MATH 1B03/1ZC3 Winter 2019

# Lecture 4: More matrix operations

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## Covered in last lecture:

- Matrices: rows, columns, entries
- Matrix addition
- Scalar multiplication
- Matrix multiplication
- Writing a SLE as a matrix equation

## Transpose and trace

(from Chapter 1.3 of Anton-Rorres)

As we have seen, matrices are more interesting than numbers. On top of the familiar operations of addition and multiplication, we can also define new operations.

### **Transpose** (defined for any matrix)

**Definition 4.1: Transpose**

Let $A = \left[a_{ij}\right]$ be a matrix. The transpose of $A$ is a new matrix, denoted $A^T$, which has entries

$$(A^T)_{ij} = (A)_{ji}$$

Notice the change of order of the row and column indices.

That is, the $ij$-th entry of $A$ is the $ji$-th entry of $A^T$: the rows of $A$ are the columns of $A^T$, and vice versa.

Notice that if $A$ is $n \times m$, then $A^T$ is $m \times n$.

**Example 4.2**

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 0 \\ 4 & 5 \end{bmatrix}, \qquad A^T = \begin{bmatrix} 3 & 1 & 4 \\ 2 & 0 & 5 \end{bmatrix}$$

$$B = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}, \quad B^T = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$$

**Fact 4.3: Properties of the transpose**

Let $A$ and $B$ be matrices, and $\lambda$ a scalar. Then

1. $(A^T)^T = A$
2. $(\lambda A)^T = \lambda A^T$
3. $(AB)^T = B^T A^T$, when $AB$ and $A^T B^T$ are defined. Notice the order has swapped.
4. $(A+B)^T = A^T + B^T$, when $A+B$ is defined.

As before, keep these properties in your mind when doing calculations, and try to use them to speed up your working.

## **Trace** (defined only for square matrices)

Recall that a square matrix has the same number of rows as columns, so that it is of size $n \times n$.

### Definition 4.4: Diagonal

Let $A = \left[a_{ij}\right]$ be a square matrix. Then entries $a_{ii}$ are the diagonal entries. The set of all the $a_{ii}$ is the main diagonal of $A$.
For example, the boxed entries are on the main diagonal of this $3 \times 3$ matrix

$$\begin{bmatrix} \boxed{a_{11}} & a_{12} & a_{13} \\ a_{21} & \boxed{a_{22}} & a_{23} \\ a_{31} & a_{32} & \boxed{a_{33}} \end{bmatrix}$$

### Definition 4.5: Trace

$A = \left[a_{ij}\right]$ be an $n \times n$ square matrix. The trace of $A$ is a number, written $tr(A)$, and defined

$$\begin{aligned} tr(A) &= \sum_{i=1}^{n} a_{ii} \\ &= a_{11} + a_{22} + \cdots + a_{nn}. \end{aligned}$$

The trace of $A$ is the sum of the diagonal elements of $A$.

### Example 4.6

**Question:** Compute the trace of the matrix

$$A = \begin{bmatrix} 3 & -7 & 0 & 1 \\ 2 & 0 & 9 & -3 \\ 0 & 0 & 1 & 4 \\ 4 & 4 & 0 & -3 \end{bmatrix}$$

**Answer**: The diagonal elements are

$$\begin{bmatrix} \boxed{3} & -7 & 0 & 1 \\ 2 & \boxed{0} & 9 & -3 \\ 0 & 0 & \boxed{1} & 4 \\ 4 & 4 & 0 & \boxed{-3} \end{bmatrix}$$

Then $tr(A) = 3 + 0 + 1 + -3 = 1$.

**Fact 4.7: Properties of the trace**

Let $A$ and $B$ be a square matrices of the same size, and $\lambda$ a scalar. Then

1. $tr(\lambda A + B) = \lambda tr(A) + tr(B)$
2. $tr(AB) = tr(BA)$, even if $AB \neq BA$
3. $tr(A^T) = tr(A)$

As always, keep these properties in your mind when doing calculations, and try to use them to speed up your working.

## Important properties of matrix multiplication

While there are similarities between matrix mulitiplcation and the multiplication of numbers, there are important differences to be aware of.

- Even if $AB = AC$, we cannot deduce that $B = C$. This is often known as "failure of cancellation". Here is an example: consider the matrices

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix} \quad C = \begin{bmatrix} 3 & 5 \\ 3 & 4 \end{bmatrix}$$

  It is true that

$$AB = AC = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$$

even though $B \neq C$.

- It is possible for non-zero matrices to have the zero matrix as their product. For example, if

$$M_1 = \begin{bmatrix} 4 & -2 \\ 0 & 0 \end{bmatrix} \quad M_2 = \begin{bmatrix} 0 & 8 \\ 0 & 1 \end{bmatrix}$$

then

$$M_1 M_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

**Question 4.8**

Use matrix multiplication to check the products $AB$, $AC$, and $M_1 M_2$ above.

Despite these differences, matrix multiplication does have some features which are familiar.

**Fact 4.9: Properties of matrix mulitiplication**

Let $A$ be an $m \times n$ matrix, $B$ and $C$ be $n \times k$ matrices, and $D$ a $k \times l$ matrix. Then

1. $A(B + C) = AB + AC$
2. $(B + C)D = BD + CD$ (notice the order)
3. $A(\lambda B) = (\lambda A)B = \lambda AB$, for $\lambda$ a scalar
4. $A0 = 0A = 0$, for $0$ the appropriate size $0$ matrix

As always, keep these properties in your mind when doing calculations, and try to use them to speed up your working.,

**Example 4.10**

**Question:** Let $M_1$, $M_2$ and $M_3$ be matrices such that

$$tr(M_1M_2) = 4, \quad tr(M_1M_3^T) = -12$$

Find

$$tr\left(M_1(M_2^T + M_3)^T\right)$$

(you may assume that $M_1M_2$, $M_1M_3^T$ and $M_1(M_2^T + M_3)^T$ are defined).
**Answer:** Use the properties of the transpose, the trace, and matrix multiplication (from Facts 4.3, 4.7 and 4.9).

$$\begin{aligned}
tr\left(M_1\left(M_2^T + M_3\right)^T\right) &= tr\left(M_1\left((M_2^T)^T + M_3^T\right)\right) \\
&\quad \text{as } (A+B)^T = A^T + B^T \\
&= tr\left(M_1(M_2 + M_3^T)\right) \\
&\quad \text{as } (A^T)^T = A \\
&= tr\left(M_1M_2 + M_1M_3^T\right) \\
&\quad \text{as } A(B+C) = AB + AC \\
&= tr\left(M_1M_2\right) + tr\left(M_1M_3^T\right) \\
&\quad \text{as } tr(A+B) = tr(A) + tr(B) \\
&= 4 - 12 \\
&= -8
\end{aligned}$$

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**Question:** Let $A$ be a $4 \times 7$ matrix, and $B$ an $m \times n$ matrix. Given that $B^TA^T$ is a $3 \times 4$ matrix, find $m$ and $n$.
**Answer:** Recall from Fact 4.3 that $(AB)^T = B^TA^T$. This shows that as $B^TA^T$ is defined, then $AB$ is defined also.
We can determine $m$ by looking at the size of $AB$:

$$\underset{A}{(4\times 7)} \quad \underset{B}{(m \times n)} \quad = \quad \underset{AB}{(4 \times n)}$$

as $AB$ is defined we must have $m = 7$.

To find $n$, recall that if $M$ is $k \times l$, then $M^T$ is $l \times k$.
As $AB$ is $4 \times n$, and $B^T A^T$ is $3 \times 4$, we must have that $n = 3$ (using the fact that $(AB)^T = B^T A^T$ again).

## The inverse of a matrix

(from Chapter 1.4 of Anton-Rorres)

We saw that given a matrix $A$, there is always a matrix $-A$ such that

$$A - A = 0, \qquad \text{(the zero matrix)}$$

This corresponds to the fact that given a number $x$, there exists a number $-x$ such that

$$x - x = 0, \qquad \text{(the number zero)}$$

In other words, given any matrix, we can always add another matrix to it to get back to the zero matrix.

If the number $x$ is non-zero, we can also get back to $1$:

$$x \cdot \frac{1}{x} = 1$$

Can we do this for matrices? In general, the answer is no.

We have the matrix equivalent of the number $0$: the zero matrix. We need the matrix equivalent of the number $1$.

**Definition 4.11: The identity matrix**

The identity matrix, denoted $I$, is the square matrix with $1$'s on the main diagonal, and $0$'s everywhere else. If we need to specify the size, we use $I_n$ to denote the $n \times n$ identity matrix.

For example, here are $I_2$ and $I_5$:

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

We can now answer the question: given a matrix $A$ does there exist another matrix $B$ such that $AB = I$?

If $A$ is an $m \times n$ matrix with $m \neq n$ (i.e. not square), then there does not exist such a matrix $B$. We will see why this is later on.

If $A$ is a square matrix, it is possible that such a matrix $B$ exists, but this is not always the case.

The matrix $B$ has a special name.

**Definition 4.12: Inverse of a matrix**

Let $A$ be an $n \times n$ matrix. We say that $A$ is invertible if there exists another $n \times n$ matrix $B$ such that

$$AB = BA = I_n.$$

If such a $B$ exists, it is unique, and we refer to it as the inverse of $A$, and denote it $A^{-1}$.

The word 'unique' in the definition above means that if there exist two matrices $B$ and $C$ such that

$$AB = BA = I_n$$
$$AC = CA = I_n$$

then $B = C$ (so there was really only one matrix all along!).

## Suggested problems

Practice the material in this lecture by attempting the following problems.

From **Chapter 1.3** of Anton-Rorres, starting on page $36$

- Questions $1$, $5$, $11$, $15$, $29$, $35$
- True/False questions $(h)$, $(i)$, $(j)$

From **Chapter 1.4** of Anton-Rorres, starting on page $49$

- Questions $3$, $11$, $17$, $33$, $46$
- True/False questions $(b)$, $(c)$, $(d)$, $(e)$