MATHEMATICS 1LT3E TEST 1

Evening Class
Duration of Examination: 75 minutes
McMaster University, 4 July 2012

E. Clements

FIRST NAME (please print): SOL^{NS}

FAMILY NAME (please print): ______

Student No.: ______

THIS TEST HAS 8 PAGES AND 8 QUESTIONS. QUESTIONS BEGIN ON PAGE 2. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE.

Total number of points is 50. Marks are indicated next to the problem number. Any non-graphing calculator is allowed.

USE PEN TO WRITE YOUR TEST. IF YOU USE A PENCIL YOUR TEST WILL NOT BE ACCEPTED FOR REMARKING (IF NEEDED).

You need to show work to receive full credit.

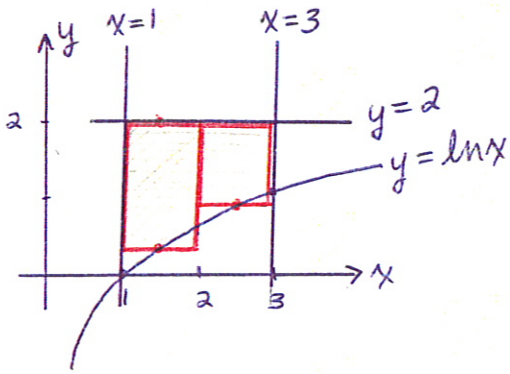
Page	Points	Mark
2	7	
3	7	
4	9	
5	6	
6	6	
7	8	
8	7	
TOTAL	50	

Continued on next page

Name:____________________

Student No.: ____________________

1\. (a) [2] Sketch the region bounded by the curves $y = \ln x$, $y = 2$, $x = 1$, and $x = 3$.



(b) [2] Approximate the area of the region in part (a) using a midpoint Riemann sum and two rectangles. Draw the two rectangles on your graph in part (a).

$$\Delta x = \frac{3-1}{2} = 1$$

$$M_2 = [2 - \ln 1.5] \cdot 1 + [2 - \ln 2.5] \cdot 1$$

$$\approx 2.7$$

(c) [3] Find the exact area of the region in part (a) by evaluating $\int_1^3 |2 - \ln x|\, dx$.

$$\int_1^3 |2-\ln x|\,dx = \int_1^3 (2-\ln x)\,dx$$

$$= \left[x(2-\ln x) + x\right|_1^3$$

$$= [3(2-\ln 3)+3] - [1(2-\ln 1)+1]$$

$$= 6 - 3\ln 3$$

$$\approx 2.7$$

<u>Aside:</u>

$u = 2 - \ln x \qquad dv = dx$

$du = \frac{-1}{x}\,dx \qquad v = x$

$$\int (2-\ln x)\,dx = x(2-\ln x) - \int x \cdot \frac{-1}{x}$$

$$= x(2-\ln x) + x + C$$

Name:________________

Student No.: ________________

2. Evaluate the following improper integrals to determine whether they are convergent or divergent.

(a) [3] $\int_4^\infty \frac{1}{(3x+1)^4}\,dx$

$$= \lim_{T\to\infty} \int_4^T (3x+1)^{-4}\,dx$$

$$= \lim_{T\to\infty} \left[\frac{(3x+1)^{-3}}{-3(3)} \right|_4^T$$

$$= \lim_{T\to\infty} \left[-\frac{1}{9(3T+1)^3} - \left(-\frac{1}{9(3(4)+1)^3}\right)\right]$$

$$= \cancel{\frac{-1}{\infty}}^{\,0} + \frac{1}{19773}$$

$$= \frac{1}{19773} \quad \text{(CONVERGENT)}$$

(b) [4] $\int_1^\infty \frac{\ln x}{x}\,dx$

$$= \lim_{T\to\infty} \int_1^T \frac{\ln x}{x}\,dx$$

$$= \lim_{T\to\infty} \left[\frac{(\ln x)^2}{2} \right|_1^T$$

$$= \lim_{T\to\infty} \left[\frac{(\ln T)^2}{2} - \frac{(\ln 1)^2}{2} \right]$$

$$= \frac{(\infty)^2}{2} - 0$$

$$= \infty \quad \text{(DIVERGENT)}$$

Aside:

$u = \ln x$

$du = \frac{1}{x}\,dx$

$$\Rightarrow \int \frac{\ln x}{x}\,dx = \int u\,du = \frac{u^2}{2} + C = \frac{(\ln x)^2}{2} + C$$

3. [4] Suppose that the concentration of a toxin in a cell is increasing at a rate of $50e^{-2t}$ micromoles per litre per second, starting from a concentration of 10μ mol/L. If the cell is poisoned when the concentration exceeds 30μ mol/L, could this cell survive?

$C(0)$

$$C(T) - C(0) = \int_0^T 50e^{-2t}\,dt$$
$$= \frac{50e^{-2t}}{-2}\Big|_0^T$$
$$= -25e^{-2T} + 25$$
$$C(T) = -25e^{-2T} + 25 + C(0) = -25e^{-2T} + 35$$
$$\lim_{T\to\infty} C(T) = -25\underbrace{e^{-\infty}}_{\to 0} + 35 = 35\mu mol/L$$

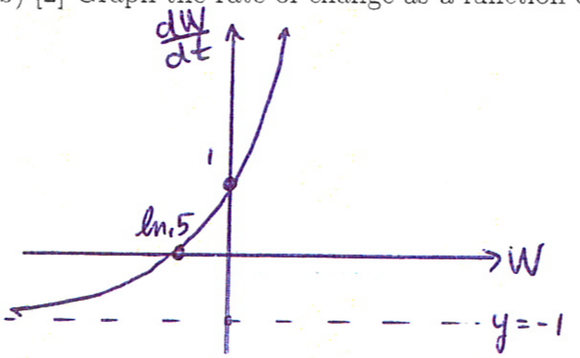
$\Rightarrow$ Eventually, the concentration will exceed $30\mu mol/L$ and the cell will not survive

4. Consider the autonomous differential equation $\frac{dW}{dt} = 2e^W - 1$.

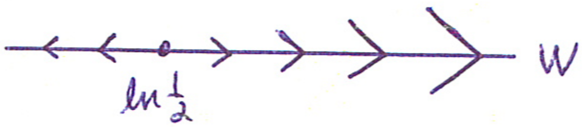
(a) [1] Find the equilibrium solution.

$$\frac{dW}{dt} = 0 \text{ when } 2e^W - 1 = 0 \Rightarrow e^W = \tfrac{1}{2} \Rightarrow W = \ln\tfrac{1}{2}$$

(b) [2] Graph the rate of change as a function of the state variable.



(c) [2] Draw a phase-line diagram for W.



Name:_______________

Student No.: _______________

5. A basic model for the spread of a disease within a population is given by

$$\frac{dI}{dt} = \alpha I(1 - I) - \mu I$$

where I denotes the fraction of individuals infected with the disease at time t and α, $\mu > 0$ are parameters.

(a) [1] What does the term $-\mu I$ represent?

The recovery rate.

(b) [2] Suppose that $\alpha = 0.4$ and $\mu = 0.1$. Determine equilibrium solutions for this model.

$$\frac{dI}{dt} = 0{,}4 I(1-I) - 0{,}1 I$$

$\frac{dI}{dt} = 0$ when $0{,}4I(1-I) - 0{,}1I = 0$

$$I\left[0{,}4(1-I) - 0{,}1\right] = 0$$

$\boxed{I = 0}$ or $1 - I = \frac{0{,}1}{0{,}4}$

$$\Rightarrow \boxed{I = 0{,}75}$$

(c) [3] Determine the stability of each equilibrium in part (b) using the Stability Theorem.

$$f(I) = 0{,}4I - 0{,}4I^2 - 0{,}1I$$
$$= 0{,}3I - 0{,}4I^2$$
$$f'(I) = 0{,}3 - 0{,}8I$$

$f'(0) = 0{,}3 > 0 \Rightarrow I^* = 0$ is UNSTABLE.

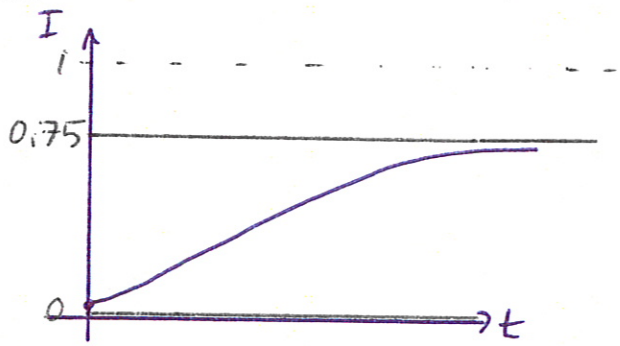
$f'(0{,}75) = -0{,}3 < 0 \Rightarrow I^* = 0{,}75$ is STABLE.

5. continued...

(d) [1] Describe what happens to the fraction of infected individuals over time if initially 1% of the population is infected with this disease.

The fraction infected will increase and approach 0.75.

(e) [2] Sketch the equilibrium solutions and the solution $I(t)$ starting from the initial condition $I(0) = 0.01$.



6. The solution of the logistic equation $\frac{dP}{dt} = 0.08P\left(1 - \frac{P}{4000}\right)$ when $P(0) = 500$ is given by $P(t) = \frac{4000}{1 + 7e^{-0.08t}}$.

[3] Find the time needed for the population to reach 95% of its carrying capacity.

$$\frac{4000}{1+7e^{-0.08t}} = 0.95 \times 4000$$ carrying capacity

$$\Rightarrow \frac{1}{0.95} = 1 + 7e^{-0.08t}$$

$$\Rightarrow \frac{5}{95} = 7e^{-0.08t}$$

$$\Rightarrow \frac{5}{95} \times \frac{1}{7} = e^{-0.08t}$$

$$\Rightarrow t = \frac{\ln \frac{1}{133}}{-0.08} \approx 61 \text{ time units}$$

Name:________________
Student No.: ________________

7. Use the separation of variables technique to solve each differential equation.

(a) [4] $\frac{dy}{dx} = \frac{2xy^2}{1+x^2}$, $y(0) = -0.25$

$$\int y^{-2}\,dy = \int \frac{2x}{1+x^2}\,dx$$

$$\begin{cases} u = 1+x^2 \\ du = 2x\,dx \\ \Rightarrow \int \frac{2x}{1+x^2}\,dx = \int \frac{1}{u}\,du = \ln|u| + C \\ \qquad = \ln(1+x^2) + C \end{cases}$$

$$-y^{-1} = \ln(1+x^2) + C$$

$$\frac{1}{y} = -\ln(1+x^2) - C$$

$$\Rightarrow y = \frac{-1}{\ln(1+x^2) + C}$$

$$y(0) = -0.25 \Rightarrow -0.25 = \frac{-1}{\underbrace{\ln 1}_{=0} + C} \Rightarrow C = 4$$

$$\therefore y(t) = \frac{-1}{\ln(1+x^2) + 4}$$

(b) (i) [3] $\frac{dP}{dt} - 2P\sqrt{t} = 0$

$$\frac{dP}{dt} = 2P\sqrt{t}$$

$$\int \frac{1}{P}\,dP = \int 2t^{1/2}\,dt$$

$$\ln|P| = \frac{4}{3}t^{3/2} + C$$

$$|P| = e^C \cdot e^{\frac{4}{3}t^{3/2}}$$

$$P = \pm e^C \cdot e^{4/3\,t^{3/2}}$$

$$P = Ke^{4/3\,t^{3/2}} \quad \text{where } K = \pm e^C$$

(ii) [1] Are there any other solutions to this differential equation **not** covered by the equation you found in part (i)?

YES! The equilibrium sol^{n} $P=0$ is a sol^{n} to the original eq^{n} but cannot be obtained by the formula $P(t) = \pm e^C \cdot e^{4/3\,t^{3/2}}$.

Name:____________________
Student No.: ______________

8. The following pair of equations represent the population growth of two different species where one is the predator and the other is the prey.

$$\frac{dA}{dt} = 0.2A - 0.001AB$$

$$\frac{dB}{dt} = -0.05B + 0.00005AB$$

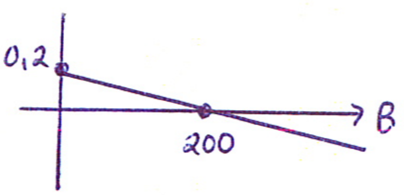
(a) [2] Which of the variables represents the predator population and which represents the prey population? Explain.

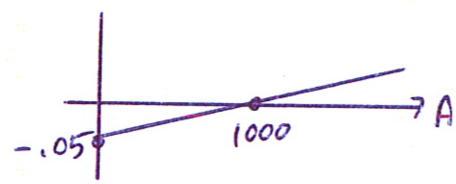
A represents the prey since when $B=0$, it will grow exponentially but when $B>0, A>0$, interactions decrease growth rate.

B represents the predator since it will die out when $A=0$ but benefits from interact^ns ($+.00005AB$)

(b) [3] Sketch the per capita production rates for both A and B. What are the equilibrium solutions for this system?

$$\frac{\frac{dA}{dt}}{A} = 0.2 - 0.001B \qquad \frac{\frac{dB}{dt}}{B} = -0.05 + 0.00005A$$





Eq^m Sol^ns:

$\begin{cases} A=0 \\ B=0 \end{cases}$

or

$\begin{cases} A=1000 \\ B=200 \end{cases}$

(c) [2] If $A_0 = 40$ and $B_0 = 15$, approximate the size of both populations after one year using Euler's method and a step size of 6 months. Here, t is measured in months.

$h=6$

$t_0=0$

$A_0=40$

$B_0=15$

$t_1=6$

$A_1 = 40 + [0.2(40) - .001(40)(15)]6 \approx 84.4$

$B_1 = 15 + [-0.05(15) + .00005(40)(15)]6 \approx 10.68$

$t_2=12$

$A_2 = 84.4 + [0.2(84.4) - .001(84.4)(10.68)]6 \approx 180$

$B_2 = 10.68 + [-.05(10.68) + .00005(84.4)(10.68)]6 \approx 7.7$

THE END