12C3

Last Day Triple product: $\vec{u}\cdot(\vec{v}\times\vec{w})$

- always a scalar quantity

determinant!

$$\vec{u}\cdot(\vec{v}\times\vec{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \det\left[\vec{u}\,|\,\vec{v}\,|\,\vec{w}\right]$$

If any of $\vec{u},\vec{v},\vec{w} = \vec{0} \Rightarrow \vec{u}\cdot(\vec{v}\times\vec{w}) = 0$

If any of $\vec{u},\vec{v},\vec{w}$ parallel $\Rightarrow \vec{u}\cdot(\vec{v}\times\vec{w}) = 0$

Swap any 2 vector $\Leftrightarrow$ swap 2 rows $\Leftrightarrow$ sign change

$$\vec{u}\cdot(\vec{v}\times\vec{w}) = -\vec{v}\cdot(\vec{u}\times\vec{w}) = \vec{w}\cdot(\vec{u}\times\vec{v})$$
$$= -\vec{w}\cdot(\vec{v}\times\vec{u})$$

& don't forget $|\vec{u} \cdot (\vec{v} \times \vec{w})| =$ ~~area~~ volume of parallelepiped generated by $\vec{u}, \vec{v}, \vec{w}$.

eg. Find the volume of the parallelepiped generated by

$\vec{u} = (1, -1, 2) \quad \vec{v} = (0, 1, 3) \quad \vec{w} = (2, 0, 0)$

solution

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} 1 & -1 & 2 \\ 0 & 1 & 3 \\ 2 & 0 & 0 \end{vmatrix}$$

$$= 2 \cdot \begin{vmatrix} -1 & 2 \\ 1 & 3 \end{vmatrix} = 2 \cdot (-3 - 2)$$

$$= -10$$

$$\text{Area} = |\vec{u} \cdot (\vec{v} \times \vec{w})| = |-10| = 10$$

Real Vector Space Axioms

V is a set $\quad \vec{u}, \vec{v}, \vec{w} \in V, \quad k, l \in \mathbb{R}.$

V is a real vector space (& its elements are vectors) if.

Addition

1) $\vec{u} + \vec{v} \in V$

"closed under addn."

2) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$

3) $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
$= \vec{u} + \vec{v} + \vec{w}$

4) There exists a unique $\vec{0}$
such that $\vec{u} + \vec{0} = \vec{u}$

5) For each $\vec{u}$, there exists a "$-\vec{u}$"
such that $\vec{u} + (-\vec{u}) = \vec{0}$

Multiplication (by scalar)

6) $k\vec{u} \in V$

"Closed under scalar multn."

7) $k(\vec{u} + \vec{v}) = k\vec{u} + k\vec{v}$

8) $(k + l)\vec{u} = k\vec{u} + l\vec{u}$

9) $(kl)\vec{u} = k(l(\vec{u}))$

10) $1\vec{u} = \vec{u}$

Examples $\mathbb{R}^3$, $\mathbb{R}^2$, $\mathbb{R}^4$, $\mathbb{M}_{mn}$ (space of $m \times n$ matrices!)

$\mathbb{R}^1$, $\{\vec{0}\}$

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eg $\mathbb{C}$ - usual addition
- scalar multn. by real k

Check by explicitly testing each of 10 axiom! (checklist!)

eg test $k(\vec{u}+\vec{v}) = k\vec{u} + k\vec{v}$

$\vec{u} = a+ib \in \mathbb{C}$, $\vec{v} = c+id \in \mathbb{C}$, $k \in \mathbb{R}$

$$LS = k(\vec{u}+\vec{v}) = k\,(a+ib+c+id) = k\,(a+c+i(b+d))$$
$$= ka + kc + ikb + ikd.$$

notice computed in order given

$RS = k\vec{u} + k\vec{v} = k(a+ib) + k(c+id)$

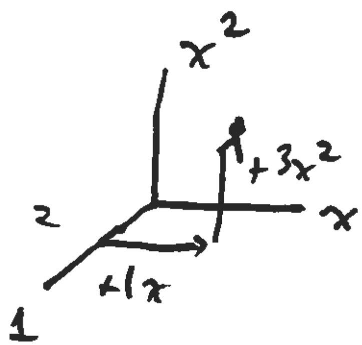
$= ka + ikb + kc + ikd = LS$

$RS = LS$ #7 holds

9. Parabolas are a vector space!

(and lower order)

ie $\mathbb{P}_2 = \{ ax^2+bx+c,\ a,b,c \in \mathbb{R} \}$



$(c, b, a) = c \cdot 1 + bx + ax^2$

$(2, 1, 3) = 2 + x + 3x^2$

e.g. $\mathbb{P}_1$, $\mathbb{P}_4$, $\mathbb{P}_n$ = polynomial space order n.

$\mathbb{P}_\infty$ = Taylor series space!

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eg Let $V = \mathbb{R}^+ = \{ x \mid x > 0 \}$

Let $\vec{x} = x \in \mathbb{R}^+$, $\vec{y} = y \in \mathbb{R}^+$

Define $\vec{x} + \vec{y} = xy$ Define $k\vec{x} = x^k$

Check axiom 5 " Is the a unique $\vec{0}$? "

Let $\vec{b} = \vec{0}$, if it exists

$\Rightarrow \quad \vec{u} + \vec{b} = \underline{u}\underline{b} = \underline{u} = \vec{u}$

$b = 1$

$1 \in \mathbb{R}^+$ ✓

Uniquely !

$1 = \vec{0}$