MATHEMATICS 1LT3 TEST 1

Evening Class
Duration of Test: 60 minutes
McMaster University

E. Clements

29 January 2015

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40

FIRST NAME (please print): Sol^{N}s

FAMILY NAME (please print): ____________

Student No.: ____________

THIS TEST HAS 8 PAGES AND 8 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE.

Total number of points is 40. Marks are indicated next to the problem number in square brackets. You may use the McMaster standard calculator, Casio fx991 MS+, on this test.

USE PEN TO WRITE YOUR TEST. IF YOU USE A PENCIL, YOUR TEST WILL NOT BE ACCEPTED FOR REMARKING (IF NEEDED).

You need to show work to receive full credit, except for Multiple Choice.

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1. State whether each statement is true or false. Explain your reasoning.

(a) [2] Consider the autonomous differential equation $\frac{dx}{dt} = x^2 + 6x + 9$. In the phase-line diagram for $x(t)$, all arrows point to the right.

$\frac{dx}{dt} = (x+3)^2 \geq 0 \quad \forall x \in \mathbb{R}$ so $x(t)$ is always increasing (except at eq^n sol^n $x^* = -3$)

$\therefore$ TRUE

(b) [2] Consider the selection model, $\frac{dp}{dt} = (\mu - \lambda)p(1-p)$. If initially the proportions of type a and of type b are equal, i.e., $p_0 = 0.5$, then $p(t) = 0.5$ for all $t \geq 0$.

FALSE.
Suppose $\mu \neq \lambda$. Then the only equilibrium sol^{n}s are $p^* = 0$ and $p^* = 1$. If $p_0 = 0.5$, then $p(t)$ will either increase or decrease when $t > 0$.

2. Multiple Choice. Clearly circle the one correct answer.

(a) [3] Consider the population model, $\frac{dP}{dt} = 0.08P\left(1 - \frac{P}{4000}\right)\left(1 - \frac{180}{P}\right)$. Which of the following statements is/are true?

✗ (I) If $P(0) > 0$, then the population will increase. not if $P \leq 180$ or $P \geq 4000$

✗ (II) The carrying capacity of the population is 180. no, this is 4000

✓ (III) 4000 is a stable equilibrium. yes, carrying capacity is stable.

(A) none	(B) I only	(C) II only	(D) III only
(E) I and II	(F) I and III	(G) II and III	(H) all three

(b) [3] Consider the model for the spread of a disease, $\frac{dI}{dt} = \overset{\alpha}{0.4}I(1-I) - \overset{\mu}{0.1}I$, where I represents the proportion of infected individuals in the population. Which of the following statements is/are true?

✓ (I) There are two biologically plausible equilibria. true, since $\alpha > \mu$

✗ (II) If initially 2% of the population is infected, then I will decrease. $I_0 = 0.02$

✓ (III) If initially 80% of the population is infected, then I will decrease. $I_0 = 0.8$

(A) none	(B) I only	(C) II only	(D) III only
(E) I and II	(F) I and III	(G) II and III	(H) all three

$\frac{dI}{dt} = 0$ when $I(0.4(1-I) - 0.1) = 0 \rightarrow \underset{\text{unstable}}{I = 0}$ or $\underset{\text{stable}}{I = 0.75}$.

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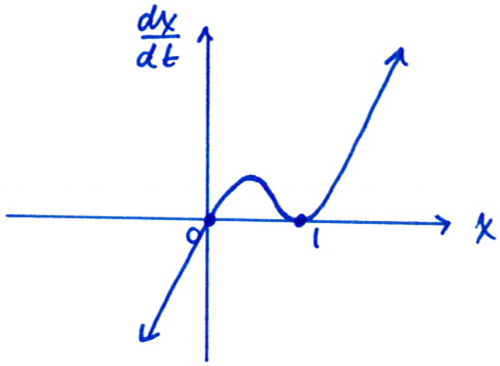
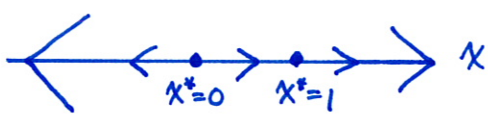
3. Consider the autonomous differential equation $\frac{dx}{dt} = \underbrace{x(x-1)^2}_{\text{cubic}}$.

(a) [2] Sketch the graph of the rate of change, $\frac{dx}{dt}$, as a function of the state variable, x.

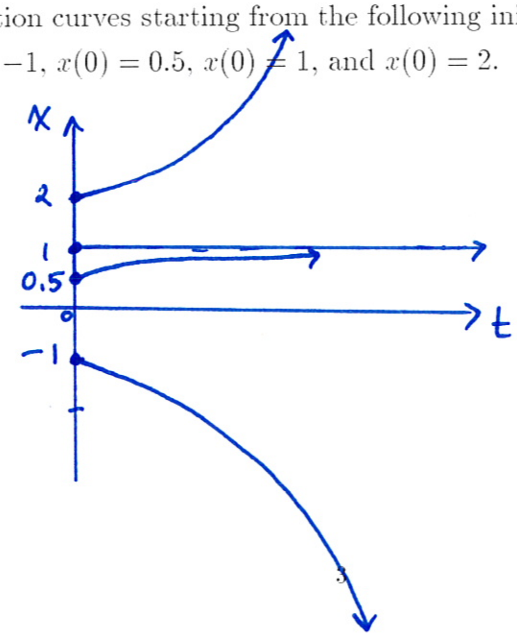


(b) [3] Draw a phase-line diagram for solutions $x(t)$.



(c) [2] Sketch solution curves starting from the following initial values:

$x(0) = -1$, $x(0) = 0.5$, $x(0) = 1$, and $x(0) = 2$.



4. Consider the differential equation $\frac{dy}{dt} = ye^{-\beta y} - \alpha y$.

(a) [2] Find the equilibrium solutions.

$\frac{dy}{dt} = 0$ when $y(e^{-\beta y} - \alpha) = 0 \Rightarrow \boxed{y=0}$ or $e^{-\beta y} - \alpha = 0$

$e^{-\beta y} = \alpha \mid \ln$

$(\alpha > 0)$

$\boxed{y = \frac{\ln\alpha}{-\beta}}$

(b) [4] Use the Stability Theorem to determine the stability of each equilibria in part (a).

$f(y) = y(e^{-\beta y} - \alpha)$ *note: $\boxed{\alpha > 0}$

$f'(y) = 1\cdot(e^{-\beta y} - \alpha) + y(e^{-\beta y}(-\beta)) =$

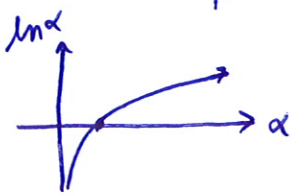
$= e^{-\beta y} - \alpha - \beta y e^{-\beta y}$

$= e^{-\beta y}(1 - \beta y) - \alpha$

$f'(0) = 1 - \alpha$ ... $y=0$ is stable when $1-\alpha < 0$ ie when $\alpha > 1$

$y = 0$ is unstable when $1-\alpha > 0$ ie when $\alpha < 1$

$f'(-\frac{\ln\alpha}{\beta}) = e^{-\beta(\frac{\ln\alpha}{-\beta})}(1 - \beta(\frac{\ln\alpha}{-\beta})) - \alpha = \alpha(1 + \ln\alpha) - \alpha = \alpha\ln\alpha$



$y = \frac{\ln\alpha}{-\beta}$ is stable when $\underbrace{\alpha}_{\oplus}\underbrace{\ln\alpha}_{\oplus \text{ or } \ominus} < 0$ ie when $0 < \alpha < 1$

$y = \frac{\ln\alpha}{-\beta}$ is unstable when $\alpha\ln\alpha > 0$ ie when $\alpha > 1$

5. Suppose that a population, $P(t)$, grows at a rate proportional to the square of its size.

(a) [1] Write a differential equation to model this population.

$\frac{dP}{dt} \propto P^2 \Rightarrow \frac{dP}{dt} = kP^2$ where k is a parameter

(b) [2] Solve the differential equation in part (a) using the Separation of Variables technique.

$$\int P^{-2}\,dP = \int k\,dt$$

$$-P^{-1} = kt + C$$

$$\frac{1}{P} = -kt - C$$

$$P = \frac{-1}{kt+C}$$

(c) [2] Does the formula you found in part (b) give all possible solutions? Explain.

No. $P=0$ is also a soln to the DE in part (a) but since $\frac{-1}{kt+C} \neq 0$, the formula in part (b) misses this soln.

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6. Consider the differential equation $\dfrac{dx}{dt} = \dfrac{x+1}{1+t^2}$ and the initial condition $x(0) = 1$.

(a) [2] Use Euler's Method with a step size of 0.5 to approximate $x(1)$.

$t_0 = 0 \qquad x_0 = 1 \qquad h = 0.5$

$t_1 = 0.5$

$t_2 = 1$

$$x_1 = 1 + \frac{1+1}{1+0^2}(0.5) = 2$$

$$x_2 = 2 + \frac{2+1}{1+(0.5)^2}(0.5) = 3.2$$

$\therefore x(1) \approx 3.2$

(b) [4] Find an explicit solution to the initial value problem by using the Separation of Variables technique. Use this solution to find the actual value of $x(1)$.

$$\int \frac{1}{x+1}\,dx = \int \frac{1}{1+t^2}\,dt$$

$$\ln|x+1| = \arctan t + C \quad | e$$

$$|x+1| = e^C \cdot e^{\arctan t}$$

$$x = \pm e^C \cdot e^{\arctan t} - 1$$

$$= Ae^{\arctan t} - 1 \qquad \text{where } A = \pm e^C.$$

$$x(0) = 1 \Rightarrow 1 = A\underbrace{e^{\overbrace{\arctan 0}^{=0}}}_{=1} - 1 \Rightarrow A = 2$$

$$\therefore x(t) = 2e^{\arctan t} - 1$$

$$x(1) = 2e^{\arctan 1} - 1$$

$$= 2e^{\frac{\pi}{4}} - 1$$

$$\approx 3.39$$

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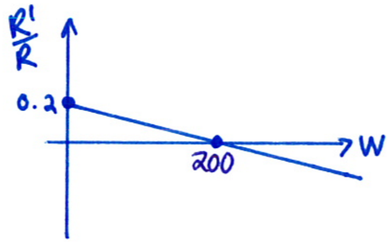
7. The following system of equations represents the growth rate of a population of rabbits, R, and a population of wolves, W, interacting in the same habitat.

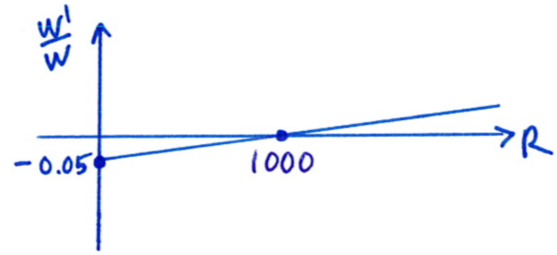
$$\frac{dR}{dt} = 0.2R - 0.001RW \qquad \frac{dW}{dt} = -0.05W + 0.00005RW$$

(a) [2] Sketch the per capita growth rates for both R and W.

$\frac{R'}{R} = 0.2 - 0.001W$



$\frac{W'}{W} = -0.05 + 0.00005R$



(b) [2] Find the equilibrium solutions for this system.

$\begin{cases} \frac{dR}{dt} = 0 \\ \frac{dW}{dt} = 0 \end{cases}$ when $\begin{cases} R[0.2 - 0.001W] = 0 \\ W[-0.05 + 0.00005R] = 0 \end{cases} \Rightarrow \begin{cases} R = 0 \text{ or } W = 200 \\ W = 0 \text{ or } R = 1000 \end{cases}$

$\therefore \begin{cases} R = 0 \\ W = 0 \end{cases}$ and $\begin{cases} R = 1000 \\ W = 200 \end{cases}$ are the eq^{n} sol^{n}s.

8. [2] Suppose that a certain habitat contains two predators that must eat each other to survive. Write a system of differential equations to describe this situation. You may make up parameter values as needed.

Let P_1 and P_2 represent predator 1 and predator 2, respectively.

$$\frac{dP_1}{dt} = -0.1P_1 + 0.04P_1P_2$$

$$\frac{dP_2}{dt} = -0.5P_2 + 0.008P_1P_2$$