MATH 1B03/1ZC3 Winter 2019

Lecture 10: Eigenvalues and eigenvectors

Instructor: Dr Rushworth February 5th

Matrices as transformations

(from Chapter 5.1 of Anton-Rorres)

We have spent the first part of this course thinking about matrices abstract algebraic objects i.e. arrays of numbers.

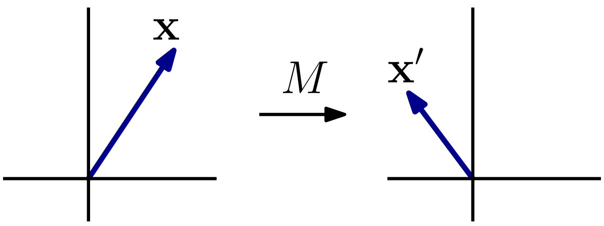
However, matrices may be realized as geometric objects. Many of the concepts we have defined can be better understood in this way.

Let M be a 2×2 matrix, and $\mathbf{x}$ a 2×1 matrix. Then

$$\underset{2\times 2}{M} \;\; \underset{2\times 1}{\mathbf{x}} = \underset{2\times 1}{\mathbf{x}'}$$

where $\mathbf{x}'$ is also a 2×1 matrix. That is, we can think of M as taking as input a 2×1 matrix, and outputting another 2×1 matrix.

We can represent this diagrammatically by thinking of $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ as a vector in the x-y plane:



The transformation described by M is the effect of multiplying every vector in the plane by M.

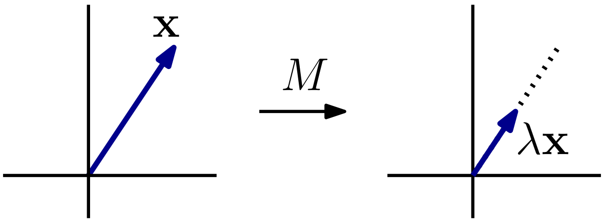
In general, the transformation described by M may rotate, translate, and stretch the vectors. In fact, any transformation of the x-y plane can be described by a

2×2 matrix. This realization of matrices as transformations extends to matrices of any size.

Given a 2×2 matrix M, suppose there exists a vector $\mathbf{x}$ and a scalar λ such that

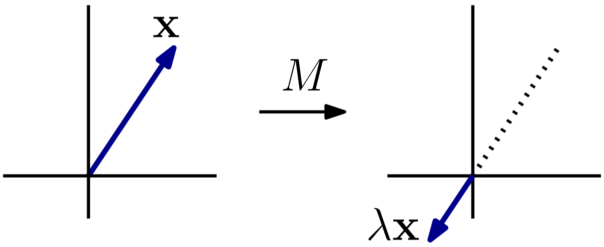
$$M\mathbf{x} = \lambda\mathbf{x}$$

What does this mean diagrammatically?



If $\mathbf{x}$ satisfies the equation above, then multiplying by M does not change its direction, only its magnitude.

Notice that λ may be negative: in that case we would have



Vectors like this are very special, and can be used to understand the matrix M in great detail. They are called eigenvectors, and the scalars λ are called eigenvalues.

Eigenvalues and eigenvectors

Definition 10.1: Eigenvalues and eigenvectors

Let A be an $n \times n$ matrix. An eigenvector of A is a non-zero $n \times 1$ matrix $\mathbf{x}$ such that

$$A\mathbf{x} = \lambda\mathbf{x}$$

The scalar λ is an eigenvalue of A. The vector $\mathbf{x}$ is an eigenvector associated to λ.

The word *eigen* is German for *proper*: the eigenvectors are the proper vectors to be used with A.

Notice that the vector $0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is prohibited from being an eigenvector of any matrix, by definition. This is because

$$A0 = 0$$

for any matrix A, so that it would always be eigenvector, if we allowed it.

On the other hand, $\lambda = 0$ is a permitted eigenvalue.

Example 10.2

Let

$$A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$$

Then

$$\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 15 \\ 20 \end{bmatrix} = 5 \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

so that $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ is an eigenvector associated to the eigenvalue 5.

Computing eigenvalues

Finding the eigenvalues and eigenvectors of a matrix will tell us a large amount of information about the matrix. How do we do this?

Fact 10.3: Characteristic equation

Let A be an $n \times n$ matrix. The scalar λ is an eigenvalue of A if and only if it is a solution to the equation

$$det(\lambda I - A) = 0$$

(for I the identity matrix).
The equation

$$det(\lambda I - A) = 0$$

is known as the characteristic equation of A.

You may see the characteristic equation defined as $det(A - \lambda I) = 0$ in other sources.

Example 10.4: Using the characteristic equation

Question: Find the eigenvalues of

$$A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$$

Answer: Consider

$$\lambda I - A = \begin{bmatrix} \lambda - 1 & 3 \\ -4 & \lambda - 2 \end{bmatrix}$$

Then

$$\begin{aligned} det(\lambda I - A) &= \begin{vmatrix} \lambda - 1 & 3 \\ -4 & \lambda - 2 \end{vmatrix} \\ &= (\lambda - 1)(\lambda - 2) - 12 \\ &= \lambda^2 - 3\lambda - 10 \\ &= (\lambda - 5)(\lambda + 2) \end{aligned}$$

The equation

$$det(\lambda I - A) = 0$$

becomes

$$(\lambda - 5)(\lambda + 2) = 0$$

which has solutions $\lambda = 5$, $\lambda = -2$. Therefore the eigenvalues of A are $\lambda_1 = 5$, $\lambda_2 = -2$.

When using the characteristic equation, we will always encounter a polynomial in λ. If A is $n \times n$, when expanding we will obtain

$$det(\lambda I - A) = \lambda^n + c_1\lambda^{n-1} + \cdots + c_n$$

for c_i a constant. The polynomial

$$\lambda^n + c_1\lambda^{n-1} + \cdots + c_n$$

is known as the characteristic polynomial of A. The eigenvalues of A are exactly the roots of its characteristic polynomial.

Recall that a degree n polynomial has exactly n roots, when counted with multiplicity. That is, the polynomial

$$(x-2)^2$$

has the root $x = 2$, with multiplicity 2.

Therefore, an $n \times n$ matrix has exactly n eigenvalues, when counted with multiplicity.

Further, consider the polynomial $(y+4)^2$. It has a root $y = 2i$ of multiplicity 2. It follows that it is possible for a matrix to have complex eigenvalues, even if it only has real entries.

Example 10.5

Question: Compute the eigenvalues of the matrix

$$B = \begin{bmatrix} 1 & 4 & -2 \\ 0 & 8 & 0 \\ -3 & 11 & 7 \end{bmatrix}$$

Answer: Consider

$$\lambda I - B = \begin{bmatrix} \lambda - 1 & -4 & 2 \\ 0 & \lambda - 8 & 0 \\ 3 & -11 & \lambda - 7 \end{bmatrix}$$

then

$$\begin{aligned} det(\lambda I - B) &= \begin{vmatrix} \lambda - 1 & -4 & 2 \\ 0 & \lambda - 8 & 0 \\ 3 & -11 & \lambda - 7 \end{vmatrix} \\ &= (\lambda - 8) \begin{vmatrix} \lambda - 1 & 2 \\ 3 & \lambda - 7 \end{vmatrix} \\ &= (\lambda - 8)((\lambda - 1)(\lambda - 7) - 6) \end{aligned}$$

and the characteristic equation is

$$(\lambda - 8)((\lambda - 1)(\lambda - 7) - 6) = 0$$

We see that $\lambda = 8$ is a solution. If $\lambda \neq 8$ then

$$\begin{aligned} (\lambda - 1)(\lambda - 7) - 6 &= 0 \\ \lambda^2 - 8\lambda - 6 &= 0 \\ \lambda^2 - 8\lambda + 1 &= 0 \end{aligned}$$

Applying the quadratic equation, we obtain the solutions $\lambda = 4 + \sqrt{15}$ and $\lambda = 4 - \sqrt{15}$.
Therefore the eigenvalues of B are $\lambda_1 = 8$, $\lambda_2 = 4 + \sqrt{15}$, and $\lambda_3 = 4 - \sqrt{15}$.

Finding the roots of polynomials gets harder as the degree increases. As we saw with the inverse and determinant, this means that finding the eigenvalues of a matrix gets more difficult the bigger the matrix is.

We saw that the determinant of a triangular matrix was very easy to determine, however, no matter how big it is. The same holds for the eigenvalues.

Fact 10.6: Eigenvalues of a triangular matrix

Let A be a triangular matrix. Then the eigenvalues of A are listed on its main diagonal.

Example 10.7

Question: Find the eigenvalues of the matrix

$$M = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 \\ 3 & 3 & 0 & 0 & 0 \\ -7 & 2 & 3 & 0 & 0 \\ 11 & -5 & 6 & 2 & 0 \\ 1 & -4 & -3 & 2 & 0 \end{bmatrix}$$

Answer: As M is lower triangular, its eigenvalues are

$$\begin{aligned} \lambda_1 &= 5 \\ \lambda_2 &= 3 \\ \lambda_3 &= 3 \\ \lambda_4 &= 2 \\ \lambda_5 &= 0 \end{aligned}$$

so that 3 is a repeated eigenvalue.

In summary, λ is an eigenvalue of the matrix A if

- there exists a vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$
- λ is a solution to the equation $det(\lambda I - A) = 0$

These conditions are completely equivalent. Sometimes it is convenient to use the first condition, and other times the second. As such, we should be familiar with both of them.

Computing eigenvectors

Now that we know how to compute eigenvalues we can move on to computing their associated eigenvectors.

Let A be an $n \times n$ matrix and λ an eigenvalue of A. Recall that an eigenvector associated to λ is an $n \times 1$ column vector $\mathbf{x}$ satisfying

$$A\mathbf{x} = \lambda\mathbf{x}$$

We can rearrange this equation to obtain

$$\begin{aligned} A\mathbf{x} &= \lambda\mathbf{x} \\ 0 &= \lambda\mathbf{x} - A\mathbf{x} \\ 0 &= (\lambda I - A)\mathbf{x} \end{aligned}$$

Therefore, to find an eigenvector $\mathbf{x}$ associated to λ we can solve the equation

$$A\mathbf{x} = \lambda\mathbf{x}$$

or the equation

$$(\lambda I - A)\mathbf{x} = 0.$$

In general an eigenvalue has more than one associated eigenvector. In fact, there is often an infinite number of eigenvectors associated to a given eigenvalue. To make sense of this we introduce the following terminology.

Definition 10.8: Linear combination

Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ be a collection of column vectors. A linear combination of $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ is a vector

$$\mathbf{x} = a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_n\mathbf{x}_n$$

for $a_1, a_2, \ldots, a_n$ scalars.

Example 10.9

As

$$\begin{bmatrix} 3 \\ 7 \\ -9 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 7\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - 9\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

we see that the vector $\begin{bmatrix} 3 \\ 7 \\ -9 \end{bmatrix}$ can be written as a linear combination of the vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Definition 10.10: Eigenspace

Let λ be an eigenvalue of the matrix A. The eigenspace of λ is the collection of all eigenvectors associated to λ i.e. all the vectors $\mathbf{x}$ such that

$$A\mathbf{x} = \lambda\mathbf{x}.$$

Definition 10.11: Basis of an eigenspace

A basis of the eigenspace of λ is a collection of eigenvectors associated to λ such that every eigenvector in the eigenspace of λ can be written as a linear combination of the vectors in the collection.

We will cover the notion of basis in more detail later on in the course.

Finding the eigenvectors of a matrix boils down to finding the eigenvalues, then computing a basis for the eigenspace of each eigenvalue.

Example 10.12

Question: Find the eigenvectors of

$$A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$$

Answer: Recall from a previous example that the eigenvalues of A are $\lambda_1 = 5$ and $\lambda_2 = -2$.
• $(\lambda_1 = 5)$: We will solve the equation

$$(5I - A)\mathbf{x} = 0$$

We have

$$5I - A = \begin{bmatrix} 4 & -3 \\ -4 & 3 \end{bmatrix}$$

Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. The equation becomes

$$\begin{bmatrix} 4 & -3 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This yields the simultaneous equations

$$\begin{aligned} 4x_1 - 3x_2 &= 0 \\ -4x_1 + 3x_2 &= 0 \end{aligned}$$

which have solution

$$x_1 = \frac{3}{4}x_2$$

Let $x_2 = t$, for t a variable, then $x_1 = \frac{3}{4}t$ and

$$\mathbf{x} = \begin{bmatrix} \frac{3}{4}t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{3}{4} \\ 1 \end{bmatrix}$$

Therefore any eigenvector associated to $\lambda_1 = 5$ can be written as a multiple of $\begin{bmatrix} \frac{3}{4} \\ 1 \end{bmatrix}$, so that it forms a basis of the eigenspace.

• $(\lambda_2 = -2)$: We repeat the process, but solving the equation

$$(-2I - A)\mathbf{x} = 0$$

We have

$$-2I - A = \begin{bmatrix} -3 & -3 \\ -4 & -4 \end{bmatrix}$$

and the equation becomes

$$\begin{bmatrix} -3 & -3 \\ -4 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

yielding the simultaneous equations

$$\begin{aligned} -3x_1 - 3x_2 &= 0 \\ -4x_1 - 4x_2 &= 0 \end{aligned}$$

which has solution

$$x_1 = -x_2$$

Setting $x_2 = t$, we see that $\mathbf{x} = \begin{bmatrix} -t \\ t \end{bmatrix}$ and that $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is a basis for the eigenspace.

A piece of evidence showing that the eigenvalues and eigenvectors of a matrix contain important information about the matrix is provided by the following fact.

Fact 10.13

A square matrix A is invertible if and only if 0 is not an eigenvalue of A.

Example 10.14

Question: Compute the eigenvalues and eigenvectors of the matrix

$$B = \begin{bmatrix} 4 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Answer: As B is lower triangular the eigenvalues are $\lambda_1 = 4$, $\lambda_2 = -2$ (this eigenvalue has multiplicity 2).

• $(\lambda_1 = 4)$: We have

$$4I - B = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

The equation

$$\begin{bmatrix} 0 & -1 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

yields the simultaneous equations

$$\begin{aligned} -x_2 &= 0 \\ 6x_2 &= 0 \\ 6x_3 &= 0 \end{aligned}$$

which has solution $x_2 = x_3 = 0$, with x_1 free. Therefore $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is a basis of the eigenspace of $\lambda_1 = 4$

• $(\lambda_2 = -2)$: We have

$$-2I - B = \begin{bmatrix} -6 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The equation

$$\begin{bmatrix} -6 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

yields the equation

$$-6x_1 - x_2 = 0$$

which has solution $-6x_1 = x_2$ with x_3 free. Then let $x_2 = t$ and $x_3 = s$, to obtain

$$\begin{bmatrix} s \\ -6s \\ t \end{bmatrix} = s \begin{bmatrix} 1 \\ -6 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and we see that $\begin{bmatrix} 1 \\ -6 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is a basis for the eigenspace of $\lambda_2 = -2$.

Question 10.15: Suggested problem 33

Let A be an invertible matrix. Prove that if λ is an eigenvalue of A then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.

Example 10.16

Question: Show that A and A^T have the same eigenvalues.

Answer: Consider the characteristic polynomial of A

$$\begin{aligned} det(\lambda I - A) &= det((\lambda I - A)^T), \text{ as } det(A) = det(A^T) \\ &= det(\lambda I^T - A^T), \text{ as } (A+B)^T = A^T + B^T \\ &= det(\lambda I - A^T), \text{ as } I^T = I \end{aligned}$$

therefore the characteristic polynomial of A is equal to the characteristic polynomial of A^T, and the eigenvalues of A must be equal to the eigenvalues of A^T.

Question: Show that A and A^T do not always have the same eigenvectors.
Answer: We need to find an example of a matrix which does not have the same eigenvectors as its transpose. Recall that the matrix

$$A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$$

has eigenvalue $\lambda_1 = 5$, with eigenvector $\mathbb{x}_1 = \begin{bmatrix} \frac{3}{4} \\ 1 \end{bmatrix}$, and $\lambda_2 = -2$, with eigenvector $\mathbb{x}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.
But

$$A^T = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$$

and the eigenvectors are

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -\frac{4}{3} \end{bmatrix}$$

Suggested Problems

Practice the material covered in this lecture by attempting the following questions from Chapter 5.1 of Anton-Rorres, starting on page 300

- Questions 7, 9, 11, 28, 33, 34, 35

- True/False questions (a), (b), (c)