1Z C3

Last Day The Megatheorem (version 1)

If A is a square (ie. $n \times n$) matrix, the following are all equivalent

1) A is invertible (ie $A^{-1}$ exists)
2) A has RREF of I (row equivalent to I)
3) A is a product of elementary matrices
4) $A\vec{x} = \vec{b}$ has unique solution

new

5) $A\vec{x} = \vec{0}$ has only the trivial solution

ie $A\vec{x} = \vec{0}$ has only $\vec{x} = \vec{0}$ as a solution

"The trivial solution"

Note (4) $\Rightarrow$ (5) ie $\vec{b} = \vec{0}$ case, specifically

5 $\Rightarrow$ 2 If $A\vec{x}=\vec{0}$ has $\vec{x}=\vec{0}$ only as solution

all $n$

no parameter!

$$\left\{\begin{array}{l} x_1 = 0 \\ x_2 = 0 \\ \vdots \end{array}\right\} \Rightarrow \left\{\underbrace{\left[\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 \cdots & 0 & 0 \\ \cdots & & & & \vdots \end{array}\right]}_{I_n}\right\} \text{RREF}$$

$A$ is row-equivalent to $I$ ✓

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Fun things we can prove

Let's prove you can't have 2, 3, 4 etc. solutions!
(ie only 0, 1 or $\infty$)

Proof Say $A\vec{x}=\vec{b}$ & $A\vec{y}=\vec{b}$ ; $\vec{x}\neq\vec{y}$

Subtract we get $A\vec{x} - A\vec{y} = \cancel{\vec{b} - \vec{b}} \; \vec{0}$

$$A(\underbrace{\vec{x} - \vec{y}}) = \vec{0}$$

$\vec{x} \neq \vec{y} \Rightarrow \underline{\vec{x} - \vec{y} \neq 0}$

(& A not invertible!)

If call $\vec{x} - \vec{y} = \vec{v} \Rightarrow A\vec{v} = \vec{0}$

$$tA\vec{v} = \vec{0}, \text{ all } t \in \mathbb{R}$$

$$A(t\vec{v}) = \vec{0}$$

$$\underline{A(\vec{x} + t\vec{v})} = A\vec{x} + A(t\vec{v}) = \vec{b} + \vec{0} = \underline{\vec{b}}$$

$$A\underbrace{(\vec{x} + t\vec{v})} = \vec{b} \text{ for } \underline{\text{all}} \; \underline{t \in \mathbb{R}}$$

$\infty$ possible solution!

Note we often generalize! If $\vec{x}$ & $\vec{y}$ solve $A\_ = \vec{b}$

then $A(\vec{x} - \vec{y}) = \vec{0}$ always!

& any $A(\vec{x} + (\text{homog. solution})) = A\vec{x} + \vec{0}$
$= \vec{b}$

So all solutions to $A\vec{x} = \vec{b}$ can be expressed

as $\vec{x}_0 + \vec{y}_h$

a particular arbitrary choice of solution

any homog. solution! ie $A\vec{y}_h = \vec{0}$

Next If, BA = I (or AB = I) only, then $B = A^{-1}$

we know

Proof If $A^{-1}$ exists & $BA = I$

$A^{-1} = IA^{-1} = BAA^{-1} = BI = B$

Remember $A^{-1}$, if exists, has property

$A^{-1}A = AA^{-1} = I$

But If $A\vec{x} = \vec{0}$

$BA\vec{x} = B\vec{0} = \vec{0}$

& if BA = I $\Rightarrow$ $= I\vec{x} = \vec{x} = \vec{0}$ Only!

$\Rightarrow$ So $BA = I$ means $A\vec{x} = \vec{0}$ has $\vec{x} = 0$ soln. only!

So $A^{-1}$ exists & $B = A^{-1}$ (from above!)

If $BA = I$

So if $AB = I$ $\Rightarrow$ by above $B^{-1}$ exists

$$B^{-1} = A$$

$$\Rightarrow \boxed{A^{-1} = B}$$

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So If $A$ & $B$ are $n \times n$ matrices & either $AB = I$ or $BA = I$

Proof

$$\underbrace{(AB)^{-1} AB}\ = I$$

$$= (\underbrace{(AB)^{-1} A}) B = I$$

must be $B^{-1}$ (by previous!)

$$AB\,(AB)^{-1} = I$$

$$A\,(\underbrace{B(AB)^{-1}}) = I$$

This is $A^{-1}$, also by above!

So both $A$ & $B$ are inv. if $AB$ is inv

To prove

then $A^{-1}$ & $B^{-1}$ exist!

Lets add more to our Big Theorem!

Remarks

1) $A$ has RREF $I$

2) $A$ is a product of $E$'s (elementary matrices)

3) $A^{-1}$ exists

$A\vec{x} = \vec{b}$
$\vec{x} = A^{-1}b$

4) $A\vec{x} = \vec{b}$ has unique solution

5) $A\vec{x} = \vec{0}$ has trivial ($\vec{x} = \vec{0}$) soln. only.

4) really says any $A\vec{x} = \vec{b}$ soln. is unique!

but got for free $A\vec{x} = \vec{b}$ always has a solution

Is having a soln. for all possible $\vec{b}$ sufficient to show rest?

Yes! Let's show it!

Proof: Say $A\vec{x} = \vec{b}$ always has a solution

Then $A\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$ } 1 in first position, 0 elsewhere!

has a solution! call it $\vec{x}_1$

Similarly $A\vec{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ for some $\vec{x}_2$

$A\vec{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix}$ for some $\vec{x}_3$

Then $A\left[\vec{x}_1 \mid \vec{x}_2 \mid \vec{x}_3 \dots\right] = \left[A\vec{x}_1 \mid A\vec{x}_2 \dots\right]$

$$= \begin{bmatrix} 1 & 0 & 0 & \\ 0 & 1 & 0 & \\ 0 & 0 & 1 & \\ 0 & \vdots & \vdots & \ddots \end{bmatrix} = I$$

$$\Rightarrow A[\bar{x}_1 \dots \bar{x}_n] = I \Rightarrow \underline{\underline{A^{-1} \text{ exists}}}$$