Second Derivative in Partial Derivatives Using Chain Rule

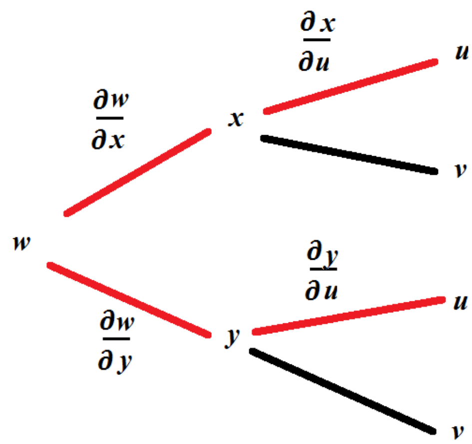
Let's say that:

$$w = f(x,y) \quad \text{and} \quad x = g(u,v), \quad y = h(u,v)$$

And let's say instead of finding a first derivative, we want a second derivative. For example, let's consider w_{uu}.

To calculate a formula for this, we need to first get the first derivative in u, using the regular chain rule. Let's sketch our tree:

The variable w depends on x and y, and each of these depends on u and v.
So we follow along each branch from w to u. At each transition we construct a derivative, and we multiply along the branches and add between the branches to get our formula:



$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u}$$

And to remind us about which expression is written in which variables, let's be a bit explicit about it:

$$\frac{\partial w}{\partial u} = \frac{\partial}{\partial x} f(x,y) \cdot \frac{\partial x}{\partial u} + \frac{\partial}{\partial y} f(x,y) \cdot \frac{\partial y}{\partial u}$$

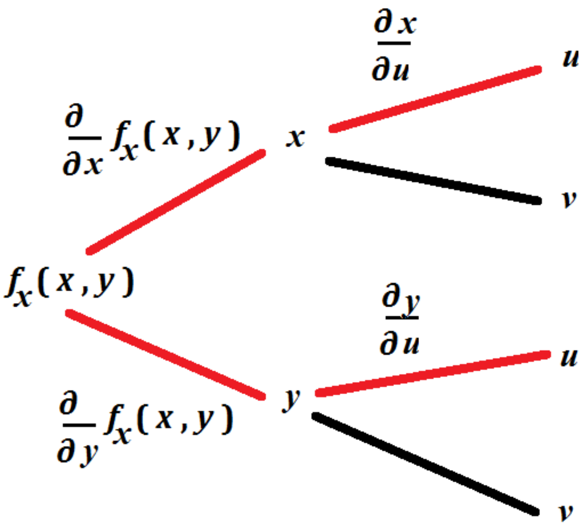
Now, we want a 2nd derivative in u, so we differentiate, and, since we have products, we carefully use the product rule on each of the two products:

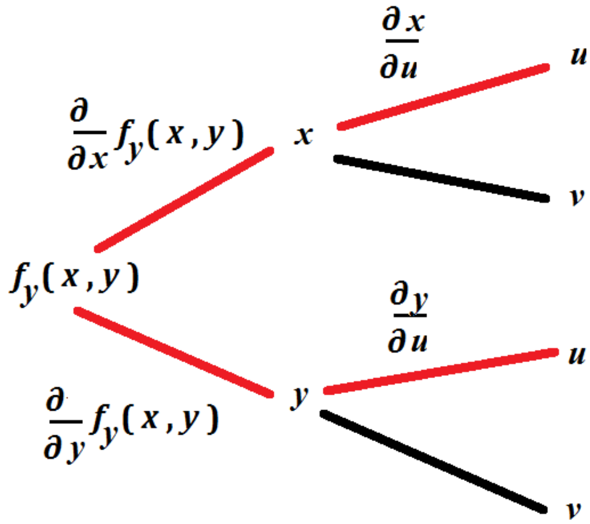
$$\frac{\partial^2 w}{\partial u^2} = \frac{\partial}{\partial u}\left(\frac{\partial}{\partial x} f(x,y)\right) \cdot \frac{\partial x}{\partial u} + \frac{\partial}{\partial x} f(x,y) \cdot \frac{\partial^2 x}{\partial u^2} + \frac{\partial}{\partial u}\left(\frac{\partial}{\partial y} f(x,y)\right) \cdot \frac{\partial y}{\partial u} + \frac{\partial}{\partial y} f(x,y) \cdot \frac{\partial^2 y}{\partial u^2}$$

Great!

But what do we do about the u-partial derivatives acting on the x and y derivatives of f?

Since each of the first partials of f above are functions of x and y we'll have to use chain rules on these too!





$$\frac{\partial}{\partial u} f_x(x,y) = \frac{\partial}{\partial x} f_x(x,y) \cdot \frac{\partial x}{\partial u} + \frac{\partial}{\partial y} f_x(x,y) \cdot \frac{\partial y}{\partial u} \qquad \frac{\partial}{\partial u} f_y(x,y) = \frac{\partial}{\partial x} f_y(x,y) \cdot \frac{\partial x}{\partial u} + \frac{\partial}{\partial y} f_y(x,y) \cdot \frac{\partial y}{\partial u}$$

Now, insert these into our previous calculation:

$$\begin{aligned} \frac{\partial^2 w}{\partial u^2} &= \left(\frac{\partial}{\partial x} f_x(x,y) \cdot \frac{\partial x}{\partial u} + \frac{\partial}{\partial y} f_x(x,y) \cdot \frac{\partial y}{\partial u} \right) \cdot \frac{\partial x}{\partial u} + \frac{\partial}{\partial x} f(x,y) \cdot \frac{\partial^2 x}{\partial u^2} \\ &\quad + \left(\frac{\partial}{\partial x} f_y(x,y) \cdot \frac{\partial x}{\partial u} + \frac{\partial}{\partial y} f_y(x,y) \cdot \frac{\partial y}{\partial u} \right) \cdot \frac{\partial y}{\partial u} + \frac{\partial}{\partial y} f(x,y) \cdot \frac{\partial^2 y}{\partial u^2} \end{aligned}$$

And pretty up the notation a bit:

$$\begin{aligned} \frac{\partial^2 w}{\partial u^2} &= \left(f_{xx}(x,y) \cdot x_u + f_{xy}(x,y) \cdot y_u \right) \cdot x_u + f_x(x,y) \cdot x_{uu} \\ &\quad + \left(f_{yx}(x,y) \cdot x_u + f_{yy}(x,y) \cdot y_u \right) \cdot y_u + f_y(x,y) \cdot y_{uu} \\ &= f_{xx}(x,y) x_u^2 + f_{xy}(x,y) x_u y_u + f_x(x,y) x_{uu} \\ &\quad + f_{yx}(x,y) x_u y_u + f_{yy}(x,y) \cdot y_u^2 + f_y(x,y) y_{uu} \end{aligned}$$

And, using Clairaut's theorem about the equality of mixed partial derivatives:

$$\frac{\partial^2 w}{\partial u^2} = f_{xx}(x,y) x_u^2 + 2 f_{xy}(x,y) x_u y_u + f_{yy}(x,y) \cdot y_u^2 + f_x(x,y) x_{uu} + f_y(x,y) y_{uu}$$