1ZA3

Last Day Chain Rule

eg. $$\underbrace{\frac{d}{dx}}_{\frac{d}{dx}\sqrt{u}} \sqrt{\underbrace{\cos x + 1}_{u}} = \frac{d}{du}\sqrt{u} \cdot \frac{du}{dx}$$

$u \Rightarrow f(u) = \sqrt{u}$
$= u^{1/2}$

$$= \frac{1}{2} u^{-\frac{1}{2}} \cdot \left[-\sin x + 0\right]$$

$$= \frac{-\sin x}{2\sqrt{\cos x + 1}}$$

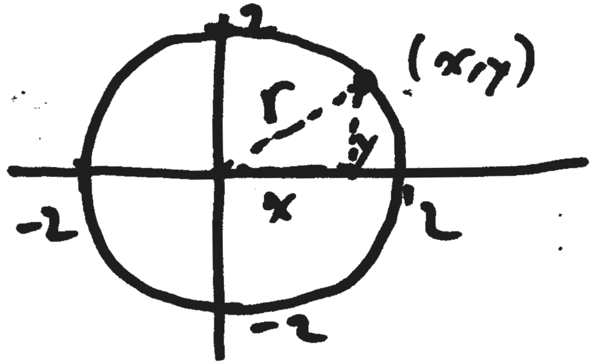
eg $$\frac{d}{dx} \underset{f}{x} \underset{g}{e^{\sin x}} = \underbrace{1 e^{\sin x} + \left[\frac{d}{dx} e^{\sin x}\right] \cdot x}_{f'g + g'f}$$

$$= e^{\sin x} + x e^{\sin x} \cdot \cos x$$

$$= e^{\sin x}(1 + x\cos x)$$

Implicit Differentiation

eg



$x^2 + y^2 = r^2 = 4 = 2^2$

For **this** circle.

Say I want slope of tangent to circle at $(\sqrt{2}, \sqrt{2})$.

Solution Old "Explicit" way!

$x^2 + y^2 = 4$ **not** **function** $\Rightarrow$ cheat!

but $\quad y^2 = 4 - x^2 \Rightarrow \quad y = \pm\sqrt{4 - x^2}$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\pm\sqrt{4 - x^2}\right) = \pm\frac{1}{2}(4 - x^2)^{-\frac{1}{2}}(-2x)$$

$$y' = \mp \frac{x}{\sqrt{4-x^2}}$$ } now: if $(x,y) = (\sqrt{2}, \sqrt{2})$

$\Rightarrow x = \sqrt{2},\ y > 0$

$$\underline{\text{so}}\quad \frac{dy}{dx} = -\frac{\sqrt{2}}{\sqrt{4-(\sqrt{2})^2}} = -\frac{\sqrt{2}}{\sqrt{4-2}} = -1.$$

<u>New</u> "Implicit" way:

$$\frac{d}{dx}\left(x^2 + y^2\right) = \cancel{\frac{d}{dx}4} \to 0$$

Always get a $\underline{y'}$ if diff. y function

$$\Rightarrow 2x + \frac{d}{dx}y^2 = 0$$

$$\frac{d}{dx}y^2 \to 2yy'$$

notice: if $y = f(x)$

$$\underline{\text{:}}\quad \frac{d}{dx}y^2 = \frac{d}{dx}(f(x))^2 = 2(f(x))\cdot f'(x) = 2y\underline{y'}.$$

Solve for y'

$$x + \cancel{2}yy' = 0$$

$$\boxed{y' = -x/y}$$

so $y'\Big|_{(x,y)=(\sqrt{2},\sqrt{2})} = -\frac{\sqrt{2}}{\sqrt{2}} = \underline{\underline{-1}}$

<u>Implicit Diff</u>

<u>Good</u>: Generally simpler to do

Don't have to solve for y (may be impossible!)

<u>Bad</u>: Need <u>both</u> x & y value for a point to get y'

- May mess up & plug in nonsense points!
- Final answer often hard to <u>write/simplify</u>!

eg. Given $e^{xy^3} = y\cos x + y$, find an expression for $\underline{y'}$.

Solution

$$\frac{d}{dx} e^{xy^3} = \frac{d}{dx} y\cos x + \frac{d}{dx} y$$

$$e^{xy^3}\cdot \frac{d}{dx}(xy^3) = \left(\cancelto{y'}{\frac{d}{dx} y}\right)\cos x + y(-\sin x) + \cancelto{y'}{\frac{dy}{dx}}$$

$$e^{xy^3}\cdot\left[1y^3 + x\cdot 3y^2\cdot y'\right] = y'\cos x - y\sin x + y'$$

Always easy to isolate!

Let's isolate y'

$$3xy^2 e^{xy^3}\,\underline{y'} - (\cos x)\,\underline{y'} - \underline{y'} = \underline{-y^3 e^{xy^3}} \underline{- y\sin x}$$

$$\boxed{y' = \frac{-y^3 e^{xy^3} - y\sin x}{3xy^2 e^{xy^3} - \cos x - 1}}$$

Inverse & Derivatives

I want $\frac{d}{dx} f^{-1}(x)$, only know $f(f^{-1}(x)) = x$

So let's Implicit it!

$$\frac{d}{dx} f(f^{-1}(x)) = \cancel{\frac{d}{dx} x} \; 1$$

$$f'(f^{-1}(x)) \cdot \left(\frac{d}{dx} f^{-1}(x)\right) = 1.$$

so $$\boxed{\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}}$$

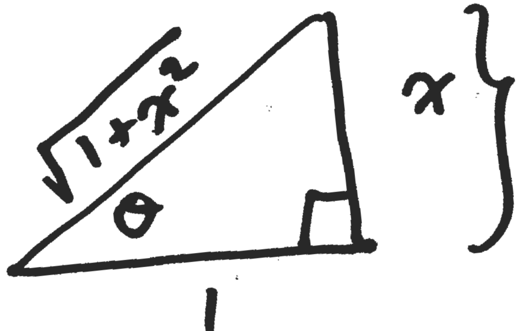
so $$\frac{d}{dx}\tan^{-1}x = \frac{d}{dx}\arctan x = \frac{1}{\sec^2(\tan^{-1}(x))}$$

$$\boxed{\frac{d}{dx}\tan x = \sec^2 x}$$

$$= \cos^2(\tan^{-1}(x))$$

so let $\theta = \tan^{-1}(x) \Rightarrow \tan\theta = x = x/1$

$\theta \in (-\pi/2, \pi/2)$



$\Rightarrow \cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{1}{\sqrt{1+x^2}}.$

$$\frac{d}{dx}\tan^{-1}(x) = \cos^2(\tan^{-1}(x)) = \left(\frac{1}{\sqrt{1+x^2}}\right)^2$$

$$= \frac{1}{1+x^2}$$ $\Leftarrow$ memorize

Similarly

$$\boxed{\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}}$$ } Know!

$$\boxed{\frac{d}{dx}\cos^{-1}(x) = -\frac{1}{\sqrt{1-x^2}}}$$ } not so relevant.

Let's get $\frac{d}{dx}\ln x$

$$\frac{d}{dx}e^{\ln x} = \frac{dx}{dx} \rightsquigarrow \cancelto{x}{e^{\ln x}}\left(\frac{d}{dx}\ln x\right) = 1$$

$\underline{\underline{\omega}}$ $\boxed{\frac{d}{dx}\ln x = 1/x}$ $\underline{\underline{x>0}}$.

$x>0$

$\underline{\underline{\text{Only}}}$

$\underline{\underline{\text{So}}}$ $\boxed{\frac{d}{dx}e^x = e^x}$ $\boxed{\frac{d}{dx}\ln x = \frac{1}{x}}$

$$\frac{d}{dx}a^x = \frac{d}{dx}\left(e^{\ln a}\right)^x = \frac{d}{dx}e^{x\cdot\ln a}$$

$$= e^{x\cdot\ln a}\cdot\frac{d}{dx}\left(\cancel{x}^{\ln a}\ln a\right) = \boxed{a^x\ln a = \frac{d}{dx}a^x}$$

$$\frac{d}{dx}\log_a x = \frac{d}{dx}\frac{\ln x}{\ln a} = \underbrace{\frac{1}{\ln a}}_{\text{constant!}}\frac{d}{dx}\ln x$$

$$= \frac{1}{x\ln a}$$