MATH 1LT3E

A#1 - SOL$^N$S

#1. (a) $\int_0^1 \left(e^{0.2x} - 5x^4 + \frac{10}{(1+x)^2}\right) dx$

$= \left[\frac{e^{0.2x}}{0.2} - x^5 - \frac{10}{1+x}\right|_0^1$

$= \left[5e^{0.2} - 1^5 - \frac{10}{2}\right] - \left[5\underbrace{e^{0}}_{=1} - 0^5 - \frac{10}{1+0}\right]$

$= 5e^{0.2} - 1$

(b) $\int \frac{3x^2 + 5x + x^{1/2}}{x} dx$

$= \int \left(\frac{3x^2}{x} + \frac{5x}{x} + \frac{x^{1/2}}{x}\right) dx$

$= \int \left(3x + 5 + x^{-1/2}\right) dx$

$= \frac{3x^2}{2} + 5x + 2x^{1/2} + C$

(c) $\int_1^2 x \ln x \, dx$

$\left.\begin{matrix} u = \ln x & dv = x\,dx \\ du = \frac{1}{x}dx & v = \frac{x^2}{2} \end{matrix}\right\}$ $\int x \ln x \, dx = \frac{x^2}{2}\ln x - \int \frac{x^2}{2}\cdot\frac{1}{x} dx$

$= \frac{x^2}{2}\ln x - \frac{1}{2}\int x \, dx$

$= \frac{x^2}{2}\ln x - \frac{1}{4}x^2 + C$

$\Rightarrow \int_1^2 x \ln x \, dx = \left[\frac{x^2}{2}\ln x - \frac{1}{4}x^2\right|_1^2$

$= \left[\frac{2^2}{2}\ln 2 - \frac{1}{4}(2)^2\right] - \left[\frac{1^2}{2}\underbrace{\ln 1}_{=0} - \frac{1}{4}(1)^2\right]$

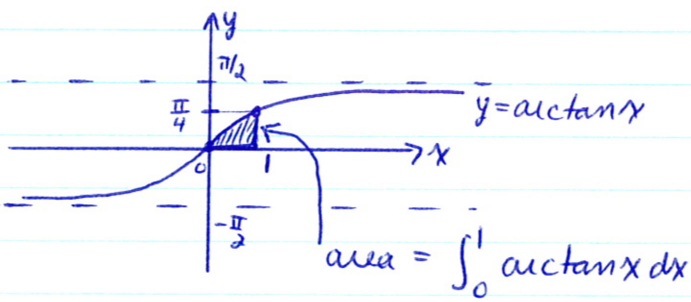
$= 2\ln 2 - \frac{3}{4}$

#1. (d) $\int x e^{5x^2}\,dx$

Let $u = 5x^2$. Then $\frac{du}{dx} = 10x \Rightarrow dx = \frac{du}{10x}$.

$$\int x e^{5x^2}\,dx = \int x e^{u} \frac{du}{10x}$$
$$= \frac{1}{10}\int e^{u}\,du$$
$$= \frac{1}{10} e^{u} + C$$
$$= \frac{1}{10} e^{5x^2} + C$$

#2. (a)




(b) $$\underset{a=0}{T_3}(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$$

$f(x) = \arctan x \quad \ldots \quad f(0) = \arctan 0 = 0$

$f'(x) = \frac{1}{1+x^2} \quad \ldots \quad f'(0) = \frac{1}{1+0^2} = 1$

$f''(x) = \frac{-2x}{(1+x^2)^2} \quad \ldots \quad f''(0) = \frac{-2(0)}{(1+0^2)^2} = 0$

$$f'''(x) = \frac{-2(1+x^2)^{\cancel{2}} - (-2x)\cdot 2(\cancel{1+x^2})(2x)}{(1+x^2)^{\cancel{4}\,3}}$$
$$= \frac{-2(1+x^2) + 8x^2}{(1+x^2)^3}$$
$$= \frac{6x^2-2}{(1+x^2)^3} \quad \ldots \quad f'''(0) = \frac{6(0)^2-2}{(1+0^2)^3} = -2$$

So, $T_3(x) = 0 + \frac{1}{1!}x + \frac{0}{2!}x^2 + \frac{-2}{3!}x^3$

$$= x - \frac{1}{3}x^3$$

Since $\arctan x \approx x - \frac{1}{3}x^3$ when $x$ is near 0,

(c) $\int_0^1 \arctan x \, dx \approx \int_0^1 (x - \frac{1}{3}x^3)\,dx$

$$\int_0^1 (x - \tfrac{1}{3}x^3)\,dx = \left[\frac{x^2}{2} - \frac{1}{3}\cdot\frac{x^4}{4}\right]_0^1$$
$$= \left[\frac{1}{2} - \frac{1}{12}\right] - \left[0\right]$$
$$= \frac{5}{12} \quad (\approx 0.42)$$

So, $\int_0^1 \arctan x \, dx \approx \frac{5}{12}$

To improve this approximation, use a higher degree Taylor polynomial to approximate $y = \arctan x$ near 0.

(d) $\int_0^1 \arctan x \, dx$ could also be approximated by interpreting this number as the area under the graph of $y = \arctan x$ from $x = 0$ to $x = 1$ and then using Riemann sums to approximate this area.

(e) $\int_0^1 \arctan x \, dx$

$u = \arctan x \qquad dv = dx$

$du = \frac{1}{1+x^2}\,dx \qquad v = x$

$$\Rightarrow \int \arctan x \, dx = x \arctan x - \underbrace{\int \frac{x}{1+x^2}\,dx}$$

$u = 1 + x^2$

$du = 2x\,dx \Rightarrow dx = \frac{du}{2x}$

$$\int \frac{x}{1+x^2}\,dx = \int \frac{x}{u}\,\frac{du}{2x}$$
$$= \frac{1}{2}\int \frac{1}{u}\,du$$
$$= \frac{1}{2}\ln|u| + C$$
$$= \frac{1}{2}\ln(1+x^2) + C$$

So, $\int \arctan x \, dx = x \arctan x - \frac{1}{2}\ln(1+x^2) + C$

$$\Rightarrow \int_0^1 \arctan x \, dx = \left[ x\arctan x - \frac{1}{2}\ln(1+x^2) \right|_0^1$$

$$= \left[ 1\arctan 1 - \frac{1}{2}\ln(1+1^2) \right] - \left[ 0 \right]$$

$$= \frac{\pi}{4} - \frac{1}{2}\ln 2$$

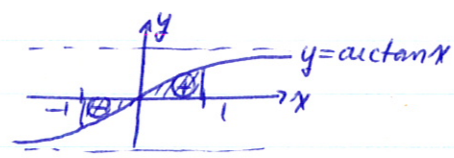
$$\approx 0.44$$

(f) $\int_{-1}^{1} \arctan x \, dx$ = net area under $f(x) = \arctan x$ over $x \in [-1, 1]$.

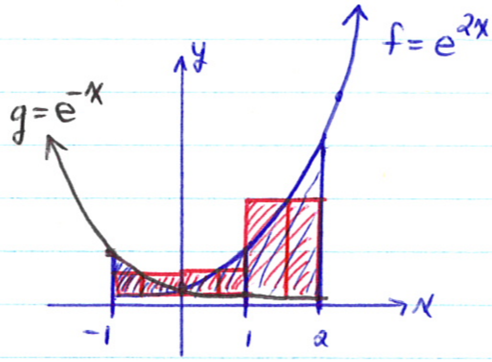
= area above $x$-axis − area below $x$-axis.

Since $\arctan x$ is odd, it is symmetric about the origin so area above the $x$-axis matches the area below the $x$-axis over $[-1, 1]$ so the net area is 0, i.e.,

$$\int_{-1}^{1} \arctan x \, dx = 0$$




#3. (a)




$$\Delta x = \frac{2-(-1)}{3} = 1$$

(b) Using a midpoint Riemann Sum:

area $\approx M_3 \approx [g(-0.5) - f(-0.5)] \cdot 1 + [f(0.5) - g(0.5)] \cdot 1 + [f(1.5) - g(1.5)] \cdot 1$

$\approx [e^{0.5} - e^{-1}] + [e - e^{-0.5}] + [e^3 - e^{-1.5}]$

$\approx 23.25$

#3. (c) $\int_{-1}^{2} |e^{2x} - e^{-x}|\,dx$

$= \int_{-1}^{0} (e^{-x} - e^{2x})\,dx + \int_{0}^{2} (e^{2x} - e^{-x})\,dx$

$= \left[ -e^{-x} - \frac{e^{2x}}{2} \right|_{-1}^{0} + \left[ \frac{e^{2x}}{2} + e^{-x} \right|_{0}^{2}$

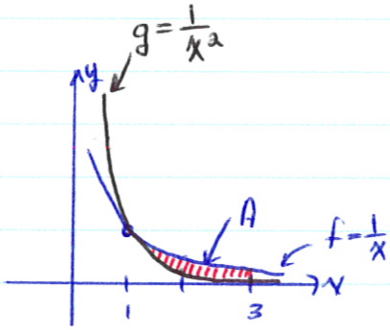
$= \left(-e^{0} - \frac{e^{0}}{2}\right) - \left(-e^{1} - \frac{e^{-2}}{2}\right) + \left(\frac{e^{4}}{2} + e^{-2}\right) - \left(\frac{e^{0}}{2} + e^{0}\right)$

$= \left(-1 - \frac{1}{2}\right) - \left(-e - \frac{1}{2e^{2}}\right) + \left(\frac{e^{4}}{2} + \frac{1}{e^{2}}\right) - \left(\frac{1}{2} + 1\right)$

$= -3 + e + \frac{3}{2e^{2}} + \frac{e^{4}}{2}$

$\approx 27.22$

#4. (a)




Area $A = \int_{1}^{3} \left( \frac{1}{x} - \frac{1}{x^2} \right) dx$

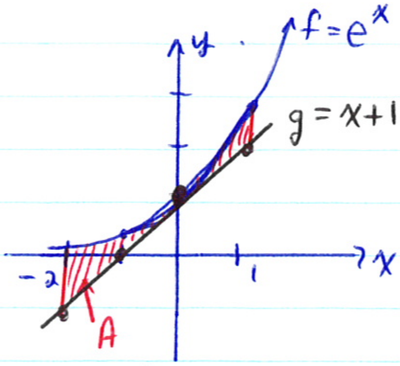
$= \left[ \ln|x| + \frac{1}{x} \right|_{1}^{3}$

$= \left[ \ln 3 + \frac{1}{3} \right] - \left[ \ln 1 + \frac{1}{1} \right]$

$= \ln 3 - \frac{2}{3}$

$\approx 0.43$

#4. (b)




$$\text{Area } A = \int_{-2}^{1} \left( e^{x} - (x+1) \right) dx$$

$$= \left[ e^{x} - \frac{x^2}{2} - x \right|_{-2}^{1}$$

$$= \left[ e - \tfrac{1}{2} - 1 \right] - \left[ e^{-2} - \tfrac{4}{2} + 2 \right]$$

$$= e - e^{-2} - \frac{3}{2}$$

$$\approx 1.08$$

#5. (a)

$$\int_0^{\infty} \frac{1}{(1+2x)^{3/2}}\, dx = \lim_{T\to\infty} \int_0^{T} \frac{1}{(1+2x)^{3/2}}\, dx$$

$$= \lim_{T\to\infty} \left[ -\frac{1}{\sqrt{1+2x}} \right|_{0}^{T}$$

$$= \lim_{T\to\infty} \left[ -\frac{1}{\sqrt{1+2T}} - \left( -\frac{1}{\sqrt{1+2(0)}} \right) \right]$$

$$= \lim_{T\to\infty} \left[ 1 - \frac{1}{\sqrt{1+2T}} \right]$$

$$= 1 - \cancel{\frac{1}{\sqrt{1+2\infty}}}^{\,\to 0}$$

$= 1$ (integral is convergent)

#5.(b) $\int_{10}^{\infty} \frac{1}{x^2}\,dx = \lim_{T\to\infty} \int_{10}^{T} x^{-2}\,dx$

$= \lim_{T\to\infty} \left[ -\frac{1}{x} \right|_{10}^{T}$

$= \lim_{T\to\infty} \left[ -\frac{1}{T} - \left(-\frac{1}{10}\right) \right]$

$= \lim_{T\to\infty} \left[ \frac{1}{10} - \frac{1}{T} \right]$

$= \frac{1}{10} - \cancel{\frac{1}{\infty}}^{\to 0}$

$= \frac{1}{10}$

(c) $\int_{1}^{\infty} e^{-0,5x}\,dx = \lim_{T\to\infty} \int_{1}^{T} e^{-0,5x}\,dx$

$= \lim_{T\to\infty} \left[ -2e^{-0,5x} \right|_{1}^{T}$

$= \lim_{T\to\infty} \left[ -2e^{-0,5T} - \left(-2e^{-0,5}\right) \right]$

$= \lim_{T\to\infty} \left[ 2e^{-0,5} - 2e^{-0,5T} \right]$

$= 2e^{-0,5} - 2\underbrace{e^{-0,5(\infty)}}_{\to 0}$

$= \frac{2}{e^{0,5}}$

#5. (d) $\displaystyle\int_1^\infty \frac{\ln x}{x^4}\,dx = \lim_{T\to\infty}\int_1^T \frac{\ln x}{x^4}\,dx$

<u>Aside</u>: Let $u = \ln x$ and $dv = x^{-4}dx$.
Then $du = \frac{1}{x}dx$ and $v = -\frac{x^{-3}}{3}$

$$\Rightarrow \int \frac{\ln x}{x^4}dx = -\frac{\ln x}{3x^3} - \int -\frac{1}{3x^3}\cdot\frac{1}{x}\,dx$$

$$= -\frac{\ln x}{3x^3} + \frac{1}{3}\int x^{-4}dx$$

$$= -\frac{\ln x}{3x^3} - \frac{1}{9x^3} + C$$

So, $$\int_1^T \frac{\ln x}{x^4}dx = \left[-\frac{\ln x}{3x^3} - \frac{1}{9x^3}\right|_1^T$$

$$= \left[-\frac{\ln T}{3T^3} - \frac{1}{9T^3}\right] - \left[-\frac{\overbrace{\ln 1}^{=0}}{3(1)^3} - \frac{1}{9(1)^3}\right]$$

$$= \frac{1}{9} - \frac{\ln T}{3T^3} - \frac{1}{9T^3}$$

$$\int_1^\infty \frac{\ln x}{x^4}dx = \lim_{T\to\infty}\left(\frac{1}{9} - \frac{\ln T}{3T^3} - \frac{1}{9T^3}\right)$$

$$= \frac{1}{9} - \frac{\ln\infty}{3\infty^3} - \frac{1}{9\infty^3}\to 0$$

$\hookrightarrow$ <u>aside</u>: $\displaystyle\lim_{T\to\infty} -\frac{\ln T}{3T^3}$

$$\overset{LH}{=} \lim_{T\to\infty}\frac{-\frac{1}{T}}{9T^2}$$

$$= \lim_{T\to\infty} -\frac{1}{9T^3}$$

$$= \frac{-1}{9(\infty)^3}$$

$$= 0$$

So, $$\int_1^\infty \frac{\ln x}{x^4}dx = \frac{1}{9}$$

$$\#6.\ (a)\quad \overbrace{A(24)-A(0)}^{\text{total change in amount of pollutant over } 1^{ST} \text{ 24 hours}} = \int_0^{24} 1.2t\, e^{-0.2t}\, dt$$

Aside:

$$u = 1.2t \qquad dv = e^{-0.2t}\, dt$$

$$du = 1.2\, dt \qquad v = -5e^{-0.2t}$$

$$\Rightarrow \int 1.2t e^{-0.2t}\, dt = -6te^{-0.2t} - \int -6e^{-0.2t}\, dt$$

$$= -6te^{-0.2t} - 30e^{-0.2t} + C$$

$$\text{so, } \int_0^{24} 1.2te^{-0.2t}\, dt = \left[e^{-0.2t}(-6t-30)\right|_0^{24}$$

$$= e^{-4.8}(-174) - e^0(-30)$$

$$= 30 - 174e^{-4.8}$$

$$\approx 28.57g$$

$$(b)\quad A(T) - \overbrace{A(0)}^{=0} = \int_0^T 1.2te^{-0.2t}\, dt$$

$$= \left[e^{-0.2t}(-6t-30)\right|_0^T$$

$$= e^{-0.2T}(-6T-30) + 30$$

$$A(T) = e^{-0.2T}(-6T-30) + 30$$

$$\lim_{T\to\infty} A(T) = \underbrace{e^{-\infty}}_{=0} \cdot \underbrace{(-6\cdot\infty - 30)}_{-\infty} + 30$$

Aside:

$$\lim_{T\to\infty} \frac{-6T-30}{e^{0.2T}}$$

$$\stackrel{LH}{=} \lim_{T\to\infty} \frac{-6}{0.2e^{0.2T}}$$

$$= \frac{-6}{0.2e^{\infty}} = \frac{-6}{\infty} = 0$$

$$\therefore \lim_{T\to\infty} A(T) = 30\ g.$$

As time goes on, the total amount of pollutant leaking into lake approaches 30 grams.