MATHEMATICS 1LT3 TEST 2

Day Class
Duration of Test: 60 minutes
McMaster University, 8 March 2012

E. Clements

FIRST NAME (please print): ______________________

FAMILY NAME (please print): ______________________

Student No.: ______________________

THIS TEST HAS 8 PAGES AND 7 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE.

Total number of points is 40. Marks are indicated next to the problem number. Any non-graphing calculator is allowed.

USE PEN TO WRITE YOUR TEST. IF YOU USE A PENCIL YOUR TEST WILL NOT BE ACCEPTED FOR REMARKING (IF NEEDED).

You need to show work to receive full credit.

Problem	Points	Mark
1	6	
2	6	
3	5	
4	6	
5	4	
6	7	
7	6	
TOTAL	40	

Continued on next page

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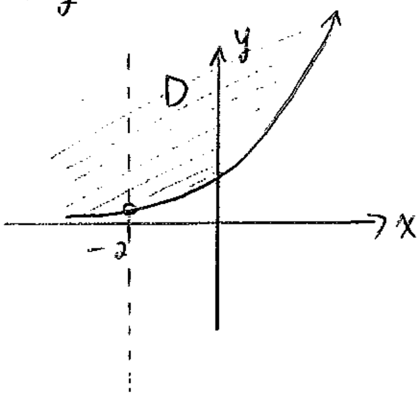
1. (a) [2] Determine the domain of $f(x,y) = \dfrac{\sqrt{y-e^x}}{x+2}$. Sketch this region.

$y - e^x \geq 0$ AND $x+2 \neq 0$

$y \geq e^x$ AND $x \neq -2$



(b) [2] State the range of $g(x,y) = 5 - x^2 - y^2$. (Note: You do not have to provide a formal proof like we did in class, just explain your reasoning.)

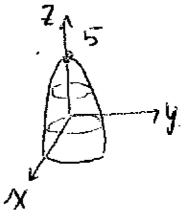
$g(x,y) = 5 - (x^2+y^2)$

$x^2+y^2 \geq 0$

$x^2+y^2+5 \geq 5$

$5 \geq \underbrace{5-x^2-y^2}_{g(x,y)}$

So, $g(x,y) \leq 5$



(c) [2] Sketch a contour map for $h(x,y) = (xy)^{\frac{1}{3}}$. Include at least 5 level curves.

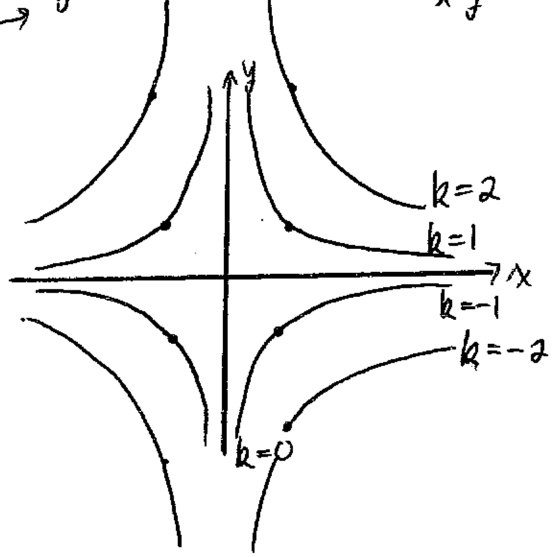
$(xy)^{1/3} = k \Rightarrow xy = k^3 \Rightarrow y = k^3 \cdot \frac{1}{x}$

domain $= \mathbb{R}^2$

$k=0 \rightarrow xy=0 \Rightarrow x=0 \wedge y=0$

$k=1 \Rightarrow y=\frac{1}{x}$

$k=2 = y = \frac{8}{x}$



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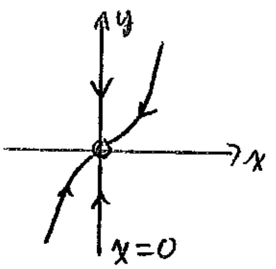
2. (a) [2] In your own words, describe what is meant by $\lim_{(x,y)\to(a,b)} f(x,y) = L$. What must L be in order for $f(x,y)$ to be continuous at (a,b)?

$\lim_{(x,y)\to(a,b)} f(x,y) = L$ means that the z-values approach L as $(x,y)\to(a,b)$ along every path in the domain of f.

If f is continuous at (a,b), then $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.

(b) [2] Show that the limit of $g(x,y) = \dfrac{x^3 y}{x^6 + y^2}$ as $(x,y)\to(0,0)$ does not exist.

$g(0,y) = 0$ so $g(x,y)\to 0$ as $(x,y)\to(0,0)$ along $x=0$.



$$g(x,x^3) = \frac{x^3\cdot x^3}{x^6+(x^3)^2} = \frac{x^6}{2x^6} = \frac{1}{2}$$

so $g(x,y)\to \frac{1}{2}$ as $(x,y)\to(0,0)$ along $y=x^3$

$\Rightarrow \lim_{(x,y)\to(0,0)} g(x,y)$ D.N.E.

(c) [2] Use the definition of continuity to show that

$$h(x,y) = \begin{cases} 4 - e^{-x-y+2} & \text{if } (x,y)\neq(1,1) \\ 3 & \text{if } (x,y)=(1,1)\end{cases}$$

is continuous at $(1,1)$.

$$\lim_{(x,y)\to(1,1)} h(x,y) = \lim_{(x,y)\to(1,1)} \left(4 - e^{-x-y+2}\right) = 4 - e^0 = 3$$

$$h(1,1) = 3$$

$\therefore \lim_{(x,y)\to(1,1)} h(x,y) = 3 = h(1,1)$ $\therefore$ h is continuous @ $(1,1)$.

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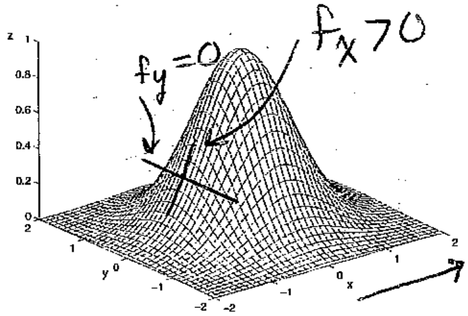
3. (a) [1] Write the definition of $\dfrac{\partial f}{\partial x}(a,b)$.

$$\frac{\partial f}{\partial x}(a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

(b) [1] Given the graph of f below, determine whether the partial derivatives, f_x and f_y, are positive, negative or zero at the point $(-1,0)$.

$f_x(-1,0) > 0$

$f_y(-1,0) = 0$



(c) [3] The table below shows the values of the wind chill index $W(T,v)$, with temperatures measured in degrees Celsius and wind speed in kilometres per hour. Estimate $W_T(-15,30)$, and interpret the result.

	$T=-20$	$T=-15$	$T=-10$	$T=-5$
$v=40$	-34.1	-27.4	-20.8	-14.1
$v=30$	-32.6	-26.0	-19.5	-13.0
$v=20$	-30.5	-24.2	-17.9	-11.6

$$W_T(-15,30) \approx \frac{W(-10,30) - W(-15,30)}{-10-(-15)} \approx \frac{-19.5-(-26.0)}{5} \approx 1.3$$

$$W_T(-15,30) \approx \frac{W(-20,30) - W(-15,30)}{-20-(-15)} \approx \frac{-32.6-(-26.0)}{-5} \approx 1.32$$

avg: $W_T(-15,30) \approx \dfrac{1.3+1.32}{2} \approx 1.31$ wind chill indices/°C

∴ When temp is -26.0°C and wind speed is 30km/h, the wind chill index is increasing at about 1.32 indices/°C

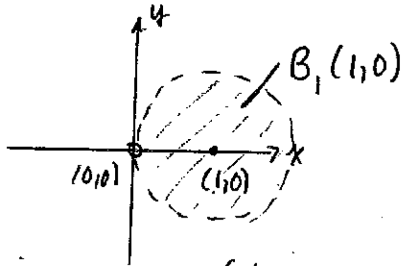
4. (a) [3] Show that the function $f(x,y) = \ln(\overbrace{x^2+y^2}^{>0})$ is differentiable at $(1,0)$. What is the largest open disk centred at $(1,0)$ on which f is differentiable?

$f_x = \dfrac{2x}{x^2+y^2}$

$f_y = \dfrac{2y}{x^2+y^2}$

$\text{dom}(f) = \{(x,y) \in \mathbb{R}^2 \mid (x,y) \neq (0,0)\}$

domain of f_x and f_y:



f_x and f_y are continuous on $B_1(1,0)$ (ball is in domain of each) $\Rightarrow f(x,y)$ is differentiable at $(1,0)$.

$B_1(1,0) = \{(x,y) \in \mathbb{R}^2 \mid (x-1)^2 + y^2 < 1\}$ is the largest disk.

(b) [1] Verify the linear approximation $\ln(x^2+y^2) \approx 2(x-1)$ near $(1,0)$.

$f_x(1,0) = 2$

$f_y(1,0) = 0$

$L_{(1,0)}(x,y) = \ln(1^2+0^2) + 2(x-1) + 0(y-0) = 2(x-1)$

$\because f$ is differentiable at $(1,0)$, this linear approximation is valid.

(c) [2] Compute $T_2(x,y)$ for $f(x,y)$ near $(1,0)$.

$f_{xx} = \dfrac{2(x^2+y^2) - 2x(2x)}{(x^2+y^2)^2} = \dfrac{2y^2-2x^2}{(x^2+y^2)^2}$ $\qquad f_{xx}(1,0) = -2$

$f_{yy} = \dfrac{2x^2-2y^2}{(x^2+y^2)^2}$ $\qquad f_{yy}(1,0) = 2$

$f_{xy} = \dfrac{0 - 2x(2y)}{(x^2+y^2)^2} = \dfrac{-4xy}{(x^2+y^2)^2}$ $\qquad f_{xy}(1,0) = 0$

$T_2(x,y) = 2(x-1) - \dfrac{2}{2!}(x-1)^2 + \dfrac{2}{2!}(y-0)^2$

$= 2(x-1) - (x-1)^2 + y^2$

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5. The number of whales N depends on the availability of plankton P and the ocean temperature T. Suppose that in a certain region, the average temperature of the ocean is rising at a rate of 0.15^oC/year and the amount of plankton is decreasing at a rate of 0.1 plankton/year. Suppose that currently $\frac{\partial N}{\partial T} = -0.02$ and $\frac{\partial N}{\partial P} = 0.08$.

(a) [2] What is the significance of the signs of these partial derivatives?

$\frac{\partial N}{\partial T} = \ominus$ means that the # of whales decreases as the temperature increases

$\frac{\partial N}{\partial P} = \oplus$ means that the # of whales increases as the # plankton increases.

(b) [2] Estimate the current rate of change of in the whale population, $\frac{dN}{dt}$.

$$\frac{dN}{dt} = \frac{\partial N}{\partial T} \cdot \frac{dT}{dt} + \frac{\partial N}{\partial P} \cdot \frac{dP}{dt}$$

$$= (-0.02)(0.15) + (0.08)(-0.1)$$

$$= -0.011$$

The whale pop^N is decreasing over time at a rate of 0.011 whales / year.

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6. (a) [4] Compute the directional derivative of the function $f(x,y) = y\sqrt{x}+y^3$ at the point $(1,2)$ in the direction of the vector $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j}$. Are the values of the function increasing or decreasing at $(1,2)$ as we move in the direction $\mathbf{v}$?

$$\|\vec{v}\| = \sqrt{3^2+4^2} = 5$$

$$\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v} = \underbrace{\frac{3}{5}}_{u_1}\hat{\imath} + \underbrace{\frac{4}{5}}_{u_2}\hat{\jmath}$$

$$f_x = \frac{y}{2\sqrt{x}} \qquad f_x(1,2) = 1$$

$$f_y = \sqrt{x} + 3y^2 \qquad f_y(1,2) = 13$$

$$D_{\vec{u}} f(1,2) = f_x(1,2)\cdot u_1 + f_y(1,2)u_2$$
$$= 1\left(\tfrac{3}{5}\right) + 13\left(\tfrac{4}{5}\right)$$
$$= 11 > 0 \Rightarrow$$ f is increasing at $(1,2)$ as we move in the direction $\vec{v}$.

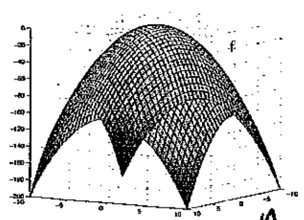
(b) [2] Is there a direction $\mathbf{v_2}$ such that $D_{\mathbf{v_2}}f(1,2) = 14$? Explain.

$$\nabla f(1,2) = 1\hat{\imath} + 13\hat{\jmath}$$

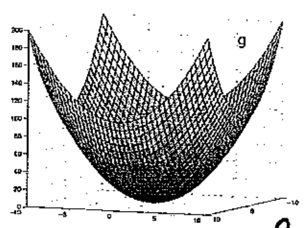
$$\|\nabla f(1,2)\| = \sqrt{1^2+13^2} \doteq 13.04$$

↑ the magnitude of the gradient is the max. rate of change in this f^n at $(1,2)$ so NO, there is no other direction at $(1,2)$ such that the rate of change in that direction is 14.

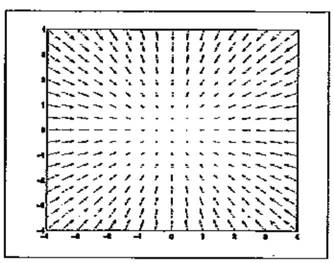
(c) [1] Match each function with its gradient vector field.



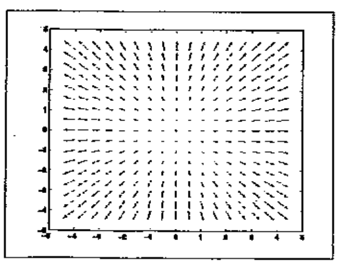
f corresponds to gradient vector field A



g corresponds to gradient vector field B



A



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7. (a) [2] Verify that the only critical point of $f(x,y) = x + y + \frac{1}{xy}$ is $(1,1)$.

$f_x = 1 - \frac{1}{x^2y} = \frac{x^2y-1}{x^2y}$ $\quad f_y = \frac{y^2x-1}{y^2x}$ $\quad \boxed{\begin{matrix} x\neq 0 \\ y \neq 0\end{matrix}}$

$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \Rightarrow \begin{cases} x^2y-1=0 \\ y^2x-1=0 \end{cases} \Rightarrow \begin{cases} y = \frac{1}{x^2} & ① \\ x = \frac{1}{y^2} & ② \end{cases}$

sub ① into ②: $x = \frac{1}{(\frac{1}{x^2})^2} \Rightarrow x = x^4 \Rightarrow x^4 - x = 0 \Rightarrow x(x^3-1) = 0$ $\quad \cancel{\boxed{x=0}}$ reject or $\boxed{x=1}$

sub $x=1$ into ①: $y = \frac{1}{1^2} = 1$ $\quad \therefore (1,1)$ is the only crit point.

(b) [2] Use the second derivatives test to determine whether f has a local maximum, local minimum, or a saddle point at $(1,1)$.

$f_{xx} = \frac{2}{x^3y}$ $\quad f_{yy} = \frac{2}{y^3x}$ $\quad f_{xy} = \frac{1}{x^2y^2}$

$$D(1,1) = f_{xx}(1,1)\, f_{yy}(1,1) - [f_{xy}(1,1)]^2$$
$$= (2)(2) - 1^2$$
$$= 3 > 0$$

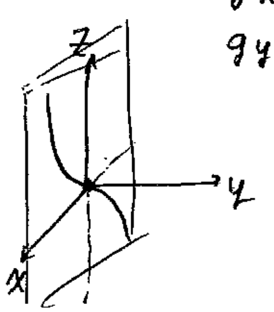
$f_{xx}(1,1) = 2 > 0 \Rightarrow f \cup$ in x-dir.

$\Rightarrow f$ has a local min. at $(1,1)$.

(c) [2] Reason geometrically, (i.e., without using the second derivatives test) to show that the function $g(x,y) = x^3 - 3xy^2$ has a saddle point at $(0,0)$.

$g_x = 3x^2 - 3y^2$ $\quad \begin{cases} g_x(0,0) = 0 \\ g_y(0,0) = 0 \end{cases} \Rightarrow (0,0)$ is a critical point

$g_y = -6xy$



when $y=0$, $g(x,0) = x^3$

when $x>0$, $g(x,0) > g(0,0)$

So $g(0,0)$ cannot be a max.

when $x<0$, $g(x,0) < g(0,0)$

$\Rightarrow g(0,0)$ cannot be a min

$\Rightarrow g$ has a saddle point at $(0,0)$.

THE END