# Sample Graph Sketching Problem

Let's do a graph sketching problem. We'll look at the graph of the function: $y = e^{1/x} = f(x)$

**Step #1 : Domain**

It's probably easier to find what isn't in the domain. Functions like $e^x$ are continuous on all $x$. The only possible problem here is with $1/x$.

This function is undefined at $x = 0$, so our composite function will also be undefined at $x = 0$.

Our domain then is: $\{x | x \neq 0\}$, or in other words, all $x$ such that $x \in (-\infty, 0) \cup (0, \infty)$

**Step #2 : Symmetry**

Let's check the usual suspects.

1) *Periodic* : Does $f(x) = f(x + a)$ for some real $a$-value? *NO*

2) *Even / Odd* : We look at $f(-x) = e^{1/(-x)} = e^{-1/x}$,

$$f(-x) = e^{-1/x} \neq f(-x), \text{ so not odd}$$

$$e^{-1/x} \neq f(x), \text{ so not even}$$

In all, no usable symmetries.

**Step #3 : Intercepts**

Only bother with these if they are reasonably easy to calculate.

Here, they are especially easy:

***y - intercept*** **:** $x = 0$ isn't in the domain, so no intercept here.

***x - intercept*** **:** $y = 0 = e^{1/x}$, that never happens. Real exponentials are > 0.

**Step #4 : Asymptotes**

***Horizontal Asymptotes ("HA")*** **:**

Notice: $\lim_{x \to \infty} e^{1/x} = \text{"}e^{1/\infty}\text{"} = e^0 = 1 \Rightarrow$ HA: $y = 1$ as $x \to \infty$

$\lim_{x \to -\infty} e^{1/x} = \text{"}e^{1/-\infty}\text{"} = e^0 = 1 \Rightarrow$ HA: $y = 1$ as $x \to -\infty$

So we have the same $y = 1$ horizontal asymptote in both directions.

***Vertical Asymptotes ("VA")*** **:**

These are only going to occur at possible discontinuities, so the only place they *could* exist on our graph is at $x = 0$.

$$\lim_{x \to 0^+} e^{1/x} = \text{"}e^{1/0^+}\text{"} = \text{"}e^{+\infty}\text{"} = \infty, \quad \text{but} \quad \lim_{x \to 0^-} e^{1/x} = \text{"}e^{1/0^-}\text{"} = \text{"}e^{-\infty}\text{"} = 0$$

So we do have a vertical asymptote, but *ONLY* as we approach from the right!

**Step #5 : Examine $f\,'(x)$**

$$f(x) = e^{1/x} \Rightarrow \; f\,'(x) = e^{1/x} \times \left( \frac{d}{dx} \frac{1}{x} \right) = e^{1/x} \times \left( -\frac{1}{x^2} \right) = -\frac{e^{1/x}}{x^2}$$

Notice, that exponentials are all positve, as is $x^2$, so $f\,'(x) < 0$ (*where it's defined...*)
This means our function is decreasing everywhere, and there are no critical points.

But let's grind this through formally for the practice.

***Critical Points*** **:**

***Case#1*** **: $\boldsymbol{f\,'(x) = 0}$**

$$f\,'(x) = -\frac{e^{1/x}}{x^2} = 0 \Rightarrow \quad e^{1/x} = 0$$

But again, real exponentials are always positve, so this never happens.

***Case#2*** **: $\boldsymbol{f\,'(x)}$ = DNE,** ***(but $\boldsymbol{f(x)}$ exits.)***

$$f\,'(x) = -\frac{e^{1/x}}{x^2} = DNE \Rightarrow \quad x = 0, \text{ but that's not in the domain.}$$

So we have no critical points (c.p.'s) whatsoever. But we do have a gap at zero, so when we look for the intervals of increasing and decreasing, we'll still keep $x = 0$ in mind.

***Intervals of Increasing and Decreasing*** **:**

By inspection, as discussed above, or by plugging in a sample $x$-value from each interval, we can find the sign on each interval:

| | $(-\infty, 0)$ | $(0, \infty)$ |
|---|---|---|
| $f\,'(x)$ | " − " | " − " |
| $f(x)$ | *decreasing* | *decreasing* |

***Local maximums and mimimums*** **:**

Normally we could use the information from the chart above to do a first derivative test. But no c.p.'s means <u>*no local maximum or minimum*</u> either.

**Step #6 : Examine $f''(x)$**

$$f'(x) = -\frac{e^{1/x}}{x^2} = -x^{-2}e^{1/x}$$

$$\Rightarrow\ f''(x) = \frac{-\left(\frac{d}{dx}e^{1/x}\right)\times x^2 + 2xe^{1/x}}{\left(x^2\right)^2} = \frac{\frac{e^{1/x}}{x^2}\times x^2 + 2xe^{1/x}}{x^4}$$

$$= \frac{e^{1/x} - 2xe^{1/x}}{x^4} = \frac{e^{1/x}}{x^4}(1+2x)$$

Again, exponentials are all positve, as is $x^4$.

We, then, have two factors: $\frac{e^{1/x}}{x^4} > 0$ (*again, where it's defined...*)

and $(1+2x)$, which changes sign.

***Inflection Points (i.p.'s)*:**

As before, we look for these as we did the critical points, but using the 2nd derivative.

***Case#1*: $f''(x) = 0$**

$$f''(x) = \frac{e^{1/x}}{x^4}(1+2x) = 0 \Rightarrow \quad (1+2x) = 0 \Rightarrow \ x = -\tfrac{1}{2}$$

***Case#2* : $f''(x)$ = DNE, *(but $f(x)$ exits.)***

$f''(x) = \frac{e^{1/x}}{x^4}(1+2x) = DNE \Rightarrow \quad x = 0$, but, as before, that's not in the domain.

This means we have one <u>*possible*</u> inflection point, the one at $x = -\tfrac{1}{2}$.

We'll do the concavity chart, and at the same time we can verify if this is, in fact an i.p.

***Intervals of Concavity*:**

We use both our potential i.p. at $x = -\tfrac{1}{2}$ and our discontinuity to set our boundries.

This gives us the intervals: $(-\infty, -\tfrac{1}{2})$, $(-\tfrac{1}{2}, 0)$, $(0, \infty)$

Let's examine $f''(x)$ on each interval: ( *and remember*, $\frac{e^{1/x}}{x^4} > 0$ )

$$x \in (-\infty, -\tfrac{1}{2}) \Rightarrow (1+2x) < 0 \Rightarrow \frac{e^{1/x}}{x^4}(1+2x) < 0$$

$$x \in (-\tfrac{1}{2}, 0) \Rightarrow (1+2x) > 0 \Rightarrow \frac{e^{1/x}}{x^4}(1+2x) > 0$$

$$x \in (0, \infty) \Rightarrow (1+2x) > 0 \Rightarrow \frac{e^{1/x}}{x^4}(1+2x) > 0$$

And we get a chart:

| | $(-\infty,-\frac{1}{2})$ | $(-\frac{1}{2},0)$ | $(0,\infty)$ |
|---|---|---|---|
| $f''(x)$ | " - " | " + " | " + " |
| $f(x)$ | *concave* down (C.D.) | *concave up* (C.U.) | *concave up* (C.U.) |

Notice, that at $x = -\frac{1}{2}$, we get a *change in concavity*. So yes, indeed it is an i.p.

**Step #7 : Graph It**