MATH 1B03/1ZC3 Winter 2019

Lecture 19: General vector spaces

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General vector spaces

(from Chapter 4.1 of Anton-Rorres)

The n-dimensional spaces $\mathbb{R}^n$ have very useful algebraic structure: addition of two vectors to produce a new vector, and scalar multiplication.

This structure is not unique to $\mathbb{R}^n$. It turns out that many other collections of objects possess this structure; they are known as vector spaces.

We will now study vector spaces in abstraction. This is the process of progress in mathematics: we identity a structure present in a number of different instances, and then study that structure in abstraction. This allows us better understand this structure, and the situations in which it appears.

Henceforth, the term *vector* takes on a more general meaning: arrows with direction and magnitude are examples of vectors, but not all vectors are such arrows.

Definition 19.1: Sets and elements

A set is a collection of objects. The objects making up a set are known as its elements.
If X is a set with elements x, y and z we write

$$X = \{x, y, z\}$$

If x is an element of the set X we write $x \in X$.
If t is not an element of X we write $t \notin X$.

Definition 19.2: Vector space

Let V be a set equipped with two operations:

- **Addition:** takes as input $\mathbf{u}, \mathbf{v} \in V$ and outputs $\mathbf{u} + \mathbf{v}$
- **Scalar multiplication:** takes as input $\mathbf{u} \in V$ and k a real number, and outputs $k\mathbf{u}$

The set V equipped with these operations is a vector space if the following axioms (a.k.a. rules) are satisfied for all $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in V:

1. V is closed under addition: $\mathbf{u} + \mathbf{v} \in V$ i.e. $\mathbf{u} + \mathbf{v}$ is an element of V
2. Addition is commutative: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. Addition is associative: $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{v} + \mathbf{u}) + \mathbf{w}$
4. Additive identity: there exists a unique element of V, denoted $\mathbf{0}$, such that $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$. The vector $\mathbf{0}$ is known as the zero vector.
5. Additive inverses: given $\mathbf{u}$, there exists a unique element $-\mathbf{u}$ such that $\mathbf{u} - \mathbf{u} = \mathbf{0}$
6. V is closed under scalar multiplication: for any scalar k, $k\mathbf{u} \in V$
7. For any scalar k, $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$
8. For any scalars k and m, $(k + m)\mathbf{u} = k\mathbf{u} + m\mathbf{u}$
9. For any scalars k and m, $(km)\mathbf{u} = k(m\mathbf{u})$
10. $1\mathbf{u} = \mathbf{u}$

You might be thinking 'don't we know all this already?'. And we do if $V = \mathbb{R}^n$. The point is that we can talk about vector spaces which **are not** $\mathbb{R}^n$.

Given a set V, we can attempt to turn it into a vector space by **defining** an addition and a scalar multiplication. Therefore, for the remainder of this course symbols like $+$, $\mathbf{0}$ and $-\mathbf{u}$ can mean many different things, depending on the context. It is important to be aware of this as we learn about abstract vector spaces.

Example 19.3

The trivial vector space: Let $V = \{\mathbf{0}\}$ be a set containing only one element. Then V can be turned into a (very boring) vector space by defining

$$\begin{aligned} \mathbf{0} + \mathbf{0} &= \mathbf{0} \\ k\mathbf{0} &= \mathbf{0} \end{aligned}$$

for all scalars k. We can quickly verify that these definitions satisfy the axioms of Definition 19.2.
This vector space is known as the trivial vector space.

The space of polynomials: Let P be the set of all polynomials in the variable x with real coefficients. That is, given $\mathbf{u} \in P$ we have

$$\mathbf{u} = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

Then P can be turned into a vector space by defining the following operations. Given $\mathbf{u} = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ and $\mathbf{v} = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0$ (notice they need not have the same degree), define $\mathbf{u} + \mathbf{v}$ via collecting like terms. For example,

$$\begin{aligned} \mathbf{u} &= 4x^5 - 2x^2 + 7 \\ \mathbf{v} &= 3x^2 + 6x - 5 \\ \mathbf{u} + \mathbf{v} &= 4x^5 - 2x^2 + 7 + 3x^2 + 6x - 5 \\ &= 4x^5 + x^2 + 6x + 2 \end{aligned}$$

and we see that P is closed under addition. Scalar multiplication is defined as follows. Given $\mathbf{u} = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ and k a scalar, then

$$k\mathbf{u} = ka_n x^n + ka_{n-1}x^{n-1} + \cdots + ka_1 x + ka_0$$

These operations turn P into a vector space. Unlike the spaces $\mathbb{R}^n$, it is *infinite dimensional* (we will return to this later on).

The space of matrices: The set of $m \times n$ matrices equipped with matrix addition and the usual scalar multiplication is a vector space.

Under these definitions

$$\mathbf{0} = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$$

i.e. it is the $m \times n$ zero matrix.

The space of functions of one real variable: Let F be the set of functions $f : \mathbb{R} \longrightarrow \mathbb{R}$. Then F forms a vector space when equipped with the following operations.
For $f, g \in F$ define $f + g$ as

$$(f + g)(x) = f(x) + g(x)$$

and given k a scalar define kf as

$$(kf)(x) = kf(x)$$

A counter-intuitive vector space: The space $\mathbb{R}$ is a vector space when equipped with the standard definitions of addition and multiplication.
However, we can turn it into a vector space in another way, by defining different addition and scalar multiplication operations.
Define addition as

$$\mathbf{u} + \mathbf{v} = \mathbf{uv}, \text{ the product of two real numbers}$$

and scalar multiplication as

$$k\mathbf{u} = \mathbf{u}^k$$

Therefore, under these **new definitions** of addition and scalar multiplication, we have:

$$7 + 2 = 14$$
$$(2)4 = 4^2 = 16$$

where we have treated 2 as a scalar and 4 as a vector in the second line.
These new definitions satisfy the axioms of a vector space. This is an example of the fact that the same set can be turned into two distinct vector spaces, by defining different operations.

Question 19.4

Check that the axioms are satisfied for the various vector spaces given in the example above.
Find the additive inverses for each of the vector spaces.

Why are the axioms what they are? There are many different viewpoints, but one answer is that they are the smallest set of rules we need to generalise the behaviour of $\mathbb{R}^n$.

It is important to notice that other desirable properties that we know from our previous experience with $\mathbb{R}^n$ follow directly from the axioms (so that we did not need to specify them as rules).

Fact 19.5

Let V be a vector space, $\mathbf{u} \in V$ and k a scalar. Then

1. $0\mathbf{u} = \mathbf{0}$
2. $k\mathbf{0} = \mathbf{0}$
3. $(-1)\,\mathbf{u} = -\mathbf{u}$
4. If $k\mathbf{u} = \mathbf{0}$ then either $k = 0$ or $\mathbf{u} = \mathbf{0}$.

Proof: We will prove 1. and 2. leaving 3. and 4. as exercises.

1. : we have

$$\begin{aligned} 0\mathbf{u} &= (k-k)\,\mathbf{u} \\ &= k\mathbf{u} - k\mathbf{u} \\ &= \mathbf{0} \end{aligned}$$

2. : we have

$$\begin{aligned} k\mathbf{0} &= k\,(\mathbf{u} - \mathbf{u}) \\ &= k\mathbf{u} - k\mathbf{u} \\ &= \mathbf{0} \end{aligned}$$

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Example 19.6

Question: Let X be the set of all 2×2 invertible matrices.
Is X a vector space when equipped with the standard matrix addition and scalar multiplication?
Answer: To determine whether or not X is a vector space under the given operations, we need to check if all of the axioms are satisfied. Let's begin with the first axiom: closure under addition.
Let $A \in X$. Then $\det(A) \neq 0$ as A is invertible. The matrix $-A$ is invertible also (where $-A$ is the matrix obtained by multiplying the entries of A by -1) as

$$\det(-A) = (-1)^2 \det(A) = \det(A) \neq 0$$

Therefore $-A \in X$ also.
But

$$A - A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$\begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} = 0$$

so $A - A$ is not invertible and $(A - A) \notin X$.
Therefore the first axiom of Definition 19.2 is not satisfied and X with these operations is not a vector space.

Subspaces

(from Chapter 4.2 of Anton-Rorres)

Recall that the set of all 2×2 matrices is a vector space when equipped with the standard matrix addition and scalar multiplication. Example 19.6 shows that the collection of 2×2 invertible matrices are not a vector space, however.

Collections of vectors within a vector space that **do** define vector spaces themselves are therefore special.

Definition 19.7: Subset

Let X be a set. A subset of X is a collection of some of the elements of X. If Y is a subset of X we write $Y \subset X$.

Notice that under this definition X is a subset of itself i.e. $X \subset X$.

Example 19.8

The set of even whole numbers is a subset of the real numbers.

The set of 2×2 invertible matrices is a subset of the set of 2×2 matrices.

The set of quadratic polynomials in x is a subset of the set of all polynomials in x.

Definition 19.9: Subspace

Let V be a vector space. A subset $W \subset V$ is a subspace if it is a vector space itself.

Example 19.10

$\mathbb{R}^3$ is a subspace of $\mathbb{R}^5$. In fact, if $l < k$, then $\mathbb{R}^l$ is a subspace of $\mathbb{R}^k$.

When first presented with a set and some operations we must check all of the axioms given in Definition 19.2 to determine if the set forms a vector space. It is much less work to check if a subset is a subspace, however.

Fact 19.11: When is a subset a subspace?

Let V be a vector space. The subset $W \subset V$ is a subspace if and only if

1. W is closed under addition: if $\mathbf{u} \in W$ and $\mathbf{v} \in W$, then $\mathbf{u} + \mathbf{v} \in W$
2. W is closed under scalar multiplication: if $\mathbf{u} \in W$ then $k\mathbf{u} \in W$ for all scalars k

In Example 19.6 we saw that subset of 2×2 invertible matrices is not a subspace of the vector space of 2×2 matrices, because it is not closed under addition.

We can use Fact 19.11 to check if a subset is a subspace by in the following steps.

Recipe 19.12: Checking if a subset is a subspace

Let V be a vector space, and $W \subset V$ a subset. Use this recipe to check if W is a subspace of V.

Step 1: Check if W is closed under addition: take $\mathbf{u} \in W$ and $\mathbf{v} \in W$, two general vectors in W. Compute their sum $\mathbf{u} + \mathbf{v}$, and determine if $\mathbf{u} + \mathbf{v}$ is an element of W.
If it is not, then W is not a subspace.
If it is, go to Step 2.

Step 2: Check if W is closed under scalar multiplication: take $\mathbf{u} \in W$ a general vector in W, and a scalar k. Compute $k\mathbf{u}$, and determine if it is in W.
If it is not, then W is not a subspace.
If it is, then W is a subspace.

When using Recipe 19.12 we cannot just check closure for *particular vectors*; we must check closure for *every vector* in W, by taking the general form of such vectors.

Example 19.13

Question: Determine if P_n, the subset of polynomials of degree at most n, is a subspace of P, the vector space of polynomials as defined in Example 19.3.
Answer: We are required to check if $P_n \subset P$ is closed under addition and scalar multiplication.
A general vector $\mathbf{u} \in P_n$ has the form

$$\mathbf{u} = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

for a_i a real number. Pick another general vector in W

$$\mathbf{v} = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0$$

and compute their sum:

$$\begin{aligned}\mathbf{u} + \mathbf{v} &= a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \\ &\quad + b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0 \\ &= (a_n + b_n)x^n + (a_{n-1} + b_{n-1})x^{n-1} + \cdots + (a_1 + b_1)x + a_0 + b_0\end{aligned}$$

We observe that $\mathbf{u} + \mathbf{v}$ is also a polynomial of degree at most n, so that $\mathbf{u} + \mathbf{v} \in P_n$.
As we picked $\mathbf{u}$ and $\mathbf{v}$ to be general vectors in P_n this proves that P_n is closed under addition.
Step 2. is to check if P_n is closed under scalar multiplication. Let $\mathbf{u}$ be as above, and k a scalar. Then

$$\begin{aligned}k\mathbf{u} &= k\left(a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0\right) \\ &= ka_n x^n + ka_{n-1}x^{n-1} + \cdots + ka_1 x + ka_0\end{aligned}$$

We observe that $k\mathbf{u}$ is also a polynomial of degree at most n, and P_n is closed under scalar multiplication.
Therefore P_n is a subspace of P.

We can produce subspaces of vector spaces on demand using the following definitions.

Definition 19.14: Linear combination

Let V be a vector space and $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}$ be a collection of vectors in V. A linear combination of these vectors is a new vector

$$\mathbf{v} = a_1\mathbf{v_1} + a_2\mathbf{v_2} + \cdots + a_k\mathbf{v_k}$$

for $a_1, a_2, \ldots, a_k$ scalars.

Definition 19.15: Span

Let V be a vector space and $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}$ be a collection of vectors in V. The span of $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}$ is denoted

$$\text{span}\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}\}$$

and is defined to be the set of all possible linear combinations of $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}$.

If we let the set $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}\}$ then we may also write

$$\text{span}\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}\} = \text{span}(S)$$

Given a vector $\mathbf{u} \in V$, then $\mathbf{u} \in \text{span}\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}\}$ if and only if there exist scalars $a_1, a_2, \ldots, a_k$ such at

$$\mathbf{v} = a_1\mathbf{v_1} + a_2\mathbf{v_2} + \cdots + a_k\mathbf{v_k}$$

Fact 19.16: Spans are subspaces

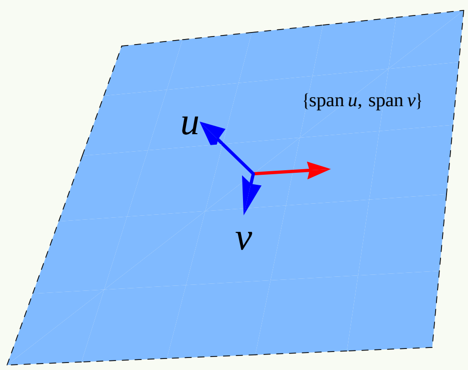
Let V be a vector space and $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}\}$. Then $\text{span}(S)$ is a subspace of V.

Question 19.17

Prove this fact.
Hint: Use the fact that if $\mathbf{u} \in \text{span}(S)$ then it can be written as a linear combination of the vectors in S.

Example 19.18

If $V = \mathbb{R}^3$ and $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$, then span$\{\mathbf{u}, \mathbf{v}\}$ is a plane:



If $V = \mathbb{R}^4$ and $S = \{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}$, then

$$\text{span}(S) = \mathbb{R}^4$$

that is, we recover all of $\mathbb{R}^4$ as the span of S. To see this, pick $\mathbf{u} = (a, b, c, d) \in \mathbb{R}^4$. Then we have

$$(a, b, c, d) = a\,(1, 0, 0, 0) + b\,(0, 1, 0, 0) + c\,(0, 0, 1, 0) + d\,(0, 0, 0, 1)$$

Let V be the vector space of 2×2 matrices, and

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Then span(S) is the subspace of all upper triangular 2×2 matrices. To see this, let M be an upper triangular 2×2 matrix i.e.

$$M = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$$

Then we have

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

However, notice that

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \notin \text{span}(S)$$

so that span(S) is not the entire vector space of 2×2 matrices.

Let $V = P$, the vector space of polynomials in x with real coefficients, and $S = \{1, x, x^2\}$.
Then span(S) is the subspace of all polynomials $\mathbf{u} = ax^2 + bx + c$ i.e. the subspace of quadratic polynomials.
Again, notice that there are polynomials which are not in span(S), such as $\mathbf{v} = x^3$.

Suggested Problems

Practice the material covered in this lecture by attempting the following questions from Chapter 4.1 of Anton-Rorres, starting on page 190

- Questions $1, 2, 5, 7, 11$

and from Chapter 4.2 of Anton-Rorres, starting on page 200

- Questions $1, 2, 3, 7, 10, 11, 13$
- True/False (e), (g), (h), (j)