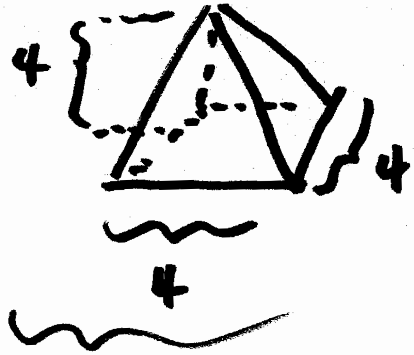
12A3 Volume (Basic Version!)

eg. Say I have a pyramid

4 by 4 square base

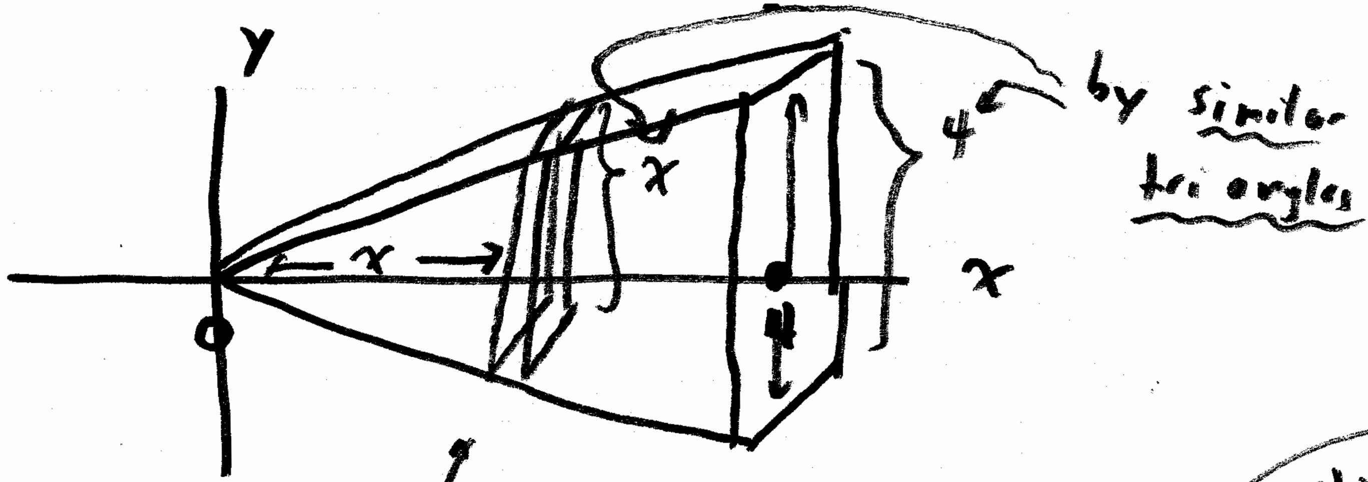
4 units tall.



$$\text{Vol} = \frac{1}{3} \cdot \text{base} \cdot \text{height}$$
$$= \frac{1}{3} \cdot 4^2 \cdot 4 = 4^3/3$$

(From grade school!)

Instead of memorizing, let's derive!



$$V_{slice} = \text{width} \cdot \text{Area}$$
$$= \Delta x \quad x^2$$

slice area!

$$\sum_{i=1}^{n} A(x_i)\,\Delta x$$

$$\text{Net vol} \simeq \sum \text{slice volume} = \sum_{i=1}^{n} x_i^2\,\Delta x$$

$$\text{Actual vol} = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i)\,\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} x_i^2\,\Delta x$$

$$\underline{\text{Vol}} = \int_a^b A(x)\,dx$$

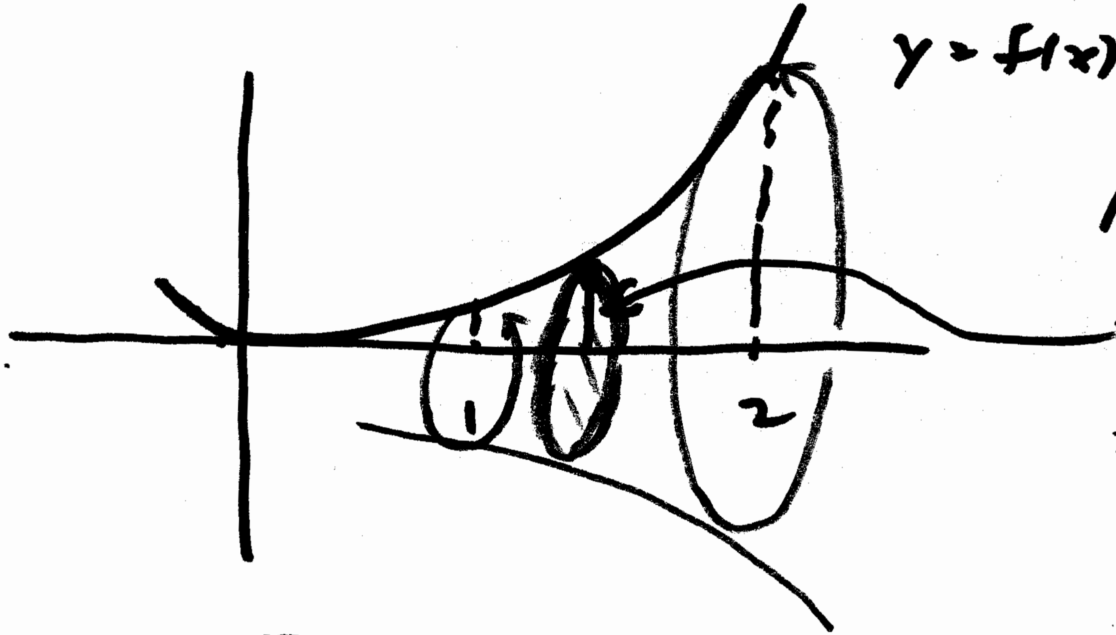
$$= \int_0^4 x^2\,dx = \frac{1}{3}x^3\Big|_0^4$$

$$= \frac{1}{3}(4^3 - 0^3) = \frac{4^3}{3} \checkmark$$

Special Case Volumes of Revolution

eg. Say I have $f(x) = x^2$, $x \in [1, 2]$

Find the vol. of revolution if we rotate about x-axis



$y = f(x) = x^2$

$A_{\text{slice}} = \pi r^2$ $r = |f(x)|$

$= \pi (f(x))^2$

$$\text{Vol} \approx \sum \text{slices} = \sum A(x_i)\,\Delta x$$

$$= \sum_{i=1}^{n} \pi (f(x_i))^2 \,\Delta x$$

$$\text{Vol} = \int_a^b \pi (f(x))^2 \, dx = \int_1^2 \pi (x^2)^2 \, dx$$

in this case!

$$= \int_1^2 \pi x^4 \, dx = \frac{1}{5} \pi x^5 \Big|_1^2$$

$= \frac{31}{5}\pi$

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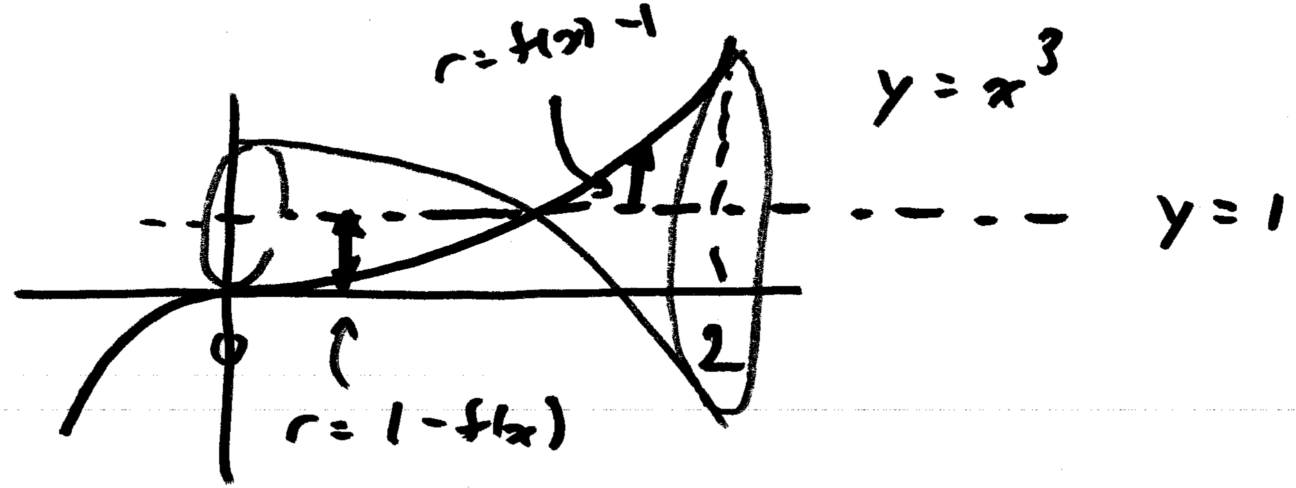
eg. Find the vol. of <u>revolution</u> of

$y = x^3$, $x \in [0, 2]$ about $y = 1$.

<u>Solution</u>

$r = |f(x) - 1|$



$$Vol = \int_a^b \underbrace{\pi r^2}_{\text{cross section area}} dx = \int_0^2 \pi (x^3 - 1)^2 \, dx$$

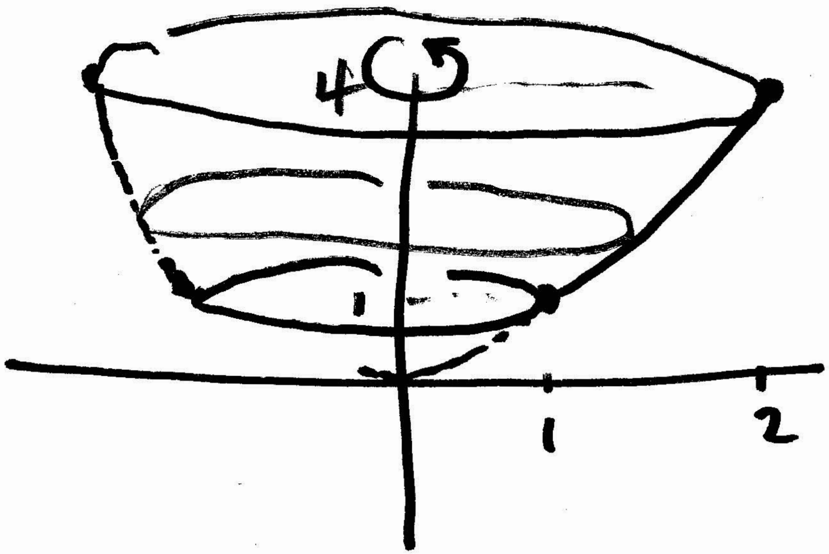
In general rotate $f(x)$ about $y = k$

$$Vol = \int_a^b \pi r^2 \, dx = \int_a^b \pi (f(x) - k)^2 \, dx$$

$$r = |f(x) - k|$$

$$\int_0^2 \pi (x^6 - 2x^3 + 1) \, dx = \#.$$

eg. Find the vol. of revolution of $y = x^2$, $x \in [1,2]$ about y-axis



$y = x^2$

$x \in [1,2]$

$x = \sqrt{y}$

$$\text{Vol} \approx \sum \text{slice} = \sum \text{Area} \cdot \text{thick}$$

$$= \sum A(y)\ \Delta y = \sum \pi r^2 \Delta y$$

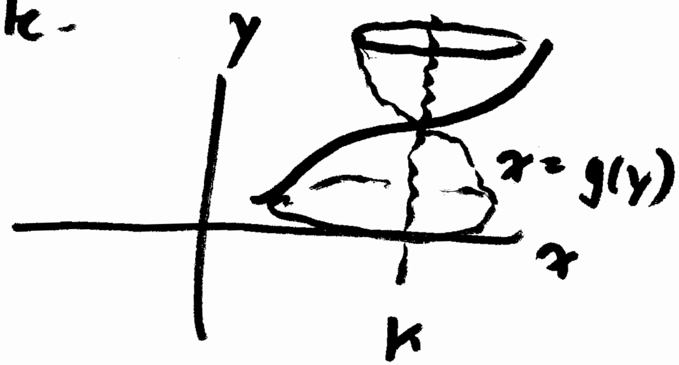
$$\text{Vol} = \int_c^d \pi r^2\, dy = \int_c^d \pi (g(y))^2\, dy$$

Here

$$\text{Vol} = \int_1^4 \pi \underbrace{\cancel{(\sqrt{y})^2}}_{y}\, dy = \int_1^4 \pi y\, dy$$

$$= \frac{\pi}{2} y^2 \Big|_1^4 = \frac{15}{2}\pi$$

$\underline{\underline{\text{so}}}$ What if I want to rotate $x = g(y)$ about $x = k$?



$$\text{Vol} = \int_a^b \pi r^2 \, dy$$

$$= \int_a^b \pi \left(g(y) - k \right)^2 dy$$

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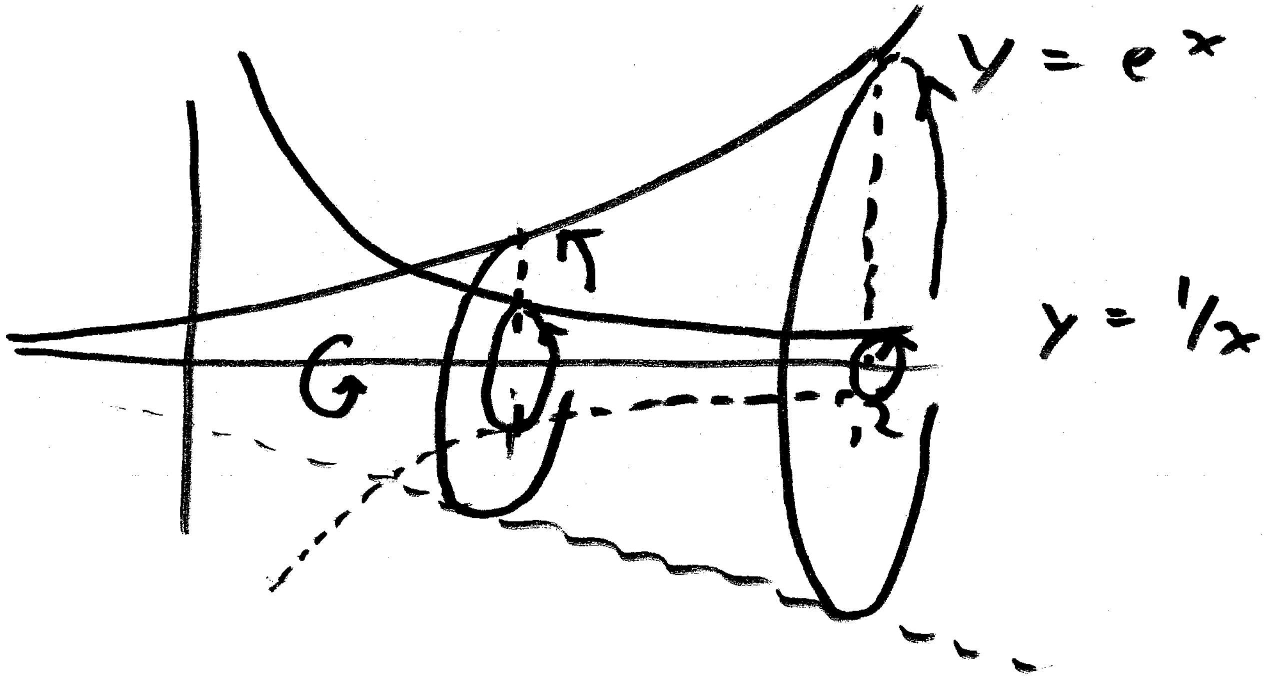
eg. Given $f(x) = e^x$, $h(x) = \frac{1}{x}$, $x \in [1,2]$

Find vol. of revolution of region between $y = e^x$ & $y = 1/x$, rotated abt. x-axis



Vol. of Revolution = Outer Vol − Inner Vol.

$$= \int_a^b \pi R^2\,dx - \int_a^b \pi r^2\,dx$$

$$= \int_a^b \pi (R^2 - r^2)\,dx$$

Here we get $V = \int_1^2 \pi \left(e^{2x} - \frac{1}{x^2} dx\right.$

$$= \frac{\pi}{2} e^{2x} + \pi/x \Big|_1^2$$

$$= \pi/2 \left(e^4 - e^2\right) + \pi/2 - \pi$$

$$= \frac{\pi}{2} \left(e^4 - e^2 - 1\right)$$