MATH 1B03/1ZC3 Winter 2019

Lecture 14: Complex numbers I

Instructor: Dr Rushworth February 26th

Introduction to complex numbers

(from Chapter 10.1 and 10.2 of Anton-Rorres 9th edition))

What's the point?

In earlier mathematics courses you have encountered quadratic polynomials that do not possess real roots. For example

$$x^2 + 1 = 0$$

has a root exactly when $x^2 = -1$. But if we restrict to the real numbers, such an x does not exist (as $x^2 \geqq 0$ for x real). We encountered this earlier in this course, and saw that it is possible for matrices with real entries to have complex eigenvalues.

What happens if we produce a new number, by <u>defining</u> it to be a solution to

$$x^2 = -1$$

that is, to be the square root of -1? We denote this new number by i, so that

$$i^2 = -1$$

The solutions to the equation

$$x^2 = -1$$

are therefore $x = \pm i$.

It turns out that defining this new number yields a rich and deep new theory, known as the theory of complex numbers, which appears in many surprising situations in the natural world.

Why should you care?

If you are a mathematician, you should be interested in complex numbers as they represent your first step into a gigantic new realm of number systems distinct to the real numbers. This is the field of abstract algebra, where 0 can be equal to 1, ab is not necessarily ba, and you can add a number to itself and get back to 0, among many other things.

If you are an engineer, a physicist, or studying another science, you will need to understand complex numbers as they are essential tools in the physics of waves, optics, signals processing, electronics, quantum physics, and a great many other physical situations.

Misleading names

The name *complex numbers* does not mean that this topic is complicated or more difficult than any others we have been studying. In this context, the word *complex* means

> “composed of interconnected parts, formed by a combination of simple elements"

(It comes from the Latin *plectere*: to weave, braid, or intertwine.)

The number i is often referred to as an *imaginary number*. This does not mean that complex numbers don’t appear in the real world: the name was chosen a long time ago, and it’s too late to change it!

Definition

Definition 14.1: The imaginary unit

Denote by i the imaginary unit, defined $i = \sqrt{-1}$.

Definition 14.2: Complex number

A complex number z is a number of the form

$$z = a + ib$$

for a, b real numbers.

The collection of all complex numbers is often denoted $\mathbb{C}$ (collection of all real numbers is denoted $\mathbb{R}$).

There are multiple ways to express the same complex number. For example, we can express complex numbers as ordered pairs of real numbers. That is,

$$z = (a, b) = a + ib$$

The word ordered here means that it matters which order a and b appear in, so that

$$a + ib = (a, b) \neq (b, a) = b + ia$$

in general.

Definition 14.3

Let $z = a + ib$ be a complex number. The real part of z is

$$\text{Re}(z) = a.$$

The imaginary part of z is

$$\text{Im}(z) = b.$$

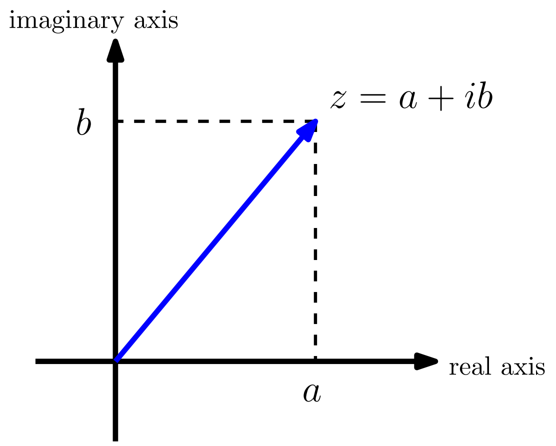
For example, if $z = 17 - 3i$ then $\text{Re}(z) = 17$, and $\text{Im}(z) = -3$.

If $\text{Im}(z) = 0$, then z is a real number. If $\text{Re}(z) = 0$, then we say that z is an imaginary number. Notice that every real number is a complex number, but not every complex number is a real number.

In this light we see why complex numbers have the name they do: they are made up of a real part, and an imaginary part.

Complex numbers as vectors

We can represent complex numbers as vectors (this is often the best way to understand them). Let $z = a + ib$, then:



This set of axes, one real and one imaginary, produces the Argand plane a.k.a an Argand diagram.

Treating complex numbers as special types of vectors often lets us work faster and lets us introduce new concepts.

Operations on complex numbers

The theory of complex numbers would be very boring if we couldn't combine them, or have any other operations.

Definition 14.4

Let $z = a + ib$ and $w = c + id$ be complex numbers. Then $z = w$ if and only if

$$\begin{aligned} \mathrm{Re}(z) &= \mathrm{Re}(w) \\ \mathrm{Im}(z) &= \mathrm{Im}(w). \end{aligned}$$

Definition 14.5: Operations on complex numbers

Let $z = a + ib$ and $w = c + id$ be complex numbers.

- Addition:
$$z + w = (a + c) + i(b + d)$$
- Subtraction:
$$z - w = (a - c) + i(b - d)$$
- Multiplication by real numbers: let k be a real number. Then
$$kz = ka + ikb$$

Example 14.6

Let $z = -2 + i$, $w = 2 + 3i$. Then

$$\begin{aligned} z + w &= -2 + 2 + i(1 + 3) \\ &= 4i \\ z - w &= -4 - 2i \end{aligned}$$

If $z = 12 - 4i$, then $\frac{1}{4}z = 3 - i$.

It is very important to note that it is not possible to order the complex numbers, as we can order real numbers. That is, the symbols $<$, $>$, $\geq$, $\leq$ do not make sense.

While complex numbers can be treated as vectors in the plane there are operations which are specific to complex numbers. The first such operation is multiplication. It is not possible to multiply two 2×1 matrices together, but we can define a special kind of multiplication for complex numbers.

Definition 14.7: Multiplication of complex numbers

Let $z = a + ib$ and $w = c + id$ be complex numbers. Define the product

$$\begin{aligned} zw &= (a + ib)(c + id) \\ &= ac + iad + ibc + i^2bd \\ &= ac - bd + i(ad + bc) \end{aligned}$$

(as $i^2 = -1$).

Notice that $zw = wz$, so that multiplication of complex numbers is commutative.

Example 14.8

Let $z = -2 + i$, $w = 2 + 3i$. Then

$$\begin{aligned} zw &= (-2 + i)(2 + 3i) \\ &= -4 - 6i + 2i - 6i^2 \\ &= 6 - 4 + i(2 - 6) \\ &= 2 - 4i \end{aligned}$$

Let $z = 4 - 4i$ and $w = 4 + 4i$. Then

$$\begin{aligned} zw &= (4 - 4i)(4 + 4i) \\ &= 16 - 16i^2 \\ &= 32 \end{aligned}$$

Here we see that it is possible to multiply two complex numbers together to obtain a real number.

We can define powers of complex numbers exactly as they are defined for real numbers.

Definition 14.9: Powers of a complex number

Let z be a complex number, and k a positive integer. The k-th power of z is denote z^k and is defined

$$z^k = \underbrace{zz\cdots z}_{k \text{ times}}.$$

For example, if $z = 2 - i$, then

$$\begin{aligned} z^3 &= (2-i)(2-i)(2-i) \\ &= (2-i)(4-4i+i^2) \\ &= (2-i)(3-4i) \\ &= 6-8i-3i+4i^2 \\ &= 2-11i \end{aligned}$$

We saw in Example 14.8 that if $z = 4 - 4i$ and $w = 4 + 4i$ then $zw = 32$. This was a special case of the following concept.

Definition 14.10: Complex conjugate

Let $z = a + ib$ be a complex number. The complex conjugate of z is denote $\overline{z}$, and is defined

$$\overline{z} = a - ib$$

We say "z-bar" for $\overline{z}$. The complex conjugate is often simply called the conjugate.

For example, if $z = 18 - 7i$ then $\overline{z} = 18 + 7i$.

Fact 14.11

Let z be a complex number and $\overline{z}$ its conjugate. Then

$$z\overline{z} = \overline{z}z$$

is a real number.

Proof: Let $z = a + ib$, so that $\overline{z} = a - ib$ and

$$\begin{aligned} z\overline{z} &= (a + ib)(a - ib) \\ &= a^2 + iab - iab - b^2 i^2 \\ &= a^2 + b^2 \end{aligned}$$

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It is not possible to directly compare the size of complex numbers. However, we can indirectly compare their sizes in the following way.

Definition 14.12: Modulus

Let $z = a + ib$ be a complex number. The modulus of z is denoted $|z|$ and is defined

$$|z| = \sqrt{a^2 + b^2}$$

The modulus is also known as the absolute value.

Fact 14.13

Let z be a complex number. Then

$$|z|^2 = z\overline{z}$$

Proof: Let $z = a + ib$. Then

$$\begin{aligned} z\overline{z} &= (a + ib)(a - ib) \\ &= a^2 + b^2 \\ &= (\sqrt{a^2 + b^2})^2 \\ &= |z|^2 \end{aligned}$$

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Notice that if z is a real number, then $|z| = z$.

Using the modulus we can define division of complex numbers.

Definition 14.14: Reciprocal of a complex number

Let $z = a + ib$ be a non-zero complex number. We define

$$z^{-1} = \frac{1}{z} = \frac{1}{|z|^2}\overline{z}$$

Why is this definition the correct one? If a is a real number, then

$$a\frac{1}{a} = 1.$$

We want to replicate this for complex numbers:

$$\begin{aligned} z\frac{1}{z} &= z\frac{1}{|z|^2}\overline{z} \\ &= \frac{1}{|z|^2}z\overline{z} \\ &= \frac{1}{|z|^2}|z|^2, \quad \text{by Fact 14.13} \\ &= 1 \end{aligned}$$

as desired.

Using this, we can manipulate complex fractions. If z and w are complex numbers, then

$$\frac{z}{w} = z\left(\frac{1}{w}\right) = \frac{1}{|w|^2}z\overline{w}.$$

Example 14.15

Question: Express the complex number

$$\frac{7-3i}{-2+5i}$$

in the form $a + ib$.
Answer: Let $z = 7 - 3i$ and $w = -2 + 5i$.

Then

$$|w|^2 = (-2)^2 + 5^2 = 29$$

and

$$\overline{w} = -2 - 5i$$

Then

$$\begin{aligned}
\frac{7-3i}{-2+5i} &= \frac{z}{w} \\
&= \frac{1}{|w|^2} z\overline{w} \\
&= \frac{1}{29}(7-3i)(-2-5i) \\
&= \frac{1}{29}(-14-35i+6i+15i^2) \\
&= \frac{1}{29}(-29-29i) \\
&= -1-i
\end{aligned}$$

We conclude by giving some other important properties of the complex conjugate.

Fact 14.16: Properties of the complex conjugate

Let z and w be complex numbers. Then

- $\overline{z+w} = \overline{z} + \overline{w}$
- $\overline{z-w} = \overline{z} - \overline{w}$
- $\overline{zw} = (\overline{z})(\overline{w})$
- $\overline{\frac{z}{w}} = \frac{\overline{z}}{\overline{w}}$
- $\overline{\overline{z}} = z$

Suggested Problems

Practice the material covered in this lecture by attempting the following questions from Chapter 10.1 of Anton-Rorres 9th Edition (available on the coursepage)

- Questions $5, 11, 17, 19, 21, 22$

and the questions from Chapter 10.2 of Anton-Rorres 9th Edition

- Questions $9, 11, 15, 16, 19, 21$