MATH 1B03/1ZC3 Winter 2019

# Lecture 11: Diagonalization

Instructor: Dr Rushworth February 5th

## Motivating diagonalization

(from Chapter 5.2 of Anton-Rorres)

*This discussion is not fully detailed, as we don't have the required definitions yet. We will come back to these concepts later on in the course and cover them in more detail. You do not need to understand the following discussion in order to understand diagonalization, and can skip to the section titled 'How to diagonalize a matrix' if you wish.*

Matrices are not just lists of numbers: they describe geometric transformations. That is, they are abstract geometric objects. In order to write a matrix down, we need to pick a coordinate system. For example, the collection

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

is a coordinate system for $3$-dimensional space, because any point can be reached by taking a linear combination. For example

$$\begin{bmatrix} 7 \\ 4 \\ -1 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

If we pick another coordinate system, the space we are describing will not change. However, how vectors are written down will change, in general. For example, the collection

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

is another coordinate system for $3$-dimensional space (we will prove this later on). Now we have

$$\begin{bmatrix} 7 \\ 4 \\ -1 \end{bmatrix} = \frac{46}{9}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{7}{9}\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} + \frac{5}{9}\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

We have expressed the same vector in two different ways, by picking two different coordinate systems.

We can do the same for matrices. Schematically:

$$\begin{tikzcd} & \text{A matrix } M \arrow[dl, "\text{pick coord. system}"'] \arrow[dr, "\text{pick another coord. system}"] & \\ M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} & & M = \begin{bmatrix} x & y \\ v & w \end{bmatrix} \end{tikzcd}$$

A matrix $M$ may have different representatives in different coordinate systems. However, we consider them to be the same matrix as they describe the same transformation!

We have seen that diagonal matrices are very easy to work with: it is easy to find their inverse, their determinant, and to take powers of them. As such, an obvious question to ask is

Does there exist a coordinate system in which $M$ is diagonal?

If there does exist such a coordinate system, then we can use it to speed up our calculations when working with $M$.

## How to diagonalize a matrix

*You do not need to understand the previous section in order to understand the process of diagonalizing a matrix. However, you **do** need to understand eigenvalues and eigenvectors.*

Given a matrix $A$, the question 'can $A$ be diagonalized?' means: does there exist an invertible matrix $P$ such that

$$D = P^{-1}AP$$

is a diagonal matrix? If such a $P$ exists, we say that $A$ is diagonalizable, and that $P$ diagonalizes $A$.

Not every matrix is diagonalizable, as we shall we later.

If $A$ is diagonalizable, then calculating its determinant is easy:

$$\begin{aligned} det(P^{-1}AP) &= det(P^{-1})det(A)det(P) \\ &= \frac{1}{det(P)}det(P)det(A) \\ &= det(A) \end{aligned}$$

therefore the matrix $D = P^{-1}AP$ has the same determinant as $A$. But $D$ is diagonal, so it's determinant is the product of its diagonal entries!

This is an example of a property which interacts nicely with diagonalization, allowing us to speed up calculation. There are a number of other properties like this:

| Property | Details |
|---|---|
| Determinant | $A$ and $P^{-1}AP$ have the same determinant |
| Invertibility | $A$ is invertible if and only if $P^{-1}AP$ is invertible |
| Trace | $A$ and $P^{-1}AP$ have the same trace |
| Characteristic polynomial | $A$ and $P^{-1}AP$ have the same char. polynomial |
| Eigenvalues | $A$ and $P^{-1}AP$ have the same eigenvalues |

The relationship between $A$ and $P^{-1}AP$ is so important it gets a special name.

**Definition 11.1: Matrix similarity**

Let $A$ and $B$ be square matrices of the same size. We say that $A$ and $B$ are similar if there exists an invertible matrix $P$ such that

$$B = P^{-1}AP$$

We can rephrase the definition of diagonalizable using this term.

**Definition 11.2**

Let $A$ be a square matrix. We say that $A$ is diagonalizable if it is similar to a diagonal matrix.

As we have seen, if a matrix is diagonalizable we can determine a number of important quantities very quickly. However, diagonalizing matrices - that is, finding the matrix $P$ - is a difficult task. Later on in the course we shall see how to determine exactly when a matrix is diagonalizable. For now, we will use the following procedure.

**Recipe 11.3: Diagonalization**

Use this process to check if an $n \times n$ matrix $A$ is diagonalizable, and diagonalize it if it is.

**Step 1:** Compute the eigenvalues of $A$.
If $A$ has no repeated eigenvalues, then it is diagonalizable.
If $A$ has a repeated eigenvalue, then it may not be diagonalizable. We need to make a further check to determine if it is.

**Step 2:** Pick an eigenvalue and find a basis for its eigenspace. Repeat until you have found a basis for every eigenspace.
If the total number of vectors across all of the eigenspaces is less than $n$, then $A$ is not diagonalizable.
If the total number of vectors across all of the eigenspaces is equal to $n$ (it

cannot be greather than $n$), then go to the next step.

**Step 3:** Form a matrix by placing the vectors which make up the bases of the eigenspaces side by side. That is, if $\mathbf{x}_1$ to $\mathbf{x}_n$ are the vectors, form the matrix

$$P = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

Notice that $P$ is $n \times n$.
If $det(P) = 0$ then $A$ is not diagonalizable.
If $det(P) \neq 0$ then $A$ is diagonalizable and

$$D = P^{-1}AP$$

is a diagonal matrix.

## Example 11.4

**Question:** Determine whether or not the matrix

$$A = \begin{bmatrix} 2 & 4 & -1 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

is diagonalizable, and diagonalize it if it is.
**Answer:** As $A$ is upper triangular the eigenvalues can by determined by inspection:

$$\lambda_1 = 2,\ \lambda_2 = 2,\ \lambda_3 = 1$$

As we have a repeated eigenvalue $A$ may not be diagonalizable: we need to compute its eigenvectors to find out.
• $(\lambda = 2)$: We have

$$2I - A = \begin{bmatrix} 0 & -4 & 1 \\ 0 & 0 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} 0 & -4 & 1 \\ 0 & 0 & -3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Which yields the simultaneous equations

$$\begin{aligned} -4x_2 + x_3 &= 0 \\ -3x_2 &= 0 \\ x_3 &= 0 \end{aligned}$$

with solution $x_2 = 0$, $x_3 = 0$, and $x_1$ free. A basis for the eigenspace is

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

• $(\lambda = 1)$: We have

$$I - A = \begin{bmatrix} -1 & -4 & 1 \\ 0 & -1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} -1 & -4 & 1 \\ 0 & -1 & -3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Which yields the simultaneous equations

$$\begin{aligned} -x_1 + 4x_2 + x_3 &= 0 \\ -x_2 - 3x_3 &= 0 \end{aligned}$$

Solving these equations we obtain $x_1 = 13x_3$, $x_2 = 3x_3$. Setting $x_3 = t$, we obtain

$$\mathbf{x} = \begin{bmatrix} 13t \\ 3t \\ t \end{bmatrix}$$

so that

$$\begin{bmatrix} 13 \\ 3 \\ 1 \end{bmatrix}$$

is a basis for the eigenspace.
The total number of vectors in the bases of the eigenspaces is $2$, and as $A$ is $3 \times 3$ we can conclude that $A$ is not diagonalizable.

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**Question:** Determine whether or not the matrix

$$B = \begin{bmatrix} 4 & 1 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & -2 \end{bmatrix}$$

is diagonalizable, and diagonalize it if it is.
**Answer:** Computing the eigenvalues and eigenvectors using the method given in the previous lecture yields

$$\text{eigenvalue}: \quad \lambda_1 = -2 \quad \lambda_2 = 2 \quad \lambda_3 = 3$$

$$\text{basis:} \qquad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 4 \\ -8 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 5 \\ -5 \\ 1 \end{bmatrix}$$

Form the matrix $P$ of the basis vectors

$$P = \begin{bmatrix} 0 & 4 & 5 \\ 0 & -8 & -5 \\ 1 & 1 & 1 \end{bmatrix}$$

Computing the determinant we obtain $det(P) = 20$. Therefore $B$ is diagonalizable.

## Question 11.5

Compute $P^{-1}$ and $P^{-1}BP$ and verify that it is diagonal. What are its diagonal entries?

## Suggested Problems

Practice the material covered in this lecture by attempting the following questions from Chapter 5.2 of Anton-Rorres, starting on page $311$

- Questions $5$, $7$, $14$
- True/False questions (c), (d), (e), (f)