Discrete Mathematics with Applications I

COMPSCI&SFWRENG 2DM3

McMaster University, Fall 2019

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“Read”!

$$Aos \,\llparenthesis\, (C \cap \sim(B \fatsemi C^{\smile}))^{\smile} \setminus B \,\rrparenthesis\, Jun$$

$$B \fatsemi (\{Jun\} \times \llcorner P \lrcorner) \quad \cap \quad (C \fatsemi C^{\smile}) \quad \subseteq \quad \mathrm{Id}$$

$$
\begin{array}{ll}
 & Aos \,\llparenthesis\, (C \cap \sim(B \fatsemi C^{\smile}))^{\smile} \setminus B \,\rrparenthesis\, Jun \\
= & \langle \text{ Right residual, relation converse } \rangle \\
 & \forall\, p \;\mid\; Aos \,\llparenthesis\, C \cap \sim(B \fatsemi C^{\smile}) \,\rrparenthesis\, p \;\bullet\; p \,\llparenthesis\, B \,\rrparenthesis\, Jun \\
= & \langle \text{ Relation intersection, Relation complement } \rangle \\
 & \forall\, p \;\mid\; Aos \,\llparenthesis\, C \,\rrparenthesis\, p \wedge \neg(Aos \,\llparenthesis\, B \fatsemi C^{\smile} \,\rrparenthesis\, p \;\bullet\; p \,\llparenthesis\, B \,\rrparenthesis\, Jun \\
= & \langle \text{ (9.3b) Trading for } \forall\text{, Double negation } \rangle \\
 & (\forall\, p \;\mid\; Aos \,\llparenthesis\, C \,\rrparenthesis\, p \;\bullet\; p \,\llparenthesis\, B \,\rrparenthesis\, Jun \vee Aos \,\llparenthesis\, B \fatsemi C^{\smile} \,\rrparenthesis\, p) \\
= & \langle \text{ (14.20) Relation composition } \rangle \\
 & (\forall\, p \;\mid\; Aos \,\llparenthesis\, C \,\rrparenthesis\, p \;\bullet\; p \,\llparenthesis\, B \,\rrparenthesis\, Jun \vee (\exists\, s \;\bullet\; Aos \,\llparenthesis\, B \,\rrparenthesis\, s \,\llparenthesis\, C^{\smile} \,\rrparenthesis\, p)) \\
= & \langle \text{ (14.18) Relation converse } \rangle \\
 & (\forall\, p \;\mid\; Aos \,\llparenthesis\, C \,\rrparenthesis\, p \;\bullet\; p \,\llparenthesis\, B \,\rrparenthesis\, Jun \vee (\exists\, s \;\bullet\; Aos \,\llparenthesis\, B \,\rrparenthesis\, s \wedge p \,\llparenthesis\, C \,\rrparenthesis\, s))
\end{array}
$$

“Everybody Aos called is a brother of Jun or called a sibling of Aos.”
“Aos called at most Jun’s brothers or people who called his siblings.”

$$(B ⨾ (\{Jun\} \times \llcorner P \lrcorner)) \cap (C ⨾ C^{\smile}) \subseteq \mathrm{Id}$$

$= \langle$ Relation inclusion $\rangle$

$$(\forall\, b, p : P \mid b \,(\!|\, B ⨾ (\{Jun\} \times \llcorner P \lrcorner)) \cap (C ⨾ C^{\smile}) \,|\!)\, p \bullet b \,(\!|\, \mathrm{Id} \,|\!)\, p)$$

$= \langle$ Relation intersection, Identity relation $\rangle$

$$(\forall\, b, p : P \mid b \,(\!|\, B ⨾ (\{Jun\} \times \llcorner P \lrcorner)) \,|\!)\, p \wedge b \,(\!|\, C ⨾ C^{\smile} \,|\!)\, p \bullet b = p)$$

$= \langle$ Relation composition $\rangle$

$$(\forall\, b, p : P \mid (\exists\, q \bullet b \,(\!|\, B \,|\!)\, q \wedge \langle q, p \rangle \in (\{Jun\} \times \llcorner P \lrcorner)) \wedge (\exists\, r \bullet b \,(\!|\, C \,|\!)\, r \wedge r \,(\!|\, C^{\smile} \,|\!)\, p) \bullet b = p)$$

$= \langle$ (14.4) Cart. prod. membership, (14.18) Relation converse $\rangle$

$$(\forall\, b, p : P \mid (\exists\, q \bullet b \,(\!|\, B \,|\!)\, q \wedge q \in \{Jun\} \wedge p \in \llcorner P \lrcorner) \wedge (\exists\, r \bullet b \,(\!|\, C \,|\!)\, r \wedge p \,(\!|\, C \,|\!)\, r) \bullet b = p)$$

$= \langle$ Universal set (since $\llcorner P \lrcorner = \mathbf{U}$), Identity of $\wedge$, $x \in \{y\} \equiv x = y$ $\rangle$

$$(\forall\, b, p : P \mid (\exists\, q \bullet b \,(\!|\, B \,|\!)\, q \wedge q = Jun) \wedge (\exists\, r \bullet b \,(\!|\, C \,|\!)\, r \wedge p \,(\!|\, C \,|\!)\, r) \bullet b = p)$$

$= \langle$ (9.19) Trading for $\exists$, (8.14) One-point rule $\rangle$

$$(\forall\, b, p : P \mid b \,(\!|\, B \,|\!)\, Jun \wedge (\exists\, r \bullet b \,(\!|\, C \,|\!)\, r \wedge p \,(\!|\, C \,|\!)\, r) \bullet b = p)$$

$= \langle$ (8.20) Quantification nesting $\rangle$

$$(\forall\, b \mid b \,(\!|\, B \,|\!)\, Jun \bullet (\forall\, p \mid (\exists\, r \bullet b \,(\!|\, C \,|\!)\, r \wedge p \,(\!|\, C \,|\!)\, r) \bullet b = p))$$

"Each brother of Jun called only people whom nobody else called."

Domain- and Range-Restriction and -Antirestriction

Given types t_1, t_2 : Type, sets A : **set** t_1 and B : **set** t_2, and relation $R : t_1 \leftrightarrow t_2$:

- **Domain restriction**: $A \lhd R \;=\; R \cap (A \times \mathbf{U})$
- **Domain antirestriction**: $A \ntriangleleft R \;=\; R \cap ({\sim} A \times \mathbf{U})$
- **Range restriction**: $R \rhd B \;=\; R \cap (\mathbf{U} \times B)$
- **Range antirestriction**: $R \ntriangleright B \;=\; R \cap (\mathbf{U} \times {\sim} B)$

$$B ⨾ (\{Jun\} \times \llcorner P \lrcorner) \;\cap\; (C ⨾ C^{\smile}) \;\subseteq\; \mathrm{Id}$$

$\equiv \langle$ Domain- and range restriction properties $\rangle$

$$Dom(B \rhd \{Jun\}) \lhd (C ⨾ C^{\smile}) \;\subseteq\; \mathrm{Id}$$

Still no quantifiers, and no x, y of element type — but not only relations, also sets!

(The abstract version of this is called **Peirce algebra**, after Chales Sanders Peirce.)

Relational Image and Relation Overriding

Given types t_1, t_2 : Type, sets A : **set** t_1 and B : **set** t_2, and relations $R, S : t_1 \leftrightarrow t_2$:

- **Relational image**: $R (\!| A |\!) \;=\; Ran(A \lhd R)$

$$B ⨾ (\{Jun\} \times \llcorner P \lrcorner) \;\cap\; (C ⨾ C^{\smile}) \;\subseteq\; \mathrm{Id}$$

$\equiv \langle$ Domain- and range restriction properties $\rangle$

$$Dom(B \rhd \{Jun\}) \lhd (C ⨾ C^{\smile}) \;\subseteq\; \mathrm{Id}$$

$\equiv \langle$ Relational image $\rangle$

$$(B^{\smile} (\!| \{Jun\} |\!)) \lhd (C ⨾ C^{\smile}) \;\subseteq\; \mathrm{Id}$$

- **Relation overriding**: $R \oplus S \;=\; (Dom\, S \ntriangleleft R) \cup S$

Plan for Today

- **Operators involving relations and sets** $\surd$
- **Simple Graphs**

(Graphs), Simple Graphs

A **graph** consists of:

- a set of “nodes” or “vertices”
- a set of “edges” or “arrows”
- “incidence” information specifying how edges connect nodes

— ***more details another day.***

A **simple graph** consists of:

- a set of “nodes”, and
- a set of “edges”, which **are** pairs of nodes.

(A simple graph has no “parallel edges”.)

Formally: A **simple graph** (N, E) is a pair consisting of

- a set N, the elements of which are called “nodes”, and
- a relation $E \subseteq N \times N$, the element pairs of which are called “edges”.

Simple Graphs: Example

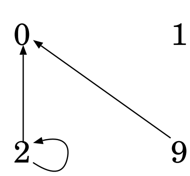
Formally: A **simple graph** (N, E) is a pair consisting of

- a set N, the elements of which are called “nodes”, and
- a relation $E \subseteq N \times N$, the element pairs of which are called “edges”.

Example:

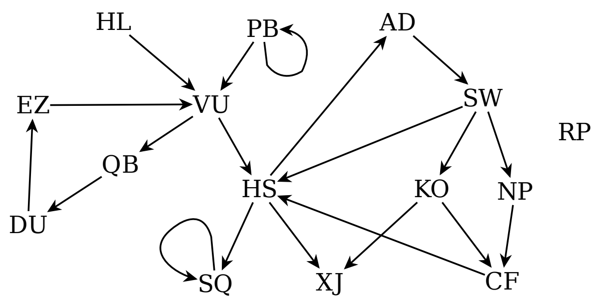
$$G_1 = (\{2, 0, 1, 9\}, \{\langle 2, 0\rangle, \langle 9, 0\rangle, \langle 2, 2\rangle\})$$

Graphs are normally visualised via **graph drawings**:

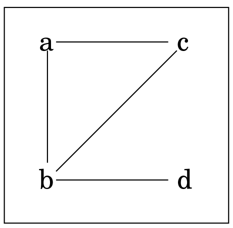


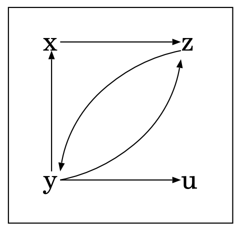
Reachability in graph $G = (V, E)$ — 1a

- No edge ends at node s
  $s \notin \mathit{Ran}\ E$ or $s \in \sim(\mathit{Ran}\ E)$ — s is called a **source** of G
- No edge starts at node s
  $s \notin \mathit{Dom}\ E$ or $s \in \sim(\mathit{Dom}\ E)$ — s is called a **sink** of G
- Node n_2 is reachable from node n_1 via a three-edge path
  $n_1 \mathrel{(\!|} E \mathbin{;} E \mathbin{;} E \mathrel{|\!)} n_2$



Directed versus Undirected Graphs





- Edges in undirected graphs can be considered as “unordered pairs” (two-element sets, or one-to-two-element sets)
- The **associated relation** of an undirected graph relates two nodes if there is an edge between them
- **The associated relation of an undirected graph is always symmetric**
- In a **simple** graph, no two edges have the same source and the same target. (No “parallel edges”.)
- Relations directly represent simple graphs.

Symmetric Closure

Relation $Q : B \leftrightarrow B$ is the **symmetric closure** of $R : B \leftrightarrow B$
iff Q is the smallest symmetric relation containing R,

or, equivalently, iff

- $R \subseteq Q$
- $Q = Q^{\smile}$
- $(\forall P : B \leftrightarrow B \mid R \subseteq P = P^{\smile} \bullet Q \subseteq P)$

Theorem: The symmetric closure of $R : B \leftrightarrow B$ is $R \cup R^{\smile}$.

Fact: If R represents a simple directed graph, then the symmetric closure of R is the associated relation of the corresponding simple undirected graph.