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10.2
DIVISION OF COMPLEX NUMBERS

In the last section we defined multiplication of complex numbers. In this section we shall define division of complex numbers as the inverse of multiplication.

We begin with some preliminary ideas.

Complex Conjugates

If $z = a + bi$ is any complex number, then the ***complex conjugate*** of z (also called the ***conjugate*** of z) is denoted by the symbol $\bar{z}$ (read " z bar" or " z conjugate") and is defined by

$$\bar{z} = a - bi$$

In words, $\bar{z}$ is obtained by reversing the sign of the imaginary part of z. Geometrically, $\bar{z}$ is the reflection of z about the real axis (Figure 10.2.1).

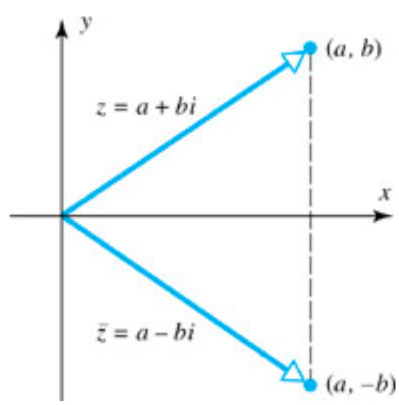


Figure 10.2.1
The conjugate of a complex number.

EXAMPLE 1 Examples of Conjugates

$$\begin{array}{ll} z=3+2i & \bar{z}=3-2i \\ z=-4-2i & \bar{z}=-4+2i \\ z=i & \bar{z}=-i \\ z=4 & \bar{z}=4 \end{array}$$

Remark The last line in Example 1 illustrates the fact that a real number is the same as its conjugate. More precisely, it can be shown (Exercise 22) that $z=\bar{z}$ if and only if z is a real number.

If a complex number z is viewed as a vector in R^2, then the norm or length of the vector is called the modulus of z. More precisely:

DEFINITION

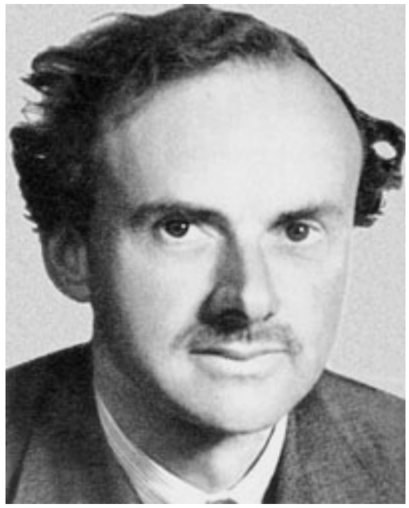
The ***modulus*** of a complex number $z=a+bi$, denoted by $|z|$, is defined by

$$|z|=\sqrt{a^2+b^2} \tag{1}$$

If $b=0$, then $z=a$ is a real number, and

$$|z|=\sqrt{a^2+0^2}=\sqrt{a^2}=|a|$$

so the modulus of a real number is simply its absolute value. Thus the modulus of z is also called the ***absolute value*** of z.



Paul Adrien Maurice Dirac *(1902–1984)* was a British theoretical physicist who devised a new form of quantum mechanics and a theory that predicted electron spin and the existence of a fundamental atomic particle called a positron. He received the Nobel Prize for physics in 1933 and the medal of the Royal Society in 1939.

EXAMPLE 2 Modulus of a Complex Number

Find $|z|$ if $z = 3 - 4i$.

Solution

From 1, with $a = 3$ and $b = -4$, $|z| = \sqrt{(3)^2 + (-4)^2} = \sqrt{25} = 5$.

The following theorem establishes a basic relationship between $\bar{z}$ and $|z|$.

THEOREM 10.2.1

For any complex number z,

$$z\bar{z} = |z|^2$$

Proof If $z = a + bi$, then

$$z\bar{z} = (a + bi)(a - bi) = a^2 - abi + bai - b^2 i^2 = a^2 + b^2 = |z|^2$$

■

Division of Complex Numbers

We now turn to the division of complex numbers. Our objective is to define division as the inverse of multiplication. Thus, if $z_2 \neq 0$, then our definition of $z = z_1 / z_2$ should be such that

$$z_1 = z_2 z \tag{2}$$

Our procedure will be to prove that 2 has a unique solution for z if $z_2 \neq 0$, and then to define z_1 / z_2 to be this value of z. As with real numbers, division by zero is not allowed.

THEOREM 10.2.2

If $z_2 \neq 0$, then Equation 2 has a unique solution, which is

$$z = \frac{1}{|z_2|^2} z_1 \bar{z}_2 \tag{3}$$

Proof Let $z = x + iy$, $z_1 = x_1 + iy_1$, and $z_2 = x_2 + iy_2$. Then 2 can be written as

$$x_1 + iy_1 = (x_2 + iy_2)(x + iy)$$

or

$$x_1 + iy_1 = (x_2 x - y_2 y) + i(y_2 x + x_2 y)$$

or, on equating real and imaginary parts,

$$x_2x - y_2y = x_1$$
$$y_2x + x_2y = y_1$$

or

$$\begin{bmatrix} x_2 & -y_2 \\ y_2 & x_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \tag{4}$$

Since $z_2 = x_2 + iy_2 \neq 0$, it follows that x_2 and y_2 are not both zero, so

$$\begin{vmatrix} x_2 & -y_2 \\ y_2 & x_2 \end{vmatrix} = x_2^2 + y_2^2 \neq 0$$

Thus, by Cramer's rule (Theorem 2.1.4), system 4 has the unique solution

$$x = \frac{\begin{vmatrix} x_1 & -y_2 \\ y_1 & x_2 \end{vmatrix}}{\begin{vmatrix} x_2 & -y_2 \\ y_2 & x_2 \end{vmatrix}} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} = \frac{x_1x_2 + y_1y_2}{|z_2|^2}$$

$$y = \frac{\begin{vmatrix} x_2 & x_1 \\ y_2 & y_1 \end{vmatrix}}{\begin{vmatrix} x_2 & -y_2 \\ y_2 & x_2 \end{vmatrix}} = \frac{y_1x_2 - x_1y_2}{x_2^2 + y_2^2} = \frac{y_1x_2 - x_1y_2}{|z_2|^2}$$

Therefore,

$$\begin{aligned} z = x + iy &= \frac{1}{|z_2|^2}[(x_1x_2 + y_1y_2) + i(y_1x_2 - x_1y_2)] \\ &= \frac{1}{|z_2|^2}(x_1 + iy_1)(x_2 - iy_2) = \frac{1}{|z_2|^2} z_1\bar{z}_2 \end{aligned}$$

Thus, for $z_2 \neq 0$, we define

$$\frac{z_1}{z_2} = \frac{1}{|z_2|^2} z_1\bar{z}_2 \tag{5}$$

Remark To remember this formula, multiply the numerator and denominator of z_1 / z_2 by $\bar{z}_2$:

$$\frac{z_1}{z_2} = \frac{z_1\bar{z}_2}{z_2\bar{z}_2} = \frac{z_1\bar{z}_2}{|z_2|^2} = \frac{1}{|z_2|^2} z_1\bar{z}_2$$

EXAMPLE 3 Quotient in the Form $a+bi$

Express

$$\frac{3+4i}{1-2i}$$

in the form $a+bi$.

Solution

From 5 with $z_1=3+4i$ and $z_2=1-2i$,

$$\begin{aligned}\frac{3+4i}{1-2i} &= \frac{1}{|1-2i|^2}(3+4i)(\overline{1-2i}) = \frac{1}{5}(3+4i)(1+2i)\\ &= \frac{1}{5}(-5+10i) = -1+2i\end{aligned}$$

Alternative Solution

As in the remark above, multiply numerator and denominator by the conjugate of the denominator:

$$\frac{3+4i}{1-2i} = \frac{3+4i}{1-2i}\cdot\frac{1+2i}{1+2i} = \frac{-5+10i}{5} = -1+2i$$

Systems of linear equations with complex coefficients arise in various applications. Without going into detail, we note that all the results about linear systems studied in Chapters 1 and 2 carry over without change to systems with complex coefficients. Note, however, that a few results studied in other chapters *will* change for complex matrices.

EXAMPLE 4 A Linear System with Complex Coefficients

Use Cramer's rule to solve

$$\begin{aligned} ix+2y &= 1-2i\\ 4x-iy &= -1+3i\end{aligned}$$

Solution

$$x = \frac{\begin{vmatrix} 1-2i & 2 \\ -1+3i & -i \end{vmatrix}}{\begin{vmatrix} i & 2 \\ 4 & -i \end{vmatrix}} = \frac{(-i)(1-2i)-2(-1+3i)}{i(-i)-2(4)} = \frac{-7i}{-7} = i$$

$$y = \frac{\begin{vmatrix} i & 1-2i \\ 4 & -1+3i \end{vmatrix}}{\begin{vmatrix} i & 2 \\ 4 & -i \end{vmatrix}} = \frac{(i)(-1+3i)-4(1-2i)}{i(-i)-2(4)} = \frac{-7+7i}{-7} = 1-i$$

Thus the solution is $x = i$, $y = 1-i$.

We conclude this section by listing some properties of the complex conjugate that will be useful in later sections.

THEOREM 10.2.3

Properties of the Conjugate

For any complex numbers z, z_1*, and* z_2*:*

(a) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

(b) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$

(c) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

(d) $\overline{(z_1 / z_2)} = \bar{z}_1 / \bar{z}_2$

(e) $\bar{\bar{z}} = z$

We prove (*a*) and leave the rest as exercises.

Proof (a) Let $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$; then

$$\begin{aligned}\overline{z_1 + z_2} &= \overline{(a_1 + a_2) + (b_1 + b_2)i} \\ &= (a_1 + a_2) - (b_1 + b_2)i \\ &= (a_1 - b_1 i) + (a_2 - b_2 i) \\ &= \bar{z}_1 + \bar{z}_2\end{aligned}$$

Remark It is possible to extend part (*a*) of Theorem 10.2.3 to *n* terms and part (*c*) to *n* factors. More precisely,

$$\overline{z_1 + z_2 + \cdots + z_n} = \bar{z}_1 + \bar{z}_2 + \cdots + \bar{z}_n$$
$$\overline{z_1 z_2 \cdots z_n} = \bar{z}_1 \bar{z}_2 \cdots \bar{z}_n$$

Exercise Set 10.2

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1. In each part, find $\bar{z}$.

(a) $z = 2 + 7i$

(b) $z = -3 - 5i$

(c) $z = 5i$

(d) $z = -i$

(e) $z = -9$

(f) $z = 0$

2. In each part, find $|z|$.

(a) $z = i$

(b) $z = -7i$

(c) $z = -3 - 4i$

(d) $z = 1 + i$

(e) $z = -8$

(f) $z = 0$

3. Verify that $z\bar{z} = |z|^2$ for

(a) $z = 2 - 4i$

(b) $z = -3 + 5i$

(c) $z = \sqrt{2} - \sqrt{2}i$

4. Given that $z_1 = 1 - 5i$ and $z_2 = 3 + 4i$, find

(a) z_1 / z_2

(b) $\bar{z}_1 / z_2$

(c) $z_1 / \bar{z}_2$

(d) $\overline{(z_1 / z_2)}$

(e) $z_1 / |z_2|$

(f) $|z_1 / z_2|$

5. In each part, find $1 / z$.

(a) $z = i$

(b) $z = 1 - 5i$

(c) $z = \frac{-i}{7}$

6. Given that $z_1 = 1 + i$ and $z_2 = 1 - 2i$, find

(a) $z_1 - \left(\frac{z_1}{z_2}\right)$

(b) $\frac{z_1 - 1}{z_2}$

(c) $z_1^2 - \left(\frac{iz_1}{z_2}\right)$

(d) $\frac{z_1}{iz_2}$

In Exercises 7–14 perform the calculations and express the result in the form $a+bi$.

7. $\frac{i}{1+i}$

8. $\frac{2}{(1-i)(3+i)}$

9. $\frac{1}{(3+4i)^2}$

10. $\frac{2+i}{i(-3+4i)}$

11. $\frac{\sqrt{3}+i}{(1-i)(\sqrt{3}-i)}$

12. $\frac{1}{i(3-2i)(1+i)}$

13. $\frac{i}{(1-i)(1-2i)(1+2i)}$

14. $\frac{1-2i}{3+4i} - \frac{2+i}{5i}$

15. In each part, solve for z.

(a) $iz = 2 - i$

(b) $(4-3i)\overline{z} = i$

16. Use Theorem 10.2.3 to prove the following identities:

(a) $\overline{\bar{z}+5i}=z-5i$

(b) $\overline{iz}=-i\bar{z}$

(c) $\dfrac{\overline{i+\bar{z}}}{i-z}=-1$

17. In each part, sketch the set of points in the complex plane that satisfies the equation.

(a) $|z|=2$

(b) $|z-(1+i)|=1$

(c) $|z-i|=|z+i|$

(d) $\mathrm{Im}\,(\bar{z}+i)=3$

18. In each part, sketch the set of points in the complex plane that satisfies the given condition(s).

(a) $|z+i|\leq 1$

(b) $1<|z|<2$

(c) $|2z-4i|<1$

(d) $|z|\leq|z+i|$

19. Given that $z = x + iy$, find

(a) $\operatorname{Re}(\overline{iz})$

(b) $\operatorname{Im}(\overline{iz})$

(c) $\operatorname{Re}(i\bar{z})$

(d) $\operatorname{Im}(i\bar{z})$

20.

(a) Show that if n is a positive integer, then the only possible values for $(1/i)^n$ are 1, −1, i, and $-i$.

(b) Find $(1/i)^{2509}$.

Hint See Exercise 23(b) of Section 10.1.

21. Prove:

(a) $\frac{1}{2}(z + \bar{z}) = \operatorname{Re}(z)$

(b) $\frac{1}{2i}(z - \bar{z}) = \operatorname{Im}(z)$

22. Prove: $z = \bar{z}$ if and only if z is a real number.

23. Given that $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2 \neq 0$, find

(a) $\operatorname{Re}\left(\frac{z_1}{z_2}\right)$

(b) $\operatorname{Im}\left(\frac{z_1}{z_2}\right)$

24. Prove: If $(\bar{z})^2 = z^2$, then z is either real or pure imaginary.

25. Prove that $|z| = |\bar{z}|$.

26. Prove:

(a) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$

(b) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

(c) $\overline{(z_1 / z_2)} = \bar{z}_1 / \bar{z}_2$

(d) $\bar{\bar{z}} = z$

27.

(a) Prove that $\overline{z^2} = (\bar{z})^2$.

(b) (b) Prove that if n is a positive integer, then $\overline{z^n} = (\bar{z})^n$.

(c) Is the result in part (b) true if n is a negative integer? Explain.

In Exercises 28–31 solve the system of linear equations by Cramer's rule.

28. $$\begin{aligned} ix_1 - ix_2 &= -2 \\ 2x_1 + x_2 &= i \end{aligned}$$

29. $$\begin{aligned} x_1 + x_2 &= 2 \\ x_1 - x_2 &= 2i \end{aligned}$$

30. $$\begin{aligned} x_1 + x_2 + x_3 &= 3 \\ x_1 + x_2 - x_3 &= 2 + 2i \\ x_1 - x_2 + x_3 &= -1 \end{aligned}$$

31. $$\begin{aligned} ix_1 + 3x_2 + (1+i)x_3 &= -i \\ x_1 + ix_2 + \quad 3x_3 &= -2i \\ x_1 + x_2 + \quad x_3 &= 0 \end{aligned}$$

In Exercises 32 and 33 solve the system of linear equations by Gauss–Jordan elimination.

32. $$\begin{bmatrix} -1 & -1-i \\ -1+i & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

33. $$\begin{bmatrix} 2 & -1-i \\ -1+i & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

34. Solve the following system of linear equations by Gauss–Jordan elimination.

$$\begin{aligned} x_1 + \quad ix_2 - ix_3 &= 0 \\ -x_1 + (1-i)x_2 + 2ix_3 &= 0 \\ 2x_1 + (-1+2i)x_2 - 3ix_3 &= 0 \end{aligned}$$

35. In each part, use the formula in Theorem 1.4.5 to compute the inverse of the matrix, and check your result by showing that $AA^{-1} = A^{-1}A = I$.

(a) $A = \begin{bmatrix} i & -2 \\ 1 & i \end{bmatrix}$

(b) $A = \begin{bmatrix} 2 & i \\ 1 & 0 \end{bmatrix}$

36. Let $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ be a polynomial for which the coefficients a_0, a_1, a_2, …, a_n are real. Prove that if z is a solution of the equation $p(z) = 0$, then so is $\bar{z}$.

37. Prove: For any complex number z, $|\operatorname{Re}(z)| \leq |z|$ and $|\operatorname{Im}(z)| \leq |z|$.

38. Prove that

$$\frac{|\operatorname{Re}(z)| + |\operatorname{Im}(z)|}{\sqrt{2}} \leq |z|$$

Hint Let $z = x + iy$ and use the fact that $(|x| - |y|)^2 \geq 0$.

39. In each part, use the method of Example 4 in Section 1.5 to find A^{-1}, and check your result by showing that $AA^{-1} = A^{-1}A = I$.

(a) $A = \begin{bmatrix} 1 & 1+i & 0 \\ 0 & 1 & i \\ -i & 1-2i & 2 \end{bmatrix}$

(b) $A = \begin{bmatrix} i & 0 & -i \\ 0 & 1 & -1-4i \\ 2-i & i & 3 \end{bmatrix}$

40. Show that $|z - 1| = |\bar{z} - 1|$. Discuss the geometric interpretation of the result.

41.

(a) If $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$, find $|z_1 - z_2|$ and interpret the result geometrically.

(b) Use part (a) to show that the complex numbers 12, $6 + 2i$, and $8 + 8i$ are vertices of a right triangle.

42. Use Theorem 10.2.3 to show that if the coefficients *a*, *b*, and *c* in a quadratic polynomial are real, then the solutions of the equation $az^2 + bz + c = 0$ are complex conjugates. What can you conclude if *a*, *b*, and *c* are complex?