MATHEMATICS 1LT3 TEST 1

Day Class
Duration of Test: 60 minutes
McMaster University

E. Clements

28 January 2013

FIRST NAME (please print): ____________________

FAMILY NAME (please print): ____________________

Student No.: ____________________

THIS TEST HAS 8 PAGES AND 7 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE.

Total number of points is 40. Marks are indicated next to the problem number in square brackets. Any Casio fx991 (or lower, non-graphing) calculator is allowed.

USE PEN TO WRITE YOUR TEST. IF YOU USE A PENCIL, YOUR TEST WILL NOT BE ACCEPTED FOR REMARKING (IF NEEDED).

You need to show work to receive full credit, except for Multiple Choice.

Problem	Points	Mark
1	4	
2	7	
3	6	
4	6	
5	8	
6	5	
7	4	
TOTAL	40	

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1. **Multiple Choice**. Clearly **circle** the one correct answer.

(a) [2] Consider the differential equation $\frac{dy}{dx} = x + y$, where $y(0) = 1$. Using Euler's Method with step size $h = 1$, the approximate value of $y(2)$ is

(A) 4 (B) 4.5 (C) 5 (D) 5.5

(b) [2] In the basic model for the spread of a disease, $\frac{dI}{dt} = \alpha I(1 - I) - \mu I$, which of the following statements is/are true?

F ~~(I) $I^* = 0$ is a stable equilibrium.~~ not necessarily!

T (II) If $\mu > \alpha$, then the disease will eventually die out.

F (III) If $\mu < \alpha$ and $I(0) > 0$, then $I(t) \to 1$ as $t \to \infty$.

(A) none (B) I only (C) II only (D) III only

(E) I and II (F) I and III (G) II and III (H) all three

2. A population of elephants is described by the logistic differental equation

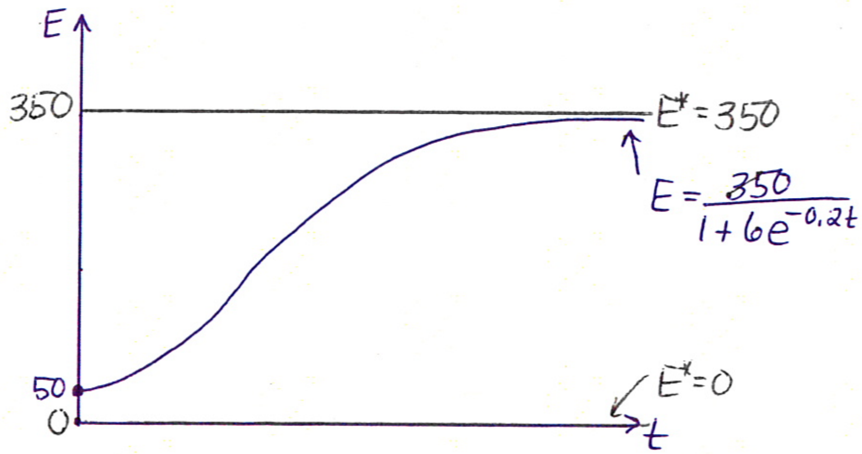
$$\frac{dE}{dt} = 0.2E\left(1 - \frac{E}{350}\right)$$

(a) [2] Find the equilibria of this equation. What do these numbers represent?

$\frac{dE}{dt} = 0$ when $E_1^* = 0$ or $E_2^* = 350$

350 is the carrying capacity for this pop^n

(b) [2] The solution of this equation when E(0)=50 is $E(t) = \frac{350}{1+6e^{-0.2t}}$, where t is measured in months. Sketch this solution curve. Include equilibrium solutions from part (a).



(c) [3] Determine when the population will reach 95% of its carrying capacity.

Set $E(t) = 0.95 \times 350$:

$$0.95 \times \cancel{350} = \frac{\cancel{350}}{1+6e^{-0.2t}}$$

$$1 + 6e^{-0.2t} = \frac{1}{0.95}$$

$$e^{-0.2t} = \frac{\frac{100}{95} - 1}{6}$$

$$t = \frac{\ln 114}{0.2} \approx 23.7 \text{ months}$$

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3. Consider the following autonomous differential equation $\frac{dx}{dt} = x - x^3$.

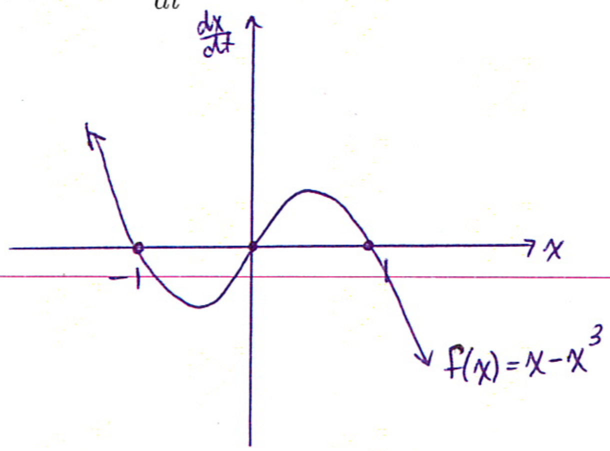
(a) [2] Determine the equilibria for this differential equation.

$\frac{dx}{dt} = 0$ when $x - x^3 = 0$

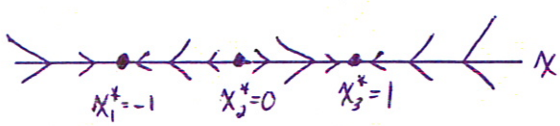
$x(1-x^2) = 0$

$x_1^* = 0, \; x_2^* = -1, \; x_3^* = +1$

(b) [2] Graph $\frac{dx}{dt}$ as a function of x.



(c) [2] Draw a phase-line diagram for $\frac{dx}{dt} = x - x^3$.



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4. The selection equation, $\frac{dp}{dt} = (\mu - \lambda)p(1-p)$, models the dynamics of two variants of a population, type A and type B, where p represents the proportion of type A and μ, $\lambda \geq 0$.

(a) [1] Find the equilibria for this differential equation.

$\frac{dp}{dt} = 0$ when $p_1^* = 0$ or $p_2^* = 1$

(b) [3] Using the stability theorem, determine the stability of each equilibrium you found in part (a). (Note: Since stability depends on parameters μ and λ, you will need to consider different cases.)

$f(p) = (\mu - \lambda)\cdot(p - p^2)$
$f'(p) = (\mu - \lambda)\cdot(1 - 2p)$

$f'(0) = \mu - \lambda \Rightarrow p_1^* = 0$ is stable when $\mu < \lambda$ and unstable when $\mu > \lambda$.
$f'(1) = \lambda - \mu \Rightarrow p_2^* = 1$ is stable when $\lambda < \mu$ and unstable when $\lambda > \mu$.

(c) [2] Describe what will happen to the proportion of type A for each case in part (b).

When $\mu > \lambda$, 0 is an unstable eq'm and 1 is a stable eq'm
$\Rightarrow$ type A is "stronger" and the proportion of type A approaches 1.
When $\lambda > \mu$, 0 is a stable eq'm + 1 is an unstable eq'm
$\Rightarrow$ type B is "stronger" and the proportion of type A approaches 0.

5. Use the separation of variables technique to solve each differential equation.

(a) [4] $\dfrac{dy}{dx} = \dfrac{\ln x}{xy}$, where $y(1) = 2$

$\int y\,dy = \int \frac{\ln x}{x}\,dx$

$\frac{y^2}{2} = \frac{(\ln x)^2}{2} + C$

$\Rightarrow y^2 = (\ln x)^2 + K$ (where $K = 2C$)

when $x = 1$, $y = 2 \Rightarrow 2^2 = (\ln 1)^2 + K \Rightarrow K = 4$

$\therefore y^2 = (\ln x)^2 + 4$

$y = +\sqrt{(\ln x)^2 + 4}$ (not $-\sqrt{(\ln x)^2+4}$ since $y(1) = +2$)

Aside:

Let $u = \ln x$. Then $du = \frac{1}{x}dx$.

So $\int \frac{\ln x}{x}\,dx = \int u\,du$
$= \frac{u^2}{2} + C$
$= \frac{(\ln x)^2}{2} + C$

(b) (i) [3] $y' = x^2 y$

$\frac{dy}{dx} = x^2 y$

$\int \frac{1}{y}\,dy = \int x^2\,dx$

$\ln|y| = \frac{x^3}{3} + C$

$|y| = e^C \cdot e^{\frac{x^3}{3}}$

$y = \pm e^C \cdot e^{x^3/3}$

$\therefore y = Ke^{x^3/3}$, where $K = \pm e^C$.

(ii) [1] Are there any other solutions to this differential equation **not** covered by the equation you found in part (i)?

YES! $y = 0$ is a sol^n of the original DE.

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6. The following pair of equations represent the population growth of two different species where one is the predator and the other is the prey.

$$\frac{dx}{dt} = (0.2 - 0.004y)x = 0.2x - 0.004xy$$

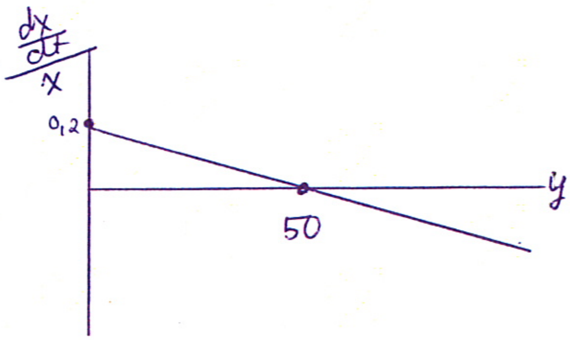
$$\frac{dy}{dt} = (0.001x - 0.07)y = -0.07y + 0.001xy$$

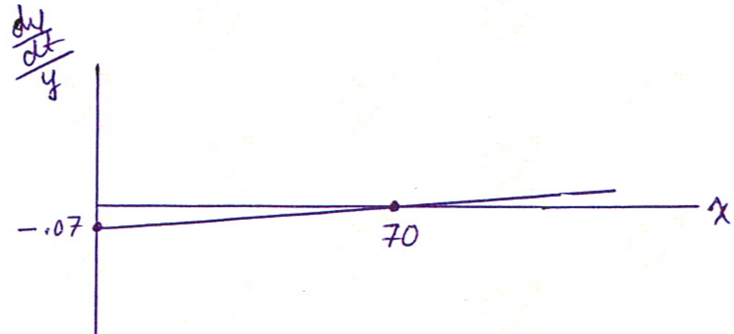
(a) [2] Which of the variables represents the predator population and which represents the prey population? Explain.

x represents the prey since in the absence of predators, $\frac{dx}{dt} = 0.2x$, ie the pop^n would grow exponentially and encounters with predators are harmful, indicated by the $-$ve coefficient of the interaction term.

y represents the predator since without prey, $\frac{dy}{dt} = -0.07y$, ie, the pop^n will die out. Also, encounters are beneficial as indicated by the $+$ve coefficient of the interact^n term.

(b) [3] Sketch the per capita production rates for both x and y. What are the equilibrium solutions for this system?





The eq^n sol^{n}s are $\begin{cases} x=0 \\ y=0 \end{cases}$ or $\begin{cases} x=70 \\ y=50 \end{cases}$

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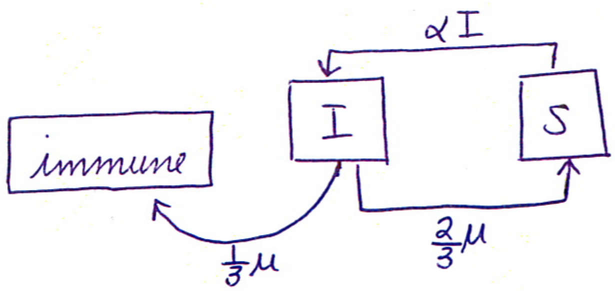
7. A system of differential equations for a disease model which measures how both the fraction of infected individuals, I, and the fraction of susceptible individuals, S, changes over time is given by

$$\frac{dI}{dt} = \alpha IS - \mu I$$
$$\frac{dS}{dt} = -\alpha IS + \mu I$$

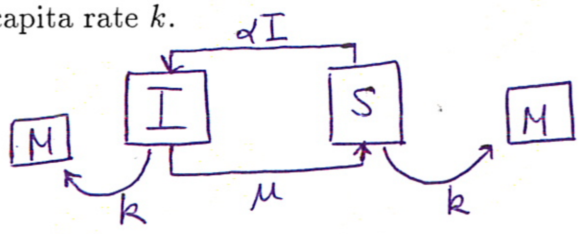
where $\alpha,\ \mu \geq 0$. Modify this system of differential equations to reflect each situation described below.

(a) [2] Suppose that one-third of the individuals who leave the infected class through recovery become permanently immune and that the other two-thirds become susceptible again.



$$\frac{dI}{dt} = \alpha IS - \mu I$$
$$\frac{dS}{dt} = -\alpha IS + \frac{2}{3}\mu I$$

(b) [2] Suppose that all individuals become susceptible upon recovery (as in the basic model above) but that there is a source of mortality, so both infected and susceptible individuals die at a per capita rate k.



$$\frac{dI}{dt} = \alpha IS - \mu I - kI$$
$$\frac{dS}{dt} = -\alpha IS + \mu I - kS$$

THE END