MATH 1B03/1ZC3 Winter 2019

Lecture 13: Solving differential equations using matrices

Instructor: Dr Rushworth February 26th

THIS MATERIAL IS NOT TESTABLE ON MIDTERM 1 ON FEBRUARY 25TH.

Solving differential equations using matrices

(from Chapter 5.4 of Anton-Rorres)

Many problems in nature can be expressed as differential equations: equations involving functions and their derivatives. That is, given a function $y(x)$ of a variable x, with derivative $y' = \dfrac{dy}{dx}$, the following are examples of differential equations

1. $y'' + 7y' + 92 = y$
2. $(y')^2 + 8y = y'''$
3. $y' - 3y = 0$

A solution to a differential equation is a function which satisfies the equation. Finding solutions to a given differential equation is one of the most ubiquitous and fundamental problems in mathematics and physics.

Equations 1. and 2. in the list above are (very) hard to solve. You will learn techniques to handle equations like these in later courses. We will focus on equations which have the same form as Equation 3.

A solution to Equation 3. can be found as follows:

$$\begin{aligned}
y' - 3y &= 0 \\
y' &= 3y \\
\frac{dy}{dx} &= 3y \\
\frac{dy}{3y} &= dx, \text{ integrate to obtain} \\
\frac{1}{3}\log(y) &= x + C \\
y &= e^{3x+3C} \\
y &= Ae^{3x}, \text{ where } A = e^{3C}
\end{aligned}$$

Lets check that $y = Ae^{3x}$ is a solution to the differential equation $y' - 3y = 0$:

$$\begin{aligned}
y' &= \frac{d}{dx}\left(Ae^{3x}\right) \\
&= 3Ae^{3x} \\
&= 3y
\end{aligned}$$

so that $y' - 3y = 0$ as required.

Notice that the solution contains the constant A. For <u>any</u> choice of A the function $y = Ae^{3x}$ is a solution to the differential equation.

If we impose a condition on the solutions of the differential equation, this constant will be fixed. For example, lets impose the condition

$$y(0) = 10$$

on the solutions to $y' - 3y = 0$. We saw above that the solutions are of the form $y = Ae^{3x}$. Therefore the condition becomes

$$\begin{aligned}
y(0) &= Ae^{0} \\
&= A
\end{aligned}$$

so that $A = 10$, and the function $y = 10e^{3x}$ is the unique solution to the differential equation $y' - 3y = 0$ which satisfies the condition $y(0) = 10$.

A differential equation taken together with an intitial condition is known as an initial value problem.

In summary:

- A differential equation is an equation of the form $y' + ay = 0$, for and y a function of the variable x, and a a constant.
- A solution to a differential equation is a function which satisfies the equation.
- An initial value problem is a differential equation taken together with an initial condition $y(x_0) = b$, for x_0 and b constants.
- A solution to an initial value problem is a function which is a solution to the differential equation and satisfies the initial condition.

The differential equation

$$y' - 3y = 0$$

is an example of a homogeneous first-order constant coefficient differential equation. In nature we will often encounter systems of such equations.

Systems of first-order differential equations

Lets outline a situation in which we could encounter a system of first-order differential equations. Consider an air-ventilation system in a large building, with 8 temperature sensors in different positions around the building. Let the variable t denote time, and the function $y_i(t)$ denote the temperature recorded by the i-th sensor at time t.

In large buildings the air and temperature flow can be complicated, owing to convection currents, the conductivity of the windows, and other factors. As such, the rate of change of temperature at the i-th sensor, $y_i'(t)$, depends in general on the temperature at all the other sensors. We could model this mathematically as

$$y_i'(t) = a_{i1}y_1(t) + a_{i2}y_2(t) + \cdots + a_{i8}y_8(t)$$

for a_{ij} a constant.

As we have such an equation for all of the other temperature sensors, we obtain the following system of differential equations

$$\begin{aligned} y_1'(t) &= a_{11}y_1(t) + a_{12}y_2(t) + \cdots + a_{18}y_8(t) \\ y_2'(t) &= a_{21}y_1(t) + a_{22}y_2(t) + \cdots + a_{28}y_8(t) \\ &\vdots \\ y_8'(t) &= a_{81}y_1(t) + a_{82}y_2(t) + \cdots + a_{88}y_8(t) \end{aligned}$$

That is, a collection of first-order differential equations involving the one variable (in this case t).

To understand the temperature across the whole building we need to solve this system of differential equations i.e. find the function $y_1, y_2, \ldots, y_8$ such that every equation in the system is satisfied.

Definition 13.1: System of differential equations

A system of differential equations is a collection of n first-order differential equations

$$\begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n \\ y_2' &= a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n \\ &\vdots \\ y_n' &= a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nn}y_n \end{aligned}$$

where $y_1, y_2, \ldots, y_n$ are functions of a variable x.
A solution to the system is a choice of functions $y_1, y_2, \ldots, y_n$ which make every equation true.
An initial value problem is a system of differential equations taken together with n initial conditions, one for every function y_i:

$$\begin{aligned} y_1(x_0) &= b_1 \\ y_2(x_0) &= b_2 \\ &\vdots \\ y_n(x_0) &= b_n \end{aligned}$$

For $x_0, b_1, \ldots, b_n$ constants.
A solution to an initial value problem is a solution to the system of differential equations which also satisfies every initial condition.

As you might have guessed, we can write systems of differential equations as matrix equations. Given the system

$$\begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n \\ y_2' &= a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n \\ &\vdots \\ y_n' &= a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nn}y_n \end{aligned}$$

let

$$\mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

then

$$\mathbf{y}' = \frac{d}{dx}\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} y_1' \\ \vdots \\ y_n' \end{bmatrix}$$

We can write the system as the matrix equation

$$\mathbf{y}' = A\mathbf{y}$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Notice that A is $n \times n$. We shall refer to such matrix equations as systems of differential equations.

A solution to the system of differential equations is therefore a vector $\mathbf{y}$ which makes the above matrix equation true.

Fact 13.2: Trivial solution

Let

$$\mathbf{y}' = A\mathbf{y}$$

be a system of differential equations. Then the vector $\mathbf{y} = 0$ is solution, known as the trivial solution.

Recall that $\mathbf{y} = 0$ means the zero vector.

Proof: If $\mathbf{y} = 0$ then

$$\mathbf{y}' = 0$$

also. As $A\mathbf{y} = A0 = 0$ for any matrix A, we see that $\mathbf{y} = 0$ is a solution. ∎

As the trivial solution is always a solution to such systems, our task is to find non-trivial solutions i.e. solutions $\mathbf{y} \neq 0$.

Solving systems of differential equations via diagonalization

If A is a diagonal matrix then the system

$$\mathbf{y}' = A\mathbf{y}$$

can be solved in exactly the same way we solved the differential equation $y' - 3y = 0$. The following method works for any n, but to demonstrate the method we'll fix $n = 3$.

If A is a 3×3 diagonal matrix it must have the form

$$A = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}.$$

The matrix equation becomes

$$\begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix} = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

This yields the individual differential equations

$$\begin{aligned} y_1' &= a_{11}y_1 \\ y_2' &= a_{22}y_2 \\ y_3' &= a_{33}y_3 \end{aligned}$$

The method we used to solve $y' - 3y = 0$ can be applied again to find the solutions

$$\begin{aligned} y_1 &= c_1e^{a_{11}x} \\ y_2 &= c_2e^{a_{22}x} \\ y_3 &= c_3e^{a_{33}x} \end{aligned}$$

where c_1, c_2 and c_3 are constants. Our solution is

$$\mathbf{y} = \begin{bmatrix} c_1 e^{a_{11}x} \\ c_2 e^{a_{22}x} \\ c_3 e^{a_{33}x} \end{bmatrix}$$

Suppose we have another system of differential equations

$$\mathbf{y}' = B\mathbf{y}$$

in which B is not diagonal. However, if B is diagonalizable we can still use the above method to solve the system!

Assume that P diagonalizes B. That is

$$B = PDP^{-1}$$

for D a diagonal matrix.

The matrix equation

$$\mathbf{y}' = B\mathbf{y}$$

becomes

$$\mathbf{y}' = PDP^{-1}\mathbf{y}$$

multiplying from the left by P^{-1} we obtain

$$P^{-1}\mathbf{y}' = DP^{-1}\mathbf{y}$$

Now define $\mathbf{u} = P^{-1}\mathbf{y}$. Then $\mathbf{u}' = P^{-1}\mathbf{y}'$ (as P does not depend on the variable x).

The matrix equation

$$P^{-1}\mathbf{y}' = DP^{-1}\mathbf{y}$$

becomes

$$\mathbf{u}' = D\mathbf{u}$$

But D is diagonal, so we can use the method we saw above to solve this new system. Once we find a solution $\mathbf{u}$ to the system $\mathbf{u}' = D\mathbf{u}$ we can find a solution $\mathbf{y}$ to the original system using the relationship

$$\mathbf{y} = P\mathbf{u}$$

This method is summarised in the following recipe.

Recipe 13.3: Solving a system of differential equations via diagonalization

Step 1: Let $\mathbf{y}' = A\mathbf{y}$ be a system of differential equations or an initial value problem. Check if A is diagonalizable. If it is not, this method cannot be used. If it is, go to Step 2.

Step 2: If A is diagonalizable find P and P^{-1} such that

$$A = PDP^{-1}$$

for D a diagonal matrix.

Step 3: Solve the new system

$$\mathbf{u}' = D\mathbf{u}$$

Step 4: A solution to the original system $\mathbf{y}' = A\mathbf{y}$ is given by

$$\mathbf{y} = P\mathbf{u}$$

If you were solving an initial value problem, go to Step 5.

Step 5: Impose the initial conditions given on the solution $\mathbf{y}$ to find the unique solution to the initial value problem.

Example 13.4

Question: Solve the system

$$\begin{aligned} y_1' &= 4y_1 + y_3 \\ y_2' &= -2y_1 + y_2 \\ y_3' &= -2y_1 + y_3 \end{aligned}$$

subject to the initial conditions

$$y_1(0) = -1$$
$$y_2(0) = 1$$
$$y_3(0) = 0$$

Answer: Write the system as a matrix equation

$$\mathbf{y}' = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \mathbf{y}$$

So that

$$A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

To check if A is diagonalizable we need to compute its eigenvalues and eigenvectors. Using the method given in Lecture 10 we see that

$$\text{eigenvalue}: \quad \lambda_1 = 1 \quad \lambda_2 = 2 \quad \lambda_3 = 3$$

$$\text{basis:} \qquad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} \qquad \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

so that A is diagonalizable with

$$P = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

and

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Now we solve the system

$$\mathbf{u}' = D\mathbf{u}$$
$$\begin{bmatrix} u_1' \\ u_2' \\ u_3' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

This yields the equations

$$\begin{aligned} u_1' &= u_1 \\ u_2' &= 2u_2 \\ u_3' &= 3u_3 \end{aligned}$$

with solutions

$$\begin{aligned} u_1 &= c_1 e^x \\ u_2 &= c_2 e^{2x} \\ u_3 &= c_3 e^{3x} \end{aligned}$$

for c_1, c_2 and c_3 constants.
Now find the solution $\mathbf{y} = P\mathbf{u}$

$$\begin{aligned} \mathbf{y} &= P\mathbf{u} \\ &= \begin{bmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} c_1 e^x \\ c_2 e^{2x} \\ c_3 e^{3x} \end{bmatrix} \\ &= \begin{bmatrix} -c_2 e^{2x} - c_3 e^{3x} \\ c_1 e^x + 2c_2 e^{2x} + c_3 e^{3x} \\ 2c_2 e^{2x} + c_3 e^{3x} \end{bmatrix} \end{aligned}$$

so that

$$\begin{aligned} y_1 &= -c_2 e^{2x} - c_3 e^{3x} \\ y_2 &= c_1 e^x + 2c_2 e^{2x} + c_3 e^{3x} \\ y_3 &= 2c_2 e^{2x} + c_3 e^{3x} \end{aligned}$$

is a solution to the original system of differential equations.
Now apply the initial conditions:

$$\begin{aligned} y_1(0) &= -c_2 - c_3 = -1 \\ y_2(0) &= c_1 + 2c_2 + c_3 = 1 \\ y_3(0) &= 2c_2 + c_3 = 0 \end{aligned}$$

The equation

$$2c_2 + c_3 = 0$$

implies that $-2c_2 = c_3$. Plugging this into the equation

$$-c_2 - c_3 = -1$$

we obtain

$$\begin{aligned} -c_2 - c_3 &= -1 \\ -c_2 + 2c_2 &= -1 \\ c_2 &= -1 \end{aligned}$$

As $-2c_2 = c_3$ and $c_2 = -1$ we obtain

$$c_3 = 2$$

Plugging these values into the equation

$$c_1 + 2c_2 + c_3 = 1$$

we obtain

$$c_1 - 2 + 2 = 1$$

so that $c_1 = 1$.
Therefore the unique solution to the initial value problem is

$$\mathbf{y} = \begin{bmatrix} e^{2x} + -2e^{3x} \\ e^{x} - 2e^{2x} + 2e^{3x} \\ -2e^{2x} + 2e^{3x} \end{bmatrix}$$

or equivalently

$$\begin{aligned} y_1 &= e^{2x} + -2e^{3x} \\ y_1 &= e^{x} - 2e^{2x} + 2e^{3x} \\ y_3 &= -2e^{2x} + 2e^{3x} \end{aligned}$$