

12C3

- Matlab Due! Today!

- TAs available in B513 (2nd floor) (see web!)

Last Day Eigenstuff!

Remember If A is an $n \times n$ (i.e. square) matrix

& $\boxed{A \vec{x} = \lambda \vec{x}}$ for a non-zero vector \vec{x}

then λ is an eigenvalue of A & \vec{x} is an eigenvector
for A , for that λ .

Notice $\left\{ \begin{array}{l} A \vec{x} = \lambda \vec{x} \Rightarrow A^2 \vec{x} = \lambda^2 \vec{x} \text{ \& in general } \end{array} \right. \underline{\underline{A^n \vec{x} = \lambda^n \vec{x}}}$

\Rightarrow any λ eigenvector of A is a λ^n eigenvector of A^n .

Notice

If A^{-1} exists $\Rightarrow A^{-1}A\vec{x} = A^{-1}\lambda\vec{x}$

$$I\vec{x} = \vec{x} = \lambda \cdot (A^{-1})(\vec{x})$$

\hookrightarrow so $\lambda \neq 0$

& $\frac{1}{\lambda}\vec{x} = A^{-1}\vec{x}$ or $A^{-1}\vec{x} = \left(\frac{1}{\lambda}\right)\vec{x}$

$\Rightarrow A^{-1}$ has same eigenvectors but now corresponds to $\frac{1}{\text{old } \lambda}$ i.e. new $\lambda = \frac{1}{\text{original } \lambda}$

Last Day we found eigenvalues & eigenvectors of A

Eigenvalue, = roots of $C_A(z) = \underbrace{|A - zI|}_{\text{"characteristic polynomial"}} = \text{polynomial in } \underline{z}$.

(equivalent, alternate form? $|zI - A|$ \leftarrow same roots either way!)

& To get eigenvectors solve linear system

aug. matrix $\rightarrow \left\{ \underbrace{[A - \lambda I | \vec{0}]} \right\} \left(\text{re } (A - \lambda I) \vec{x} = \vec{0} \right)$

eg. Last day

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$C_A(z) = |A - zI| = \begin{vmatrix} 1-z & 0 & 0 \\ 1 & 2-z & 0 \\ 0 & 0 & 2-z \end{vmatrix}$$

$$= (1-z)(2-z)(2-z)$$

$$\lambda = 1, \underline{2, 2}$$

alg. mult. $\leq \underline{1}$

alg. mult. = 2

Alg. mult. of λ
is # repeated factors!

→ solve for eigenvector

say $\underline{\lambda = 2}$ $[A - 2I | \vec{0}]$

& solve! $\Rightarrow \vec{x} = \begin{bmatrix} 0 \\ t \\ s \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

two params \Rightarrow geo. mult = # params = 2

& 2 basis eigenvectors in $\lambda = 2$
eigenspace!

Always $1 \leq \text{geo. mult} \leq \text{alg. mult.} \leq n$

Proposition 1) notice $C_A(z) = |A - zI|$

\swarrow order n

$= \dots \pm z^n$

can be factored
into monomials

$\swarrow \left\{ \begin{aligned} &= \\ &= (\lambda_1 - z)(\lambda_2 - z)(\lambda_2 - z) \dots (\lambda_n - z) \end{aligned} \right.$

\Rightarrow roots = λ_i = eigenvalues!

\Rightarrow

$C_A(z) = (\lambda_1 \lambda_2 \lambda_3 \dots \lambda_n) + ()z + ()z^2 + \dots \pm z^n$

$|A - zI|$

\swarrow

product of the eigenvalues
each repeated to alg. multiplicity

So plug in $z=0$

$$\det A = C_A(0) = \lambda_1 \lambda_2 \dots \lambda_n$$

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$$\det A = \text{product of eigenvalue (to their alg. multiplicity)}$$
$$= \lambda_1 \lambda_2 \dots \lambda_n$$

(\hookrightarrow sum of all alg. multi. is n)

And

A^{-1} exists iff $\det A \neq 0$ iff $\lambda \neq 0$ for any
eigenvalue of A

Finally

Cayley - Hamilton Theorem

If $C_A(z) = |A - zI| = \text{char. polynomial of } A$

$$\text{then } \underline{\underline{C_A(A) = 0}}$$

eg. Say $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

$$\det(A - zI) = \begin{vmatrix} 1-z & 2 \\ 2 & 1-z \end{vmatrix}$$

$$= (1-z)^2 - 4 = z^2 - 2z - 3$$

$$= (z-3)(z+1) \Rightarrow \underline{\underline{\lambda = 3, -1}}$$

by Cayley-Hamilton.

$$C_A(z) = z^2 - 2z - 3 \quad \swarrow \text{ plug in } A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

we'll get $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}^2 - 2 \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} - 3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} - \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}}$$

Diagonalizability

Consider diag. matrices!

A diag. matrix has: - entries on diag are λ 's

$$\begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \\ & & & \lambda_n \end{bmatrix}$$

- base eigenvectors are i, j, k , in general $\vec{e}_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$

1 only in spot i

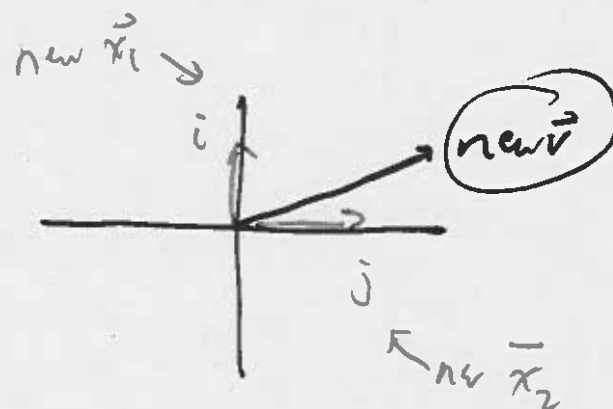
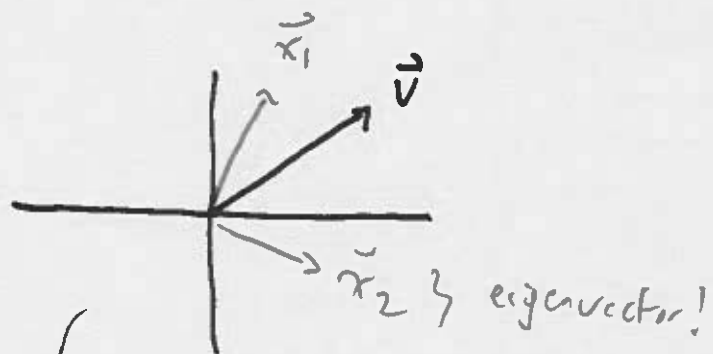
$$\text{eg } \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \lambda_2 \end{bmatrix} = \lambda_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ etc.}$$

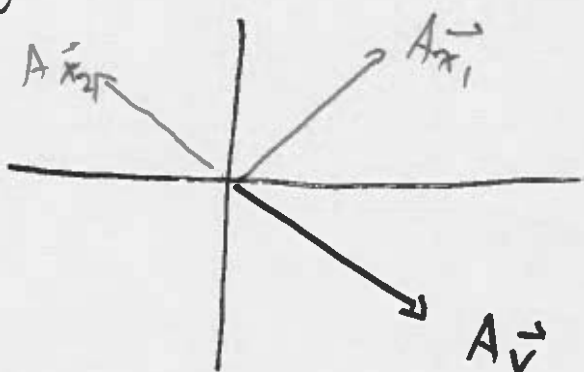
- geo. mult. = alg. mult. for all λ

pretty!

Goal: Make matrix "act" diagonal



A



mult. by diag. $\rightarrow \underline{\underline{D}}$

