Announcements

Topics:

- Solving Separable DEs (8.4)
- Systems of DEs (8.5)
- The Phase Plane (8.6)
- Solutions in the Phase Plane (8.7)

To Do:

- Read sections 8.4, 8.5, 8.6, and 8.7 in the textbook
- Work on Assignments and Suggested Practice Problems assigned on the webpage under the SCHEDULE + HOMEWORK link

Predator-Prey Model

Previously, we studied a variety of models for the growth of a single species that lives alone in an environment.

Now, we will consider two species interacting in the same habitat.

One species, called the **prey**, has an ample food supply and the second species, call the **predator**, feeds on the prey.

Predator-Prey Model

For example, consider a population of rabbits (prey) and wolves (predators) in an isolated forest.

Let

$R(t) = \#$ rabbits at time t

$W(t) = \#$ wolves at time t

Two dependent variables, both functions of time

Predator-Prey Model

<u>Assumption 1</u>:

Without predators, prey will grow exponentially.

$$\frac{dR}{dt} = kR, \quad k > 0$$

Without prey, predators will die out exponentially.

$$\frac{dW}{dt} = -rW, \quad r > 0$$

Predator-Prey Model

<u>Assumption 2</u>:

There will be more encounters between the two species if the population of either increases:

$$\#\ \text{of encounters}\ \propto RW$$

Predator-Prey Model

<u>Assumption 3</u>:

Encounters are bad for prey:

$$\frac{dR}{dt} = kR - aRW, \quad a > 0$$

Encounters are good for predators:

$$\frac{dW}{dt} = -rW + bRW, \quad b > 0$$

Predator-Prey Model

Predator-Prey Equations:

$$\frac{dR}{dt} = kR - aRW$$

$$\frac{dW}{dt} = -rW + bRW$$

W (for wolves) represents the predator

R (for rabbits) represents the prey

k, r, a, and b are positive constants

Predator-Prey Model

Example:
Suppose that the populations of rabbits and wolves are described by the predator-prey equations

$$\frac{dR}{dt} = 0.08R - 0.001RW$$

$$\frac{dW}{dt} = -0.02W + 0.00002RW$$

where time *t* is measured in months.

Predator-Prey Model

Example:
(a) Graph the per capita growth rates for each species.

(b) Suppose that initially there are 1000 rabbits and 40 wolves. What will happen to these populations after 2 months? (use Euler's Method)

Euler's Method for a Pair of Linked Autonomous DEs

Algorithm:

$$t_{n+1} = t_n + h$$
$$x_{n+1} = x_n + f(x_n, y_n)h$$
$$y_{n+1} = y_n + g(x_n, y_n)h$$

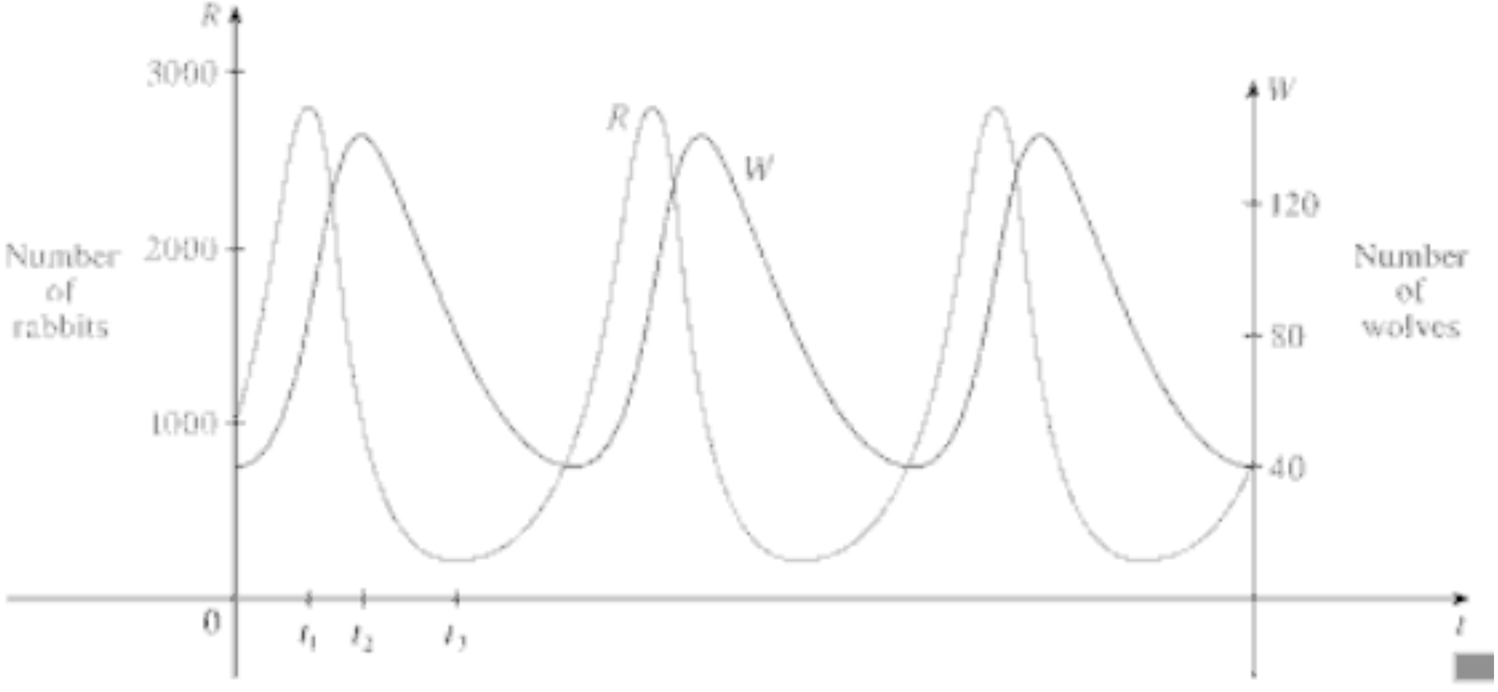
Algorithm In Words:

next time step = previous time step + step size

next approximation = previous approximation + rate of change of the function at previous values x step size

Predator-Prey Model

Graphs of the rabbit and wolf populations as functions of time



Systems of Differential Equations

The predator-prey model is an example of a system of **coupled** (or **linked**) autonomous differential equations.

Coupled Autonomous Differential Equations:
A pair of differential equations in which the rate of change of each state variable depends on its own value and on the value of the other state variable.

$$\frac{dx}{dt} = f(x,y) \quad \text{and} \quad \frac{dy}{dt} = g(x,y)$$

Competitive Selection Model

Recall: **Selection Model**

When two variations of a certain population grow at a rate proportional to their size, we can write a pair of *uncoupled* autonomous DEs:

$$\frac{da}{dt} = \mu a \qquad\qquad \frac{db}{dt} = \lambda b$$

a(t)=population size of type *a* at time *t*;
μ =per capita production rate of type *a*;

b(t)=population size of type *b* at time *t*;
λ =per capita production rate of type *b*.

Competitive Selection Model

Consider the case in which these two types interact and compete for the same resources.

As the size of the total population increases, so does competition for resources, which has a negative effect on the growth rate for each type.

Competitive Selection Model

Suppose that the per capita growth rate of each type decreases *linearly* as a function of the total population, *a*+*b*:

$$\text{per capita growth rate of type } a = \mu\left(1 - \frac{a+b}{K_a}\right)$$

$$\text{per capita growth rate of type } b = \lambda\left(1 - \frac{a+b}{K_b}\right)$$

$$K_a = \text{carrying capacity of type } a$$

$$K_b = \text{carrying capacity of type } b$$

Competitive Selection Model

The coupled autonomous DEs for a competitive selection model are given by

$$\frac{da}{dt} = \mu\left(1 - \frac{a+b}{K_a}\right)a \quad \text{and} \quad \frac{db}{dt} = \lambda\left(1 - \frac{a+b}{K_b}\right)b$$

where $K_a = \text{carrying capacity of type } a$

$K_b = \text{carrying capacity of type } b$

Newton's Law of Cooling

Recall:

Newton's law of cooling expresses the rate of change of the temperature, *T*, of an object as a function of the ambient temperature, *A*, by the equation

$$\frac{dT}{dt} = \alpha(A - T)$$

where α depends on the the size, shape, and material of the object.

Newton's Law of Cooling

If the object is large relative to its environment, it will also have an effect on the ambient temperature.

Newton's Law of Cooling can then be applied to describe the rate of change of the ambient temperature by the equation

$$\frac{dA}{dt} = \alpha_2 (T - A)$$

where α_2 depends on the the size, shape, and heat properties of the environment the object is in.

Newton's Law of Cooling

The rates of change of the temperature of the object and its environment are given by the following system of **coupled autonomous DEs:**

$$\frac{dT}{dt} = \alpha\left(A - T\right) \quad \text{and} \quad \frac{dA}{dt} = \alpha_2\left(T - A\right)$$

In general, α_2 will be smaller as the environment becomes larger.

A Model for a Disease

Recall: **Basic Model for a Disease**

Suppose a disease is circulating in a population. Individuals recover from this disease unharmed but are susceptible to reinfection.

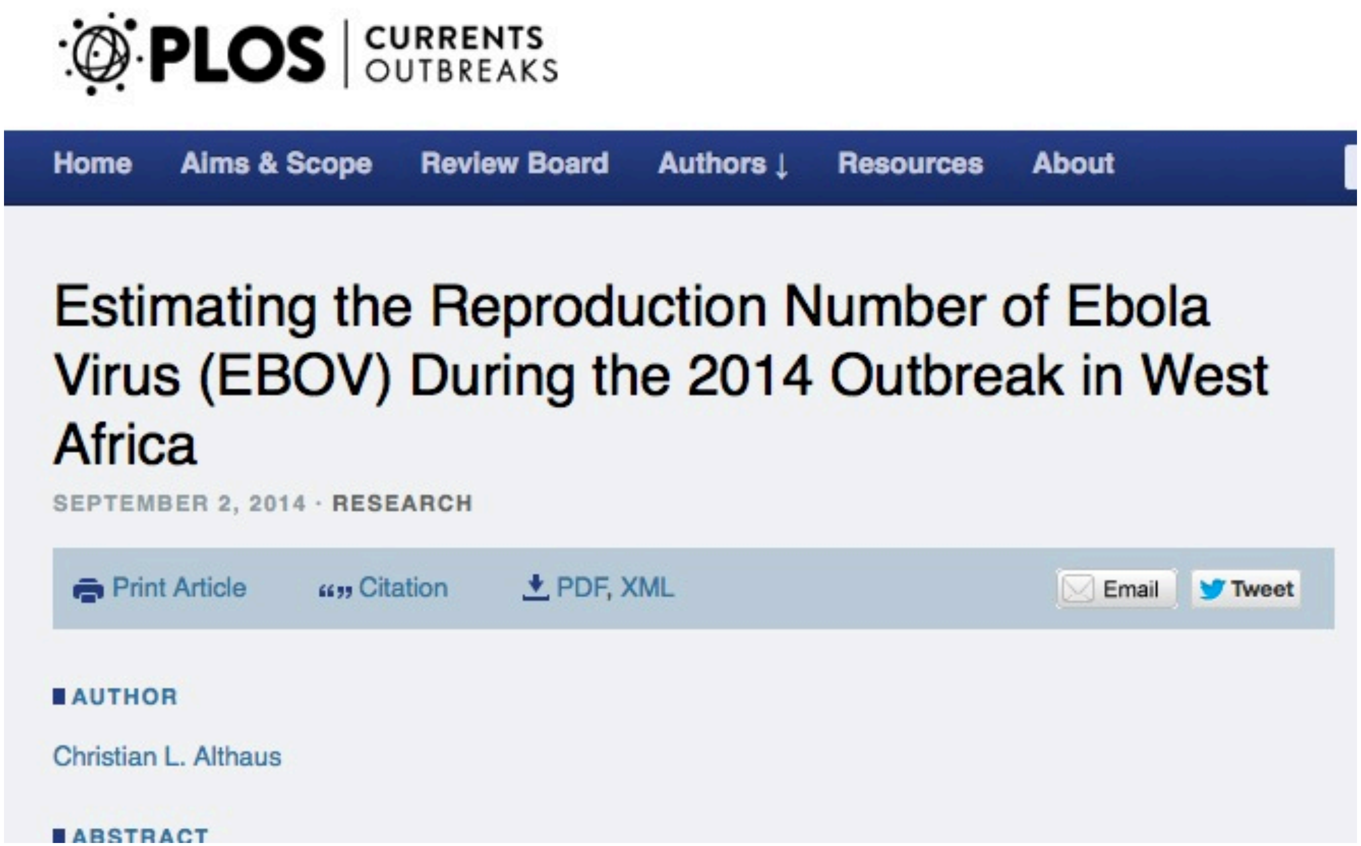
Let I denote the fraction of infected individuals in a population. Then, the rate at which the fraction of infected individuals is changing is given by

$$\frac{dI}{dt} = \alpha I(1 - I) - \mu I$$

where α and μ are positive constants.

A Model for a Disease

__Recall__: This journal article on the Ebola virus (EBOV) outbreak in West Africa was studied in Math 1LS3 (full article posted on our course webpage)



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Estimating the Reproduction Number of Ebola Virus (EBOV) During the 2014 Outbreak in West Africa

SEPTEMBER 2, 2014 · RESEARCH

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ABSTRACT

A Model for a Disease

METHODS

The transmission of EBOV follows SEIR (susceptible-exposed-infectious-recovered) dynamics and can be described by the following set of ordinary differential equations (ODEs):[2]

$$\frac{dS}{dt} = -\beta(t)SI/N,$$
$$\frac{dE}{dt} = \beta(t)SI/N - \sigma E,$$
$$\frac{dI}{dt} = \sigma E - \gamma I,$$
$$\frac{dR}{dt} = (1-f)\gamma I.$$

After transmission of the virus, susceptible individuals S enter the exposed class E before they become infectious individuals I that either recover and survive (R) or die. $1/\sigma$ and $1/\gamma$ are the average durations of incubation and infectiousness. The case fatality rate is given by f. The transmission rate in absence of control interventions is constant, i.e., $\beta(t) = \beta$. After control measures are introduced at time $\tau \leq t$, the transmission rate was assumed to decay exponentially at rate k:[3]

$$\beta(t) = \beta e^{-k(t-\tau)},$$

i.e., the time until the transmission rate is at 50% of its initial level is $t_{1/2} = \ln(2)/k$. Assuming the epidemic

The Phase Plane

The **phase plane** for a system of two autonomous DEs is a coordinate plane with the axes representing the values of the two state variables.

A **phase-plane diagram** is a graphical display of the qualitative behaviour of solutions to a system of two autonomous DEs.

Nullclines

A **nullcline** is a set of points for which a state variable does not change (i.e., for which the rate of change of the state variable is zero).

For the system $\dfrac{dx}{dt} = f(x, y)$ and $\dfrac{dy}{dt} = g(x, y)$

the solutions of $\dfrac{dx}{dt} = f(x, y) = 0$ define the ***x*-nullcline** and

the solutions of $\dfrac{dy}{dt} = g(x, y) = 0$ define the ***y*-nullcline**.

Nullclines

Example:

Find and graph the *R*- and *W*-nullclines in the phase plane for the predator-prey model

$$\frac{dR}{dt} = 0.08R - 0.001RW, \quad \frac{dW}{dt} = -0.02W + 0.00002RW$$

Equilibria

An **equilibrium** of a two-dimensional system of autonomous DEs is a point where the rate of change of *both* state variables is zero.

Equilibria can be found by solving the system

$$\frac{dx}{dt} = f(x,y) = 0 \quad \text{and} \quad \frac{dy}{dt} = g(x,y) = 0$$

Equilibria

Example:

Identify the equilibria of the predator-prey model:

$$\frac{dR}{dt} = 0.08R - 0.001RW, \quad \frac{dW}{dt} = -0.02W + 0.00002RW$$

Finding the Nullclines and Equilibria of Coupled Autonomous DEs

Algorithm:

1. Decide which variable is represented by the horizontal axis and which one by the vertical axis in the phase plane.
2. Write the equations for the nullclines and solve them.
3. Graph each solution in the phase plane.
4. Identify the intersections of nullclines belonging to different variables, as these are the equilibria of the system.

Finding the Nullclines and Equilibria of Coupled Autonomous DEs

Example: Graph the nullclines and find the equilibria for the modified competition equations:

$$\frac{da}{dt} = \mu\left(1 - \frac{a+b}{K_a}\right)a, \quad \frac{db}{dt} = \lambda\left(1 - \frac{b}{K_b}\right)b$$

Solutions

Our ultimate goal is to find *solutions*, that is, descriptions of how the state variables change over time.

Since systems of autonomous differential equations are generally impossible to solve exactly, we need to use alternative methods to determine the behaviour of solutions.

Solutions in the Phase Plane

One approach is to approximate values of solutions using Euler's method.

Example:

Approximate values of the solutions b(t) and p(t) for the predator-prey equations

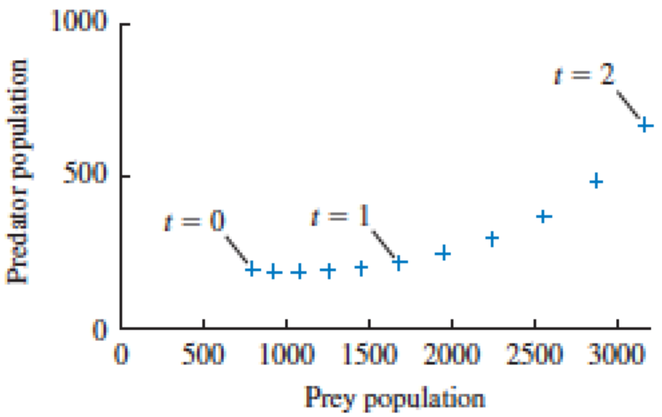
$$\frac{db}{dt} = (1 - 0.001p)b, \quad \frac{dp}{dt} = (-1 + 0.001b)p$$

Solutions in the Phase Plane

Table

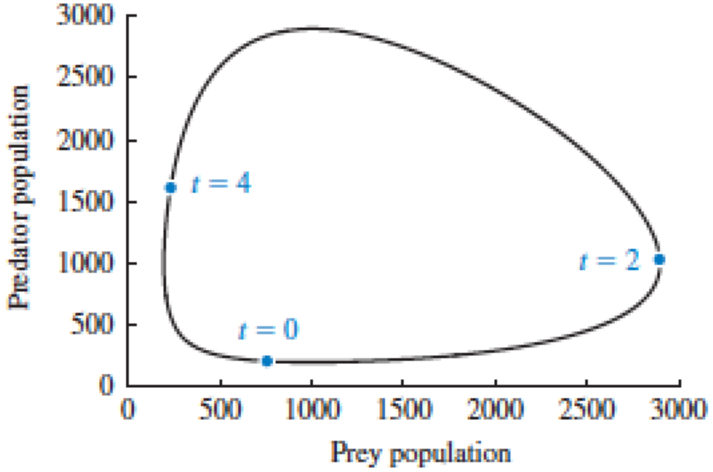
t	Approximation for b	Approximation for p
0	800	200
0.2	928	192
0.4	1078	189.2
0.6	1252.8	192.2
0.8	1455.2	201.9
1	1687.4	220.3
1.2	1950.6	250.6
1.4	2242.9	298.2
1.6	2557.8	372.3
1.8	2878.8	488.3
2	3173.4	671.8

Results from Euler's method plotted in the phase plane



Solutions in the Phase Plane

The graph of a solution in the phase plane is called a **phase-plane trajectory**.



Solutions in the Phase Plane

To describe solutions qualitatively, and avoid the calculations required for Euler's method, we can add *direction arrows* to our phase-plane diagram.

Using these direction arrows, we can sketch a phase-plane trajectory, starting from given initial conditions.

Direction Arrows

In a phase-plane diagram, nullclines divide the plane into regions. We can determine the behaviour of the state variables in each region and draw a **direction arrow** to simultaneous represent these behaviours.

Note: Direction arrows on nullclines are either vertical or horizontal.

Direction Arrows

Drawing Direction Arrows:

Method 1: Pick a pair of values, (b,p), in the region and substitute into the DE. Check whether db/dt and dp/dt are positive or negative.

Method 2: Manipulate the inequalities algebraically.

Method 3: Reason about the equations.

Solutions in the Phase Plane

Example:

(a) Add direction arrows to the phase-plane diagram for the predator-prey equations

$$R' = 0.08R - 0.001RW$$

$$W' = -0.02W + 0.00002RW$$

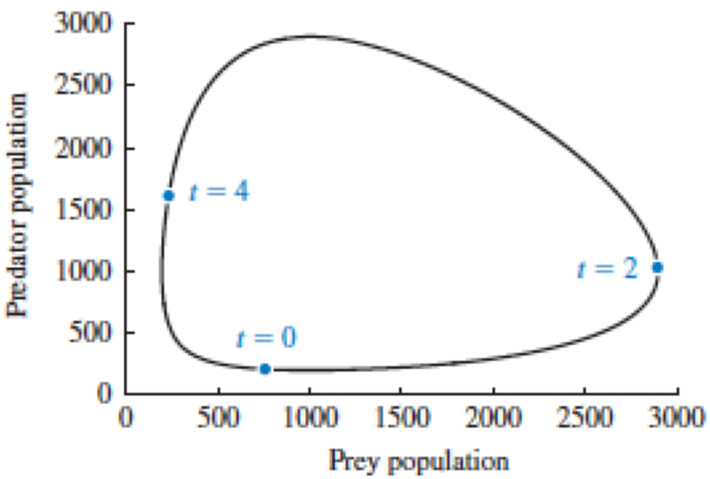
(b) Use the direction arrows to sketch a phase-plane trajectory starting from (R,W) = (1000, 40).

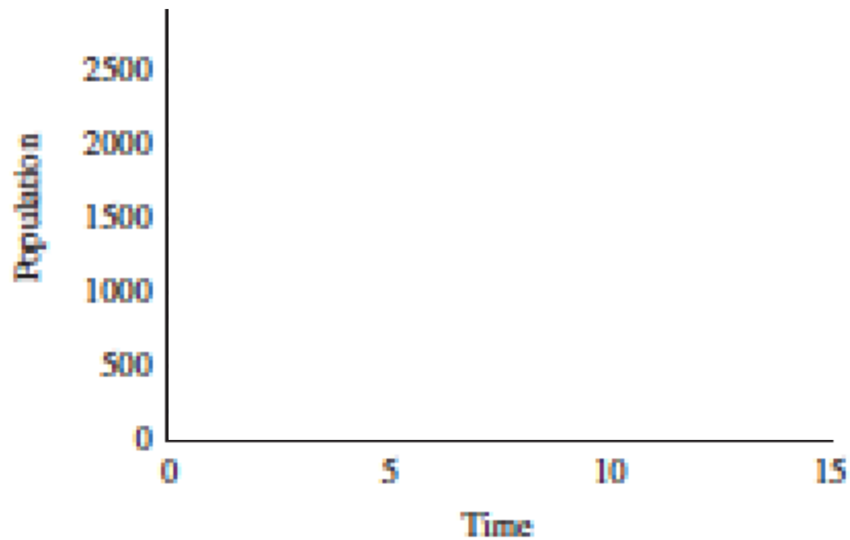
Phase-Plane Trajectory ➡ Graph of a Solution

Starting from the phase-plane trajectory, we can sketch the solutions by tracing along the graph at a constant speed. The horizontal location of the pencil gives the prey population, or the height of the graph b(t). The vertical location of the pencil gives the predator population, or the height of the graph p(t).

Phase-Plane Trajectory → Graph of a Solution

Example:





Graph of a Solution ➡ Phase-Plane Trajectory

Starting from the solution, we can plot the number of predators against the number of prey at several times (such as t=0, t=1, etc.) and connect the dots.

Graph of a Solution ➡ Phase-Plane Trajectory

Example: