2.3

If A is an $n \times n$ (ie square) matrix & $BA = AB = I$ then $B = A^{-1}$ } definition of inverse of A

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If a matrix exists for A with above "B" then A is invertible

If no A^{-1} exists then A is non-invertible (or singular)

If A is a 2x2 matrix, & $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $A^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \frac{1}{ad-bc}$

eg. $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \Rightarrow A^{-1} = \begin{bmatrix} 5 & -2 \\ -3 & 1 \end{bmatrix} \frac{\overset{-1}{\cancel{1}}}{1(5) - (2)3} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$

eg $B = \begin{bmatrix} 2 & 0 \\ 4 & -1 \end{bmatrix} \Rightarrow B^{-1} = \begin{bmatrix} -1 & 0 \\ -4 & 2 \end{bmatrix} \frac{1}{ad-bc}$ $\quad \frac{1}{-2-0} = -\frac{1}{2}$

$= \begin{bmatrix} \frac{1}{2} & 0 \\ 2 & -1 \end{bmatrix}$

Check $B B^{-1} = \begin{bmatrix} 2 & 0 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1+0 & 0+0 \\ 2-2 & 0+1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ ✓

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eg $\underbrace{C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}$ $\Rightarrow$ $\cancel{C^{-1} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \cdot \frac{1}{0\cdot 0 - 0(-1)}}$

$1/0$ = Bad!

non-invertible! Singular!

note if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, A^{-1} DNE iff $ad - bc = 0$

if $\begin{cases} ad - bc \neq 0 \Rightarrow A^{-1} \text{ exists! Invertible!} \\ ad - bc = 0 \Rightarrow A^{-1} \text{ DNE! Singular!} \end{cases}$

"$ad-bc$" is the determinant of A (if A is 2×2)

$= \det A$

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eg. $2x + 3y = 7$ Solve using inverse!

$x - y = 2$

Solution

$A =$ coeff. matrix: $\begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}$ $\qquad \vec{x} =$ variables $= \begin{bmatrix} x \\ y \end{bmatrix}$

$\vec{b} =$ constants $= \begin{bmatrix} 7 \\ 2 \end{bmatrix}$

$$\begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix} \iff A\vec{x} = \vec{b}$$

so

$$A^{-1}A\vec{x} = A^{-1}\vec{b}$$

$$I\vec{x} = A^{-1}\vec{b}$$

$$\vec{x} = A^{-1}\vec{b}$$

$$A^{-1}b = \underbrace{\begin{bmatrix} -1 & -3 \\ -1 & 2 \end{bmatrix} \cdot \cancel{\frac{1}{(-2-3)}}^{-\frac{1}{5}} \cdot \begin{bmatrix} 7 \\ 2 \end{bmatrix}}$$

$$= -\frac{1}{5} \begin{bmatrix} -1 & -3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 7 \\ 2 \end{bmatrix}$$

$$= -\frac{1}{5} \begin{bmatrix} -7-6 \\ -7+4 \end{bmatrix} = -\frac{1}{5} \begin{bmatrix} -13 \\ -3 \end{bmatrix} = \begin{bmatrix} 13/5 \\ 3/5 \end{bmatrix}$$

so $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 13/5 \\ 3/5 \end{bmatrix}$ or $x = 13/5$

$y = 3/5$

What about 3×3 & larger A's & their inverses?

We need more tools & background

Now **Elementary Matrices**

An Elementary Matrix is result of a single elementary row operation applied to an identity matrix

eg. Row 1 $\leftrightarrow$ Row 2 on I_2, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

eg. Row 2 $\to \frac{1}{5}$ Row 2 on I_4 $\Rightarrow$ $E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

eg Row 1 $\to$ Row 1 $- 4$ Row 2 on I_3 $\Rightarrow$ $E = \begin{bmatrix} 1 & -4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

<u>Note 1</u>: Can be shown if E is an elementary matrix

EA = row op. on A

eg.

Row 1 $\leftrightarrow$ Row 2 $\Rightarrow$ $E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$E \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix}$$

Note 2: Each row op. has an inverse row op. that "undoes" it!

eg $\text{Row}_3 \leftrightarrow \text{Row}_4$ reverse $\text{Row}_3 \leftrightarrow \text{Row}_4$

eg $7 \cdot \text{Row } 2$ reverse $\frac{1}{7} \cdot \text{Row } 2$.

eg $\text{Row } 1 \to \text{Row } 1 + 4 \cdot \text{Row } 2$ reverse $\text{Row } 1 \to \text{Row } 1 - 4 \text{ Row } 2$

Say E_1 = elem. matrix for a given op.

& E_2 = elem. matrix for reverse row op.

$E_2 \cdot (E_1 \cdot I) = E_2$ (row op on I) = undoes row op on (op. on I)

= I

$$\boxed{E_2 E_1 = I}$$

& Similarly $\boxed{E_2 E_1 = I}$ $\Rightarrow$ $E_2 = E_1^{-1}$

$\Rightarrow$ All elem. matrices are invertible!

10 Two matrices are "Row equivalent"
if row ops turn one into other

$\Rightarrow$ If R is RREF of A $\Rightarrow$ R is row equivalent to A

Now If A has RREF, $R = I$
i.e. A is row equivalent to I.

then $\ldots E_3 E_2 E_1 A = I$ for some set E_1, E_2 etc.
of elem. matrices!

e_3 $\cancel{E}$ $E_3 E_2 E_1 A = I$

$$E_2 E_1 A = E_3^{-1}$$

$$E_1 A = E_2^{-1} E_3^{-1}$$

$$A = E_1^{-1} E_2^{-1} E_3^{-1}$$

A is a product of elementary matrices