McMaster University
Math 1LT3E

ASSIGNMENT 5

Sections 3, 4, and 5 in the Red Module

1. (a) In your own words, explain what is meant by $\lim_{(x,y)\to(a,b)} f(x,y) = L$.

The z-values approach L more and more closely as (x, y) approaches (a, b) more and more closely along every path in the domain of f.

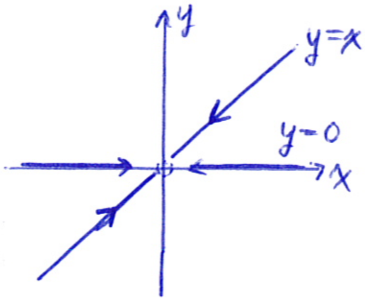
(b) Explain how you would show that $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist.

you must find two paths P_1 and P_2 in the domain of f such that

$$\lim_{(x,y)\to(a,b)} f(x,y) \text{ along } P_1 \neq \lim_{(x,y)\to(a,b)} f(x,y) \text{ along } P_2$$

2. Show that the following limits do not exist. Sketch the domains and paths involved.

(a) $\lim_{(x,y)\to(0,0)} \underbrace{\frac{(x-y)^2}{x^2+y^2}}_{f}$



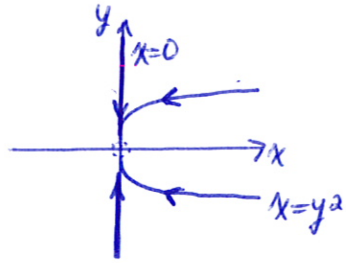
domain : $\mathbb{R}^2 \setminus \{(0,0)\}$

Along $y=0$:
$f(x,0) = \frac{x^2}{x^2} = 1 \Rightarrow f(x,y) \to 1$ as $(x,y)\to(0,0)$ along $y=0$

Along $y=x$:
$f(x,x) = \frac{0^2}{2x^2} = 0 \Rightarrow f(x,y)\to 0$ as $(x,y)\to(0,0)$ along $y=x$

$\therefore \lim_{(x,y)\to(0,0)} f(x,y)$ D.N.E.

(b) $\lim_{(x,y)\to(0,0)} \underbrace{\frac{2xy^2}{x^2+y^4}}_{f}$



domain : $\mathbb{R}^2 \setminus \{(0,0)\}$

Along $x=0$:
$f(0,y) = \frac{0}{y^4} = 0 \Rightarrow f(x,y)\to 0$ as $(x,y)\to(0,0)$ along $x=0$

Along $x=y^2$:
$f(y^2,y) = \frac{2y^2\cdot y^2}{(y^2)^2+y^4} = \frac{2y^4}{2y^4} = 1 \Rightarrow f(x,y)\to 1$ as $(x,y)\to(0,0)$ along $x=y^2$

$\therefore \lim_{(x,y)\to(0,0)} f(x,y)$ D.N.E.

3. (a) Explain what you would have to show in order to prove that a function $f(x, y)$ is continuous at (a, b).

you would have to show that the limit of the function f as (x,y) approaches (a,b) exists and is equal to the value of the function at (a,b), ie,

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

(b) Find a function g such that $\lim_{(x,y)\to(5,4)} g(x, y)$ exists but g is not continuous at $(5, 4)$.

Many possible answers... here's one:

$$g(x,y) = \begin{cases} x+y & \text{if } (x,y) \neq (5,4) \\ 10 & \text{if } (x,y) = (5,4) \end{cases}$$

so, $\lim_{(x,y)\to(5,4)} g(x,y) = \lim_{(x,y)\to(5,4)} (x+y) = 9$ (limit exists)

but $g(5,4) = 10 \Rightarrow$ by the def[n] of continuity, g is not continuous at $(5,4)$.

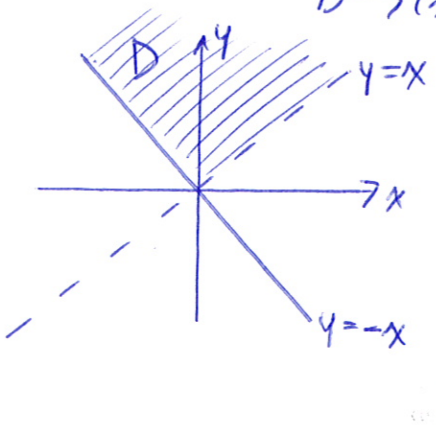
(c) Find and sketch the largest domain on which $z = \ln(y - x) + \sqrt{y + x}$ is continuous.

$y - x > 0$ AND $y + x \geq 0$

$\Rightarrow$ $y > x$ AND $y \geq -x$

Since z is a combination of continuous functions, it is continuous on its domain, ie, continuous on

$$D = \{(x,y) \in \mathbb{R}^2 \mid y > x \text{ and } y \geq -x\}$$



4. Use the definition of continuity to show that

$$h(x,y) = \begin{cases} 4 - e^{-x-y+2} & \text{if } (x,y) \neq (1,1) \\ 3 & \text{if } (x,y) = (1,1) \end{cases}$$

is continuous at $(1,1)$.

① $\lim_{(x,y)\to(1,1)} h(x,y) = \lim_{(x,y)\to(1,1)} (4 - e^{-x-y+2}) = 4 - e^{0} = 3$

② $h(1,1) = 3$

③ Since $\lim_{(x,y)\to(1,1)} h(x,y) = 3 = h(1,1)$, by the def^{n} of continuity, h is continuous at $(1,1)$.

5. Assume that the function $T(x,y,t)$ models the temperature (in degrees Celsius) at time t in a city located at a longitude of x degrees and a latitude of y degrees. The time t is measured in hours. What is the meaning of the partial derivative $T_t(x,y,t)$? What are its units? What is most likely going to be the sign of $T_y(x,y,t)$ for Winnipeg, Manitoba in January?

$T_t = \frac{dT}{dt}$ measures the rate of change in temperature with respect to time.

UNITS: $°C/\text{hour}$

$T_y = \frac{dT}{dy}$ measures the rate of change in temperature with respect to. an increase in latitude (heading north). Since temperature is likely decreasing as we move north from Winnipeg in January, the sign of this partial derivative is most likely negative.

6. Below is an excerpt from a table of values of I, the temperature-humidity index, which is the perceived air temperature when the actual temperature is T (degrees fahrenheit), and the relative humidity is h (percent).

$T \downarrow$ $h \rightarrow$	20	30	40	50	60	70
80	74	76	78	82	83	86
85	81	82	84	86	90	94
90	86	90	93	96	101	106
95	94	94	98	107	111	125
100	99	101	109	122	129	138

(a) Write the definition (equation) of the partial derivative of $I(T, h)$ with respect to h.

$$\frac{\partial I}{\partial h} = \lim_{\Delta h \to 0} \frac{I(T, h+\Delta h) - I(T,h)}{\Delta h}$$

(b) Approximate $I_h(95, 40)$ and interpret your answer, i.e., write a statement to explain what this number represents.

$$\frac{\partial I}{\partial h}(95,40) \approx \frac{I(95, 40+\Delta h) - I(95,40)}{\Delta h}$$

take $\Delta h = 10\%$: $\frac{\partial I}{\partial h}(95,40) \approx \frac{I(95,50) - I(95,40)}{10} \approx \frac{107-98}{10} \approx 0.9$

take $\Delta h = -10\%$: $\frac{\partial I}{\partial h}(95,40) \approx \frac{I(95,30) - I(95,40)}{-10} \approx \frac{94-98}{-10} \approx 0.4$

avg $= \frac{0.9+0.4}{2} = 0.65$ humidex points/% humidity

This partial derivative tells us that when temperature is 95°F and humidity is 40%, the humidex is INCREASING at a rate of approximately 0.65 humidex points per 1% increase in humidity.

7. Compute the indicated partial derivatives.

(a) $f(x,y) = \dfrac{4x - xy}{x^2 + y^2}$; $f_x(x,y)$

$$= \frac{(4-y)x}{x^2+y^2}$$

$$f_x = \frac{(4-y)(x^2+y^2) - (4-y)x \cdot 2x}{(x^2+y^2)^2}$$

$$= \frac{(4-y)(x^2+y^2-2x^2)}{(x^2+y^2)^2}$$

$$= \frac{(4-y)(y^2-x^2)}{(x^2+y^2)^2}$$

(b) $h(x,t) = e^{\sqrt{x-4t^2}}$; $h_t(5,1)$

$$h_t = e^{\sqrt{x-4t^2}} \cdot \frac{1}{2\sqrt{x-4t^2}} \cdot -8t$$

$$h_t(5,1) = e^{\sqrt{1}} \cdot \frac{1}{2\sqrt{1}} \cdot -8(1) = -4e$$

8. A hiker is standing at the point (2, 1, 21) on a hill whose shape is given by the graph of the function $f(x, y) = 24 - (x-3)^2 - 2(y-2)^2$.

(a) In which of the two directions (x-direction or y-direction) is the hill steeper?

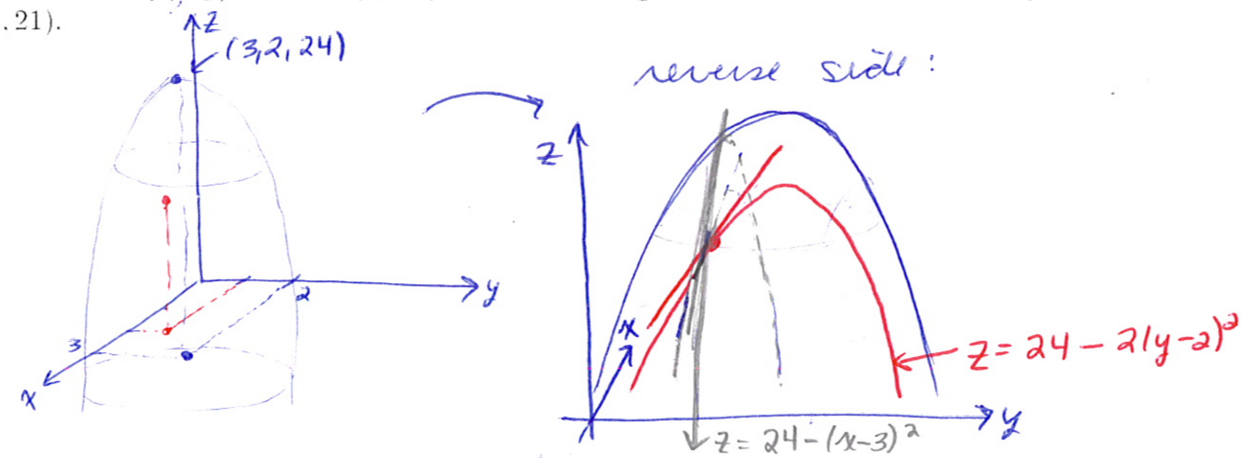
$f_x = -2(x-3) \ldots \quad f_x(2,1) = -2(2-3) = 2$

$f_y = -4(y-2) \ldots \quad f_y(2,1) = -4(1-2) = 4$

Since f_y and f_x represent the slope of the mountain in the y and x directions, respectively, the hill is steeper in the y-direction at (2,1,21) since

$$f_y(2,1) = 4 > 2 = f_x(2,1)$$

(b) Sketch a graph of the function $f(x, y) = 24 - (x-3)^2 - 2(y-2)^2$. On the graph, draw the curves $z = f(2, y)$ and $z = f(x, 1)$. Draw the tangent line to each curve at the point (2, 1, 21).



(c) For what x- and y-coordinates will the hiker reach the top of the hill? What are the values of f_x and f_y at this point?

$$z = 24 - [\underbrace{(x-3)^2 + 2(y-2)^2}_{\geq 0}]$$

so, $\underline{z \leq 24}$

The top of the hill corresponds to the maximum of the function which is $z = 24$. This maximum value is attained when $x = 3$ and $y = 2$.
The partial derivatives are both zero here.

9. Let $f(x, y) = \ln(3x - y + 1)$.

(a) Compute the partial derivatives of f.

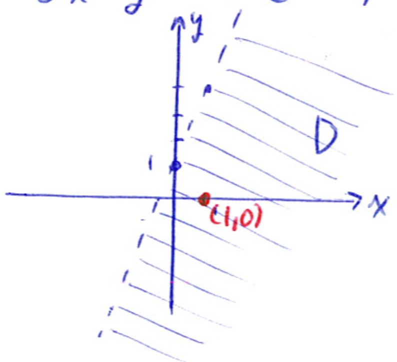
$$f_x = \frac{1}{3x - y + 1}(3)$$

$$f_y = \frac{1}{3x - y + 1}(-1)$$

(b) Find and sketch the domains of f_x and f_y.

* recall : domain of any derivative of $f \subseteq$ domain of f

so, $3x - y + 1 > 0 \Rightarrow y < 3x + 1$



(c) Is f differentiable at $(1, 0)$? Explain.

YES! Since f_x and f_y are continuous on their domain and $B, (1,0) \in$ domain of f_x and f_y, $f(x,y)$ is differentiable at $(1,0)$ (by Thm).

10. Consider the function $f(x, y) = \sqrt{y + \cos^2 x}$.

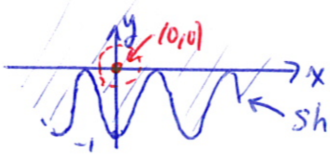
(a) Show that the function is differentiable at $(0, 0)$.

$$f_x = \frac{1}{2\sqrt{y+\cos^2 x}}(2\cos x(-\sin x)) = \frac{-\sin 2x}{2\sqrt{y+\cos^2 x}}$$

$$f_y = \frac{1}{2\sqrt{y+\cos^2 x}}(1)$$

domain of f_x & f_y: $y + \cos^2 x > 0 \Rightarrow y > -\cos^2 x$

There exists some small number $\varepsilon > 0$ such that $B_\varepsilon(0,0)$ is inside the domains of f_x and f_y. Since f_x and f_y are continuous on their domains, they are continuous on $B_\varepsilon(0,0)$
$\Rightarrow f$ is differentiable at $(0,0)$.



(b) Verify the linear approximation $\sqrt{y + \cos^2 x} \approx 1 + \frac{1}{2}y$ at (0.0).

Since f is differentiable at $(0,0)$, this approximation is valid if $1 + \frac{1}{2}y$ is the linearization of f at $(0,0)$.

$$L_{(0,0)}(x,y) = f(0,0) + \overbrace{f_x(0,0)}^{=0}(x-0) + f_y(0,0)(y-0)$$

$$= \sqrt{0 + \underbrace{\cos^2 0}_{=1}} - \frac{\overbrace{\sin 2(0)}}{2\sqrt{0+\cos^2 0}}x + \frac{1}{2\underbrace{\sqrt{0+\cos^2 0}}_{=1}}y$$

$$= 1 + \frac{1}{2}y \checkmark$$

So then this approximation is valid.

THE END