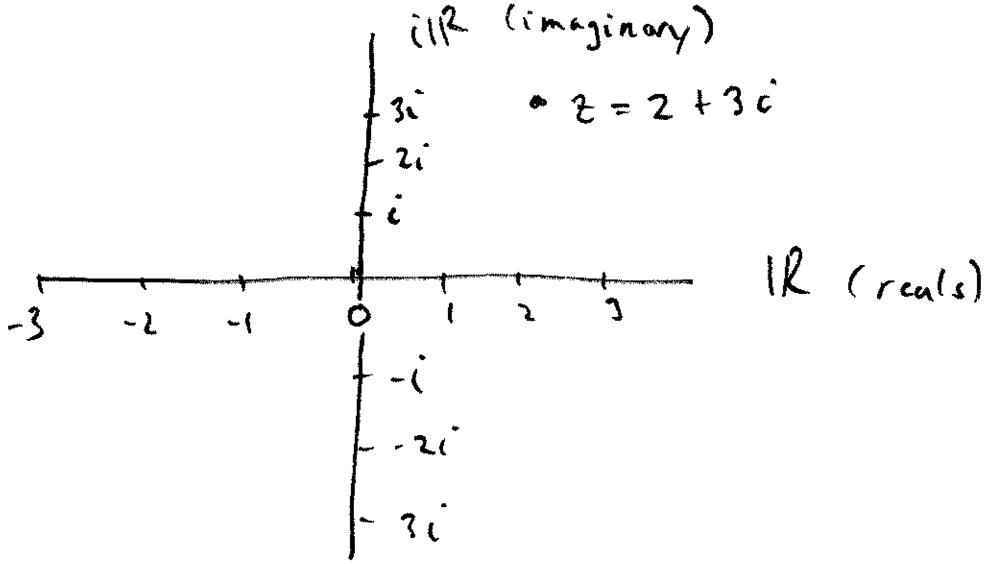
1ZC3

Last Day

$i^2 = -1$

Imaginary & Complex Numbers

Note Extra Chapter 10.1, 10.2, 10.3 on website



$\mathbb{C}$ complex plane

$z = a + ib$

is a complex number!

if $z \in \mathbb{C}$, $z = a + ib \Rightarrow \mathrm{Re}(z)$ = real part = a

$\mathrm{Im}(z)$ = imaginary part = b "no i"

eg if $\underline{z = 2 + 3i}$ $\Rightarrow$ $Re(z) = 2$, $Im(z) = 3$

<u>watch out</u>: sometimes we use letter $\underline{j}$, $\underline{j^2 = -1}$

(engineering <u>only</u>)

$\mathbb{C}$ Arithmetic

$$\left.\begin{aligned}(2+3i) + (7-i) &= (2+7) + (3-1)i \\ &= \underline{9 + 2i}\end{aligned}\right\}$$

real & im add separately (like <u>vectors in $\mathbb{R}^2$</u>)

if $z \in \mathbb{C}$, $w \in \mathbb{C}$, $z + w \in \mathbb{C}$ ✓

$$\begin{aligned}(2+3i)(1-5i) &= 2(1) + 2(-5)i + (3i)(1) + (3i)(-5i) \\ &= 2 - 10i + 3i - 15\cancel{i^2}^{\,-1} \\ &= \underline{17 - 7i}\end{aligned}$$

note $z \in \mathbb{C}, w \in \mathbb{C}, zw \in \mathbb{C}$ ✓

complex conjugate

Division is **odd**: eg $\dfrac{2+5i}{1-2i} \cdot \dfrac{1+2i}{1+2i} = \dfrac{(2+5i)(1+2i)}{1^2 - 2^2 i^2}$

$= 1^2 + 2^2 = 5$

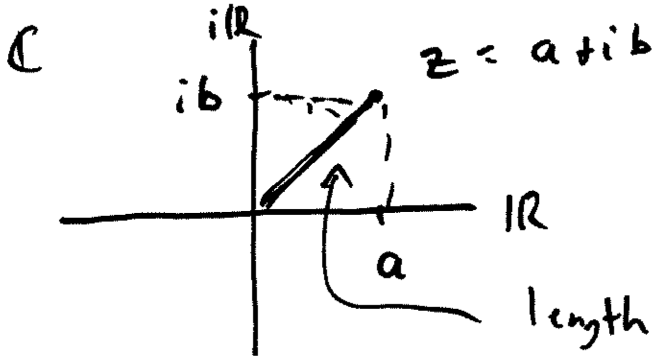
$= \dfrac{2+4i+5i-10}{5} = -\dfrac{8}{5} + \dfrac{9}{5}i$

In general! $\dfrac{z}{w} \cdot \dfrac{\overline{w}}{\overline{w}} = \dfrac{z\overline{w}}{w\overline{w}} = \dfrac{z\overline{w}}{|w|^2}$

$(z, w \in \mathbb{C}, w \neq 0)$

if $w = c+id$

$\overline{w} = c-id$ = complex conjugate



length $= \sqrt{a^2+b^2}$ $=$ **modulus** of $\underline{z}$ $= |z|$

Generalizes absolute value to $\mathbb{C}$

$|5| = \sqrt{5^2+0^2} = 5$, $|-7| = \sqrt{(-7)^2+0^2} = 7$

so for example:

$$\frac{1}{i} = \frac{1 \cdot \overline{i}}{|i|^2} = \frac{1 \cdot \overline{(0+1i)}}{|0+i|^2} = \frac{0-i}{0^2+1^2} = -i$$

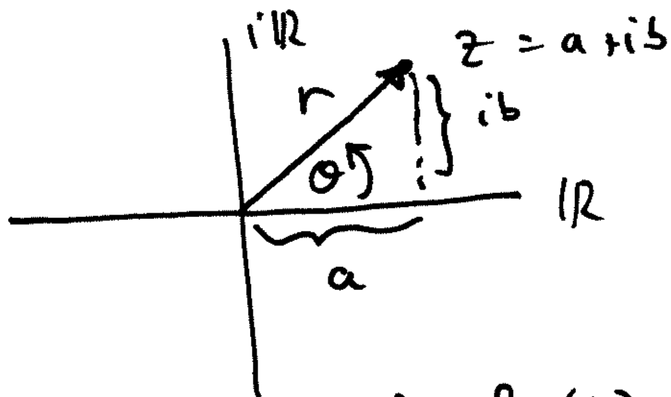
Almost forgot!.. <u>Properties of conjugate</u>

$$z = a+ib \Rightarrow \overline{z} = a-ib \Rightarrow \overline{\overline{z}} = a-(-i)b = a+ib = z$$

$$\overline{z+w} = \overline{a+ib+c+id} = \overline{(a+c)+(b+d)i} = (a+c)-(b+d)i = (a-bi)+(c-di) = \overline{z}+\overline{w}$$

$\overline{zw} = \bar{z}\,\bar{w}$, $\overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\bar{w}}$ Similarly!

Complex Numbers in Polar Form



$r = |z| = \sqrt{a^2 + b^2}$

$\theta = \arg(z)$ = argument of z

= angle to +ve real axis CCW

$a = \mathrm{Re}(z) = r\cos\theta$

$b = \mathrm{Im}(z) = r\sin\theta$

$\Rightarrow \quad z = a + ib = r\cos\theta + ir\sin\theta$

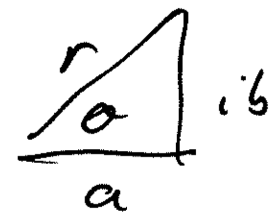
$= r(\cos\theta + i\sin\theta)$ polar form.

$z = r \cdot \operatorname{cis}\theta$, $(\operatorname{cis}\theta = \cos\theta + i\sin\theta)$

e.g. write: $z = 1 + i$ & $w = -\sqrt{3} - i$ in <u>polar form</u>

<u>Solution</u> $z = 1 + i \Rightarrow$ modulus $= \sqrt{1^2 + 1^2} = \sqrt{2}$

$$\tan\theta = \frac{b}{a} = \frac{1}{1} = 1$$

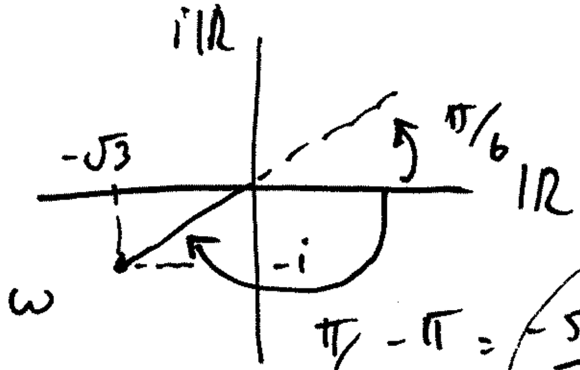


$\theta = \pi/4$ generally try to use "<u>principal argument</u>" $\Rightarrow \theta \in [-\pi, \pi]$

$$\Rightarrow z = r \operatorname{cis}\theta = \underline{\underline{\sqrt{2}\operatorname{cis}(\pi/4)}}$$

$$w = -\sqrt{3} - i \Rightarrow |w| = r = \sqrt{(\sqrt{3})^2 + (-1)^2} = 2$$

$$\tan\theta = b/a = \frac{-1}{-\sqrt{3}} = \frac{1}{\sqrt{3}} \Rightarrow \theta = \pi/6 \underline{+ \pi} \Big\} \text{ Q. \#3}$$



$\pi/6 - \pi = -\frac{5\pi}{6}$

or $\pi/6 - \pi$

$= -\frac{5\pi}{6} \in (-\pi, \pi]$

$\Rightarrow$ principal argument

$\Rightarrow$ $w = r \operatorname{cis}\theta = 2 \operatorname{cis}(-5\pi/6)$

So in $Q2, Q3 \Rightarrow \pm\pi$ to $\theta \Rightarrow \theta = \tan^{-1}\left(\frac{b}{a}\right) \pm \pi$

in $Q1, Q4 \Rightarrow \theta = \tan^{-1}(b/a)$

Polar Multiplication

Let $z = r \operatorname{cis}\theta$, $w = \rho \operatorname{cis}\phi$

$$zw = r\rho \operatorname{cis}\theta \operatorname{cis}\phi = r\rho(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)$$

$$= r\rho\,(\cos(\theta+\varphi) + i\sin(\theta+\phi))$$ (by trig. identities)

$$= r\rho\,(\underline{\underline{\operatorname{cis}(\theta+\phi)}})$$

& $\underline{\underline{z^n = r^n \operatorname{cis}(n\theta)}}$

So

$\operatorname{cis}\theta = \cos(\theta) + i\sin\theta)$ is <u>wierd!</u>

$$(\operatorname{cis}\theta)^n = \operatorname{cis}(n\theta)$$

$$\operatorname{cis}(\theta+\phi) = \operatorname{cis}\theta \cdot \operatorname{cis}(\phi)$$

next day

$$e^{i\theta} = \cos\theta + i\sin\theta$$