McMaster University
Math 1LT3

ASSIGNMENT 2

Sections 7.2, 7.3, 7.4

1. A population of sharks is described by the logistic differental equation

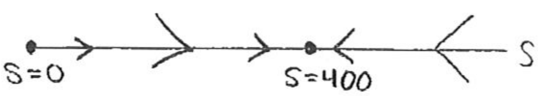
$$\frac{dS}{dt} = 0.8S\left(1 - \frac{S}{400}\right)$$

(a) Find the equilibria of this equation. What do these numbers represent?

$\frac{dS}{dt} = 0$ when $S = 0$ or $S = 400$

400 represents the carrying capacity for this popⁿ

(b) Draw a phase-line diagram for this differential equation.



(c) Sketch the equilibrium solutions and solution curves corresponding to the initial conditions $\cancel{P}(0) = 20$, $\cancel{P}(0) = 300$, and $\cancel{P}(0) = 500$.

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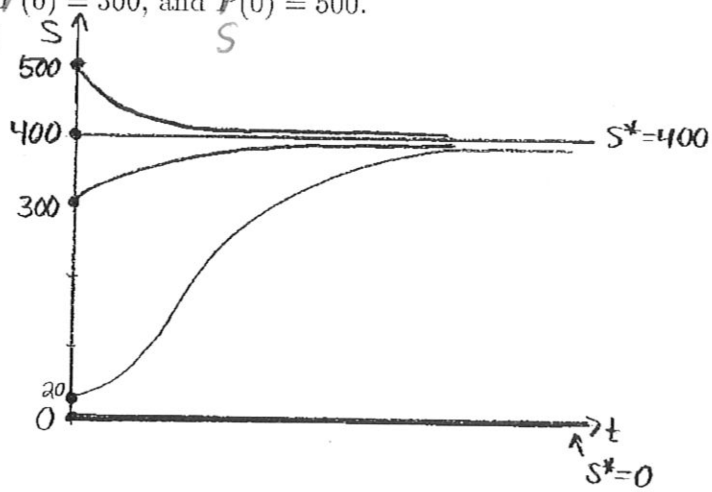
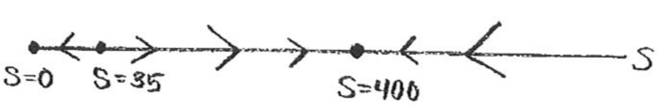


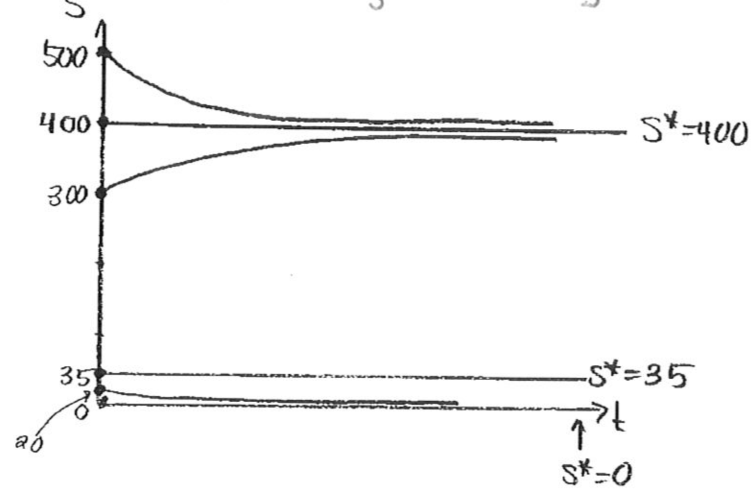
(d) Suppose it is known that the population will die out if it ever falls below 35 sharks. Write a modified logistic differential equation to illustrate this.

$$\frac{dS}{dt} = 0.8\,S\left(1 - \frac{S}{400}\right)\left(1 - \frac{35}{S}\right)$$

(e) Draw a phase-line diagram for this modified differential equation.



(f) For the equation in part (d), sketch the equilibrium solutions and solution curves corresponding to the initial conditions $\underset{S}{\cancel{P}}(0) = 20$, $\underset{S}{\cancel{P}}(0) = 300$, and $\underset{S}{\cancel{P}}(0) = 500$.



2. Assuming that there is competition within a population of bacteria for resources, a model for limited bacterial growth is given by

$$\frac{db}{dt} = \lambda(b)b$$

where the per capita production rate, λ, is a decreasing function of the population size, b. Suppose that the per capita production rate is a linear function of population size with a maximum of $\lambda(0) = 1$ and a slope of -0.002.

(a) Find $\lambda(b)$ and write the differential equation for b.

$$\lambda(b) = -0.002b + 1$$

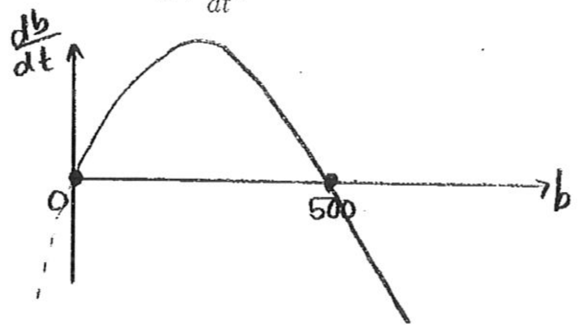
$$\frac{db}{dt} = (-0.002b+1)b = -0.002b^2 + b$$

(b) Determine the equilibrium solutions.

$$\frac{db}{dt} = 0 \quad \text{when} \quad -0.002b + 1 = 0 \quad \text{or} \quad \boxed{b=0}$$

$$\boxed{b = 500}$$

(c) Graph the rate of change, $\frac{db}{dt}$, as a function of b.



(d) Describe, in words, the dynamics of a population of bacteria modeled by the differential equation found in part (a).

When the pop^N is between 0 and 500, the pop^N will increase. It will increase the fastest when there are 250 bacteria. When the pop^N is greater than 500, the pop^N will decrease. For very large pop^n sizes, the rate of decrease will be large as well.

When pop^N is 0 or 500, it will remain this size *theoretically* forever.

3. Consider the differential equation $\frac{dy}{dt} = ye^{-\beta y} - ay$, where a and β are parameters.

(a) Find the equilibria.

$\frac{dy}{dt} = 0$ when $ye^{-\beta y} - ay = 0 \Rightarrow y(e^{-\beta y} - a) = 0$

$\Rightarrow y^* = 0$ or $e^{-\beta y} = a$

$y^* = \frac{\ln a}{-\beta}$

NOTE: $\boxed{a>0}$

(b) Use the stability theorem to determine the stability of the equilibria. (Note: The stability depends on the parameter a so you will need to consider different cases.)

$f(y) = y(e^{-\beta y} - a)$

$f'(y) = 1\cdot(e^{-\beta y} - a) + y\cdot(e^{-\beta y}\cdot(-\beta)) = e^{-\beta y}(1 - \beta y) - a$

$f'(0) = e^{-\beta\cdot 0}(1 - \beta 0) - a = 1 - a$

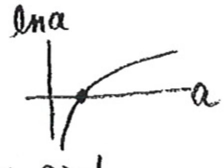
$y^* = 0$ is stable when $1 - a < 0 \Rightarrow a > 1$

" unstable " $1 - a > 0 \Rightarrow a < 1$

$f'(-\frac{\ln a}{\beta}) = e^{-\beta(-\frac{\ln a}{\beta})}\cdot(1 - \beta(-\frac{\ln a}{\beta})) - a = a(1 + \ln a) - a = \underbrace{\overset{\oplus}{a}\cdot\ln a}$

$\oplus$ when $a > 1$

$\ominus$ when $0 < a < 1$



$\therefore y^* = -\frac{\ln a}{\beta}$ is stable when $0 < a < 1$

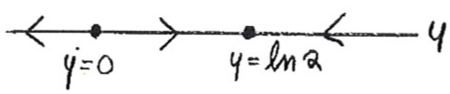
" " unstable " $a > 1$

(c) Suppose that $a = 0.5$ and $\beta = 1$. Draw the phase-line diagram for $\frac{dy}{dt} = ye^{-\beta y} - ay$.

$y_1^* = 0$, $y_2^* = -\frac{\ln 0.5}{1} = \ln 2 \approx 0.7$

unstable

stable

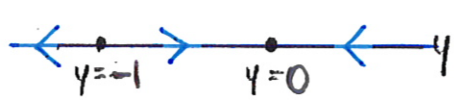


(d) Suppose that $a = e$ and $\beta = 1$. Draw the phase-line diagram for $\frac{dy}{dt} = ye^{-\beta y} - ay$.

$y_1^* = 0$, $y_2^* = -\frac{\ln e}{1} = -1$

stable

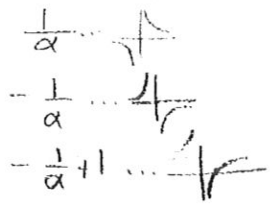
unstable

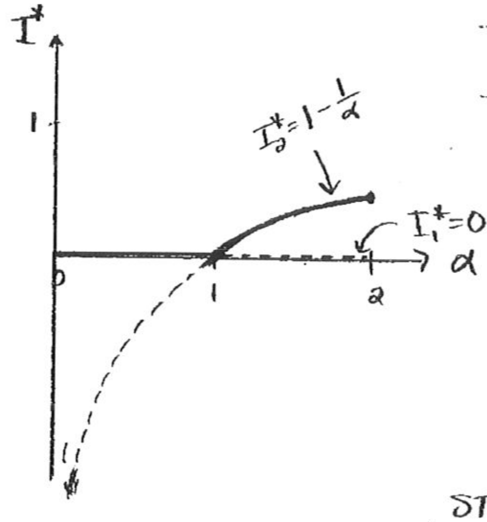


4. Exercise 42 on page 543 in your text.

$$\frac{dI}{dt} = \alpha I(1-I) - \mu I \quad , \qquad I_1^* = 0$$
$$I_2^* = 1 - \frac{\mu}{\alpha} \text{ (only valid if } \mu < \alpha)$$

$$\mu = 1 \rightarrow \frac{dI}{dt} = \alpha I(1-I) - I \quad , \qquad I_1^* = 0$$
$$I_2^* = 1 - \frac{1}{\alpha} \text{ (only valid if } \alpha > 1)$$





STABILITY ANALYSIS

$$f'(I) = \alpha - 2\alpha I - \mu = \alpha - 2\alpha I - 1$$
$f'(0) = \alpha - 1 \ldots$ $I_1^* = 0$ is stable when $\alpha - 1 < 0$ ie $\alpha < 1$
" " unstable " $\alpha - 1 > 0$ ie $\alpha > 1$

$$f'\left(1 - \frac{1}{\alpha}\right) = \alpha - 2\alpha\left(1 - \frac{1}{\alpha}\right) - 1 = 1 - \alpha$$

$I_2^* = 1 - \frac{1}{\alpha}$ is stable when $1 - \alpha < 0$ ie $\alpha > 1$
" " unstable " $1 - \alpha > 0$ ie $\alpha < 1$

5. First, determine the equilibrium solutions of each differential equation. Then, use the separation of variables technique to find remaining solutions for each differential equation.

(a) $\frac{dy}{dx} = \frac{y^2 \cos x}{1+y^2}$

$\frac{dy}{dx} = 0$ when $y^2 \cos x = 0 \Rightarrow \boxed{y^* = 0}$ (eqᴺ solᴺs are values of y for which $\frac{dy}{dx} = 0$)

$\frac{1+y^2}{y^2}\,dy = \cos x\,dx$

$\int (y^{-2}+1)\,dy = \int \cos x\,dx$

$-y^{-1} + y = \sin x + C$

$y - \frac{1}{y} = \sin x + C$ ← general, implicit solᴺ (cannot solve explicitly for y).

(b) $(x^2+1)\frac{dy}{dx} = xy$

$\int \frac{1}{y}\,dy = \int \frac{x}{x^2+1}\,dx$

$\ln|y| = \frac{1}{2}\ln(x^2+1) + C$

$\ln|y| = \ln\sqrt{x^2+1} + C \quad \Big|\, e$

$|y| = e^C \cdot \sqrt{x^2+1}$

$y = \pm e^C \cdot \sqrt{x^2+1}$

$\therefore y = K\sqrt{x^2+1}$ where $K = \pm e^C$

also, eqᴺ solᴺ: $\boxed{y^* = 0}$

aside: $\int \frac{x}{x^2+1}\,dx$

use substitution:
let $u = x^2+1$. Then $du = 2x\,dx$
$dx = \frac{1}{2x}\,du$

$\Rightarrow \int \frac{x}{u} \cdot \frac{1}{2x}\,du$

$= \frac{1}{2}\ln|u| + C_1$

$= \frac{1}{2}\ln(x^2+1) + C_1$

6. Use separation of variables to find the solution of the differential equation that satisfies the given initial condition.

(a) $P'(t) = P(t) + tP(t) + t + 1, \ P(0) = 50$

common factoring

$$\frac{dP}{dt} = P(1+t) + t+1 = (t+1)(P+1)$$

$$\int \frac{1}{P+1}\,dP = \int (t+1)\,dt$$

$$\ln|P+1| = \frac{t^2}{2} + t + C$$

$$|P+1| = e^{C} \cdot e^{\frac{t^2}{2}+t}$$

$$P = \pm e^{C} \cdot e^{\frac{t^2}{2}+t} - 1$$

$$P = K e^{t^2/2+t} - 1$$

$$P(0) = 50 \Rightarrow 50 = K \cdot e^{0} - 1 \Rightarrow K = 51$$

$$\therefore P(t) = 51 e^{t^2/2+t} - 1$$

(b) $(2y + e^{3y})y' = x\cos x, \ y(0) = 0$

$y' = \frac{dy}{dx}$

Int. by Parts:
$u = x \quad dv = \cos x$
$du = dx \quad v = \sin x$

$$\int (2y + e^{3y})\,dy = \int x \cdot \cos x\,dx$$

$$y^2 + \frac{1}{3}e^{3y} = x \cdot \sin x - \int \sin x\,dx$$

$$= x \cdot \sin x + \cos x + C$$

$$y(0) = 0 \Rightarrow 0^2 + \underbrace{\frac{1}{3}e^{0}}_{=1} = 0 \cdot \overset{0}{\sin 0} + \overset{=1}{\cos 0} + C \Rightarrow C = \frac{1}{3} - 1 = -\frac{2}{3}$$

$$\therefore y^2 + \frac{1}{3}e^{3y} = x\sin x + \cos x - \frac{2}{3}$$

or

$$3y^2 + e^{3y} = 3x\sin x + 3\cos x - 2$$

implicit sol^n.

7. Consider a special case of the "selection equation", $\frac{dp}{dt} = p(1-p)$.

(a) Solve the selection equation using separation of variables and integration by partial fractions as outlined in exercises 45-50 on pages 551 and 552 in your textbook.

$$\int \frac{1}{p(1-p)}\,dp = \int 1\,dt$$

$$\Rightarrow \int \left(\frac{1}{p} + \frac{1}{1-p}\right)dp = \int 1\,dt \quad \text{(using partial fraction decomposition)}$$

$$\Rightarrow \ln|p| - \ln|1-p| = t + C$$

$$\Rightarrow \ln\left|\frac{p}{1-p}\right| = t + C$$

$$\Rightarrow \left|\frac{p}{1-p}\right| = e^{c}\cdot e^{t}$$

$$\Rightarrow \frac{p}{1-p} = \pm e^{c}\cdot e^{t}$$

$$\Rightarrow \frac{1-p}{p} = \frac{1}{\pm e^{c}\cdot e^{t}}$$

$$\Rightarrow \frac{1}{p} - 1 = Ke^{-t}$$

where $K = \frac{1}{\pm e^{c}}$

$$\frac{1}{p} = Ke^{-t} + 1$$

$$\Rightarrow p = \frac{1}{1+Ke^{-t}}$$

(b) Exercise 52 on page 552. Using your solution from part (a) and the initial condition $p(0) = 0.5$, find the value of the constant. Evaluate the limit of the solution as t approaches infinity.

$$p(0) = 0.5 \Rightarrow \frac{1}{2} = \frac{1}{1+Ke^{0}} \Rightarrow 1+K = 2 \Rightarrow K = 1$$

$$\therefore P(t) = \frac{1}{1+e^{-t}}$$

$$\lim_{t\to\infty} P(t) = \frac{1}{1+\underbrace{e^{-\infty}}_{\to 0}} = 1$$

THE END