Math 1LS3 Week 12: Integral Applications

Owen Baker

McMaster University

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Week 12: Sections 6.6,6.7 (the last section we'll cover in the course)
Monday will be the last day of class. No new material then. Come prepared with questions.

Area Between Curves

If $y = f(x)$ is on top of $y = g(x)$, the area between f and g over $[a, b]$ is

$$\int_a^b (f(x) - g(x))dx.$$

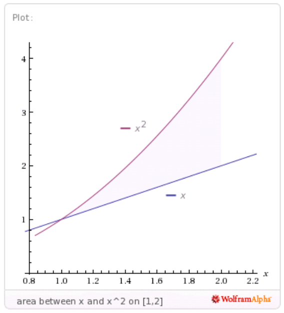
Why? Two reasons:

1. (area under f) - (area under g)
2. Riemann Sum (this reason generalizes to other applications)

Area Between Curves: Example 1

Problem

Find the area between x *and* x^2 *above* $[1, 2]$.



Solution

Which curve is on top? x^2, *so evaluate:*

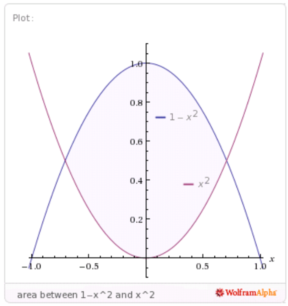
$$\int_1^2 (x^2 - x)dx = \left(\frac{x^3}{3} - \frac{x^2}{2}\right)\Bigg|_1^2 = \left(\frac{2^3}{3} - \frac{2^2}{2}\right) - \left(\frac{1^3}{3} - \frac{1^2}{2}\right) = \boxed{\frac{5}{6}}$$

Area Between Curves: Example 2

Problem

*Find the area **bounded** between the parabolas $1 - x^2$ and x^2.*

Solution



First find limits of integration.

$$1 - x^2 = x^2 \implies 2x^2 = 1 \implies x = \pm\frac{1}{\sqrt{2}}$$

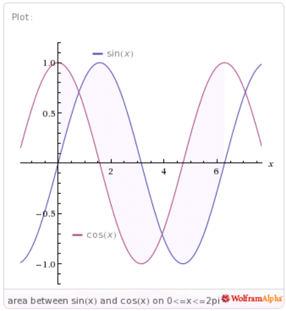
$$\int_{-1/\sqrt{2}}^{1/\sqrt{2}} (1 - x^2) - x^2 dx = \boxed{\frac{2\sqrt{2}}{3}}$$

Area Between Curves: Example 3

Problem

Find the area between $\sin(x)$ *and* $\cos(x)$ *over 1 period.*

Solution



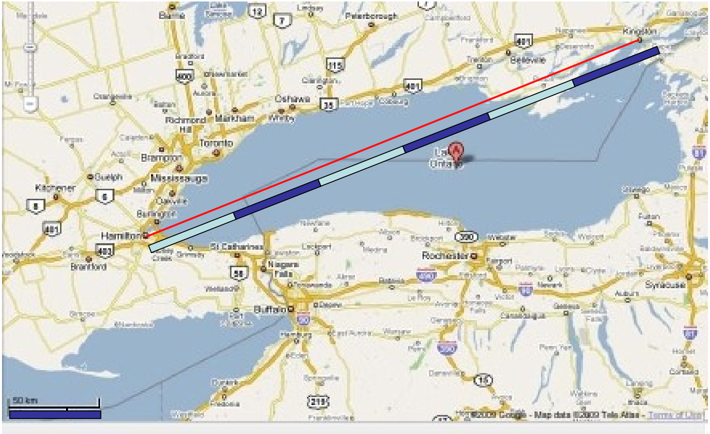
$$\int_0^{\pi/4} \cos(x) - \sin(x)dx + \int_{\pi/4}^{5\pi/4} \sin(x) - \cos(x)dx + \int_{5\pi/4}^{2\pi} \cos(x) - \sin(x)dx$$

Slicker solution:

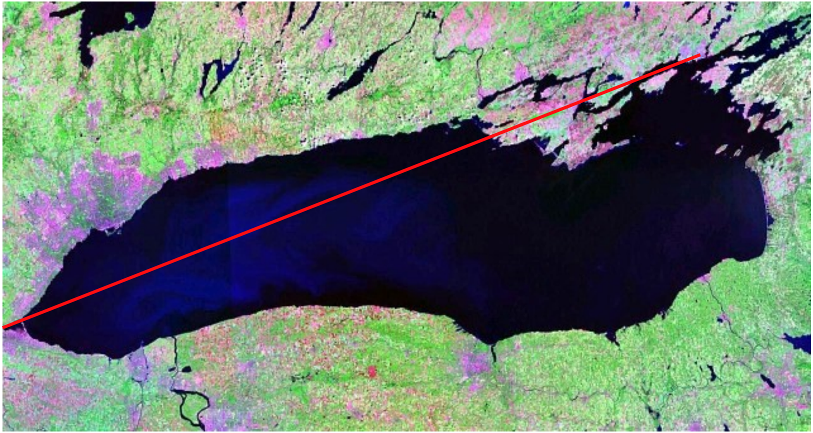
$$2\int_{\pi/4}^{5\pi/4} \sin(x) - \cos(x)dx = 4\sqrt{2}$$

Area of Lake Ontario

estimate of the surface area of lake ontario

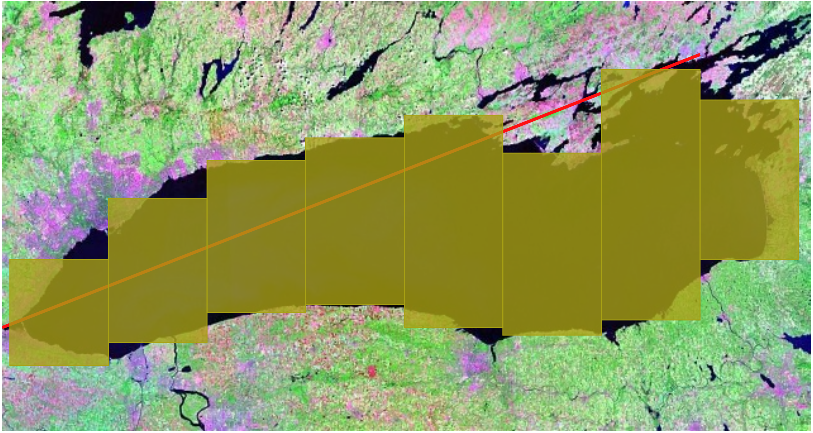


aerial distance hamilton - kingston approx. 290 km



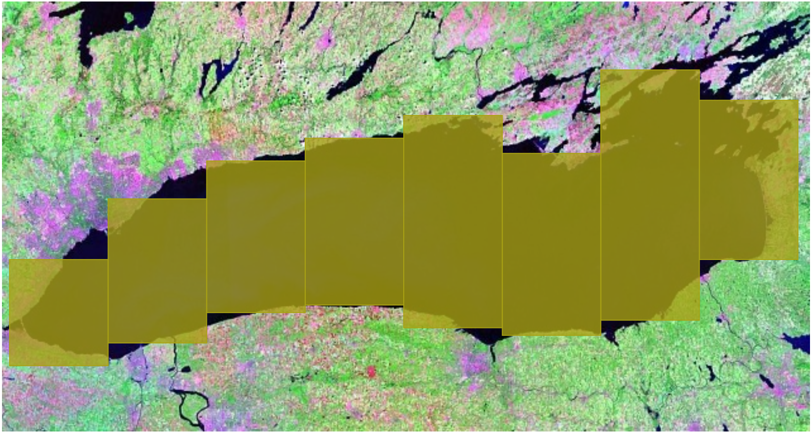
aerial distance hamilton - kingston = 290km

represents approximately 38km



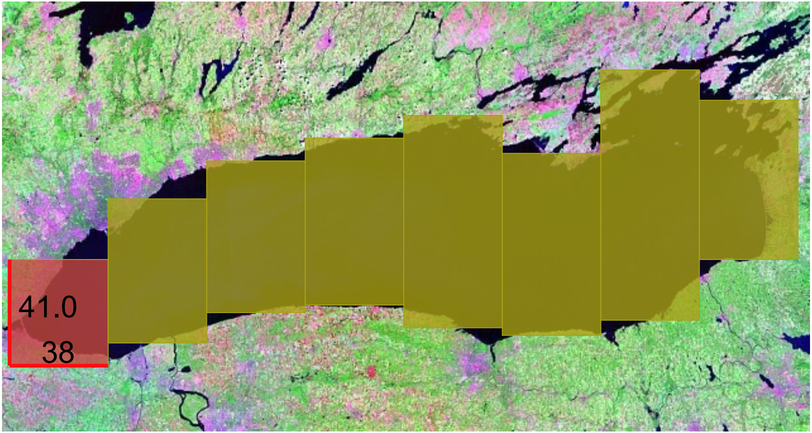
aerial distance hamilton - kingston = 290km

represents approximately 38km

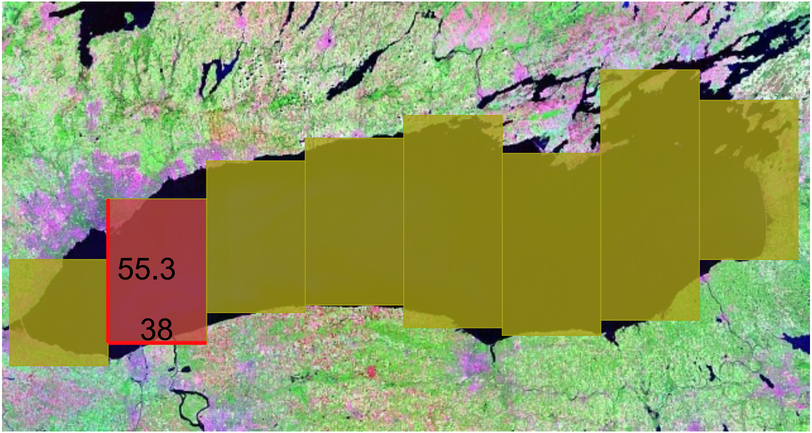


aerial distance hamilton - kingston = 290km

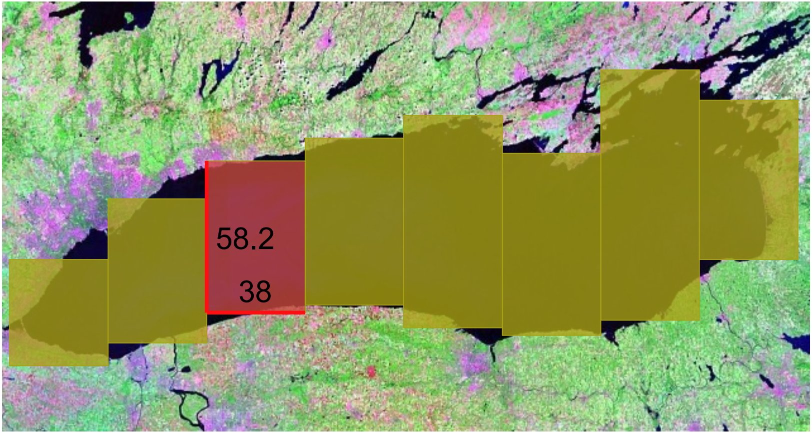
represents approximately 38km



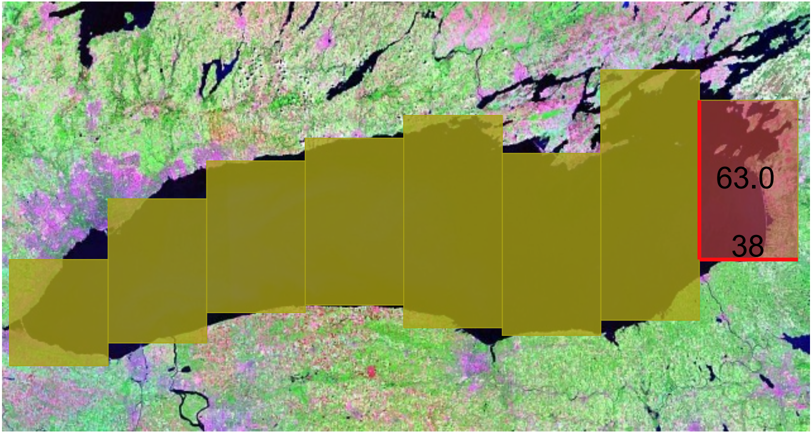
area = 38*41=1558 km^2



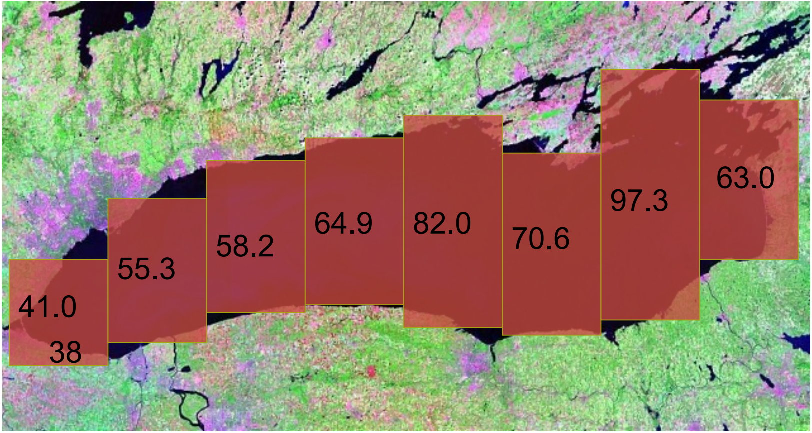
area = 38*55.3=2101.4 km^2



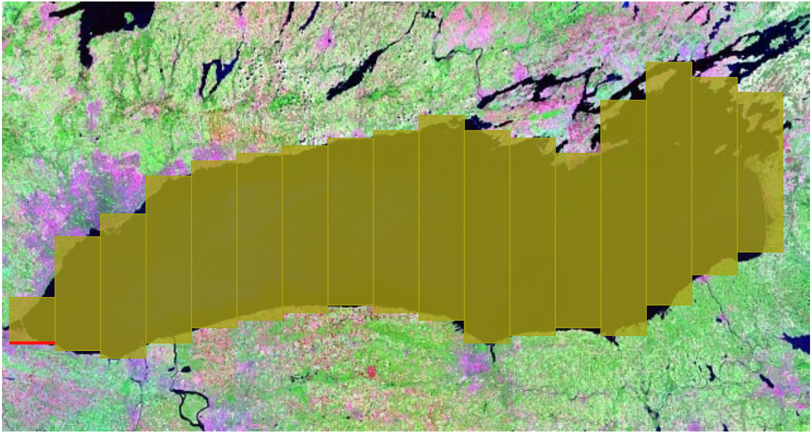
area = 38*58.2=2211.6 km^2



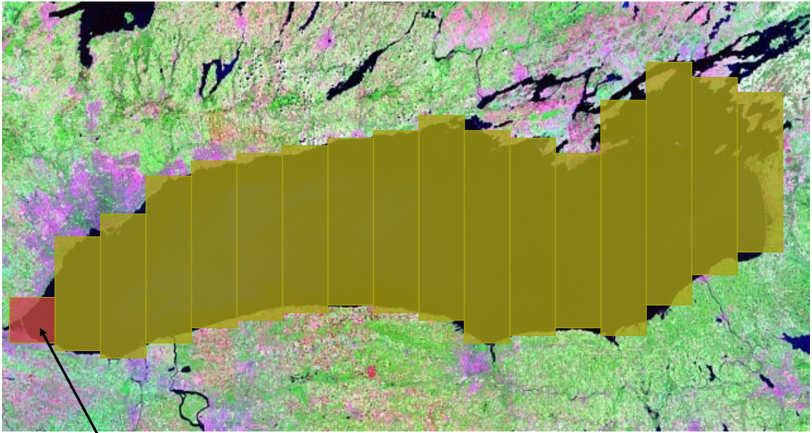
area = 38*63.0=2394.0 km^2



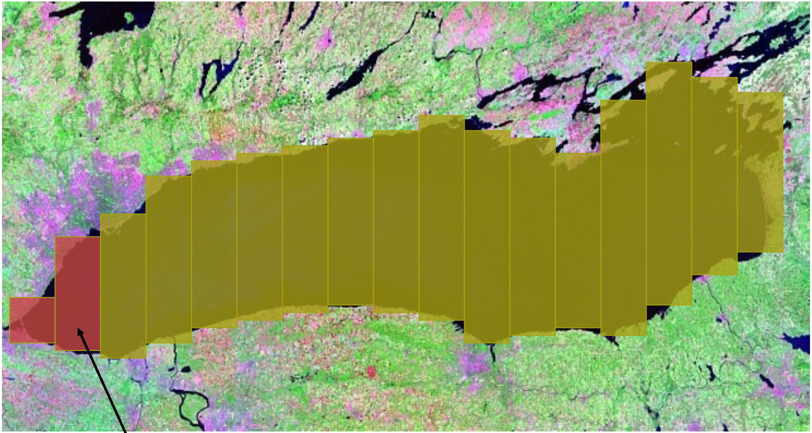
total area = 38*41.0+ 38*55.3+ 38*58.2+ 38*64.9+ 38*82.0+ 38*70.6+ 38*97.3+ 38*63.0= 20,227.4 km^2



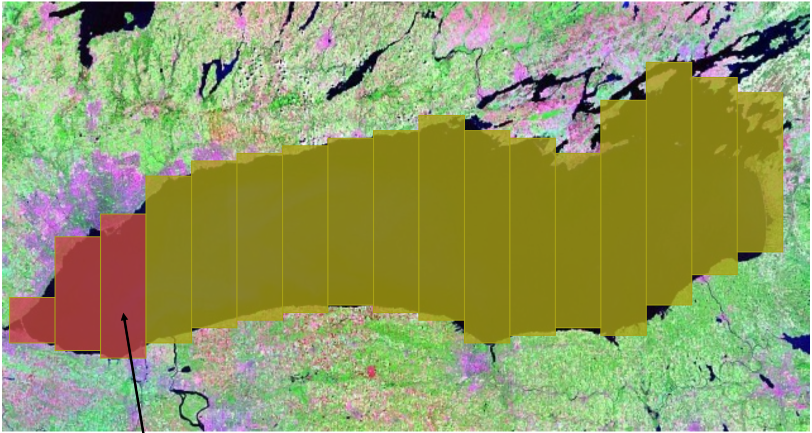
represents approximately 17km



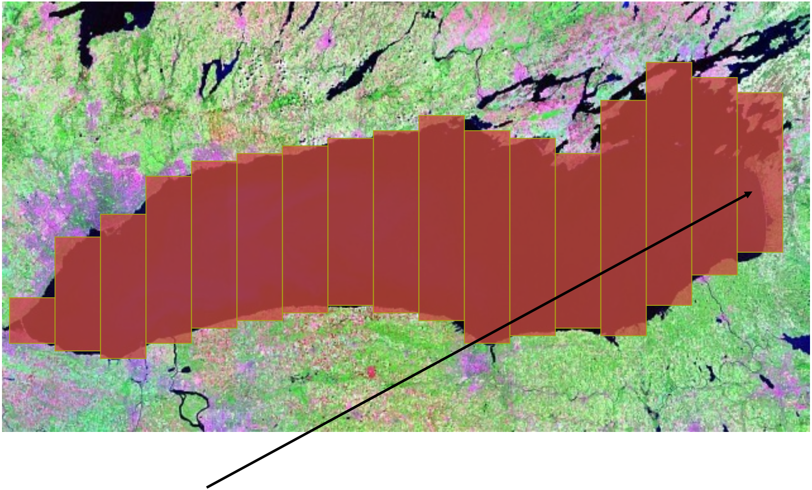
area = 17.2*17.2=295.8 km^2



area = 17.2*43.9=764.8 km^2



area = 17.2*55.3=951.7 km^2



area = 17.2*61.0=1050.1 km^2

total area = 19,233.5

average of the two estimates = (20,227.4 + 19,233.5)/2= 19,730.5 km^2 our estimate = 19,730.5 km^2

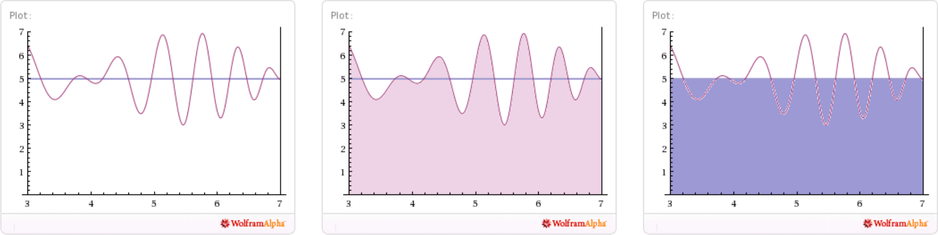
New York Times Almanac … 19,500 km^2

NOAA (National Oceanographic and Atmospheric Administration, U.S.) … 19,009 km^2

EPA (Environmental Protection Agency) … 18,960 km^2

Britannica Online … 19,011km^2

How should we define the average value of a function?



Define the average of $f(x)$ on $[a, b]$ to be the number f_{avg} such that

$$\text{Area under } f = \text{Area under } f_{\text{avg}}$$

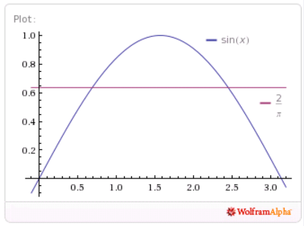
Since the area of the rectangle is (width)·(height)= $(b - a) \cdot f_{\text{avg}}$:

$$\boxed{f_{\text{avg}} := \frac{1}{b-a} \int_a^b f(x)dx}$$

Average value of $\sin(x)$ on $[0, \pi]$

Problem

What is the average value of $\sin(x)$ *on* $[0, \pi]$*?*



Solution

$$\textit{Average Value} = \frac{1}{\pi - 0}\int_0^\pi \sin(x)dx = \frac{1}{\pi}(-\cos(x))\Big|_0^\pi = \frac{1+1}{\pi} = \boxed{\frac{2}{\pi}}$$

Multiplication vs. Integration

- Multiplication is repeated addition of the **same** amount.
- Integration is like repeated addition of **variable** amounts.
- This lets us generalize quantities you usually compute with multiplication!

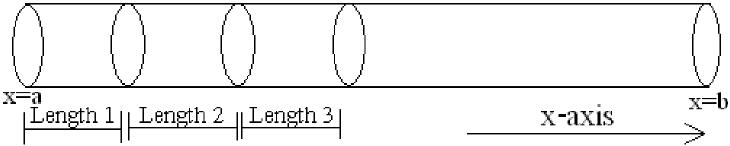
$$\boxed{\text{Mass} = \text{Density} \times \text{Length}} \Rightarrow \boxed{\text{Mass} = \int \text{of Density w.r.t. Length}}$$

$$\boxed{\text{Volume} = \text{Area} \times \text{Height}} \Rightarrow \boxed{\text{Volume} = \int \text{of Area w.r.t. Height}}$$

Let's see this is a little more detail. . .

$$\boxed{\text{One-Dimensional Density } \rho = \frac{\text{Mass}}{\text{Length}}}$$

A rod of variable density $\rho(x)$ (placed at $[a, b]$ on x-axis):



$$\begin{aligned}\text{Total Mass} = (\text{Density } 1)(\text{Length } 1) &+ (\text{Density } 2)(\text{Length } 2)\\ &+ (\text{Density } 3)(\text{Length } 3) + \cdots\end{aligned}$$

The expression on the right is a **Riemann Sum**, so

$$\boxed{\text{Mass of rod} = \int_a^b \rho(x)dx}$$

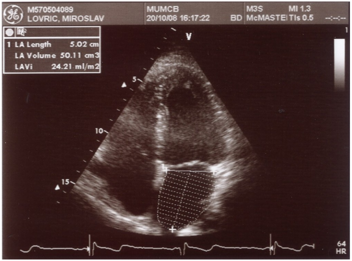
Density Example

Problem (6.6.8)

A certain 100cm vertical bar is denser near the ground. Find its mass if $\rho(z) = e^{-0.01z}$ g/cm.

Solution

$$\begin{aligned} Mass &= \int_0^{100} density\; dz = \int_0^{100} e^{-0.01z} dz \\ &= \left.\frac{e^{-0.01z}}{-0.01}\right|_0^{100} = -100(e^{-1} - 1) = \boxed{100(1 - 1/e) \approx 63.21} \end{aligned}$$

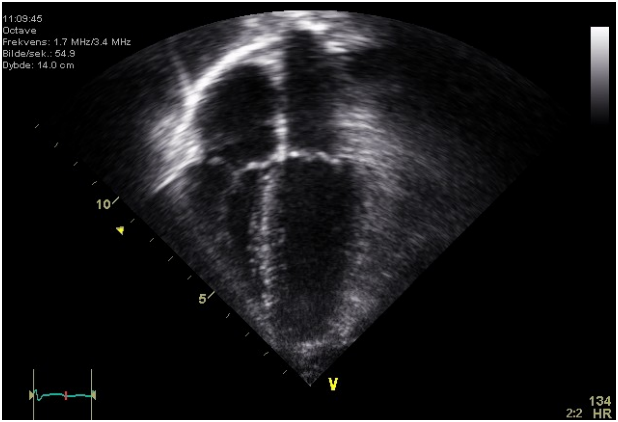


estimate of the volume
of a heart chamber
from echocardiogram

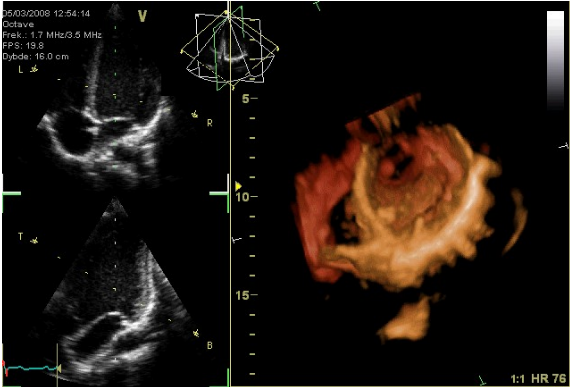
echocardiogram (ECHO, cardiac ultrasound) is a sonogram of the heart

ECHO uses standard ultrasound techniques to image two-dimensional slices of the heart

latest ultrasound systems now employ 3D real-time imaging



ECHO uses standard ultrasound techniques to image two-dimensional slices of the heart



latest ultrasound systems now employ 3D real-time imaging uses of ECHO

(a) creating 2D picture of the cardiovascular system (shape of the heart)

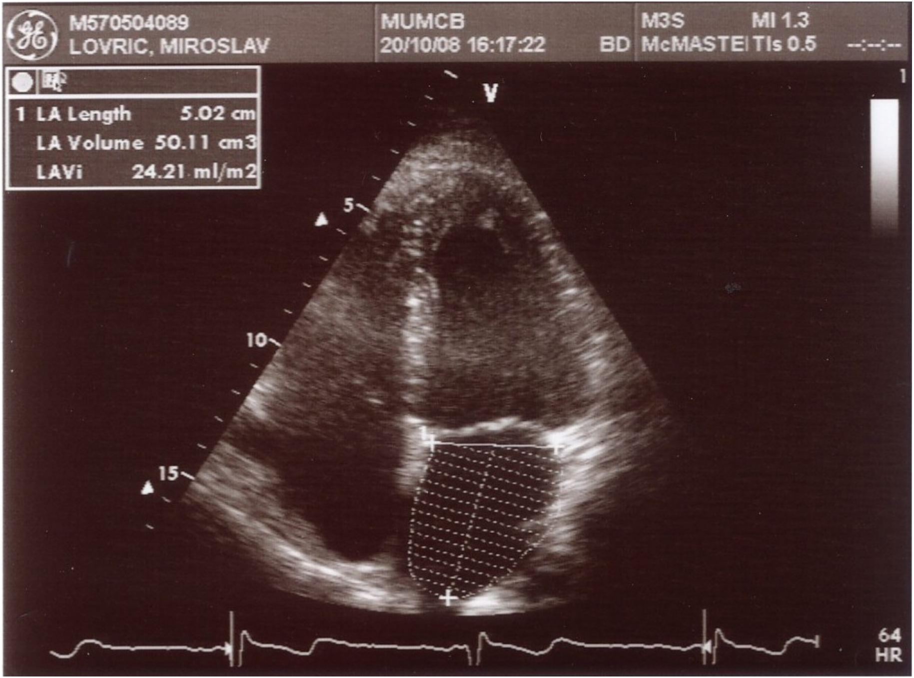
(b) assessment of quality of cardiac tissue (damage, thickening of walls within the heart)

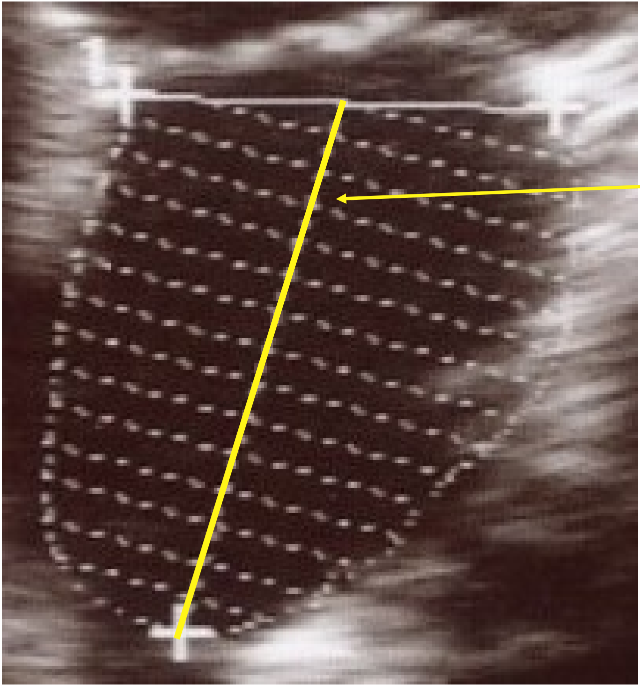
(c) estimate of the velocity of blood

uses of ECHO

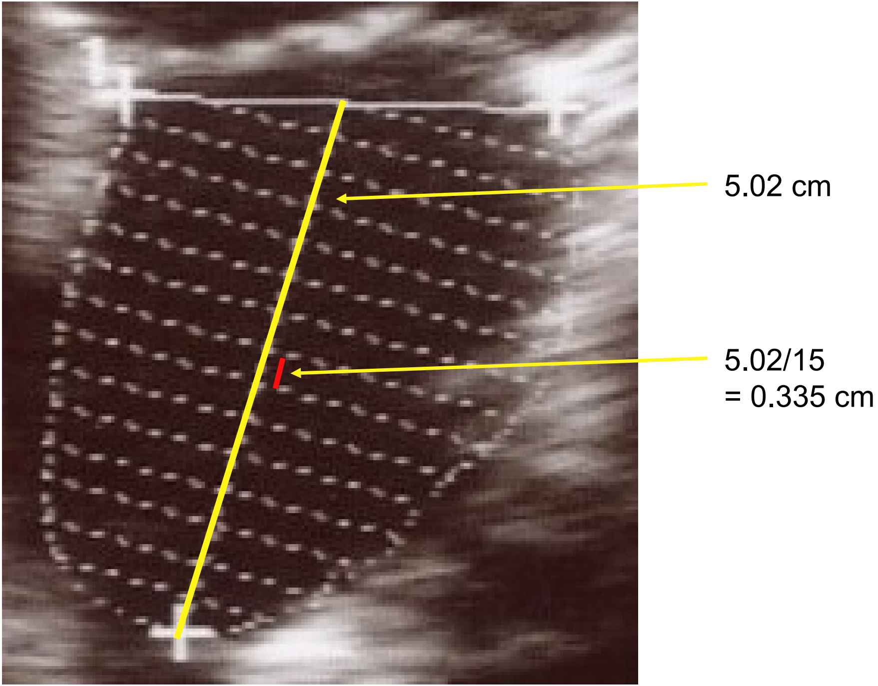
(d) investigate features of blood flow

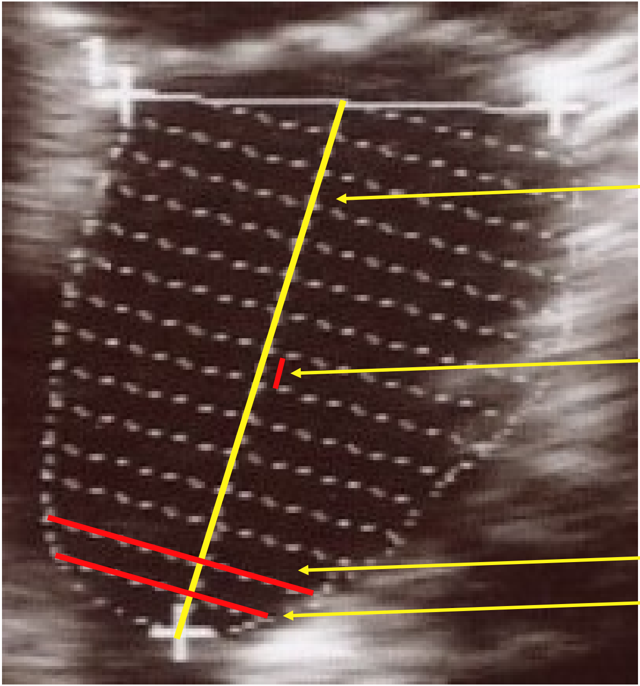
- functioning of cardiac valves
- detection of abnormal communication between the left and right side of the heart
- leaking of blood through the valves
- strength at which blood is pumped out of heart (cardiac output)


M570504089
LOVRIC, MIROSLAV
MUMCB
20/10/08 16:17:22
BD
M3S
MI 1.3
McMASTEI TIs 0.5
1 LA Length 5.02 cm
LA Volume 50.11 cm3
LAVi 24.21 ml/m2
V
5
10
15
64
HR



5.02 cm


5.02 cm
5.02/15
= 0.335 cm

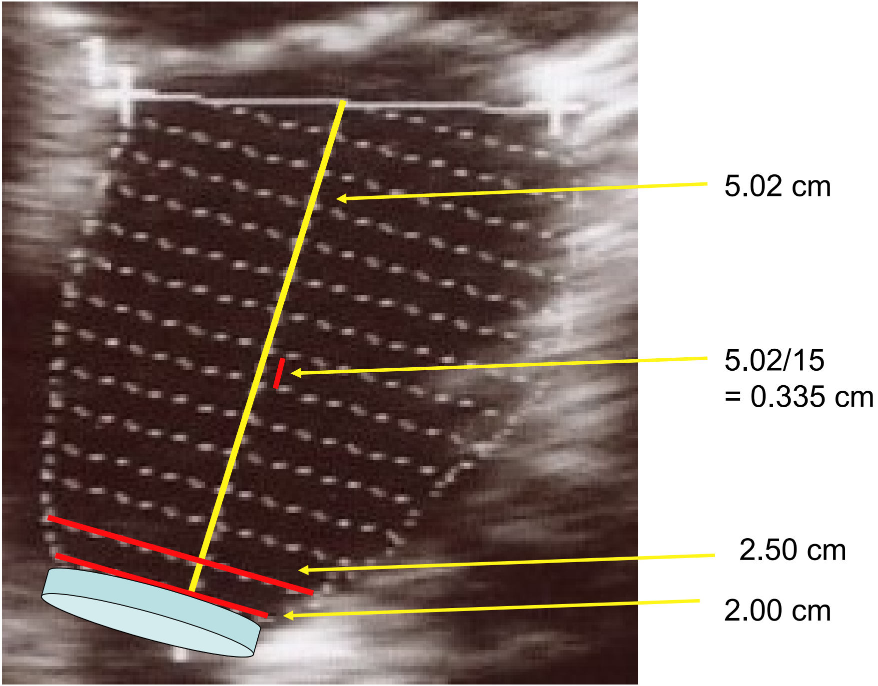


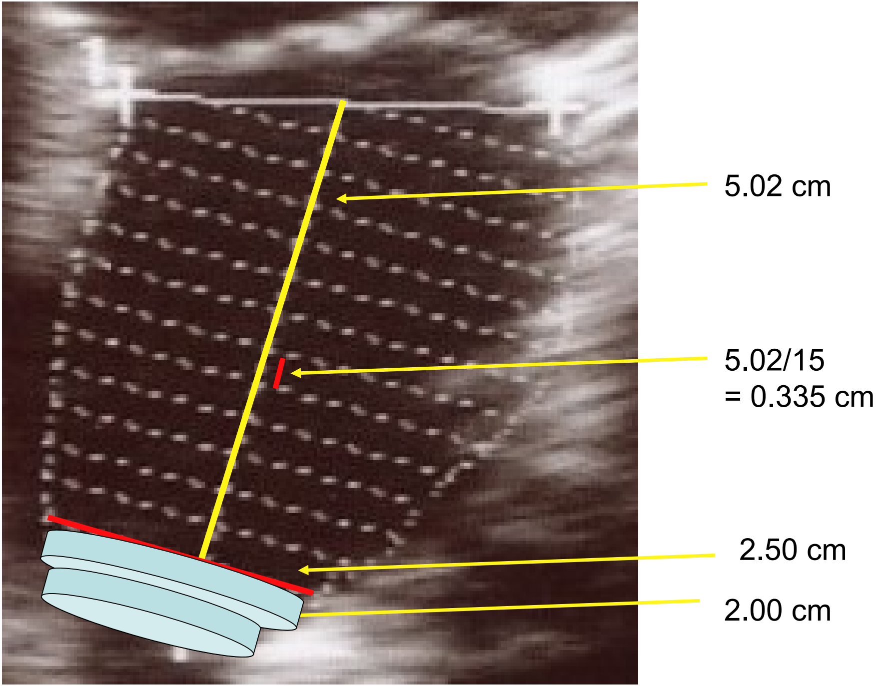
5.02 cm

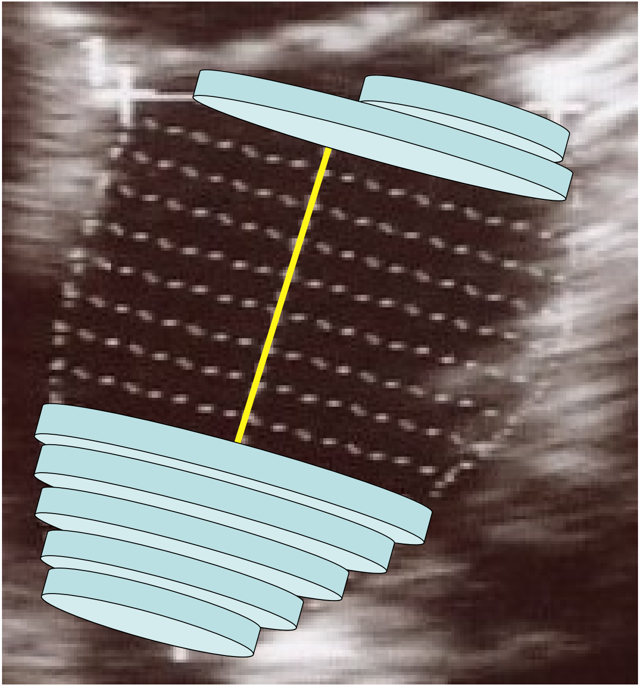
5.02/15
= 0.335 cm

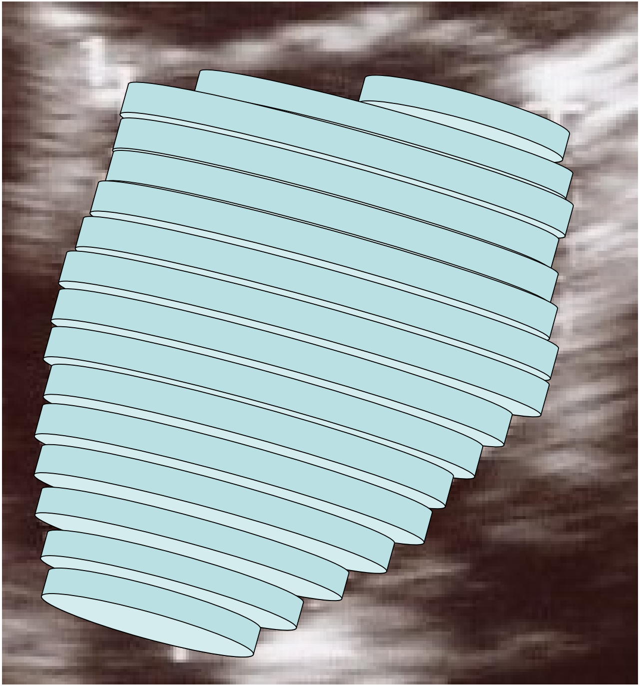
2.50 cm

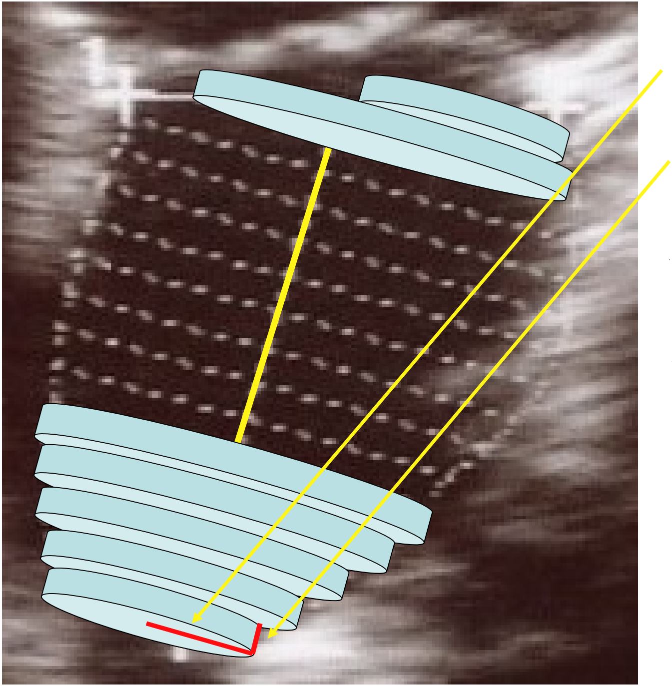
2.00 cm


5.02 cm
5.02/15
= 0.335 cm
2.50 cm
2.00 cm


5.02 cm
5.02/15
= 0.335 cm
2.50 cm
2.00 cm







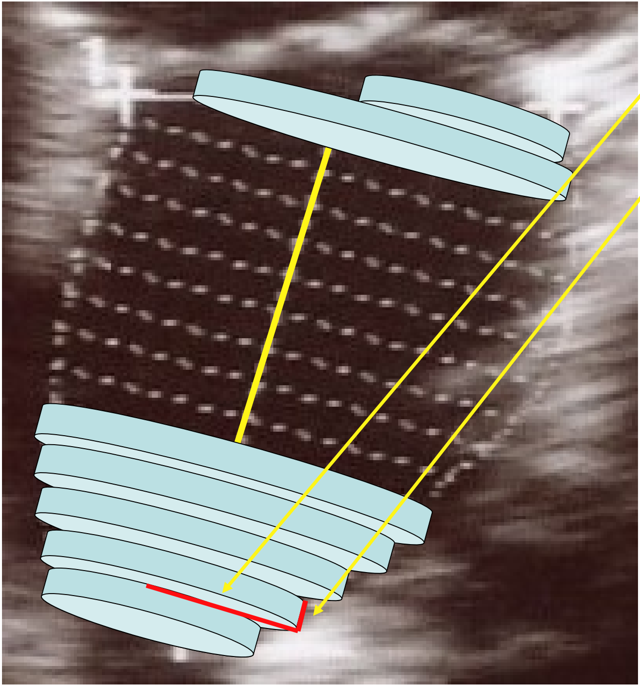
1.00 cm

0.335 cm

volume =

$p\,(1)^2\,0.335 =$

$1.052\ cm^3$



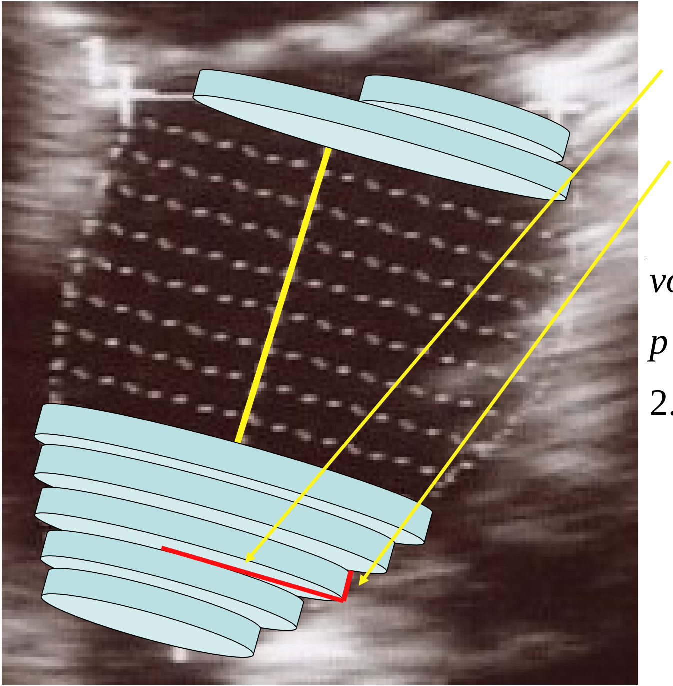
1.25 cm

0.335 cm

$volume =$

$p\,(1.25)^2 0.335 =$

$1.644\ cm^3$



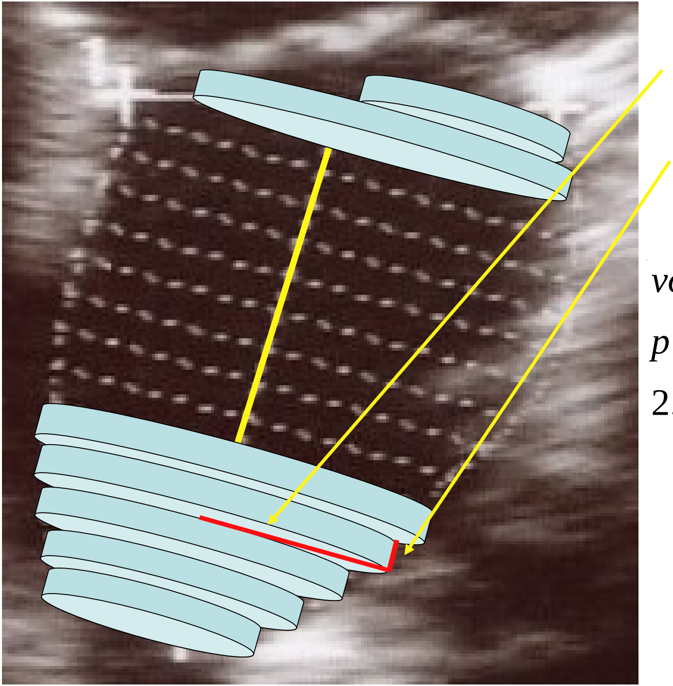
1.50 cm

0.335 cm

$$volume =$$

$$p\,(1.50)^2 0.335 =$$

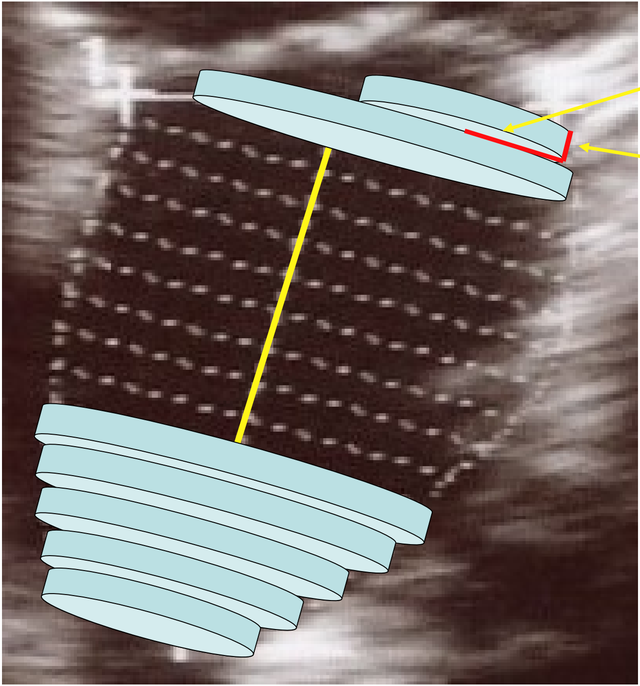
$$2.368\ cm^3$$



$$volume =$$

$$p\,(1.61)^2\,0.335 =$$

$$2.728\;cm^3$$



1.07 cm

0.335 cm

$$volume =$$

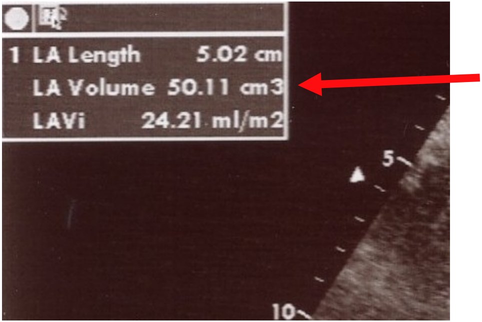
$$p\,(1.07)^2 0.335 =$$

$$1.205\ cm^3$$

volume of heart chamber =

$$1.052 + 1.644 + 2.368 + 2.728 + \ldots + 1.205$$

$$= 50.342\ cm^3$$

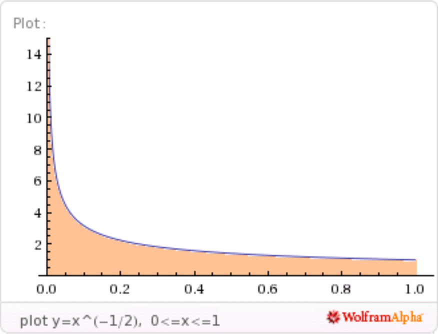


all diameters, from bottom to top:
all in cm

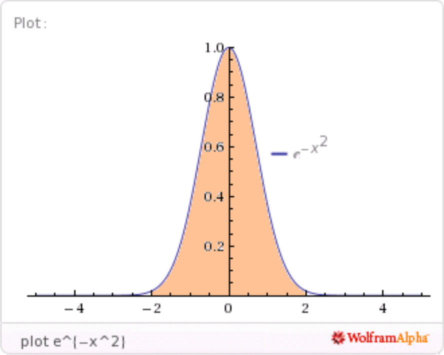
2.00, 2.50, 3.00, 3.21, 3.43, 3.57,
3.93, 4.07, 4.29, 4.29, 4.29, 4.14,
4.07, 3.43, 2.14

height: 5.02/15 cm

The graph of an **unbounded function** or of a function on an **unbounded domain** can sometimes enclose finite area!



Area under $\frac{1}{\sqrt{x}}$ over $[0, 1]$ is 2.
(unbounded function)



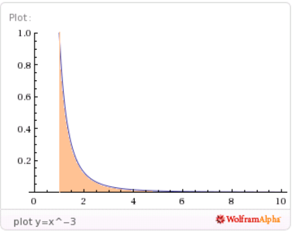
Area under e^{-x^2} on $(-\infty, \infty)$ is $\sqrt{\pi}$
(unbounded domain)

Caution: Area under $\frac{1}{\sqrt{x}}$ over $[1, \infty)$ is infinite.

Problem

Find the area $\int_1^\infty \frac{1}{x^3}\,dx$ *under* $\frac{1}{x^3}$ *over* $[1,\infty)$*.*

Solution



Improper integral. Take area under curve on $[1,b]$ *and let* $b\to\infty$*.*

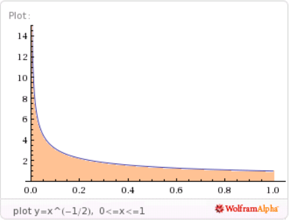
$$\int_1^\infty \frac{1}{x^3}\,dx := \lim_{b\to\infty}\int_1^b \frac{1}{x^3}\,dx = \lim_{b\to\infty} \left.\frac{1}{-2x^2}\right|_1^b$$

$$= \lim_{b\to\infty}\frac{1}{-2b^2}+\frac{1}{2} = 0+\frac{1}{2} = \boxed{\frac{1}{2}}$$

Problem

Find the area $\int_0^1 \frac{1}{\sqrt{x}}dx$ *under* $\frac{1}{\sqrt{x}}$ *over* $(0,1]$.

Solution



Improper integral. Take area on $[a,1]$ *and let* $a \to 0^+$.

$$\int_0^1 \frac{1}{\sqrt{x}}dx := \lim_{a\to 0^+} \int_a^1 \frac{1}{\sqrt{x}}dx = \lim_{a\to 0^+} \frac{1}{\frac{1}{2}} x^{1/2}\Big|_a^1$$

$$= \lim_{a\to 0^+} 2\sqrt{x}\Big|_a^1 = \lim_{a\to 0^+} 2\sqrt{1} - 2\sqrt{a} = \boxed{2}$$

A Divergent Improper Integral

Problem

Find the area $\int_1^\infty \frac{1}{\sqrt{x}} dx$ *under* $\frac{1}{\sqrt{x}}$ *over* $[1, \infty)$.

Solution

Improper integral. Take area on $[1, b]$ *and let* $b \to \infty$.

$$\int_1^\infty \frac{1}{\sqrt{x}} dx := \lim_{b\to\infty} \int_1^b \frac{1}{\sqrt{x}} dx = \lim_{b\to\infty} \frac{1}{\frac{1}{2}} x^{1/2} \Big|_1^b$$

$$= \lim_{b\to\infty} 2\sqrt{x}\Big|_1^b = \lim_{b\to\infty} 2\sqrt{b} - 2\sqrt{1} = \boxed{\infty}$$

There is infinite area under the graph.

Convergence/Divergence

An improper integral involving finite area **converges**.

An improper integral involving infinite area **diverges**.

Simplifying assumption: function≥ 0, so we don't have to deal with *signed* area.

Area under $1/x^p$

p	$\int_0^1 \frac{1}{x^p}\,dx$	$\int_1^\infty \frac{1}{x^p}\,dx$
$p < 1$	$\frac{1}{1-p}$	∞
$p = 1$	∞	∞
$p > 1$	∞	$\frac{1}{p-1}$

- Homework: verify this table (compute as in last three slides).
- Memorize this table for use in comparisons (next few slides).

Leading Behaviour

Suppose f is continuous on $[2, \infty)$ (say).
Recall: f_∞ is leading behaviour at infinity.

$$\boxed{\int_2^\infty f(x)dx \text{ converges} \iff \int_2^\infty f_\infty(x)dx \text{ converges}}$$

(To see if the tail contains ∞ area, only the largest term matters.)

Example

Does $\int_3^\infty \frac{2}{x^2} + e^{-x}dx$ converge? $f_\infty(x) = \frac{2}{x^2}$, so:

$$\int_3^\infty f(x)dx \text{ converges} \iff \int_3^\infty \frac{2}{x^2}dx \text{ converges}$$

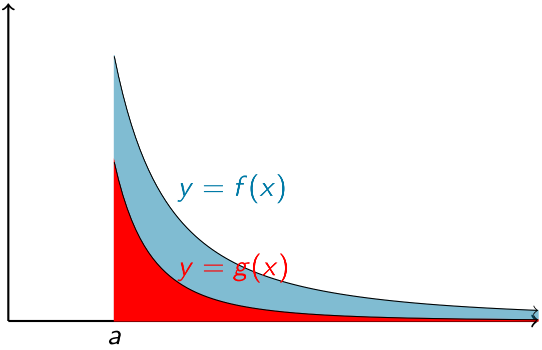
From our study of $\frac{1}{x^p}$, it **does** ($p = 2 > 1$).

Note: use leading behaviour *at vertical asymptote(s)* instead of at ∞ for **unbounded** functions.

The Comparison Test

Suppose $f(x) \geq g(x) \geq 0$ on $[a, \infty)$.

- If $\int_a^\infty f(x)dx$ converges then $\int_a^\infty g(x)dx$ converges (less area).
- If $\int_a^\infty g(x)dx$ diverges then $\int_a^\infty f(x)dx$ diverges (more area).



If $\int_a^\infty f(x)dx$ diverges, must $\int_a^\infty g(x)dx$ diverge? No!

Exercise: formulate the analogue for unbounded improper integrals.

The Comparison Test: Example

Problem

Does $\int_0^\infty e^{-x^2}$ *converge?*

Solution

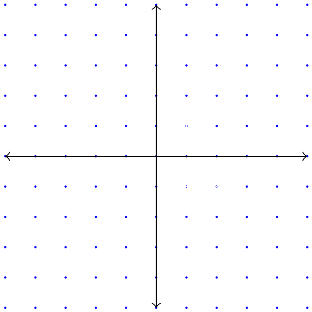
- *As* $x \to \infty$, $e^{x^2} \gg e^x$, *so* $e^{-x^2} \ll e^{-x}$.
- $\int_0^\infty e^{-x} dx = -e^{-x}\Big|_0^\infty = -e^{-\infty} + e^{-0} = 1.$
- *So* $\int_0^\infty e^{-x} dx$ *converges.*
- *By the comparison test,* $\int_0^\infty e^{-x^2} dx$ *therefore converges.*

Factoid: $\int_0^\infty e^{-x^2} dx = \sqrt{\pi}/2.$

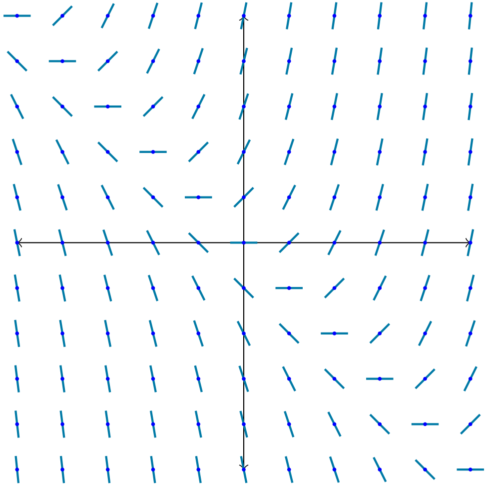
$$\text{Goal: solve } \frac{dy}{dx} = x + y$$

Observation: we can already tell what the *tangent lines* to solutions must look like!

- If a solution passes through $(2, -1)$, what's the slope there? 1
- If a solution passes through $(1, 1)$, what's the slope there? 2
- If a solution passes through $(1, -1)$, what's the slope there? 0



Goal: approximately solve $\dfrac{dy}{dx} = x + y, \qquad y(1) = -1$



Euler's Method

Given:

- DiffEq describing $y'(t)$ in terms of $t, y(t)$.
- Initial value $y(t_0)$.
- Step size Δt.

To *approximate* the solution, compute:

$$\begin{aligned} y(t_0 + \Delta t) &\approx y(t_0) + y'(t_0) \cdot \Delta t, \\ y(t_0 + 2\Delta t) &\approx y(t_0 + \Delta t) + y'(t_0 + \Delta t) \cdot \Delta t, \\ y(t_0 + 3\Delta t) &\approx y(t_0 + 2\Delta t) + y'(t_0 + 2\Delta t) \cdot \Delta t, \qquad \text{etc.} \end{aligned}$$

- Each step above just recalculates the current tangent line.
- Shorter $\Delta t \longrightarrow$ more accurate approximate solution.
- No need to compute entire slope field in advance.