MATHEMATICS 1LS3 TEST 2

Day Class
Duration of Examination: 60 minutes
McMaster University, 31 October 2016

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First name (PLEASE PRINT): SOLUTIONS

Family name (PLEASE PRINT): ____________

Student No.: ____________

THIS TEST HAS 8 PAGES AND 6 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE.

Total number of points is 40. Marks are indicated next to the problem number. Any non-graphing calculator is allowed.

EXCEPT ON QUESTIONS 1 AND 2, YOU MUST SHOW CORRECT WORK TO EARN CREDIT.

USE PEN TO WRITE YOUR TEST. IF YOU USE A PENCIL YOUR TEST WILL NOT BE ACCEPTED FOR REMARKING (IF NEEDED).

You must show work to receive full credit.

Problem	Points	Mark
1	4	
2	6	
3	7	
4	6	
5	6	
6	11	
TOTAL	40	

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1. **Multiple choice questions: circle ONE answer. No justification is needed.**

(a)[2] If $f(x) = \ln(ax)\ln(bx)$ then $f'(1)$ is equal to

(A) $\ln a \ln b$ (B) $\ln(a+b)$ (C) $\ln(ab)$ (D) $\frac{\ln(a+b)}{a+b}$

(E) $\frac{\ln(ab)}{a+b}$ (F) $\frac{\ln a \ln b}{a+b}$ (G) $\frac{1}{ab}$ (H) $\frac{1}{a}+\frac{1}{b}$

$$f' = \frac{1}{\cancel{a}x}\cancel{a}\ln(bx) + \ln(ax)\cdot\frac{1}{\cancel{b}x}\cdot\cancel{b}$$

$$= \frac{\ln(bx)+\ln(ax)}{x}$$

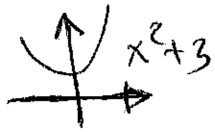
$$\to f'(1) = \ln b + \ln a = \underline{\underline{\ln ab}}$$

(b)[2] Which of the following functions has/have **no critical points**?

(I) $f(x) = 2.3x + 5$ ✓

(II) $f(x) = x^2 + 3$

(III) $f(x) = e^{0.04x}$ ✓



(I) and (III) are increasing functions

(A) none (B) I only (C) II only (D) III only

(E) I and II (F) I and III (G) II and III (H) all three

increasing function : $f' > 0 \to$ no cps

(I) $f' = 2.3 > 0$

(III) $f' = e^{0.04x} \cdot 0.04 > 0$

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2. **Identify each statement as true or false (circle your choice). You do not need to justify your answer.**

(a)[2] The function $y = -1$ is the linear approximation of $f(x) = \sec x$ at $x = \pi$.

TRUE FALSE

$$L(x) = \underbrace{f(\pi)}_{-1} + \underbrace{f'(\pi)}_{0}(x-\pi) = -1$$

$$f(\pi) = \sec\pi = -1$$

$$f'(x) = \sec x \tan x \longrightarrow f'(\pi) = 0$$

(b)[2] From $f''(x) = e^{-x-2}(3-x)$ we conclude that the graph of $f(x)$ is concave down on the interval $(0, 3)$.

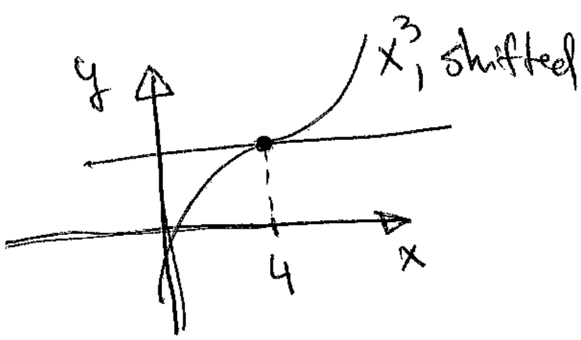
TRUE FALSE

$$f''(x) = \underbrace{e^{-x-2}}_{\oplus}\ \underbrace{(3-x)}_{\oplus} \text{ for } x \text{ in } (0,3)$$

$$\longrightarrow f''(x) > 0$$

(c)[2] The function $f(x)$ has a horizontal tangent at $x = 4$. Therefore, it must have a local maximum or a local minimum at $x = 4$.

TRUE FALSE



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Questions 3-6: You must show CORRECT work to receive full credit.

3. (a)[3] Using L'Hôpital's rule, calculate $\lim_{x\to 0^+} x^4 \ln x$. $= 0 \cdot (-\infty)$

$$= \lim_{x\to 0^+} \frac{\ln x}{x^{-4}} \overset{LH}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-4x^{-5}} =$$

$$= \lim_{x\to 0^+} \left(-\frac{1}{4}\right) \cdot x^4 = \underline{\underline{0}}$$

(b)[4] Find $y'(x)$ if $x^3 \ln y = x - e^y + e$. Compute y' when $x = 0$ and $y = 1$.

$$3x^2 \cdot \ln y + x^3 \cdot \frac{1}{y} \cdot y' = 1 - e^y y' + 0$$

$$y'\left(\frac{x^3}{y} + e^y\right) = 1 - 3x^2 \ln y$$

$$y' = \frac{1 - 3x^2 \ln y}{\frac{x^3}{y} + e^y}$$

$$x=0, y=1 \to y' = \frac{1-0}{0+e} = \underline{\underline{\frac{1}{e}}}$$

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4. (a)[2] In the article *Migration behaviour of grizzly bears in Northern British Columbia: contribution to a modelling approach.* G. Brown et al, Bear Science 4 (June 2012), we find the formula

$$P(t) = \arctan(1.7t) + 4.7$$

where t represents time.

Next, we read "initially, $P(t) \approx 1.7t + 4.7$, which gives a linear relationship." Explain why this statement is correct. [Hint: Think in terms of the linear approximation at $t = 0$.]

$$L(t) = P(0) + P'(0) \cdot t$$

$$P(0) = \arctan(0) + 4.7 = 4.7$$

$$P'(t) = \frac{1}{1+(1.7t)^2} \cdot 1.7 + 0 \rightarrow P'(0) = 1.7$$

thus $\underline{L(t) = 4.7 + 1.7t \approx P(t)}$ near $\underline{t=0}$

(b)[4] A simple model of diffusion states that the concentration of a substance diffusing in air is given by

$$c(x) = e^{-x^2+0.2}$$

where x is the distance from the source. This formula is sometimes simplified using a quadratic approximation near $x = 0$. Find that approximation.

$$T_2(x) = c(0) + c'(0)x + \frac{c''(0)}{2}x^2$$

$$c(0) = e^{0.2}$$

$$c'(x) = e^{-x^2+0.2} \cdot (-2x) \rightarrow c'(0) = 0$$

$$c''(x) = e^{-x^2+0.2}(-2x)(-2x) + e^{-x^2+0.2}(-2)$$

$$\rightarrow c''(0) = e^{0.2}(-2)$$

thus $T_2(x) = e^{0.2} - e^{0.2}x^2$

or: $= 1.22 - 1.22x^2$

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5. The resistance R of the flow of blood through a blood vessel (assumed to have the shape of a cylindrical tube) is given by

$$R = \frac{K^{0.96} L(\gamma + 1)^2}{d^4}$$

where L is the length of the tube, d is its diameter and $\gamma \geq 0$ is the curvature. The positive constant K represents the viscosity of the blood (viscosity is a measure of the resistance of fluid to stress; water has low viscosity, honey has high viscosity).

(a)[3] Find the derivative of R with respect to K and interpret your answer, i.e., explain what your answer implies for the dependence of R on the viscosity of the blood.

$$\frac{dR}{dK} = 0.96 \cdot K^{-0.04} \cdot \frac{L(\gamma+1)^2}{d^4}$$

$$= \underbrace{\frac{0.96}{K^{0.04}}}_{\oplus} \cdot \underbrace{\frac{L(\gamma+1)^2}{d^4}}_{\oplus} > 0$$

so R is increasing

i.e., as viscosity increases, so does the resistance

(b)[3] Find the derivative of R with respect to d and interpret your answer, i.e., explain what your answer implies for the dependence of R on the diameter of a blood vessel.

$$R = K^{0.96} L(\gamma+1)^2 d^{-4}$$

$$\rightarrow \frac{dR}{dd} = \underbrace{K^{0.96} \cdot L(\gamma+1)^2}_{\oplus} \underbrace{(-4)}_{\ominus} \underbrace{d^{-5}}_{\oplus} < 0$$

i.e., as the vessel becomes wider (larger diameter), the resistance decreases

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6. (a)[3] The function $f(x) = x^2e^{4x}$ has two critical points. Find them.

$$f'(x) = 2xe^{4x} + x^2 \cdot e^{4x} \cdot 4$$
$$= 2xe^{4x}(1+2x)$$

$f'=0 \Rightarrow x=0$

$\quad \searrow 1+2x=0 \rightarrow x=-\frac{1}{2}$

f' dne $\rightarrow$ no need to check as we are told that there are two critical points

(b)[2] Give a statement of the Extreme Value Theorem. Make sure to clearly identify assumptions and conclusions.

IF $f(x)$ continuous, $[a,b]$ closed interval } assumptions

THEN $f(x)$ has an absolute max. and absolute minimum in $[a,b]$ } conclusion

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(c)[3] Find the absolute maximum and the absolute minimum of the function $f(x) = x^2e^{4x}$ on the interval $[-1, 1]$. In each case, state what the value is, and where it occurs.

x	$f(x)=x^2e^{4x}$	
0	0	$\longrightarrow$ abs. min. $=0$, at $x=0$
$-\frac{1}{2}$	$\frac{1}{4}e^{-2} \approx 0.0339$	
-1	$e^{-4} \approx 0.0183$	
1	$e^4 \approx 54.598$	$\longrightarrow$ abs max. $= e^4$, at $x=1$

(d)[3] You have to find the absolute maximum and the absolute minimum of the function $f(x) = x^2e^{4x}$, this time on the interval $[1, 10]$. Without repeating the routine as in part (c), find the absolute maximum and the absolute minimum of $f(x)$, and explain why your answer makes sense.

$f(x)$ is an increasing function
so abs. min. at left end: $f(1) = e^4$
and abs. max. at right end: $f(10) = 100e^{40}$

$\rightarrow$ as product of two increasing functions
or: from (a) $f'(x) = \underbrace{2x}_{\oplus}\,\underbrace{e^{4x}}_{\oplus}\,\underbrace{(1+2x)}_{\oplus} > 0$

THE END