1ZC3

Test Day Today!

Last Day Systems of Linear Diff. Equations

Recap

$$\underbrace{y = Ce^{kx} \qquad\qquad y' = cke^{kx}}_{\Updownarrow}$$

$$y' = ky$$

diff. equation initial condition

So If I say $\boxed{y' = ky}$, $\boxed{y(0) = 2}$

$$\Rightarrow \quad y = c\underset{\uparrow}{e}^{kx} \qquad \& \quad y(0) = ce^0 = c = 2$$

general solution

c arbitrary!

$\Rightarrow$ particular solution $\boxed{y = 2e^{kx}}$ $\Big\}$ Satisfies diff. eqn. & <u>initial condition</u>

"initial value problem"

← 1 bunny!

← 1 lemming!

↑ y_1 ↑ y_2

$$y_1' = \lambda_1 y_1 \rightsquigarrow y_1 = c_1 e^{\lambda_1 x}$$

$$y_2' = \lambda_2 y_2 \rightsquigarrow y_2 = c_2 e^{\lambda_2 x}$$

$\Rightarrow$ Let $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \Rightarrow \vec{y}\,' = \begin{bmatrix} \lambda_1 \cdot y_1 \\ \lambda_2 \cdot y_2 \end{bmatrix}$

$$\Rightarrow y' = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

eg $\vec{y}' = D\vec{y}$, $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

⇒ general solution: $\vec{y} = \begin{bmatrix} c_1 e^{\lambda_1 x} \\ c_2 e^{\lambda_2 x} \end{bmatrix}$

What if ~~D~~ our matrix A is not diagonal! But diagonalizable!?

$$y' = Ay$$

$$\Updownarrow$$

$$P^{-1} y' = P^{-1} P D P^{-1} y$$

$$P^{-1} y' = D P^{-1} y$$

eg

$$y_1' = 6y_1 - y_2$$

$$y_2' = 2y_1 + 3y_2$$

Linear eqn. in y_i

$$\Rightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Let $\boxed{\vec{u} = P^{-1}\vec{y}}$ $\Rightarrow$ $\underline{\underline{\vec{u}\,' = D\vec{u}}}$

$\Rightarrow$ by prev. example $\vec{u} = \begin{bmatrix} c_1 e^{\lambda_1 x} \\ c_2 e^{\lambda_2 x} \end{bmatrix}$

$\Rightarrow$ $\vec{y} = P\vec{u}$

$= P\left(c_1 e^{\lambda_1 x}\vec{i} + c_2 e^{\lambda_2 x}\vec{j} \right)$

$$\boxed{\vec{y} = c_1 e^{\lambda_1 x}\vec{x}_1 + c_2 e^{\lambda_2 x}\vec{x}_2}$$

In <u>general</u>

$$\boxed{\vec{y} = \sum_{i=1}^{n} c_i e^{\lambda_i x}\vec{x}_i}$$

<u>Remember</u>

if $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

$P = [\vec{x}_1 \mid \vec{x}_2]$ corresponding basis

↑ eigenvector

$P\vec{i} = P\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \vec{x}_1$

$P\vec{j} = P\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \vec{x}_2$

eg. Let $y_1' = 6y_1 - y_2$ $\quad y_1(0) = 3$

$y_2' = 2y_1 + 3y_2$ $\quad y_2(0) = 4$

} solve the linear system of Diff. eqn.

<u>Solution</u>

$$\vec{y}\,' = \begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix} \vec{y}$$

⇒ can show

$\lambda_1 = 4 \qquad \vec{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\lambda_2 = 5 \qquad \vec{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

⇒ <u>diagonalizable!</u>

$$\text{Let } D = \begin{bmatrix} \overset{\lambda_1}{\overset{\downarrow}{4}} & 0 \\ 0 & \underset{\lambda_2}{\underset{\uparrow}{5}} \end{bmatrix} \Rightarrow P = \begin{array}{c} \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \\ \begin{array}{cc} \uparrow & \uparrow \\ \vec{x}_1 & \vec{x}_2 \end{array} \end{array}$$

$$\underline{P^{-1}} = \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix} \frac{1}{1^2 - 2}$$

$$= \underline{\underline{\begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}}}$$

$$\Rightarrow \quad \vec{y} = c_1 e^{\lambda_1 x}\vec{x}_1 + c_2 e^{\lambda_2 x}\vec{x}_2$$

$$= c_1 e^{4x}\begin{bmatrix}1\\2\end{bmatrix} + c_2 e^{5x}\begin{bmatrix}1\\1\end{bmatrix}$$

$$\left.\begin{array}{l}\boxed{y_1 = c_1 e^{4x} + c_2 e^{5x}}\\ \boxed{y_2 = 2c_1 e^{4x} + c_2 e^{5x}}\end{array}\right\}$$

But!

we also had an initial condition

$$\boxed{y_1(0) = 3 \,, \quad y_2(0) = 4}$$

Shortcut to do Initial Condition

$$P^{-1}\vec{y}(0) = P^{-1}\begin{bmatrix}3\\4\end{bmatrix} = P^{-1}\left(c_1 \overset{\vec{x}_1}{\cancel{\begin{bmatrix}1\\2\end{bmatrix}}} + c_2 \overset{\vec{x}_2}{\cancel{\begin{bmatrix}1\\1\end{bmatrix}}}\right)$$

$$P^{-1}\begin{bmatrix}3\\4\end{bmatrix} = c_1\vec{i} + c_2\vec{j} = \underline{\underline{\begin{bmatrix}c_1\\c_2\end{bmatrix}}}$$

In general $\quad P^{-1}\bar{y}(0) = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \end{bmatrix}$ our unknown const. coefficients!

here $\quad y(0) = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \qquad P^{-1} = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -3+4 \\ 6-4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

so $\quad y_1 = c_1 e^{4x} + c_2 e^{5x}, \quad y_2 = 2c_1 e^{4x} + c_2 e^{5x}$

$$\Rightarrow \boxed{y_1 = e^{4x} + 2e^{5x}, \quad y_2 = 2e^{4x} + 2e^{5x}}$$

Next Day

Complex Plane

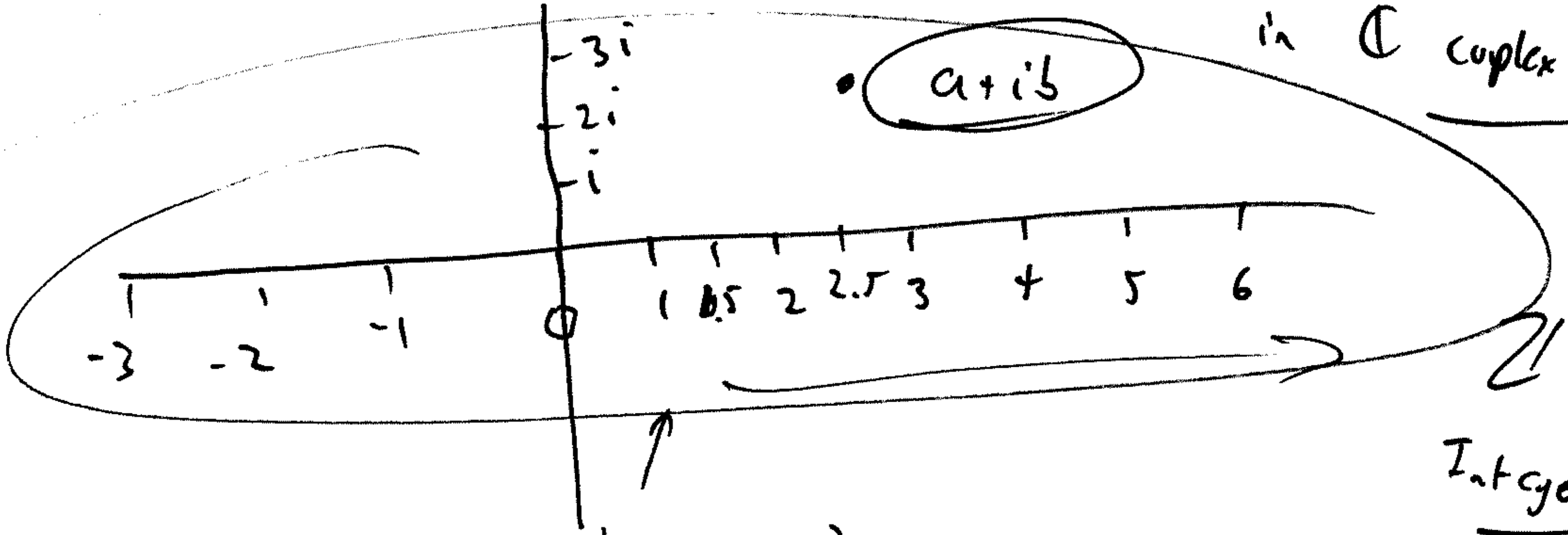
Complex #

in $\mathbb{C}$ complex plane!

$\mathbb{R}$

//

Reals!



Integers!

$\mathbb{Q} = \left\{ \frac{a}{b} \;\middle|\; \begin{matrix} a, b \in \mathbb{Z} \\ b \neq 0 \end{matrix} \right\}$

rationals!

$\overline{\mathbb{Q}}$ = non-repeating decimals! Irrationals!

$i = \sqrt{-1}$

$i\mathbb{R}$ imaginaries

$1i$

$2i$

$4i$