COMPSCI/SFWRENG 2FA3
Discrete Mathematics with Applications II
Winter 2020

Week 02 Exercises with Solutions

Dr. William M. Farmer
McMaster University

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Background Definitions

1. The notation $\sum_{i=m}^{n} f(i)$ is defined by:

$$\sum_{i=m}^{n} f(i) = \begin{cases} 0 & \text{if } m > n \\ f(n) + \sum_{i=m}^{n-1} f(i) & \text{if } m \leq n \end{cases}$$

2. The Fibonacci sequence $\mathsf{fib} : \mathbb{N} \rightarrow \mathbb{N}$ is defined by:

$$\mathsf{fib}(n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ \mathsf{fib}(n-1) + \mathsf{fib}(n-2) & \text{if } n \geq 2 \end{cases}$$

3. Let $a, b \in \mathbb{Z}$. a *divides* b, written $a \mid b$, if $b = ac$ for some $c \in \mathbb{Z}$.

Exercises

1. Prove the following statements:

   a. The sum of two odd integers is an even integer.

   Solution:

   Proof Let m and n be arbitrary odd integers. Since m and n are odd, there are integers i and j such that $m = 2i + 1$ and $n = 2j + 1$. Then

   $$m + n = (2i + 1) + (2j + 1) = 2(i + j + 1),$$

   which show $m + n$ is even. $\square$

b. If x is an even integer, then x^2 is also even.

Solution:

Proof Let x be an arbitrary even integer. Since x is even, there is some integer i such that $x = 2i$. Then

$$x^2 = (2i)^2 = 4i^2 = 2(2i^2),$$

which shows x^2 is even. $\square$

c. Let $a, b, c, d \in \mathbb{Z}$. If $a \mid b$ and $c \mid d$, then $ac \mid bd$.

Solution:

Proof Let $a, b, c, d \in \mathbb{Z}$ and assume $a \mid b$ and $c \mid d$. We must show $ac \mid bd$. Since $a \mid b$ and $c \mid d$, there are integers i and j such that $b = ai$ and $d = cj$. Then

$$bd = aicj = (ac)(ij),$$

which shows $ac \mid bd$. $\square$

d. The square root of 2 is an irrational number.

Solution:

Proof Assume the statement is false, i.e., that $\sqrt{2}$ is a rational number. This implies that there are integers m and n with no common divisor other than ± 1 and with $n \neq 0$ such that $\sqrt{2} = \frac{m}{n}$. Then

$$2 = (\sqrt{2})^2 = \left(\frac{m}{n}\right)^2 = \frac{m^2}{n^2}.$$

Hence $2n^2 = m^2$ since $n \neq 0$. This implies $2 \mid m^2$ which implies $2 \mid m$. Hence there is some integer k such that $m = 2k$ and so $2n^2 = m^2 = (2k)^2 = 2(2k^2)$. This implies $2 \mid n^2$ which implies $2 \mid n$. Therefore $2 \mid m$ and $2 \mid n$ which contradicts our assumption that m and n have no common divisor other than ± 1. $\square$

2. Prove the following statements by weak induction:

a. $\sum_{i=0}^{n} 2i = n(n+1)$ for all $n \in \mathbb{N}$.

Solution:

Proof Let $P(n) \equiv \sum_{i=0}^{n} 2 * i = n * (n+1)$. We will prove $P(n)$ for all $n \in \mathbb{N}$ by weak induction.

Base case: $n = 0$. Prove $P(0)$.

$$\begin{aligned}
&\sum_{i=0}^{0} 2 * i && \langle \text{LHS of } P(0) \rangle \\
&= 2 * 0 && \langle \text{definition of } \textstyle\sum_{i=m}^{n} f(i) \text{ when } m = n \rangle \\
&= 0 * (0 + 1) && \langle \text{arithmetic; RHS of } P(0) \rangle
\end{aligned}$$

This shows that $P(0)$ holds.

Induction step: $n \geq 0$. Assume $P(n)$. Prove $P(n+1)$.

$$\begin{aligned}
&\sum_{i=0}^{n+1} 2 * i && \langle \text{LHS of } P(n+1) \rangle \\
&= 2 * (n+1) + \sum_{i=0}^{n} 2 * i && \langle \text{definition of } \textstyle\sum_{i=m}^{n} f(i) \rangle \\
&= 2 * (n+1) + n * (n+1) && \langle \text{induction hypothesis: } P(n) \rangle \\
&= n^2 + 3 * n + 2 && \langle \text{put in standard form} \rangle \\
&= (n+1) * (n+2) && \langle \text{factor; RHS of } P(n+1) \rangle
\end{aligned}$$

This shows that $P(n+1)$ holds.

Therefore, $P(n)$ holds for all $n \in \mathbb{N}$ by weak induction. $\square$

b. $\sum_{i=1}^{n}(2i - 1) = n^2$ for all $n \in \mathbb{N}$.

Solution:

Similar to 1a.

c. $\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ for all $n \in \mathbb{N}$.

Solution:

Proof Let $P(n) \equiv \sum_{i=0}^{n} i^2 = \frac{n*(n+1)*(2*n+1)}{6}$. We will prove $P(n)$ for all $n \in \mathbb{N}$ by weak induction.

Base case: $n = 0$. Prove $P(0)$.

$$\begin{aligned}
&\sum_{i=0}^{0} i^2 && \langle \text{LHS of } P(0) \rangle \\
&= 0^2 && \langle \text{definition of } \textstyle\sum_{i=m}^{n} f(i) \rangle \\
&= \frac{0 * (0+1) * (2 * 0 + 1)}{6} && \langle \text{arithmetic; RHS of } P(0) \rangle
\end{aligned}$$

This shows that $P(0)$ holds.

Induction step: $n \geq 0$. Assume $P(n)$. Prove $P(n+1)$.

$$\sum_{i=0}^{n+1} i^2$$

$\langle$LHS of $P(n+1)\rangle$

$$= (n+1)^2 + \sum_{i=0}^{n} i^2$$

$\langle$definition of $\sum_{i=m}^{n} f(i)\rangle$

$$= (n+1)^2 + \frac{n*(n+1)*(2*n+1)}{6}$$

$\langle$induction hypothesis: $P(n)\rangle$

$$= \frac{6*(n+1)^2 + n*(n+1)*(2*n+1)}{6}$$

$\langle$addition of fractions$\rangle$

$$= \frac{(n+1)*(6*(n+1) + n*(2*n+1))}{6}$$

$\langle$factor out $n+1\rangle$

$$= \frac{(n+1)*(2*n^2 + 7*n + 6)}{6}$$

$\langle$multiply out second factor and collect like terms$\rangle$

$$= \frac{(n+1)*(n+2)*(2*n+3)}{6}$$

$\langle$factor second factor$\rangle$

$$= \frac{(n+1)*(n+2)*(2*(n+1)+1)}{6}$$

$\langle$arithmetic; RHS of $P(n+1)\rangle$

This shows that $P(n+1)$ holds.

Therefore, $P(n)$ holds for all $n \in \mathbb{N}$ by weak induction. □

d. $\sum_{i=0}^{n-1} 2^i = 2^n - 1$ for all $n \in \mathbb{N}$.

Solution:

See the lecture slides for the 1 Mathematical Proof topic.

e. $\sum_{i=0}^{n} \mathsf{fib}(i) = \mathsf{fib}(n+2) - 1$ for $n \in \mathbb{N}$.

Solution:

Proof Let $P(n) \equiv \sum_{i=0}^{n} \mathsf{fib}(i) = \mathsf{fib}(n+2) - 1$. We will prove $P(n)$ for all $n \in \mathbb{N}$ by weak induction.

Base case: $n = 0$. Prove $P(0)$.

$$\begin{aligned}
&\sum_{i=0}^{0} \mathsf{fib}(i) && \langle \text{LHS of } P(0) \rangle \\
&= \mathsf{fib}(0) && \langle \text{def. of } \textstyle\sum_{i=m}^{n} f(i) \text{ when } m = n \rangle \\
&= \mathsf{fib}(0) + \mathsf{fib}(1) - 1 && \langle \text{arithmetic and definition of } \mathsf{fib} \rangle \\
&= \mathsf{fib}(0 + 2) - 1 && \langle \text{definition of } \mathsf{fib}\text{; RHS of } P(0) \rangle
\end{aligned}$$

This shows that $P(0)$ holds.

Induction step: $n \geq 0$. Assume $P(n)$. Prove $P(n + 1)$.

$$\begin{aligned}
&\sum_{i=0}^{n+1} \mathsf{fib}(i) && \langle \text{LHS of } P(n+1) \rangle \\
&= \mathsf{fib}(n + 1) + \sum_{i=0}^{n} \mathsf{fib}(i) && \langle \text{def. of } \textstyle\sum_{i=m}^{n} f(i) \rangle \\
&= \mathsf{fib}(n + 1) + \mathsf{fib}(n + 2) - 1 && \langle \text{ind. hypo.: } P(n) \rangle \\
&= \mathsf{fib}((n + 1) + 2) - 1 && \langle \text{def. of } \mathsf{fib}\text{; RHS of } P(n + 1) \rangle
\end{aligned}$$

This shows that $P(n + 1)$ holds.

Therefore, $P(n)$ holds for all $n \in \mathbb{N}$ by weak induction. $\square$

f. $\sum_{i=0}^{n} (\mathsf{fib}(i))^2 = \mathsf{fib}(n) * \mathsf{fib}(n + 1)$ for all $n \in \mathbb{N}$.

Solution:

Proof Let $P(n) \equiv \sum_{i=0}^{n} (\mathsf{fib}(i))^2 = \mathsf{fib}(n) * \mathsf{fib}(n + 1)$. We will prove $P(n)$ for all $n \in \mathbb{N}$ by weak induction.

Base case: $n = 0$. Prove $P(0)$.

$$\begin{aligned}
&\sum_{i=0}^{0} (\mathsf{fib}(i))^2 && \langle \text{LHS of } P(0) \rangle \\
&= (\mathsf{fib}(0))^2 && \langle \text{definition of } \textstyle\sum_{i=m}^{n} f(i) \text{ when } m = n \rangle \\
&= 0^2 && \langle \text{definition of } \mathsf{fib} \rangle \\
&= 0 * 1 && \langle \text{arithmetic} \rangle \\
&= \mathsf{fib}(0) * \mathsf{fib}(0 + 1) && \langle \text{definition of } \mathsf{fib}\text{; RHS of } P(0) \rangle
\end{aligned}$$

This shows that $P(0)$ holds.

Induction step: $n \geq 0$. Assume $P(n)$. Prove $P(n+1)$.

$$\begin{aligned}
& \sum_{i=1}^{n+1} (\mathsf{fib}(i))^2 && \langle \text{LHS of } P(n+1) \rangle \\
&= (\mathsf{fib}(n+1))^2 + \sum_{i=1}^{n} (\mathsf{fib}(i))^2 && \langle \text{def. of } \textstyle\sum_{i=m}^{n} f(i) \rangle \\
&= (\mathsf{fib}(n+1))^2 + \mathsf{fib}(n) * \mathsf{fib}(n+1) && \langle \text{ind. hypo.: } P(n) \rangle \\
&= \mathsf{fib}(n+1) * (\mathsf{fib}(n+1) + \mathsf{fib}(n)) && \langle \text{factor out } \mathsf{fib}(n+1) \rangle \\
&= \mathsf{fib}(n+1) * \mathsf{fib}(n+2) && \langle \text{def. of } \mathsf{fib} \rangle \\
&= \mathsf{fib}(n+1) * \mathsf{fib}((n+1)+1) && \langle \text{RHS of } P(n+1) \rangle
\end{aligned}$$

This shows that $P(n+1)$ holds.

Therefore, $P(n)$ holds for all $n \in \mathbb{N}$ by weak induction. $\square$

3. Prove the following statements by strong induction:

   a. If $n \in \mathbb{N}$ with $n \geq 2$, then n is a product of prime numbers.

   Solution:

   See the lecture slides for the 1 Mathematical Proof topic.

   b. $\mathsf{fib}(n) < 2^n$ for all $n \in \mathbb{N}$.

   Solution:

   Proof Let $P(n) \equiv \mathsf{fib}(n) < 2^n$. We will prove $P(n)$ for all $n \in \mathbb{N}$ by strong induction.

   Base case 1: $n = 0$. Prove $P(0)$.

$$\begin{aligned}
& \mathsf{fib}(0) && \langle \text{LHS of } P(0) \rangle \\
&= 0 && \langle \text{definition of } \mathsf{fib} \rangle \\
&< 1 && \langle \text{arithmetic} \rangle \\
&= 2^0 && \langle \text{arithmetic; RHS of } P(0) \rangle
\end{aligned}$$

   This shows that $P(0)$ holds.

   Base case 2: $n = 1$. Prove $P(1)$.

$$\begin{aligned}
& \mathsf{fib}(1) && \langle \text{LHS of } P(1) \rangle \\
&= 1 && \langle \text{definition of } \mathsf{fib} \rangle \\
&< 2 && \langle \text{arithmetic} \rangle \\
&= 2^1 && \langle \text{arithmetic; RHS of } P(1) \rangle
\end{aligned}$$

This shows that $P(1)$ holds.

Induction step: $n \geq 2$. Assume $P(m)$ for all $m < n$. Prove $P(n)$.

$$
\begin{aligned}
& \mathsf{fib}(n) && \langle \text{LHS of } P(n) \rangle \\
&= \mathsf{fib}(n-1) + \mathsf{fib}(n-2) && \langle \text{definition of } \mathsf{fib} \rangle \\
&< 2^{n-1} + \mathsf{fib}(n-2) && \langle \text{induction hypothesis: } P(n-1) \rangle \\
&< 2^{n-1} + 2^{n-2} && \langle \text{induction hypothesis: } P(n-2) \rangle \\
&< 2^{n-1} + 2^{n-1} && \langle \text{arithmetic} \rangle \\
&= 2^n && \langle \text{arithmetic; RHS of } P(n) \rangle
\end{aligned}
$$

This shows that $P(n)$ holds.

Therefore, $P(n)$ holds for all $n \in \mathbb{N}$ by strong induction. $\square$

c. It takes $n-1$ divisions to break up a rectangular chocolate bar containing n squares into individual squares.

Solution:

Proof For a positive integer n, let $P(n)$ hold iff every rectangular chocolate bar containing n squares needs $n-1$ divisions to be broken into individual squares. We will prove $P(n)$ for all $n \in \mathbb{N}$ with $n \geq 1$ by strong induction.

Base case: $n = 1$. Prove $P(1)$. A chocolate bar containing 1 square is already broken into individual squares, and so 0 divisions are needed to break it up. This shows that $P(1)$ holds.

Induction step: $n \geq 2$. Assume $P(m)$ for all $m < n$. Prove $P(n)$. Suppose we have a $a \times b$ chocolate bar containing $n = ab$ squares. W.l.o.g., we may assume that the first division of the chocolate bar breaks it into $(a-c) \times b$ and $c \times b$ chocolate bars. By the induction hypothesis, each of these chocolate bars can be broken up into individual squares with $(a-c)b - 1$ and $cb - 1$ divisions, respectively. Then the number of division needed to break up the original chocolate bar is

$$1 + (a-c)b - 1 + cb - 1 = 1 + ab - 1 - 1 = n - 1.$$

This shows that $P(n)$ holds.

Therefore, $P(n)$ holds for all $n \in \mathbb{N}$ with $n \geq 1$ by strong induction. $\square$

4. Let t_n, s_n, o_n be the nth triangle, square, and oblong numbers, respectively, where $n \in \mathbb{N}$.

   a. Define t_n, s_n, o_n by recursion.

b. Prove by induction that every triangle number is exactly half of an oblong number.

c. Prove by induction that the sum of every two consecutive triangle numbers is a square number.

Solution:

a. $t_n = \begin{cases} 0 & \text{if } n = 0 \\ t_{n-1} + n & \text{if } n > 0 \end{cases}$

$s_n = \begin{cases} 0 & \text{if } n = 0 \\ s_{n-1} + 2 * n - 1 & \text{if } n > 0 \end{cases}$

$o_n = \begin{cases} 0 & \text{if } n = 0 \\ o_{n-1} + 2 * n & \text{if } n > 0 \end{cases}$

b. **Theorem** $t_n = o_n/2$ for all $n \in \mathbb{N}$.

Proof Let $P(n) \equiv t_n = o_n/2$. We will prove $P(n)$ for all $n \in \mathbb{N}$ by weak induction.

Base case: $n = 0$. Prove $P(0)$.

$$\begin{aligned} & t_0 && \langle \text{LHS of } P(0) \rangle \\ =\ & 0 && \langle \text{definition of } t_n \rangle \\ =\ & 0/2 && \langle \text{arithmetic} \rangle \\ =\ & o_0/2 && \langle \text{definition of } o_n\text{; RHS of } P(0) \rangle \end{aligned}$$

This shows that $P(0)$ holds.

Induction step: $n \geq 0$. Assume $P(n)$. Prove $P(n+1)$.

$$\begin{aligned} & t_{n+1} && \langle \text{LHS of } P(n+1) \rangle \\ =\ & t_n + n + 1 && \langle \text{definition of } t_n \rangle \\ =\ & o_n/2 + n + 1 && \langle \text{induction hypothesis: } P(n) \rangle \\ =\ & (o_n + 2 * (n+1))/2 && \langle \text{arithmetic} \rangle \\ =\ & o_{n+1}/2 && \langle \text{def. of } o_n\text{; RHS of } P(n+1) \rangle \end{aligned}$$

This shows that $P(n+1)$ holds.

Therefore, $P(n)$ holds for all $n \in \mathbb{N}$ by weak induction. □

c. **Theorem** $t_n + t_{n+1} = s_{n+1}$ for all $n \in \mathbb{N}$.

Proof Let $P(n) \equiv t_n + t_{n+1} = s_{n+1}$. We will prove $P(n)$ for all $n \in \mathbb{N}$ by weak induction.

Base case: $n = 0$. Prove $P(0)$.

$$\begin{aligned} & t_0 + t_1 && \langle \text{LHS of } P(0) \rangle \\ =\ & 0 + 1 && \langle \text{definition of } t_n \rangle \\ =\ & 1 && \langle \text{arithmetic} \rangle \\ =\ & s_1 && \langle \text{definition of } s_n\text{; LHS of } P(0) \rangle \end{aligned}$$

This shows that $P(0)$ holds.

Induction step: $n \geq 0$. Assume $P(n)$. Prove $P(n+1)$.

$$
\begin{array}{ll}
\quad t_{n+1} + t_{n+2} & \langle \text{LHS of } P(n+1) \rangle \\
= t_n + n + 1 + t_{n+1} + n + 2 & \langle \text{definition of } t_n \rangle \\
= s_{n+1} + n + 1 + n + 2 & \langle \text{ind. hypo.: } P(n) \rangle \\
= s_{n+1} + 2 * (n+2) - 1 & \langle \text{arithmetic} \rangle \\
= s_{n+2} & \langle \text{def. of } s_n\text{; RHS of } P(n+1) \rangle
\end{array}
$$

This shows that $P(n+1)$ holds.

Therefore, $P(n)$ holds for all $n \in \mathbb{N}$ by weak induction. $\square$