MATHEMATICS 1LT3 TEST 3

Evening Class
Duration of Examination: 60 minutes
McMaster University, 27 March 2014

Dr. M. Lovrić

FIRST NAME (please print): SOLUTIONS

FAMILY NAME (please print): ______

Student No.: ______

THIS TEST HAS 8 PAGES AND 7 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE.

Total number of points is 40. Marks are indicated next to the problem number. Any non-graphing calculator is allowed.

USE PEN TO WRITE YOUR TEST. IF YOU USE A PENCIL YOUR TEST WILL NOT BE ACCEPTED FOR REMARKING (IF NEEDED).

You must show work to receive full credit.

Problem	Points	Mark
1	6	
2	6	
3	6	
4	6	
5	4	
6	6	
7	6	
TOTAL	40	

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Student No.: ________________

1. **Multiple choice questions: circle ONE answer. No justification is needed.**

(a)[3] You know that $\nabla f(2,3) = 3\mathbf{i} + \mathbf{j}$. Which of the following statements is/are true?

(I) The directional derivative of $f(x,y)$ in the direction $\mathbf{v} = 3\mathbf{i} + \mathbf{j}$ is zero ✗

(II) The largest rate of change of $f(x,y)$ at $(2,3)$ is $\sqrt{10}$ ✓

(III) $f(x,y)$ has a local extreme value at $(2,3)$ ✗

(A) none	(B) I only	(C) II only	(D) III only
(E) I and II	(F) I and III	(G) II and III	(H) all three

(b)[3] Which of the following facts is/are true for any two independent events A and B?

(I) $P(A|B) = P(B)$ ✗

(II) $A \cap B = \emptyset$ ✗

(III) $P(A \cap B) = P(A)P(B)$ ✓

(A) none	(B) I only	(C) II only	(D) III only
(E) I and II	(F) I and III	(G) II and III	(H) all three

$P(A|B) = P(A)$

or

$P(A \cap B) = P(A) \cdot P(B)$

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Questions 2-7: You must show work to receive full credit.

2. Consider the function $f(x,y) = x^3 e^{xy} - y$.

(a)[2] Determine whether $f(x,y)$ is increasing or decreasing at the point $(1,1)$ in the direction of the vector $\mathbf{v} = -\mathbf{i} - \mathbf{j}$.

$$\left.\begin{aligned} f_x &= 3x^2 e^{xy} + x^3 e^{xy} \cdot y \\ f_y &= x^3 e^{xy} \cdot x - 1 \end{aligned}\right\} \quad \nabla f(1,1) = (3e+e, e-1) = (4e, e-1)$$

$$\vec{v} = -\vec{i} - \vec{j} \rightarrow \vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = -\frac{1}{\sqrt{2}}\vec{i} - \frac{1}{\sqrt{2}}\vec{j}$$

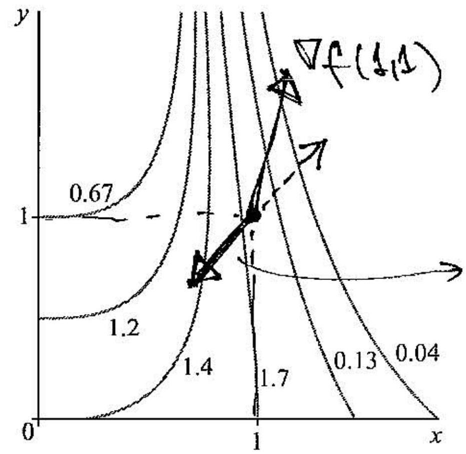
$$D_{\vec{u}} f = \nabla f(1,1) \cdot \vec{u} = 4e\left(-\frac{1}{\sqrt{2}}\right) + (e-1)\left(-\frac{1}{\sqrt{2}}\right) = -\frac{5e}{\sqrt{2}} + \frac{1}{\sqrt{2}} < 0$$

decreasing

(b)[2] Determine whether or not $(0,1)$ is a critical point of $f(x,y)$.

$$\left.\begin{aligned} f_x(0,1) &= 0 \\ f_y(0,1) &= -1 \end{aligned}\right\} \text{NO}$$

(c)[2] Could the picture below be a contour diagram of $f(x,y)$? Explain why or why not. NO



$\nabla f(1,1) = (4e, e)$ is the direction of largest increase $\rightarrow$ contradicts contour curves

OR:

in (a) we showed that f decreases in this direction $\rightarrow$ so it must increase in the opposite direction contradicts contour curves

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3. Consider the function $f(x,y) = x^3 - 2y^2 + 3xy + 1$.

(a)[2] Verify that $(0,0)$ and $(-3/4, -9/16)$ are the only critical points of $f(x,y)$.

$f_x = 3x^2 + 3y = 0 \rightarrow x^2 + y = 0 \rightarrow y = -x^2$

substitute into f_y

$f_y = -4y + 3x = 0$

$4x^2 + 3x = 0 \rightarrow x(4x+3) = 0 \rightarrow x = 0$ (hence $y=0$)

$x = -3/4$ ($y = -x^2 = -9/16$)

(b)[2] Use the second derivative test to classify $(-3/4, -9/16)$ a local minimum, local maximum, or a saddle point.

$$\left.\begin{array}{l} f_{xx} = 6x \\ f_{xy} = 3 \\ f_{yy} = -4 \end{array}\right\} \quad D = f_{xx}f_{yy} - (f_{xy})^2 = -24x - 9$$

$D(-\frac{3}{4}, -\frac{9}{16}) = -24(-\frac{3}{4}) - 9 = 9 > 0$

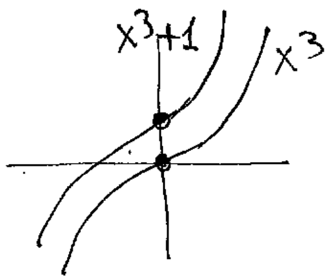
$f_{xx}(-\frac{3}{4}, -\frac{9}{16}) = 6(-\frac{3}{4}) = -\frac{9}{2} < 0$

$\Rightarrow$ LOCAL MAX.

(c)[2] Without using the second derivative test, determine whether $(0,0)$ is a local minimum, local maximum, or a saddle point.

$f(x,0) = x^3 + 1$

so $f(0,0)$ cannot be min. nor max.



$\Rightarrow$ $(0,0)$ is a saddle point

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4. The average incidence of $\overbrace{\text{autism spectrum disorder}}^{\text{ASD}}$ is 5 cases per 1,000. A test for the disorder shows a positive result in 96% of people who have the disorder, and in 1% of people who do not have it.

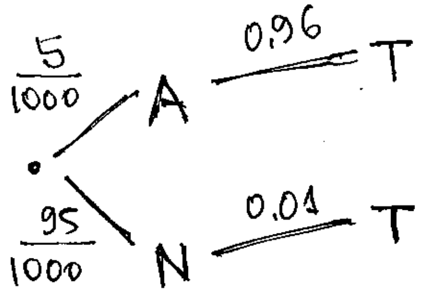
(a)[3] What is the probability that a randomly selected person tests positive for the disorder?

A = " person has autism sp. disorder" .. $P(A) = 5/1000$

N = " person does not have ASD" .. $P(N) = 995/1000$

T = " test is positive "

Given: $P(T|A) = 0.96$, $P(T,N) = 0.01$



By the law of total probability (or from the tree)

$$P(T) = P(T|A)P(A) + P(T|B)P(B)$$
$$= 0.96 \cdot \frac{5}{1000} + 0.01 \cdot \frac{995}{1000}$$
$$= 0.01475$$

(b)[3] If a person tests positive for the disorder, what is the probability that they have it?

$P(A|T)$ = Bayes' formula

$$= \frac{P(T|A)P(A)}{P(T)}$$

$P(T|A)P(A)$ ← first term in $P(T)$

$P(T)$ ← 0.01475 from (a)

$$= \frac{0.96 \cdot 5/1000}{0.01475} \approx \underline{0.325}$$

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5. [4] One way to get rid of most of the house dust mites (which are the most commom cause of allergic reactions and asthma) is to wash laundry in hot water. It has been determined that the chance that a house dust mite survives in laundry washed at 60°C is 0.01. What is the probability that, in a sample of 100 house dust mites, at least one will survive?

Define H_i = "i-th house dust mite survives in laundry washed at 60°C"

$i = 1, 2, \ldots, 100$; it is given that $P(H_i) = 0.01$

Let A = "at least house dust mite survives"

Then A^c = "no house dust mites survive

$$A^c = H_1^c \cap H_2^c \cap \ldots \cap H_{100}^c$$

By independence,

$$P(A^c) = P(H_1^c) \cdot P(H_2^c) \cdots P(H_{100}^c)$$
$$= 0.99 \cdot 0.99 \cdots 0.99$$
$$= 0.99^{100}$$

So $P(A) = P$(at least one house dust mites survives)

$$= 1 - 0.99^{100} \approx 0.634 \text{, ie about } 63.4\%$$

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6. Given below is the cumulative distribution function $F(x)$ of a random variable X.

$$F(x) = \begin{cases} 0 & x < 2 \\ 0.1 & 2 \leq x < 4 \\ 0.5 & 4 \leq x < 5 \\ 0.8 & 5 \leq x < 7 \\ 1 & x \geq 7 \end{cases}$$

(a)[2] Find the probability mass function of X.

x	p(x)
2	0.1
4	0.4
5	0.3
7	0.2

(b)[2] Let $Y = 7X$. Find the expected value $E(Y)$. $= 7 \cdot E(X)$

$$E(X) = (2)(0.1) + (4)(0.4) + (5)(0.3) + (7)(0.2)$$
$$= 4.7$$

so $E(Y) = 32.9$

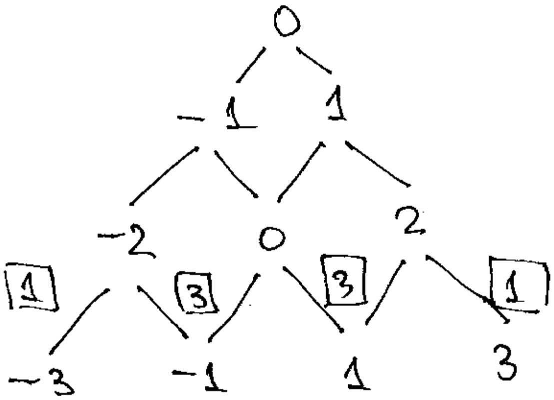
(c)[2] Let $Z = X + 3$. You need to compute $\text{var}(Z)$, but are running out of time. Your very reliable friend tells you that $\text{var}(X) = 2.01$ How can this help you figure out $\text{var}(Z)$? Explain why.

$\text{Var}(Z) = \text{var}(X) = \underline{2.01}$

by shifting the whole distribution we do not change its spread

7. The experiment consists of observing the third step in the random walk (i.e., a particle starts at the origin, and in each step moves by one unit either left or right, with equal probability).

(a)[1] What is the sample space of the locations of the particle after three steps?



$S = \{-3, -1, 1, 3\}$

→ there is a total of 8 paths from the origin to a location in third step

(b) [2] Define A="The particle ends to the right of the origin after three steps" and C="in the first step, the particle moves right." Find $P(A|C)$.

$$P(A|C) = \frac{P(A \cap C)}{P(C)}$$

$P(C) = 1/2$ (given, by def. of random walk)

$A \cap C$ ... first step is R and the particle ends at 1 or 3

$A \cap C = \{ \underbrace{RRR}_{\text{ends at 3}}, \underbrace{RRL}_{\text{at 1}}, \underbrace{RLR}_{\text{at 1}}, \cancel{\underbrace{RLL}_{\text{at -1}}} \} \rightarrow P(A \cap C) = 3/8$

$$\Rightarrow P(A|C) = \frac{3}{4}$$

(c) [3] Define A="The particle ends to the right of the origin after three steps" and B="in the first step, the particle moves left." Are A and B independent events? Justify your answer.

$A = \{RRR, RLR, LRR, RRL\} \rightarrow P(A) = 4/8 = 1/2$

$B = \{LLL, LRR, LRL, LLR\} \rightarrow P(B) = 4/8 = 1/2$

$A \cap B = \{LRR\} \rightarrow P(A \cap B) = 1/8$

$$P(A \cap B) \stackrel{?}{=} P(A) \cdot P(B)$$

$$\frac{1}{8} \neq \frac{1}{2} \cdot \frac{1}{2}$$

NOT INDEP.

THE END