Lecture 6: Elementary matrices and more about SLEs

Instructor: Dr Rushworth January 22nd

Covered in last lecture:

- Powers of matrices
- The inversion algorithm for $n \times n$ matrices
- Elementary matrices

Elementary matrices continued

Recall that an elementary matrix is a matrix which can be obtained from the identity matrix by applying exactly one elementary row operation.

Fact 6.1

If E is the elementary matrix obtained by applying the elementary row operation O to the identity. Let A be another matrix. The product EA is the matrix obtained from A by applying the elementary row operation O.

Example 6.2

Let

$$E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

be the elementary matrix obtained from the identity by swapping $R1$ and $R2$.

Then if

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$$

we have

$$EA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}$$

which is the matrix obtained from A by swapping $R1$ and $R2$.

Fact 6.3

Every elementary matrix is invertible, and the inverse is also an elementary matrix.

In fact, the inverse of an elementary matrix is simply the matrix associated to the reverse elementary row operation.

Example 6.4

Let

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 17 \end{bmatrix}$$

be the elementary matrix obtained from the identity by multiplying $R2$ by 17. Then

$$E^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{17} \end{bmatrix}$$

Check:

$$\begin{bmatrix} 1 & 0 \\ 0 & 17 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{17} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

In mathematics it is often useful to express the same thing in different ways. Elementary matrices allow us to express invertibility of a matrix in a number of different ways.

Fact 6.5

Let A be an $n \times n$ matrix. The following are equivalent (that is, either all of them are true, or all of them are false):

1. A is invertible.
2. The matrix equation $A\mathbf{x} = 0$ has the unique solution $\mathbf{x} = 0$ (where 0 is the $n \times 1$ zero matrix).
3. The RREF of A is I_n.
4. A may be represented as a product of elementary matrices. That is $A = E_1 E_2 \cdots E_n$, where E_i is an elementary matrix.

For our purposes 2. is the most important. What does it mean?

A solution to the matrix equation

$$A\mathbf{x} = 0$$

is an $n \times 1$ matrix $\mathbf{x}$ (a column vector) such that $A\mathbf{x} = 0$. The trivial column vector

$$\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = 0_{n\times 1}$$

is always a solution to this matrix equation, as

$$A0 = 0$$

for any matrix A. Point 2. of Fact 6.5 states that the matrix A is invertible if and only if $\mathbf{x} = 0$ is the unique solution to the matrix equation.

Why is this useful? Remember that we can write SLEs as matrix equations: the fact above lets us understand the solutions of SLEs by understand the invertibility of matrices, and vice versa.

Example 6.6

Given the SLE

$$\begin{aligned} 6x_1 + 9x_2 &= 0 \\ 2x_1 + 3x_2 &= 0 \end{aligned}$$

write the associated matrix equation

$$\begin{bmatrix} 6 & 9 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

By inspection we can determine that

$$\mathbf{x} = \begin{bmatrix} -\frac{3}{2} \\ 1 \end{bmatrix}$$

is a non-trivial solution to this matrix equation as

$$\begin{bmatrix} 6 & 9 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -\frac{3}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Therefore $\mathbf{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is <u>not</u> the unique solution, and we know that A is not invertible!

SLEs and invertible matrices

(from Chapter 1.6 of Anton-Rorres)

Writing SLEs as matrix equations can be very useful. For instance, it allows us to prove the fact we saw in the first lecture.

Fact 6.7

A system of linear equations has either

1. A unique solution

2. Infinitely many solutions

3. No solutions

We could prove this for SLEs of 2 equations in 2 variables by looking at the possible ways straight lines can intersect in the plane. How do we go about proving this for more variables with more equations? Writing the SLE as a matrix equation lets us prove this fact very quickly.

Proof: Let a SLE have the matrix equation

$$A\mathbf{x} = \mathbf{b}$$

for A an $n \times m$ matrix, $\mathbf{x}$ an $m \times 1$ matrix, and $\mathbf{b}$ an $n \times 1$ matrix. Then the equation has either

1. a unique solution
2. more than one solution
3. no solutions

To prove Fact 6.7 we need to show that if the equations has more than one solution, then it must have infinitely many solutions. Assume the matrix equation has more than one solution. Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be two solutions, with $\mathbf{x}_1 \neq \mathbf{x}_2$. That is

$$\begin{aligned} A\mathbf{x}_1 &= \mathbf{b} \\ A\mathbf{x}_2 &= \mathbf{b} \end{aligned}$$

Define $\mathbf{x}_0 = \mathbf{x}_1 - \mathbf{x}_2$. Then

$$\begin{aligned} A\mathbf{x}_0 &= A(\mathbf{x}_1 - \mathbf{x}_2) \\ &= A\mathbf{x}_1 - A\mathbf{x}_2 \\ &= \mathbf{b} - \mathbf{b} \\ &= 0 \end{aligned}$$

so that $A\mathbf{x}_0 = 0$. Consider the matrix $\mathbf{x}_1 + \lambda\mathbf{x}_0$, for a scalar λ:

$$\begin{aligned} A(\mathbf{x}_1 + \lambda\mathbf{x}_0) &= A\mathbf{x}_1 + A(\lambda\mathbf{x}_0) \\ &= A\mathbf{x}_1 + \lambda A\mathbf{x}_0 \\ &= \mathbf{b} + \lambda 0 \\ &= \mathbf{b} \end{aligned}$$

Therefore $\mathbf{x}_1 + \lambda\mathbf{x}_0$ is a solution to the equation for any choice of λ, so there is an infinite number of solutions. ■

Solving SLEs via matrix inversion

We have just seen that matrices can be very useful in the study of SLEs. In particular, that the matrix equation

$$A\mathbf{x} = 0$$

has the trivial solution as its unique solution if and only if the matrix A is invertible. This is an example of an homogeneous SLE.

Definition 6.8

A SLE is homogenous if it has 0 on the right hand side of the associated matrix equation i.e.

$$A\mathbf{x} = 0$$

We can also use matrix inversion to determine the solutions of nonhomogeneous SLEs i.e. systems that look like

$$A\mathbf{x} = \mathbf{b}$$

for $\mathbf{b} \neq 0$.

First, we can use matrix inversion to determine the solutions of SLEs which have the same number of equations as variables.

Fact 6.9

Let $A\mathbf{x} = \mathbf{b}$ be an SLE for A a square matrix. The system has a unique solution if and only if A is invertible.
If A is invertible then the unique solution is $\mathbf{x} = A^{-1}\mathbf{b}$.

We need to get used to manipulating and solving matrix equations. The proof of this fact is a good example of this. Recall that matrix multiplication is noncommutative: that is, $AB \neq BA$. Therefore we need to specify which direction we multiply from when we multiply by matrices.

Let A, B, and C be matrices. If

$$A = B$$

we multiply by C from the left to obtain

$$CA = CB$$

We multiply by C from the right to obtain

$$AC = BC$$

Proof: (Sketch) Consider

$$A\mathbf{x} = \mathbf{b}$$

We want to solve for $\mathbf{x}$. If A^{-1} exists, we can multiply by A^{-1} from the left to obtain

$$\begin{aligned} A^{-1}A\mathbf{x} &= A^{-1}\mathbf{b} \\ \mathbf{x} &= A^{-1}\mathbf{b} \end{aligned}$$

as $A^{-1}A = I$, and $I\mathbf{x} = \mathbf{x}$. ∎

Example 6.10

Question: Solve the SLE

$$\begin{aligned} 2x_1 + 1x_2 + 1x_3 &= 4 \\ 4x_1 + 2x_2 + 1x_3 &= 2 \\ 4x_1 + 1x_2 + 2x_3 &= 5 \end{aligned}$$

Answer: Write down the associated matrix equation

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & 2 & 1 \\ 4 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix}$$

Determine if A is invertible, and find A^{-1} if it is. Place A on the left of I_3, and apply the inversion algorithm described in Lecture 5 to determine if A is invertible. In this case, A^{-1} exists and is given by

$$\begin{bmatrix} -\frac{3}{2} & \frac{1}{2} & \frac{1}{2} \\ 2 & 0 & -1 \\ 2 & -1 & 0 \end{bmatrix}$$

Then by Fact 6.9 the SLE has a unique solution, found by

$$
\begin{aligned}
A^{-1}\begin{bmatrix}4\\2\\5\end{bmatrix} &= \begin{bmatrix}-\frac{3}{2} & \frac{1}{2} & \frac{1}{2}\\ 2 & 0 & -1\\ 2 & -1 & 0\end{bmatrix}\begin{bmatrix}4\\2\\5\end{bmatrix}\\
&= \begin{bmatrix}-\frac{15}{2}\\ 3\\ 6\end{bmatrix}
\end{aligned}
$$

Suggested problems

Practice the material in this lecture by attempting the following problems in **Chapter 1.6** of Anton-Rorres, starting on page 66

- Questions $1,\ 3,\ 15,\ 17$
- True/False questions (b), (c), (e), (f)

Not all of these questions can be completed using material from this lecture - some require material from next lecture.