1ZC3

Assignment: Due Thurs.

Last Day: Determinants of $n \times n$ Matrices (by Cofactor Expansion)

$$\det A = \sum_{i \text{ or } j=1}^{n} a_{ij}\, C_{ij}$$

} sum along any single row or column of A, of (entries)·(cofactors)

$$= \sum_{i=1}^{n} a_{ij} \overbrace{\underbrace{(-1)^{i+j}}_{\text{pos. sign}} \underbrace{M_{ij}}_{\text{minor determinant}}}^{C_{ij}}$$

or

$$\underbrace{\sum_{j=1}^{n} a_{ij} (-1)^{i+j} M_{ij}}$$

For a given $\underline{A}$ same value no matta row or column chosen!

Note If A has a row <u>or</u> a col. of <u>only</u> 0 entries $\Rightarrow$ $\det(A) = 0$

Special Case

eg $\det B = \begin{vmatrix} 2 & 4 & 7 & 1 \\ 0 & -1 & 2 & 3 \\ 0 & 0 & 5 & 7 \\ 0 & 0 & 0 & 3 \end{vmatrix} = 2(-1)(5)(3)$

$= -30$

Triangular & diagonal matrices are <u>special</u>

det (triang. matrix) = <u>product along principal diagonal</u>

Goal: To use row ops to simplify $\det A$ into something triangular!

$\Rightarrow$ First understand Elem. row ops!

How do row ops. change $\det A$?

eg $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $|A| = 1(4) - 2(3) = \underline{\underline{-2}}$

$\downarrow$ 2 · Row 2

$\begin{bmatrix} 1 & 2 \\ 6 & 8 \end{bmatrix} \Rightarrow \begin{vmatrix} 1 & 2 \\ 6 & 8 \end{vmatrix} = 1(8) - 2(6)$

$= \underline{\underline{-4}}$

In general $k \cdot \text{Row } i \Rightarrow$ new det $= k \cdot$ old det eg $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \rightsquigarrow |A| = 1(4) - 2(3) = \underline{\underline{-2}}$

$\Big\downarrow$ Row$_1 \leftrightarrow$ Row$_2$

$\begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} \rightsquigarrow \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = 3(2) - 1(4)$

$= \underline{\underline{+2}}$

In general: Row$_i \leftrightarrow$ Row$_j \Rightarrow$ new det $= -$ old det

eg $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \rightsquigarrow |A| = 1(4) - 2(3) = -2$

$\Big\downarrow$ Row2 $-$ 2Row1

$\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \rightsquigarrow \begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} = 1(0) - 2(1) = \underline{\underline{-2}}$

In general $Row_i \to Row_i + k\,Row_j \Rightarrow$ new det = old det

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eg. Use row ops. to compute the det. of

$$A = \begin{bmatrix} 2 & 4 & 6 \\ 3 & 6 & 10 \\ 3 & 8 & 9 \end{bmatrix}$$

Goal: turn triangular & mult. along principal diagonal

Solution

$$\det A = \begin{vmatrix} 2 & 4 & 6 \\ 3 & 6 & 10 \\ 3 & 8 & 9 \end{vmatrix} \underset{Row\,1 \cdot \frac{1}{2}}{=} 2 \cdot \begin{vmatrix} 1 & 2 & 3 \\ 3 & 6 & 10 \\ 3 & 8 & 9 \end{vmatrix}$$

Row 2 - 3Row1 & Row3 - 3Row1 $\to$

$$= 2 \begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 2 & 0 \end{vmatrix} \underset{R_2 \leftrightarrow R_3}{=} -2 \begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= -2 \cdot (1)(2)(1) = \boxed{-4}$$

If A^{-1} exists $\Rightarrow$ A reduces to I $\Rightarrow$ $\det A \neq 0$

(since row ops cannot make a det. $= 0$)

If A^{-1} does not exist $\Rightarrow$ A does not reduce to I

$\Rightarrow$ RREF of A has a row of 0's

$\Rightarrow$ $\det A = 0$

$\rightarrow$ A invertible (i.e. A^{-1} exists) iff $\det A \neq 0$

for all $n \times n$ matrices!

Elementary Matrices & Determinants

Say $\text{Row}_i \leftrightarrow \text{Row}_j \Rightarrow |E| = |\,I$ with 2 rows swapped$\,|$

$= -|I| = -1$

Say $\text{Row}_i \to k\,\text{Row}_i \Rightarrow |E| = |\,I$ with 1 row mult. by $k\,|$

$$= \begin{vmatrix} 1 & & & & & \\ & 1 & & & & 0 \\ & & \ddots & & & \\ & & & k & & \\ & 0 & & & 1 & \\ & & & & & \ddots\ 1 \end{vmatrix} = k$$

Say $\text{Row}_i \to \text{Row}_i + k\,\text{Row}_j \Rightarrow |E| = \|\,I$ with row j added k times to row $i\,\|$

$= |I| = 1$

eg I_3 with $R_1 \to R_1 + 3R_2$

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So Say E is elementary

$$|EA| = |A \text{ after } E\text{'s single row op}|$$
$$= |E|\,|A|$$

So Say A, B invertible $\Rightarrow A = E_1 \cdots E_n \quad B = E_{b_1} \cdots E_{b_n}$

$$\det(AB) = |AB| = |E_1 \cdots E_n E_{b_1} \cdots E_{b_n}|$$
$$= \underbrace{|E_1|\,|E_2| \cdots |E_n|}\ \underbrace{|E_{b_1}| \cdots |E_{b_n}|}$$
$$= |A|\,|B|$$

So if A, B inv. $\quad |AB| = |A|\,|B|$

Say A or B (or both!) <u>not inv.</u> (singular)

$\Rightarrow$ AB singular $\Rightarrow$ $|AB| = 0$ equal!

but $|A| = 0$ or $|B| = 0$ $\}$ $\Rightarrow$ $|A||B| = 0$

$\Rightarrow$ $|AB| = |A||B|$ Always