# MATHEMATICS 1LS3 TEST 4

Day Class
Duration of Examination: 60 minutes
McMaster University, 19 November 2012

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FIRST NAME (please print): SOLUTIONS

FAMILY NAME (please print): ____________

Student No.: ____________

THIS TEST HAS 8 PAGES AND 7 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE.

Total number of points is 40. Marks are indicated next to the problem number. Any non-graphing calculator is allowed.

USE PEN TO WRITE YOUR TEST. IF YOU USE A PENCIL YOUR TEST WILL NOT BE ACCEPTED FOR REMARKING (IF NEEDED).

**You need to show work to receive full credit.**

| Problem | Points | Mark |
|---|---|---|
| 1 | 6 | |
| 2 | 6 | |
| 3 | 5 | |
| 4 | 6 | |
| 5 | 6 | |
| 6 | 5 | |
| 7 | 6 | |
| TOTAL | 40 | |

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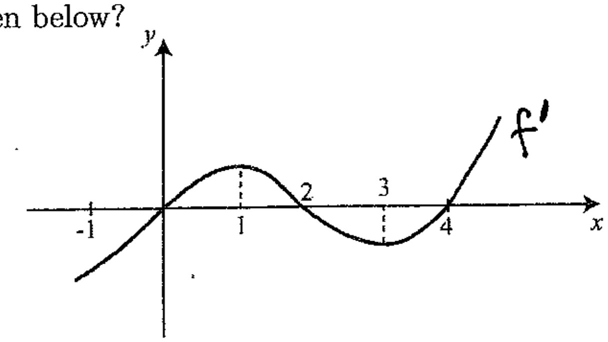
1. (a)[3] Which of the following statements is/are true for the **antiderivative** of the function given below?




(I) Decreasing on the interval $(1, 3)$ ✗

(II) Decreasing on the interval $(2, 4)$ → yes because $f' < 0$

(III) Concave down on the interval $(1, 3)$ → yes because $f'$ decreases

(A) none (B) I only (C) II only (D) III only

(E) I and II (F) I and III (G) II and III (H) all three

(b)[3] Which of the following equations is/are pure-time differential equations?

(I) $y' = 3y \sin x + x^2$ ✗

(II) $y' = 3y \sin y + y^2$ ✗

(III) $y' = 3x \sin x + x^2$ ✓

cannot have y on the right side

(A) none (B) I only (C) II only (D) III only

(E) I and II (F) I and III (G) II and III (H) all three

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2. **Identify each statement as true or false (circle your choice). No justification is needed.**

(a)[2] $\int \frac{1}{x^2}\,dx = \ln(x^2) + C$.

TRUE FALSE

$$(\ln(x^2)+C)' = \frac{1}{x^2}\cdot 2x = \frac{2}{x} \neq \frac{1}{x^2}$$

(b)[2] The leading behaviour of $y = 3x + e^{-x} - 7x^2 - 3x^3$ at $\infty$ is $e^{-x}$.

$\infty \quad 0 \quad \infty \quad \infty$

TRUE FALSE

(c)[2] The functions $y = 3x + e^{-x} - 7x^2 - 3x^3$ and $y = e^{-x}$ have the same leading behaviour at $x = 0$.

$0 \quad 0 \quad 0$

TRUE FALSE

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3. The rate at which new influenza cases occured in 2011 in Greater Vancouver Area follows the formula $125.4e^{0.3t}$ people/day. By $t$ we represent the time in days measured from 1 December 2011 (so $t = 0$ represents 1 December 2011). On 1 December 2011 there were 56 cases of influenza.

(a)[2] Write a differential equation and the initial condition for the number $N(t)$ of influenza cases.

$$N'(t) = 125.4\, e^{0.3t}$$

$$N(0) = 56$$

(b)[3] Find the formula for the total number of influenza cases by day $t$.

$$N(t) = \int 125.4 e^{0.3t}\, dt$$

$$= 125.4 \cdot \frac{1}{0.3} e^{0.3t} + C$$

$$= 418 e^{0.3t} + C$$

$$N(0) = 56 \rightarrow \quad 56 = 418 \cdot e^{0} + C$$

$$C = -362$$

$$N(t) = 418 e^{0.3t} - 362$$

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4. (a)[3] Find $\lim_{x\to 0} \frac{e^{x^3} - 1 - x^3}{x^6} = \frac{0}{0} \overset{LH}{=} \lim_{x\to 0} \frac{e^{x^3}\cdot 3x^2 - 3x^2}{6x^5}$

$= \lim_{x\to 0} \frac{e^{x^3} - 1}{2x^3} = \frac{0}{0} \overset{LH}{=} \lim_{x\to 0} \frac{e^{x^3}\cdot \cancel{3x^2}}{\cancel{6x^2}} = \lim_{x\to 0} \frac{e^{x^3}}{2} = \frac{1}{2}$

or, can use Taylor : $e^A \approx 1 + A + \frac{A^2}{2}$

so $e^{x^3} \approx 1 + x^3 + \frac{x^6}{2}$ and

$\lim_{x\to 0} \frac{e^{x^3} - 1 - x^3}{x^6} = \lim_{x\to 0} \frac{\cancel{1} + \cancel{x^3} + \frac{x^6}{2} - \cancel{1} - \cancel{x^3}}{x^6}$

$= \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}$ (no LH needed)

(b)[3] Find $\lim_{x\to 0^+} x^4 \ln x = 0\cdot(-\infty) = \lim_{x\to 0+} \frac{\ln x}{\frac{1}{x^4}} = \frac{-\infty}{\infty}$

$\overset{LH}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{4}{x^5}} = \lim_{x\to 0^+}\left(-\frac{x^4}{4}\right) = \underline{0}$

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5. Find the following antiderivatives:

(a)[2] $\int \frac{6}{35x}\,dx$ $= \frac{6}{35}\int \frac{1}{x}\,dx = \frac{6}{35}\ln|x| + C$

(b)[2] $\int \left(\frac{5}{1+x^2} + \frac{1+x^2}{5}\right) dx$ $= 5\int \frac{1}{1+x^2}\,dx + \frac{1}{5}\int (1+x^2)\,dx$

$= 5\arctan x + \frac{1}{5}\left(x + \frac{x^3}{3}\right) + C$

(c)[2] $\int (\sec x \tan x + \pi)\,dx$ $= \sec x + \pi x + C$

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6. Consider a modified logistic dynamical system $x_{t+1} = rx_t(1-x_t)^2$, where $r > 0$.

(a)[2] Identify all equilibrium points.

$$x^* = rx^*(1-x^*)^2$$

$$x^*(1-r(1-x^*)^2) = 0$$

$$\underline{\underline{x^*=0}} \quad \text{or} \quad 1 = r(1-x^*)^2$$

$$(1-x^*)^2 = \frac{1}{r}$$

$$1-x^* = \pm\sqrt{\frac{1}{r}}$$

$$\underline{\underline{x^* = 1 \pm \sqrt{\frac{1}{r}}}}$$

(b)[1] Take $r = 1/4$. Find the largest equilibrium point.

$$x^* = 1 + \sqrt{1/_{1/4}} = \underline{\underline{3}}$$

(c)[2] Determine whether the equilibrium point from (b) is stable or not.

$$f(x) = \frac{1}{4}x(1-x)^2 = \frac{1}{4}(x - 2x^2 + x^3)$$

$$f'(x) = \frac{1}{4}(1-4x+3x^2)$$

$$f'(3) = \frac{1}{4}(1-12+27) = 4$$

$$|f'(3)| = 4 > 1 \implies \underline{\underline{x^*=3 \text{ is unstable}}}$$

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7. Consider the function $f(x) = \dfrac{x^3 + 2x^2 + 6x + 3}{x + 2x^2}$.

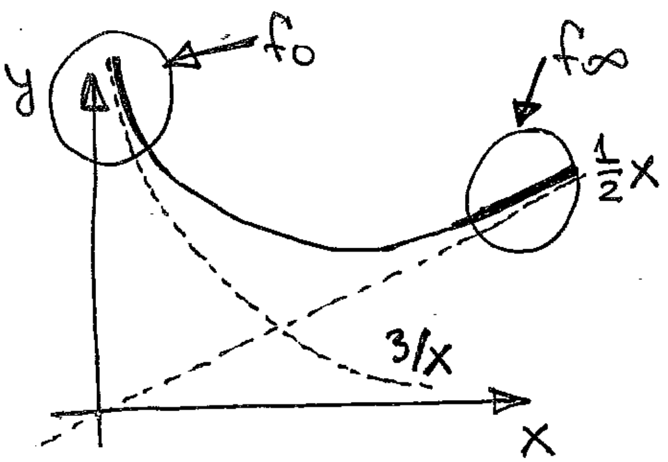
(a)[2] Find the leading behaviour of $f(x)$ at 0.

$$f_0(x) = \frac{3}{x}$$ ← all other terms → 0

(b)[2] Find the leading behaviour of $f(x)$ at $\infty$.

$$f_\infty(x) = \frac{x^3}{2x^2} = \frac{1}{2}x$$

(c)[2] Based on your answers to (a) and (b), sketch a graph that matches the leading behaviour of $f(x)$ at 0 and approaches the leading behaviour of $f(x)$ at $\infty$.

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