD^kP^{-1}$$

using only three matrix products.

This method is summarised as follows.

Recipe 12.2: Computing the powers of diagonalizable matrix

Let A be a diagonalizable matrix. Follow these steps to compute A^k.

Step 1: Compute the eigenvalues and eigenvectors of A. Form the matrix P, and find P^{-1}.

Step 2: The matrix D has the eigenvalues of A on its diagonal. Compute the power D^k.

Step 3: The matrix A^k is found by computing $A^k = PD^kP^{-1}$.

Example 12.3:

Given the matrix

$$B = \begin{bmatrix} 4 & 1 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & -2 \end{bmatrix}$$

compute B^5.

Answer: Previously in this lecture we saw that B has eigenvalues $\lambda_1 = -2$, $\lambda_2 = 2$, and $\lambda_3 = 3$, and is diagonalized by the matrix

$$P = \begin{bmatrix} 0 & 4 & 5 \\ 0 & -8 & -5 \\ 1 & 1 & 1 \end{bmatrix}$$

where

$$P^{-1} = \begin{bmatrix} -\frac{3}{20} & \frac{1}{4} & 1 \\ -\frac{1}{4} & \frac{1}{4} & 1 \\ \frac{2}{5} & 0 & 0 \end{bmatrix}.$$

Therefore we have

$$B = PDP^{-1}$$

where

$$D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Then

$$D^5 = \begin{bmatrix} -32 & 0 & 0 \\ 0 & 32 & 0 \\ 0 & 0 & 243 \end{bmatrix}$$

and we can compute

$$\begin{aligned} B^5 &= PD^5P^{-1} \\ &= \begin{bmatrix} 0 & 4 & 5 \\ 0 & -8 & -5 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -32 & 0 & 0 \\ 0 & 32 & 0 \\ 0 & 0 & 243 \end{bmatrix} \begin{bmatrix} -\frac{3}{20} & \frac{1}{4} & 1 \\ -\frac{1}{4} & \frac{1}{4} & 1 \\ \frac{2}{5} & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 454 & 211 & 0 \\ -422 & -179 & 0 \\ 94 & 39 & -32 \end{bmatrix} \end{aligned}$$

The following fact is related to this method of computing matrix powers using diagonalization.

Fact 12.4

Let A be a square matrix and k a positive integer. If λ is an eigenvalue of A with associated eigenvector $\mathbf{x}$, then λ^k is an eigenvalue of A^k, with associated eigenvector $\mathbf{x}$.

Notice that the eigenvalues are raised to the power of k, while the eigenvectors remain the same.

This fact allows us to determine the eigenvalues and eigenvectors of A^k from those of A.