Math 1LS3 Week 11: Integration Techniques

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Week 11: Sections 6.4,6.5

Thought Experiment from Earlier

You're in a car that travels along x-axis in the positive direction. You have:

- Spedometer (measures speed $v = dx/dt$)
- Stopwatch
- Notebook, Pencil, Coffee

How can you accurately determine distance travelled over a given time interval $[0, t]$?

Continue the Thought Experiment

$$\begin{aligned}\text{Distance travelled} &\approx \text{(first speed measure)(first time interval)}\\ &+ \text{(second speed measure)(second time interval)}\\ &+ \text{(third speed measure)(third time intveral)}+\cdots.\end{aligned}$$

$$\begin{aligned}\text{Area under } v(x) \text{ above } [0,t] &\approx \text{(speed 1)(time width 1)}\\ &+ \text{(speed 2)(time width 2)}\\ &+ \text{(speed 3)(time width 3)}+\cdots\end{aligned}$$

Theorem (Fundamental Theorem of Calculus)

distance travelled = area under speed curve

$$x(t) - x(0) = \int_0^t v(t)dt$$

Recall: $x(t) = \int v(t)dt$ is an **antiderivative** of v.

We've related antiderivatives to definite integrals!

Fundamental Theorem of Calculus

Theorem (Fundamental Theorem of Calculus, p.456)

Suppose $f(x)$ is continuous on $[a, b]$ and $F(x)$ is any antiderivative of $f(x)$. (That means $F(x)$ is any function whose derivative is $f(x)$). Then:

$$\int_a^b f(x)dx = F(b) - F(a)$$

We can use this to compute definite integrals!

Notation:

$$F(x)|_a^b := F(b) - F(a), \quad \text{so} \quad \int_a^b f(x)dx = F(x)|_a^b.$$

Find your favorite antiderivative, and then just plug in the **limits of integration** and subtract.

Evaluating a Definite Integral

Problem

Evaluate

$$\int_0^2 x^3 dx$$

Solution

$$\int_0^2 x^3 dx = \left.\frac{x^4}{4}\right|_0^2 = \frac{2^4}{4} - \frac{0^4}{4} = 4$$

Note, you get the same answer with any antiderivative:

$$\left.\frac{x^4}{4} + 17\right|_0^2 = \left(\frac{2^4}{4} + 17\right) - \left(\frac{0^4}{4} + 17\right) = 4$$

Another Definite Integral

Problem

What's the area under $y = 1/x$ above $[1, 2]$?

Solution

$$\int_1^2 \frac{1}{x} dx = \ln(x)\Big|_1^2 = \ln(2) - \ln(1) = \ln(2)$$

And another

Problem

$$\int_0^1 (e^x - 2x)dx$$

Solution

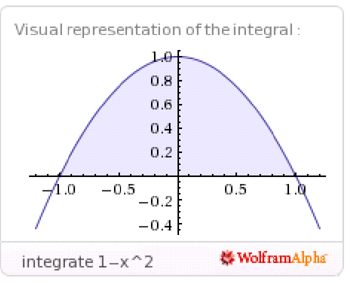
$$\int_0^1 (e^x - 2x)dx = (e^x - x^2)\Big|_0^1 = (e^1 - 1) - (e^0 - 0) = e - 2.$$

And another

Problem

What's the area under $y = 1 - x^2$ above the x-axis?

Solution



When is $1 - x^2$ above x-axis?
$1 - x^2 = 0$ *at* $x = \pm 1$.

$$\int_{-1}^{1} (1 - x^2) dx = \left(x - \frac{x^3}{3} \right) \Big|_{-1}^{1} = \frac{4}{3}$$

$$= 2\left(x - \frac{x^3}{3} \right) \Big|_{0}^{1} = 2\left(1 - \frac{1}{3} \right) = \frac{4}{3}$$

Checking if you're awake...

Example

$$\int_{-10}^{2} \frac{1}{x^2} dx = \left(-\frac{1}{x}\right)\Big|_{-10}^{2} = -\frac{1}{2} - \frac{1}{10} = -\frac{3}{5}$$

What went wrong?

f is not continuous at 0.

f must be continuous on $[a, b]$ to apply FTC to $\int_a^b f(x)dx$.

Note: In this example, the actual area is infinite: see 4.7.

Integrals by Guessing

Problem

Compute

$$\int_0^{\pi/4} \sin(2x)dx$$

Solution

Need to find an antiderivative of $\sin(2x)$.

Guess $\cos(2x)$. *But* $(\cos(2x))' = -2\sin(2x)$, *not* $\sin(2x)$.

Instead use $-\frac{1}{2}\cos(2x)$. *Indeed,*
$(-\frac{1}{2}\cos(2x))' = -\frac{1}{2} \cdot -2\sin(2x) = \sin(2x)$. *So:*

$$\int_0^{\pi/4} \sin(2x)dx = \left(-\frac{1}{2}\cos(2x)\right)\Big|_0^{\pi/4} = -\frac{1}{2}\cos(2\pi/4) + \frac{1}{2}\cos(0) = \boxed{\frac{1}{2}}$$

Measuring Growth as a Definite Integral

Problem (6.4.13)

Find how much a fish grows from year 2 to year 5 if growth rate is governed by the DiffEq

$$\frac{dL}{dt} = 6.48e^{-0.09t}$$

(length L in cm, time t in yrs).

Solution

Guess method: since $(e^{-0.09t})' = -0.09e^{-0.09t}$, *we find:*

$$\left(\frac{6.48}{-0.09}e^{-0.09t}\right)' = 6.48e^{-0.09t}.$$

$$L(5) - L(2) = \int_2^5 \frac{dL}{dt}dt = \int_2^5 6.48e^{-0.09t}dt = \frac{6.48}{-0.09}e^{-0.09t}\Big|_2^5 \approx \boxed{14.2}$$

Alternate Techniques

Note:

- Instead of guessing, you could memorize p.459 formulae:

$$\int e^{ax}dx = \frac{1}{a}e^{ax} + C$$

$$\int \sin(ax)dx = -\frac{1}{a}\cos(ax) + C$$

$$\int \cos(ax)dx = \frac{1}{a}\sin(ax) + C$$

- Substitution method (next topic) gives same result as guessing.
- Instead of using FTC twice in last problem, you could solve the DiffEq for $L(t)$ with your favorite initial value, then subtract $L(5) - L(2)$. You'd be doing essentially the same computation plus added work to get the constant right, but it might be conceptually easier.

Substitution Method

Problem

$$\int \cos(x/2)dx$$

- Introduce a new variable u. $u = x/2$
- Find $\frac{du}{dx}$=(some function of x). $\frac{du}{dx} = \frac{1}{2}$
- "Solve for dx in terms of du". $dx = 2du$
- Write everything in terms of u and du, no x or dx. $\int \cos(u) \cdot 2du$
- Integrate. $2\sin(u) + C$
- Substitute back: the answer should involve only x. $2\sin(x/2) + C$

$$\int \cos(x/2)dx = 2\sin(x/2) + C$$

This trick works because of the chain rule: $\int f'(u(x))\frac{du}{dx}dx = f(u(x))$.

Substitution: Example

Choosing u is something of an art.

Problem

$$\int x e^{x^2+1} dx$$

Solution

Use the "inner function" $u = x^2 + 1$ *, especially since you see its derivative* $2x$ *hiding outside.*

$$u = x^2 + 1 \implies du = 2x dx \implies x dx = du/2$$

$$\int x e^{x^2+1} dx = \int e^{x^2+1}(x dx) = \int e^{u}(du/2) = \frac{1}{2}e^{u} + C = \boxed{\frac{1}{2}e^{x^2+1} + C}$$

Note: $\int e^{x^2} dx$ and $\int e^{-x^2} dx$ are not **elementary** integrals.

Substitution: Another Example

Problem

$$\int \frac{\arcsin(x)}{\sqrt{1-x^2}}dx$$

Solution

Use $u = \arcsin(x)$ *since* $\arcsin(x)$ *is ugly and you see its derivative* $\frac{1}{\sqrt{1-x^2}}$ *hanging around.*

$$u = \arcsin(x) \implies du = \frac{1}{\sqrt{1-x^2}}dx$$

$$\int \frac{\arcsin(x)}{\sqrt{1-x^2}}dx = \int u\,du = \frac{u^2}{2} + C = \boxed{\frac{(\arcsin(x))^2}{2} + C}$$

Substitution with Definite Integrals

With definite integrals, you have a choice: you can avoid replacing u with x in the last step **if** you change the limits of integration.

Problem (6.5.6)

$$\int_0^1 x^3(x^4+2)^4 dx = ?$$

Solution

Substitute $u = x^4 + 2 \implies du = 4x^3 dx \implies x^3 dx = du/4.$

Change limits: when $x = 0$, $u = 2$. *When* $x = 1$, $u = 3$.

$$\int_0^1 x^3(x^4+2)^4 dx = \int_2^3 u^4 du/4 = \frac{1}{4} \cdot \frac{u^5}{5}\bigg|_2^3 = \frac{3^5 - 2^5}{20} = 10.55$$

Revisiting the Fish Growth Problem

Problem

A fish grows from length $L(0) = 0$ according to

$$\frac{dL}{dt} = 6.48e^{-0.09t}.$$

When will the fish reach a mature 45cm? How big will the fish get?

Solution

Use substitution to show $L(t) = -72e^{-0.09t} + C$.
Since $L(0) = 0$, *get* $0 = -72e^{-0.09\cdot 0} + C \implies C =$72. *So:*

$$L(t) = 72(1 - e^{0.09t}).$$

The fish length approaches 72, but never reaches it. Maturity occurs at:

$$45 = 72(1 - e^{0.09t}) \implies \boxed{t = 10.9 \text{ years}}$$

Integration by Parts: The Product Rule in Reverse

$$d(uv) = udv + vdu \implies uv = \int udv + \int vdu \implies \boxed{\int udv = uv - \int vdu}$$

Example

$$\int x\cos(x)dx$$

$$\boxed{\begin{array}{c} u = x \\ dv = \cos(x)dx \end{array}} \implies \boxed{\begin{array}{c} du = 1dx \\ v = \int \cos(x)dx = \sin(x) \end{array}}$$

$$\int x\cos(x)dx = \int udv = uv - \int vdu = x\sin(x) - \int \sin(x)1dx$$

$$= x\sin(x) - (-\cos(x)) + C = \boxed{x\sin(x) + \cos(x) + C}$$

Hint: take u with a nice derivative, dv with a nice integral

Integration by Parts: Another Example

Example

$$\int ln(x)dx$$

$$\boxed{\begin{array}{c} u = \ln(x) \\ dv = dx \end{array}} \implies \boxed{\begin{array}{c} du = \frac{1}{x}dx \\ v = \int dx = x \end{array}}$$

$$\int \ln(x)dx = \int udv = uv - \int vdu = \ln(x)x - \int x\frac{1}{x}dx$$

$$= x\ln(x) - \int 1dx = \boxed{x\ln(x) - x + C}$$

Idea: You didn't know the integral of $\ln(x)$, so you had to put it in the u factor (not dv).

Choosing u and dv

Good choices for u:

- $x, x^2, x^3, \dots$
- $\ln(x)$
- inverse trig functions
- functions with derivatives *simpler* than themselves

Good choices for dv:

- functions with integrals *no more complicated* than themselves
- $e^x dx, \sin(x)dx, \cos(x)dx$
- sometimes dx (even though x is "more complicated" than 1)

Integration by Parts: Repeated Use

Example

$$\int x^2 \sin(x)dx \quad \boxed{\begin{array}{c} u = x^2 \\ dv = \sin(x)dx \end{array}} \Longrightarrow \boxed{\begin{array}{c} du = 2xdx \\ v = -\cos(x) \end{array}}$$

$$\int x^2 \sin(x)dx = x^2(-\cos(x)) - \int -\cos(x)2xdx$$

$$= -x^2\cos(x) + 2\int x\cos(x)dx$$

$$\boxed{\begin{array}{c} u = x \\ dv = \cos(x)dx \end{array}} \Longrightarrow \boxed{\begin{array}{c} du = dx \\ v = \sin(x) \end{array}}$$

$$\int x^2 \sin(x)dx = -x^2\cos(x) + 2\left(x\sin(x) - \int \sin(x)dx\right)$$

$$= -x^2\cos(x) + 2\left(x\sin(x) - -\cos(x)\right) + C$$

$$= -x^2\cos(x) + 2x\sin(x) + 2\cos(x) + C$$

Another Integration by Parts Trick

Example

$$I = \int e^x \cos(x)dx \quad \boxed{\begin{array}{c} u = e^x \\ dv = \cos(x)dx \end{array}} \implies \boxed{\begin{array}{c} du = e^x dx \\ v = \sin(x) \end{array}}$$

$$I = e^x \sin(x) - \int \sin(x) e^x dx \quad \boxed{\begin{array}{c} u = e^x \\ dv = \sin(x)dx \end{array}} \implies \boxed{\begin{array}{c} du = e^x dx \\ v = -\cos(x) \end{array}}$$

$$I = e^x \sin(x) - \left(e^x(-\cos(x)) - \int -\cos(x) e^x dx \right) =$$

$$e^x \sin(x) + e^x \cos(x) - I$$

Solve for I! $2I = e^x \sin(x) + e^x \cos(x) \implies I = \frac{1}{2} e^x \sin(x) + \frac{1}{2} e^x \cos(x)$

$$\boxed{\int e^x \cos(x)dx = \frac{1}{2} e^x \sin(x) + \frac{1}{2} e^x \cos(x) + C}$$

Taylor Approximation and Integration

If a function can't be easily integrated, a Taylor approximation can sometimes be integrated in its place.

Problem

Approximate $\int_0^1 e^{-x^2}dx$ using the degree 6 Taylor polynomial for e^{-x^2} (at $x = 0$).

Solution

$$e^u \approx 1 + u + \frac{u^2}{2} + \frac{u^3}{3!} \quad \Longrightarrow \quad e^{-x^2} \approx T_6(x) = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{3!}$$

$$\int_0^1 e^{-x^2}dx \approx \int_0^1 \left(1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6}\right)dx = \left(x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42}\right)\Big|_0^1$$

$$= 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} \approx .743 \textit{(actual value} \approx .747\textit{)}$$

Approximation and Integration

Problem

Approximate $\int_{0.1}^{1} \frac{e^x}{x} dx$.

Solution

$$e^x \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} \implies \frac{e^x}{x} \approx \frac{1}{x} + 1 + \frac{x}{2} + \frac{x^2}{6}$$

$$\int_{0.1}^{1} \frac{e^x}{x} dx \approx \int_{0.1}^{1} \left(\frac{1}{x} + 1 + \frac{x}{2} + \frac{x^2}{6} \right) dx = \left(\ln(x) + x + \frac{x^2}{4} + \frac{x^3}{18} \right) \Big|_{0.1}^{1}$$

$$\approx 3.506 (\textit{actual value} \approx 3.518)$$