1ZC3

Don't Forget your Matlab!

Last Day Eigenvectors & Eigenvalues

If A is an $n \times n$ matrix & $\vec{x}$ is a non-zero column vector

If $A\vec{x} = \lambda \vec{x}$ for some scalar λ

then $\vec{x}$ is an eigenvector of A, with eigenvalue λ

- The λ-eigenspace of A is set of all eigenvectors for a given λ, & $\vec{x} = \vec{0}$

- Includes all sums & multiples of those eigenvectors.
"all linear combinations"

eg. Given $A = \begin{bmatrix} 2 & 1 \\ 6 & 1 \end{bmatrix}$ let's get all eigenvectors & eigenvalues!

Solution! λ's are roots of the characteristic polynomial of A

$$C_A(\lambda) = \det(A - \lambda I)$$

here $|A - \lambda I| = \left| \begin{bmatrix} 2 & 1 \\ 6 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right|$

$$= \begin{vmatrix} 2-\lambda & 1 \\ 6 & 1-\lambda \end{vmatrix}$$

$$= (2-\lambda)(1-\lambda) - 1(6) = \lambda^2 - 3\lambda - 4 = (\lambda - 4)(\lambda + 1)$$

$\lambda = 4, -1$ if $C_A(\lambda) = 0$

To get vectors solve $A\vec{x} = \lambda\vec{x}$ for $\vec{x}$

$$\Rightarrow (A - \lambda I)\vec{x} = \vec{0} \quad \text{solve!}$$

Say $\lambda = -1$ $\Rightarrow$ $A - \lambda I = \begin{bmatrix} 2-\lambda & 1 \\ 6 & 1-\lambda \end{bmatrix}\Bigg|_{\lambda = -1}$

$$= \begin{bmatrix} 2+1 & +1 \\ +6 & 1+1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix}$$

$\Rightarrow$ solve : $\left[\begin{array}{cc|c} 3 & 1 & 0 \\ 6 & 2 & 0 \end{array}\right]$

$\Rightarrow$ Row 2 - 2 Row 1 $\Rightarrow$ $\left[\begin{array}{cc|c} 3 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $\begin{array}{l} \rightsquigarrow 3x + y = 0 \\ \rightsquigarrow 0 = 0 \end{array}$

$$\left.\begin{array}{l} x = -\frac{1}{3}t \\ y = t \end{array}\right\} \Rightarrow \boxed{\begin{pmatrix} x \\ y \end{pmatrix} = t\begin{pmatrix} -\frac{1}{3} \\ 1 \end{pmatrix}}$$

All $\lambda = -1$ eigenvectors have $\uparrow$ this form!

Represents the entire $\lambda = -1$ eigenspace for our A

$\begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix}$ is the basis eigenvector for our eigenspace

$\Downarrow$

generates all rest of $\lambda = -1$ eigenvectors

Note: Any non-zero scalar multiple will equivalently generate the same space! $\Rightarrow$ Pick pretty when possible!

eg $\lambda = -1 \Rightarrow$ equivalent $\lambda = -1$ eigenspace: $\begin{pmatrix} x \\ y \end{pmatrix} = t \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

note $A = \begin{bmatrix} 2 & 1 \\ 6 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 6 & -3 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

$= (-1) \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

Yes! it's a $\lambda = -1$ eigenvector!

$\left\{\right.$ $\lambda = 4$ eigenspace works the same way! We'll leave it as an exercise! (posted later!) $\left.\right)$

What about repeated roots?

eg $B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Let's find eigenvalues!

$c_B(\lambda) = |B - \lambda I| = 0$

$\Rightarrow \quad |B - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & 2-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix}$ ← since triangular!

$= (1-\lambda)(2-\lambda)(2-\lambda) = 0$

$\Rightarrow \quad \lambda = 1, 2, 2$ $\left.\right\}$ For triangular matrix λ's on principal diagonal.

# of repeats of a given root in $C_A(\lambda)$ is "<u>algebraic multiplicity</u>"

here $\lambda = 1$ has mult. of $\underline{\underline{1}}$

$\lambda = 2$ has mult. of $\underline{\underline{2}}$: <u>repeated root</u>

Let's find $\lambda = 2$ eigenspace for $\underline{\underline{B}}$.

<u>Solve</u> $[B - 2I \mid \vec{0}\,]$

$$\left[\begin{array}{ccc|c} \cancel{1}\overset{-1}{2} & 0 & 0 & 0 \\ 1 & \cancel{2}\overset{0}{2} & 0 & 0 \\ 0 & 0 & \cancel{2}\overset{0}{2} & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x = 0 \\ y = t \leftarrow \underline{\text{two}} \\ z = s \leftarrow \underline{\text{params!}} \end{array}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ t \\ s \end{bmatrix} = t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

<u>Two</u> <u>basis</u> <u>eigenvectors</u> for our eigenspace <u>$\lambda = 2$</u>

two distinct directions (non-parallel!)

A <u>two parameter</u> solution.

<u># parameter</u> in eigenspace = "<u>geometric</u> <u>multiplicity</u>"

of our <u>eigenvalue</u>!

for <u>any</u> <u>λ</u> <u>geo. multiplicity ≥ 1</u> always.

& <u>geo. multiplicity</u> $\leq$ <u>algebraic mult.</u>

Compare to: $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$ $\quad c_C(\lambda) = (1-\lambda)(2-\lambda)^2$ as before!

Alg. mult. of $\lambda = 1$ is $\underline{1}$ $\quad$ Alg. mult. of $\lambda = 2$ is $\underline{2}$ again!

But look at $\lambda = 2$ eigenspace of this matrix!

Solve $(C - \lambda I)\vec{x} = \vec{0}$ for $\lambda = 2$

$$[\, C - 2I \mid \vec{0} \,] = \left[\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \quad \begin{aligned} x &= 0 \\ y &= 0 \\ z &= t \end{aligned} \quad \Rightarrow$$ eigenspace for $\lambda = 2$ & matrix $\underline{C}$

is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

So $\begin{pmatrix} \text{geometric mult.} \\ \text{for } \lambda = 2 \end{pmatrix} = 1 \le 2 = \begin{pmatrix} \text{alg. mult} \\ \text{for } \lambda = 2. \end{pmatrix}$

<u>properties</u> if λ is an eigenvalue of A

$$1 \le \begin{pmatrix} \text{Geo. multiplicity} \\ = \#\text{ parameters in eigenspace} \\ = \#\text{ distinct basic eigenvectors.} \end{pmatrix} \le \begin{pmatrix} \text{alg. multiplicity} \\ = \#\text{ repeats of the} \\ \text{root } \lambda \text{ in} \\ c_A(\lambda) \end{pmatrix} \le \underline{\underline{n}}$$

Sum of <u>all</u> <u>alg. multiplicities</u> for all $\underline{\underline{\lambda}}$ for a given A <u>must</u> <u>add</u> <u>to</u> $\underline{\underline{n}}$ = <u>#variables</u>!