MATHEMATICS 1LT3 TEST 3

Evening Class
Duration of Test: 60 minutes
McMaster University

E. Clements

27 March 2017

FIRST NAME (please print): Sol^{NS}

FAMILY NAME (please print): __________

Student No.: __________

THIS TEST HAS 8 PAGES AND 8 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE.

Total number of points is 40. Marks are indicated next to the problem number in square brackets. You may use the McMaster standard calculator, Casio fx991 MS+, on this test.

USE PEN TO WRITE YOUR TEST. IF YOU USE A PENCIL, YOUR TEST WILL NOT BE ACCEPTED FOR REMARKING (IF NEEDED).

You need to show work to receive full credit, except for Question 1.

Problem	Points	Mark
1	9	
2	6	
3	6	
4	6	
5	4	
6	3	
7	3	
8	3	
TOTAL	40	

MATH 1LT3 * Test 3 * 27 March 2017

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1. **Multiple Choice**. Clearly **circle** the one correct answer.

(a) [3] Which of the following is/are true for the function $f(x, y) = \cos x - \sin y$?

(I) $(0, 0)$ is a critical point ✗
(II) $f(x, y) \leq 1$ for all $(x, y) \in \mathbb{R}^2$ ✗
(III) $D(x, y) = -\cos x \sin y$ ✓

$f_x = -\sin x \quad f_y = -\cos y$
$f_{xx} = -\cos x \quad f_{yy} = \sin y$
$f_{xy} = f_{yx} = 0$

(I) $f_y(0,0) = -\cos 0 = -1$
(II) $f(0, -\frac{\pi}{2}) = 2$
(III) $D = f_{xx} f_{yy} - (f_{xy})^2 = -\cos x \cdot \sin y$

(A) none (B) I only (C) II only (D) III only
(E) I and II (F) I and III (G) II and III (H) all three

(b) [3] Suppose that a family has three children. Assuming that female and male children are equally likely to be born, determine which of the following statements is/are true?

(I) The probability that one child is a girl is 0.125. ✗
(II) The probability that at least one child is a girl is 0.875. ✓
(III) The probability that at least two children are boys is 0.5. ✓

$S = \{GGG, GGB, GBG, BGG, GBB, BGB, BBG, BBB\}$

(I) $P(1 \text{ girl}) = 3/8 = 0.375$
(II) $P(\geq 1 \text{ girl}) = 7/8 = 0.875$
(III) $P(\geq 2 \text{ boys}) = 4/8 = 0.5$

(A) none (B) I only (C) II only (D) III only
(E) I and II (F) I and III (G) II and III (H) all three

(c) [3] Consider the sample space $S = \{1, 2, 3, 4, 5\}$ and the subsets $A = \{1, 2, 4, 5\}$ and $B = \{4, 5\}$. Suppose that $P(1) = 0.1$, $P(2) = 0.3$, $P(3) = 0.2$, $P(4) = 0.3$, and $P(5) = 0.1$. Which of the following statements is/are true?

(I) $P(A \cap B) = 0.4$ ✓ (II) $P(A \cup B) = 0.8$ ✓ (III) $P(A|B) = 0$ ✗

$A \cap B = \{4,5\} (= B)$
$P(A \cap B) = P(4) + P(5)$
$= 0.3 + 0.1$
$= 0.4$

$A \cup B = \{1,2,4,5\} (= A)$
$P(A \cup B) = 1 - P(3)$
$= 1 - 0.2$
$= 0.8$

$B \subseteq A \Rightarrow P(A|B) = 1$

$\boxed{B \quad A}$

(A) none (B) I only (C) II only (D) III only
(E) I and II (F) I and III (G) II and III (H) all three

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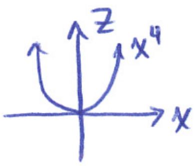
2. State whether each statement is **true or false**. **Explain** your reasoning.

(a) [2] The function $f(x, y) = x^4 - y^4$ has a local maximum at $(0, 0)$.

FALSE.



$f(0,0) = 0$

Consider $f(x,0) = x^4 \geq 0 \ \forall x \in \mathbb{R}$.

$f(x,0) > f(0,0)$ for all $x \neq 0 \Rightarrow f(0,0)$ is not a local max

(b) [2] It is possible to assign probabilities to sets A and B in the following way:

$$P(A) = 0.5,\ P(B) = 0.2, \text{ and } P(A \cap B) = 0.4$$

Here, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= 0.5 + 0.2 - 0.4$

$= 0.3$

But $P(A \cup B) \geq P(A)$ (and $\geq P(B)$) since $A \subseteq A \cup B$

$\therefore$ FALSE, probabilities cannot be assigned as above.

(c) [2] Consider the sample space $S = \{1, 2, 3, 4\}$. Assume that $P(1) = 0.1$, $P(2) = 0.4$, $P(3) = 0.1$, and $P(4) = 0.4$. If $A = \{1, 3\}$ and $B = \{1, 4\}$, then A and B are independent.

A and B are independent $\Leftrightarrow P(A \cap B) = P(A) \cdot P(B)$

$$\left.\begin{array}{l} P(A) = P(1) + P(3) = 0.1 + 0.1 = 0.2 \\ P(B) = P(1) + P(4) = 0.1 + 0.4 = 0.5 \end{array}\right\} \Rightarrow P(A) \cdot P(B) = (0.2)(0.5) = \underline{\underline{0.1}}$$

$P(A \cap B) = P(1) = \underline{\underline{0.1}}$

$\therefore$ TRUE. A and B are independent since $P(A \cap B) = P(A) P(B)$

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3. Consider the function $f(x, y) = x^3 - 6xy + y^2$.

(a) [3] Find the critical points of f.

$f_x = 3x^2 - 6y = 3(x^2 - 2y)$ $\quad f_y = -6x + 2y = 2(y - 3x)$

$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \Rightarrow \begin{cases} x^2 - 2y = 0 \\ y - 3x = 0 \end{cases} \Rightarrow \begin{cases} x^2 - 2y = 0 & ① \\ y = 3x & ② \end{cases}$

sub ② into ①: $x^2 - 2(3x) = 0$

$x(x-6) = 0$

$x = 0$ or $x = 6$

sub $x = 0$ into ②: $y = 0$

sub $x = 6$ into ②: $y = 18$

$\therefore$ $(0,0)$ and $(6,18)$ are the critical points.

(b) [3] Using the second derivatives test, classify the critical points from part (a).

$f_{xx} = 6x \quad f_{yy} = 2 \quad f_{xy} = -6 \quad f_{yx} = -6$

$D = (6x)(2) - (-6)^2 = 12x - 36$

$D(0,0) = -36 \Rightarrow (0,0)$ is a saddle point of f

$\left.\begin{array}{l} D(6,18) = 36 \\ f_{xx}(6,18) = 36 \end{array}\right\} \Rightarrow f$ has a local minimum value at $(6,18)$

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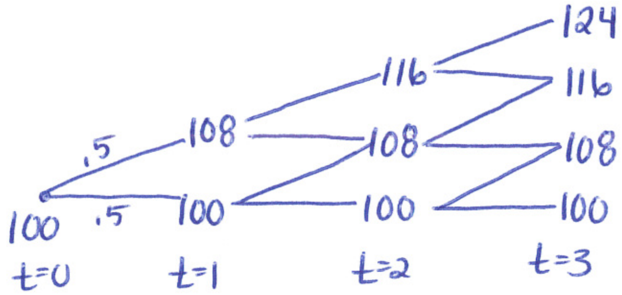
4. Consider a population of 100 elephants. Suppose that within any given year, there is a 50% chance that the population will increase by 8 elephants and a 50% chance that it will stay the same.

(a) [2] Write the sample space for the population of elephants after 3 years.



$S = \{100, 108, 116, 124\}.$

(b) [2] What is the probability that the population will have increased after 3 years?

$$P(p_3 > 100) = P(p_3 = 108 \text{ or } 116 \text{ or } 124)$$
$$= P(p_3 = 108) + P(p_3 = 116) + P(p_3 = 124)$$
$$= \frac{3}{8} + \frac{3}{8} + \frac{1}{8}$$
$$= \frac{7}{8}$$

(c) [2] Suppose that conditions changed and now there is a 60% chance that the population will increase by 8 and a 40% it will decrease by 10 in a given year. What is more likely to happen to the number of elephants over time? A net increase or a decrease? Explain.

Consider net pop^{N} change likely over 10 years:

$$6 \times (+8) + 4 \times (-10) = 8$$

6 years   increase by 8   4 years   decrease by 10

∴ It is likely that the population will experience a net increase over time.

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5. Two, fair six-sided dice are rolled. $S = \{(1,1), (1,2), \ldots, (6,6)\}$ $|S| = 36$

(a) [2] Find the probability that one die is a 4.

$A = \{(1,4), (2,4), (3,4), (5,4), (6,4), (4,1), (4,2), (4,3), (4,5), (4,6)\}$

$$P(A) = \frac{|A|}{|S|} = \frac{10}{36} = \frac{5}{18}$$

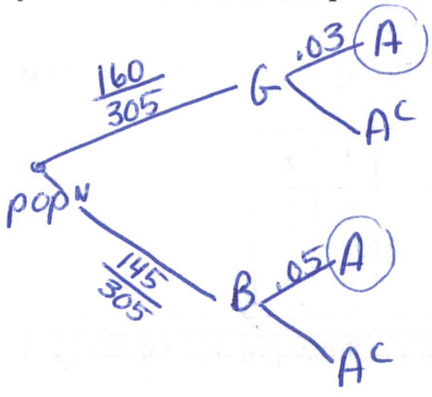
(b) [2] Using *conditional probability*, find the probability that one die is a 4 given that the sum is 6.

$C = \{(1,5), (2,4), (3,3), (4,2), (5,1)\}$

$A \cap C = \{(2,4), (4,2)\}$

$$P(A|C) = \frac{P(A \cap C)}{P(C)} = \frac{\frac{|A \cap C|}{|S|}}{\frac{|C|}{|S|}} = \frac{\frac{2}{36}}{\frac{5}{36}} = \frac{2}{5}$$

6. [3] Consider a population of school aged children comprised of 160 girls and 145 boys. Suppose that 3% of girls and 5% of boys within this population are estimated to be affected by ADHD. What is the probability that a randomly chosen child will be affected by ADHD?



$$P(A) = P(A|G)P(G) + P(A|B)P(B)$$
$$= (0.03)\left(\frac{160}{305}\right) + (0.05)\left(\frac{145}{305}\right)$$
$$\approx 0.03951$$

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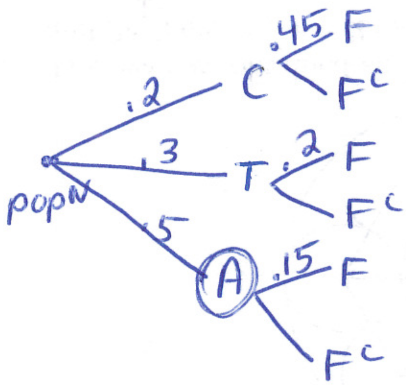
7. [3] A certain population consists of 20% children, 30% adolescents, and 50% adults. The probabilities that a certain member of this population catches the flu are 0.45 for a child, 0.2 for an adolescent, and 0.15 for an adult. What is the probability that a randomly selected person with the flu is an adult?



$$P(A|F) = \frac{P(F|A)P(A)}{P(F|A)P(A) + P(F|T)P(T) + P(F|C)P(C)}$$

$$= \frac{0.15(0.5)}{0.15(0.5) + 0.2(0.3) + 0.45(0.2)}$$

$$\approx 0.3333$$

8. [3] A medical test for a certain disease gives a false-positive result with a probability of 0.002. (A false positive describes the situation where the test turns out positive although the person tested does not have the disease.) What is the probability that in a group of 100 people, at least one false positive will occur?

F_i = false +ve result for person i

F = at least one false +ve occurs

F^c = no false +ves occur $= \bigcap_{i=1}^{100} F_i^c$

$P(F_i) = 0.002 \Rightarrow P(F_i^c) = 1 - 0.002 = 0.998$ for $1 \le i \le 100$

$$P(F) = 1 - P(F^c)$$
$$= 1 - P\left(\bigcap_{i=1}^{100} F_i^c\right)$$
$$\overset{(*)}{=} 1 - \prod_{i=1}^{100} P(F_i^c)$$
$$= 1 - (0.998)^{100}$$
$$\approx 0.1814$$

(*) F_i^c are independent events