12C3

Last Day   Row, Column, Null Space

A is $m \times n$

eg $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \\ -2 & -4 \end{bmatrix}$ ← $A \in M_{32}$, a 3x2 matrix

& $A\vec{x} = \vec{b}$ has $\vec{x} \in \mathbb{R}^2$, $\vec{b} \in \mathbb{R}^3$

$\mathbb{R}^2$ ← n, $\mathbb{R}^3$ ← m

row(A) = Span of rows of A = Span([1,2] [3,6] [-2,-4])

$\subseteq \mathbb{R}^2 = \mathbb{R}^n$

note: row(A) = row(RREF of A), here = Span([1,2])

dim(row(A)) = rank = # leading 1's in REF ⇒ $\begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

here rank = 1

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \\ -2 & -4 \end{bmatrix}$$

null(A) = all vectors such that $A\vec{x} = \underline{\underline{\vec{0}}}$

$\subseteq \mathbb{R}^2 = \mathbb{R}^n$

<u>here</u> $\text{null}(A) = \text{Span}\left(\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right)$

dim(null(A)) = <u>nullity</u> = # parameters in $A\vec{x} = \vec{0}$

here = 1 = nullity

nullity + rank = (# variables in $A\vec{x} = \vec{b}$) = n = <u>2 here</u>

<u>note</u> row(A) & null(A) are <u>orthogonal</u>

(using dot product / Euclid. inner product).

eg $A = \begin{bmatrix} 1 & 1 & 2 & 5 \\ 2 & 3 & 1 & 0 \\ 2 & 2 & 4 & 10 \end{bmatrix}$ Let's put in <u>RREF</u>.

$R_2 - 2R_1$ & $R_3 - 2R_1$ $\Rightarrow$ $\begin{bmatrix} 1 & 1 & 2 & 5 \\ 0 & 1 & -3 & -10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$R_1 - R_2$ $\Rightarrow$ $\begin{bmatrix} 1 & 0 & 5 & 15 \\ 0 & 1 & -3 & -10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

<u>Notice</u>

$$\left.\begin{bmatrix} 5 \\ -3 \\ 0 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 3\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\} \text{ in RREF}$$

$$\left.\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - 3\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}\right\} \text{ in original } A.$$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \\ -2 & -4 \end{bmatrix} \qquad \text{col}(A) = \text{span of col. of } A$$

$$= \text{span}\left(\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} \begin{bmatrix} 2 \\ 6 \\ -4 \end{bmatrix} \right) \leq \mathbb{R}^3 = \underline{\mathbb{R}^m}$$

If $A\vec{x} = \vec{b} \Rightarrow \vec{b} \in \text{col}(A)$

here $\begin{bmatrix} 1 & 2 \\ 3 & 6 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \vec{b} \Rightarrow \underline{x \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} + y \begin{bmatrix} 2 \\ 6 \\ -4 \end{bmatrix} = \vec{b}}$

<u>Note</u>

<u>Row ops</u> change <u>col. vectors</u> & <u>col. space</u> <u>but</u> do <u>not change relations between columns!</u>

Similarly $\quad 3 \cdot (\text{col}\ \#3) + (-1)\,\text{col}(\#2) = \underline{\text{col}\ \#4}$

in both $\underline{A}$ & RREF $\underline{\text{of } A}$

$$A = \begin{bmatrix} 1 & 1 & 2 & 5 \\ 2 & 3 & 1 & 0 \\ 2 & 2 & 4 & 10 \end{bmatrix} \qquad \text{RREF} \begin{bmatrix} 1 & 0 & 5 & 15 \\ 0 & 1 & -3 & -10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

<u>In A</u> col. corresponding to <u>leading 1's</u> in RREF $\Leftarrow$
- <u>Span</u> col(A)
- form a <u>basis</u> of <u>col(A)</u>

- <u>Col. with leading 1's</u> span all col. in RREF
- Leading 1 col. form a <u>basis</u> of RREF col. space.

<u>Here</u> $\quad \text{col}(A) = \text{Span}\left(\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \right)$

$$\underline{\dim(\text{col}(A))} = \underline{\text{rank}(A)} = \underline{2}$$

So notice row(A) & col(A) both have dimension = rank(A)

rank(A) ≤ min(m, n) for M_{mn}.

eg. Find the dimension of span of columns of the given matrix!

$$\begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \\ 2 & 1 & 6 \\ 5 & 5 & 20 \\ 3 & 7 & 20 \end{bmatrix} = A.$$

Soln Could row reduce as it is!

leading 1's = rank = dim(col(A))

Shorter Reduce A^T

$$\begin{bmatrix} 1 & 1 & 2 & 5 & 3 \\ 0 & 1 & 1 & 5 & 7 \\ 2 & 4 & 6 & 20 & 20 \end{bmatrix}$$

row(A) = col(A^T), col(A) = row(A^T)

$\dim(\text{row}(A^T)) = \text{rank}(A^T)$ **faster!**

A^T take $R_3 - 2R_1 \Rightarrow \begin{bmatrix} 1 & 1 & 2 & 5 & 3 \\ 0 & 1 & 1 & 5 & 7 \\ 0 & 2 & 2 & 10 & 14 \end{bmatrix}$

$R_3 - 2R_2 \rightarrow \begin{bmatrix} 1 & 1 & 2 & 5 & 3 \\ 0 & 1 & 1 & 5 & 7 \\ & & & & \end{bmatrix}$ Rank = 2

$\dim(\text{col}(A)) = \dim(\text{row}(A^T)) = \text{rank} = 2$

Remember our Mega theorem

If A is a square $n \times n$ matrix the following are equivalent!

1) A^{-1} exists
2) A is a product of elementary E
3) A has RREF of I

9) nullity $= 0$
10) $\text{null}(A) = \{\vec{0}\}$

4) $A\vec{x} = \vec{0} \Rightarrow \vec{x} = \vec{0}$ only

5) $A\vec{x} = \vec{b}$ has unique soln.

6) $A\vec{x} = \vec{b}$ always has soln.

7) $\det(A) \neq 0$

8) $\lambda \neq 0$

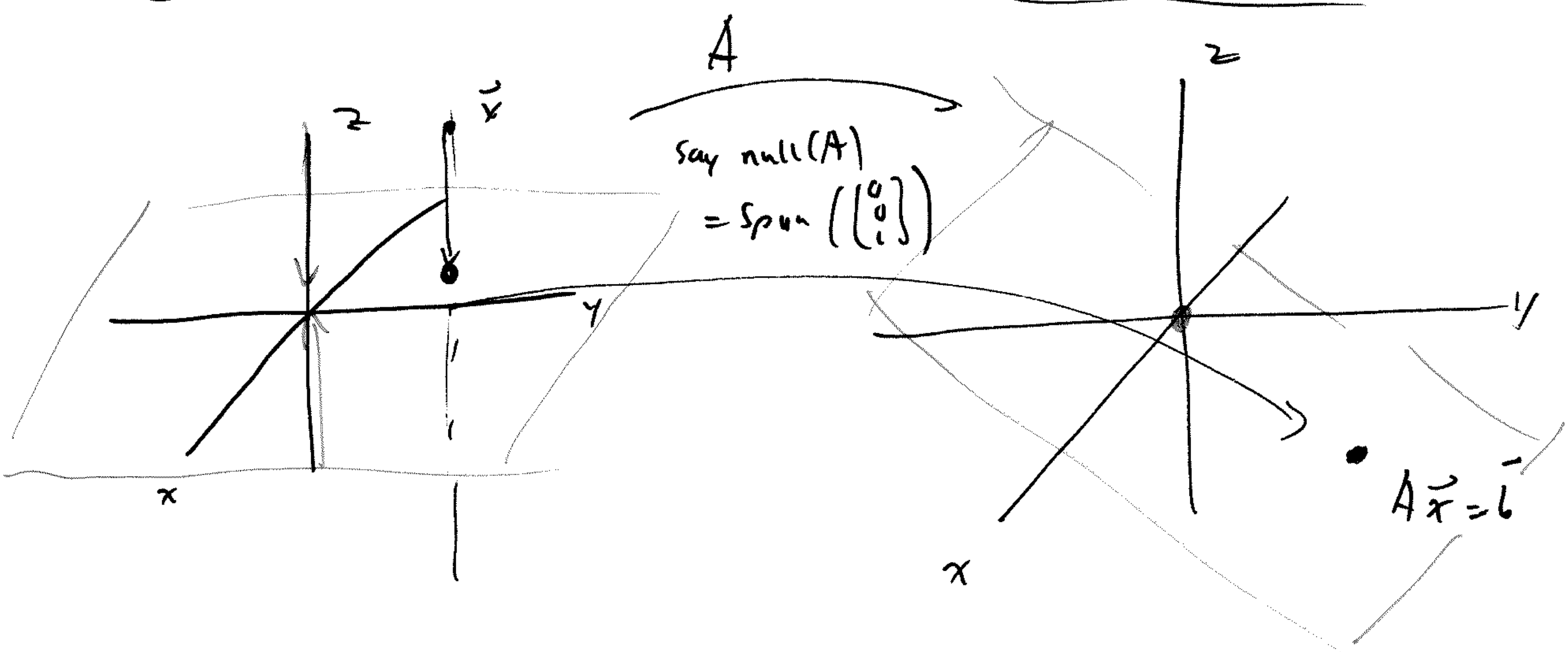
11) $\text{col}(A) = \mathbb{R}^n$

12) col(A) L. I.

13) rank = n

14) $\text{row}(A) = \mathbb{R}^n$

15) rows L. I.



Modular Arithmetic

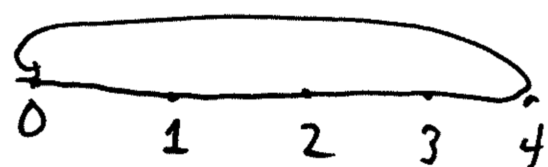
$$a \equiv b \mod p \iff a = b + np, \quad n \in \mathbb{Z}$$

e.g. Let's work "mod 5"

$$2 \equiv 7 \equiv -3 \equiv 12 \equiv 102 \mod 5$$

$2+5 \qquad 2-5 \qquad 2+2\cdot 5 \qquad 2+20\cdot 5$

In mod 5   number line



+1   +1   +1   +1   +1

$$5 = 0 + 1\cdot 5$$
$$\equiv 0 \mod 5$$