MATHEMATICS 1LT3 SAMPLE TEST 3

1. A population of lions p_t, where $t = 0, 1, 2...$, is modelled by $p_{t+1} = p_t + I_t$. The immigration term is equal to $I_t = 25$ with a 80% chance and $I_t = -30$ with a 20% chance. Assume that $p_0 = 100$.

(a) What is the deterministic part of this model? What is the stochastic part?

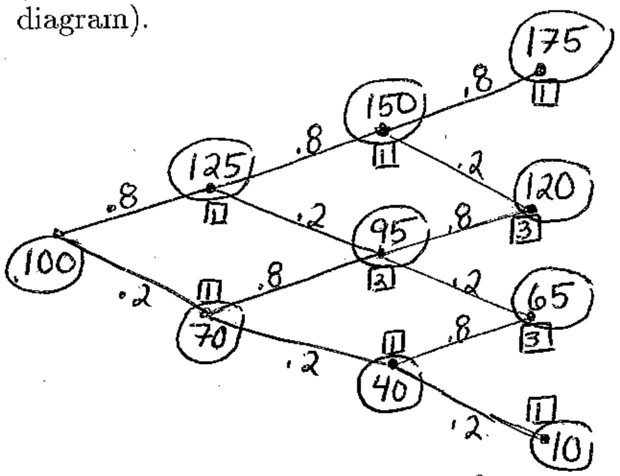
$$p_{t+1} = p_t + I_t$$

stochastic part

deterministic part

$$I_t = \begin{cases} 25 & 80\% \text{ chance} \\ -30 & 20\% \text{ chance} \end{cases}$$

(b) Write the sample space for the population of lions after 3 years. (Hint: Draw a tree diagram).



$S = \{10, 65, 120, 175\}$

(c) Assuming that immigration from year to year is independent, determine probabilities for each outcome in the sample space in part (b).

Let X be the pop^N size of lions after 3 years.

x	$p(x) = P(X = x)$
10	$(.2)^3 = 0.008$
65	$3(.2)^2(.8) = 0.096$
120	$3(.8)^2(.2) = 0.384$
175	$(.8)^3 = 0.512$

Check:

$\sum_x p(x) = .008 + .096 + .384 + .512 = 1$

Name: ______________________

Student No.: ______________________

1. continued...

(d) What is your prediction for the population of lions in 10 years? Will the population increase, decrease, or remain about the same? Explain.

In 8 out of 10 years, we expect an increase of 25 lions / year.
In 2 out of 10 years, we expect a decrease of 30 lions both years.

Expected net lions in 10 years:

$$8 \times 25 + 2 \times (-30) = 140$$

∴ After 10 years, we expect a net increase of 140 lions

2. Let $S = \{1, 2, 3, 4, 5\}$ be the sample space for a random experiment where $P(1) = 0.4$, $P(2) = 0.15$, $P(3) = 0.2$, and $P(5) = 0.2$.

(a) Find $P(4)$.

$$\begin{aligned} P(4) &= 1 - P(1) - P(2) - P(3) - P(5) \\ &= 1 - 0.4 - 0.15 - 0.2 - 0.2 \\ &= 0.05 \end{aligned}$$

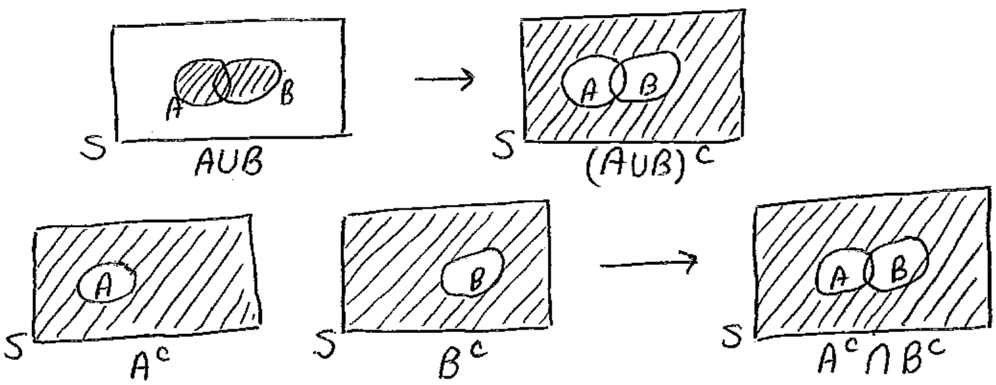
(b) If $A = \{1, 2\}$ and $B = \{2, 3, 4\}$, find $P(A)$, $P(B)$, and $P(A \cup B)$.

$$P(A) = P(1) + P(2) = 0.4 + 0.15 = 0.55$$

$$P(B) = P(2) + P(3) + P(4) = .15 + .2 + .05 = 0.4$$

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= 0.55 + 0.4 - P(2) \\ &= 0.55 + 0.4 - 0.15 \\ &= 0.8 \end{aligned}$$

3. Use Venn diagrams to show that $(A \cup B)^C = A^C \cap B^C$.



MATH 1LT3 * Sample Test 3 *

Name:____________________

Student No.: ____________________

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4. A family has 5 children. Assume that female and male children are equally likely to be born.

(a) What is the probability that at least one child is a girl?

$G = \geq 1$ is a girl

$G^c = <1$ is a girl

$= \{BBBBB\}$

$P(G^c) = \frac{|G^c|}{|S|} = \frac{1}{2^5} = 0.03125$

∵ equally likely

$\therefore P(G) = 1 - P(G^c)$

$= 1 - 0.03125$

$= 0.96875$

∴ There is about a 97% chance that at least one child is a girl

(b) What is the probability that exactly one child is a girl?

$G =$ exactly one is a girl

$= \{GBBBB, BGBBB, BBGBB, BBBGB, BBBBG\}$

$P(G) = \frac{|G|}{|S|} = \frac{5}{2^5} = 0.15625$

∴ There is about a 16% that exactly one is a girl.

(c) If it is known that at least one child is a boy, what is the probability that at least one child is a girl?

$B =$ at least one is a boy

$B^c =$ zero boys

$= \{GGGGG\}$

$P(B) = 1 - P(B^c)$

$= 1 - \frac{|B^c|}{|S|}$

$= 1 - \frac{1}{2^5} = 0.96875$

$G = \geq 1$ is a girl

$P(G|B) = \frac{P(G \cap B)}{P(B)}$ at least one boy and at least one girl

$(G \cap B)^c = \{GGGGG, BBBBB\}$

$= \frac{\frac{30}{32}}{\frac{31}{32}}$

$= \frac{30}{31} \approx 0.9677$

∴ There is still about a 97% chance at least one is a girl.

5. For the purposes of a study, university students were divided into two categories: those who work (at a paid job) throughout the school year and those who only work during the summer. Is this a partition of the sample space? If not, suggest a way in which the sample space could be partitioned based on work habits of students.

This is NOT a partition!

① The two sets could intersect/overlap: students who work during the school year AND the summer + students who only work in the summer.

② The union of the sets may not cover whole pop^N:
what about students who don't work at all?

One possible partition:

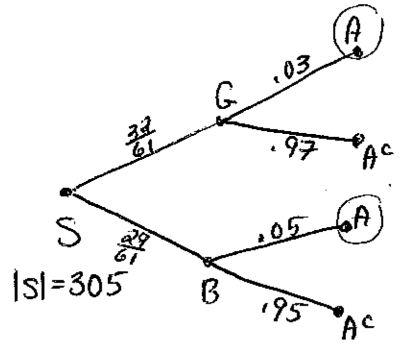
$E_1 =$ students who work at some point during a 12 month period

$E_2 =$ " " do not work at all " " " " "

Name:____________________
Student No.: ________________

6. It is estimated that ADHD affects 3 to 5 percent of school aged children globally, with males being diagnosed more frequently than females. Consider a population of school aged children that consists of 160 girls and 145 boys. Suppose that 3% of girls and 5% of boys within this population are estimated to be affected by ADHD.

(a) What is the probability that a randomly chosen child will be affected by ADHD?



A = affected by ADHD

$$P(A) = P(A|G)\cdot P(G) + P(A|B)\cdot P(B)$$
$$= (0.03)\left(\tfrac{32}{61}\right) + (0.05)\left(\tfrac{29}{61}\right)$$
$$\approx 0.0395$$

$\therefore$ There is about a 3.95% chance a randomly selected child will be affected by ADHD.

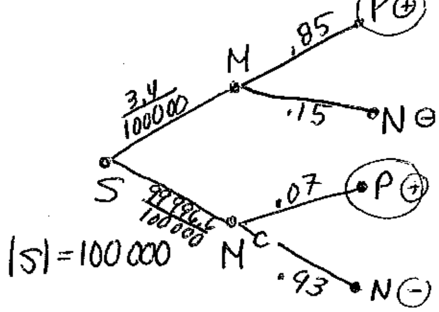
(b) What is the probability that a child with ADHD is a girl?

$$P(G|A) = \frac{P(A|G)P(G)}{P(A)} = \frac{(0.03)\left(\frac{32}{61}\right)}{(0.03)\left(\frac{32}{61}\right)+(0.05)\left(\frac{29}{61}\right)}$$
$$\approx 0.3983$$

$\therefore$ There is about a 39.83% chance that a child with ADHD is a girl.

7. The incidence of bacterial meningitis within a certain population was estimated to be about 3.4 cases per 100,000 people during 2012. A test for meningitis shows a positive result in 85% of people who have it and in 7% of people who do not have it (false-positive). If you belong to this population and test positive for bacterial meningitis, what is the probability that you actually have it?

M = you have meningitis



$$P(M|P) = \frac{P(P|M)\cdot P(M)}{P(P|M)\cdot P(M) + P(P|M^c)\cdot P(M^c)}$$
$$= \frac{(0.85)\left(\frac{3.4}{100\,000}\right)}{(.85)\left(\frac{3.4}{100\,000}\right) + (0.07)\left(\frac{99\,996.6}{100\,000}\right)}$$
$$\approx 0.0004127$$

$\therefore$ There is only about a 0.04% chance you actually have meningitis given that you tested positive for it.

8. In roulette, a wheel with numbered slots is spun and a ball is rolled in the opposite direction around the wheel. Players can bet on a single number or range of numbers based on where they expect the ball will stop. In American roulette, the wheel is numbered from 0 to 37 and the ball has an equally likely chance of stopping on any one of these numbers. If you always bet on 13, what is the probability of the ball stopping on 13 at least once in ten rolls? What is the probability of the ball stopping on 13 for all 10 rolls?

$S = \{0,1,2,\dots,37\}$ $|S| = 38$ prob. of each outcome in $S = \frac{1}{38}$

Let X_i be the event the ball stops on 13 on the i^{th} roll.

$P(X_i) = \frac{1}{38}$ for $i = 1,2,\dots$

X = stops on 13 at least once in 10 rolls

X^c = does not stop on 13 at all in 10 rolls.

$= X_1^c \cap X_2^c \cap \dots \cap X_{10}^c \quad \left(= \bigcap_{i=1}^{10} X_i^c\right)$

∵ rolls are independent

$P(X^c) = P(X_1^c) \cdot P(X_2^c) \cdot \dots \cdot P(X_{10}^c) \quad \left(= \prod_{i=1}^{10} P(X_i^c)\right)$

$= \left(\frac{37}{38}\right)^{10}$

≈ 0.7659

$\Rightarrow P(X) = 1 - P(X^c) = 1 - \left(\frac{37}{38}\right)^{10} \approx 0.2341$

Y = stops on 13 for all 10 rolls

$= X_1 \cap X_2 \cap \dots \cap X_{10}$

$P(Y) = P(X_1) \cdot P(X_2) \cdot \dots \cdot P(X_{10}) = \left(\frac{1}{38}\right)^{10} \approx 1.59 \times 10^{-16}$

∴ There is about a 23% chance the ball will stop on 13 at least once in 10 rolls but an almost 0% chance it will stop on 13 on all 10 rolls.

9. An online dating site claims that 1 out of 4 blind dates end in disappointment. To avoid disappointment, you decide to limit yourself to 3 blind dates in a year. What is wrong with this reasoning?

Each blind date is independent of the other blind dates. Each date, first, second, third, forth, etc. has an equally likely chance of ending badly. (25% chance).

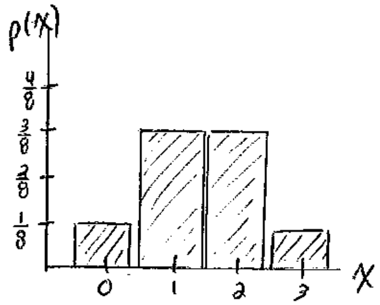
Name: ____________________
Student No.: ____________________

10. Consider a virus that appears and disappears randomly within a population. Suppose that each month the virus has a 50/50 chance of being present within the population during that month. Let X be the discrete random variable that counts the number of virus-free months over the next 3 month period.

(a) Determine the probability mass function, $p(x)$, for X. Draw a histogram for $p(x)$.

range of X: $\{0,1,2,3\}$ $\quad S = \{NNN, NNV, NVN, VNN, NVV, VNV, VVN, VVV\}$

x	$p(x)$
0	$\frac{1}{8}$
1	$\frac{3}{8}$
2	$\frac{3}{8}$
3	$\frac{1}{8}$



(b) Compute the expected value and standard deviation for X.

$$E(X) = \sum_x x\,p(x) = 0\cdot\frac{1}{8} + 1\cdot\frac{3}{8} + 2\cdot\frac{3}{8} + 3\cdot\frac{1}{8} = \frac{12}{8} = 1.5 \text{ months}$$

$$Var(X) = \sum_x (x-1.5)^2 p(x) = (0-1.5)^2\cdot\frac{1}{8} + (1-1.5)^2\cdot\frac{3}{8} + (2-1.5)^2\cdot\frac{3}{8} + (3-1.5)^2\cdot\frac{1}{8} = 0.75$$

$$\text{Std Dev} = \sqrt{Var(X)} = \sqrt{0.75} \approx 0.9 \text{ months}$$

(c) What is the probability that the number of virus-free months will be at least one standard deviation above its mean?

$$P(X \geq 1.5+0.9) = P(X \geq 2.4) = P(X=3) = \frac{1}{8}$$

Name:____________
Student No.: ____________

11. The blood glucose level measurments (mmol/L) for two samples of adults are given below:

(1) Sample of 12 healthy adults:

4.2, 3.6, 3.5, 4, 4.1, 4.2, 4.2, 4, 3.8, 3.8, 3.8, 3.4

(2) Sample of 12 adults who have experienced problems related to increased blood glucose levels:

4, 4.4, 6, 5.8, 6, 4.4, 5, 6.2, 6.8, 6, 6, 6.2

(a) Determine the mean, median, and mode for both samples.

(1) 3.4, 3.5, 3.6, 3.8, 3.8, 3.8, 4, 4, 4.1, 4.2, 4.2, 4.2

$\text{mean} = 3.4\left(\frac{1}{12}\right) + 3.5\left(\frac{1}{12}\right) + 3.6\left(\frac{1}{12}\right) + 3.8\left(\frac{3}{12}\right) + 4\left(\frac{2}{12}\right) + 4.1\left(\frac{1}{12}\right) + 4.2\left(\frac{3}{12}\right)$

≈ 3.9 mmol/L

$\text{median} = \frac{3.8+4}{2} = 3.9$ mmol/L

mode = 3.8, 4.2 mmol/L

(2) 4, 4.4, 4.4, 5, 5.8, 6, 6, 6, 6, 6.2, 6.2, 6.8

mean ≈ 5.6 mmol/L

$\text{median} = \frac{6+6}{2} = 6$ mmol/L

mode = 6 mmol/L

(b) For each sample, calculate the five-number summary and draw box plots for comparison.

(1) min = 3.4

$Q_1 = \frac{3.6+3.8}{2} = 3.7$

$M = 3.9$

$Q_3 = \frac{4.1+4.2}{2} = 4.15$

max = 4.2

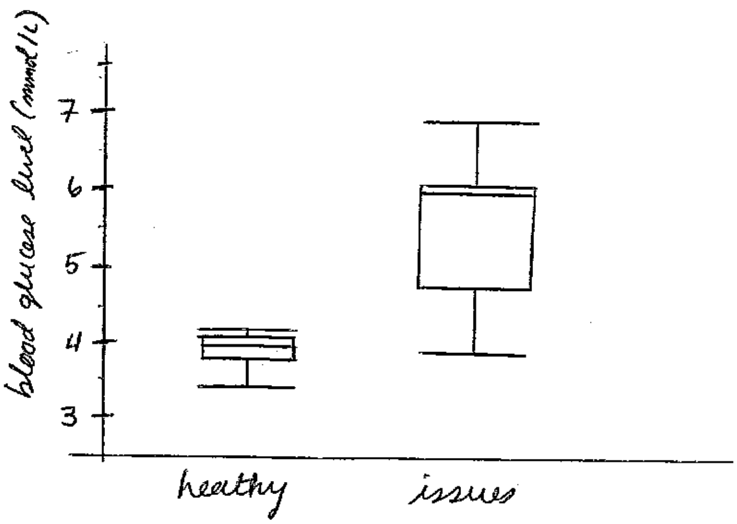
(2) min = 4

$Q_1 = \frac{4.4+5}{2} = 4.7$

$M = 6$

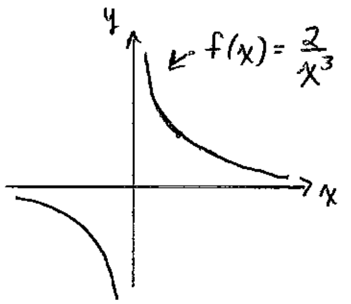
$Q_3 = \frac{6+6.2}{2} = 6.1$

max = 6.8



12. Consider the function $f(x) = \dfrac{2}{x^3}$ on the interval $[1, \infty)$.

(a) Verify that f can be a probability density function for a continuous random variable X.



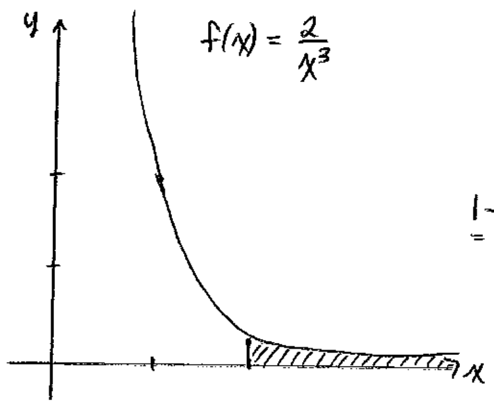
$f(x) = \frac{2}{x^3} > 0$ on $(0, \infty)$ ✓ (so, on $[1, \infty)$ as required)

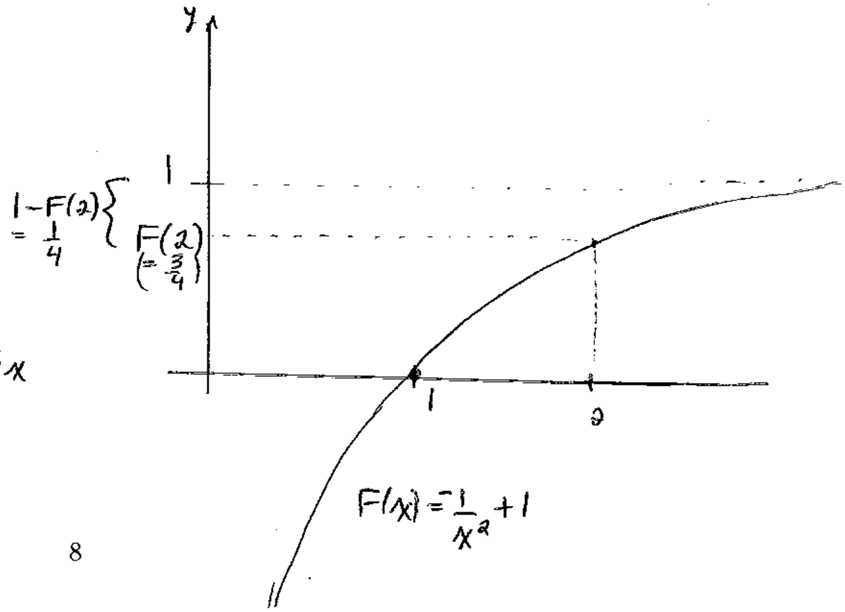
$$\int_1^\infty \frac{2}{x^3}\,dx = \lim_{T\to\infty} \int_1^T 2x^{-3}\,dx$$
$$= \lim_{T\to\infty} \left[-\frac{1}{x^2} \right|_1^T$$
$$= \lim_{T\to\infty} \left[-\frac{1}{T^2} - \left(-\frac{1}{1^2}\right)\right]$$
$$= 1 - \frac{1}{\infty^2} \to 0$$
$$= 1 \checkmark$$

(b) Determine the corresponding cumulative distribution function, $F(x)$.

$$F(x) = \int_1^x \frac{2}{t^3}\,dt = \left[-\frac{1}{t^2} \right|_1^x = -\frac{1}{x^2} - \left(-\frac{1}{1^2}\right) = 1 - \frac{1}{x^2}$$

(c) Graph both $f(x)$ and $F(x)$.





12. continued...

(d) Calculate $P(X > 2)$. Illustrate on both graphs in (c) what this number represents geometrically.

$$P(X > 2) = 1 - P(X \leq 2)$$
$$= 1 - F(2)$$
$$= 1 - \left[1 - \frac{1}{2^2}\right]$$
$$= \frac{1}{4}$$

(e) Compute the expected value and the standard deviation of X.

$$\mu = E(X) = \int_1^\infty x \cdot \frac{2}{x^3}\,dx = \int_1^\infty \frac{2}{x^2}\,dx = \lim_{T\to\infty} \int_1^T 2x^{-2}\,dx = \lim_{T\to\infty} \left[-\frac{2}{x}\right|_1^T = \ldots$$

$$\ldots = \lim_{T\to\infty} \left[-\frac{2}{T} - \left(-\frac{2}{1}\right)\right] = 2 - \frac{2}{\infty}^{\to 0} = 2$$

$$E(X^2) = \int_1^\infty x^2 \cdot \frac{2}{x^3}\,dx = \lim_{T\to\infty} \int_1^T \frac{2}{x}\,dx = \lim_{T\to\infty} \left[2\ln|x|\right|_1^T = \lim_{T\to\infty}\left[2\ln T - 2\ln 1\right] = \ldots$$

$$\ldots = 2\ln\infty - 0 = \infty$$

$\because$ this integral is divergent, we are not able to calculate variance or standard deviation

(f) What is the probability that a value of X will be within one standard deviation of its mean?

Can't calculate

Name:________________
Student No.: ________________

13. Find the following using Table 14.4.

(a) Given that X is normally distributed with mean 3 and variance 4, find the probability that X is less than 4.1.

$X \sim N(3, 2^2)$

$$P(X < 4.1) = P\left(\frac{X-3}{2} < \frac{4.1-3}{2}\right)$$
$$= P(Z < 0.55)$$
$$= F(0.55)$$
$$= 0.708840$$

(b) Let $X \sim N(-2, \underset{3^2}{9})$; find $P(1 \le X \le 5)$.

$$P(1 \le X \le 5) = P\left(\frac{1-(-2)}{3} \le \frac{X-(-2)}{3} \le \frac{5-(-2)}{3}\right)$$
$$\simeq P(1 \le Z \le 2.3)$$
$$= F(2.3) - F(1)$$
$$= 0.989276 - 0.841345$$
$$= 0.147931$$

(c) Suppose that $X \sim N(2, \overset{12^2}{144})$.
Find an approximate value of x such that (i) $P(X \le x) = 0.95$ and (ii) $P(X > x) = 0.3$.

(i) $P(Z \le z) = F(z) = 0.95$ when $z \approx 1.65$

$z = \frac{X-2}{12} \Rightarrow X = 12z + 2$

so, $P(X \le x) = 0.95$ when $x \approx 12(1.65) + 2 \approx 21.8$

(ii) $P(Z > z) = 1 - P(Z \le z) = 1 - F(z) = 0.3 \Rightarrow F(z) = 1 - 0.3 = 0.7$

$F(z) = 0.7$ when $z \approx 0.55$

So, $P(X > x) = 0.3$ when $x \approx 12(0.55) + 2 \approx 8.6$

Name:

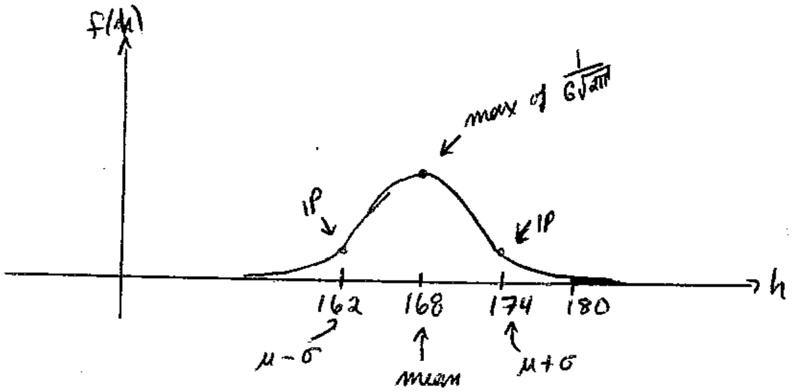
Student No.:

14. Let H represent the height of a university student. Assume that heights are approximately normally distributed with a mean of 168 cm and a standard deviation of 6 cm.

(a) Write the formula for the probability density function for H and draw a rough sketch, labelling the mean, maximum, and location of inflection points. Approximately what proportion of students are within one standard deviation from the mean?

$H \sim N(168, 6^2)$

$$f(h) = \frac{1}{6\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{h-168}{6}\right)^2}$$

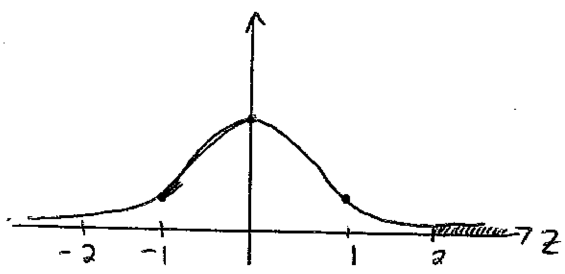


Approximately 68% of students are within one standard deviation of the mean (most are 168cm ± 6cm).

(b) What is the probability that a randomly chosen student is taller than 180 cm? Shade the area representing this probability on your sketch in part (a). Sketch the probability density function for the standard normal random variable Z and shade in the area representing the equivalent probability.

$$\begin{aligned} P(H > 180) &= 1 - P(H \le 180) \\ &= 1 - P\left(\frac{H-168}{6} \le \frac{180-168}{6}\right) \\ &= 1 - P(Z \le 2) \\ &= 1 - F(2) \\ &= 1 - 0.977250 \\ &= 0.02275 \end{aligned}$$

∴ The probability is about 2.3%.



Name:
Student No.:

15. (a) Determine the Taylor polynomial of degree 6 for $f(x) = e^{-\frac{x^2}{2}}$ near $x = 0$. (Hint: Find the Taylor polynomial of degree 3 for $g(x) = e^x$ then replace x by $\frac{-x^2}{2}$ in the formula).

$$T_3(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} \quad \text{for } e^x$$

$$e^x \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{6} \quad \text{near } 0$$

$$e^{-\frac{x^2}{2}} \approx 1 + \left(-\frac{x^2}{2}\right) + \frac{1}{2}\left(-\frac{x^2}{2}\right)^2 + \frac{1}{6}\left(-\frac{x^2}{2}\right)^3 \approx 1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{48}$$

(b) Use your approximation in part (a) to estimate $P(0 \le Z \le 0.5)$. Compare this to the value obtained using Table 14.4.

$$P(0 \le Z \le 0.5) = \int_0^{0.5} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\, dz$$

$$\approx \frac{1}{\sqrt{2\pi}} \int_0^{0.5} \left(1 - \frac{z^2}{2} + \frac{z^4}{8} - \frac{z^6}{48}\right) dz$$

$$\approx \frac{1}{\sqrt{2\pi}} \left[z - \frac{z^3}{6} + \frac{z^5}{40} - \frac{z^7}{336} \right]_0^{0.5}$$

$$\approx 0.19146224$$

$$P(0 \le Z \le 0.5) = F(0.5) - F(0)$$

$$= 0.691462 - 0.5$$

$$= 0.191462$$

THE END