MATH 1B03/1ZC3 Winter 2019

# Lecture 17: More about vectors

Instructor: Dr Rushworth March 5th

## Dot products and norms continued

(from Chapter 3.2 of Anton-Rorres )

Recall that the dot product of two vectors $\mathbf{u}$ and $\mathbf{v}$, denoted $\mathbf{u} \bullet \mathbf{v}$, is a scalar. It satisfies the following properties.

**Fact 17.1: Properties of the dot product**

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in $\mathbb{R}^n$, and $k$ a scalar. Then

- $\mathbf{u} \bullet \mathbf{v} = \mathbf{v} \bullet \mathbf{u}$
- $\mathbf{u} \bullet (\mathbf{v} + \mathbf{w}) = \mathbf{u} \bullet \mathbf{v} + \mathbf{u} \bullet \mathbf{w}$
- $k (\mathbf{u} \bullet \mathbf{v}) = (k\mathbf{u}) \bullet \mathbf{v}$
- $\mathbf{u} \bullet \mathbf{u} \geq 0$, and $\mathbf{u} \bullet \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$

As usual, use these properties when working with vector equations involving the dot product.

### Important identities and inequalities

The norm and the dot product satisfy the following important identities and inequalities. Use these Facts to speed up calculations and solve problems.

### Fact 17.2: The Cauchy-Schwarz Inequality

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$. Then

$$|\mathbf{u} \bullet \mathbf{v}| \leq ||\mathbf{u}||||\mathbf{v}||$$

Here $|\mathbf{u} \bullet \mathbf{v}|$ means the absolute value of the scalar $\mathbf{u} \bullet \mathbf{v}$.

### Example 17.3

**Question:** Verify that the vectors $\mathbf{u} = (3, 0, 1)$ and $\mathbf{v} = (-1, 1, 0)$ satisfy the Cauchy-Schwarz Inequality.
Answer: Compute

$$||\mathbf{u}|| = \sqrt{10}$$

$$||\mathbf{v}|| = \sqrt{2}$$

$$|\mathbf{u} \bullet \mathbf{v}| = |-3| = 3$$

Then

$$3 \leq \sqrt{2}\sqrt{10} = \sqrt{20} \approx 4.47\ldots$$

### Fact 17.4: The Triangle Inequality

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$. Then

$$||\mathbf{u} + \mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||$$

### Question 17.5

Prove the triangle identity.
**Hint:** Start with $||\mathbf{u} + \mathbf{v}||^2$ and use the properties of the norm.

**Fact 17.6: Parallelogram Rule**

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$. Then

$$||\mathbf{u}+\mathbf{v}||^2 + ||\mathbf{u}-\mathbf{v}||^2 = 2\left(||\mathbf{u}||^2 + ||\mathbf{v}||^2\right)$$

**Fact 17.7**

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$. Then

$$\mathbf{u} \bullet \mathbf{v} = \frac{1}{4}||\mathbf{u}+\mathbf{v}||^2 - \frac{1}{4}||\mathbf{u}-\mathbf{v}||^2$$

Both Fact 17.6 and Fact 17.7 can be proved directly using the properties of the norm.

## Orthogonality

(from Chapter 3.4 of Anton-Rorres )

We have used the norm to generalize the notion of distance to $\mathbb{R}^n$.

Now we can generalize the notion of two vectors being "at right angles" to $\mathbb{R}^n$ using the dot product.

**Definition 17.8: Orthogonal**

Two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ are orthogonal if

$$\mathbf{u} \bullet \mathbf{v} = 0$$

We may also use the term perpendicular instead of orthogonal.

Two non-zero vectors in $\mathbb{R}^n$ are orthogonal if and only if they are at right-angles. Recall that

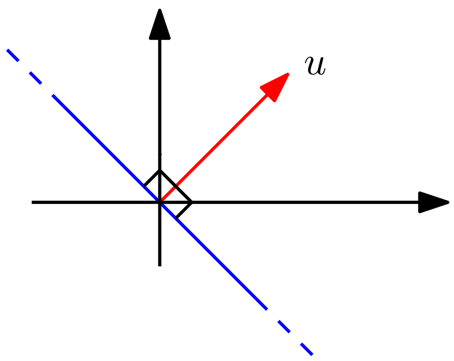
$$\mathbf{u} \bullet \mathbf{v} = ||\mathbf{u}||\,||\mathbf{v}||\cos(\theta)$$

if $\mathbf{u} \bullet \mathbf{v} = 0$ then

$$||\mathbf{u}||||\mathbf{v}|| \cos(\theta) = 0$$
$$\cos(\theta) = 0, \text{ as } \mathbf{u} \text{ and } \mathbf{v} \text{ are nonzero}$$
$$\theta = \frac{\pi}{2}$$

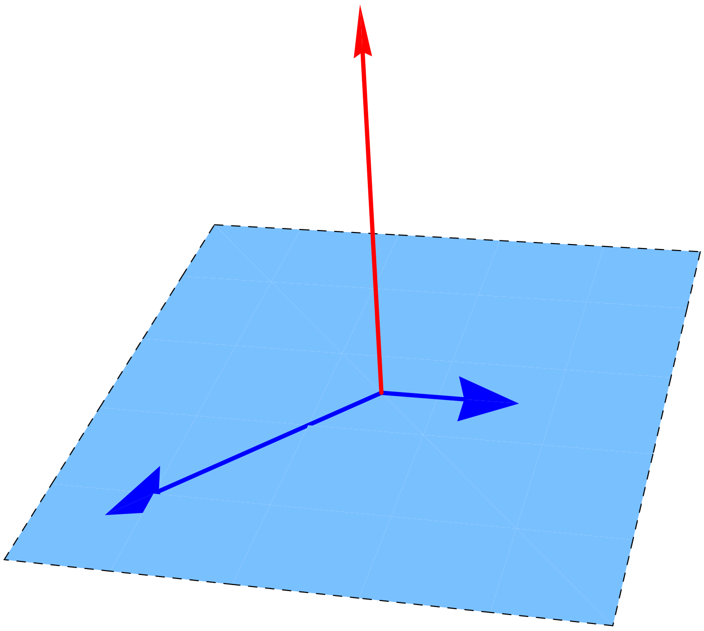
Given a fixed vector $\mathbf{u}$, what does the collection of vectors orthogonal to $\mathbf{u}$ look like?

In $\mathbb{R}^2$:




any vector which lies in the blue line is orthogonal to $\mathbf{u}$.

In $\mathbb{R}^3$:



any vector which lies on the blue plane is orthogonal to $\mathbf{u}$.

In both cases the red vector is said to be normal to the line or to the plane.

From these diagrams we observe that given a point in $\mathbb{R}^2$ and a normal vector, we can describe a line. In $\mathbb{R}^2$, a point and a normal vector, we can describe a plane.

**Definition 17.9: Point-Normal equations**

Let $a$ and $b$ be non-zero constants. The equation

$$ax + by + c = 0$$

represents a line in $\mathbb{R}^2$ with normal vector $\mathbf{n} = (a, b)$.
Let $a$, $b$ and $c$ be non-zero constants. The equation

$$ax + by + cz + d = 0$$

represents a plane in $\mathbb{R}^3$ with normal vector $\mathbf{n} = (a, b, c)$.

## Orthogonal projection

Using the concept of orthogonality we can decompose a vector in terms of another. That is, given a vector $\mathbf{u}$ and another vector $\mathbf{a}$, we can write

$$\mathbf{u} = (\text{a part parallel to } \mathbf{a}) + (\text{a part orthogonal to } \mathbf{a})$$

these parts are known as the projections of $\mathbf{u}$ with respect to $\mathbf{a}$.

**Definition 17.10**

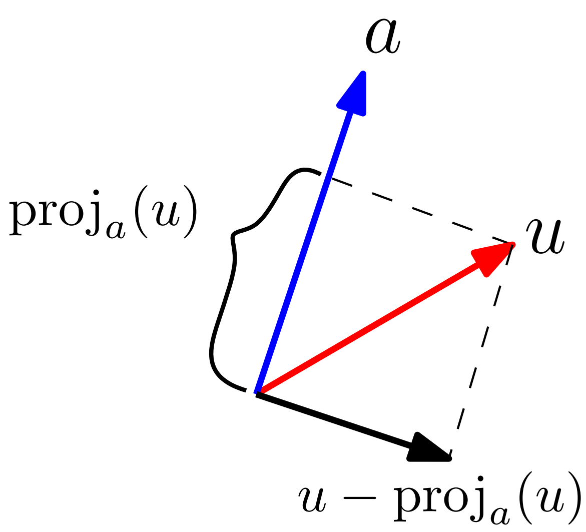
Let $\mathbf{u}$ and $\mathbf{a}$ be vectors in $\mathbb{R}^n$. The vector component of $\mathbf{u}$ in the direction of $\mathbf{a}$ is given by

$$\text{proj}_{\mathbf{a}}(\mathbf{u}) = \frac{1}{||\mathbf{a}||^2}(\mathbf{u} \bullet \mathbf{a})\,\mathbf{a}$$

The vector component of $\mathbf{u}$ orthogonal to $\mathbf{a}$ is given by

$$\mathbf{u} - \text{proj}_{\mathbf{a}}(\mathbf{u})$$

What does this mean geometrically? We are asking "how much of $\mathbf{u}$ is in the direction of $\mathbf{a}$?", and "how much of $\mathbf{u}$ is orthogonal to $\mathbf{a}$?".




Notice that these formulae work in any dimension.

### Example 17.11

**Question:** Given $\mathbf{u} = (4, -4, 1, 2)$ and $\mathbf{v} = (2, -1, 3, -3)$, compute the vector component of $\mathbf{u}$ in the direction of $\mathbf{v}$, and the vector component of $\mathbf{v}$ orthogonal to $\mathbf{u}$.
**Answer:** Compute

$$\begin{aligned} ||\mathbf{u}|| &= \sqrt{37} \\ ||\mathbf{v}|| &= \sqrt{23} \\ \mathbf{v} \bullet \mathbf{v} &= 8 + 4 + 3 - 6 = 9 \end{aligned}$$

Then the vector component of $\mathbf{u}$ in the direction of $\mathbf{v}$ is given by

$$\begin{aligned} \text{proj}_{\mathbf{v}}(\mathbf{u}) &= \frac{1}{||\mathbf{v}||^2}(\mathbf{u} \bullet \mathbf{v})\,\mathbf{v} \\ &= \frac{1}{23}(9)\,\mathbf{v} \\ &= (\frac{18}{23}, -\frac{9}{23}, \frac{27}{23}, -\frac{278}{23}) \end{aligned}$$

The vector component of $\mathbf{v}$ orthogonal to $\mathbf{u}$ is given by

$$\begin{aligned} \mathbf{v} - \text{proj}_{\mathbf{u}}(\mathbf{v}) = \mathbf{v} - \frac{1}{||\mathbf{u}||^2}(\mathbf{u} \bullet \mathbf{v})\,\mathbf{u} \\ &= (2, -1, 3, -3) - \frac{1}{37}(9)(4, -4, 1, 2) \\ &= (2 - \frac{36}{37}, -1 + \frac{36}{37}, 3 - \frac{9}{37}, -3 - \frac{18}{37}) \end{aligned}$$

### Fact 17.12

Let $\mathbf{u}$ and $\mathbf{v}$ be orthogonal vectors in $\mathbb{R}^n$. Then

$$||\mathbf{u} + \mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2$$

Recall that $||\mathbf{u} + \mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||$ if the vectors are not orthogonal.

## Calculating distances

Although we will not cover the proof, the following formulae can be proved using the concept of orthogonal projection.

### Fact 17.13: Calculating distance

Let $ax + by + c = 0$ be a line in $\mathbb{R}^2$, and $(x_0, y_0)$ a point. The shortest distance from $(x_0, y_0)$ to the line is given by

$$D = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}} \tag{1}$$

If $x + by + cz + d = 0$ is a plane in $\mathbb{R}^3$, and $(x_0, y_0, z_0)$ is a point, then he shortest distance from $(x_0, y_0, z_0)$ to the plane is given by

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}} \tag{2}$$

### Example 17.14

**Question:** Calculate the shortest distance between the point $(8, -1)$ and the line $y = \frac{2}{3}x - \frac{5}{3}$.
**Answer:** Rearrange the equation of the line:

$$\begin{aligned} y &= \frac{2}{3}x - \frac{5}{3} \\ 3y &= 2x - 5 \\ 2x - 3y - 5 &= 0 \end{aligned}$$

Now apply the formula given in Fact 17.13:

$$\begin{aligned} D &= \frac{|(2)(8) + (-3)(-1) - 5|}{\sqrt{4+9}} \\ &= \frac{|16 + 3 - 5|}{\sqrt{15}} \\ &= \frac{14}{\sqrt{15}} \end{aligned}$$

## More vector geometry

(from Chapter 3.4 of Anton-Rorres )

We started this course by looking at ways to solve systems of linear equations. We can understand solutions to such systems in geometric terms. To do this we need to describe lines and planes in $\mathbb{R}^2$ and $\mathbb{R}^3$. parametrically.

### Equation of a line

If $L$ is a line passing through the point $\mathbf{x}_0$, which is parallel to the vector $\mathbf{v}$, then the vector equationof $L$ is given by

$$\mathbf{x} = \mathbf{x}_0 + t\mathbf{v}$$

for $t$ a scalar parameter.

This holds in both $\mathbb{R}^2$ and $\mathbb{R}^3$.

### Equation of a plane

If $W$ is a plane passing through the point $\mathbf{x}_0$, which is parallel to the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ (where $\mathbf{v}_1$ is not parallel to $\mathbf{v}_2$), then the vector equationof $W$ is given by

$$\mathbf{x} = \mathbf{x}_0 + t\mathbf{v}_1 + s\mathbf{v}_2$$

for $s$ and $t$ scalar parameters.

This holds only in $\mathbb{R}^3$.

### Parametric equations

Given the vector equation of a line or a plane, we can expand the equation into a system of equations, one for each variable. The resulting equations are known as the parametric equations of the line or plane.

**Example 17.15**

**Question:** Find the parametric equations of the line $L$ in $\mathbb{R}^3$ that passes through the points $\mathbf{x}_0 = (3, 12, -2)$ and $\mathbf{x}_1 = (-4, 5, 1)$.
**Answer:** The vector $\mathbf{d} = \mathbf{x}_1 - \mathbf{x}_0$ lies on the line $L$ (draw a picture to see this). We have

$$\begin{aligned} \mathbf{d} &= (-4, 5, 1) - (3, 12, -2) \\ &= (-7, -7, 3) \end{aligned}$$

The vector $\mathbf{d}$ is certainly parallel to $L$, and we know that the point $\mathbf{x}_0$ lies on $L$. By definition, the vector equation of $L$ is therefore

$$\begin{aligned} \mathbf{x} &= \mathbf{x}_0 + t\mathbf{d} \\ &= (3, 12, -2) + t(-7, -7, 3) \\ &= (3 - 7t, 12 - 7t, 3t - 2) \end{aligned}$$

We can extract the parametric equation of $L$ by letting $\mathbf{x} = (x, y, z)$

$$\begin{aligned} x &= 3 - 7t \\ y &= 12 - 7t \\ z &= 3t - 2 \end{aligned}$$

## Suggested Problems

Practice the material covered in this lecture by attempting the following questions from Chapter 3.2 of Anton-Rorres, starting on page 153

- Questions 7, 13, 15, 17, 27

- True/False (d), (e), (g), (h)

the following questions from Chapter 3.3 of Anton-Rorres, starting on page $162$

- Questions $15$, $21$, $27$, $31$, $33$, $38$
- True/False (b), (c), (d), (f)

and the following questions from Chapter 3.4 of Anton-Rorres, starting on page $170$

- Questions $1$, $9$, $13$, $15$, $23$
- True/False (d), (f)