MATH 1B03/1ZC3 Winter 2019

Lecture 18: Even more about vectors

Instructor: Dr Rushworth March 12th

The geometry of linear systems (continued)

(from Chapter 3.4 of Anton-Rorres)

We can geometrically describe the solutions to systems of linear equations, using the notion of orthogonality.

This is easiest to visualize in the case of systems with 3 variables, but it extends to systems with an arbitrary number of variables. In this section we will be freely changing between writing vectors as (x_1, x_2, x_3) and as a column vectors, such as $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

Let A be an $m \times 3$ matrix, and consider solutions to the homogeneous equation

$$A\mathbf{x} = \mathbf{0}$$

(recall that this is the matrix form of a system of linear equations).

Let $\mathbf{a}_k = (a_1, a_2, a_3)$ be a row vector of A. Recall that the collection of vectors orthogonal to $\mathbf{a}_k$ is a plane in $\mathbb{R}^3$. A vector $\mathbf{x}$ lies in this plane if and only if

$$\mathbf{a}_k \bullet \mathbf{x} = 0$$

Notice that when computing the matrix product $A\mathbf{x}$ we will compute the dot product $\mathbf{a}_k \bullet \mathbf{x}$ in the k-th entry. That is

$$\begin{bmatrix} & \vdots & \\ a_1 & a_2 & a_3 \\ & \vdots & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \vdots \\ \mathbf{a}_k \bullet \mathbf{x} \\ \vdots \end{bmatrix}$$

If $\mathbf{x}$ is a solution to $A\mathbf{x} = \mathbf{0}$, then $\mathbf{a}_k \bullet \mathbf{x} = 0$ for each k, and $\mathbf{x}$ lies in the plane described by every row vector $\mathbf{a}_k$.

This is summarised as follows:

1. Each row $\mathbf{a}_k$ of A describes a plane in $\mathbb{R}^3$ (the collection of vectors orthogonal to $\mathbf{a}_k$)
2. A vector $\mathbf{x}$ lies in the plane described by $\mathbf{a}_k$ if and only if $\mathbf{a}_k \bullet \mathbf{x} = 0$
3. The vector $\mathbf{x}$ is a solution to the equation $A\mathbf{x} = \mathbf{0}$ if and only if it lies in all of the planes described by the rows of A

We can produce a very similar argument if A is $m \times n$ (rather than just $m \times 3$), and arrive at the following fact.

Fact 18.1

Let A be a an $m \times n$ matrix. An $n \times 1$ vector $\mathbf{x}$ is a solution to the equation

$$A\mathbf{x} = \mathbf{0}$$

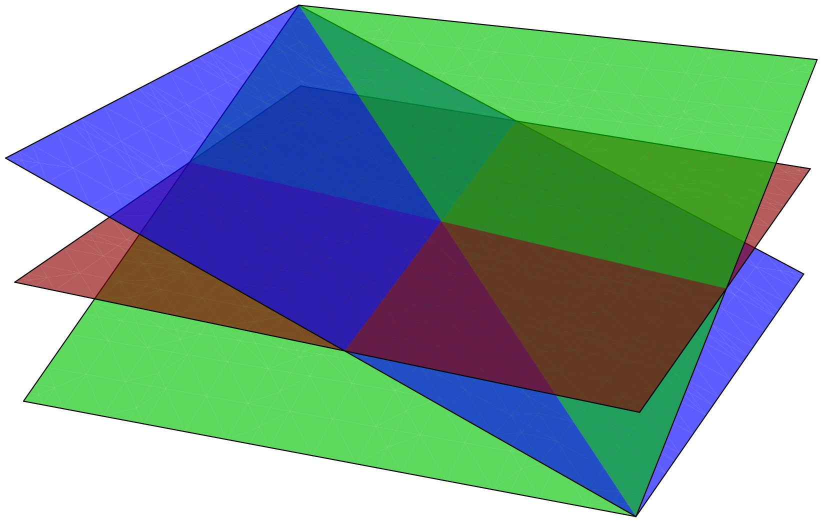
if and only if $\mathbf{x}$ is orthogonal to every row vector of A.

Let's see some examples in $\mathbb{R}^3$. Let A be 3×3; there are 3 row vectors of A, and each of them defines a plane in $\mathbb{R}^3$. Recall that there are exactly 3 possibilities for solutions of the equation:

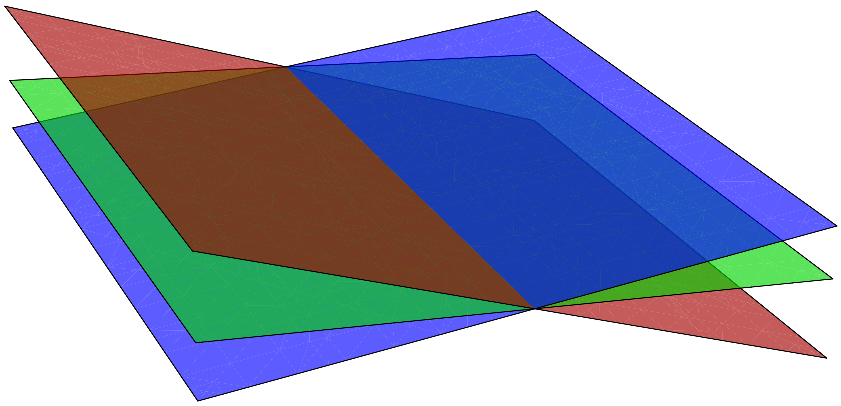
1. A unique solution
2. Infinitely many solutions
3. No solutions

We can realize these possibilities geometrically.

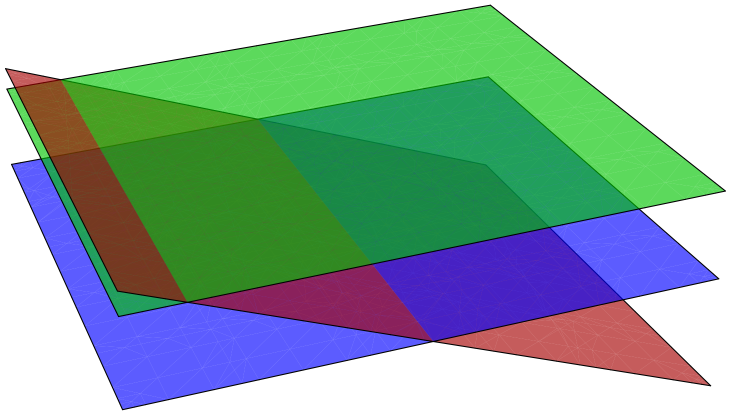
A unique solution: the 3 planes defined by the row vectors intersect in exactly one point:



Infinitely many solutions: the 3 planes defined by the row vectors intersect in a line:



No solutions: the 3 planes defined by the row vectors do not all intersect:



Notice that there are intersections between pairs of the planes, but there is no point at which all 3 of the planes intersect.

When A is larger than 3×3, the associated diagrams are harder to draw, but the principle remains the same: each row vector of A defines a geometric object, and solutions to $A\mathbf{x} = \mathbf{0}$ are vectors which lie in the intersection of all of these geometric objects.

Nonhomogeneous equations

The discussion above focused on homogeneous equations $A\mathbf{x} = \mathbf{0}$. It does not apply directly to nonhomgeneous equations $A\mathbf{x} = \mathbf{b}$.

The solutions to a nonhomogeneous equation $A\mathbf{x} = \mathbf{b}$ may be found by first solution the associated homogeneous equation $A\mathbf{x} = \mathbf{0}$, however.

Therefore if $\mathbf{x}$ is a solution to $A\mathbf{x} = \mathbf{b}$, then $\mathbf{x}$ may be written as

$$\mathbf{x} = \mathbf{x}_0 + \mathbf{x}_1$$

where $\mathbf{x}_0$ is a solution to $A\mathbf{x} = \mathbf{0}$, and so is orthogonal to every row vector of A. The term $\mathbf{x}_1$ is harder to visualize, however.

The cross product

(from Chapter 3.5 of Anton-Rorres)

Previously we have defined the norm and the dot product for vectors in $\mathbb{R}^n$.

In this section we shall define a new operation, the cross product for vectors in $\mathbb{R}^3$: it takes as input two vectors and outputs a new vector. It is unique to $\mathbb{R}^3$.

Operation	Input	Output	Defined
Norm, $\|\|\mathbf{u}\|\|$	one vector	one scalar	every $\mathbb{R}^n$
Dot product, $\mathbf{u} \bullet \mathbf{v}$	two vectors	one scalar	every $\mathbb{R}^n$
Cross product, $\mathbf{u} \times \mathbf{v}$	two vectors	one vector	only $\mathbb{R}^3$

(There is a cross product defined for $\mathbb{R}^7$, in fact, but we will not study it in this course. There are no cross products in dimensions other than 3 or 7.)

The cross product is interesting mathematically and appears in a diverse array of physics and engineering applications.

Definition 18.2: The cross product

Let $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$ be vectors in $\mathbb{R}^3$. The cross product of $\mathbf{u}$ and $\mathbf{v}$ is denoted $\mathbf{u} \times \mathbf{v}$, and defined

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2,\ u_3v_1 - u_1v_3,\ u_1v_2 - u_2v_1)$$

Example 18.3

Let $\mathbf{u} = (7, -3, 0)$, $\mathbf{v} = (-1, 2, 1)$. Then

$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= ((-3)(1) - (0)(2),\ (0)(-1) - (7)(1),\ (7)(2) - (-3)(-1)) \\ &= (-3, -7, 11)\end{aligned}$$

Fact 18.4: Properties of the cross product

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in $\mathbb{R}^3$, and k a scalar. Then

1. $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$
2. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$
3. $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w}$
4. $k\,(\mathbf{u} \times \mathbf{v}) = (k\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (k\mathbf{v})$
5. $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$
6. $\mathbf{u} \times \mathbf{u} = \mathbf{0}$

Fact 18.5: Relation to other operations

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in $\mathbb{R}^3$. Then

1. $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. That is

$$\mathbf{u} \bullet (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \bullet (\mathbf{u} \times \mathbf{v}) = 0$$

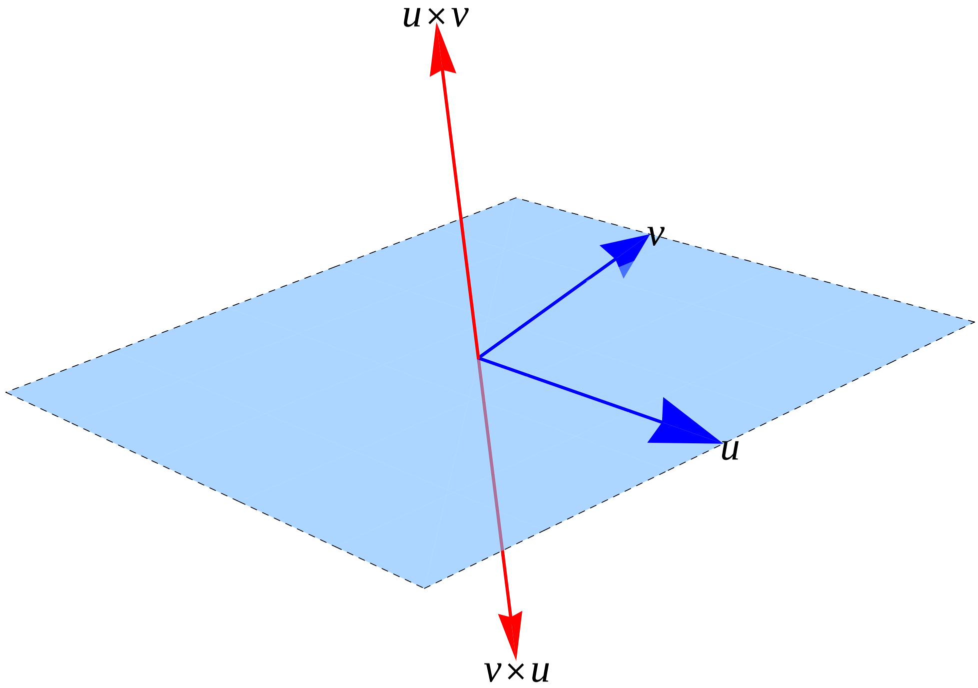
2. $||\mathbf{u} \times \mathbf{v}||^2 = ||\mathbf{u}||^2||\mathbf{v}||^2 - (\mathbf{u} \bullet \mathbf{v})^2$
3. $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \bullet \mathbf{w})\,\mathbf{v} - (\mathbf{u} \bullet \mathbf{v})\,\mathbf{w}$

Notice that the cross product is not commutative: $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$.

It is also not associative. That is, it matters where you put the brackets:

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \neq (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$$

The following diagram sums up some of the properties of the cross product:



The cross product as a determinant

Recall the standard unit vectors of $\mathbb{R}^3$:

$$\mathbf{i} = (1, 0, 0)$$
$$\mathbf{j} = (0, 1, 0)$$
$$\mathbf{k} = (0, 0, 1)$$

We can use these unit vectors to express the cross product as a determinant.

Fact 18.6

Let $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$ be vectors in $\mathbb{R}^3$. Their cross product may be computed

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

When using this formula it is easiest to expand along the top row.

Example 18.7

Let $\mathbf{u} = (7, -3, 0)$, $\mathbf{v} = (-1, 2, 1)$ as in Example 18.3. Then

$$\begin{aligned} \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 7 & -3 & 0 \\ -1 & 2 & 1 \end{vmatrix} &= \mathbf{i}(-3) - \mathbf{j}(7) + \mathbf{k}(14 - 3) \\ &= (-3, -7, 11) \end{aligned}$$

as required.

Finding areas and volumes via the cross product

We can use the cross product to compute areas and volumes of shapes in $\mathbb{R}^3$.

Definition 18.8: Scalar triple product

Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ be vectors in $\mathbb{R}^3$. Define their scalar triple product as

$$\mathbf{u} \bullet (\mathbf{v} \times \mathbf{w})$$

Let $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$, and $\mathbf{w} = (w_1, w_2, w_3)$. Notice that using the definition of the cross product as a determinant, we have

$$\mathbf{u} \bullet (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Example 18.9

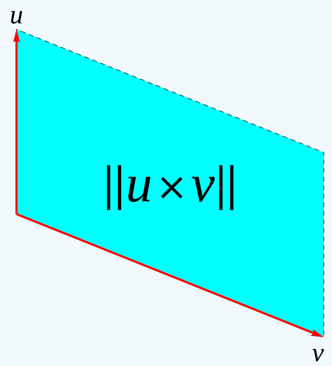
Let $\mathbf{u} = (7, -3, 0)$, $\mathbf{v} = (-1, 2, 1)$, and $\mathbf{w} = (0, 4, -5)$. Then

$$\begin{aligned}\mathbf{u} \bullet (\mathbf{v} \times \mathbf{w}) &= \begin{vmatrix} 7 & -3 & 0 \\ -1 & 2 & 1 \\ 0 & 4 & -5 \end{vmatrix} \\ &= 7(-10-4) + 15 \\ &= -83\end{aligned}$$

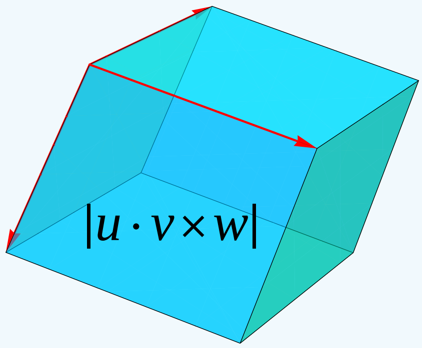
Using the scalar triple product we can compute the area or volume of shapes in $\mathbb{R}^3$.

Fact 18.10

1. Let $\mathbf{u}, \mathbf{v}$ be vectors in $\mathbb{R}^3$. Then $||\mathbf{u}\times\mathbf{v}||$ is the area of the parallelogram defined by $\mathbf{u}$ and $\mathbf{v}$.



2. Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ be vectors in $\mathbb{R}^3$. Then $|\mathbf{u} \bullet (\mathbf{v} \times \mathbf{w})|$ is the area of the parallelepiped defined by $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.



Fact 18.11

Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ be vectors in $\mathbb{R}^3$. If $|\mathbf{u} \bullet (\mathbf{v} \times \mathbf{w})| = 0$ then $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ lie in a plane.

Example 18.12

Question: Find the area of the triangle with vertices $(0, 4, -2)$, $(1, 3, -1)$ and $(2, 0, 1)$.
Answer: The triangle has sides given by the vectors

$$\begin{aligned} \mathbf{u} &= (1, 3, -1) - (0, 4, -2) \\ &= (1, -2, 1) \\ \mathbf{v} &= (2, 0, 1) - (0, 4, -2) \\ &= (2, -4, 3) \end{aligned}$$

Then the area of the parallelogram defined by $\mathbf{u}$ and $\mathbf{v}$ is

$$\begin{aligned} A &= ||\mathbf{u} \times \mathbf{v}|| \\ &= \left|\left| \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 2 & -4 & 3 \end{vmatrix} \right|\right| \\ &= ||\,(1, -1, -2)\,|| \\ &= \sqrt{6} \end{aligned}$$

The area of the triangle T is given by half the area of the parallelogram

$$\begin{aligned} T &= \frac{1}{2}A \\ &= \frac{1}{2}\sqrt{6} \\ &= \sqrt{\frac{3}{2}} \end{aligned}$$

Question: Find the volume of the parallelepiped defined by the vectors

$$\begin{aligned} \mathbf{u} &= (1, 0, -2) \\ \mathbf{v} &= (2, 5, -7) \\ \mathbf{w} &= (-2, 3, -2) \end{aligned}$$

Answer: Applying the formula given above we have

$$\begin{aligned} \text{Volume} &= |\mathbf{u} \bullet (\mathbf{v} \times \mathbf{w})\,| \\ &= \begin{vmatrix} 1 & 0 & -2 \\ 2 & 5 & -7 \\ -2 & 3 & -2 \end{vmatrix} \\ &= |(-10 + 21) - 2\,(6 + 10)| \\ &= |11 - 32| \\ &= 21 \end{aligned}$$

Suggested Problems

Practice the material covered in this lecture by attempting the following questions from Chapter 3.4 of Anton-Rorres, starting on page 170

- Questions $1, 9, 13, 15, 23$
- True/False (d), (f)

and from Chapter 3.5 of Anton-Rorres, starting on page 179

- Questions 15, 17, 25, 29, 34, 26