12A3

Last Day Antiderivatives

$F(x)$ is an antiderivative of $f(x)$ if $F'(x) = f(x)$

eg. $\frac{d}{dx} x = 1 \Rightarrow$ x is an antiderivative of 1

$\frac{d}{dx} x^2 = 2x$ $\qquad$ $\frac{d}{dx} x^2+10 = 2x$

$\frac{d}{dx}\left(x^2+59^{59}\right) = 2x$ $\qquad$ $\frac{d}{dx}\left(x^2+c\right) = 2x$

If $F(x)$ is an antiderivative of $f(x)$

then any $F(x) + c$ is also an antiderivative of $f(x)$

(for all real constant c)

Notice Say $F(x)$ & $G(x)$ are both antiderivs of $f(x)$

$\Rightarrow F'(x) = f(x) = G'(x)$

$\Rightarrow F'(x) - G'(x) = 0 = \frac{d}{dx}\left(F(x) - G(x)\right)$

always!

but by MVT on any interval $[a,b]$

if F, G are cont. & diff $\Rightarrow$ ~~F(x)~~ diffs is constantly zero!

Why? Let $\Delta = F - G$

$$\underbrace{\frac{\Delta(b) - \Delta(a)}{b - a} = \Delta'(c)}_{\text{MVT}} = 0$$

$$\Delta(b) - \Delta(a) = 0 \Rightarrow \Delta(b) = \Delta(a) = \underline{\underline{\text{const}}}$$

$$\Rightarrow \underline{\underline{F(x) - G(x)}} = \Delta(x) = \underline{\underline{C}}$$

$\Rightarrow$ **Punchline** or **TL;DR**

If $F(x)$ & $G(x)$ are both antideriv. of $f(x)$

$$\Rightarrow F(x) - G(x) = \text{const.}$$

So we can define: General Antiderivative of $f(x)$

Notation: If $F'(x) = f(x)$ (i.e. F is any antiderivative of $f(x)$)

then general antiderivative

$$\int f(x)\,dx = F(x) + C$$

aka. "The Indefinite Integral"

e.g. $\int 1\,dx = x + C$ $\}$ ⇒ family of antiderivatives! (arbitrary constant!)

$$\int \sin x\,dx = -\cos x + C$$

$$\int \sec^2 x\,dx = \tan x + C$$

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1}(x) + C$$

$$\int x\,dx = \frac{x^2}{2} + C \quad \Big\} \Leftarrow \frac{1}{2}\frac{d}{dx} x^2 = \frac{2x}{2}$$

$$\frac{d}{dx}\frac{x^2}{2} = x$$

because $\frac{1}{2}$ = const. multiple!

$$\int x^2\,dx = \frac{x^3}{3} + C$$

$$\int x^5\,dx = \frac{x^6}{6} + C$$

$$\boxed{\int x^p\,dx = \frac{x^{p+1}}{p+1} + C \quad p \neq -1}$$

Power Rule of Integration

$$\int x^{101}\,dx = x^{102}/102 + C$$

$$\int \sqrt{x}\,dx = \frac{x^{\frac{1}{2}+1}}{1+\frac{1}{2}} + c = \frac{2}{3}x^{3/2} + C$$

$$\int \frac{1}{x}\,dx = \int x^{-1}\,dx$$

$\underline{\underline{x>0}}$

$$= \ln x + C, \ \underline{\underline{x>0}}$$

$$\frac{d}{dx}\, ? = \frac{1}{x}$$

$$? = \ln x$$

$$\int \frac{1}{\text{mowa}}\, d\,\text{mower} = \ln(\text{mowa}) + C$$

$(\text{mowa} > 0)$

what about $\underline{\underline{-\text{ve } x\text{'s}}}$?

$\ln x + \underline{\underline{\ln k}}$

$$\frac{d}{dx}\ln x = \frac{1}{x}, \quad \frac{d}{dx}\ln(kx) = \frac{1}{kx}\cdot k = \frac{1}{x}$$

$$\frac{d}{dx}\ln(-x) = \frac{1}{(-x)}(-1) = \frac{1}{x}$$

domain: $x < 0$ only!

so

$$\frac{d}{dx}\ln x = \frac{1}{x}, \quad x > 0 \qquad \frac{d}{dx}\ln(-x) = \frac{1}{x}, \quad x < 0$$

so $\frac{d}{dx}\ln|x| = \frac{1}{x}$ all $x \neq 0$

$$\Rightarrow \int \frac{1}{x}\,dx = \ln|x| + C \quad \text{all } x \neq 0$$

Integration Laws: If $\int f(x)\,dx = F(x)+c$, $\int g(x)\,dx = G(x)+c$

1) $\int f(x) \pm g(x)\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

$= (F(x) + c_1) \pm (G(x) + c_2)$

$= F(x) + G(x) + C$

2) $\int k f(x)\,dx$, $k \in \mathbb{R}$

$= k \int f(x)\,dx = k(F(x) + c)$

$= kF(x) + c$

c arbitrary c

eg. $\int 7x^3 + 2x - \cosh x \, dx$

$= \int 7x^3 \, dx + \int 2x \, dx - \int \cosh x \, dx$

$= 7 \int x^3 \, dx + 2 \int x \, dx - \int \cosh x \, dx$

$= 7 \cdot \left(\frac{x^4}{4} \right) + 2 \left(\frac{x^2}{2} \right) - \sinh(x) + C$

$= \left(\frac{7}{4} \right) x^4 + x^2 - \sinh(x) + C$

eg. $\left. \int \frac{\sqrt{x} + x^9 + 4}{x} \, dx \right\}$ Quotients Evil

Kill with fire!

(Divide!)

$= \int \frac{x^{1/2} + x^9 + 4}{x} \, dx$

$$= \int x^{-\frac{1}{2}} + x^{8} + 4 \cdot \frac{1}{x} \, dx$$

$$= \frac{x^{1/2}}{(1/2)} + \frac{x^{9}}{9} + 4 \ln|x| + C$$

$$= 2\sqrt{x} + \frac{1}{9} x^{9} + \ln(x^{4}) + C$$

eg. $\int 10^{x} dx \longrightarrow \frac{d}{dx} 10^{x} = 10^{x} \cdot \underline{\ln 10}$

$$= \frac{10^{x}}{\ln 10} + C$$

$\Rightarrow \left(\frac{1}{\ln 10} \frac{d}{dx} 10^{x}\right) = 10^{x}$

$\Rightarrow \frac{d}{dx}\left(\frac{10^{x}}{\ln 10}\right) = 10^{x}$

constant

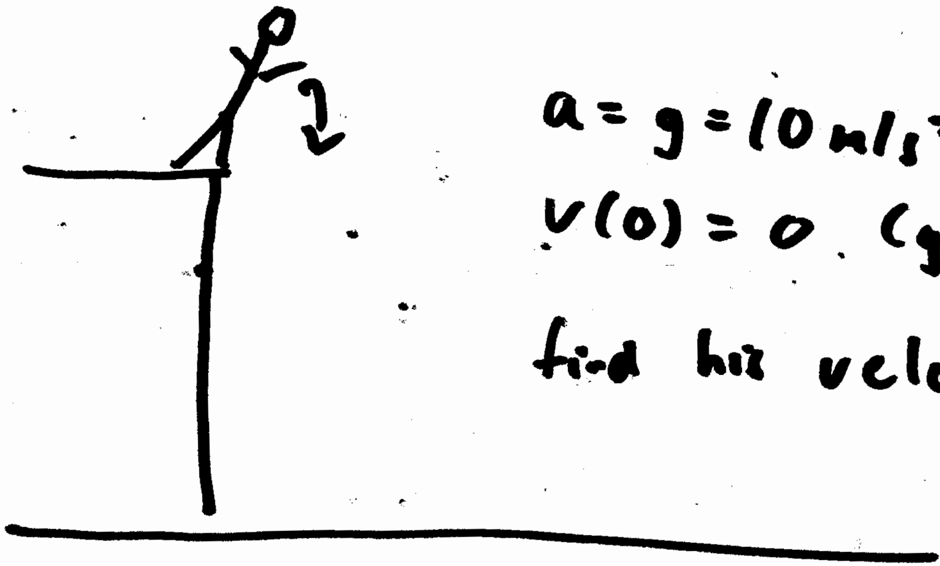
eg $\int 2xe^{x^2}dx$ ← notice $\frac{d}{dx} e^{x^2} = 2xe^{x^2}$

$= \underline{e^{x^2} + c}$

we'll reverse chain & product next week

Today Let's talk App: "Initial Value Problem" (IVP)

eg



$a = g = 10\, m/s^2$

$v(0) = 0$. (given)

find his velocity at time t

Solution

$a = \text{const}$

$a = 10 = v' \Rightarrow v = \int a\, dt$

$= a \int 1\, dt$

$v = at + c$

$= 10t + c$

but I. C.

$v(0) = 0$

$v(0) = 0 = 10(0) + c \Rightarrow c = 0$

Our particular solution for our initial condition is

$$\boxed{v(t) = 10t}$$

Part b) If at $t = 0$, $p(0) = 0$ = position)

find $p(t)$

Solution

$$p'(t) = v(t) = 10t$$

$$p = \int 10t\,dt = 10 \cdot \underline{\tfrac{1}{2}t^2 + C}$$

If $p(0) = 0 \rightsquigarrow$

$$0 = 5\underset{0}{\cancel{t^2}} + C \rightsquigarrow \underline{C = 0}$$

$$\boxed{p(t) = 5t^2}$$