MATHEMATICS 1LS3 TEST 3

Day Class
Duration of Examination: 60 minutes
McMaster University, 11 November 2015

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First name (PLEASE PRINT): SOLUTIONS

Family name (PLEASE PRINT): ____________________

Student No.: ____________________

THIS TEST HAS 8 PAGES AND 7 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE.

Total number of points is 40. Marks are indicated next to the problem number. Any non-graphing calculator is allowed.

USE PEN TO WRITE YOUR TEST. IF YOU USE A PENCIL YOUR TEST WILL NOT BE ACCEPTED FOR REMARKING (IF NEEDED).

You must show work to receive full credit.

Problem	Points	Mark
1	4	
2	6	
3	6	
4	6	
5	6	
6	6	
7	6	
TOTAL	40	

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1. **Multiple choice questions: circle ONE answer. No justification is needed.**

(a)[2] Which of the following functions has/have **no critical points**?

(I) $f(x) = 3 - 7x$ → line → no cps

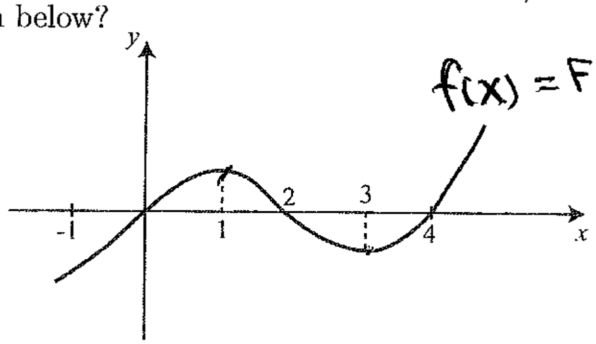
(II) $f(x) = x^2 + 11$

(III) $f(x) = e^{-3x}$ → always decreasing → no cps

(A) none (B) I only (C) II only (D) III only

(E) I and II (F) I and III (G) II and III (H) all three

(b)[2] Which of the following statements is/are true for the **antiderivative** of the function given below?



$f(x) = F'(x)$

F(x) such that $F'(x) = f(x)$

so the picture shows $F'(x)$, and we need to deduce properties of F(x)

(I) Decreasing on the interval $(1, 3)$ X

$F' < 0$ (II) Decreasing on the interval $(2, 4)$ ✓

F' is decreasing (III) Concave down on the interval $(1, 3)$ ✓

(A) none (B) I only (C) II only (D) III only

(E) I and II (F) I and III (G) II and III (H) all three

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2. Identify each statement as true or false, or yes or no (circle your choice). You do not need to justify your answer.

(a)[2] $P'(t) = \underbrace{3P(t) - e^{-P(t)}}$ is an autonomous differential equation

TRUE FALSE

t does not appear explicitly

(b)[2] $\int \ln t \, dt = \frac{1}{t} + C$

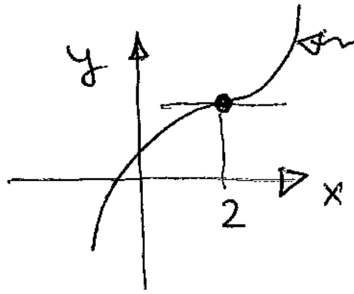
TRUE FALSE

$\left(\frac{1}{t}\right)' = -\frac{1}{t^2} \neq \ln t$

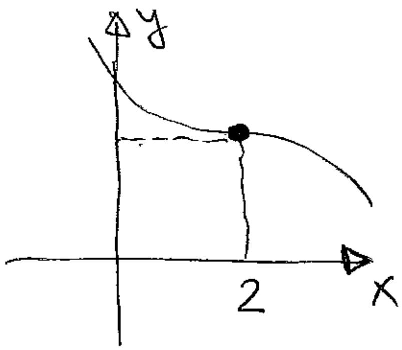
(c)[2] If $f'(2) = 0$ then $f(x)$ has a local extreme value at $x = 2$.

TRUE FALSE

graph like x^3



or



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Questions 3-7: You must show work to receive full credit.

3. (a)[3] Find $\lim_{x\to 0} \frac{\sin x - x}{x^3}$ $= \frac{0}{0}$

$$\overset{LH}{=} \lim_{x\to 0} \frac{\cos x - 1}{3x^2} = \frac{0}{0}$$

$$\overset{LH}{=} \lim_{x\to 0} \frac{-\sin x}{6x} = \frac{0}{0}$$

$$\overset{LH}{=} \lim_{x\to 0} \frac{-\cos x}{6} = -\frac{1}{6}$$

(b)[3] Find $\lim_{x\to 0^+} x^4 \ln x$ $= 0\cdot(-\infty) = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x^4}}$

$$\overset{LH}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{4}{x^5}}$$

$$= \lim_{x\to 0^+} \left(-\frac{x^4}{4}\right) = 0$$

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4. The function $c(t) = t^2e^{-6t}$ has been used to model the absorption of a drug (such as morphine); $c(t)$ is the concentration (in milligrams per millilitre, mg/mL) of the drug in the bloodstream, and $t \geq 0$ is time (in hours).

(a)[2] The function $c(t)$ has two critical points such that $t \geq 0$. Find them.

$$c'(t) = 2te^{-6t} + t^2e^{-6t}(-6)$$
$$= 2te^{-6t}(1-3t) = 0 \longrightarrow \underline{\underline{t=0}},\ \underline{\underline{t=\tfrac{1}{3}}}$$

($c'(t)$ dne ... no such t)

(b)[2] When does the concentration reach its maximum, and what is that maximum value? Justify your answer.

t	0		1/3	
$c'(t)$		$\oplus$		$\ominus$
$c(t)$		$\nearrow$		$\searrow$

rel. max when $t = 1/3$ (i.e., 20 minutes after the drug was given)

max. value $= \left(\frac{1}{3}\right)^2 e^{-6(1/3)} = (1/9)(e^{-2}) = 1/9e^2 \approx 0.015$

(c)[2] Find the absolute maximum and the absolute minimum values that the concentration $c(t)$ reaches during the first hour after the drug is administered, i.e., over the interval $[0, 1]$.

t	$c(t) = t^2e^{-6t}$
0	0
1	$e^{-6} \approx 0.002$
1/3	$\frac{1}{9e^2} \approx 0.015$

$\longrightarrow$ abs. min. at $t=0$
value $= 0$ mg/mL
(makes sense!)

$\longrightarrow$ abs. max. at $t = 1/3$
value ≈ 0.015 mg/mL

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5. Consider the initial value problem $f'(t) = 4t + 1$, $f(0) = 1$.

(a)[3] Compute the first two steps of Euler's Method with step size $\Delta t = 0.5$.

$t_{n+1} = t_n + \Delta t$ ---- values of t

$y_{n+1} = y_n + G(t_n)\Delta t$

$= y_n + (4t_n+1)(0.5)$ --- approximations

$t_0 = 0$

$y_0 = 1$

$t_1 = t_0 + \Delta t = \underline{0.5}$

$y_1 = y_0 + (1)(0.5) = 1 + 0.5 = \underline{1.5}$

$t_2 = t_1 + \Delta t = \underline{1}$

$y_2 = \underbrace{y_1}_{1.5} + (4\underbrace{(0.5)+1)}_{3}(0.5) = \underline{3}$

(b)[2] Solve the given initial value problem algebraically, and find $f(1)$.

$f(t) = 2t^2 + t + C$

$f(0) = 1 \longrightarrow 1 = 0 + 0 + C$, so $C = 1$

thus $f(t) = 2t^2 + t + 1$

and $f(1) = 4$

(c)[1] What is the meaning of your answer in (a) in relation to your answer in (b)?

$\underset{3}{\overset{\shortparallel}{y_2}}$ is an approximation of $f(1) = 4$

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6. (a)[2] Find $\int Me^{-(K+n)t}\,dt$ $= M \int e^{-(K+n)t}\,dt$

$$= M \cdot \frac{1}{-(K+n)} e^{-(K+n)t} + C$$

(b)[2] Find $\int \left(\frac{2}{1+x^2} + \frac{1+x^2}{3}\right) dx$ $= 2 \int \frac{1}{1+x^2}\,dx + \int \frac{1}{3}\,dx + \int \frac{x^2}{3}\,dx$

$$= 2\arctan x + \frac{1}{3}x + \frac{1}{9}x^3 + C$$

(c)[2] Describe the following event as an initial value problem (i.e., write down a differential equation and an initial condition). Do not solve the equation.
A sample of dangerous bacteria, initially at the temperature of 15^oC, is put into a -75^oC refrigerator. Let $T(t)$ be the temperature of the sample at time t. The temperature of the sample changes proportionally to the square of the difference between the temperature of the sample and the temperature of the refrigerator.

$$T'(t) = k\left(T(t) - (-75)\right)^2 \qquad k = \text{constant}$$

$$T(0) = 15$$

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7. The change in the number of people infected with Ebola virus in Liberia in 2014 has been modelled by the initial value problem

$$I'(t) = 10.5\sqrt{t} + 2e^{-0.1t}, \ \ I(0) = 26$$

Time t is measured in days, and $t = 0$ represents 1 August 2014.
(a)[4] Find a formula for $I(t)$.

$$I(t) = \int (10.5\sqrt{t} + 2e^{-0.1t})\,dt$$

$$= 10.5 \cdot \frac{t^{3/2}}{3/2} + 2 \cdot \frac{1}{-0.1} e^{-0.1t} + C$$

$\frac{2}{3} \cdot 10.5 = 7$ $$= 7t^{3/2} - 20e^{-0.1t} + C$$

$$I(0) = 26 \rightarrow \ 26 = 0 - 20 + C$$

so $C = 46$

and $I(t) = 7t^{3/2} - 20e^{-0.1t} + 46$

(b)[2] According to this model, what is the number of infected people on 11 August 2014? Round off to the nearest integer.

$t = 10$

$$I(10) = 7 \cdot 10^{3/2} - 20 \cdot e^{-0.1(10)} + 46$$

$$\approx 260$$

THE END