MATH 1LT3E

A#2 SOL^{N}S

#1. (a) Let V be the volume in μm^3 and let t represent time.

$$\frac{dV}{dt} = -a\frac{1}{t} \qquad t>0 \qquad (a>0)$$

pure-time

$$V(0) = 100\mu m^3$$

(b) Let T represent temperature of mud in °C and let t represent time.

$$\frac{dT}{dt} = a(A-T) \qquad (a>0)$$

A = constant temp. of surrounding air

$$T(0) = 125°C$$

#2. (a) $y' = 4x^3 - \sqrt{x} + \frac{1}{x}$ where $y(1)=5$

$$y = \int \left(4x^3 - x^{1/2} + \frac{1}{x}\right)dx$$

$$= x^4 - \frac{2}{3}x^{3/2} + \ln|x| + C$$

$y(1)=5 \Rightarrow 5 = 1 - \frac{2}{3} + \ln 1 + C \Rightarrow C = \frac{14}{3}$

So, $y(x) = x^4 - \frac{2}{3}x^{3/2} + \ln|x| + \frac{14}{3}$

#2. (b) $\frac{dP}{dt} = 10te^{0.5t}$ where $P(0)=500$

$$P(t) = \int 10te^{0.5t}\,dt$$

$$\begin{cases} u = 10t & dv = e^{0.5t}\,dt \\ du = 10\,dt & v = 2e^{0.5t} \end{cases}$$

$$= 20te^{0.5t} - \int 20e^{0.5t}\,dt$$

$$= 20te^{0.5t} - 40e^{0.5t} + C$$

$$P(0) = 500 \Rightarrow 500 = 0 - 40e^{0} + C \Rightarrow C = 540$$

$$\therefore P(t) = 20te^{0.5t} - 40e^{0.5t} + 540$$

#3. $y = \frac{1+e^x}{1-e^x}$

$$y' = \frac{e^x(1-e^x) - (1+e^x)(-e^x)}{(1-e^x)^2}$$

$$= \frac{2e^x}{(1-e^x)^2}$$

<u>DE</u>: $y' = \frac{1}{2}y^2 - \frac{1}{2}$

LS: y'

$$= \frac{2e^x}{(1-e^x)^2}$$

RS: $\frac{1}{2}y^2 - \frac{1}{2}$

$$= \frac{1}{2}\left(\frac{1+e^x}{1-e^x}\right)^2 - \frac{1}{2}$$

$$= \frac{1}{2}\cdot\frac{1+2e^x+e^{2x}}{(1-e^x)^2} - \frac{1}{2}\cdot\overbrace{\frac{(1-e^x)^2}{(1-e^x)^2}}^{\text{to create a common denominator}}$$

$$= \frac{1}{2}\left[\frac{\cancel{1}+2e^x+\cancel{e^{2x}} - (\cancel{1} - 2e^x + \cancel{e^{2x}})}{(1-e^x)^2}\right]$$

$$= \frac{1}{2}\left[\frac{4e^x}{(1-e^x)^2}\right]$$

$$= \frac{2e^x}{(1-e^x)^2}$$

LS = RS $\therefore y = \frac{1+e^x}{1-e^x}$ is a sol^n to the DE.

#4. $\dfrac{dP}{dt} = \dfrac{P}{1+t^2}$ $\quad P(1) = 100 \quad h = 0{,}5$

Euler's Method: $\quad t_{n+1} = t_n + h$

$$P_{n+1} = P_n + \left.\frac{dP}{dt}\right|_{\substack{t=t_n \\ P=P_n}} \cdot h$$

$t_0 = 1$
$P_0 = 100$

$t_1 = t_0 + h = 1 + 0{,}5 = 1{,}5$

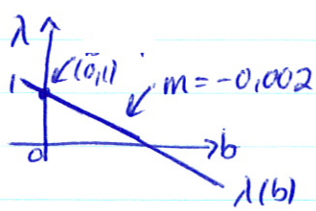
$$P_1 = P_0 + \left.\frac{dP}{dt}\right|_{\substack{t=t_0 \\ P=P_0}} \cdot h = 100 + \frac{100}{1+1^2}(0{,}5) = 125$$

$t_2 = t_1 + h = 1{,}5 + 0{,}2 = 2$

$$P_2 = P_1 + \left.\frac{dP}{dt}\right|_{\substack{t=t_1 \\ P=P_1}} \cdot h = 125 + \frac{125}{1+(1{,}5)^2}(0{,}5) \simeq 144.$$

$\therefore \; P(2) \simeq 144.$

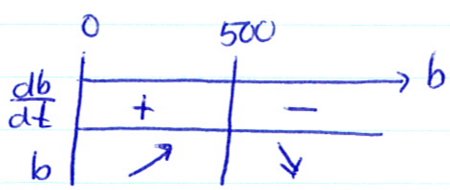
#5.



$\lambda(b) = -0.002b + 1$

(a) $\dfrac{db}{dt} = (-0{,}002b + 1)\,b$

(b) $\dfrac{db}{dt} = 0$ when $b = 0$ or $b = \dfrac{1}{0{,}002} = 500$



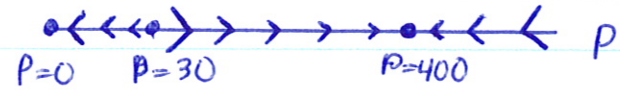
If $b = 0$ or 500, the pop^n will remain constant over time. If b is between 0 and 500, the pop^n will increase. If b is larger than 500, the pop^n will decrease.

#6. (a) $P_1^* = 0$ ← logical equilibrium
(0 sharks ⇒ 0 growth rate)

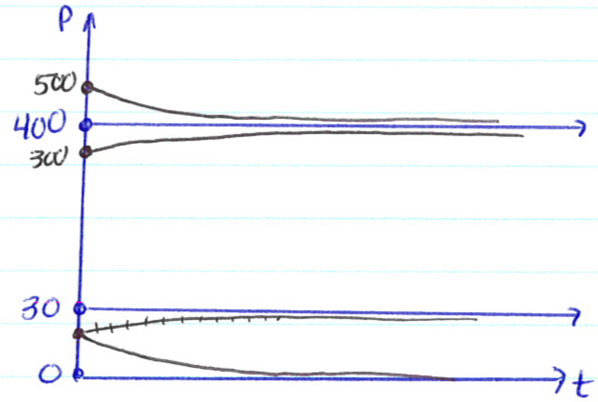
$P_2^* = 400$ ← carrying capacity
(max. # of sharks environment is capable of sustaining long-term)

$P_3^* = 30$ ← existential threshold
(min. # of sharks required for survival of the pop^N)

(b)



(c)



(d) The pop^N of sharks will remain constant over time if the pop^N is 0, 30, or 400.
If the pop^N falls between 0 and 30, ie under the existential threshold, then it will die out.
If the pop^N is between 30 and 400 (under its carrying capacity but over its existential threshold), then the pop^N will increase towards 400 with smaller pop^Ns increasing at a greater rate.
If the pop^N is ever above its carrying capacity of 400, it will decrease towards 400.

\# 7. $\dfrac{dy}{dt} = y e^{-\beta y} - ay$

(a) $\dfrac{dy}{dt} = 0$ when $y(e^{-\beta y} - a) = 0$

$$\Rightarrow \boxed{y=0} \quad \text{or} \quad e^{-\beta y} = a \qquad (a>0)$$

$$\Rightarrow -\beta y = \ln a$$

$$\Rightarrow \boxed{y = -\frac{\ln a}{\beta}}$$

(b) $f(y) = y(e^{-\beta y} - a)$

$$f'(y) = 1 \cdot (e^{-\beta y} - a) + y(-\beta e^{-\beta y})$$

$$= e^{-\beta y} - a - \beta y e^{-\beta y}$$

$$= e^{-\beta y}(1 - \beta y) - a$$

$$f'(0) = 1 - a$$

$y=0$ is $\begin{cases} \text{stable if } & 1-a<0 \Rightarrow a>1 \\ \text{unstable if } & 1-a>0 \Rightarrow a<1 \quad (0<a<1) \end{cases}$

stability test cannot be used when $a=1$.

$$f'\left(-\frac{\ln a}{\beta}\right) = e^{-\beta\left(-\frac{\ln a}{\beta}\right)}\left(1 - \beta\left(-\frac{\ln a}{\beta}\right)\right) - a$$

$$= a(1 + \ln a) - a$$

$$= a \ln a$$

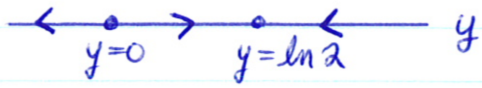
$y = -\dfrac{\ln a}{\beta}$ is $\begin{cases} \text{stable if } & \overset{\oplus}{a}\,\overbrace{\ln a}^{\oplus \text{ or } \ominus} < 0 \Rightarrow \ln a < 0 \Rightarrow 0<a<1 \\ \text{unstable if } & a \ln a > 0 \Rightarrow a>1 \end{cases}$

again, test does not apply when $a=1$.

#7. (c) $\alpha = 0.5,\ \beta = 1$

$y=0$ is an unstable eq^n

$y = -\frac{\ln 0.5}{1} = \ln 2$ is a stable eq^n.



(d) $\alpha = e,\ \beta = 1.$

$y=0$ is now a stable eq^n

$y = -\frac{\ln e}{1} = -1$ is an unstable eq^n