MATHEMATICS 1LS3 FINAL EXAMINATION
version 1

Day Class
Duration of Examination: 3 hours
McMaster University
17 December 2014

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FIRST NAME (PRINT CLEARLY): SOLUTIONS

FAMILY NAME (PRINT CLEARLY): ____________

Student No.: ____________

THIS EXAMINATION PAPER HAS 16 PAGES AND 10 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE. BRING ANY DISCREPANCY TO THE ATTENTION OF THE INVIGILATOR.

Total number of points is 80. Marks are indicated next to the problem number. Any Casio fx991 calculator is allowed. Write your answers in the space provided. EXCEPT ON QUESTIONS 1 AND 2, YOU MUST SHOW WORK TO GET FULL CREDIT. Good luck.

Problem	Points	Mark
1	20	
2	6	
3	6	
4	5	
5	11	
6	6	
7	6	
8	7	
9	7	
10	6	
TOTAL	80	

Name:_______________

Student No.: _______________

Question 1: Circle the correct answer. No justification is needed.

1. (a)[2] Demerol is a pain reliever used for moderate to severe pain. It is considered safer than morphine, and carries less risk of addiction. The half-life of demerol is about 4 hours. Which of the following statements is/are true?

(I) A patient is given one tablet. After 8 hours, 25% of the initial amount of Demerol is still present (unabsorbed) in the patient's body. ✓

(II) A patient is given two tablets. After 8 hours, 50% of the initial amount of Demerol is still present (unabsorbed) in the patient's body.

(III) A patient is given two tablets. After 8 hours, 25% of the initial amount of Demerol is still present (unabsorbed) in the patient's body. ✓

(A) none	(B) I only	(C) II only	(D) III only
(E) I and II	(F) I and III	(G) II and III	(H) all three

(b)[2] Identify all correct interpretations of the definite integral $\int_2^3 (2x-1)\,dx$.

(I) Total change of the function $f(x) = 2x - 1$ from $x = 2$ to $x = 3$. X

(II) Area of the region under the graph of $f(x) = 2x - 1$, above the x-axis, from $x = 2$ to $x = 3$. ✓

(III) Average value of the function $f(x) = 2x - 1$ on the interval $[2, 3]$. ✓

(A) none	(B) I only	(C) II only	(D) III only
(E) I and II	(F) I and III	(G) II and III	(H) all three

(c)[2] If $f(x) = e^{\sec(\pi x)\tan(\pi x)}$, then $f'(1) =$

(A) 0 (B) 1 (C) -1 (D) π

(E) $-\pi/4$ (F) $-\pi$ (G) $-\pi/4$ (H) 2

$$f'(x) = e^{\sec(\pi x)\tan(\pi x)}\left(\sec(\pi x)\tan(\pi x)\cdot\pi\cdot\tan(\pi x) + \sec(\pi x)\sec^2(\pi x)\pi\right)$$

$$f'(1) = 1\cdot\left(0 + (-1)(1)\pi\right) = -\pi$$

(d)[2] $\lim_{x\to 0}\frac{\cos x - 1 + \frac{x^2}{2}}{x^4} = \frac{0}{0}$

(A) 0 (B) ∞ (C) $-1/3$ (D) $-1/24$

(E) $1/6$ (F) $1/3$ (G) $1/24$ (H) $-\infty$

$$= \lim_{x\to 0}\frac{-\sin x + x}{4x^3} = \lim_{x\to 0}\frac{-\cos x + 1}{12x^2} = \lim_{x\to 0}\frac{\sin x}{24x}$$

$$= 1/24$$

(e)[2] The linear-quadratic (LQ) model for the percent S of cancer cells surviving radiation treatment states that

$$S = e^{-\alpha n d - \beta n d^2 + \gamma T}$$

where $d > 0$ is the dose (in Gray) per treatment of radiation, n is the number of treatments applied, γ is the tumour cell repopulation rate (i.e., new tumour cells growing during treatment) and $T > 0$ is the total treatment duration. The parameters α, β and γ are positive.
Which of the following statements is/are true?

(I) The semilog graph of S as a function of d is a line ✗

(II) The semilog graph of S as a function of T is a line ✓

(III) The semilog graph of S as a function of n is a line ✓

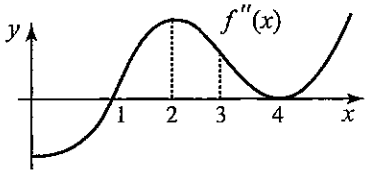
(A) none (B) I only (C) II only (D) III only
(E) I and II (F) I and III (G) II and III (H) all three

(f)[2] It is known that the function $f(x)$ is defined for all real numbers, and its derivative is given by $f'(x) = \dfrac{\left(3x^3 - x^4\right) e^{-2x}}{(4-x)^{1/3}}$. Find all its critical points.

$$\frac{x^3(3-x)}{(4-x)^{1/3}} \cdot e^{-2x}$$

(A) no critical points (B) 0 only (C) 3 only (D) 4 only
(E) 0 and 3 (F) 0 and 4 (G) 3 and 4 (H) 0, 3 and 4

(g)[2] The graph of the second derivative $f''(x)$ of a function $f(x)$ is given. Which statements is/are true?



(I) $x = 4$ is a point of inflection of $f(x)$ X

(II) The graph of $f(x)$ is concave up on $(1, 4)$ ✓

(III) The rate of change of $f(x)$ is increasing on $(2, 4)$ ✓

(A) none (B) I only (C) II only (D) III only

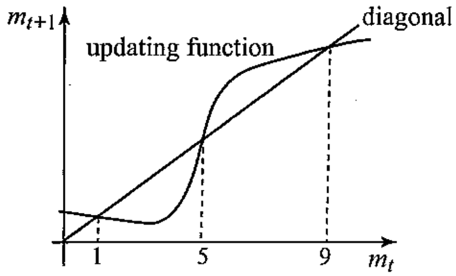
(E) I and II (F) I and III (G) II and III (H) all three

(h)[2] Turbidity T is a measure of cloudiness or haziness in water, and is used to assess the quality of drinking water. It is known that turbidity is proportional to the natural logarithm of the number of phytoplankton N, proportional to the amount of sediment S, and inversely proportional to the square of the depth d. Which formula represents the turbidity? (k is a constant)

(A) $T = k\dfrac{Sd^2}{N}$ (B) $T = k\dfrac{S\ln N}{d^2}$ (C) $T = k\dfrac{\ln N}{Sd^2}$ (D) $T = k\dfrac{\ln N}{d^2}$

(E) $T = k\dfrac{Sd}{N}$ (F) $T = k\dfrac{Sd^2}{\ln N}$ (G) $T = k\dfrac{Sd}{\ln N}$ (H) $T = k\dfrac{S\ln N}{d}$

(i)[2] Consider the dynamical system $m_{t+1} = f(m_t)$, whose updating function is drawn in the diagram below. Which equilibrium points are unstable?



$|f'(m^*)| > 1$

(A) none (B) 1 only (C) 5 only (D) 9 only
(E) 1 and 5 (F) 1 and 9 (G) 5 and 9 (H) all three

(j)[2] Which of the following improper integrals are divergent?

(I) $\int_1^\infty x^{-2}\,dx$ (II) $\int_1^\infty x^{-1}\,dx$ (III) $\int_1^\infty x^{-1.1}\,dx$

(A) none (B) I only (C) II only (D) III only
(E) I and II (F) I and III (G) II and III (H) all three

$\int_1^\infty \frac{1}{x^p}\,dx \quad 0 \le p \le 1$

Question 2: Circle the correct answer. No justification is needed.

2. (a)[2] If $f(x) = g(x)h(x)$, then by the product rule, $f''(x) = g''(x)h(x) + g(x)h''(x)$.

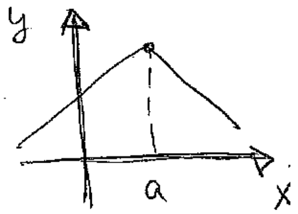
TRUE FALSE

$f' = g'h + gh'$

$f'' = g''h + g'h' + g'h' + gh''$

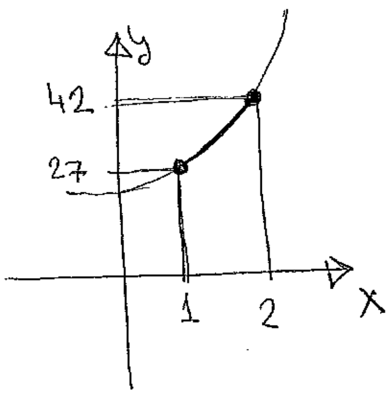
(b)[2] If a continuous function $f(x)$ has a relative extreme value at a, then $f'(a) = 0$.

TRUE FALSE



(c)[2] The average value of the function $f(x) = 5x^2 + 22$ on $[1, 2]$ is 27.

TRUE FALSE



$\bar{f}$ must be larger than the minimum value of 27

Questions 3-10: You must show work to obtain full credit

3. (a)[2] Evaluate the definite integral $\int_0^1 \frac{2}{5^x}\,dx$. $= \int_0^1 2 \cdot 5^{-x}\,dx$

$$= 2\,(-1)\,\frac{5^{-x}}{\ln 5}\Big|_0^1$$

$$= \left(\frac{-2}{\ln 5}\cdot 5^{-1}\right) - \left(\frac{-2}{\ln 5}\right) = \frac{2}{\ln 5} - \frac{2}{5\ln 5}$$

$$= \frac{8}{5\ln 5} \approx 0.994$$

(b)[2] Find $\int \frac{5x^2}{1+x^2}\,dx$ (Hint: Use long division.)

$$= \int \left(5 - \frac{5}{1+x^2}\right) dx$$

$$= 5x - 5\arctan x + C$$

(c)[2] Prove that $F(x) = -\left(5+4x+x^2\right)e^{-x}$ is an antiderivative of $f(x) = (x+1)^2 e^{-x}$.

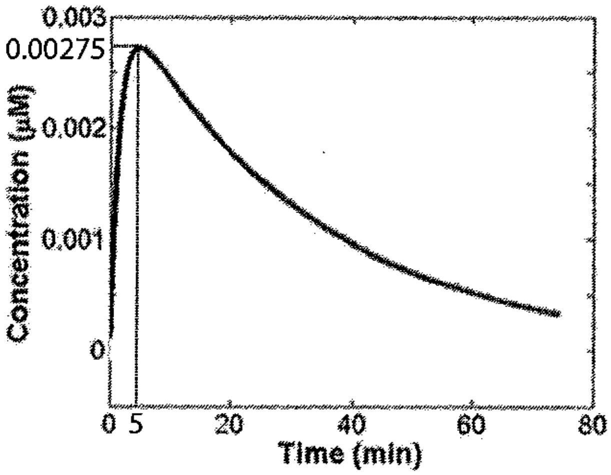
$$F'(x) = -(4+2x)\,e^{-x} - (5+4x+x^2)\,e^{-x}\,(-1)$$

$$= e^{-x}\left(-4-2x+5+4x+x^2\right)$$

$$= e^{-x}\left(x^2+2x+1\right)$$

$$= e^{-x}\,(x+1)^2 = f(x)$$

4. The following graph is taken from *Modeling of pharmacokinetics of cocaine in human reveals the feasibility for development of enzyme therapies for drugs of abuse.* C.G. Zhan and F. Zheng. PLoS Computational Biology. 8.7 (July 2012).



Based on this graph, you need to find a formula for the concentration as a function of time. Going through your textbook, you notice that the graph of $f(t) = Ate^{-\beta t}$ (with $A > 0$ and $\beta > 0$) matches the shape of the given graph.

(a)[3] Find the relative maximum of $f(t)$ and the value of t where it occurs. (Your answer will contain A and β.)

$$f'(t) = Ae^{-\beta t} + Ate^{-\beta t}(-\beta) = Ae^{-\beta t}(1-\beta t)$$

$$f'(t) = 0 \rightarrow t = 1/\beta$$

$$f\left(\frac{1}{\beta}\right) = A \cdot \frac{1}{\beta} e^{-\beta(1/\beta)} = \frac{A}{\beta} e^{-1} = \frac{A}{\beta e}$$

(b)[2] Using the given graph and your answer to (a), identify the values of β and A, and write the formula for the concentration.

$$t = 5 \rightarrow \frac{1}{\beta} = 5 \quad | \quad \beta = 0.2$$

$$\frac{A}{\beta e} = \frac{A}{0.2e} = 0.00275 \rightarrow A = 0.00275\,(0.2e) \approx 0.0015$$

$$\rightarrow c(t) = 0.0015 \cdot te^{-0.2t}$$

5. (a)[2] Find $\int x^3 \ln x \, dx$. $= \left\{ \begin{array}{l} u = \ln x \to u' = 1/x \\ v' = x^3 \to v = x^4/4 \end{array} \right\}$

$$= \frac{x^4}{4} \ln x - \int \frac{1}{4} x^3 dx$$

$$= \frac{x^4}{4} \ln x - \frac{1}{16} x^4 + C$$

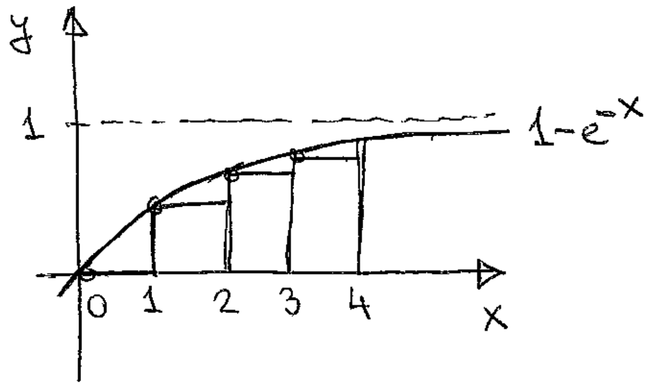
(b)[3] Find $\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}}$. $= \left\{ \begin{array}{l} u = \ln x \\ \frac{du}{dx} = \frac{1}{x} \to \frac{1}{x} dx = du \end{array} \right\}$

$$= \int_1^4 u^{-1/2} du = 2u^{1/2} \Big|_1^4 = 2\sqrt{4} - 2\sqrt{1} = 2$$

or $$= \int u^{-1/2} du = 2u^{1/2} = 2\sqrt{\ln x} \Big|_{e}^{e^4} = 2\sqrt{\ln e^4} - 2\sqrt{\ln e}$$

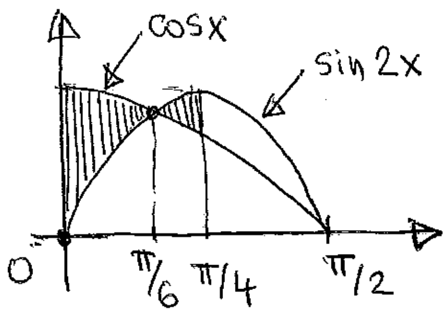
$$= 2$$

(c)[3] Find an approximation of the area of the region below the graph of $y = 1 - e^{-x}$ and over the interval $[0, 4]$, using L_4 (i.e., left sum with four rectangles). Sketch the function and the four rectangles involved.



$$L_4 = 1 \cdot 0 + 1 \cdot (1 - e^{-1}) + 1 \cdot (1 - e^{-2}) + 1 \cdot (1 - e^{-3})$$
$$= 3 - e^{-1} - e^{-2} - e^{-3} \approx 2.447$$

(d)[3] Sketch (shade) the region bounded by the graphs of $y = \cos x$, $y = \sin 2x$, $x = 0$ and $x = \pi/4$ and set up the formula for its area. (Recall that $\sin 2x = 2 \sin x \cos x$.) **Do not evaluate the integral.**



$$\cos x = \sin 2x = 2 \sin x \cos x$$
$$\cos x (1 - 2 \sin x) = 0$$
$$\cos x = 0 \rightarrow \underbrace{x = \frac{\pi}{2} + \pi k}_{\text{not relevant}}$$
$$\sin x = \frac{1}{2} \rightarrow x = \frac{\pi}{6} + \pi k$$

$$A = \int_0^{\pi/6} (\cos x - \sin 2x)\,dx + \int_{\pi/6}^{\pi/4} (\sin 2x - \cos x)\,dx$$

6. (a)[2] Find the Taylor polynomial $T_2(x)$ for the function $f(x) = (x+1)^2e^{-x}$ near $x = 0$.

$$f(x) = (x+1)^2e^{-x} \longrightarrow f(0) = 1$$

$$f'(x) = 2(x+1)e^{-x} + (x+1)^2e^{-x}(-1)$$

$$= e^{-x}(\cancel{2x}+2-x^2-\cancel{2x}-1)$$

$$= e^{-x}(-x^2+1) \longrightarrow f'(0) = 1$$

$$f''(x) = -e^{-x}(-x^2+1) + e^{-x}(-2x) \longrightarrow f''(0) = -1$$

$$T_2(x) = 1 + x - \frac{1}{2}x^2$$

(b)[2] Use your answer to (a) to find an estimate for $\int_0^1 (x+1)^2e^{-x}\,dx$.

$$\int_0^1 (x+1)^2e^{-x} \approx \int_0^1 (1 + x - \tfrac{1}{2}x^2)\,dx$$

$$= x + \frac{x^2}{2} - \frac{x^3}{6}\Big|_0^1 = 1 + \frac{1}{2} - \frac{1}{6} = \frac{4}{3} \approx 1{,}333$$

(c)[2] Using Fundamental Theorem of Calculus and question 3(c), find the value of the definite integral $\int_0^1 (x+1)^2e^{-x}\,dx$ rounded to three decimal places.

$$\int_0^1 (x+1)^2e^{-x}\,dx = -(5+4x+x^2)e^{-x}\Big|_0^1$$

$$= (-10e^{-1}) - (-5)$$

$$= -10e^{-1} + 5 \approx 1{,}321$$

7. According to Von Bertalanffy model, the rate of growth of pacific salmon is given by $dL/dt = 12.6e^{-0.15t}$, where the length L is in centimetres and the time t is in years.

(a)[1] Explain why $L(t)$ is an increasing function.

$$\frac{dL}{dt} = 12.6\,\underbrace{e^{-0.15t}}_{\oplus} > 0$$

(b)[2] Given that $L(0) = 0$ (i.e., at the moment of fertilization it is assumed that the length is zero), find a formula for the length $L(t)$ of pacific salmon.

$$L(t) = \int 12.6\, e^{-0.15t}\, dt = -84\, e^{-0.15t} + C$$

$$L(0) = 0 \rightarrow C = 84$$

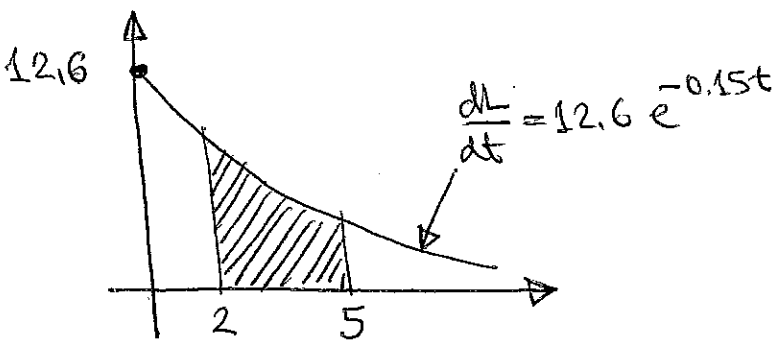
so $\quad L(t) = -84\, e^{-0.15t} + 84$

(c)[1] Compute how much the salmon grows between the ages of 2 and 5.

$$L(5) - L(2) = \left(-84\, e^{-0.15(5)} + \cancel{84}\right) - \left(-84\, e^{-0.15(2)} + \cancel{84}\right)$$

$$\approx 22.55 \text{ cm}$$

(d)[2] Sketch the graph of dL/dt and identify the region whose area is the same as your answer to (c).



$$\int_2^5 \frac{dL}{dt}\, dt = L(5) - L(2)$$

8. Consider the discrete-time dynamical system $p_{t+1} = \dfrac{4.4p_t}{1+0.0002p_t}$, where p_t is the number of bacteria at time t.

(a)[2] Find all equilibrium points of the system.

$$p^* = \frac{4.4\,p^*}{1+0.0002p^*}$$

$$p^*\left(1 - \frac{4.4}{1+0.0002p^*}\right) = 0$$

$$\text{so } \underline{p^*=0} \quad \text{or} \quad 1+0.0002p^* = 4.4$$

$$p^* = \underline{17{,}000}$$

(b)[3] Use the Stability Theorem to check whether each equilibrium point you found in (a) is stable or unstable.

$$f(p) = \frac{4.4p}{1+0.0002p}$$

$$\rightarrow f'(p) = \frac{4.4(1+0.0002p) - 4.4p(0.0002)}{(1+0.0002p)^2} = \frac{4.4}{(1+0.0002p)^2}$$

$p^*=0 \ldots f'(p^*) = 4.4 > 1 \ldots$ unstable

$$p^* = 17{,}000 \ldots f'(p^*) = \frac{4.4}{(\underbrace{1+0.0002(17000)}_{3.4})^2} = \frac{1}{4.4} < 1$$

... stable

(c)[2] Explain what the stability or instability of the largest equilibrium point in (b) means for the population of bacteria.

populations which start near $p^* = 17{,}000$ will approach 17,000 and remain near 17,000

or: growth is limited to (around) 17,000

9. A commonly used model of the spread of an infectious disease (logistic growth model) states that $I(t)$, the number of people (in thousands) infected with a virus, is given by

$$I(t) = \frac{10}{1+9e^{-t}} = 10(1+9e^{-t})^{-1}$$

where t is time in days.

(a)[2] Show that $I(t)$ is increasing for $t > 0$ (this justifies the use of the word "growth").

$$I'(t) = 10(-1)(1+9e^{-t})^{-2}\, 9e^{-t}(-1)$$
$$= 90e^{-t}(1+9e^{-t})^{-2} > 0$$

(b)[2] Find the limit of $I(t)$ at $t \to \infty$. Interpret what your answers to (a) and (b) mean in the context of the spread of the virus.

$$\lim_{t\to\infty} I(t) = \frac{10}{1+0} = 10$$

number of people increases over time, but it is bounded $\to$ approaches the limit of 10 (ie 10,000)

(c)[3] Find the point of inflection of $I(t)$. How does the number of infected people at the point of inflection relate to the number you obtained in (b)?

$$I''(t) = -90e^{-t}(1+9e^{-t})^{-2} + 90e^{-t}(-2)(1+9e^{-t})^{-3}(-9e^{-t})$$
$$= -90e^{-t}(1+9e^{-t})^{-3}(1+9e^{-t}-18e^{-t})$$
$$I''(t) = 0 \to 1-9e^{-t} = 0, \quad e^{-t} = 1/9$$
$$\text{so } t = -\ln(1/9) = \ln 9$$

check: I'' changes sign at $\ln 9$

$$I(\ln 9) = \frac{10}{1+9e^{-\ln 9}} = \frac{10}{2} = 5$$

value at inflection $= \frac{1}{2}$ of the HA $= \frac{1}{2}$ of the value in (b)

10. (a)[2] Find the indefinite integral $\int xe^{-0.1x}\,dx$. $= \left\{ \begin{array}{l} u = x \longrightarrow u' = 1 \\ v' = e^{-0.1x} \rightarrow v = -10e^{-0.1x} \end{array} \right\}$

$$= -10xe^{-0.1x} + 10\int e^{-0.1x}\,dx$$

$$= -10xe^{-0.1x} - 100e^{-0.1x} + C$$

(b)[2] The function $f(x) = xe^{-0.1x}$ models the density of the seeds dispersed from a tree (measured in grams per metre), where x represents the distance in metres from the tree (the tree is located at $x = 0$). Find the value of the definite integral $\int_2^\infty xe^{-0.1x}\,dx$.

$$\int_2^\infty xe^{-0.1x}\,dx = \lim_{T\to\infty} \left(-10xe^{-0.1x} - 100e^{-0.1x}\right)\Big|_2^T$$

$$= \lim_{T\to\infty} \left(-10\cancel{Te^{-0.1T}} - 100\cancel{e^{-0.1T}}\right) - \left(-20e^{-0.2} - 100e^{-0.2}\right) = 120e^{-0.2}$$

$0 \quad 0$

$$\approx 98.25$$

$$\lim_{T\to\infty} Te^{-0.1T} = \lim_{T\to\infty} \frac{T}{e^{0.1T}} = 0$$

(c)[1] What are the units of $\int_2^\infty xe^{-0.1x}\,dx$? $\frac{g}{m} \cdot m$ = grams

(d)[1] What does the integral $\int_2^\infty xe^{-0.1x}\,dx$ represent?

total mass of seeds dispersed 2 metres or more from the tree

THE END