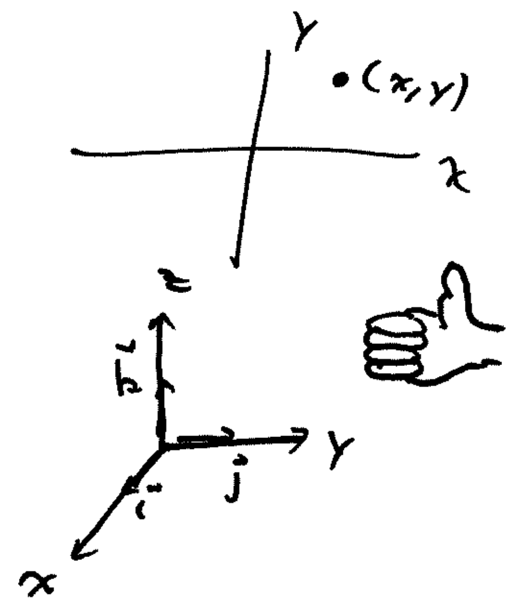
1ZC3 Last Day Vectors (& Oranges)

As long as scalar multn. & addition act "normally"

$\Rightarrow$ we have vectors!

eg. $\mathbb{R}^2$ space of numerical couplets (x, y)



$\mathbb{R}^3$ triplets (x, y, z)

$\mathbb{R}^4$ quadruples (x, y, z, w)

$\mathbb{R}^{17}$ 17-tuples $(x_1, x_2 \ldots\ldots x_{17})$

$\mathbb{R}^n$ n-tuples of numbers $(x_1 \ldots\ldots x_n)$.

no matter the $\underline{n}$ of our <u>n-tuple</u>

no matter # co-ords per point!

Scalar multn. & addn. <u>work the same</u>

eg $7(1, 0, -1, 2) = (7, 0, -7, 14)$

$(1, 5, -1, 6, 12) + (2, -1, 1, 5, -8)$

$= \underline{(3, 4, 0, 11, 4)}$

<u>obeys</u> all llgrm law properties!

all prop. of scalar multn. at all

<u>n-values!</u>

If $\vec{u} = (u_1, u_2, \dots u_n)$

$$\|\vec{u}\| = \underbrace{\sqrt{u_1^2 + u_2^2 + u_3^2 + u_4^2 + \dots + u_n^2}}$$

"Euclidean Norm" (from pythagorean theorem)

Concept of Length / Magnitude: "Norms"

A "Norm" has following properties

$$\begin{cases} 1) & \|k\vec{u}\| = |k|\,\|\vec{u}\| \\ 2) & \|\vec{u}\| \geq 0 \\ 3) & \|\vec{0}\| = 0 \text{ , only!} \end{cases}$$

Dot Product: $\vec{u} = (u_1, u_2)$, $\vec{v} = (v_1, v_2)$

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2$$

In $\mathbb{R}^3$ $\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3)$

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

for n-tuple vectors!

$\vec{u} = (u_1 \dots u_n)$

$\vec{v} = (v_1 \dots v_n)$

$$\vec{u} \cdot \vec{v} = \sum_{i=1}^{n} u_i v_i$$

← dot product in n-tuples! in $\mathbb{R}^n$.

This is a Special Case of the "Inner Product"

Notation $\vec{u}, \vec{v}$ vector, inner product $\langle \vec{u}, \vec{v} \rangle$

or $\langle \vec{u} \mid \vec{v} \rangle$

What is an inner product?

"Scalar product"

Rules 0) takes 2 vectors, $\vec{u}$ & $\vec{v}$, returns a real value symmetric $\rightarrow$ 1) $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$

2) $\langle k\vec{u}, \vec{v} \rangle = k \langle \vec{u}, \vec{v} \rangle$

$\left(= \langle u, k\vec{v} \rangle \right)$

3) $\langle \vec{u} + \vec{w}, \vec{v} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{w}, \vec{v} \rangle$

4) $\langle \vec{u}, \vec{u} \rangle \geq 0$

5) $\langle \vec{u}, \vec{u} \rangle = 0$ iff $\underline{\underline{\vec{u} = \vec{0}}}$.

$\langle \vec{u}, \vec{u} \rangle = \|\vec{u}\|^2$ for a related norm!

$\Downarrow$ eg.

Euclid. norm $\quad \|\vec{u}\| = \sqrt{u_1^2 + u_2^2 + \dots} = \sqrt{\sum u_i^2}$

Dot. product $\quad \underline{\underline{\vec{u} \cdot \vec{u}}} = u_1 u_1 + u_2 u_2 \dots = \sum u_i^2 = \underline{\underline{\|\vec{u}\|^2}}$

Notice Find $\vec{u} \cdot \vec{v}$ if $\vec{u} = (1, 0, -1, 2)$
$\vec{v} = (0, 5, 1, 1)$

Solutn $\vec{u} \cdot \vec{v} = 1(0) + 0(5) + (-1)(1) + 2(1) = \underline{1}$

Find $\|\vec{u}\|$ solutn $\sqrt{1^2 + 0^2 + (-1)^2 + 2^2} = \sqrt{\langle \vec{u}, \vec{u} \rangle}$

$= \sqrt{1 + 1 + 4} = \sqrt{6} = \|\vec{u}\|$

by Euclid. norm

Notice for example, if we let $\vec{u} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$\vec{u} \cdot \vec{v} = 3(-1) + 5(2) = \begin{bmatrix} 3 & 5 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

or $\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v}$ as matrix arithmetic

Also if A is a symmetric matrix (ie $A = A^T$)

& A has +ve eigenvalues

$\vec{u}^T A \vec{v} = \langle \vec{u}, \vec{v} \rangle_A$ is a alternate inner product of $\mathbb{R}^n$.

Save for 2nd or 3rd year!).

All inner products (dot product included!) have

Cauchy - Schwarz Inequality (ie C-S inequality)

$$| \langle \vec{u}, \vec{v} \rangle | \leq \|\vec{u}\| \, \|\vec{v}\|$$

for dot product in $\mathbb{R}^2$ can be shown

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos\theta$$

because $|\cos\theta| \leq 1 \Rightarrow |\vec{u} \cdot \vec{v}| \leq \|\vec{u}\| \, \|\vec{v}\|$

ie C-S holds for dot!

but also can define $\dfrac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \, \|\vec{v}\|} = \cos\theta$

if $\theta = \pm \pi/2 \Rightarrow \langle \vec{u}, \vec{v} \rangle = 0$ in $\mathbb{R}^2$ $\Leftrightarrow$ $\vec{u}$ is $\perp$ to $\vec{v}$

So in general define "orthogonality"

$\vec{u}$ & $\vec{v}$ orthogonal under inner product iff $\langle \vec{u}, \vec{v} \rangle = \vec{0}$

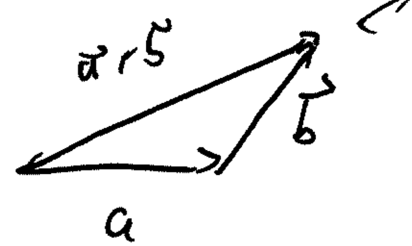
Besides orth. & $\cos\theta$, C-S. is powerful!

prove many geometric features from it!

eg. Use C-S to show "The Triangle Inequality"

ie $\|\vec{a}+\vec{b}\| \le \|\vec{a}\| + \|\vec{b}\|$

Solution



Proof

$$
\begin{aligned}
\|\vec{a}+\vec{b}\|^2 &= (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b}) \\
&= \vec{a}\cdot\vec{a} + \vec{b}\cdot\vec{a} + \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{b} \\
&= \|\vec{a}\|^2 + \|\vec{b}\|^2 + 2\,\vec{a}\cdot\vec{b} \qquad \le \|\vec{a}\|\,\|\vec{b}\| \\
&\le \|\vec{a}\|^2 + \|b\|^2 + 2\|\vec{a}\|\,\|\vec{b}\|
\end{aligned}
$$

$$= \left(\|\vec{a}\| + \|\vec{b}\| \right)^2$$

$$\|\vec{a}+\vec{b}\|^2 \leq \left(\|\vec{a}\| + \|\vec{b}\| \right)^2$$

$$\underline{\underline{\|\vec{a}+\vec{b}\| \leq \|\vec{a}\| + \|\vec{b}\|}} \quad \underline{\underline{\text{QED}}}$$