Day Class
Duration of Test: 60 minutes
McMaster University

E. Clements

13 February 2013

$\frac{\quad}{40}$

FIRST NAME (please print) : SOL^NS

FAMILY NAME (please print) :

Student No.:

THIS TEST HAS 7 PAGES AND 8 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE.

Total number of points is 40. Marks are indicated next to the problem number in square brackets. Any Casio fx991 (or lower, non-graphing) calculator is allowed.

USE PEN TO WRITE YOUR TEST. IF YOU USE A PENCIL, YOUR TEST WILL NOT BE ACCEPTED FOR REMARKING (IF NEEDED).

You need to show work to receive full credit, except for Multiple Choice.

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1. State whether each statement is **true or false** and then **explain** your reasoning.

(a) [2] The range of $y = 2\arctan x$ is $(0, \pi)$. True or false? Explain.

FALSE.

$-\frac{\pi}{2} < \arctan x < \frac{\pi}{2}$ so $-\pi < 2\arctan x < \pi$

ie the range of $y = 2\arctan x$ is $(-\pi, \pi)$.

(b) [2] If $b_{t+1} = 2b_t$ and $b_0 = 3$, then $b_{14} = 49152$. True or false? Explain.

$b_t = b_0 \cdot 2^t = 3 \cdot 2^t$ ← sol^N

$b_{14} = 3 \cdot 2^{14} = 49152$

so, TRUE.

2. **Multiple Choice.** Clearly **circle** the one correct answer.

(a) [3] For the discrete-time dynamical system $M_{t+1} = 0.5M_t + 1$, which of the following statements is/are true?

(I) The updating function is increasing. ✓
(II) $M_{t+2} = 0.25M_t + 1.5$ ✓
(III) $M_{t-1} = 2(M_t - 1)$ ✓

(A) none (B) I only (C) II only (D) III only

(E) I and II (F) I and III (G) II and III (H) all three

(b) [3] Which of the three points is/are the equilibrium points of the discrete-time dynamical system $m_{t+1} = 4m_t^2 - 3m_t - 8$?

(I) $m^* = 0$ ✗ (II) $m^* = 1$ ✗ (III) $m^* = 2$ ✓

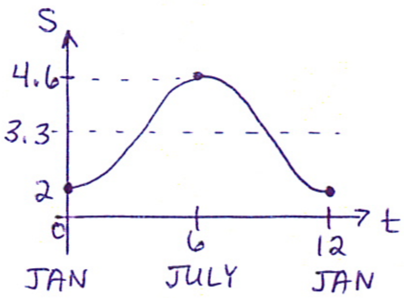
(A) none (B) I only (C) II only (D) III only

(E) I and II (F) I and III (G) II and III (H) all three

(c) [3] The value of $\lim_{x \to 4} \dfrac{\sqrt{x^2+9}-5}{x-4}$ is

(A) $\frac{4}{5}$ (B) $\frac{5}{4}$ (C) 0 (D) D.N.E.

3. [3] A population of salmon changes periodically with a period of 12 months. In January, it reaches its minimum of 2 million, and in July it reaches its maximum of 4.6 million. Find a formula that describes how the population changes with time. (*Hint:* It might help to sketch the graph first.)



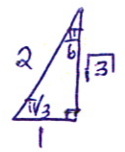
average value $= \dfrac{4.6+2}{2} = 3.3$

amplitude $= 1.3$

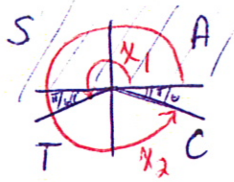
period $= 12 \Rightarrow \dfrac{2\pi}{k} = 12 \Rightarrow k = \dfrac{\pi}{6}$

$\therefore\ S(t) = -1.3\cos\dfrac{\pi}{6}t + 3.3$

4. (a) [2] Find all solutions of $\sin x = -\dfrac{1}{2}$ on $[0, 2\pi]$.



$x' = \dfrac{\pi}{6}$



$x_1 = \pi + \dfrac{\pi}{6} = \dfrac{7\pi}{6}$

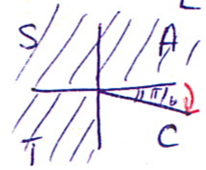
$x_2 = 2\pi - \dfrac{\pi}{6} = \dfrac{11\pi}{6}$

(b) [2] Describe what it means to compute $\arcsin\left(-\frac{1}{2}\right)$. Find the value of $\arcsin\left(-\frac{1}{2}\right)$.

To compute $\arcsin(-\frac{1}{2})$ we must find the angle θ between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ such that $\sin\theta = -\frac{1}{2}$.

$\arcsin(-\frac{1}{2}) = \theta \Leftrightarrow \sin\theta = -\frac{1}{2}$ for $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

$\theta' = \dfrac{\pi}{6}$



$\theta = -\dfrac{\pi}{6}$

5. The dynamics of caffeine absorption and replacement can be described by the discrete time dynamical system

$$c_{t+1} = 0.87c_t + d$$

where c_t denotes the amount of caffeine (in mg) present in your body at time t (in hours) and d is the amount of caffeine taken every hour. Suppose that every hour you consume a small coffee (about 50mg of caffeine).

(a) [1] Determine the equilibrium amount of caffeine.

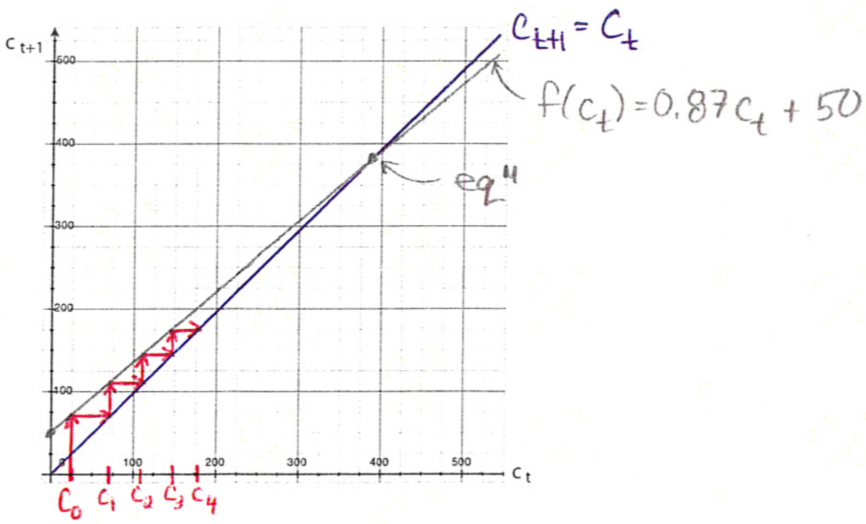
$C_{t+1} = 0.87C_t + 50$

$C^* = 0.87C^* + 50$

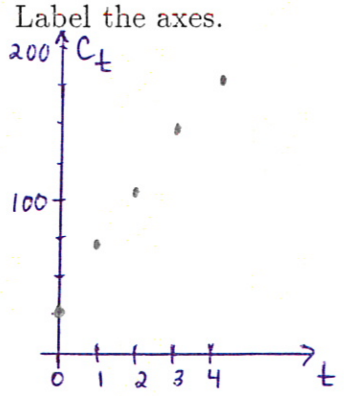
$0.13C^* = 50$

$C^* = \frac{50}{0.13} \simeq 385 \text{ mg}$

(b) [2] Graph the updating function and the diagonal. Label the equilibrium point. Then, cobweb for 4 steps starting from $c_0 = 25$ mg.



(c) [2] Roughly (no calculations required!) plot the solution points you found in part (b). Label the axes.



6. The discrete-time dynamical system $l_{t+1} = \frac{120}{45+l_t} l_t$ describes the growth of a certain population of lions where l_t represents the number of lions present at time t in years.

(a) [2] Identify the per capita production rate, $r(l_t)$. Describe, in words, how the rate, r, depends on the population size, l_t.

$$r(l_t) = \frac{120}{45+l_t}$$

For small pop^N sizes, the rate will be high.
For large pop^N sizes, the rate will be low.

(b) [2] Solve for the equilibria of this system algebraically.

$$l^* = \frac{120}{45+l^*} l^*$$

$$(l^*)^2 + 45 l^* = 120 l^*$$

$$(l^*)^2 - 75 l^* = 0$$

$$l^* [l^* - 75] = 0$$

$$l^* = 0 \quad \text{or} \quad l^* = 75$$

(c) [2] In general, what does an equilibrium of a dynamical system represent? Give an example of a dynamical system which has **no** equilibria.

An equilibrium is a value of the measurement which remains unchanged over time.

The DTDS $h_{t+1} = h_t + 0.8$ has no equilibria.

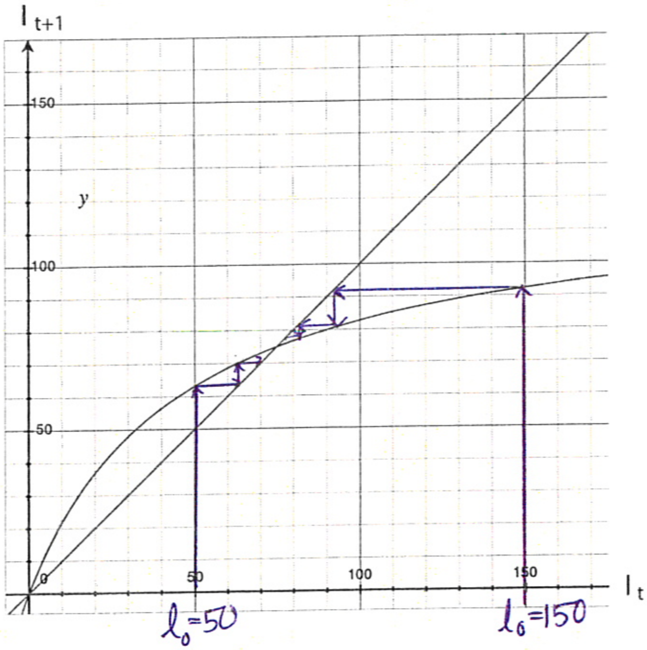
Try to find equilibria:

$$h^* = h^* + 0.8$$

$$h^* - h^* = 0.8$$

$$0 = 0.8 \quad ?$$

(d) [4] Below is the graph of the updating function and the diagonal for $l_{t+1} = \dfrac{120}{45 + l_t} l_t$.



(i) Suppose initially there are 50 lions. Cobweb for several steps to determine what will happen to the population of lions over time. Summarize your results in one sentence.

The pop^N of lions will increase towards the eq^M of 75 lions (Also, the amount of increase each year will be smaller than the increase in the previous year.)

(ii) Now, suppose initially there are 150 lions. Cobweb for several steps to determine what will happen to the population of lions over time. Summarize your results in one sentence.

The pop^N of lions will decrease and approach the eq^M of 75 lions.

7. A population changes according to the formula $P(t) = 5e^{-t}$, where t is time in years.

(a) [2] Find the average rate of change starting from time $t = 2$ and lasting (i) 1 year and then (ii) 0.1 years. $\Delta t = 1$ $\Delta t = 0.1$

(i) $\frac{P(3) - P(2)}{3 - 2} = \frac{5e^{-3} - 5e^{-2}}{1} \approx -0.43 \left(\frac{\text{individuals}}{\text{year}}\right)$

(ii) $\frac{P(2.1) - P(2)}{2.1 - 2} = \frac{5e^{-2.1} - 5e^{-2}}{0.1} \approx -0.64 \left(\frac{\text{individuals}}{\text{year}}\right)$

(b) [1] Which of the two rates found in part (a) better describe how $P(t)$ changes at $t = 2$? Why?

The rate of -0.64 individuals/year better describes how the pop^N changes at $t=2$ since the point' $(2.1, P(2.1))$ is closer to the base point $(2, P(2))$ than $(3, P(3))$ and so the slope of the secant line better approximates the slope of the tangent line at $t=2$

8. Let $f(x) = \begin{cases} e^{-2x} & \text{when} \quad x < 0 \\ \frac{1}{x^2+1} & \text{when} \quad x > 0 \end{cases}$

[2] Compute $\lim_{x\to 0^-} f(x)$ and $\lim_{x\to 0^+} f(x)$. What can we conclude about $\lim_{x\to 0} f(x)$?

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} e^{-2x} = e^0 = 1$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{1}{x^2+1} = \frac{1}{0+1} = 1$

$\therefore \lim_{x\to 0} f(x) = 1$

THE END