MATHEMATICS 1LS3 TEST 4

Day Class
Duration of Examination: 60 minutes
McMaster University, 19 November 2014

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FIRST NAME (please print): SOLUTIONS

FAMILY NAME (please print): ____________________

Student No.: ____________________

THIS TEST HAS 8 PAGES AND 7 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE.

Total number of points is 40. Marks are indicated next to the problem number. Any non-graphing calculator is allowed.

USE PEN TO WRITE YOUR TEST. IF YOU USE A PENCIL YOUR TEST WILL NOT BE ACCEPTED FOR REMARKING (IF NEEDED).

You must show work to receive full credit.

Problem	Points	Mark
1	6	
2	6	
3	5	
4	5	
5	6	
6	7	
7	5	
TOTAL	40	

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1. **Multiple choice questions: circle ONE answer. No justification is needed.**

(a)[3] Which of the following limits is/are indeterminate form(s)?

$$\text{(I)} \lim_{x\to 0} \frac{\cos x - x}{x} = \frac{1}{0} \quad \text{(II)} \lim_{x\to 0} \frac{e^x - 1 - x}{x^2} = \frac{0}{0} \quad \text{(III)} \lim_{x\to 0} \frac{\sec x - 1}{\tan x} = \frac{0}{0}$$

X

(A) none (B) I only (C) II only (D) III only

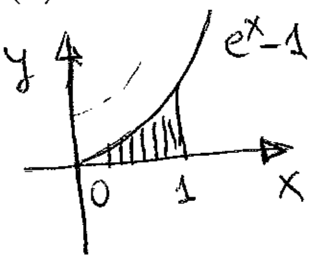
(E) I and II (F) I and III (G) II and III (H) all three

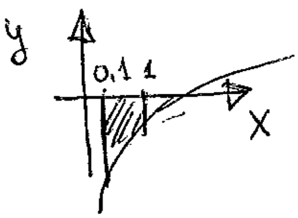
(b)[3] Which of the following numbers is/are positive?

$$\text{(I)} \int_0^1 (e^x - 1)\,dx \checkmark \quad \text{(II)} \int_{0.1}^1 (\ln x - 1)\,dx \times \quad \text{(III)} \int_0^1 (\sec x - 1)\,dx \checkmark$$

(A) none (B) I only (C) II only (D) III only

(E) I and II (F) I and III (G) II and III (H) all three





$$= \frac{1}{\cos x} - 1$$

$$= \frac{1-\cos x}{\cos x} \geq 0$$

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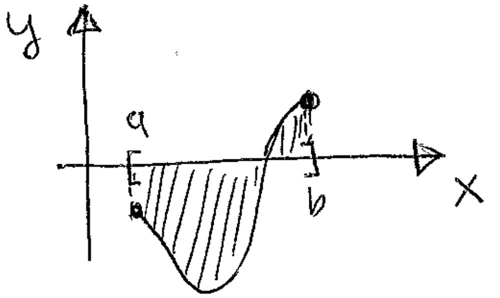
2. Identify each statement as true or false, or yes or no (circle your choice). You do not need to justify your answer.

(a)[2] It is known that $\int_a^b f(x)dx < 0$. This implies that the function $f(x)$ is negative, i.e., $f(x) < 0$ for all x in $[a, b]$.

TRUE FALSE

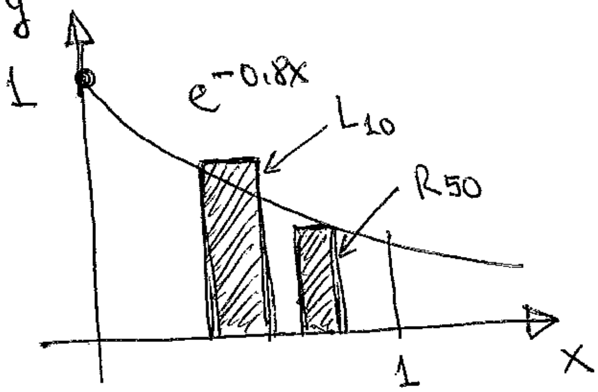


$\int_a^b f(x)dx$ = area above − area below < 0

but f(x) is not negative for all x in [a,b]

(b)[2] Let $f(x) = e^{-0.8x}$ on [0,1]. The left sum L_{10} is larger than the right sum ~~R_{50}~~.

TRUE FALSE



$L_{10} >$ area under $e^{-0.8x}$

$R_{50} <$ area under $e^{-0.8x}$

(c)[2] The function $f(x) = \dfrac{1}{4-x}$ is ~~an~~ one possible antiderivative of $g(x) = \dfrac{1}{(4-x)^2}$.

TRUE FALSE

ie $\int \frac{1}{(4-x)^2} dx \overset{?}{=} \frac{1}{4-x}$

$\left(\frac{1}{4-x}\right)' = (-1)(4-x)^{-2}(-1) = \frac{1}{(4-x)^2}$ ✓

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Questions 3-7: You must show work to receive full credit.

3. The rate at which new influenza cases occured in 2013 in Greater Vancouver Area follows the formula $125.4e^{0.3t} + 14.6e^{-0.1t}$ people/day. By t we represent the time in days measured from 1 December 2013 (so $t = 0$ represents 1 December 2013). On 1 December 2013 there were 56 cases of influenza.

(a)[2] Write a differential equation and the initial condition for the number $N(t)$ of influenza cases.

$$N'(t) = 125{,}4\, e^{0{,}3t} + 14{,}6\, e^{-0{,}1t}$$

$$N(0) = 56$$

(b)[3] Solve the initial value problem in (a) to find the formula for $N(t)$.

$$N(t) = \int \left(125{,}4\, e^{0{,}3t} + 14{,}6\, e^{-0{,}1t}\right) dt$$

$$= 125{,}4 \cdot \frac{1}{0{,}3} e^{0{,}3t} + 14.6 \cdot \frac{1}{-0.1} e^{-0.1t} + C$$

$$= 418\, e^{0{,}3t} - 146\, e^{-0.1t} + C$$

$N(0) = 56$

$\hookrightarrow \; 56 = 418 - 146 + C \rightarrow C = -216$

so $\quad N(t) = 418\, e^{0{,}3t} - 146\, e^{-0.1t} - 216$

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4. Consider the initial value problem $y' = (1 - t + t^2)$, $y(0) = 2$.

(a)[2] Find an approximation of $y(0.4)$ using two steps of Euler's method with the step size $\Delta t = 0.2$.

$\boxed{t_0 = 0,\ y_0 = 2}$ start

$$t_{n+1} = t_n + \Delta t$$

$$y_{n+1} = y_n + (1 - t_n + t_n^2)\,\Delta t$$

$n=0$ ... $t_1 = t_0 + \Delta t = 0.2$

$$y_1 = y_0 + (\underbrace{1 - t_0 + t_0^2}_{1}) \cdot 0.2 = 2 + 0.2 = 2.2$$

$n=1$ .. $t_2 = t_1 + \Delta t = 0.4$

$$y_2 = y_1 + (1 - t_1 + t_1^2) \cdot \Delta t$$

$$= 2.2 + (1 - 0.2 + 0.2^2) \cdot 0.2 = 2.368$$

(b)[2] Using antidifferentiation, find the exact solution of the given initial value problem.

$$y = \int (1 - t + t^2)\,dt = t - \frac{t^2}{2} + \frac{t^3}{3} + C$$

$y(0) = 2 \longrightarrow C = 2$

So $\quad y = t - \frac{t^2}{2} + \frac{t^3}{3} + 2$

(c)[1] Using (b), find the true value of $y(0.4)$ and round off to three decimal places (thus checking your approximation in (a)).

$$y(0.4) = 0.4 - \frac{0.4^2}{2} + \frac{0.4^3}{3} + 2 \approx 2.341$$

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5. Consider the differential equation

$$P'(t) = 1.1P(t)\left(1 - \frac{P(t)}{1400}\right)^{1/3}$$

where $P(t)$ represents the number of elk in Douglas Provincial Park in Saskatchewan. The variable t represents time in years, with $t = 0$ representing 2006.

(a)[1] Classify the above differential equation as pure-time, autonomous, or neither pure-time nor autonomous.

P only in the left side

(b)[2] For which values of $P(t)$ is the population increasing? Justify your answer.

$P' > 0$ when $1 - \frac{P(t)}{1400} > 0$

$\frac{P(t)}{1400} < 1 \rightarrow \underline{\underline{P(t) < 1400}}$

(c)[2] For which values of $P(t)$ is the population decreasing? Justify your answer.

$P' < 0$ when $1 - \frac{P(t)}{1400} < 0 \rightarrow P(t) > 1400$

there are other correct interpretations such as:
"1400 is the maximum number of elk"
"1400 is the carrying capacity"
"population increases if below 1400 and decreases if above 1400"
"maximum population where growth stops and decay starts"

(d)[1] What is the biological meaning of the constant 1400?

$P'(t) = 0$ when $P(t) = 1400$ → population remains unchanged, i.e. the number of elk is constant

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6. (a)[2] Find $\int_0^1 \left(\frac{5}{1+x^2} + \frac{1+x^2}{5}\right)\, dx$ $= 5\int_0^1 \frac{1}{1+x^2}dx + \int_0^1 \frac{1}{5}dx + \int_0^1 \frac{x^2}{5}dx$

$$= 5\arctan x \Big|_0^1 + \frac{1}{5}x\Big|_0^1 + \frac{1}{5}\frac{x^3}{3}\Big|_0^1$$

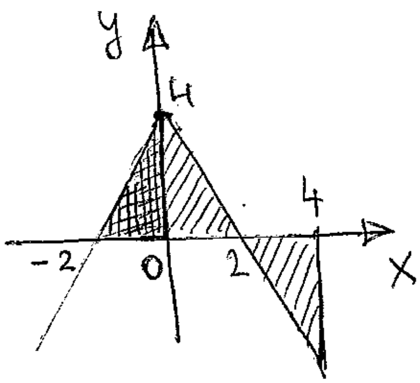
$$= 5\arctan 1 - 5\cancel{\arctan 0} + \frac{1}{5} - 0 + \frac{1}{5}\cdot\frac{1}{3} - 0$$

$$= 5\frac{\pi}{4} + \frac{1}{5} + \frac{1}{15} = \frac{5\pi}{4} + \frac{4}{15} \approx 4.19$$

(b)[2] Find $\int (\sec x \tan x + \pi)\, dx$

$$= \sec x + \pi x + C$$

(c)[3] Compute $\int_{-2}^{4} (4 - 2|x|)\, dx$ by interpreting the definite integral in terms of area(s).



$$\int_{-2}^{4} (4-2|x|)\,dx = \text{area above} - \text{area below}$$

$$= 4+4-4 = \underline{\underline{4}}$$

all three shaded triangles have the same area $= \frac{1}{2}\cdot 2 \cdot 4 = 4$

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7. Find the following limits.

(a)[3] Find $\lim_{x\to 0} \frac{e^{x^3} - 1 - x^3}{x^6}$ $= \frac{0}{0} \overset{LH}{=} \lim_{x\to 0} \frac{e^{x^3} \cdot 3x^2 - 3x^2}{6x^5}$

= simplify by cancelling by $3x^2$

$\overset{LH}{=} \lim_{x\to 0} \frac{e^{x^3} - 1}{2x^3} = \frac{0}{0}$

$= \lim_{x\to 0} \frac{e^{x^3} \cdot \cancel{3x^2}}{2 \cdot \cancel{3x^2}} = \frac{1}{2}$

(b)[2] Find $\lim_{x\to 0^+} x^4 \ln x$ $= 0 \cdot (-\infty)$

$= \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x^4}} \overset{LH}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-4}{x^5}} = -\frac{1}{4} \lim_{x\to 0^+} x^4 = 0$

THE END