1ZC3

Last Day Span

1) A L.C. (linear combination) of vectors $\{\vec{v}_1, \dots \vec{v}_n\}$

has the form $\underline{a_1\vec{v}_1 + \dots + a_n\vec{v}_n} = \sum a_i \vec{v}_i$, $a_i \in \mathbb{R}$

2) A Span of $\{\vec{v}_1, \dots \vec{v}_n\}$ = Span $(\{\vec{v}_1, \dots \vec{v}_n\})$

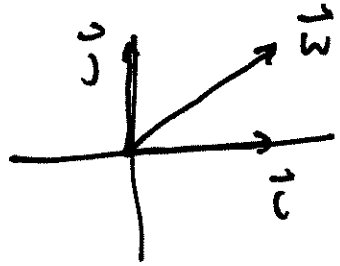
is set of all L.C. of $\{\vec{v}_1, \dots, \vec{v}_n\}$

Note Spans are always subspace of their v-space

& $\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_n\} \subseteq$ Span$(\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_n\})$.

eg Let $\vec{i} = \begin{bmatrix}1\\0\end{bmatrix}$, $\vec{j} = \begin{bmatrix}0\\1\end{bmatrix}$, $\vec{w} = \begin{bmatrix}1\\1\end{bmatrix}$

we can show $\{\vec{i}, \vec{j}, \vec{w}\}$ span $\mathbb{R}^2$



Yes! this spans $\mathbb{R}^2$ but so does $\{i, j\}$

When are spans equal?

Say I have $\{\vec{u}_1, \vec{u}_2, \dots \vec{u}_n\}$ & $\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_m\}$

$$\text{Span}(\{\vec{u}_1 \dots \vec{u}_n\}) = \text{Span}(\{\vec{v}_1, \dots \vec{v}_m\})$$

if any element of one, is in the other

equivalently if $\{\vec{v}_1, \dots \vec{v}_m\} \subseteq \text{Span}(\{\vec{u}_1 \dots \vec{u}_n\})$

$$\Rightarrow \text{Span}(\{\vec{v}_1 \dots \vec{v}_m\}) \subseteq \text{Span}(\{\vec{u}_1 \dots \vec{u}_n\})$$

Similarly if $\{\vec{u}_1, \dots \vec{u}_n\} \subseteq \text{Span}(\{\vec{v}_1 \dots \vec{v}_m\})$

$$\Rightarrow \quad \text{Span}(\{\vec{u}_1 \dots \vec{u}_n\}) \subseteq \text{Span}(\{\vec{v}_1 \dots \vec{v}_m\})$$

$$\text{so} \quad \text{Span}(\{\vec{u}\text{'s}\}) = \text{Span}(\{\vec{v}\text{'s}\})$$

eg. Let $\vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$

& $\vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ -3 \end{bmatrix}$.

Does $\text{Span}(\{\vec{u}_1, \vec{u}_2\}) = \text{Span}(\{\vec{v}_1, \vec{v}_2\})$?

Solution could soln for $\vec{u}_1 = a\vec{v}_1 + b\vec{v}_2$

$\vec{u}_2 = c\vec{v}_1 + d v_2$

& reverse for $\vec{v}_1, \vec{v}_2$ in terms of $\vec{u}$

Or by inspection

$$\left.\begin{aligned} \vec{v}_1 &= \vec{u}_1 - 2\vec{u}_2 \\ \vec{v}_2 &= -\vec{u}_1 + \vec{u}_2 \end{aligned}\right\} \Rightarrow \{\vec{v}_1, \vec{v}_2\} \subseteq \operatorname{Span}\{\vec{u}_1, \vec{u}_2\}$$

$$\left.\begin{aligned} \vec{u}_1 &= -\vec{v}_1 - 2\vec{v}_3 \\ \vec{u}_2 &= -\vec{v}_1 - \vec{v}_3 \end{aligned}\right\} \Rightarrow \{\vec{u}_1, \vec{u}_2\} \subseteq \operatorname{Span}\{\vec{v}_1, \vec{v}_2\}$$

So $\operatorname{Span}(\{\vec{u}_1, \vec{u}_2\}) = \operatorname{Span}(\{\vec{v}_1, \vec{v}_2\})$

Ahh but the Implication!

Notice: If I have $\{\vec{u}_1, \ldots \vec{u}_n\}$, and $\vec{u}_n$ is a l.c. of $\{\vec{u}_1, \ldots \vec{u}_{n-1}\}$

$\Rightarrow \operatorname{Span}(\{\vec{u}_1, \ldots \underline{\underline{\vec{u}_n}}\}) = \operatorname{Span}(\{\vec{u}_1, \underline{\underline{\vec{u}_{n-1}}})$

(i.e. we can drop $\vec{u}_n$)

eg $$\text{Span}\left(\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}1\\5\end{bmatrix}\begin{bmatrix}-1\\3\end{bmatrix}\right)$$

$$= \text{Span}\left(\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}\right)$$

$$= \text{Span}\left(\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\right)$$

Can't shrink further! Why? $\begin{bmatrix}1\\0\end{bmatrix}$ & $\begin{bmatrix}1\\0\end{bmatrix}$ are not L.C. of each other!

These are now Linearly Independent

Note a Linearly Indep. Span is a Basis

Linear Independence

A set of vectors $\{\vec{v}_1, \ldots \vec{v}_n\}$ is a Linearly independent (LI) set if no vector is a L.C of others!

or equivalently & more formally a set $\{\vec{v}_1, \ldots \vec{v}_n\}$ is L.I.

if $$a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots a_n\vec{v}_n = \vec{0}$$

iff all $a_1 = a_2 = \cdots = a_n = 0$

eg. are $\begin{bmatrix}1\\0\end{bmatrix}$ & $\begin{bmatrix}0\\1\end{bmatrix}$ L.I. ?

Check

$$\left.\begin{aligned} a\begin{bmatrix}1\\0\end{bmatrix} + b\begin{bmatrix}0\\1\end{bmatrix} &= \begin{bmatrix}0\\0\end{bmatrix} \\ \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix} &= \begin{bmatrix}0\\0\end{bmatrix} \end{aligned}\right\}$$ clearly $a = b = 0$ only!

eg. are $\begin{bmatrix}1\\1\\2\end{bmatrix}$ $\begin{bmatrix}0\\1\\-1\end{bmatrix}$ L. I.?

<u>Solutn</u>

$$a\begin{bmatrix}1\\1\\2\end{bmatrix} + b\begin{bmatrix}0\\1\\-1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}.$$

$\hookrightarrow$ $\begin{bmatrix}1 & 0\\1 & 1\\2 & -1\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$ has <u>only</u> 0 solutn! ($\Leftarrow$ L. I)

<u>Redun</u>

$$\begin{bmatrix}1 & 0\\1 & 1\\2 & -1\end{bmatrix} \xrightarrow[\text{\& } R_3-2R_1]{R_2-R_1} \begin{bmatrix}1 & 0\\0 & 1\\0 & -1\end{bmatrix} \xrightarrow{R_3+R_2} \begin{bmatrix}1 & 0\\0 & 1\\0 & 0\end{bmatrix} \Rightarrow \begin{matrix}a=0\\b=0\end{matrix} \text{ only!}$$

Above was trivial since $\vec{u}, \vec{v}$ L.I. if $\vec{u} \neq k\vec{v}$

for some k!

But are $\begin{bmatrix}1\\1\\2\end{bmatrix}$ $\begin{bmatrix}0\\1\\3\end{bmatrix}$ $\begin{bmatrix}1\\2\\5\end{bmatrix}$ L.I ?

Check

$$a\begin{bmatrix}1\\1\\2\end{bmatrix}+b\begin{bmatrix}0\\1\\3\end{bmatrix}+c\begin{bmatrix}1\\2\\5\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Check if a, b, c are only 0 .

$$\begin{bmatrix}1&0&1\\1&1&2\\2&3&5\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Here, we're "lucky" matrix square!

$a = b = c = 0$ only iff A^{-1} exists

iff $|A| \neq 0$.

but $\det(A) = 0$ for this matrix

$\Rightarrow$ A not invertible

$\Rightarrow$ vectors not linearly independent!

(Linearly dependent)

To check L.I.

Method #1 write $a_1\vec{v}_1 + \cdots + a_n\vec{v}_n = \vec{0}$ as a matrix system.

Solve & show explicitly $a_1 = \cdots = a_n = 0$.

Method #2 Instead if matrix of $\vec{v}$'s is **square**

take $\det(\vec{v}_1 \dots \vec{v}_n)$

If $\det \neq 0 \Rightarrow$ L. I.

Method #3 **Notice** one vector is a L.C. of

other by quick inspection!

Note any set containing $\vec{0}$ is **not** L. I.

Note $\{\vec{a}, \vec{b}\}$ is L.I. if $\vec{a} \neq \vec{0}$ & $\vec{b} \neq \vec{0}$

& $\vec{a} \neq k\vec{b}$ for some k