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Correctness:

Theorem

If we have that

1. *greedy choice is part of an OPT solution*
2. *$SOL = g \cup SOL_{sub}$ is a feasible solution*

then SOL is an optimal solution (i.e., greedy alg is correct).

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Proof: By induction on the input size:

- $n = k$: Up to size k, GREEDY computes OPT.

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  2. Show that from *any* OPT solution (where greedy choice g may not be the first one), we can derive another optimal solution OPT' where g *is* its first choice, performing a series of exchanges.