MATH 1B03/1ZC3 Winter 2019

Lecture 2: Simplifying matrices

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Covered in the previous lecture:

- Linear equations
- Systems of linear equations
- Solving systems of linear equations: three possibilities
  - A unique solution
  - An infinite number of solutions
  - No solutions
- Augmented matrices
- An example of how to solve a system by looking at its augmented matrix

Simplifying matrices

(corresponding to Chapter 1.2 of Anton-Rorres)

We have seen how a system of linear equations (henceforth abbreviated to SLE) can be solved using elementary row operations on its augmented matrix. But how do we know what operations to do, and what order to do them in? In this lecture we are going to see a systematic proceedure to simplify matrices and solve SLEs.

The augmented matrix of a large system

Consider the system of m equations in n variables

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

where a_{ij} is the coefficient of the variable x_j in the i-th equation. The augmented matrix of this system is

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

Simplified forms of matrices

In Example 1.10 (in Lecture 1) we simplified an augmented matrix to obtain

$$\begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \end{bmatrix}$$

This is very simple, and we could “read off" the solution to the associated SLE very easily. Matrices of this form are so important that they get special names.

Definition 2.1

A matrix is in row echelon form (REF) if

1. any all-zero rows are at the bottom of the matrix
2. the first non-zero entry (looking from left to right) in every row is a 1. These 1's are called leading 1's.
3. every leading 1 is to the right of the leading 1's above it

A matrix in REF typically looks like

$$\begin{bmatrix} 0 & 1 & * & * & * & * & * & * & * \\ 0 & 0 & 1 & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where $*$ can be any number.

Definition 2.2

A matrix is in reduced row echelon form (RREF) if

1. it is in REF
2. each leading 1 is the only non-zero entry in its column

A matrix in RREF typically looks like

$$\begin{bmatrix} 0 & 1 & 0 & * & * & 0 & * & * & * \\ 0 & 0 & 1 & * & * & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where $*$ can be any number.

Example 2.3

The following matrices are in REF but not in RREF

$$\begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The following matrices are in RREF

$$\begin{bmatrix} 1 & 3 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Question 2.4

Is the following matrix in REF, RREF, or neither?

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix}$$

Why is RREF useful?

If a matrix is in RREF we can quickly solve the associated SLE. In fact, the answer is staring us in the face! For example, take this matrix in RREF,

$$\begin{bmatrix} \boxed{1} & 2 & 0 & 2 & 5 \\ 0 & 0 & \boxed{1} & -1 & 10 \end{bmatrix}$$

The leading 1's are boxed. The variables corresponding to the leading 1's are known as <u>leading variables</u>. In this case, the SLE defined by the matrix is

$$x_1 + 2x_2 + 2x_4 = 5$$
$$x_3 - x_4 = 10$$

and the leading variables are x_1 and x_3.

The variables which are not leading variables are known as <u>free variables</u>: in this case they are x_2 and x_4.

To write down the solution to the SLE we convert the free variables to parameters, and write the leading variables in terms of these parameters. In this case, write the free variables as $x_2 = s$ and $x_4 = t$. The system becomes

$$\begin{aligned} x_1 + 2s + 2t &= 5 \quad \Rightarrow \quad x_1 = 5 - 2s - 2t \\ x_3 - t &= 10 \quad \Rightarrow \quad x_3 = 10 + t \end{aligned}$$

Therefore the solutions to the system are

$$(5 - 2s - 2t,\ s,\ 10 + t,\ t)$$

for any choice of s and t (i.e. they are arbitrary parameters). As s and t may take any value, we see that this SLE has an infinite number of solutions.

Recipe 2.5: Solving an SLE when matrix in RREF

Step (1): Identify the leading variables and the free variables.

Step (2): Convert the free variables to parameters. If there are more than two free variables, it's easiest to use the symbols $t_1, t_2, \ldots, t_n$ to denote these parameters.

Step (3): Rearrange the equations to express the leading variables in terms of the parameters. Then write down the solution, if one exists.

It is possible that every variable will be a leading variable. If so, we do not need any parameters, and the SLE has a unique solution.

Gaussian Elimination & Gauss-Jordan Elimination

Now that we know that we can use RREF to quickly solve SLEs, we would like a method of putting any matrix into RREF, using elementary row operations (remember that these operations do not change the solutions of the SLE). Gaussian elimination and Gauss-Jordan elimination are systematic ways of doing exactly this!

- Gaussian elimination puts a matrix into REF
- Gauss-Jordan elimination puts a matrix in to RREF

In fact, Gauss-Jordan elimination is simply adding an extra step to Gauss elimination. Here's how to do it.

Recipe 2.6: Gaussian Elimination & Gauss-Jordan Elimination

We shall illustrate the method by applying it to this matrix

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

Step (1): Locate the first column (from the left) with at least one non-zero entry.

$$\begin{bmatrix} \boxed{0} & 0 & -2 & 0 & 7 & 12 \\ \boxed{2} & 4 & -10 & 6 & 12 & 28 \\ \boxed{2} & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

Step (2): Swap rows (if necessary) to bring a non-zero entry to the top of this column.

$$\begin{bmatrix} 2 & 4 & -10 & 6 & 12 & 28 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix} \quad R1 \text{ and } R2 \text{ swapped}$$

Step (3): If the non-zero entry at the top of the column is c, multiply the top row of the matrix by $\frac{1}{c}$. This puts a 1 in the top entry of the column.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix} \quad R1 \text{ multiplied by } \tfrac{1}{2}$$

Step (4): Add suitable multiples of the top row to the other rows of the matrix so that all the other entries of the column are 0. When we are finished the only non-zero entry in the column will be the 1 at the top.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix} \quad -2 \times R1 \text{ added to } R3$$

Step (5): Cover the top row of the matrix, and repeat Steps (1) to (4) on the remaining part of the matrix. Keep doing this until you have covered the whole matrix.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix} \quad \text{Cover } R1$$

Multiply $R2$ by $-\frac{1}{2}$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -3.5 & -6 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

Add $-5 \times R2$ to $R3$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -3.5 & -6 \\ 0 & 0 & 0 & 0 & 0.5 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -3.5 & -6 \\ 0 & 0 & 0 & 0 & 0.5 & 1 \end{bmatrix} \quad R2 \text{ is finished, so cover it}$$

Multiply $R3$ by 2

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -3.5 & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$R3$ is finished

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -3.5 & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \quad \text{Matrix is in REF!}$$

The matrix is now in REF. If we stop here, we have done **Gaussian elimination**. If we keep going and put the matrix into RREF, we will have done **Gauss-Jordan elimination**.

Step (6): Find the last non-zero row from the top i.e. the last row which is not a row of all 0's. Use this row to remove all the non-zero entries above its leading 1. Repeat this for every row.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -3.5 & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

last non-zero row

$$\rightsquigarrow$$

Add $\frac{7}{2} \times R3$ to $R2$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\rightsquigarrow$$

Add $-6 \times R3$ to $R1$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\rightsquigarrow$$

Add $5 \times R2$ to $R1$

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 7 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

In summary: **Gaussian elimination** is **Steps (1) to (5)**, and puts a matrix into **REF**.
Gauss-Jordan elimination is **Steps (1) to (6)**, and puts a matrix into **RREF**.

Example 2.7

Question: Solve the SLE

$$\begin{aligned} x_1 + x_2 &= 2 \\ -x_1 + x_2 &= 6 \end{aligned}$$

Answer: First, form the augmented matrix

$$\begin{bmatrix} 1 & 1 & 2 \\ -1 & 1 & 6 \end{bmatrix}$$

Then put the matrix into RREF using Gauss-Jordan elimination. The top entry of the first non-zero column of this matrix is already 1 (it is the top-left entry of the matrix), so we can start at Step (4):

$$
\begin{array}{cl}
\begin{bmatrix} 1 & 1 & 2 \\ -1 & 1 & 6 \end{bmatrix} & \\
\wr\!\downarrow & \text{Add } R1 \text{ to } R2 \\
\begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 8 \end{bmatrix} & \\
\wr\!\downarrow & \text{Multiply } R2 \text{ by } \tfrac{1}{2} \\
\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix} & \\
\wr\!\downarrow & \text{Add } -1 \times R2 \text{ to } R1 \\
\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \end{bmatrix} &
\end{array}
$$

The matrix is now in RREF. Both variables are leading, and the SLE is

$$
\begin{aligned}
x_1 + 0x_2 &= -2 \\
0x_1 + x_2 &= 2
\end{aligned}
$$

so that the unique solution is $(-2, 2)$.

Question: Solve the SLE

$$
\begin{aligned}
2x + 4y &= 2 \\
x + y &= 2 \\
x - y &= 1
\end{aligned}
$$

Answer: The augmented matrix is

$$
\begin{bmatrix} 2 & 4 & 2 \\ 1 & 1 & 2 \\ 1 & -1 & 2 \end{bmatrix}
$$

Apply Gauss-Jordan elimination:

$$
\begin{bmatrix} 2 & 4 & 2 \\ 1 & 1 & 2 \\ 1 & -1 & 2 \end{bmatrix}
\rightsquigarrow \text{Multiply } R1 \text{ by } \tfrac{1}{2}
$$

$$
\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 1 & -1 & 2 \end{bmatrix}
\rightsquigarrow \text{Add } -1 \times R1 \text{ to } R2 \text{ and to } R3
$$

$$
\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 1 & -3 & 0 \end{bmatrix}
\rightsquigarrow \text{Multiply } R2 \text{ by } -1
$$

$$
\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & -3 & 0 \end{bmatrix}
\rightsquigarrow \text{Add } 3 \times R2 \text{ to } R3
$$

$$
\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -3 \end{bmatrix}
$$

We have a problem! The row $0 \quad 0 \quad -3$ corresponds to the equation

$$0x + 0y = -3.$$

But $0 \neq 3$: this means the SLE has no solutions.

As we saw in the previous example, rows which are 0 everywhere except for the right-most entry are important for us: they tell us that the SLE has no solutions. We can use the RREF of a matrix to deduce how many solutions the associated SLE has.

Fact 2.8

There are exactly three possibilities for an SLE

1. a unique solution: if every variable is a leading variable in the RREF of

the augmented matrix

2. infinitely many solutions: if there is at least one free variable in the RREF of the augmented matrix

3. no solutions: if a row

$$0 \quad 0 \quad \cdots \quad 0 \quad c$$

appears during Gauss-Jordan elimination, for c a non-zero constant.

There is a special kind of SLE for which it is easier to deduce the number of solutions.

Definition 2.9

Recall that a linear equation is homogeneous if the right hand side is 0 i.e.

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = 0.$$

A SLE is homogeneous if all of its equations are homogeneous. That is, if the SLE has m equations in n variables, then

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m1}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

An homogeneous SLE always has at least one solution: the trivial solution

$$(0, 0, \ldots, 0).$$

This makes it easier to determine how many solutions a homogeneous SLE has.

Fact 2.10

An homogeneous SLE which has more variables than equations has an infinite number of solutions.

Suggested problems

Practice the material in this lecture by attempting the following problems in **Chapter 1.2** of Anton-Rorres, starting on page 22

- Questions 1, 3, 5, 7, 25
- True/False questions (f), (h), (i)