COMPSCI/SFWRENG 2FA3 Midterm Test 1
McMaster University

Answer Key: Large arrow ($\Longleftarrow$) for correct, small ($\leftarrow$) for partially correct

Day Class CS 01, CS 02, SE 01, **Version 1** Dr. W. M. Farmer

DURATION: 2 hours February 8, 2019

Please CLEARLY print:

NAME:

Student ID:

In an addition to this examination paper, you will be given two answer sheets for this test. This examination paper includes 13 pages and 30 questions. You are responsible for ensuring that your copy of the examination paper is complete. Bring any discrepancy to the attention of your invigilator.

The examination will be conducted in two stages:

First Stage: You have 90 minutes to answer the questions in the examination paper on the first answer sheet working **by yourself**. Getting any help in any form from your fellow students and anyone else will be treated as academic dishonesty. You must submit your first answer sheet to your invigilator by the end of the 90-minute period. Your performance on the answer sheet counts for 85% of the Midterm Test 1 mark. You may want to fill out the second answer sheet as you fill out the first leaving blank those questions that you want to work on during the second stage.

Second Stage: You have 30 minutes to answer the questions in the examination paper on the second answer sheet working **with the other students in the test room**. You may walk around the test room, but you may not leave the test room. You must submit your second answer sheet and your examination paper to your invigilator by the end of the 30-minute period. Your performance on the answer sheet counts for 15% of the Midterm Test 1 mark.

Special Instructions:

1. It is your responsibility to ensure that the two answer sheets are properly completed. Your examination result depends upon proper attention to these instructions:

   - A heavy mark must be made, completely filling the circular bubble, with an HB pencil.
   - Print your name, student number, course name, course number and the date in the space provided on the top of Side 1 and fill in the corresponding bubbles underneath.
   - **Fill in the bubble corresponding to your version number.**
   - Mark only **ONE** choice from the alternatives (1, 2, 3, 4, 5 or A, B, C, D, E) provided for each question. If there is a True/False question, mark 1 (or A) for True, and 2 (or B) for False. The question number is to the left of the bubbles. Make sure that the number of the question on the scan sheet is the same as the number on the examination paper.

- Pay particular attention to the "Marking Directions" given on the scan sheet.
- Begin answering the questions using the first set of bubbles, marked "1." Answer all questions.

2. The use of notes and textbooks is **not** permitted in both stages of the test.
3. Calculators, computers, cell phones, and all other electronic devices are **not** to be utilized in both stages of the test.
4. Read each question carefully.
5. Try to allocate your time sensibly and divide it appropriately between the questions.
6. Select the **best** answer for each question.

Question 1 [1 mark]
A mathematical proof is a deductive argument that can be checked by a computer program. Is this statement true or false?

A. True.

B. False. $\Longleftarrow$

ANSWER:
A mathematical proof is a deductive argument, but a typical mathematical proof cannot be checked by a computer program.

Question 2 [1 mark]
Weak induction can be directly expressed in MSFOL. Is this statement true or false?

A. True.

B. False. $\Longleftarrow$

ANSWER:
Weak induction is a universal statement about properties of natural numbers. Quantification over properties of numbers cannot be directly expressed in MSFOL or in any other form of first-order logic.

Question 3 [1 mark]
If $T = (\Sigma, \Gamma)$ is a theory of MSFOL, then every Σ-structure is a model of T. Is this statement true or false?

A. True.

B. False. $\Longleftarrow$

ANSWER:
A Σ-structure $\mathcal{M}$ is a model of T only if the axioms in Γ are valid in $\mathcal{M}$.

Question 4 [1 mark]
Let Σ be a signature of MSFOL. A Σ-structure can be viewed as a mathematical structure M plus a mapping of the symbols in Σ to certain components of M.
Is this statement true or false?

A. True. $\Longleftarrow$

B. False.

ANSWER:
This is a good informal description of what a Σ-structure is.

Question 5 [1 mark]
First-order Peano arithmetic is an MSFOL theory of real number arithmetic. Is this statement true or false?

A. True.

B. False. $\Longleftarrow$

ANSWER:
First-order Peano arithmetic is a theory of natural number arithmetic, not real number arithmetic.

Question 6 [1 mark]
A monoid can have just one element. Is this statement true or false?

A. True. $\Longleftarrow$

B. False.

ANSWER:
A structure $(\{a\}, f)$ where the function $f : \{a\} \to \{a\}$ maps a to a is a monoid.

Question 7 [1 mark] Let $(S, <)$ be a strict total order such that S is infinite. Then it is possible that $(S, <)$ is a dense well-order. Is this statement true or false?

A. True.

B. False. $\Longleftarrow$

ANSWER:
As seen in Assignment 2, it is easy to construct an infinite descending sequence in a dense strict total order with at least two elements.

Question 8 [1 mark]
Any proof of a statement of the form $\forall\, n \in \mathbb{N} \,.\, P(n)$ using *strong induction* can be transformed into a proof using just *weak induction* by proving $\forall\, n \in \mathbb{N} \,.\, Q(n)$ as a lemma where $Q(n)$ is

$$\forall\, m \in \mathbb{N} \,.\, (m \leq n \Rightarrow P(m)).$$

Is this statement true or false?

A. True. $\Longleftarrow$

B. False.

ANSWER:
This is the trick for expressing a strong induction argument as a weak induction argument. The proof using strong induction can be easily transformed into a proof of $\forall\, n \in \mathbb{N} \,.\, Q(n)$ by weak induction. And obviously, $\forall\, n \in \mathbb{N} \,.\, Q(n) \Rightarrow P(n)$.

Question 9 [1 mark]
The main purpose of a *traditional mathematical proof* is to

A. Communicate why a mathematical statement is true. $\Longleftarrow$

B. Certify that a mathematical statement is true.

C. Organize mathematical knowledge.

D. Discovery a new mathematical fact.

ANSWER:
A is the primary purpose — and B, C, and D are secondary purposes — of a traditional mathematical proof.

Question 10 [1 mark]
What is a lemma?

A. A definition needed to complete a proof.

B. A theorem needed to complete a proof. $\Longleftarrow$

C. A theorem that follows immediately from another theorem.

D. A statement for which there is reason to believe it is true.

ANSWER:
B is the definition of a lemma.

Question 11 [1 mark]
The easiest way to prove that the Ackermann function is defined on all inputs is to use

A. Weak induction.

B. Strong induction.

C. Structural induction.

D. Ordinal induction. $\Longleftarrow$

ANSWER:
To prove that the Ackermann function is total is easy using ordinal induction and much harder using the other forms of induction.

Question 12 [1 mark]
Weak induction is *not* a special case of

A. Structural induction.

B. Ordinal induction. $\Longleftarrow$

C. Well-founded induction.

D. All of the above.

ANSWER:
We showed that weak induction is a special case of structure induction and well-founded induction. Strong induction, not weak induction, is a special case of ordinal induction.

Question 13 [1 mark]
If $R \subseteq S \times S$ is a well-founded relation, then R must be

A. Reflexive.

B. Asymmetric. $\Longleftarrow$

C. Transitive.

D. All of the above.

ANSWER:
R cannot be reflexive, can be transitive, and must be asymmetric. If R were not asymmetric, there would be two elements such that $a\ R\ b$ and $b\ R\ a$ which would imply that R is not Noetherian and hence not well-founded.

Question 14 [1 mark]
Which form of induction is the most general?

A. Natural number induction.

B. Structural induction.

C. Ordinal induction.

D. Well-founded induction. $\Longleftarrow$

ANSWER:
The other three forms of induction are all special forms of well-founded induction.

Question 15 [1 mark]
Let (S, R) be a well-order. Then

A. (S, R) is well-founded.

B. (S, R) is a strict partial order.

C. (S, R) is a strict total order.

D. All of the above. $\Longleftarrow$

ANSWER:
Every well-order is a well-founded strict total order.

Question 16 [1 mark]
Which of the following statements can be proved easier with strong induction than with weak induction?

A. Every square number is the sum of two consecutive triangle numbers.

B. It takes $n-1$ divisions to break up a rectangular chocolate bar containing n squares into individual squares. $\Longleftarrow$

C. The square root of 2 is an irrational number.

D. The number of games in a single-elimination tournament with 2^n teams is $2^n - 1$ for all $n \in \mathbb{N}$.

ANSWER:
A and D are easily proved with weak induction. B is easily proved with strong induction. And C can be proved without using induction.

Question 17 [1 mark]
Let $\Sigma = (\{\alpha, \beta, \gamma\}, \emptyset, \emptyset, \{p, q\}, \tau)$ where $\tau(p) = \alpha \times \beta \times \gamma \to \mathbb{B}$ and $\tau(q) = \alpha \times \beta \times \alpha \to \mathbb{B}$, and let A be the Σ-formula

$$\forall\, x : \alpha\ .\ ((\exists\, y : \beta\ .\ p(x : \alpha, y : \beta, z : \gamma)) \Rightarrow (\forall\, z : \beta\ .\ q(x : \alpha, y : \beta, w : \alpha))).$$

What are the values of $\mathsf{bvar}(A)$ and $\mathsf{fvar}(A)$?

A. $\{x : \alpha, y : \beta\}$ and $\{w : \alpha, z : \gamma\}$.

B. $\{x : \alpha, y : \beta, z : \beta\}$ and $\{x : \alpha, y : \beta, z : \gamma\}$.

C. $\{x : \alpha, y : \beta, z : \beta\}$ and $\{w : \alpha, z : \gamma\}$.

D. $\{x : \alpha, y : \beta, z : \beta\}$ and $\{w : \alpha, y : \beta, z : \gamma\}$. $\Longleftarrow$

ANSWER:
D correctly identifies the bound and free variables in the formula A.

Question 18 [1 mark]
Let $\Sigma = (\{\alpha\}, \{a\}, \emptyset, \{p, q\}, \tau)$ where $\tau(a) = \alpha$, $\tau(p) = \alpha \times \alpha \times \alpha \to \mathbb{B}$ and $\tau(q) = \alpha \times \alpha \to \mathbb{B}$. Let A be the Σ-formula

$$\forall\, x : \alpha\ .\ ((\exists\, z : \alpha\ .\ p(x : \alpha, y : \alpha, z : \alpha)) \Rightarrow (\exists\, y : \alpha\ .\ q(y : \alpha, z : \alpha))).$$

Which of the following substitutions does *not* cause a variable capture?

A. $A[y \mapsto x : \alpha]$.

B. $A[y \mapsto z : \alpha]$.

C. $A[z \mapsto x : \alpha]$.

D. $A[z \mapsto u : \alpha]$. $\Longleftarrow$

ANSWER:
D does not result in a variable capture since the variable $u : \alpha$ is not quantified in the formula A.

Question 19 [1 mark]
Let $T = (\Sigma, \Gamma)$ be a theory of MSFOL where $\Sigma = (\{U\}, \emptyset, \emptyset, \{<\}, \tau)$ with $\tau(<) = U \times U \rightarrow \mathbb{B}$ and Γ includes the following Σ-sentences (as well as possibly other sentences):

1. $\forall\, x : U \,.\, \neg(x < x)$.
2. $\forall\, x, y : U \,.\, x < y \Rightarrow \neg(y < x)$.
3. $\forall\, x, y, z : U \,.\, (x < y \wedge y < z) \Rightarrow x < z$.

If the Σ-sentence

$$\exists\, x : U \,.\, (\forall\, y : U \,.\, \neg(x < y)) \vee (\forall\, y : U \,.\, \neg(y < x))$$

is valid in T, then which of the following statements is true?

A. Every model of T has a minimal or maximal value. $\Longleftarrow$

B. Every model of T has either a minimal but not a maximal value or a maximal but not a minimal value.

C. Every model of T has a minimum and a maximum value.

D. Every model of T has either a minimum but not a maximum value or a maximum but not a minimum value.

ANSWER:
The sentence says there exists a minimal or a maximal value. Since it is valid in T, it must valid in each model of T.

Question 20 [1 mark]
Let $\Sigma = (\mathcal{B}, \mathcal{C}, \mathcal{F}, \mathcal{P}, \tau)$ be a signature of MSFOL and $\alpha \in \mathcal{B}$. If $k \in \mathcal{C} \cup \mathcal{F} \cup \mathcal{P}$ and $\tau(k) = \alpha \rightarrow \mathbb{B}$, then

A. $k \in \mathcal{C}$.

B. $k \in \mathcal{F}$.

C. $k \in \mathcal{P}$. $\Longleftarrow$

D. None of the above.

ANSWER:
Only predicate symbols are assigned predicate types like $\alpha \rightarrow \mathbb{B}$.

Question 21 [1 mark]
Let Σ be a signature of MSFOL and A be a Σ-formula. Then A is valid if

A. If $V_{\varphi}^{\mathcal{M}}(A) = \text{T}$ for some Σ-structure $\mathcal{M}$ and some $\varphi \in \mathsf{assign}(\mathcal{M})$.

B. If $V_{\varphi}^{\mathcal{M}}(A) = \text{T}$ for some Σ-structure $\mathcal{M}$ and all $\varphi \in \mathsf{assign}(\mathcal{M})$.

C. If $V_{\varphi}^{\mathcal{M}}(A) = \text{T}$ for all Σ-structures $\mathcal{M}$ and some $\varphi \in \mathsf{assign}(\mathcal{M})$.

D. If $V_{\varphi}^{\mathcal{M}}(A) = \text{T}$ for all Σ-structures $\mathcal{M}$ and all $\varphi \in \mathsf{assign}(\mathcal{M})$. $\Longleftarrow$

ANSWER:
If A holds, then the formula A is satisfiable. If B holds, the formula A is valid in some Σ-structure. If D holds, the formula A is valid (universally valid). If C holds, the formula does not have a property designated by a name.

Question 22 [1 mark]
Let P be a proof system for MSFOL, Σ be a signature of MSFOL, and Γ be a set of Σ-sentences. If P is sound, then

A. Every semantic consequence of Γ is a syntactic consequence of Γ in P.

B. $\Gamma \vdash_P A$ implies $\Gamma \models A$ for every Σ-formula A. $\Longleftarrow$

C. $\Gamma \models A$ implies $\Gamma \vdash_P A$ for every Σ-formula A.

D. All of the above.

ANSWER:
B says what it means for P to be sound.

Question 23 [1 mark]
Let $R_{\mathsf{suc}} \subseteq \mathbb{N} \times \mathbb{N}$ such that

$$m \; R_{\mathsf{suc}} \; n \text{ iff } n = m + 1.$$

Then $(\mathbb{N}, R_{\mathsf{suc}})$ is

A. A pre-order.

B. A partial order.

C. A well-order.

D. Well-founded. $\Longleftarrow$

ANSWER:
R_{suc} is not transitive, so it cannot be A, B, or C. However, R_{suc} is well-founded.

Question 24 [1 mark]
Let $T = (\Sigma, \Gamma)$ and $T' = (\Sigma, \Gamma')$ be theories of MSFOL. Which of the following statements is true if $\Gamma \subseteq \Gamma'$.

A. Every model of T is a model of T'.

B. Every model of T' is a model of T. $\Longleftarrow$

C. If T is satisfiable, then T' is satisfiable.

D. None of the above.

ANSWER:
Since every axiom of T is an axiom of T', if all the axioms of T' are valid in a Σ-structure, then obviously all the axioms of T will be valid in that Σ-structure.

Question 25 [1 mark]
Let $\Sigma = (\{\alpha, \beta\}, \{c, d\}, \{f, g\}, \emptyset, \tau)$ be a signature of MSFOL where $\tau(c) = \beta$, $\tau(d) = \alpha$, $\tau(f) = \alpha \to \beta$, and $\tau(g) = \alpha \times \beta \to \beta$. Which of the following is a Σ-term?

A. $f(g(d, c))$.

B. $g(c, d)$.

C. $g(d, f(d))$. $\Longleftarrow$

D. $g(g(c, c), c)$.

ANSWER:
Only C is a Σ-term. A, B, and D are not Σ-terms because the types of their subcomponents are mismatched.

Question 26 [1 mark]
Which statement about the set T_Σ of Σ-terms may be false for some signatures Σ?

A. T_Σ is infinite.

B. Some of the terms in T_Σ are open.

C. Some of the terms in T_Σ contain variable binders. $\Longleftarrow$

D. All of the above.

ANSWER:
A is true since there are infinitely many members of $\mathcal{V}$ and thus infinitely many possible variables. B is true since any variable is an open term. C is false since there are no variable binders used to construct the terms of MSFOL.

The next four questions are based on the following two definitions:

Nat is the inductive set defined by the following constructors:

1. $\mathsf{0} : \mathsf{Nat}$.
2. $\mathsf{S} : \mathsf{Nat} \to \mathsf{Nat}$.

The members of Nat are intended to represent the natural numbers.

NumExpr is the inductive set defined by the following constructors:

1. $\mathsf{Zero} : \mathsf{NumExpr}$.
2. $\mathsf{One} : \mathsf{NumExpr}$.
3. $\mathsf{Add} : \mathsf{NumExpr} \times \mathsf{NumExpr} \to \mathsf{NumExpr}$.
4. $\mathsf{Mul} : \mathsf{NumExpr} \times \mathsf{NumExpr} \to \mathsf{NumExpr}$.

The members of NumExpr are intended to represent numerical expressions constructed from 0, 1, +, and $*$ whose values are natural numbers.

Question 27 [**1 mark**]
Which statement follows from the fact that an inductive set has "no confusion"?

A. $\mathsf{Add}(\mathsf{Zero}, \mathsf{One})$ and $\mathsf{Add}(\mathsf{One}, \mathsf{Zero})$ denote different values in NumExpr. $\Longleftarrow$

B. $\mathsf{Add}(\mathsf{Zero}, \mathsf{One})$ and $\mathsf{Add}(\mathsf{One}, \mathsf{Zero})$ denote the same value in NumExpr.

C. Every value in NumExpr can be constructed using the constructors of NumExpr.

D. Some values in NumExpr cannot be constructed using the constructors of NumExpr.

ANSWER:
By "no confusion", different expressions built using the constructors of an induction set denote different values in the set.

Question 28 [1 mark]
The structural induction principle induced by the definition of $\mathsf{NumExpr}$ can be expressed as

$$(P(\mathsf{Zero}) \land P(\mathsf{One}) \land X) \Rightarrow \forall\, e : \mathsf{NumExpr}\ .\ P(e)$$

where X is _______ and P is any property of the members of $\mathsf{NumExpr}$.

A. $\forall\, e_1, e_2 : \mathsf{NumExpr}\ .\ P(\mathsf{Add}(e_1, e_2)) \lor P(\mathsf{Mul}(e_1, e_2))$.

B. $\forall\, e_1, e_2 : \mathsf{NumExpr}\ .\ (P(e_1) \land P(e_2)) \Rightarrow (P(\mathsf{Add}(e_1, e_2)) \land P(\mathsf{Mul}(e_1, e_2)))$. $\Longleftarrow$

C. $\forall\, e_1, e_2 : \mathsf{NumExpr}\ .\ (P(\mathsf{Add}(e_1, e_2)) \land P(\mathsf{Mul}(e_1, e_2))) \Rightarrow (P(e_1) \land P(e_2))$.

D. $(\forall\, e_1, e_2 : \mathsf{NumExpr}\ .\ P(\mathsf{Add}(e_1, e_2))) \land (\forall\, e_1, e_2 : \mathsf{NumExpr}\ .\ P(\mathsf{Mul}(e_1, e_2)))$. $\Longleftarrow$

ANSWER:
Only B correctly expresses the induction principle for the Add and Mul constructors.

Question 29 [1 mark]
Let NumString be the set of strings constructed from the characters 0, 1, +, and $*$, and let ++ be string concatenation. Consider the function $\mathsf{print} : \mathsf{NumExpr} \to \mathsf{NumString}$ defined by pattern matching as follows:

1. $\mathsf{print}(\mathsf{Zero}) =$ `"0"`.
2. $\mathsf{print}(\mathsf{One}) =$ `"1"`.
3. $\mathsf{print}(\mathsf{Add}(e_1, e_2)) = \mathsf{print}(e_1) +\!\!+\ \mathsf{print}(e_2) +\!\!+$ `"+"`.
4. $\mathsf{print}(\mathsf{Mul}(e_1, e_2)) = \mathsf{print}(e_1) +\!\!+\ \mathsf{print}(e_2) +\!\!+$ `"*"`.

(The function print prints a member of NumString in what is known as *reverse Polish notation*, famously used in the early Hewlett-Packard calculators.) What is the value of

$$\mathsf{print}(\mathsf{Add}(\mathsf{Mul}(\mathsf{Zero}, \mathsf{One}), \mathsf{Mul}(\mathsf{One}, \mathsf{Add}(\mathsf{One}, \mathsf{Zero}))))?$$

A. `"01+1*10*+"`.

B. `"01110*+*+"`.

C. `"01*110+*+"`. $\Longleftarrow$

D. `"110+*01*+"`

ANSWER:
Adding parentheses shows that C is correct: `"((0,1)*,(1,(1,0)+)*)+"`.

Question 30 [1 mark]
Let $f, g : \mathsf{Nat} \times \mathsf{Nat} \to \mathsf{Nat}$ be unspecified functions and $\mathsf{val} : \mathsf{NumExpr} \to \mathsf{Nat}$ be defined by pattern matching as follows:

1. $\mathsf{val}(\mathsf{Zero}) = \mathsf{0}$.
2. $\mathsf{val}(\mathsf{One}) = \mathsf{S}(\mathsf{0})$.
3. $\mathsf{val}(\mathsf{Add}(e_1, e_2)) = f(\mathsf{val}(e_1), \mathsf{val}(e_2))$.
4. $\mathsf{val}(\mathsf{Mul}(e_1, e_2)) = g(\mathsf{val}(e_1), \mathsf{val}(e_2))$.

How should f and g be defined by pattern matching so that, for all $e \in \mathsf{NumExpr}$, $\mathsf{val}(e)$ is the member of Nat that represents the value of e as a natural number.

A. $f(x, \mathsf{0}) = x$.
$f(x, \mathsf{S}(y)) = \mathsf{S}(f(x, y))$.
$g(x, \mathsf{0}) = \mathsf{0}$.
$g(x, \mathsf{S}(y)) = \mathsf{S}(g(x, y))$.

B. $f(x, \mathsf{0}) = x$.
$f(x, \mathsf{S}(y)) = g(x, y)$.
$g(x, \mathsf{0}) = \mathsf{0}$.
$g(x, \mathsf{S}(y)) = f(f(x, y), x)$.

C. $f(x, \mathsf{0}) = x$.
$f(x, \mathsf{S}(y)) = \mathsf{S}(f(x, y))$.
$g(x, \mathsf{0}) = \mathsf{0}$.
$g(x, \mathsf{S}(y)) = f(g(x, y), x)$. $\Longleftarrow$

D. $f(x, \mathsf{0}) = x$.
$f(x, \mathsf{S}(y)) = \mathsf{S}(f(x, y))$.
$g(x, \mathsf{0}) = \mathsf{0}$.
$g(x, \mathsf{S}(y)) = f(g(x, y), y)$.

ANSWER:
C recursively define f and g to be the standard addition and multiplication functions on the members of Nat. A, B, and D do not define f and g to be addition and multiplication.

THE END.