Math 1LS3 Week 1: Mathematical Models

Owen Baker

McMaster University

Sept. 10–14, 2012

This week covers sections 0.1, 0.2 and 1.1 of the textbook. Next week is 0.3, 1.2 and 1.3.

Background

- Weeks 1–2 (Chapters 0–1): math models & a ***quick*** review of prerequisite topics
- Assigments 1–4 material will ***not*** be fully covered in class
- *Consult textbook, optional textbook, my office hours, math help centre*
- Assignments 1,2: functions (definition, composition, inverses, vertical & horizontal line tests)
- Assignments 3,4: exponential, log, trig, inverse trig functions
- In first tutorial, I'll review domain/range and graphs with asymptotes, holes.
- Future tutorials – ask me about anything: current/past/background topics.

Homework

- Homework is not collected.
- Please schedule 2–3 hours outside of class for every class meeting.
- You can spend this time: reviewing your notes, working through coursepack assignments, reading the textbook, looking ahead at slides, working review problems.
- This week, make sure you know tables 0.1.1, 0.1.2, 0.1.3 and that you're comfortable with unit conversion. Practice "dimensional analysis" if this is new.

Resources

- Textbook and optional textbook
- Lecture notes (multiple versions)
- Office hours: Hamilton Hall 423 Mon,Thur after class (10:30–11:30).
- Office hours for Prof. Lovric and Dr. Clements: (course website)
- Math Help Centre walk–in hours

Some (Free) Online Resources

- ~~An entire online calculus course(!): Coursera~~
- Some calculus videos: Khan Academy
- A fancy web calculator: Wolfram Alpha

Mathematical Modelling: Where We're Headed

$$\boxed{\text{Complicated Biological System}} \rightarrow \boxed{\text{Simple Mathematical Model}}$$

Kinds of Models

- Functions
- Differential Equations "continuous–time dynamical systems"
  - Language: this semester
  - Solution tools: Math 1LT3

Mathematical Modelling: Where We're Headed

Chapter 2: discrete–time dynamical systems

- less scary version of differential equations
- just fancy name for "sequences given by a certain kind of rule"
- similar qualitative phenomena
- Note: most calculus courses omit this topic

Let's start with an example. . .

Discrete–Time Model for Flu Infections

$$\boxed{I_t = \text{number of infected people at day } t}$$

Initial Condition

$$I_0 = 2$$ (2 people are infected on day zero.)

The Rule ("updating function")

$$I_{t+1} = \frac{60 I_t}{0.5 I_t + 1.5}$$ (tells how to compute next day's value.)

Problem

How many people are infected on day 3?

Flu Infections: Solution

Problem

How many people are infected on day 3?

Solution

The rule is

$$I_{t+1} = \frac{60 I_t}{0.5 I_t + 1.5}.$$

The initial condition is $I_0 = 2$. *We want to find* I_3.

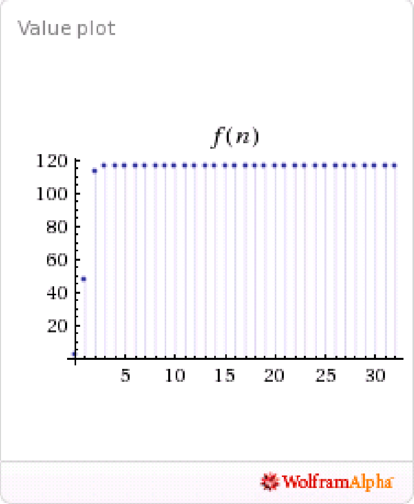
$$\textit{Plug in } t = 0 \implies I_1 = \frac{60 * I_0}{0.5 I_0 + 1.5} = \frac{60 \cdot 2}{0.5 \cdot 2 + 1.5} = 48.$$

$$\textit{Plug in } t = 1 \implies I_2 = \frac{60 * I_1}{0.5 I_1 + 1.5} = \frac{60 \cdot 48}{0.5 \cdot 48 + 1.5} \approx 113.$$

$$\textit{Plug in } t = 2 \implies I_3 = \frac{60 * I_2}{0.5 I_2 + 1.5} = \frac{60 \cdot 113}{0.5 \cdot 113 + 1.5} \approx \boxed{117}.$$

Flu Infections: Plot

`f(n+1)=60f(n)/(1.5f(n)+1.5), f(0)=2`



Graphing a discrete–time dynamical system

- Graph is isolated points, *not* a continuous curve.
- What's the most notable feature for this particular system? Stability!
- Calculus can detect stability and instability! (chap. 2)

Mathematical Models and Variables

Definition

A *model* is a mathematical system used to interpret a real-world system. It consists of mathematical objects such as functions and equations relating *variables* that have real-world meaning.

Kinds of **variables** in an *experiment*:

- **independent/input** – directly controlled (or "time"), changing
- **dependent/output** – observed, not directly controlled
- **parameters** – fixed variables ("controlled variables")

Simplification: only 1 independent & 1 dependent variable (cf. Math 1LT3)

Caution: when modelling something other than an experiment, independent/dependent/parameter depends on point of view.

Functions vs. Dynamical Systems

Dynamical system are *self–referential*:

$$I_{t+1} = \frac{60 I_t}{0.5 I_t + 1.5}, I_0 = 2$$

- Independent variable: time t
- Dependent variable: infections I_t

Dependent variable is on *both sides* of equation

Functions make simpler models (Ch. 1):

$$I_t = 117 - \frac{13455}{2 \cdot 40^t + 115}$$

Dependent variable expressed in terms of others

Solving a dynamical system means *replacing* it with a function.

Deterministic Models

All our models – functions and dynamical systems – are **deterministic**:

$$\boxed{\text{perfect initial knowledge}} \xrightarrow{\text{(in theory)}} \boxed{\text{perfect knowledge ever after}}$$

$$I_0 \to I_1 \to I_2 \to I_3 \to I_4 \to I_5 \to I_6 \to \cdots$$

Nonetheless, deterministic systems might involve:

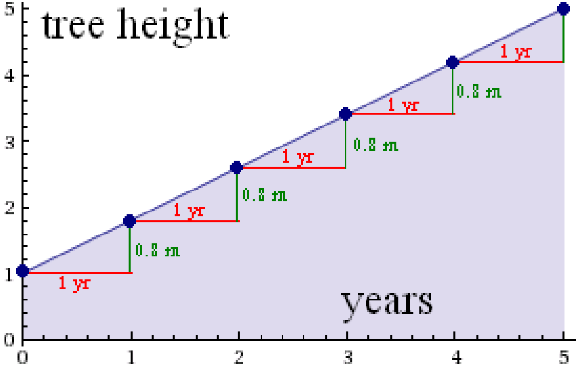
- Unfeasibly long computations (too many steps); or
- Sensitive dependence on initial conditions ("**butterfly effect**").

(We will mostly avoid this phenomenon.)

For *non*-deterministic systems, see Math 1LT3.

Slope is Rate

A tree grows 0.8 metres per year, starting at 1 metre.

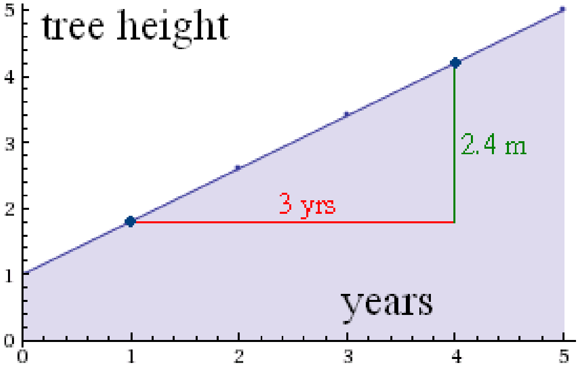


age (yrs)	height (m)
0	1.0
1	1.8
2	2.6
3	3.4
4	4.2
5	5.0

The slope is how much the tree (or graph) **rises per year**: 0.8 metres/year

Measuring Slope

On a line, the slope can be computed using any two points.



$$\textbf{Slope} = \frac{2.4}{3} = 0.8$$

$$\textbf{Slope} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

For curves, the rate of change (slope) varies from point to point.

Linear Model for the Population of Canada (p.50)

Census data:

Year	Population (in thousands)
1996	28,847
2001	30,007
2006	31,613

Problem

Use the first two data points to obtain a linear model for the population of Canada. Is it a good model?

Canada: Linear Population Model

Solution

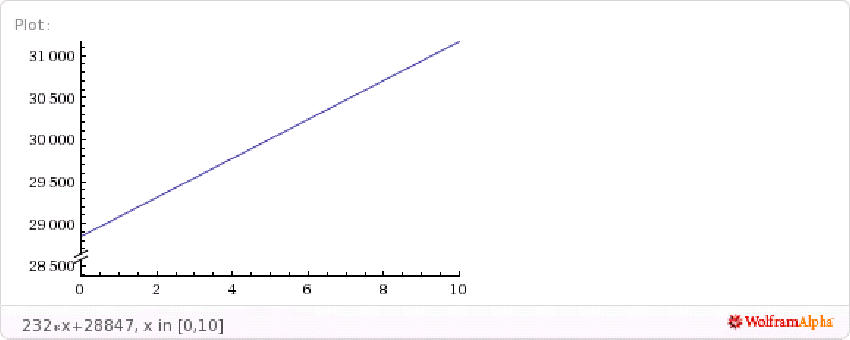
Year (t)	*Population (in thousands)* ($P(t)$)
0	*28,847*
5	*30,007*
10	*31,613*

Linear models are of the form $P(t) = m * t + b$.
Plug in $t = 0$*:* $28,847 = P(0) = m * 0 + b = b$*, so* $b = 28,847$.
Plug in $t = 5$*:* $30,007 = P(5) = m * 5 + b = m * 5 + 28,847$*. So*

$$m = \frac{30,007 - 28,847}{5 - 0} = 232.$$

Our model is $P(t) = 232 * t + 28,847$.
Is the model good? It predicts $P(10) = 232 * 10 + 28,847 = \boxed{31,167}$.
The model is not very good: population rose by 1606, not 1160.

`232*x+28847, x in [0,10]`



The graph of $y = m * x + b$ is the line passing through the y-axis at $(0, b)$ and with *slope* m.

Proportionality

Linear relationship between x and y:

$$y = mx + b.$$

Special case: y is **proportional** to x

$$y = mx.$$

The graph of a *proportional relationship* passes through ... the origin.

Example

$$\text{Mass} = \text{Density} * \text{Volume}$$

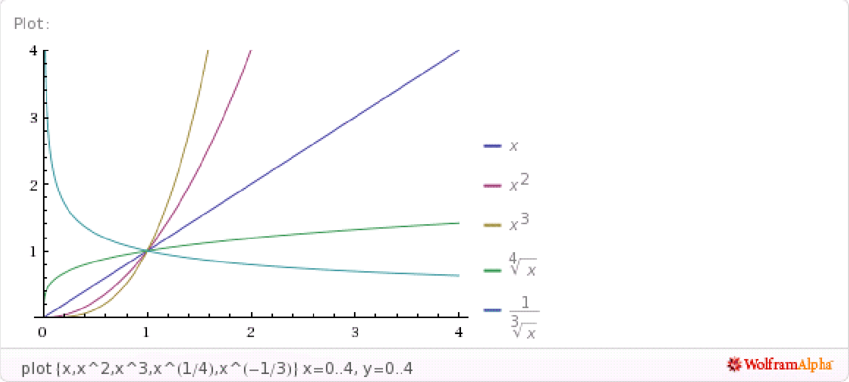
If density is constant, then Mass and Volume are proportional. We write

$$\text{Mass} \propto \text{Volume} \qquad \text{or} \qquad \text{Volume} \propto \text{Mass}.$$

If volume is constant, are mass and density proportional? Yes.
If mass is constant, are density and volume proportional? No.

Graphs of Power functions: know these shapes!

```
plot {x,x^2,x^3,x^(1/4),x^(-1/3)} x=0..4, y=0..4
```



What do you notice about exponents bigger than 1? Less than 1? Negative?

What point do they all pass through? (1,1)

Blood Circulation Time in Mammals (p.54)

Problem

Let T denote a mammal's blood circulation time and B its body mass. Use the model $T \propto \sqrt[4]{B}$. An elephant of mass 5400kg is found to have a blood circulation time of 152s. What is the blood circulation time for a mouse of mass 0.1kg? Find the proportionality constant.

Solution

Can you estimate the order of magnitude?
Let a denote the proportionality constant. The model is

$$T = a * B^{1/4}.$$

Plugging in $T = 152, B = 5400$ *we find:* $152 = a * (5400)^{1/4}$ *so*

$$a = \frac{152}{5400^{1/4}} \approx 17.73$$

Blood circulation cont'd

Solution

The model is $T = a \cdot B^{1/4}$.
We found the constant $a = 17.73$, *so* $T = 17.73 * B^{1/4}$.
The blood circulation time of the mouse is the value of T *when* $B = 0.1$, *i.e.*

$$T \approx 17.73 \cdot (0.1)^{1/4} \approx 9.97s.$$

Question: what would changing units do to the model?

Question 2: can you think of any model where one variable is proportional to a power of another, but where changing units does not merely change the proportionality constant?

Heartbeat Frequency in Mammals (p.55)

Problem

Heartbeat frequency f is inversely proportional to the fourth root of body mass B. According to this model, what happens to heartbeat frequency when mass is multiplied by 16?

Solution

Easy method: Frequency is divided by $\sqrt[4]{16} = 2$.

Always ask yourself: should it go up or down?

Details

If you're not convinced, here's a harder way:

Solution

The inverse proportionality means there is a constant k such that

$$f = kB^{-1/4}$$

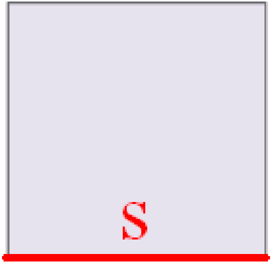
If we start with values B_0 *and* f_0 *so that* $f_0 = kB_0^{-1/4}$,
and end with values B_1 *and* f_1 *so that* $f_1 = kB_1^{-1/4}$, *then dividing these two equations yields the proportion:*

$$\frac{f_1}{f_0} = \frac{B_1^{-1/4}}{B_0^{-1/4}} = \left(\frac{B_1}{B_0}\right)^{-1/4} = 16^{-1/4} = \frac{1}{\sqrt[4]{16}} = \frac{1}{2}.$$

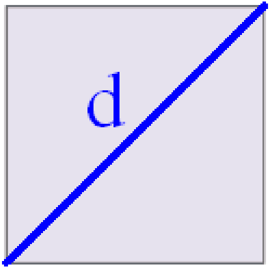
We found $\frac{f_1}{f_0} = \frac{1}{2}$. *So the heartbeat frequency is halved.*

This slide is a reference; skip during lecture.

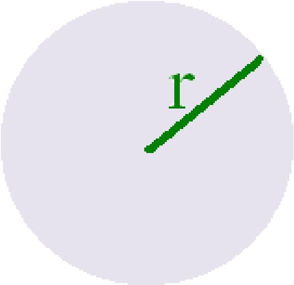
Area $\propto$ Length Measure^2.



Area $= s^2$



Area $= \frac{1}{2}d^2$



Area $= \pi r^2$

Only the constant changes!

Volume is 3-dimensional

Volume is a 3-*dimensional* measure.

- Round ball (radius r): $\frac{4}{3}\pi r^3$.
- Cube (side s): $1s^3$.
- Cow: (approximately) $\alpha \cdot \text{height}^3$, for some fixed α

Technical assumption: the shapes in question are *similar*.

Relating Surface Area to Volume

Problem

Find the surface area for a sphere in terms of its volume.

Solution

$$V = \frac{4}{3}\pi r^3 \; and \, S = 4\pi r^2$$

Intermediate step: solve first equation for r.

$$\frac{3V}{4\pi} = r^3 \implies r = \left(\frac{3V}{4\pi}\right)^{1/3}$$

Plug r into second equation:

$$S = 4\pi \left(\left(\frac{3V}{4\pi}\right)^{1/3} \right)^2 \implies \boxed{S(V) \approx 4.84 V^{2/3}}$$

Consider a Spherical Cow... (p.55)

For a **sphere**, we found:

$$\boxed{S(V) \approx 4.84V^{2/3}}$$

For another class of similar figures:

$$S(V) = \alpha V^{2/3} \text{ for some } \alpha$$

Surface area grows more *slowly* than volume, so larger animals tend to lose heat relatively slowly.

Understanding a Model

1. Recognizing independent variables (inputs), dependent variables (outputs), parameters (fixed values)
2. Verbalizing the relationship in terms of proportionality or inverse proportionality
3. Graphing and interpreting graphs: for now, increasing vs. decreasing
4. Calculating how changes in input affect changes in output (later we'll see derivatives – for now think about the effect on output in power models when doubling input)

Example

Body mass index (BMI) for a person of mass m_{kg} kilograms and height h_m metres is

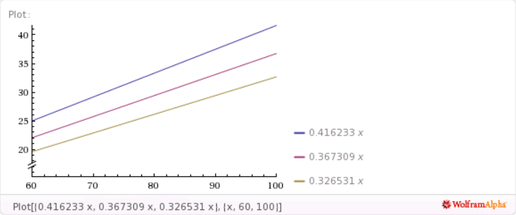
$$\mathrm{BMI} = \frac{m_{kg}}{h_m^2}.$$

Body Mass Index (p.52) as a function of mass

We study one independent variable at a time. The other is a *parameter*.

$$\mathrm{BMI} = \frac{m_{kg}}{h_m^2}.$$

1. View h_m as a parameter, m_{kg} as independent, BMI as dependent.
2. BMI is a linear function of m_{kg}. BMI is proportional to mass.
3. `f(m)=m/h^2, h in {1.55,1.65,1.75}`



Increasing lines (positive slope) for all parameters

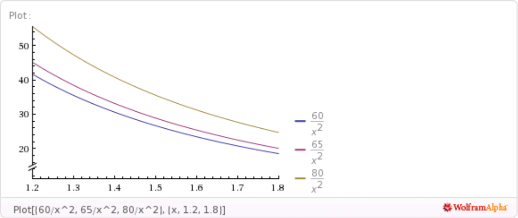
4. Doubling the independent variable does what? Doubles the BMI.

Body Mass Index (p.52) as a function of height

We study one independent variable at a time. The other is a *parameter*.

$$\mathrm{BMI} = \frac{m_{kg}}{h_m^2}.$$

1. View m_{kg} as a parameter, h_m as independent, BMI as dependent.
2. BMI is proportional to the inverse square of height h.
3. `f(h)=m/h^2, h in {60,65,80}`



Decreasing (for all parameters)

4. Doubling the independent variable does what? Quarters the BMI.

Dimensional Analysis: Book–keeping Tool

Recall, $BMI := m_{kg}/h_m^2$ (m_{kg}=mass in kg, h_m=height in metres).

Problem

You weigh m_{lbs} pounds and you are h_{in} inches tall. What is your BMI?

Solution

1kg weighs 2.2lb, so $m_{kg} = m_{lbs}/2.2$.
Since $1m = 100cm$ and $1in = 2.54cm$:

$$1m = (1\cancel{m})\left(\frac{100\cancel{cm}}{1\cancel{m}}\right)\left(\frac{1in.}{2.54\cancel{cm}}\right) \approx 39.37in.,$$

so your height in metres is $h_m = h_{in}/39$.

$$BMI = \frac{m_{kg}}{(h_m)^2} \approx \frac{m_{lbs}/2.2}{(h_{in}/39)^2} = \frac{39^2}{2.2}\frac{m_{lbs}}{(h_{in})^2} \approx 700\frac{m_{lbs}}{(h_{in})^2}$$

Alternate solution using fudge factor

Recall, $BMI := m_{kg}/h_m^2$ (m_{kg}=mass in kg, h_m=height in metres).

Problem

You weigh m_{lbs} pounds and you are h_{in} inches tall. What is your BMI?

Solution

Suppose you weigh 1kg and you are 1m tall. Then your BMI is 1.

Equivalently, you weigh 2.2lbs (the weight of a kilogram in pounds) and you have a height of 39.37in.

If you were to compute m_{lbs}/h_{in}^2, you'd get $\frac{2.2}{39.37^2} = 0.00142$.

The answer should be 1, so you must divide by 0.00142 to get the right answer.

$$BMI = \frac{1}{.00142} \frac{m_{lbs}}{h_{in}^2} \approx 700 \frac{m_{lbs}}{h_{in}^2}.$$

Domain and Range

Definition

The **domain** of a function is the set of *allowed* input values.

- **Natural domain**: input allowed if math operations are valid.
- **Restricted domain**: only some inputs declared valid.
  - e.g., maybe only some inputs correspond to real-world possibilities

Definition

The *range* of a function is the set of output values attained.

Problem

Find the domain and range of

$$f(x) = \frac{1}{\sqrt{4 - x^2}}.$$

Domain and Range

Solution

$$f(x) = \frac{1}{\sqrt{4 - x^2}}.$$

In order for the denominator make sense, we can't take the square root of a negative number. So :

$$4 - x^2 \geq 0$$

But then we need to divide by $\sqrt{4 - x^2}$, *so* $4 - x^2 \neq 0$.
The domain is therefore

$$\{x : 4 - x^2 > 0\}.$$

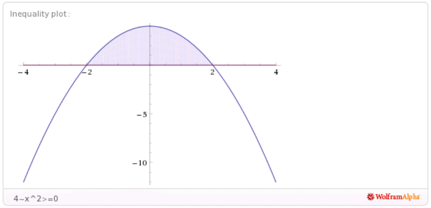
Let's see if we can write this in interval notation (remember: learn Table 0.1.2).

Solution

We want to write $4 - x^2 > 0$ *in interval notation.*

Step 1: solve $4 - x^2 = 0$.

4-x^2>0

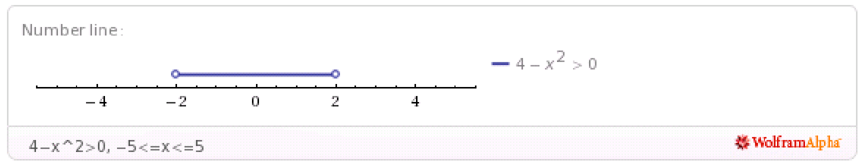


Where does the parabola cross x-axis?

$4 - x^2 = 0$

$(2 - x)(2 + x) = 0$

$x = 2$ *or* $x = -2$

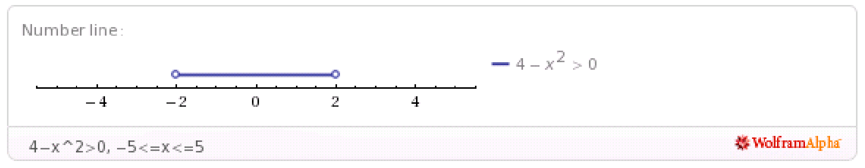


Domain of $1/\sqrt{4-x^2}$

Solution

Remember: we want to find where $4 - x^2 > 0$.

Step 2: Check each of the regions by testing a point.



If $x = -5$, *then* $4 - x^2 = 4 - 25 = -21 < 0$ *is negative.*

If $x = 0$, *then* $4 - x^2 = 4 - 0 = 4 > 0$ *is positive.*

If $x = 5$, *then* $4 - x^2 = 4 - 25 = -21 < 0$ *is negative.*

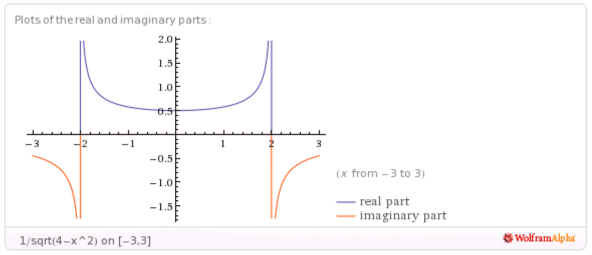
The domain of f is the *open interval* $(-2, 2)$.

Range of $1/\sqrt{4-x^2}$

Solution

What about the range (output values)?

1/sqrt{4-x^2}



$4 - x^2$ *is always at most 4 (since* x^2 *is never negative).*

$\sqrt{4-x^2}$ *is always at most 2.*

$1/\sqrt{4-x^2}$ *is always at least 1/2.*

The range is $[1/2, \infty)$.

The domain can be seen along x-axis, range along y-axis.

Summary: Models and Dynamical Systems

Just a friendly reminder about the three kinds of models.

- Continuous functions express dependent variable in terms of independent variable, parameters.
  - Graphs are "smooth"
- Discrete-time dynamical system (Chapter 2)
  - Graphs consist of isolated points
  - Dependent variable given in terms of its own previous value
- Continuous-time dynamical systems:
  - For us, these will be synonymous with **differential equations**
  - Continuous analogue of discrete-time: dependent variable is described in relation to itself (via rates of change)
  - Graphs are smooth
  - Studied in Math 1LT3 using the tools we develop in this course