Math 1LS3 Week 2: Exponents, Logs, Trig Review

Owen Baker

McMaster University

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This week covers sections 0.3, 1.2 and 1.3 of the textbook.
We won't have time to cover all the material in class. Assignments 3 and 4 cover many examples of exponential, log, trig and inverse trig questions. As well, assignment 5 checks your understanding of concepts from chapters 0 and 1.

Converting Celsius to Fahrenheit

Problem

Celsius to Fahrenheit conversion is given by the formula:

$$T_F = \frac{9}{5}T_C + 32$$

where T_C denotes Celsius temperature and T_F denotes Fahrenheit temperature. Find a formula for converting Fahrenheit to Celsius.

Solution

$$T_F - 32 = \frac{9}{5}T_C$$

$$\boxed{T_C = \frac{5}{9}(T_F - 32).}$$

What we've done is invert the function $T_C \mapsto T_F$.

Inverse Functions

Function $y = f(x)$. Given x, can compute y.

Inverse problem: Given y, solve for x. (Notation: $x = f^{-1}(y)$).

Problem

*Find the **inverse function** of $f(x) = x^3 - 1$.*

Solution

$y = x^3 - 1 \iff y + 1 = x^3 \iff x = \sqrt[3]{y+1}$.

*The **inverse function** is* $\boxed{f^{-1}(y) = \sqrt[3]{y+1}}$.

- Confusingly, we can also write this as $f^{-1}(x) = \sqrt[3]{x+1}$.
- Protip: don't change variable names without a good reason. You might forget what they mean.

A subtlety on notation

The inverse of $T_F = \frac{9}{5}T_C + 32$ is $T_C = \frac{5}{9}(T_F - 32)$

- In terms of different variables: T_C and T_F.

On the other hand, we write $f(x) = x^3$ and its inverse $f^{-1}(x) = \sqrt[3]{x}$.

- In terms of same variable: x.

Moral: when working with physically meaningful variables, invert by solving for the desired variable. Don't swap the variable names.

Thinking in Half-Lives

- A *half-life* is the amount of time it takes for half of the starting amount to remain.

Used for:

- Radioactive decay
- Drug remaining in body
- Any other exponential decay model

# Half–Lives	Percentage Remaining
0	100
1	50
2	25
3	12.5
4	6.25
5	3.125
6	1.5625

	Half-life
Tetrahydrocannabinol … Marijuana (infrequent users)	1.3-3 days
Marijuana (frequent users)	1-10 days
Marijuana (if taken orally as pills)	25-36 hours
Marijuana (smoking/inhaling)	1.6-59 hours
LSD (Lysergic acid diethylamide)	3-5 hours
MDMA … ecstasy Methylenedioxymethamphetamine	6-10 hours
Caffeine adults	4-5 hours
Caffeine infants	10-20 hours
Caffeine with oral contraceptives	5-10 hours
Caffeine (if pregnant)	9-11 hours
Caffeine (liver disease)	several days
Codeine (Tylenol 3)	3-6 hours
Demerol (pain killer)	3-5 hours
Morphine (pain killer)	2-3 hours

Doubling times of breast cancer

age	median doubling time in days (doubling time interval)
< 50	80 (44 - 147)
50-70	157 (121 - 204)
> 70	188 (120 - 295)

T = very large … ductal cancer in situ

source:PG Peer et al, *Age-dependent growth rate of primary breast cancer.* Cancer. 1993 Jun 1; 71(11): 3547-51

Doubling Times of Breast Cancer

According to your textbook, it takes 3 doublings to get from the time when a tumour is first detectable by mammogram to the time when it can be detected in a clinical breast exam.

Using the doubling time of 157 days for 50-70 year olds, this corresponds to $3 \cdot 157 = 471$ days of advance knowledge.

Radiocarbon dating, Half-life (p.69)

Problem

$^{14}\boldsymbol{C}$ *has a half-life of 5730 years. Analysis of wooden artifacts shows they contain 29.7 percent of the* $^{14}\boldsymbol{C}$ *in a living tree. How old are they?*

Solution

Radioactive decay fits an exponential model:

$$C(t) = C(0)e^{kt}$$

- t *is the amount of time after the trees died*
- $C(t)$ *is the* $^{14}\boldsymbol{C}$ *percentage at time* t
- k *is rate depending only on the particular isotope (e.g.* $^{14}\boldsymbol{C}$ *)*

Our first step is to find k *using the half-life.*

Radiocarbon dating; Half-life (p.69)

Solution

Our first step is to find k *in* $C(t) = C(0)e^{kt}$ *using half-life* $T = 5730yr$.

$$\frac{1}{2}C(0) = C(0)e^{kT}$$

$$1/2 = e^{kT}$$

$$\ln(1/2) = \ln(e^{kT}) = kT$$

$$k = \frac{\ln(1/2)}{T} = \frac{\ln(1/2)}{5730} \approx -0.00012097$$

The exponential model for $^{14}\mathbf{C}$ *decay is thus* $C(t) = C(0)e^{-0.00012097t}$.

Radiocarbon dating; Half-life (p.69)

Given: wooden artifacts contain 29.7 percent of the $^{14}\mathbf{C}$ levels of a living tree.

Solution

We obtained the model $C(t) = C(0)e^{-0.00012097t}$.
Assume that modern trees have the same $^{14}\boldsymbol{C}$ *levels as the trees from which the artifact was made.*
Then $C(T_{now}) = .297C(0)$ *where* T_{now} *is the present time.*

$$.297C(0) = C(0)e^{-0.00012097T_{now}}$$

$$\ln(.297) = \ln e^{-0.00012097T_{now}} = -.00012097T_{now}$$

The age is therefore $T_{now} = \ln(.297)/-.00012097 \approx 10,000$ *years.*

Is this a reasonable answer?

Yes. Slightly over a quarter of $^{14}\mathbf{C}$ left; age is almost two half-lives.

Log Properties (p. 64 of your text)

Memorize and be able to use them **fluently.**

$$\log_c(1) = 0$$

$$\log_c(ab) = \log_c(a) + \log_c(b)$$

$$\log_c(a/b) = \log_c(a) - \log_c(b)$$

$$\log_c(1/a) = -\log_c(a)$$

$$\log_c(a^b) = b\log_c(a)$$

$$\log_b(a) = \log_c(a)/\log_c(b)$$

If you need the blue hints on the next slide, practice these!

Do Section 1.2 textbook exercises until you're very comfortable with properties.

Another Look at Half–Life: Solve for Age

In exponential decay, if T is the half–life, then:

$$\text{fraction remaining} = \left(\frac{1}{2}\right)^{\text{Age}/T}$$

Writing this as a logarithm:

$$\log_{1/2}(\text{fraction remaining}) = \text{Age}/T$$

"Change of base": $\log_b c = \ln(c)/\ln(b)$ gives:

$$\frac{\ln(\text{fraction remaining})}{\ln(1/2)} = \text{Age}/T$$

Property $\ln(1/a) = -\ln(a)$ gives:

$$\frac{\ln(\text{fraction remaining})}{-\ln(2)} = \text{Age}/T.$$

$$\boxed{\text{Age} = -\frac{\ln(\text{fraction remaining})\,T}{\ln(2)}}$$

How old are dinosaur fossils? (p.70)

Problem

Modern tests can't distinguish between zero $^{14}\boldsymbol{C}$ *in a sample and* $< 0.0005C(0)$. *Can carbon dating be used to find the age of dinosaur fossils? (Dinosaurs went extinct 65 million years ago.)*

Solution

$$Age = -\frac{\ln(fraction\ remaining)T}{\ln(2)}$$

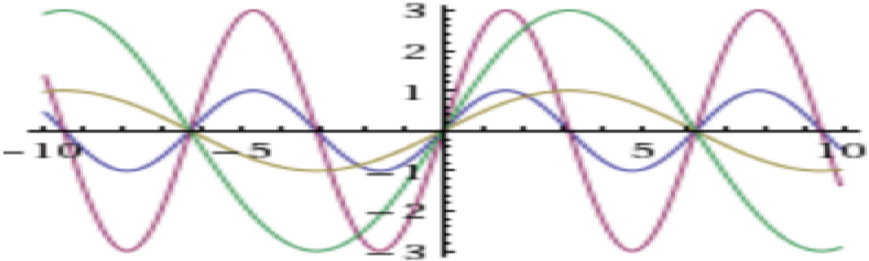
The oldest sample we can date is where $fraction\ remaining$*=.0005.*

$$Age = -\frac{\ln(.0005)\cdot 5730}{\ln 2} \approx 63,000 years \ll 65,000,000 years$$

No, we need something with a longer half–life, e.g. $^{40}\boldsymbol{K}$ *.*

Note: no need to memorize formulas for half–life, doubling time, age, etc.

How to Stretch a Graph Vertically/Horizontally



Starting from the blue curve $y = f(x)$:

- $y = f(x)$ is the original curve.
- $y = 3f(x)$ is blue curve vertically stretched by a factor of 3.
- $y = f(x/2)$ is blue curve horizontally stretched by a factor of 2.
- $y = 3f(x/2)$ is vertically and horizontally stretched.

Why? If a point (a, b) lies on the blue curve, then $b = f(a)$, so: (a, b) and $(a, 3b)$ and $(2a, b)$ and $(2a, 3b)$ must lie on their respective curves.

Problem

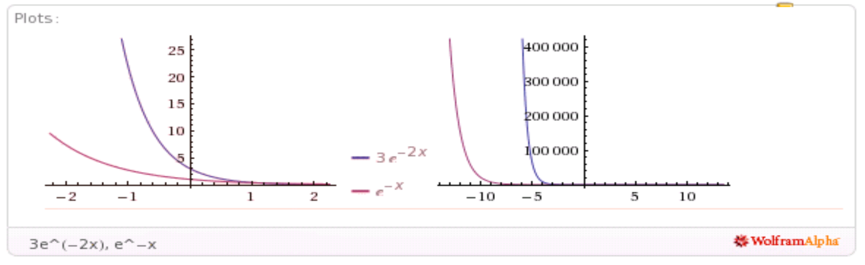
Graph $y = 3 * e^{-2t}$ and describe its behaviour.

Solution

$y = 3e^{-2t}$ *is obtained from* $y = e^{-t}$ *by*

- *stretching by 3 vertically*
- *compressing by 2 horizontally*

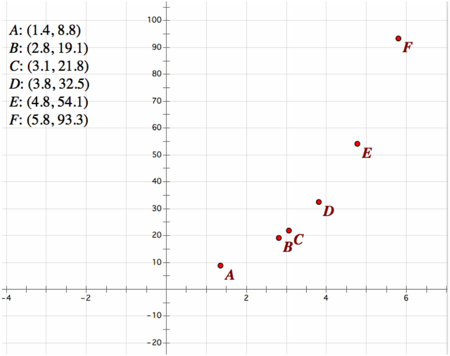
This is an exponential decay graph. It decreases for all time, approaching the horizontal asymptote $y = 0$.

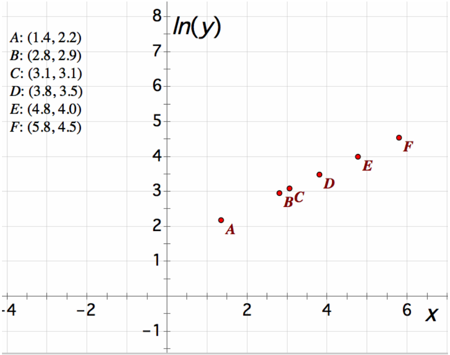


Semilog plots (p.72-74)

Goal: find explicit **model** to fit this data.

What kind of function model to use? Exponential





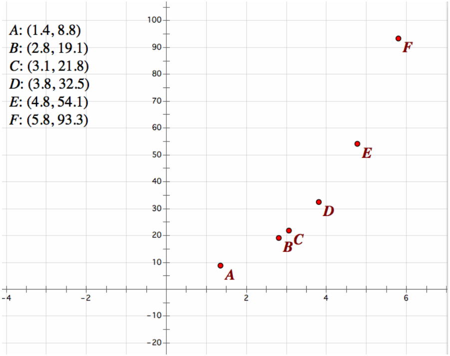
Transform data into a line:

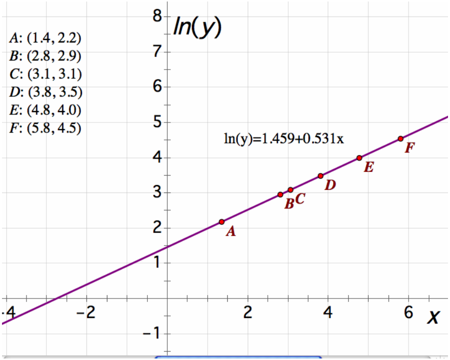
- Plot x coordinates on horizontal axis
- Plot ***logs*** of y coords on vertical axis

Semilog plots (p.72-74)

Goal: find explicit **model** fitting data.

What kind of function model fits this data? Exponential





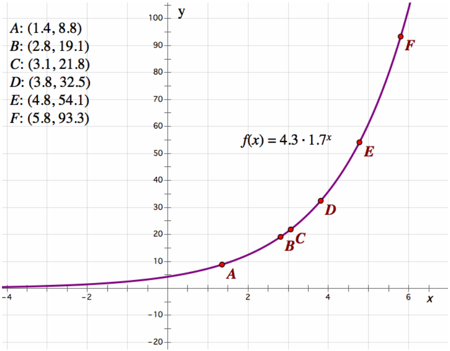
Draw a line through data. Measure slope, intercept.

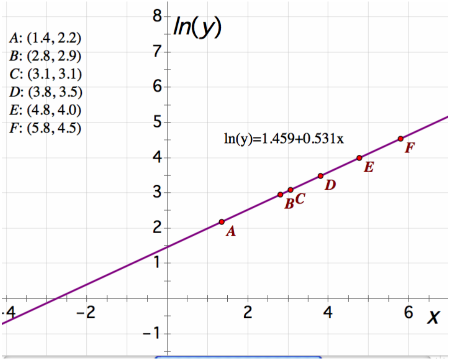
$$\ln(y) = 1.459 + 0.531x$$

Semilog plots (p.72-74)

Goal: find explicit **model** fitting data.

What kind of function model fits this data? Exponential

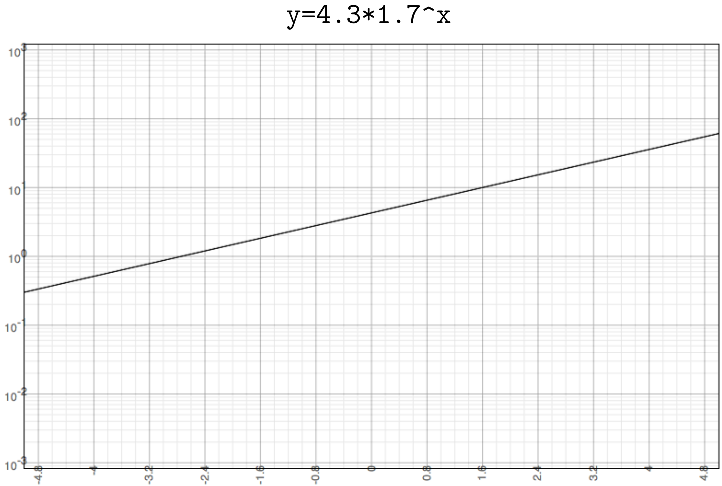




Solve for y to get model for original data set.

$$\ln(y) = 1.459 + 0.531x \implies e^{\ln(y)} = e^{1.459+0.531x} \implies \boxed{y = 4.3 \cdot 1.7^x}$$

Semilog Paper



Semilog paper exponentially distorts the y–axis.

Semilog–Plot Summary

Suppose you have a data set you think fits an exponential model.

1. Plot the data on a semilog graph.
2. Find a line $\ln(y) = mx + b$ that fits the data.
3. Then $y = e^{mx+b} = e^b \cdot e^{mx}$.

The *slope* of the line gives the *growth rate* of the exponential curve.

Similarly, log–log plots are used to fit data to curves $y = c * x^m$.

1. Plot $(\ln(x), \ln(y))$ for each data point (x, y).
2. Find a line $\ln(y) = m\ln(x) + b$ fitting the data.
3. Then $y = e^b \cdot x^m$.

The slope of the line is the power of the power curve!

Order of graph transformations

The *first* operation performed on x is the *last* transformation to be applied to the graph.

Examples

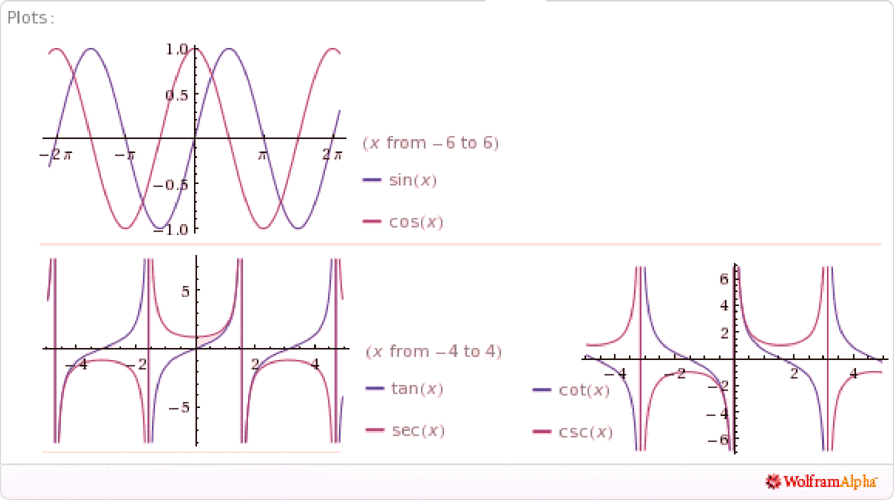
- $y = f\left(\frac{x-a}{b}\right)$. Stretch horizontally by b. Then translate right by a.
- $y = f\left(\frac{x}{b} - a\right)$. Translate right by a. Then stretch horizontally by b.
- $y = f\left(\frac{-(x-a)}{b}\right)$. Stretch. Then flip over y-axis. Then translate.
- $y = f\left(\frac{-x-a}{b}\right)$. Stretch. Translate. Then flip.

The *first* operation performed on y is the *first* transformation to be applied to the graph.

Examples

- $y = a + bf(x)$. Stretch vertically by b. Then translate up a.
- $y = b(a - f(x))$. Flip over x-axis. Translate up. Stretch vertically.

Trig functions



Memorize these. Be able to sketch any of them quickly.

Oscillation terminology

A *sinusoid* is the result of transforming $y = \sin(x)$

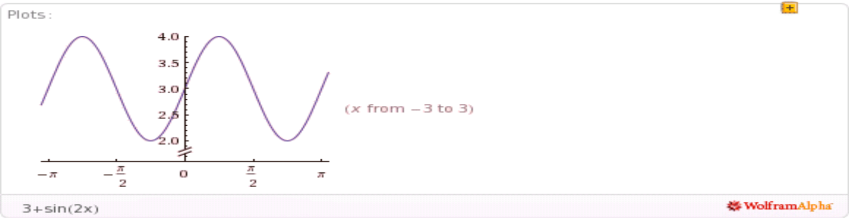
- by translations, reflections, vertical stretches, horizontal stretches.

Note: $y = \cos(x)$ is a sinusoid.

Terms describing sinusoids:

- The **minimum** is the smallest y-value achieved.
- The **maximum** is the largest y-value achieved.
- The **average** is halfway between min, max.
- The **amplitude** is the (vertical) distance between average and either extreme.
- The **period** is the time between successive peaks.
- The **frequency** is the reciprocal of the period.
- The **phase** is how much sine or cosine is shifted to the right <u>after</u> horizontally stretching. [Caution: nonstandard definition.]

Oscillation terminology



- The maximum is 4.
- The minimum is 2.
- The amplitude is 1.
- The average is 3.
- The period is π.
- The phase (as a sine curve) is 0.
- The phase (as a cosine curve) is $\pi/4$.

Transforming a trig graph(see p.83)

Starting with the graph $y = f(x)$, suppose we stretch:

- vertically by A
- horizontally by B

and *then* translate

- vertically by u
- horizontally by v.

The resulting graph is $y = u + Af(\frac{x-v}{B})$.

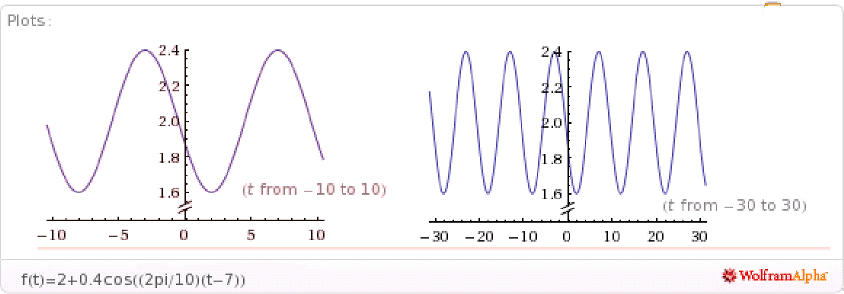
Problem

Plot:

$$f(t) = 2 + 0.4 \cos\left(\frac{2\pi}{10}(t-7)\right)$$

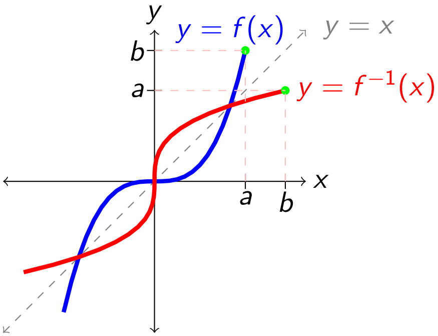
$f(t) = 2 + 0.4\cos((2\pi/10)(t-7))$

- The graph is vertically stretched by 0.4, so the amplitude is 0.4
- The graph is horizontally stretched by $10/(2\pi)$, so period is 10.
- The graph is shifted up by 2, so the average is 2.
- The maximum is average+amplitude=2.4.
- The minimum is average-amplitude=1.6.
- The phase is 7 (or -3), the amount the graph is shifted right.



Graphing Inverse Functions

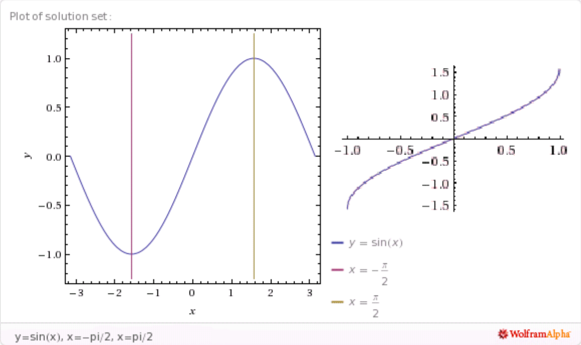
If $f(a) = b$ then $f^{-1}(b) = a$.
So $y = f(x)$ and $y = f^{-1}(x)$ are mirror images over $y = x$.



- Functions only have inverses if the mirror image is a function.
- Horizontal line test.

Trig functions are periodic. They take the same output value many times. Not invertible!

To make an invertible function, we have to restrict the domain.



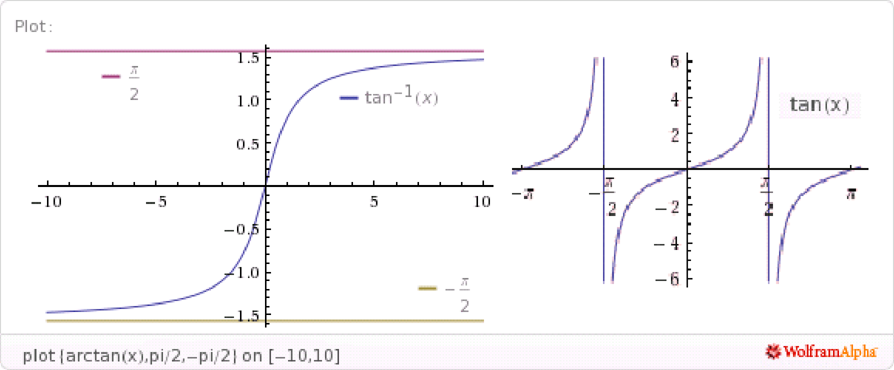
(A *piece* of *sin*(x) shown on left, *all* of $sin^{-1}(x)$ on right.)

$\sin^{-1}(x)$, also called $\arcsin(x)$, is the angle in $[-\pi/2, \pi/2]$ whose sine is x.

Caution: $\arccos(x)$ is the angle in $[0, \pi]$ whose cosine is x. Why?

Arctangent example: Find $\arctan(-\sqrt{3})$

$\arctan(x)$ is the angle in $[-\pi/2, \pi/2]$ whose tangent is x.



Solution

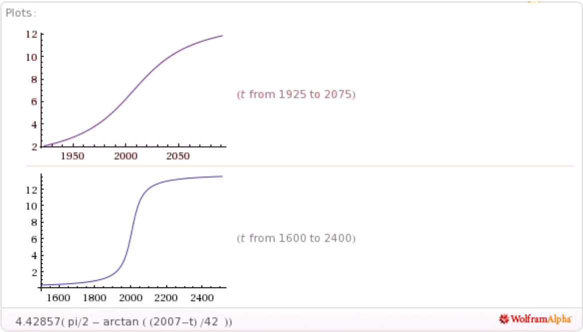
$\tan(\pi/3) = \frac{\sin(\pi/3)}{\cos(\pi/3)} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$.

So $\arctan(-\sqrt{3}) = -\pi/3$. *(Why?)*

We've finished the material for the week!

Questions? Problems to work through? Or would you like to look at pictures of blood splattering?

Also, here's an application of arctan as a human population model:



[S.P. Kapitza, The Phenomenological Theory of World Population Growth, *Physics-Uspekhi* 39(1), 57–71(1996)]. See textbook p.278.

Fun fact

Stretching an exponential graph [vertically or horizontally? you figure it out!] yields the exact same result as translating it horizontally!