MATHEMATICS 1LT3E TEST 3

Evening Class
Duration of Test: 75 minutes
McMaster University, 20 July 2011

E. Clements

$\overline{}$
50

FIRST NAME (please print) : SOL^{NS}
FAMILY NAME (please print) :
Student No.:

THIS TEST HAS 10 PAGES AND 9 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE.

Total number of points is 50. Marks are indicated next to the problem number in square brackets. Any Casio fx991 (or lower, non-graphing) calculator is allowed.

USE PEN TO WRITE YOUR TEST. IF YOU USE A PENCIL, YOUR TEST WILL NOT BE ACCEPTED FOR REMARKING (IF NEEDED).

You need to show work to receive full credit.

---

1. The following pair of equations represent the population growth of two different species where one is the predator and the other is the prey.

$$\frac{dA}{dt} = 0.1A - 0.005AB$$
$$\frac{dB}{dt} = -0.05B + 0.0001AB$$

[2] (a) Which of the variables, A or B, represents the predator population and which represents the prey population? Explain.

A represents the prey because if $B=0$, $A'=0.1A$ and A will grow exponentially. Also, the interact^{n} term "$-0.005AB$" is negative + we know interact^{n}s are bad for prey.
B represents the predator $\because$ the interact^{n} term "$.0001AB$" is positive (interact^{n}s are good for predators) and in the absence of prey $(A=0)$, $B'=-.05B \Rightarrow$ pop^{n} will die out.

[2] (b) Determine the equilibrium solutions.

$$\frac{dA}{dt}=0 \ \& \ \frac{dB}{dt}=0 \Rightarrow \begin{cases} A(0.1-0.005B)=0 \\ B(-0.05+.0001A)=0 \end{cases}$$

$$\Rightarrow \begin{cases} A=0 \text{ or } B=20 \\ B=0 \text{ or } A=500 \end{cases}$$

$\therefore$ The eq^{M} sol^{N}s are $A=0$ & $B=0$ or $A=500$ & $B=20$.

1. continued...

[3] (c) If $A_0 = 40$ and $B_0 = 15$, approximate the size of both populations after one year using Euler's method and a step size of 4 months. Here, t is measured in months.

$t_1 = 4$

$A_1 = 40 + [.1(40) - .005(40)(15)]4 = 44$

$B_1 = 15 + [-.05(15) + .0001(40)(15)]4 = 12.24$

$t_2 = 8$

$A_2 = 44 + [.1(44) - .005(44)(12.24)]4 \approx 50.8$

$B_2 = 12.24 + [-.05(12.24) + .0001(44)(12.24)]4 \approx 10$

$t_3 = 12$

$A_3 = 50.8 + [.1(50.8) - .005(50.8)(10)]4 \approx 61$

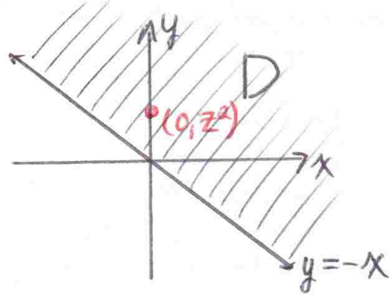
$B_3 = 10 + [-.05(10) + .0001(10)(50.8)]4 \approx 8$

$\therefore$ After 1 year, pop^N A will consist of approximately 61 individuals + pop^N B approximately 8 individuals.

2. Consider the function $f(x, y) = \sqrt{x + y}$.

[2] (a) Find and sketch the domain of f.

$x + y \geq 0 \Rightarrow y \geq -x$



[2] (b) Show that the range of f is $z \geq 0$. (i.e. given any $z \geq 0$, find a point (x, y) in the domain of f such that $f(x, y) = z$).

set $\sqrt{x+y} = z$ (where $z \geq 0$)

$x + y = z^2$

$y = z^2 - x$

let $x = 0$. Then $y = z^2$.

Note: $(0, z^2) \in$ domain(f) (see pic in part (a)).

Let $z \geq 0$.

Take $x = 0$ and $y = z^2$.

Note: $(0, z^2) \in$ domain(f).

$f(0, z^2) = \sqrt{0 + z^2}$

$= \sqrt{z^2}$

$= |z|$

$= z \quad \because z \geq 0$.

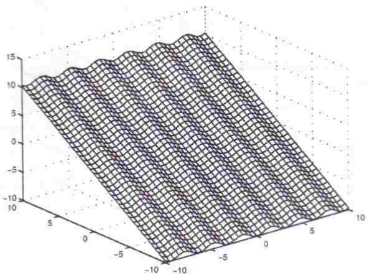
$\therefore$ Range of f is $z \geq 0$.

3. [8] Match the function (a) with its graph (labeled A-D) and (b) with its contour map (labeled I-IV).

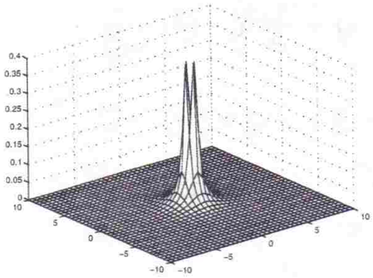
$z = x - y^2 + 4xy^2$ (a) D (b) III $z = \sin^2 x + y$ (a) A (b) IV

$z = \dfrac{1}{x^2 + y^2}$ (a) B (b) I $z = x^2 - y^2$ (a) C (b) II

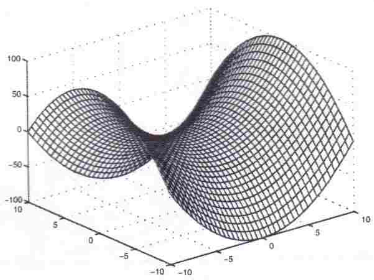
A



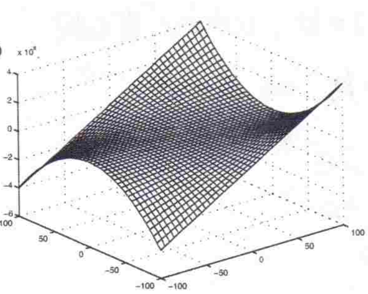
B



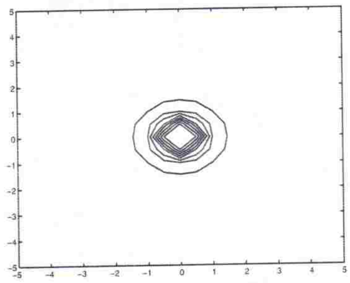
C



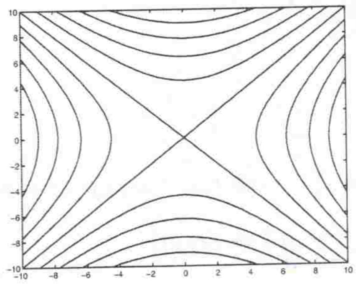
D



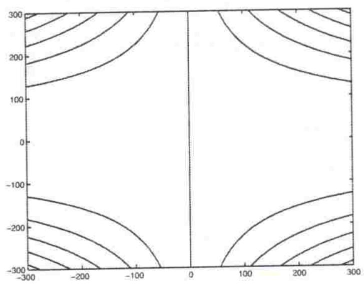
I



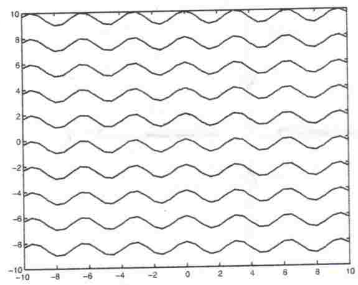
II



III



IV



4. Create a contour map for the following functions. Include atleast 5 level curves.

[2] (a) $f(x, y) = x - 2y - 6$.

set $f(x,y) = k$ where $k \in \mathbb{R}$.

$x - 2y - 6 = k$

$\Rightarrow -2y = -x + 6 + k$

$\Rightarrow y = \frac{1}{2}x - 3 - \frac{k}{2}$

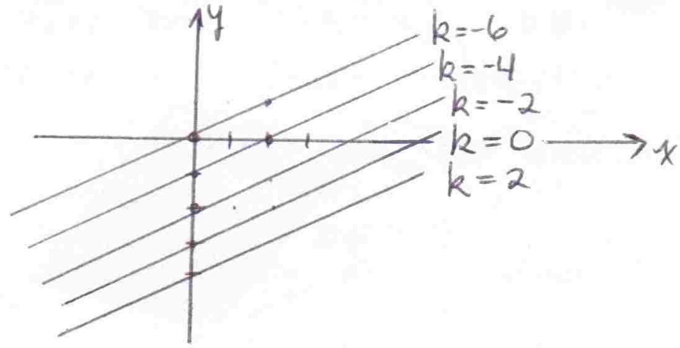
Choose $k = -2$: $y = \frac{1}{2}x - 2$

$k = 0$: $y = \frac{1}{2}x - 3$

$k = 2$: $y = \frac{1}{2}x - 4$

Choose $k = -4$: $y = \frac{1}{2}x - 1$

$k = -6$: $y = \frac{1}{2}x$



[3] (b) $f(x, y) = ye^{x}$

set $f(x,y) = k$, where $k \in \mathbb{R}$

$ye^{x} = k \Rightarrow y = ke^{-x}$

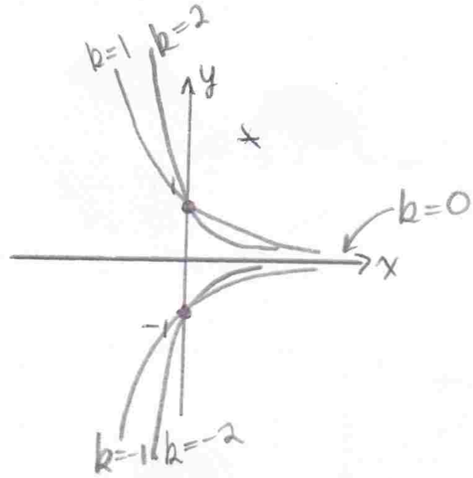
$k = -2$: $y = -2e^{-x}$

$k = -1$: $y = -1e^{-x}$

$k = 0$: $y = 0$

$k = 1$: $y = e^{-x}$

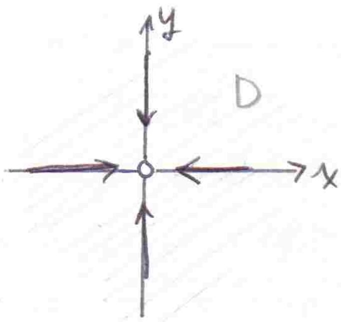
$k = 2$: $y = 2e^{-x}$



5. Consider $\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$.

[1] (a) Find and sketch the domain of this function.

$(x,y) \neq (0,0)$



5. continued...

[3] (b) Show that $\lim_{(x,y)\to(0,0)} \overbrace{\frac{x^2-y^2}{x^2+y^2}}^{f}$ does not exist. Draw the paths you used to approach $(0,0)$ on your domain picture in part (a).

$f(0,y) = \frac{-y^2}{y^2} = -1$ so $f(x,y) \to -1$ as $(x,y) \to (0,0)$ along $x=0$

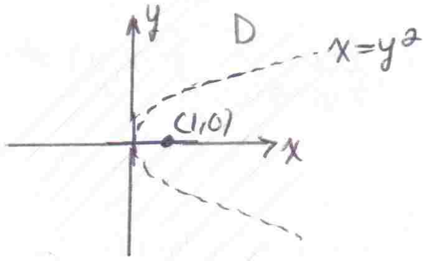
$f(x,0) = \frac{x^2}{x^2} = 1$ so $f(x,y) \to 1$ as $(x,y) \to (0,0)$ along $y=0$

$\therefore \lim_{(x,y)\to(0,0)} f(x,y)$ D.N.E.

6. Let $f(x,y) = \frac{\sin y}{x-y^2}$.

[2] (a) Find and sketch the largest domain on which f is continuous.

$x - y^2 \neq 0 \to x \neq y^2$



[1] (b) Evaluate $\lim_{(x,y)\to(1,0)} \frac{\sin y}{x-y^2}$.

$\because$ f is continuous at (1,0) $\therefore$ we can evaluate simply by direct substitution.

$\lim_{(x,y)\to(1,0)} \frac{\sin y}{x-y^2} = \frac{\sin 0}{1-0^2} = 0$

7. [3] (a) Explain in your own words what is meant by the expression $\lim_{(x,y)\to(a,b)} f(x,y) = L$. What must L be in order for $f(x,y)$ to be continous at (a,b)?

This expression means the value of the f^N (z-values) approach L as (x,y) approaches (a,b) along ANY path in the domain of f.

If f is continuous at (a,b), then $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

so, $L = f(a,b)$.

[3] (b) Using the definition of continuity, show that the function

$$f(x,y) = \begin{cases} 4 + x^2 + y^2 & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

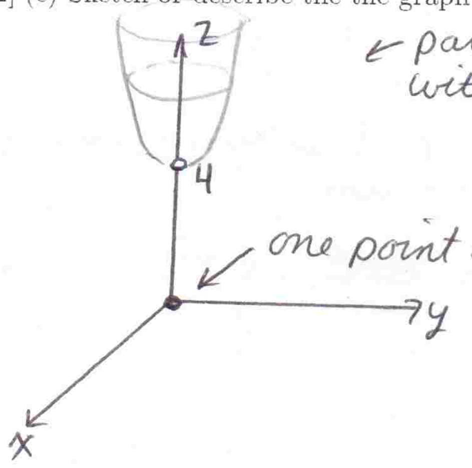
is **not** continuous at $(0,0)$.

$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} (4+x^2+y^2) = 4$

$f(0,0) = 0$

$\because \lim_{(x,y)\to(0,0)} f(x,y) \neq f(0,0)$ $\therefore$ f is not continuous at $(0,0)$

[2] (c) Sketch or describe the the graph of this function.



← paraboloid opening upwards with a hole at $(0,0,4)$

one point at $(0,0,0)$.

8. Find the indicated partial derivatives.

[3] (a) $f(x,y) = \dfrac{4x - xy}{x^2 + y^2}$; $f_y(0,1)$

$$f_y = \frac{-x(x^2+y^2) - (4x-xy)(2y)}{(x^2+y^2)^2}$$

$$f_y(0,1) = 0$$

[3] (b) $f(x,y) = x^{-3}\ln(x^2+y)$; $f_x(1,1)$

$$f_x = -3x^{-4} \cdot \ln(x^2+y) + x^{-3} \cdot \frac{1}{x^2+y}(2x)$$

$$f_x(1,1) = -3\ln 2 + 1$$

[2] (c) $f(x,y) = \dfrac{1}{4y}e^{-\frac{x^2}{y}}$; $f_x(0,1)$

$$f_x = \frac{1}{4y} e^{-x^2/y}\left(-\frac{2x}{y}\right)$$

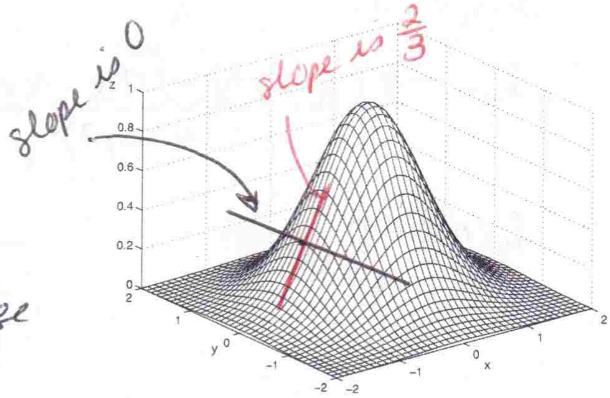
$$f_x(0,1) = 0$$

9. [3] Compute the partial derivatives, f_x and f_y, of the function $f(x, y) = e^{-x^2-y^2}$ at the point $(-1, 0)$. Interpret these numbers geometrically and illustrate on the graph of the function provided.

$f_x = e^{-x^2-y^2}(-2x)$

$f_x(-1,0) = 2e^{-1} = \frac{2}{e}$

This represents the rate of change of f in the x-direct^N at (-1,0). It is the slope of the tangent to the curve f(x,0) at (-1,0).



$f_y = e^{-x^2-y^2}(-2y)$

$f_y(-1,0) = 0$

This is the slope of the tangent to the curve f(-1,y) at (-1,0). It represents the rate of change of f in the y-direct^N at (-1,0).