MATH 1B03/1ZC3 Winter 2019

Lecture 3: More about matrices

Instructor: Dr Rushworth January 8th

Covered in last lecture:

- REF and RREF
- Solving SLEs using RREF
- Gaussian and Gauss-Jordan elimination
- How to deduce the number of solutions of a SLE from RREF

Matrix operations

(corresponding to Chapter 1.3 of Anton-Rorres)

In the previous lecture we saw that the augmented matrix of a SLE is a very useful tool for solving the SLE itself. In fact, the augmented matrix contains all the of information of the SLE. In addition to helping us solve SLEs, matrices appear in a large number of other contexts throughout science and mathematics - they come up so often because they are a great way of packaging up information, and making it easier to handle. Understanding matrices themselves helps us to better apply them in these contexts.

In this lecture we shall see a number of different ways of manipulating matrices, and combining two or more matrices.

Notation

Let's fix some notation for writing down matrices. A matrix of size $m \times n$ (we say "m by n") is a rectangular array of numbers with m rows and n columns. We write such a matrix like this

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

The term

$$a_{ij}$$

row index column index

is the ij-th entry of A i.e. the i-th entry in the j-th row of A.

We will often use the following short-hand

$$A = \left[a_{ij}\right]$$

to denote the whole matrix.

There are various different ways to represent the entries of a matrix, but they all mean the same thing! If $A = \left[a_{ij}\right]$ is a matrix, then we could also write

$$(A)_{ij} = a_{ij}.$$

In general, the notation $(M)_{ij}$ denotes the ij-th entry of the matrix M.

Example 3.1

A $1 \times n$ matrix has 1 row and n columns. We often refer to such matrices as row vectors. For example,

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{n1} \end{bmatrix}$$

A $n \times 1$ matrix has n rows and 1 column. We often refer to such matrices as column vectors. For example,

$$\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix}$$

Question 3.2

What is the size of following matrix?

$$\begin{bmatrix} 2 & 5 & 8 \\ 9 & 0 & 9 \end{bmatrix}$$

For matrices A and B, the statement $A = B$ means

1. A and B are the same size i.e. they have the same number of rows and columns
2. They have the same entries. That is, if $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ and $B = \begin{bmatrix} b_{ij} \end{bmatrix}$, then

   $$a_{ij} = b_{ij}$$

   for alll i and j.

A matrix is square if it has the same number of rows as columns i.e. if it is an $n \times n$ matrix.

Addition of matrices

Matrices are more complicated than the counting numbers 1, 2, 3, etc. However, many of things that we can do with numbers can also be done with matrices. Firstly, we shall see how to add matrices.

Definition 3.3

Let $A = \left[a_{ij}\right]$ and $B = \left[b_{ij}\right]$ be matrices of the same size. The matrix sum of A and B is defined as

$$A + B := \left[a_{ij} + b_{ij}\right].$$

Equivalently, we could write

$$(A + B)_{ij} = (A)_{ij} + (B)_{ij} = a_{ij} + b_{ij}.$$

It is clear that $A + B = B + A$.

If A and B are matrices of different sizes, then their sum is undefined.

Example 3.4

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 7 & 9 \end{bmatrix}$$

Let A and B be the following matrices

$$A = \begin{bmatrix} 1 & 0 & 5 & 3 \end{bmatrix}$$
$$B = \begin{bmatrix} 4 & 1 & 0 \\ 7 & 12 & 2 \end{bmatrix}$$

The sum $A + B$ is not defined.

The zero matrix of size $m \times n$ is the $m \times n$ matrix whose entries are all 0. For example, the 2×2 zero matrix is

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

We denote the zero matrix by 0 in matrix calculations. If we need to specify its size, we write $0_{m \times n}$.

It is clear that

$$A + 0 = 0 + A = A$$

for any matrix A (and the appropriate size zero matrix 0).

Scalar multiplication

Adding matrices is easy, and is very similar to adding numbers. Multiplying matrices is harder. In fact, there are two different kinds of multiplication. The first is scalar multiplication: multiplying an entire matrix by a scalar (a.k.a. a constant).

Definition 3.5

Let $A = \left[a_{ij}\right]$ be a matrix, and λ (the Greek letter lambda) be a constant, known as a scalar. Define the matrix λA as

$$\lambda A := \left[\lambda a_{ij}\right].$$

Equivalently,

$$(\lambda A)_{ij} = \lambda\,(A)_{ij} = \lambda a_{ij}.$$

Example 3.6

If C is the following matrix

$$C = \begin{bmatrix} 9 & -6 & 3 \\ 3 & 0 & 12 \end{bmatrix}$$

then

$$\frac{1}{3}C = \begin{bmatrix} 3 & -2 & 1 \\ 1 & 0 & 4 \end{bmatrix}$$

and

$$-2C = \begin{bmatrix} -18 & 12 & -6 \\ -6 & 0 & -24 \end{bmatrix}$$

Subtraction of matrices

We use the shorthand $(-1)M = -M$ for the negative of M i.e. the matrix with entries

$$(-M)_{ij} = -(M)_{ij}\,.$$

Definition 3.7

Let $A = \left[a_{ij}\right]$ and $B = \left[b_{ij}\right]$ be matrices of the same size. Define

$$A - B := \left[a_{ij} - b_{ij}\right]$$

Equivalently,

$$A - B = A + (-1)B$$

or

$$(A - B)_{ij} = (A)_{ij} - (B)_{ij} = a_{ij} - b_{ij}.$$

If A and B are of different sizes, then $A - B$ is undefined.

Example 3.8

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$$

Properties of matrix addition and scalar multiplication

The operations of matrix addition and scalar multiplication satisfy certain properties, which we can often use to speed up calculations.

Fact 3.9

Let A and B be matrices of the same size. Then

1. $A + B = B + A$

2. $(A + B) + C = A + (B + C)$.
   That is, it doesn't matter whether we compute $A+B$ first, or $B+C$ first.
3. $A + 0 = 0 + A = A$
4. $A - A = A + (-1)A = 0$

Let λ and μ (the Greek letter mu, pronounced "mew") be scalars. Then

5. $\lambda(A + B) = \lambda A + \lambda B$
6. $(\lambda + \mu)A = \lambda A + \mu A$
7. $\lambda(\mu A) = \lambda\mu A$
8. $1A = A$

When doing matrix calculations keep these facts in your mind, and try to use them to speed up your calculcations!

Matrix multiplication

This is the second type of multiplication: multiplying two matrices together. From now on, when we say "matrix multiplication" we always mean this type of multiplication. If we need to talk about scalar multiplication, we will make sure to say the word "scalar".

Definition 3.10

Let $A = \left[a_{ij}\right]$ be an $m \times n$ matrix and $B = \left[b_{ij}\right]$ an $n \times k$ matrix. The product of A and B is an $m \times k$ matrix, and is denoted AB. The product has entries

$$\begin{aligned}(AB)_{ij} &= \sum_{l=1}^{n} a_{il} b_{lj} \\ &= a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}.\end{aligned}$$

That is, to compute $(AB)_{ij}$ multiply the terms of the i-th row of A by those of the j-th row of B, then add up the result.

If the matrices A and B are not of the correct size, then their product is not defined. The following informal equation can help us to remember when we are allowed to take a matrix product:

$$(m \times n)(n \times k) = (m \times k).$$

For example, a the product of a 3×2 matrix and a 2×5 matrix is a 3×5 matrix:

$$(3 \times 2)(2 \times 5) = (3 \times 5)$$

but the product of a 3×2 matrix and a 4×2 matrix is undefined:

$$(3 \times 2)(4 \times 2) = \text{undefined!}$$

Example 3.11

Question: Given the matrices

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix}$$

compute the product AB.
Answer: First, check if the product is defined: A is an 2×3 matrix, and B is an 3×4 matrix. Therefore AB is defined, and as

$$(2 \times 3)\,(3 \times 4) = (2 \times 4)$$

AB will be a 2×4 matrix.
Next, compute each entry of AB (this can be done in any order you like).
To compute $(AB)_{23}$, look at the 2-nd row of A and the 3-rd column of B:

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} \Box & \Box & \Box & \Box \\ \Box & \Box & 26 & \Box \end{bmatrix}$$

as

$$(2 \cdot 4) + (6 \cdot 3) + (0 \cdot 5) = 26$$

Repeat this for $(AB)_{11}$

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 12 & \boxed{} & \boxed{} & \boxed{} \\ \boxed{} & \boxed{} & 26 & \boxed{} \end{bmatrix}$$

as

$$(1 \cdot 4) + (2 \cdot 0) + (4 \cdot 2) = 12$$

Repeating this for the remaining entries, we obtain

$$AB = \begin{bmatrix} 12 & 27 & 30 & 13 \\ 8 & -4 & 26 & 12 \end{bmatrix}$$

Question 3.12

Compute the following matrix products, if they are defined

$$\begin{bmatrix} 0 & 1 \\ -2 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 4 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

Matrix multiplication is very different from multiplication of numbers. It is **non-commutative**! This means that

$$AB \neq BA$$

in general.

Example 3.13

If

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

then

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \end{aligned}$$

but

$$\begin{aligned} BA &= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}. \end{aligned}$$

As

$$\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

we see that $AB \neq BA$.

The matrix form of a linear system

In Lectures 1 and 2 we saw how to encode a SLE into an augmented matrix. There is another way to write a SLE in terms of matrices, using matrix multiplication. These two ways are totally equivalent i.e. they contain exactly the same information. We introduce this new way because it is sometimes easier to work with than the augmented matrix.

Consider the following system of m equations in n variables

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m1}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \tag{$\star$}$$

We produce 3 matrices associated to this SLE.

The $m \times n$ matrix A

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{mn} & \cdots & a_{mn} \end{bmatrix}$$

is the matrix of the coefficients a_{ij}.

The $n \times 1$ matrix $\mathbf{x}$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

is the column vector of the variables x_i.

The $m \times 1$ matrix $\mathbf{b}$

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

is the column vector of the right hand sides b_i.

The produce $A\mathbf{x}$ is defined: as $(m \times n)(n \times 1) = (m \times 1)$, it will be a $m \times 1$ matrix. In fact,

$$A\mathbf{x} = \mathbf{b}$$

by construction! The equation $A\mathbf{x} = \mathbf{b}$ contains all of the information in Equation ($\star$). That is, we have produced a new way of packaging a SLE into a matrix equation. This matrix equation is very useful, as we shall see later.

Example 3.14

Question: write the SLE

$$\begin{aligned} x_1 - 3x_2 + 3x_3 &= 1 \\ 2x_1 + x_2 - x_3 &= 5 \end{aligned}$$

as a matrix equation.
Answer: The matrix of coefficients is

$$A = \begin{bmatrix} 1 & -3 & 3 \\ 2 & 1 & -1 \end{bmatrix}$$

The matrix of variables is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The matrix $\mathbf{b}$ is

$$\mathbf{b} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

Therefore the matrix equation is

$$\begin{bmatrix} 1 & -3 & 3 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

Suggested problems

Practice the material in this lecture by attempting the following problems in **Chapter 1.3** of Anton-Rorres, starting on page 36

- Questions $1, 5, 11, 15, 29, 35$
- True/False questions (h), (i), (j)

Not all of these questions can be completed using material from this lecture - some require material from next lecture.