Announcements

Topics:

- Review of Differential Equations and Integration Techniques (7.1, 7.2, and 7.5)
- Analysis of Autonomous DEs – Population Models (8.1)

To Do:

- Review sections 7.1, 7.2, and 7.5 in the textbook
- Read section 8.1 in the textbook
- Work on Assignment 1 posted on the webpage under the SCHEDULE + HOMEWORK link

Differential Equations

A **differential equation (DE)** is an equation that involves an unknown function and one or more of its derivatives.

Examples:

$y' = 2 + y$ $\qquad$ $y'' + 2xy = x^2$ $\qquad$ $y' = x^2 + e^x$

Differential Equations

A **solution** of a differential equation is a *function* that, along with its derivatives, satisfies the DE.

Example:

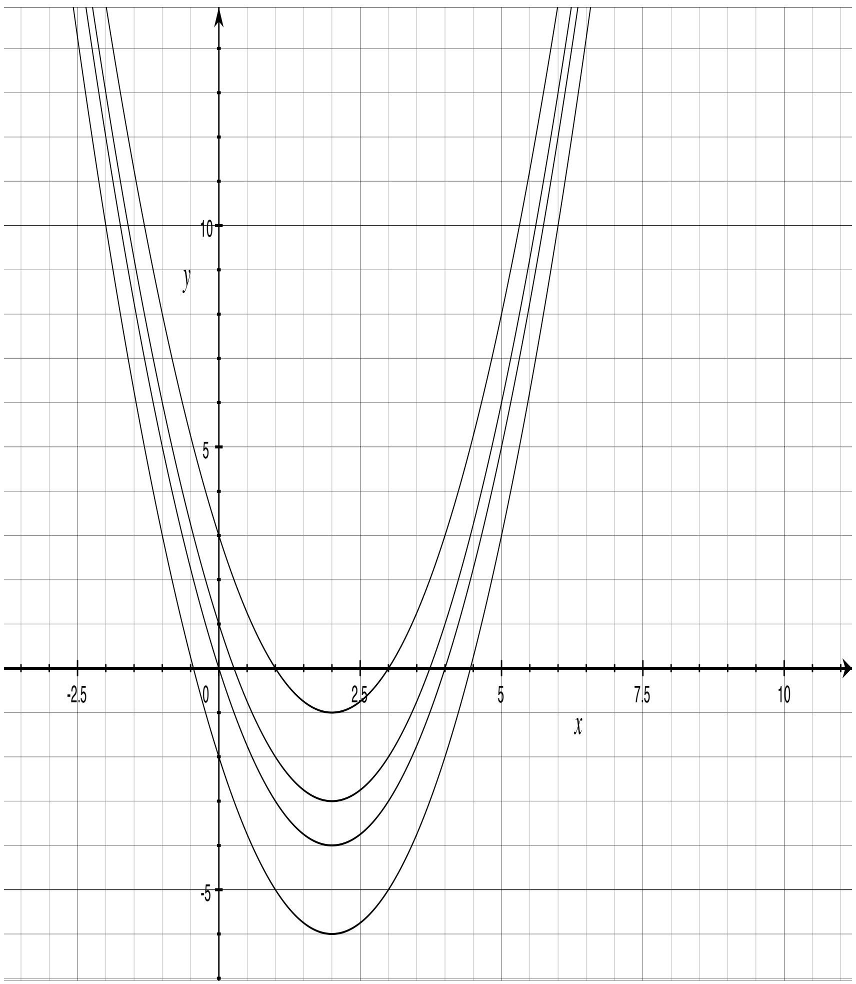
Show that $z(t) = 1 + \sqrt{1 + 2t}$ is the solution of the differential equation $\dfrac{dz}{dt} = \dfrac{1}{z - 1}$ with initial condition $z(0) = 2$.

Differential Equations

In general, a differential equation has a whole ***family*** of solutions.

Example:

Find the general solution of the DE $y' = 2x - 4$.

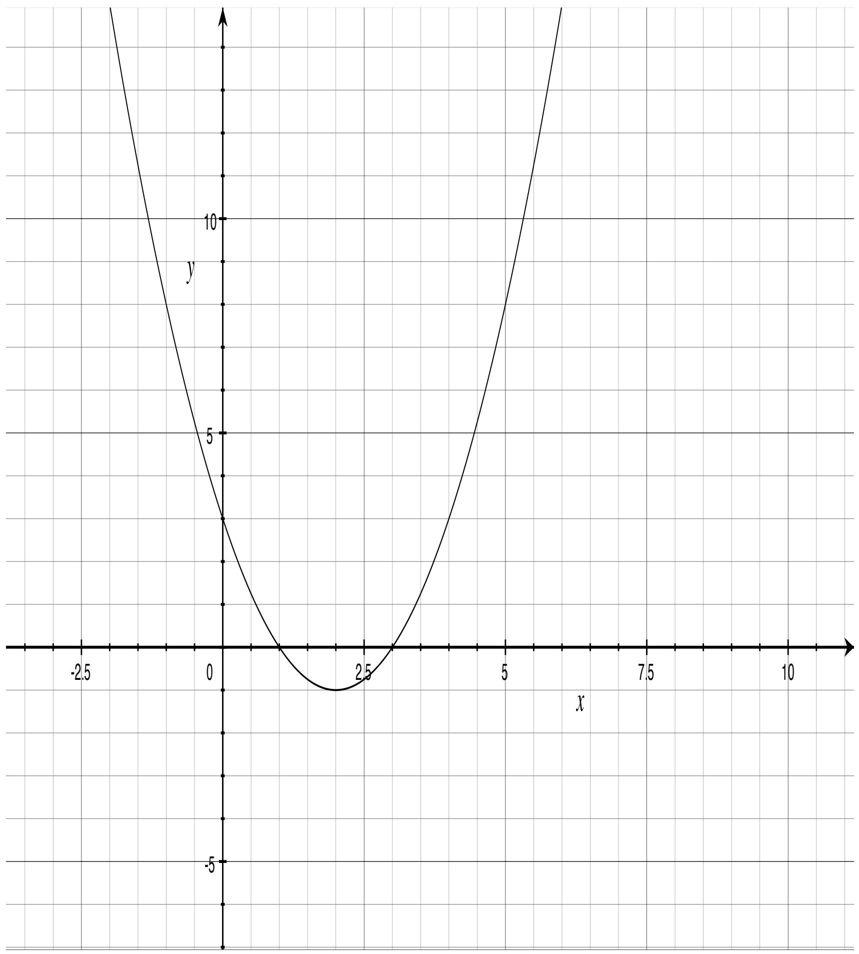


Differential Equations

An **initial value problem (IVP)** provides an initial condition so you can find a particular solution.

Example:

Find the unique solution of the IVP $y' = 2x - 4, \quad y(0) = 3.$



Modeling: Verbal Descriptions ⇨ IVPs

Example:
Write a differential equation and an initial condition to describe the following events.

(a) The relative rate of change of the population of wild foxes in an ecosystem is 0.75 baby foxes per fox per month. Initially, the population is 74 thousand.

(b) The rate of change of the thickness of the ice on a lake is inversely proportional to the square root of its thickness. Initially, the ice is 3 mm thick.

Solutions for General DEs

- <u>**Algebraic Solutions**</u>
  - an explicit formula or algorithm for the solution (often, impossible to find)
- <u>**Geometric Solutions**</u>
  - a sketch of the solution obtained from analyzing the DE
- <u>**Numeric Solutions**</u>
  - an approximation of the solution using technology and and some estimation method, such as Euler's method

Algebraic Solutions

Example 1:

Find the general solution of the pure-time DE

$$\frac{dy}{dx} = 5e^{10x} + \frac{1}{1+25x^2}$$

Example 2:

Find the general solution of the pure-time DE

$$y' = \ln x$$

Algebraic Solutions

Example 3:

Find the solution of the autonomous DE $\dfrac{dP}{dt} = 0.23P$ with initial condition $P(0) = 80$.

More Integration Practice

Example:

(a) $\int \frac{x}{1+x^2}\, dx$

(b) $\int \frac{x^2}{1+x^2}\, dx$

(c) $\int xe^{0.2x} dx$

(d) $\int xe^{-x^2} dx$

(e) $\int \frac{1}{x \ln x}\, dx$

(f) $\int x \ln x \, dx$

Geometric Solutions

Example:

Sketch the graph of the solution to the DE

$$y' = \arctan x$$

given an initial condition of $y(0) = 1$.

Euler's Method

Algorithm:

$$t_{n+1} = t_n + h$$

$$y_{n+1} = y_n + F(t_n, y_n)h$$

Algorithm In Words:

next time step = previous time step + step size

next approximation = previous approximation + rate of change of the function x step size

Euler's Method

Example:
Consider the IVP

$$\frac{dP}{dt} = e^{-t^2}, \quad P(0) = 5$$

Approximate P(1) using Euler's method and a step size of h=0.5.

<u>Note:</u> We are not able to find an exact solution for this IVP.

Euler's Method

Example:

Calculations:

Table of Approximate Values for the Solution P(t) of the IVP

$t_n = t_{n-1} + h$	P_n = approx. value of solution at t_n
$t_0 = 0$	$P_0 = 5$

Qualitative Analysis of a DE

We can analyze a DE *qualitatively* to determine important characteristics of solutions, without explicitly solving the equation.

Qualitative Analysis of a DE

Example:
Consider the following autonomous DE describing the growth of a certain population.

$$\frac{dP}{dt} = 2P(100 - P)$$

$t = \text{time}$

$P(t) = \#\text{ of individuals at time } t$

When is the population constant? When is the population increasing? When is it decreasing?

Modelling

- Start with a simple model (differential equation) to roughly explain how a system changes then modify so it fits real-life observable data as close as possible.
- If you then observe an initial condition, you can use this rule (DE) to generate a solution and use it to predict future values.

Basic Exponential Model

Model:

$$\frac{dP}{dt} = k \cdot P(t)$$

P(t) = the number of individuals at time *t*
k = proportionality constant

Solution:

$$P(t) = P_0 e^{kt}$$

Basic Exponential Model

Example:
Suppose we know that the growth rate of a population is half of its current population and the initial population is 10. Then we have the model

$$\frac{dP}{dt} = 0.5P \qquad P(0) = 10$$

Analyze the dynamics of this population, assuming that t is measured in years.

Basic Exponential Model

$$\frac{dP}{dt} = 0.5P$$

Equilibrium Solution:

Basic Exponential Model

$$\frac{dP}{dt} = 0.5P$$

Behaviour of Solutions P(t):

Basic Exponential Model

Some Solution Curves + Solution to IVP:

Basic Exponential Model

Summary:

This model describes a population that grows at a rate proportional to its size. It assumes ideal conditions, i.e. unlimited resources, no predators, no disease, etc.

Logistic Model

__Model__:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right)$$

k = positive constant

L = carrying capacity

__carrying capacity__:
the maximum population that the environment is capable of sustaining in the long run

Logistic Model

Example:

A population grows according to the logistic model

$$\frac{dP}{dt} = 0.08P\left(1 - \frac{P}{1000}\right)$$

with initial population $P(0) = 100.$

Analyze the dynamics of this population.

Logistic Model

$$\frac{dP}{dt} = 0.08P\left(1 - \frac{P}{1000}\right)$$

__Equilibrium Solutions__:

Logistic Model

$$\frac{dP}{dt} = 0.08P\left(1 - \frac{P}{1000}\right)$$

Behaviour of Solutions P(t):

Logistic Model

$$\frac{dP}{dt} = 0.08P\left(1 - \frac{P}{1000}\right)$$

Note:

Logistic Model

<u>Some Solution Curves + Solution to IVP</u>:

Logistic Model

Notes:

1. The point at which there is a change in the pattern of increase is called an inflection point.

2. The population size at the point of inflection is one-half of the horizontal asymptote, i.e., one-half of the maximum population.

Logistic Model

<u>*Summary*</u>:

This model describes a population that grows exponentially for small values of *P* but as *P* increases, the growth rate slows down and the population approaches the carrying capacity.

If the population starts above its carrying capacity, it will decrease towards the carrying capacity.

Modified Logistic Differential Equation (the Allee Effect)

Model:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right)\left(1 - \frac{m}{P}\right)$$

where

k , *m,* and *L* are positive constants and $m < L$

L = carrying capacity

m = existential threshold

Modified Logistic Differential Equation (the Allee Effect)

Example:

A population grows according to the modified logistic model

$$\frac{dP}{dt} = 0.09P\left(1 - \frac{P}{2000}\right)\left(1 - \frac{120}{P}\right)$$

Analyze the dynamics of this population.

Modified Logistic Differential Equation (the Allee Effect)

$$\frac{dP}{dt} = 0.09P\left(1 - \frac{P}{2000}\right)\left(1 - \frac{120}{P}\right)$$

Equilibrium Solutions:

Modified Logistic Differential Equation (the Allee Effect)

$$\frac{dP}{dt} = 0.09P\left(1 - \frac{P}{2000}\right)\left(1 - \frac{120}{P}\right)$$

Behaviour of Solutions P(t):

Modified Logistic Differential Equation (the Allee Effect)

__Some Solution Curves__:

Modified Logistic Differential Equation (the Allee Effect)

<u>*Summary*</u>:

This model is similar to the logistic model but includes the idea of an existential threshold – the minimum number of individuals needed to sustain a population. If the population falls below this number, it will die out (decrease to 0).

Selection Model

Consider two variations of a certain population that grow at a rate proportional to their size.

$$\frac{da}{dt} = \mu a \qquad \qquad \frac{db}{dt} = \lambda b$$

a(t) = population size of type *a* at time *t*;
μ = per capita production rate of type *a*;

b(t) = population size of type *b* at time *t*;
λ = per capita production rate of type *b*.

Selection Model

It is often difficult to count the exact number of individuals for some populations, so instead we measure the ***fraction*** or ***proportion*** of each present in the total population.

$$p = \textit{fraction of type}\, a = \frac{a}{a+b}$$

$\leftarrow$ # of individuals of type a

$\leftarrow$ total population size

$$1 - p = \textit{fraction of type}\, b = \frac{b}{a+b}$$

Selection Model

The rate of change of the fraction of type a can be expressed as a logistic (autonomous) equation:

$$\frac{dp}{dt} = (\mu - \lambda)\, p\,(1 - p)$$

a measure of the strength of selection

<u>Calculations</u>:

Selection Model

Solution:

$$p(t) = \frac{p_0 e^{\mu t}}{p_0 e^{\mu t} + (1 - p_0) e^{\lambda t}}$$

where $p_0 = \dfrac{a_0}{a_0 + b_0}$

Calculations:

Selection Model

Example:
Suppose we find two strains of bacteria, type *a* and type *b*, where the per capita production for *a* is 0.5 and for b is *0.3*.

(a) Write differential equations for the growth rate of each strain.

Selection Model

(b) Write an autonomous DE for *p*, the fraction of type a bacteria present in the sample.

(c) Given that initially 10% of the population is type *a*, write the solution for *p* and use it to find the fraction of type a bacteria present after 2 hours.

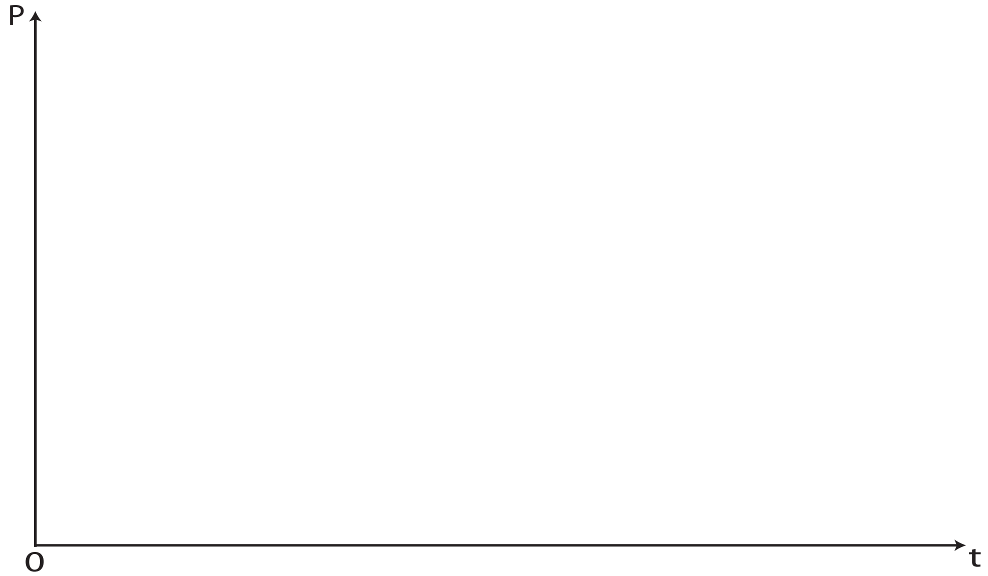
Selection Model

(d) Use Euler's Method with a step size of 1 to approximate the fraction of type *a* after 2 hours. (Compare to answer in (c))

Selection Model

(e) What happens as $t \to \infty$?

(f) Graph the solution.



Selection Model

<u>Summary</u>:

This model describes two variations of some population competing for the same resources. The rate of change of the fraction of type *a* is modeled by a logistic equation.

If the per capita production rate of type *a* is greater than that of type *b*, then type *a* will take over (i.e. the fraction of type a present will approach 1) and vice versa.