12C3

Don't Forget: Matlab due today!

Last Day! "The Megatheorem" It keeps growing!

If A is an $n \times n$ (ie square) matrix, the following are equivalent!

1) A invertible (non-singular) ie A^{-1} exists
2) A is row-equivalent to I (RREF is I)
3) A is a product of "E" elementary matrices
4) $A\vec{x} = \vec{0}$ has only trivial solution (ie $\vec{x} = \vec{0}$ only)
5) $A\vec{x} = \vec{b}$ solution will always be unique (for any $\vec{b}$).
6) $A\vec{x} = \vec{b}$ has a solution for any $\vec{b}$

eg Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 3 \\ 3 & 4 & 4 \end{bmatrix}$, A is non-invertible.

Find all $\vec{b}$ such that $A\vec{x} = \vec{b}$ has no solution

Solution $A\vec{x} = \vec{b}$ $\rightsquigarrow$ Aug. matrix is $[A \mid \vec{b}\,]$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & b_1 \\ 2 & 2 & 3 & b_2 \\ 3 & 4 & 4 & b_3 \end{array}\right]$$ & solve!

$R_2 - 2R_1$ & then $R_3 - 3R_1$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & b_1 \\ 0 & -2 & 1 & b_2 - 2b_1 \\ 0 & -2 & 1 & b_3 - 3b_1 \end{array}\right]$$

$R_3 - R_2$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & b_1 \\ 0 & -2 & 1 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 - b_1 - b_2 \end{array}\right]$$

Stop! no soln. if $0 \neq 0$

i.e. if $\boxed{b_3 - b_1 - b_2 \neq 0}$

note e.g. $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ has $b_3 - b_1 - b_2 = 2 - 1 - 1 = 0$

$A\vec{x} = \vec{b}$ has (∞) soln!

but $\vec{b} = \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix}$ has $2 - (-1) - 5 \neq 0$

$A\vec{x} = \vec{b}$ has no soln.

Special Matrices

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$I_n = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & \ddots \end{bmatrix}$$

$\leftarrow$ 0 elsewhere!

$\leftarrow$ n "1" on principal diagonal

ie $a_{11} = a_{22} = a_{33} = \cdots a_{nn} = 1$, rest $= 0$

Diagonal Matrices

eg $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ eg $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 15 \end{bmatrix}$ eg $\begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \pi & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

Zeros only for any entry a_{ij} $i \neq j$ ie off principal diagonal non-zero entries can only exist on a_{ij}, $i=j$

(principal diagonal!)

eg

$$\begin{bmatrix} 1 & & 0 \\ & 2 & \\ 0 & & 3 \end{bmatrix}$$

diagonal

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

<u>not</u> <u>diagonal</u>

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

0 ✓

0 ✓

<u>diagonal</u>

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

<u>diagonal</u>!

<u>Not</u> if <u>A diagonal</u> A^{-1} exists iff all <u>diag. entries</u> $\neq 0$

eg $\begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 4(5)+0 & 0(4)+3(0) \\ 0(5)-2(0) & 0-2(3) \end{bmatrix}$

$$= \begin{bmatrix} 4\cdot 5 & 0 \\ 0 & -2\cdot 3 \end{bmatrix} = \begin{bmatrix} 20 & 0 \\ 0 & -6 \end{bmatrix}$$

In general $A = \begin{bmatrix} a_1 & & & 0 \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_n \end{bmatrix}$ & $B = \begin{bmatrix} b_1 & & & 0 \\ & b_2 & & \\ & & \ddots & \\ 0 & & & b_n \end{bmatrix}$

$$AB = \begin{bmatrix} a_1 b_1 & & & 0 \\ & a_2 b_2 & & \\ & & \ddots & \\ 0 & & & a_n b_n \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{1}{a_1} & & & 0 \\ & \frac{1}{a_2} & & \\ & & \ddots & \\ 0 & & & \frac{1}{a_n} \end{bmatrix}$$ if all $a_i \neq 0$ only

Note if A, B diagonal

$$\Rightarrow \quad A+B, \quad A-B, \quad kA, \quad AB, \quad A^T, \quad A^{-1}, \quad A^n$$

all diagonal!

note: $A^T = \begin{bmatrix} a_1 & & 0 \\ & a_2 \ddots & \\ 0 & & a_n \end{bmatrix} = A$, $\quad A^m = \begin{bmatrix} a_1^m & & 0 \\ & a_2^m & \\ 0 & \ddots & a_n^m \end{bmatrix}$

Triangular Matrices

Triangular matrices are $n \times n$ (square) matrices such that all entries above principal diagonal are 0 (lower triangular)

or all entries ~~abov~~ below principal diagonal are 0 (upper triangular) eg

$$\begin{bmatrix} 10 & 0 & 0 \\ 3 & 1 & 0 \\ 4 & 5 & 12 \end{bmatrix}$$

Lower Triangular

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 1 & 2 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 10 & 0 & 0 \\ 5 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 & 6 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{bmatrix}$$

Upper Triangular.

$$\begin{bmatrix} 2 & 5 & 0 & 1 \\ 0 & 0 & 2 & 7 \\ 0 & 0 & 10 & -1 \\ 0 & 0 & 0 & \pi \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

zeros here => lower triangular.

zeros here => U. T.

both => diagonal

Note

If A, B are upper triangular (UT)

then $A+B$, AB, kA, A^{-1} (if exists) all U. T.

If A, B are Lower triangular (LT)

then $A+B$, AB, kA, A^{-1} all L. T.

$(UT)^T$ is LT & vice versa

A triangular matrix (LT or UT) is invertible

iff none of $a_{ii} = 0$ ie no zero on principal diagonal

A is symmetric if A is an nxn (square) matrix & $A = A^T$

eg

$$\begin{bmatrix} 3 & 2 & -6 \\ 2 & 5 & 4 \\ -6 & 4 & 7 \end{bmatrix}$$