Math 1LT3	Assignment 16

Extra practice with vectors, gradients and directional derivatives (Section 9)

1. In this exercise we review vectors and vector operations which we need for directional derivatives.

(a) Let $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j}$, $\mathbf{b} = -\mathbf{i} + \mathbf{j}$, $\mathbf{c} = 3\mathbf{j}$. Find $\mathbf{a} - 2\mathbf{b}$, $3\mathbf{c} - \mathbf{i} + 5\mathbf{b}$, $||\mathbf{a}||$, $||4\mathbf{b}||$, and $||\mathbf{a} + \mathbf{c}||\mathbf{j}$, and indicate which quantity is a vector, and which is a scalar.

(b) Find the vector from the point $(2, -1)$ to the point $(-4, -3)$, and find its length.

(c) Find the vector whose tail is at the point $(6, 0)$ and whose head is at the point $(-1, -1)$, and find its length.

(d) Find the vector whose initial point is $(0, 0)$ and terminal point is $(-4, -3)$, and find its length.

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(e) Show that $\mathbf{v} = (\mathbf{i} - \mathbf{j})/\sqrt{2}$ is a unit vector.

(f) Find the unit vector in the direction of the vector $\mathbf{v} = -4\mathbf{i} + 11\mathbf{j}$.

(g) Find the dot product $\mathbf{v} \cdot \mathbf{w}$, where $\mathbf{v} = -3\mathbf{i} + 2\mathbf{j}$ and $\mathbf{w} = 5\mathbf{i} + 4\mathbf{j}$.

(h) Find the dot product $(-\mathbf{i} - 7\mathbf{j}) \cdot (\mathbf{i} + 7\mathbf{j})$.

(i) Show that the vectors $\mathbf{a} = -3\mathbf{i} + 2\mathbf{j}$ and $\mathbf{w} = -2\mathbf{i} - 3\mathbf{j}$. are perpendicular.

(j) Find all vectors perpendicular to the vector $\mathbf{a} = \mathbf{i} + \mathbf{j}$.

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(k) Find the dot product $\mathbf{v} \cdot \mathbf{w}$ if it is known that $||\mathbf{v}|| = 4$, $||\mathbf{w}|| = 1$, and the angle between the two vectors is $\pi/3$.

(l) Show by example that $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$ does not imply that $\mathbf{v} = \mathbf{w}$. (Thus, there is no cancellation law for the dot product.)

(m) Find the dot product $\mathbf{v} \cdot \mathbf{w}$ if it is known that $||\mathbf{v}|| = 14$, $||\mathbf{w}|| = 1$, and the angle between the two vectors is π.

(n) What is the largest value of the dot product $\mathbf{a} \cdot \mathbf{b}$ if $||\mathbf{a}|| = 4$ and $||\mathbf{b}|| = 9$?

(o) What is the smallest value of the dot product $\mathbf{a} \cdot \mathbf{b}$ if $||\mathbf{a}|| = 4$ and $||\mathbf{b}|| = 9$?

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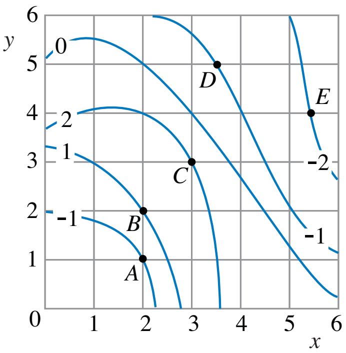
2. Consider the contour diagram of the function $f(x, y)$.



(a) Estimate the value of $D_{\mathbf{u}}f(3, 3)$ in the direction $\mathbf{v} = -\mathbf{i} - \mathbf{j}$.

(b) Estimate the value of $D_{\mathbf{u}}f$ at the point D in the direction $\mathbf{v} = -\mathbf{j}$.

(c) Estimate the value of $D_{\mathbf{u}}f$ at the point B in the direction $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j}$.

(d) Which of $||\nabla f(B)||$ or $||\nabla f(D)||$ is larger?

(e) Draw gradient vectors at several points on the contour curve of value 0

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3. Find the directional derivative of the given function at the point A in the direction of the vector $\mathbf{v}$.

(a) $f(x, y) = x(x^2 + y^2)^{-1/2}$, $A = (1, 1)$, $\mathbf{v} = 2\mathbf{i} - 5\mathbf{j}$

(b) $f(x, y) = e^{-x^2-y^2}$, $A = (0, 1)$, $\mathbf{v} = \mathbf{i} + \mathbf{j}$

(c) $f(x, y) = \sin(y + 3y^3)$, $A = (3, 1)$, $\mathbf{v} = \mathbf{i}$

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4. It is known that $\nabla f(2,3) = 4\mathbf{i} - 3\mathbf{j}$. Is there a direction $\mathbf{u}$ such that $D_{\mathbf{u}}f(2,3) = 6$? If so, find it.

5. Let $f(x,y) = x^2y - 2y$ and $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$.

(a) Compute $D_{\mathbf{u}}f(1,1)$, where $\mathbf{u}$ is the unit vector in the direction of $\mathbf{v}$.

(b) Calculate $D_{\mathbf{v}}f(1,1)$ for the vector $\mathbf{v}$ as given, ignoring the fact that we need a unit vector.

(c) What is the relation between your answers to (a) and (b)? So why do we need to use a unit vector to calculate the directional derivative?

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6. In what direction at the point $(1, 2)$ is the directional derivative of $f(x, y) = 6xy$ equal to 4? Specify the direction as an angle with respect to the gradient of $f(x, y)$ at the given point.

7. Find the maximum rate of change of the function $f(x, y) = 2ye^x + e^{-x}$ at $(0, 0)$ and the direction in which it occurs.

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8. Assume that $f(x, y)$ is a differentiable function. Identify all directions at a point (a, b) in which the rate of increase of f is at least 80% of the largest possible increase at that point.

9. Someone has calculated the following for a differentiable function f: in the direction from $(1, 1)$ toward $(2, 2)$, the directional derivative is 10; in the direction from $(1, 1)$ toward $(0, 0)$, the directional derivative is 5. How do you know that there is an error in the calculation?

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10. The temperature produced by a source located at the origin is given by $T(x,y) = 12e^{-x^2-y^2}$.

(a) Sketch the isothermal curves, i.e., the curves on which the temperature is constant. Find the gradient ∇T and add several gradient vectors to your sketch.

(b) Which point is the warmest?

(c) With (a) and (b) in mind, sketch (or describe) the gradient vector field of T.

(d) What is the direction of the most rapid decrease in temperature at the point $(1,2)$? What is the magnitude of that decrease?

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