MATH 1B03/1ZC3 Winter 2019

Lecture 21: Bases of vector spaces

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Bases of vector spaces

(from Chapter 4.4 of Anton-Rorres)

Whenever we work with abstract vector spaces we must pick a *co-ordinate system* in order to write the vectors down. One way to think of a co-ordinate system is as a set of axes, which define a grid used to describe vectors.

A co-ordinate system for a vector space is known as a basis (singular: basis, plural: bases). There are many different bases for the same vector space, as shown in the following example.

Consider the vector space $\mathbb{R}^2$. We saw earlier the standard basis vectors of $\mathbb{R}^2$, $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$. Any vector $\mathbf{v} \in \mathbb{R}^2$ can be written as a linear combination of $\mathbf{i}$ and $\mathbf{j}$:

$$(v_1, v_2) = v_1\mathbf{i} + v_2\mathbf{j}$$

Because of this, we say that the set $\{\mathbf{i}, \mathbf{j}\}$ is a basis of $\mathbb{R}^2$.

However, we can describe every vector in $\mathbb{R}^2$ using another two basis vectors. Let $\mathbf{e}_1 = (2, -1)$ and $\mathbf{e}_2 = (3, 0)$. Notice that the set $\{\mathbf{e}_1, \mathbf{e}_2\}$ is linearly independent.

Question 21.1

Check that $\{\mathbf{e}_1, \mathbf{e}_2\}$ is linearly independent.

A linearly independent pair of vectors spans a plane, so that span $(\{\mathbf{e}_1, \mathbf{e}_2\})$ =

$\mathbb{R}^2$. For example, given $\mathbf{v} = (7, 4)$ we have

$$\begin{aligned} k\,(2, -1) + m\,(3, 0) &= (7, 4) \\ (2k + 3m, -k) &= (7, 4) \end{aligned}$$

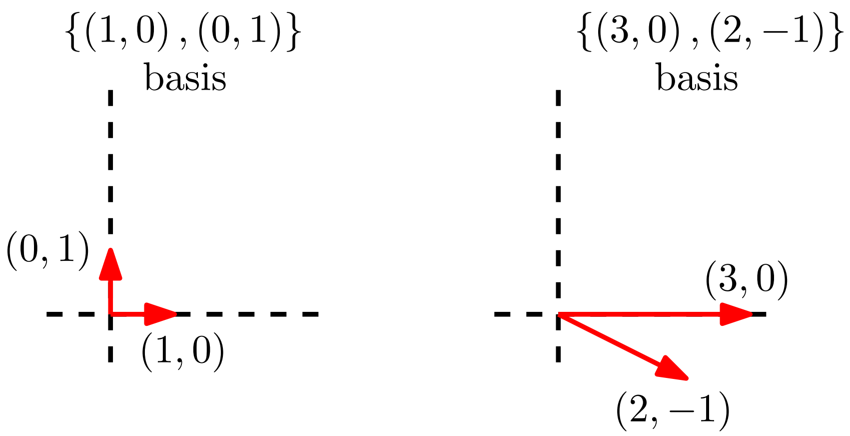
Solving the system

$$\begin{aligned} 2k + 3m &= 7 \\ -k &= 4 \end{aligned}$$

we obtain $k = -4$ and $m = 5$ so that

$$\mathbf{v} = (7, 4) = -4\,(2, -1) + 5\,(3, 0)$$

This process is equivalent to picking a new pair of axes for $\mathbb{R}^2$:



Notice that $\mathbf{e}_1$ is not orthogonal to $\mathbf{e}_2$. The set is only required to be linearly independent.

Definition 21.2: Basis of a vector space

Let V be a vector space and $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a set of vectors. Then S is a <u>basis</u> of V if

1. S is linearly independent
2. $\text{span}(S) = V$

The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are known as the <u>basis vectors</u>.

Example 21.3

The set $S = \{(1,0,0), (0,1,0), (0,0,1)\}$ is a basis for $\mathbb{R}^3$.
The set S is linearly independent as

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0$$

and $\text{span}(S) = \mathbb{R}^3$ as

$$(v_1, v_2, v_3) = v_1 (1,0,0) + v_2 (0,1,0) + v_3 (0,0,1)$$

The set S is known as the standard basis of $\mathbb{R}^3$.

The set $S = \{(1,0,0), (0,1,0), (0,0,1), (1,0,1)\}$ is **not** a basis.
Certainly $\text{span}(S) = \mathbb{R}^3$, but as S has 4 vectors in $\mathbb{R}^3$ it must be linearly dependent.

Question 21.4

Is the set $\{(1,4), (-3,-12)\}$ a basis of $\mathbb{R}^2$?

Definition 21.5: The standard basis

The set

$$\begin{aligned} S = \{ & (1,0,0,\ldots,0,0), (0,1,0,\ldots,0,0), \ldots, \\ & (0,0,0,\ldots,1,0), (0,0,0,\ldots,0,1)\} \end{aligned}$$

is a basis of $\mathbb{R}^n$, known as the standard basis.

Example 21.6

The set
$$S = \{(1,\, 0,\, 0,\, 0),\, (0,\, 1,\, 0,\, 0),\, \ldots,\\ (0,\, 0,\, 1,\, 0),\, (0,\, 0,\, 0,\, 1)\}$$
is the standard basis of $\mathbb{R}^4$.

We have been using the standard basis of $\mathbb{R}^n$ throughout the previous parts of this course, whenever we have written down the co-ordinates of a vector. For example, given $\mathbf{v} = (3,\, -2,\, 8) \in \mathbb{R}^3$ we have used the standard basis to write it down, as

$$\mathbf{v} = 3\,(1,\, 0,\, 0) - 2\,(0,\, 1,\, 0) + 8\,(0,\, 0,\, 1)$$

However, there is no specific reason to use the standard basis over any other. Given a vector expressed using one basis, we can express it in another i.e. we can *change basis*. We are describing the same vector, using different bases.

For example, we saw earlier that

$$\underbrace{-4\,(2,\, -1) + 5\,(3,\, 0)}_{\{(2,-1),(3,0)\}\text{ basis}} = (7,\, 4) = \underbrace{7\,(1,\, 0) - 4\,(0,\, 1)}_{\text{standard basis}}$$

Definition 21.7: A vector in a new basis

Let V be a vector space and $S = \{\mathbf{v}_1,\, \mathbf{v}_2,\, \ldots,\, \mathbf{v}_n\}$ a basis of V.
If a vector $\mathbf{v} \in V$ can be expressed as the linear combination

$$\mathbf{v} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n$$

then the <u>co-ordinate vector of $\mathbf{v}$ in the basis S</u> is

$$\mathbf{v} = (k_1,\, k_2,\, \ldots,\, k_n)_S$$

We also say that $(k_1,\, k_2,\, \ldots,\, k_n)_S$ is the vector $\mathbf{v}$ <u>written in the basis S</u>.

Notice that the order of the basis vectors matters!

If you see a vector $(k_1,\, k_2,\, \ldots,\, k_n)$ without the subscript $_S$, you can assume that the vector has been written in the standard basis.

Example 21.8

The set $S = \{(2, -1), (3, 0)\}$ is a basis for $\mathbb{R}^2$. We saw earlier that

$$(7, 4) = -4\,(2, -1) + 5\,(3, 0)$$

so that the co-ordinate vector of $(7, 4)$ in the basis S is

$$(7, 4) = (-4, 5)_S$$

Question: Show that the set

$$S = \{(2, -5, 0), (3, 0, 1), (0, -3, 0)\}$$

is a basis for $\mathbb{R}^3$ and find the co-ordinate vector of $(2, -6, 7)$ in the basis S.
Answer: We need to check that S is linearly independent and that $\operatorname{span}(S) = \mathbb{R}^3$.
We have 3 vectors in $\mathbb{R}^3$, so to check linear independence compute the determinant

$$\begin{vmatrix} 2 & -5 & 0 \\ 3 & 0 & 1 \\ 0 & -3 & 0 \end{vmatrix} = 3(2) = 6 \neq 0$$

Therefore S is linearly independent, and $\operatorname{span}(S) = \mathbb{R}^3$ by Fact 20.9, so that S is a basis for $\mathbb{R}^3$.
To find the co-ordinate vector of $(2, -6, 7)$ in the basis S, consider the equation

$$\begin{aligned} (2, -6, 7) &= k_1\,(2, -5, 0) + k_2\,(3, 0, 1) + k_3\,(0, -3, 0) \\ &= (2k_1 + 3k_2, -5k_1 - 3k_3, k_2) \end{aligned}$$

This yields the system of linear equations

$$\begin{aligned} 2 &= 2k_1 + 3k_2 \\ -6 &= -5k_1 - 3k_3 \\ 7 &= k_2 \end{aligned}$$

Solving this system, we obtain

$$\begin{aligned} k_1 &= -\frac{19}{2} \\ k_2 &= 7 \\ k_3 &= -\frac{107}{6} \end{aligned}$$

Therefore

$$(2, -6, 7) = -\frac{19}{2}(2, -5, 0) + 7(3, 0, 1) - \frac{107}{6}(0, -3, 0)$$

and the co-ordinate vector is

$$(2, -6, 7) = \left(-\frac{19}{2}, 7, -\frac{107}{6}\right)_S$$

We can produce bases for vector spaces which are not $\mathbb{R}^n$. Recall that P_4 is the vector space of polynomials in x with degree at most 4. Consider the set

$$S = \{1, x, x^2, x^3, x^4\}$$

We claim that S forms a basis for P_4. First we check linear independence. Consider the equation

$$k_1 1 + k_2 x + k_3 x^2 + k_4 x^3 + k_5 x^4 = 0$$

By the Fundamental Theorem of Algebra we know that a polynomial of degree 4 has exactly 4 roots (when counted with multiplicity). If there exist non-zero values $k_1 = a$, $k_2 = b$, $k_3 = c$, $k_4 = d$ and $k_5 = e$ such that

$$a1 + bx + cx^2 + dx^3 + ex^4 = 0$$

then the polynomial would have an infinite number of roots. This is because, for any integer l

$$\begin{aligned} (la1) + (lb)x + (lc)x^2 + (ld)x^3 + (le)x^4 &= l\left(a1 + bx + cx^2 + dx^3 + ex^4\right) \\ &= 0 \end{aligned}$$

But this contradicts the Fundamental Theorem of Algebra, so such values $k_1 = a$, $k_2 = b$, $k_3 = c$, $k_4 = d$ and $k_5 = e$ cannot exist.

Therefore the set S is linearly independent in P_4.

To see that $\text{span}(S) = P_4$, notice that any polynomial of degree at most 4, $\mathbf{p}(x)$ may be written

$$\mathbf{p}(x) = k_1 1 + k_2 x + k_3 x^2 + k_4 x^3 + k_5 x^4$$

This is generalised as follows.

Fact 21.9: A basis of P_n

Let P_n denote the vector space of polynomials in x of degree at most n. The set

$$S = \{1, x, x^2, \ldots, x^n\}$$

is a basis for P_n. The set S is known as the standard basis of P_n.

Proof: Repeat the argument given above for P_4. ■

Example 21.10

Question: Let S be the standard basis of P_4. Write the co-ordinate vector of $\mathbf{p}(x) = 7x^3 + 4x - 21$ relative to S.
Answer: Recall that $S = \{1, x, x^2, x^3, x^4\}$. Then

$$\mathbf{p}(x) = 7x^3 + 4x - 21 = (-21, 4, 0, 7, 0)_S$$

Notice that the order of the entries matters: it must be the same as the order the basis vectors appear in S.

Just as we did for $\mathbb{R}^n$, we can pick bases of P_n which are different to the standard basis.

Example 21.11

Question: Consider the polynomial

$$\mathbf{p}(x) = 8x^2 - 10x + 7$$

in P_2.
The set $S = \{2-x,\ x^2+3x,\ 2x^2+2\}$ is a basis for P_2. Find the co-ordinate vector of $\mathbf{p}(x)$ relative to S.
Answer: Consider the equation

$$\begin{aligned} 8x^2 - 10x + 7 &= k_1(2-x) + k_2\left(x^2+3x\right) + k_3\left(2x^2+2\right) \\ &= (k_2+2k_3)\,x^2 + (3k_2-k_1)\,x + 2\,(k_1+k_3) \end{aligned}$$

This yields the system of linear equations

$$\begin{aligned} 8 &= k_2 + 2k_3 \\ -10 &= -k_1 + 3k_2 \\ 7 &= 2k_1 + 2k_3 \end{aligned}$$

Solving this system (by Gauss-Jordon elimination) yields

$$\begin{aligned} k_1 &= -\frac{13}{5} \\ k_2 &= -\frac{21}{5} \\ k_3 &= \frac{61}{10} \end{aligned}$$

Therefore the co-ordinate vector of $\mathbf{p}(x) = 8x^2 - 10x + 7$ with respect to S is

$$\mathbf{p}(x) = \left(-\frac{13}{5}, -\frac{21}{5}, \frac{61}{10}\right)$$

We have been implicitly using the following important fact throughout our discussion of bases.

Fact 21.12

Let V be a vector space and $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ a basis. Every vector $\mathbf{v} \in V$ may be expressed as a linear combination of the basis vectors in exactly one way

$$\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 + \ldots + \mathbf{v}_n$$

Suggested Problems

Practice the material covered in this lecture by attempting the following questions from Chapter 4.4 of Anton-Rorres, starting on page 219

- Questions $13, 17, 19, 27$
- True/False (a), (b), (c), (d)