12A3

Techniques of Integration

1) Don't forget the basics!

$$\int x^p\,dx = \frac{x^{p+1}}{p+1} + C, \quad p \neq -1$$

$$\int \frac{1}{\text{mowa}}\,d\,\text{mowa} = \ln|\text{mowa}| + C.$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \cos x\,dx = \sin x + C.$$

2) Forgotten basics

$$\int \csc^2 x \, dx = -\cot x + C.$$

$$\int \csc x \cot x \, dx = -\csc x + C$$

$$\int \frac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1}(x) + C.$$

$$\int 10^x \, dx = \frac{10^x}{\ln 10} + C$$

just a #

$$\int \pi^e \, dx = \pi^e \cdot x + C$$

Substitution Tricks

let $F(x) = \int f(x)\,dx$

eg $\int f(x+k)\,dx = \int f(u)\,du = F(u) + C$

$= F(x+k) + C$

let $u = x+k$ $\quad du = 1\,dx$

eg $\int \cos(x + \ln 7)\,dx = \sin(x + \ln 7) + C$.

$\uparrow$ Just a const

Rule $\int f(x+k)\,dx = F(x+k) + C$.

a. $\int e^{7x}\,dx = \int e^{u}\cdot\frac{1}{7}\,du = \frac{1}{7}e^{u}+C$

$\uparrow u=7x \quad du=7dx, \quad dx=\frac{1}{7}du$

$= \frac{1}{7}e^{7x}+C.$

rule $\quad \int f(kx)\,dx = \frac{1}{k}F(kx)+C$

Subtle Substitution

eg $\quad \int \frac{x^2+2x+5}{\sqrt{x+2}}\,dx \qquad$ try $u=x+2$, $du=dx$

$x = u - 2$

$$\int \frac{(u-2)^2 + 2u - 4 + 5}{u^{1/2}} \, du$$

$$= \int \frac{u^2 - 2u + 5}{u^{1/2}} \, du$$

$$= \int u^{3/2} - 2u^{1/2} + 5u^{-\frac{1}{2}} \, du$$

power rule this $\uparrow$.

<u>etc.</u>

eg. $$\int \sqrt{\frac{1-x}{1+x}}\, dx \qquad 0 < x < 1$$

$$= \int \sqrt{\frac{1-x}{1+x} \cdot \frac{(1-x)}{(1-x)}}\, dx$$

$$= \int \frac{\sqrt{(1-x)^2}}{\sqrt{1-x^2}}\, dx = \int \frac{1-x}{\sqrt{1-x^2}}\, dx$$

$$= \int \frac{1}{\sqrt{1-x^2}}\, dx + \int \frac{(-1)\, x}{\sqrt{1-x^2}}\, dx.$$

$$= \sin^{-1}(x) + \frac{1}{2} \int u^{-\frac{1}{2}}\, du \qquad \underline{\underline{u = 1 - x^2}}$$

$$= \sin^{-1}(x) + \cancel{\tfrac{1}{2}}\, \cancel{2}\, u^{1/2} + C.$$

$$= \sin^{-1}(x) + \sqrt{1-x^2} + C$$

<u>In general</u> Always look for a sub. first!

<u>Especially</u> if you expect it's an <u>Int. by Parts</u>!

90% of all I by P, after substitution become.

$$\int x^n e^x \, dx, \quad \int x^n \cos x \, dx, \quad \int x^n \ln x \, dx.$$

or $\sin x$

or $\sinh x$

eg.

$$\int e^{\cos^2 x} \cdot \cos^3 x \sin x \, dx$$

$u = \cos x \qquad -du = + \sin x \, dx$

$$= -\int e^{u^2} u^3 \, du \qquad w = u^2$$

$$= -\int e^{\cancel{u^2} w} \cdot \cancel{u^2}^{\,w} \cdot \cancel{u\,du}^{\,\frac{1}{2}dw}$$

$$= -\frac{1}{2} \int w e^w \, dw$$

$$\boxed{\int u\,dv = uv - \int v\,du}$$

$u = w$

$dv = e^w \, dw$

$$= \frac{1}{2} \left[w e^w - \int 1 e^w \, dw \right]$$

$$= \frac{1}{2} \left(w e^w - e^w \right) + C$$

$$\frac{1}{2}\left(\cos^2 x\, e^{\cos^2 x} - e^{\cos^2 x}\right) + C$$

$$w = u^2 = \cos^2 x$$

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9. $$\int \frac{(\ln x)^7}{x}\,dx = \int \frac{u^7}{x}\cdot x\,du$$

$u = \ln x,\quad du = \frac{1}{x}dx$

$x\,du = dx$

$$= \int u^7\,du = \frac{1}{8}u^8 + C.$$

$$= \frac{1}{8}(\ln x)^8 + C.$$

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1) $$\int \frac{(\ln x)}{x^7}\,dx = \int x^{-7}\ln x\,dx$$ one of 3 standard I by P!

$$= \int u\,dv$$

$$= uv - \int v\,du$$

$$= (\ln x)\left(\frac{-1}{6x^6}\right) + \int \frac{1}{6x^6}\cdot\frac{1}{x}\,dx$$

$$v = \int x^{-7}\,dx$$

$$= \frac{x^{-6}}{-6}$$

$$= -\frac{\ln x}{6x^6} + \frac{1}{6}\int \frac{1}{x^7}\,dx$$

$$= -\frac{\ln x}{6x^6} - \frac{1}{36}\cdot\frac{1}{x^6} + C$$

$$= \frac{-1}{6x^6}\left(\ln x - \frac{1}{6}\right) + C$$

Mostly. New techniques have easy checklists!

But be careful!

classic P.F.

$$\int \frac{1}{x^3+4x}\,dx = \int \frac{1}{x(x^2+4)}\,dx$$

$\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$

here $a = 2$

* Get A, B, C *

$$= \int \frac{A}{x} + \frac{Bx+C}{x^2+4}\,dx$$

$$= A\ln|x| + \int \frac{B\,x}{x^2+4}\,dx + C\int\frac{1}{x^2+4}\,dx$$

$$= A\ln|x| + B\cdot\frac{1}{2}\ln(x^2+4) + \frac{C}{2}\tan^{-1}\left(\frac{x}{2}\right) + \cancel{C}$$

eg $$\int \frac{1}{x^4+2x^2+1}\,dx = \int \frac{1}{(x^2+1)^2}\,dx$$

↑ already in "PF" form! Can't break up!

Let $x = \tan t$ (trig sub!)

$dx = \sec^2 t\,dt$

$$= \int \frac{1}{(\tan^2 t+1)^2}\cdot \sec^2 t\,dt$$

$$= \int \frac{\sec^2 t}{\sec^4 t}\,dt = \int \frac{1}{\sec^2 t}\,dt$$

$$= \int \cos^2 t\,dt = \frac{1}{2}\int 1+\cos(2t)\,dt$$

etc.

4.

$$\int_{-1}^{1} \frac{x^3 + x}{x^4 + 7x^6}\, dx = 0$$

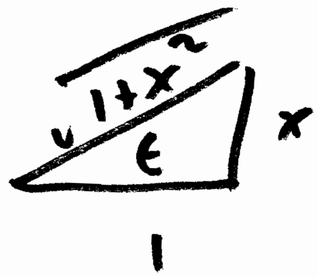
Symmetry

odd

$$= \frac{1}{2}t + \frac{1}{2}\int \cos(2t)\,dt \longrightarrow \frac{1}{2}\sin(2t) = \cancel{\tfrac{1}{2}}\cdot\cancel{2}\sin t\cos t$$

$$= \frac{1}{2}t + \frac{1}{2}\cos t\sin t + C$$

$x = \tan t$



$$= \frac{1}{2}\tan^{-1}x + \frac{1}{2}\,\frac{1\cdot x}{\sqrt{1+x^2}^{\,2}} + C$$

$$= \frac{1}{2}\tan^{-1}(x) + \frac{x}{2\sqrt{1+x^2}} + C$$

4. $$\int_{-1}^{1} \frac{x^3+x}{x^4+7x^6}\,dx = 0$$

odd

Symmetric