Math 1LS3 Week 6: The Derivative

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This week, we will cover a lot: 3.5 and 4.1-4.5. Next week, we should finish Ch.4 and start Ch.5.

Overview

By the end of the week, you should be able to:

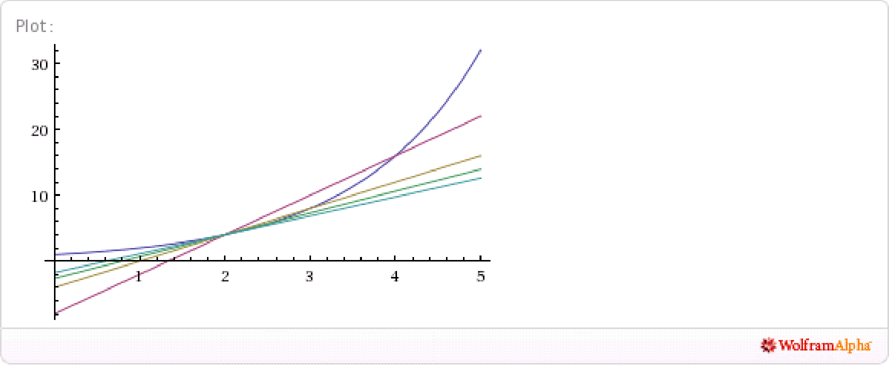
- Find the derivative of simple functions using the definition
- Graph tangent lines
- Graph of function $\rightarrow$ graph of derivative
- Interpret graph of derivative
- Understand the appropriate units for measuring a derivative
- Find the derivative of compound functions using differentiation rules

It is essential for the rest of the course (and the tests) that you:

- Memorize the derivatives of all the basic functions (ASAP).
- Practice taking derivatives of compound functions until you're fast at it (by end of this week).

Instantaneous Rates of Change

Instantaneous speed (not average speed) is what speedometers show.
How can we make sense of instantaneous speed? Geometrically:



Use shorter and shorter intervals to measure rate.

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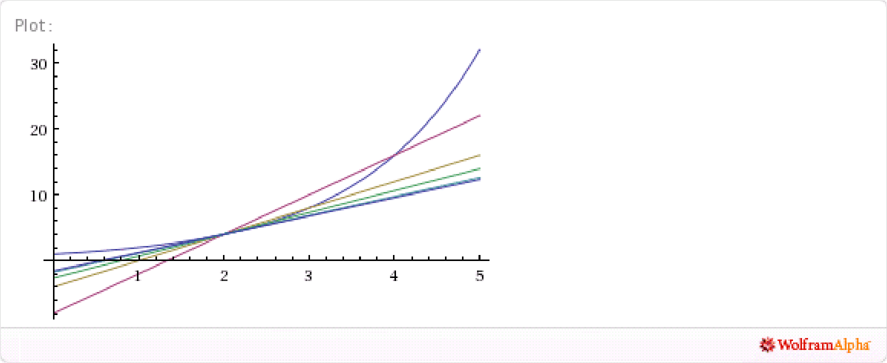

The secant lines get close to a "tangent" line.

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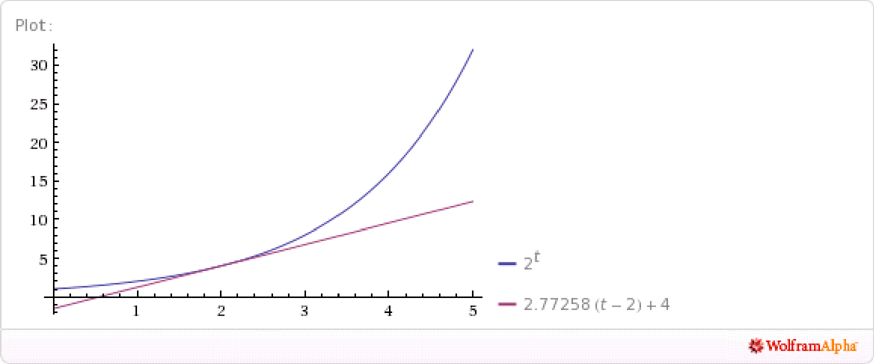
How can we make sense of instantaneous speed? Geometrically:



The slope of the tangent line is the instantaneous rate of change.

The slope *over the interval* $[x, x+h]$ is:

$$\frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{h} \quad \text{("the difference quotient")}$$

(draw picture on board).

As h gets small, we get instantaneous rate of change (at x):

$$\frac{dy}{dx} := \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{("the derivative")}$$

$\frac{dy}{dx}$=derivative=slope of tangent line=rate of change.

Finding a Derivative by Taking a Limit

Problem

Using the definition of dy/dx, find the line tangent to $y = \sqrt{x}$ at $x = 4$.

Solution

First find the slope (i.e. derivative):

$$f'(x) = \frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

Use algebra to find the limit:

$$\lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \frac{1}{2\sqrt{x}}$$

Tangent at $(4, 2)$ *has slope*$=f'(4) = \frac{1}{4}$. *Tangent:* $\boxed{y - 2 = \frac{1}{4}(x-4)}$.

Derivatives are *rates of change*.

Example

- P=population size (measured in organisms)
- t=time (measured in years, e.g.)
- $\frac{dP}{dt}$=population growth rate (in organisms/year)

A **differential equation** (continuous–time dynamical system) is an equation involving derivatives.

$$\frac{dP}{dt} = kP(L - P)$$

- P=population size (measured in organisms)
- t=time (measured in years, e.g.)
- $\frac{dP}{dt}$=population growth rate (in organisms/year)
- k=constant (initial growth rate)
- L=constant (carrying capacity)

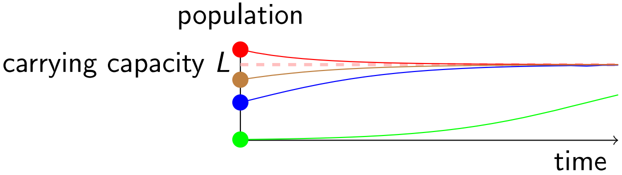
Diff. Eqs. may express proportionality between *rates* and other variables.

Solving the differential equation means replacing it with $P(t) = \cdots$.

Differential Equations and Initial Values

Different **initial values** yield different solutions.

$$\frac{dP}{dt} = kP(L - P)$$



"Initial value" means $P(0)$.

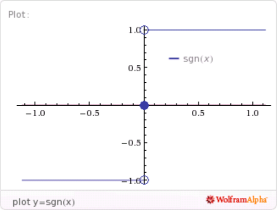
For solution techniques: see MATH 1LT3.

Differentiability

A function with a derivative is called *differentiable*.

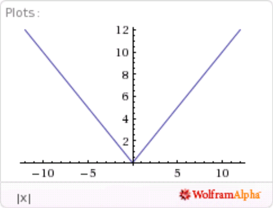
Some ways a function can fail to be differentiable:

$y = \operatorname{sgn}(x)$



discontinuity

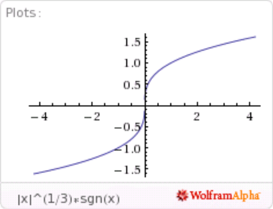
$y = |x|$



corner

$$\lim_{h\to 0^-} \frac{\Delta y}{\Delta x} \neq \lim_{h\to 0^+} \frac{\Delta y}{\Delta x}$$

$y = x^{1/3}$



vertical tangent

$dy/dx = \pm\infty$

Example of non-differentiable function

Problem

Show that $y = |x|$ is not differentiable at $x = 0$. Classify the non-differentiability.

Solution

- $\lim\limits_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = 1$ *(right slope at 0).*
- $\lim\limits_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = -1$ *(left slope at 0).*
- $\lim\limits_{h\to 0^+} \frac{f(0+h)-f(0)}{h} \neq \lim\limits_{h\to 0^-} \frac{f(0+h)-f(0)}{h}$, *so* $\lim\limits_{h\to 0} \frac{f(0+h)-f(0)}{h}$ *does not exist.*
- *So $f'(0)$ does not exist.*

This "singularity" is a corner since left slope$\neq$right slope.

Graph of Function $\rightarrow$ Graph of Derivative

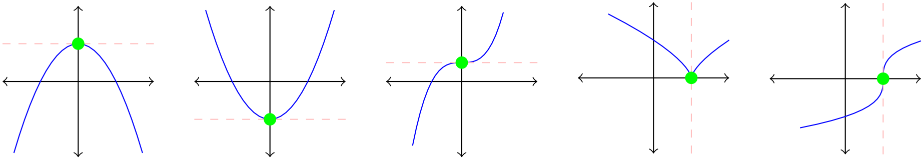
If f is differentiable, the following are roughly true:

- f is increasing $\leftrightarrow$ derivative ≥ 0
  - The steeper the increase, the bigger the derivative
- f is decreasing $\leftrightarrow$ derivative ≤ 0
  - The steeper the decrease, the more negative the derivative
- f has horizontal tangent $\leftrightarrow$ derivative $= 0$

Critical Points

A **critical point** of f is x where:

$$\boxed{f'(x) = 0 \text{ OR } f'(x) \text{ is not defined}}$$

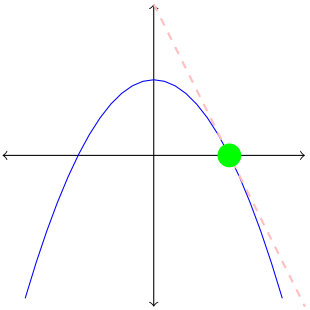


$f' = 0$	$f' = 0$	$f' = 0$	f' undef.	f' undef.
local max	local min	neither	min. (cusp)	vert. tangent

At a critical point, f might be locally maximal, minimal, or neither.

Regular Points

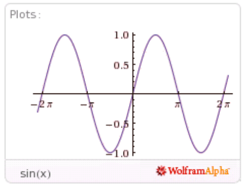
At **regular** (non–critical) points, no max or min is possible. (Why?)



Graph of Function $\to$ Graph of Derivative

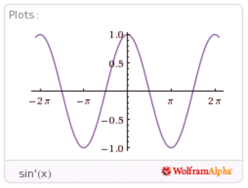
Problem

The graph of $f(x) = \sin(x)$ *is shown below. Graph* $f'(x)$.



Also, identify the critical points.

Solution



- $f' = 0$ *where* f *is flat*
- *Crit pts:* $x = \dots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \dots$.
- $f' \geq 0$ *where* f *is increasing*
- $f' \leq 0$ *where* f *is decreasing*

Why, it's the cosine function! *If there's time, I'll prove this.*

Derivatives of Important Functions

Memorize these:

Function	Derivative
17	0
$mx + b$	m
x^n	nx^{n-1}
e^x	e^x
$\ln(x)$	$1/x$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\sec^2(x)$
$\sec(x)$	$\sec(x)\tan(x)$
$\cot(x)$	$-\csc^2(x)$
$\csc(x)$	$-\csc(x)\cot(x)$
$\arcsin(x)$	$\frac{1}{\sqrt{1-x^2}}$
$\arctan(x)$	$\frac{1}{1+x^2}$
$\operatorname{arcsec}(x)$	$\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$

Why e? Why radians?

Disguised Powers

The *power rule* works even for negative and fractional n:

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Example

$$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}(x^{-1}) = -1x^{-2} = -\frac{1}{x^2}$$

Example

$$\frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}(x^{1/2}) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

Sum Rule

$$(f+g)' = f' + g'$$

Problem

Find $\frac{d}{dx}(x^3 + x^2 + 5)$.

Solution

$$(x^3 + x^2 + 5)' = (x^3)' + (x^2)' + (5)' = 3x^2 + 2x^1 + 0 = 3x^2 + 2x$$

Difference rule

$$(f - g)' = f' - g'$$

Problem

$f(x) = \sin(x) - \tan(x)$. *Find* $f'(x)$.

Solution

$$f'(x) = \cos(x) - \sec^2(x)$$

Constant Multiple Rule

If c is a constant, then:

$$\frac{d}{dx}(cf(x)) = c\frac{d}{dx}f(x)$$

Now you can find the derivative of any polynomial.

Problem

What's the derivative of $y = 5x^3 - 2x^2 + 4$?

Solution

$$\frac{dy}{dx} = 15x^2 - 4x + 0 = 15x^2 - 4x$$

The derivative of a polynomial is always one degree lower

Product Rule

$$(f \cdot g)' = f' \cdot g + f \cdot g'$$

Problem

What's the derivative of $x^2 e^x$?

Solution

$$\frac{d}{dx}(x^2 e^x) = \frac{d}{dx}(x^2) \cdot e^x + x^2 \frac{d}{dx}(e^x) = 2xe^x + x^2 e^x = (2x + x^2)e^x$$

The product rule can be understood in terms of rectangle area.

Quotient Rule

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

- If you find it easy to memorize, that's great.
- If you find it hard to memorize: don't. Just find the derivative using the product rule and the chain rule (since $\frac{f(x)}{g(x)} = f(x) \cdot (g(x))^{-1}$.)

Problem

Verify the rule $\frac{d}{dx}\tan(x) = \sec^2(x)$ *using* $\tan(x) = \frac{\sin(x)}{\cos(x)}$

Quotient Rule: $\sin(x)/\cos(x)$

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

Solution

$$\left(\frac{\sin(x)}{\cos(x)}\right)' = \frac{\sin'(x)\cos(x) - \sin(x)\cos'(x)}{\cos(x)^2}$$

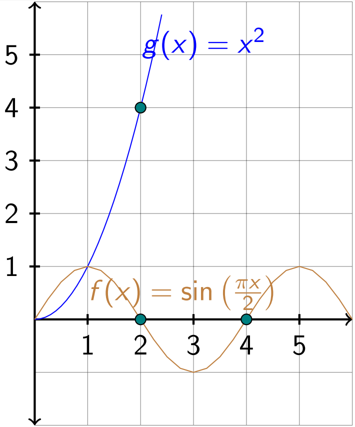
$$= \frac{\cos(x)\cos(x) - \sin(x) \cdot - \sin(x)}{\cos(x)^2}$$

$$= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)}$$

$$= \frac{1}{\cos^2 x} = \sec^2(x).$$

Problem

Let $f(x) = \sin(\pi x/2)$ *and* $g(x) = x^2$. *Is* $f(g(x))$ *increasing or decreasing near* $x = 2$?



Increasing! $g'(2)$ is positive and $f'(4)$ is positive.

The Chain Rule

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

Another way of thinking about it:

$$\frac{dz}{dx} = \frac{dz}{\cancel{dy}} \cdot \frac{\cancel{dy}}{dx}$$

The dy's "cancel".

Example

If $f(x) = \frac{1}{\sin(x)} = (\sin(x))^{-1}$, then
$f'(x) = -1(\sin(x))^{-2} \cdot \sin'(x) = -\frac{\cos(x)}{\sin^2(x)} = -\csc(x)\cot(x)$.

Note: $(f \circ g)'(x) = f'(x)g'(x)$ is wrong. In computing $(f \circ g)(x)$, the number fed into f is $g(x)$. So the rate of change of f *at* $g(x)$ is what matters.

Chain Rule: Examples

Problem

Find the derivatives of:

- $\sin(x^2)$
- $\sin(\cos(\tan(x)))$
- $\sin^3(x)$

Solution

- $\frac{d}{dx}(\sin(x^2)) = \cos(x^2) \cdot \frac{d}{dx}(x^2) = \cos(x^2) \cdot 2x = 2x\cos(x^2)$
- $\frac{d}{dx}(\sin(\cos(\tan(x)))) = \cos(\cos(\tan(x))) \cdot \frac{d}{dx}(\cos(\tan(x)))$
  $= \cos(\cos(\tan(x))) \cdot -\sin(\tan(x)) \cdot \sec^2(x)$
- $[\sin^3(x)]' = 3\sin^2(x) \cdot \cos(x)$.

Chain Rule: Implicit Differentiation

Problem

On the circle $x^2 + y^2 = 169$, what's the tangent line at $(5, -12)$?

Solution

Take the derivative of both sides. y is an (unknown) function of x, so use chain rule.

$$2x + 2yy' = 0$$

At $x = 5, y = -12$, solve for y'. $y' = 5/12$.
Then use point-slope: $y + 12 = \frac{5}{12}(x - 5)$.

Note: you could also solve for $y = -\sqrt{169 - x^2}$. But sometimes solving for y is not practical.

Chain Rule: Implicit Differentiation: Inverse Functions

Problem

Find $\frac{d}{dx}(\sin^{-1}(x))$.

Solution

$y = \sin^{-1}(x)$ *is the same as* $\sin(y) = x$. *Use implicit differentiation:*

$$\cos(y)y' = 1 \qquad \Longrightarrow \qquad y' = \frac{1}{\cos(y)}$$

How can we write $\cos(y)$ *in terms of* x*?* *Draw a triangle!*
$\cos(y) = \sqrt{1 - x^2}$, *so* $y' = \frac{1}{\sqrt{1-x^2}}$.

Good practice: figure out the derivatives of $\arctan(x), \operatorname{arcsec}(x)$.

Basic Curve–Sketching

To sketch the graph of $y = f(x)$:

1. Plot asymptotes.
2. Find $f'(x)$.
3. Find and plot critical points (where $f' = 0$ **or** f' **doesn't exist**)
4. Where is the graph increasing? Decreasing?

Next week, we'll see how to make even better sketches.

Example 4.1.7: Sketch $f(x) = 4x + \frac{10}{x^2}$.

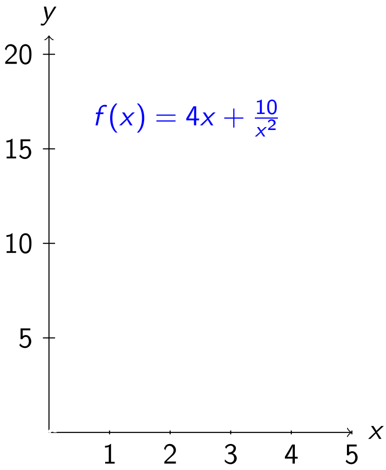
$$f(x) = 4x + \frac{10}{x^2}$$

Find Asymptotes

As $x \to 0^+$, $f(x) \to \infty$.
Vertical asymptote: $x = 0$.

As $x \to \infty$, $\frac{10}{x^2} \to 0$.
So $f(x)$ gets close to $4x$.

Oblique asymptote: $y = 4x$.



Example 4.1.7: Sketch $f(x) = 4x + \frac{10}{x^2}$.

$$f(x) = 4x + \frac{10}{x^2}$$

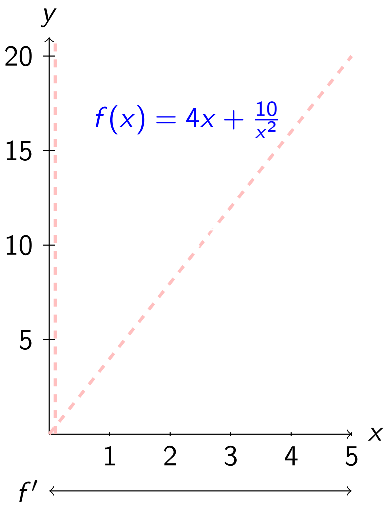
Find f':

$$f'(x) = 4 - \frac{20}{x^3}$$

Find Critical Points:

$$4 - \frac{20}{x^3} = 0 \Rightarrow x = \sqrt[3]{5} \approx 1.7$$

Plot Critical Points: $f(1.7) \approx 10.3$



Example 4.1.7: Sketch $f(x) = 4x + \frac{10}{x^2}$.

$$f(x) = 4x + \frac{10}{x^2}$$

$$f'(x) = 4 - \frac{20}{x^3}$$

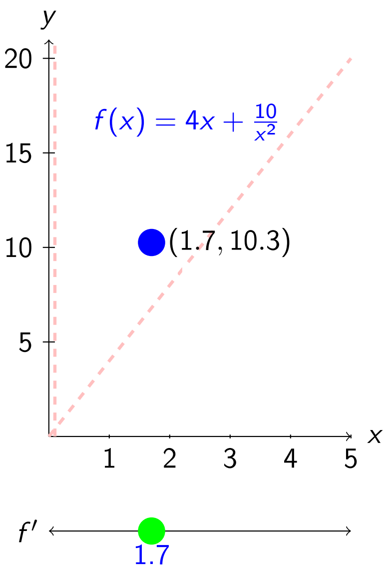
<u>Where is f increasing/decreasing?</u>

Test a point < 1.7.

$$f'(1) = 4 - 20 < 0$$

Test a point > 1.7.

$$f'(3) = 4 - \frac{20}{27} > 0$$



Proof that $\frac{d}{dx}\sin(x) = \cos(x)$

If there's time, we'll do a "proof" here. Otherwise, interested students can easily locate a proof online.