For the interested reader:

January 11, 2022 11:27 AM

Derivation of the result $\int_1^\infty \frac{1}{x^p}\,dx = \begin{cases} \frac{1}{p-1} & p>1 \quad \text{convergent} \\ \infty & p\le 1 \quad \text{divergent} \end{cases}$

We have seen that $\int_1^\infty \frac{1}{x}\,dx$ diverges $\Rightarrow$ for $p=1$ divergent

Step 1:

For $p \neq 1$,
$$\int_1^t \frac{1}{x^p}\,dx = \int_1^t x^{-p}\,dx \overset{\text{"power rule"}}{=} \frac{1}{-p+1} x^{-p+1} \Big|_1^t = \frac{t^{-p+1}}{-p+1} - \frac{\overbrace{1^{-p+1}}^{=1}}{-p+1}$$

Step 2:

$p \neq 1 \Rightarrow$
$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^p}\,dx \overset{\text{Step 1}}{=} \lim_{t\to\infty}\left[\frac{t^{-p+1}}{-p+1} - \frac{1}{-p+1}\right] = \frac{1}{-p+1}\Big(\lim_{t\to\infty} t^{-p+1}\Big) - \frac{1}{-p+1}$$

exists for all $t \geq 1$

∞ if $p<1$ $\qquad$ 0 if $p>1$

$$= \begin{cases} 0 - \frac{1}{-p+1} & p>1 \\ \infty - \frac{1}{-p+1} & p<1 \end{cases}$$

$$= \begin{cases} \frac{1}{p-1} & p>1 \\ \infty & p<1 \end{cases}$$

recall for $p=1$
$\rightarrow$ divergent

$$\Rightarrow \begin{cases} \frac{1}{p-1} & p>1 \quad \text{conv.} \\ \infty & p\le 1 \quad \text{div.} \end{cases}$$