12C3

eg. Is the set of Skew Symmetric 3x3 matrices a subspace of M_{33}?

Solution Subspaces = Vegas ie Check if closed under addn. & scalar multn.

S = set of skew-symmetric matrices.

$= \{ A \mid A^T = -A \}$

Say $\vec{u} \in S \Rightarrow u^T = -u$
$\vec{v} \in S \rightarrow v^T = -v$

$(u+v)^T$
$= u^T + v^T$
$= (-u) + (-v)$
$= -u - v = -(u+v)$ ✓

so $\vec{u}+\vec{v} \in S \Rightarrow$ closed under addn.

now $k\vec{u} \rightsquigarrow (ku)^T = ku^T = k(-u)$

(k scalar $\in \mathbb{R}$) $= -ku = -(ku)$

$k\vec{u} \in S$ closed under multn.

& $0^T = -0 \Rightarrow$ non-empty! $\Rightarrow$ It's a subspace!

Let's find a spanning set for this subspace!

If $A \in M_{33} \Rightarrow$

$$\begin{bmatrix} \cancel{a_{11}}^{0} & a_{12} & a_{13} \\ a_{21} & \cancel{a_{22}}^{0} & a_{23} \\ a_{31} & a_{32} & \cancel{a_{33}}^{0} \end{bmatrix}$$

If $A^T = -A \Rightarrow a_{11} = a_{22} = a_{33} = 0$

$a_{12} = -a_{21}$, $a_{32} = -a_{23}$, $a_{13} = -a_{31}$

$\underline{\underline{\text{So}}}$ any $A \in S$ = set of Skew. Symmetric matrices.

has $A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$

$$= \begin{bmatrix} 0 & a & 0 \\ -a & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} & & b \\ & & \\ -b & & \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & c \\ 0 & -c & 0 \end{bmatrix}$$

$$= a \underset{\substack{\uparrow \\ \vec{v}_1}}{\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}} + b \underset{\substack{\uparrow \\ \vec{v}_2}}{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}} + c \underset{\substack{\uparrow \\ \vec{v}_3}}{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}}$$

$\{v_1, v_2, v_3\}$ span S

Are these LI? Is $\{v_1, v_2, v_3\}$ LI?

LI means $a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3 = \vec{0}$ iff $a = b = c = 0$

Here $a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3 = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} = [\,0\,]$

iff $a = b = c = 0$

$\Rightarrow$ LI

(LI & Span also $\Rightarrow$ basis of S)

What is $\dim(S)$? $\dim S = n(\{\vec{v}_1, \vec{v}_2, \vec{v}_3\})$

$= \underline{\underline{3}}$ <u>elements</u> in <u>basis</u>.

ie <u>3-D</u>.

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eg <u>Consider</u>: $M = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{bmatrix} \in S$

<u>now</u> <u>consider</u> $\vec{u}_1 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\vec{u}_2 = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$

$\vec{u}_3 = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$

I claim $P = \{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ are L.I., & $n(P) = 3$

$\Rightarrow$ <u>basis</u>

express M as a vector in that basis

Solution $\quad M = a\vec{u}_1 + b\vec{u}_2 + c\vec{u}_3 \Big\}$ goal: get (a, b, c)

$$\begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -a & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & b & b \\ -b & 0 & 0 \\ -b & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & c & c \\ -c & 0 & c \\ -c & -c & 0 \end{bmatrix}$$

$1 = -a + b + c \Rightarrow -a = 1 - (\cancel{b+c}^{\,2}) \Rightarrow -a = -1$

$2 = 0 + b + c \Rightarrow b = 2 - (c) = +5 \qquad a = 1$

$-3 = 0 + 0 + c \Rightarrow c = -3$

$M = 1\vec{u}_1 + 5\vec{u}_2 - 3\vec{u}_3$

or $\quad M = (1, 5, -3)$ in P basis

eg Consider $\mathbb{R}^3$ & let $W = \text{Span}\{(1,1,3),(1,0,1)\}$

Let $\vec{v} = (1,2,3)$, find $\text{proj}_W \vec{v}$

Solution. Remember if $\{\vec{u}_1, \dots \vec{u}_n\}$ is an orthogonal basis of W

$$\rightarrow \text{proj}_W \vec{v} = \sum_{i=1}^{n} \text{proj}_{\vec{u}_i} \vec{v}$$

So we orthogonalize our basis of W.

$$\{(1,1,3),(1,0,1)\}$$

for simplicity let $u_1 = (1,0,1)$

$$\& \quad u_2 = (1,1,3) - \text{proj}_{u_1}(1,1,3)$$

$$= (1,1,3) - \frac{(1+0+3)}{(1^2+0^2+1^2)}(1,0,1)$$

$$= (-1,1,1)$$

$\rightarrow$ orth. basis of $W = \{(1,0,1),(-1,1,1)\}$

$$\text{proj}_W((1,2,3)) = \text{proj}_{u_1}(1,2,3) + \text{proj}_{u_2}(1,2,3)$$

$$= \frac{(-1+2+3)}{(1^2+1^2+1^2)}(-1,1,1) + \frac{(1+0+3)}{(1^2+0^2+1^2)}(1,0,1) = \left(\frac{2}{3}, \frac{4}{3}, \frac{10}{3}\right)$$