MATHEMATICS 1LS3E TEST 2

Evening Class
Duration of Examination: 75 minutes
McMaster University, 23 May 2012

E. Clements

FIRST NAME (please print): Sol^{NS}

FAMILY NAME (please print): ______

Student No.: ______

THIS TEST HAS 8 PAGES AND 8 QUESTIONS. QUESTIONS BEGIN ON PAGE 2. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE.

Total number of points is 40. Marks are indicated next to the problem number. Any non-graphing calculator is allowed.

USE PEN TO WRITE YOUR TEST. IF YOU USE A PENCIL YOUR TEST WILL NOT BE ACCEPTED FOR REMARKING (IF NEEDED).

You need to show work to receive full credit.

Page	Points	Mark
2	8	
3	5	
4	6	
5	4	
6	6	
7	6	
8	5	
TOTAL	40	

Continued on next page

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1. The discrete-time dynamical system for caffeine elimination from the body is given by

$$c_{t+1} = 0.87c_t + d$$

where c_t denotes the amount of caffeine (in mg) present in your body at time t (in hours) and d is the amount of caffeine taken every hour. Suppose that you start with 100mg of caffeine in your body and do not consume anymore throughout the day.

(a) [3] Compute the amount of caffeine in your body after 1, 2, 3, and 4 hours. Graph.

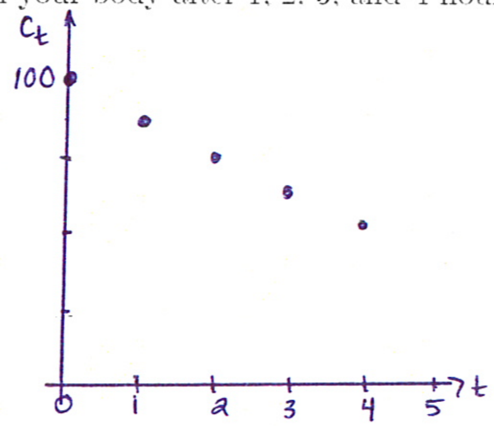
$C_0 = 100$

$C_1 = 0.87(100) = 87$

$C_2 = 0.87(87) = 75.69$

$C_3 = 0.87(75.69) = 65.8503$

$C_4 = 0.87(65.8503) \simeq 57.29$



(b) [3] Write a formula for c_t as a function of t. After how many hours will you only have 10mg left in your body?

$C_t = C_0 \cdot (0.87)^t = 100 \cdot (0.87)^t$

$10 = 100(0.87)^t$

$0.1 = (0.87)^t$

$\ln 0.1 = \ln 0.87^t$

$\ln 0.1 = t \cdot \ln 0.87$

$\rightarrow t = \frac{\ln 0.1}{\ln 0.87} \simeq 16.5$ hours

$\therefore$ After 17 hours, you will have just under 10mg in your body.

(c) [2] Determine a formula for $c_{t+2} = f(c_t)$. What does this system compute?

$C_{t+2} = (f \circ f)(C_t) = 0.87(0.87C_t + d) + d$

$= (0.87)^2 C_t + 0.87d + d$

$= 0.7569\, C_t + 1.87d$

This computes the amount of caffeine in your body 2-hours later.

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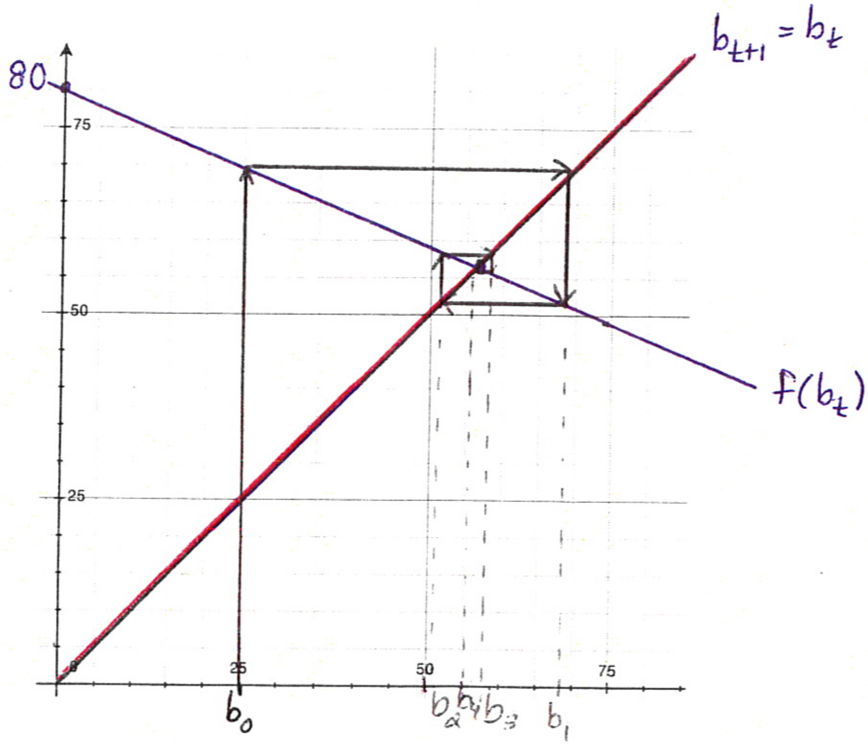
2. The dynamics of a certain population of butterflies can be described by

$$b_{t+1} = -0.4b_t + 80$$

where b_t represents the number of butterflies at time t in years.

(a) [3] Graph the updating function and the diagonal. Label the axes. Suppose that initially there are 25 butterflies. Cobweb for ~~5~~ 3 steps.



equilibrium:

$b^* = -0.4b^* + 80$

$1.4b^* = 80$

$b^* = \frac{80}{1.4} \doteq 57.1$

(b) [2] Describe what happens to the population of butterflies over time.

The population increases during the 1^{ST} year, decreases in the 2^{ND} year, increases in the 3^{RD} year, decreases in the 4^{TH} year and this alternating continues but each year the size of the increase or decrease is smaller. The pop^N converges towards an equilibrium.

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3. The discrete-time dynamical system $l_{t+1} = \dfrac{120}{45 + l_t} l_t$ describes the growth of a certain population of lions where l_t represents the number of lions present at time t in years.

(a) [2] What is the per capita production function? Describe the behaviour of the per capita production function.

$$\frac{l_{t+1}}{l_t} = \underbrace{\frac{120}{45 + l_t}}_{r(l_t)}$$

$r(l_t) = \dfrac{120}{45 + l_t}$ ← per capita production f^n

as the # of lions increases, the percapita production decreases.

(b) [2] Find the equilibria of this system algebraically.

$$l^* = \frac{120}{45 + l^*} l^*$$

$$l^*(45 + l^*) = 120\, l^*$$

$$(l^*)^2 + 45 l^* - 120 l^* = 0$$

$$(l^*)^2 - 75 l^* = 0$$

$$l^*[l^* - 75] = 0 \Rightarrow l^* = 0 \text{ or } l^* = 75$$

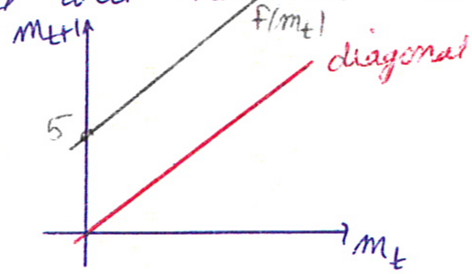
(c) [2] In general, what does an equilibrium of a dynamical system represent? Give an example of a dynamical system which has no equilibrium.

It represents a value of the measurement which remains unchanged (constant) over time.

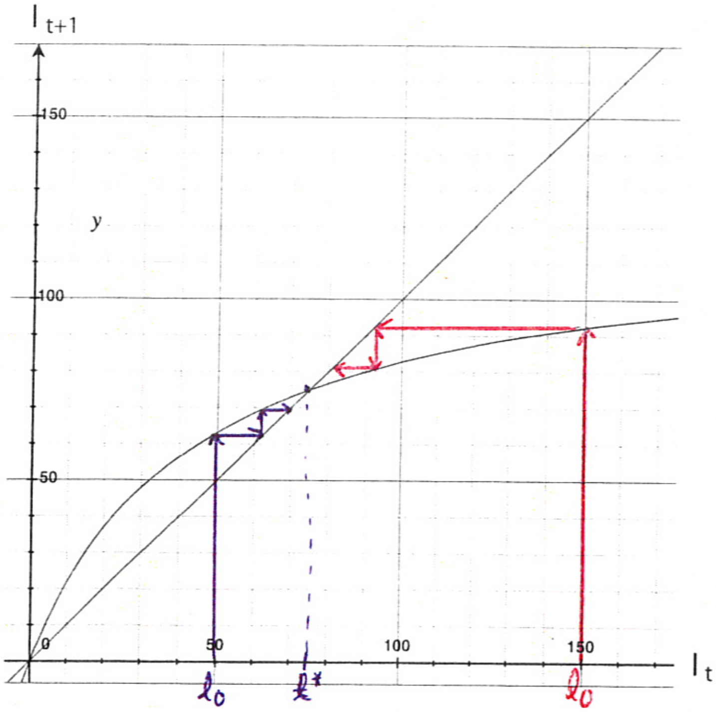
Any system in which the graph of the updating f^n does not intersect the diagonal will not have an eq^m.

For example,

$$m_{t+1} = m_t + 5$$



(e) [4] Below is the graph of the updating function and the diagonal for $l_{t+1} = \dfrac{120}{45 + l_t} l_t$.



(i) Suppose initially there are 50 lions. Cobweb for several steps to determine what will happen to the population of lions over time. Summarize your results in one sentence.

The population will increase towards the equilibrium

(ii) Now, suppose initially there are 150 lions. Cobweb for several steps to determine what will happen to the population of lions over time. Summarize your results in one sentence.

The population will decrease towards the equilibrium

4. [2] Describe the following events as discrete-time dynamical systems (i.e., state the dynamical rule and identify the initial condition).

(a) The number of deer in a forest increases by 4.5 percent per year. Initially there are 120 deer.

$$d_{t+1} = 1.045\, d_t \ , \quad d_0 = 120$$

(b) A patient's body absorbs 30% of medication per hour. Every hour, the patient is given 0.5 units of medication. The starting dosage was 2 units.

$$m_{t+1} = 0.70\, m_t + 0.5 \ , \quad m_0 = 2$$

5. Evaluate each limit, or explain why it does not exist.

(a) [2] $\lim\limits_{x \to 2} \dfrac{\frac{1}{x} - \frac{1}{2}}{x - 2}$

$$= \lim_{x \to 2} \frac{2-x}{2x} \cdot \frac{1}{x-2}$$

$$= \lim_{x \to 2} \frac{-\cancel{(x-2)}}{2x\cancel{(x-2)}}$$

$$= \frac{-1}{4}$$

(b) [2] $\lim\limits_{x \to \infty} \dfrac{x^{\frac{1}{3}}}{\ln x} \quad = \dfrac{\infty}{\infty}$

but $x^{1/3} \to \infty$ faster than $\ln x \to \infty$

so $\dfrac{x^{1/3}}{\ln x} \to \dfrac{\infty}{\text{large\#}} \to \infty$

$\lim\limits_{x \to \infty} \dfrac{x^{1/3}}{\ln x} = \infty$ (D.N.E.)

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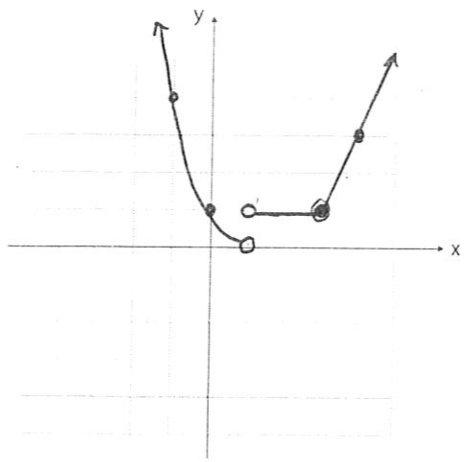
6. (a) [3] Sketch the graph of $f(x) = \begin{cases} (x-1)^2 & \text{if } x < 1 \\ 1 & \text{if } 1 < x \leq 3 \\ 2x - 5 & \text{if } x > 3 \end{cases}$.



(b) [3] From your graph in part (a), determine each limit or explain why it does not exist.

(i) $\lim_{x\to 1^-} f(x) = 0$

(ii) $\lim_{x\to 1^+} f(x) = 1$

(iii) $\lim_{x\to 1} f(x)$ D.N.E.

left limit ≠ right limit

(iv) $\lim_{x\to 3^-} f(x) = 1$

(v) $\lim_{x\to 3^+} f(x) = 1$

(vi) $\lim_{x\to 3} f(x) = 1$

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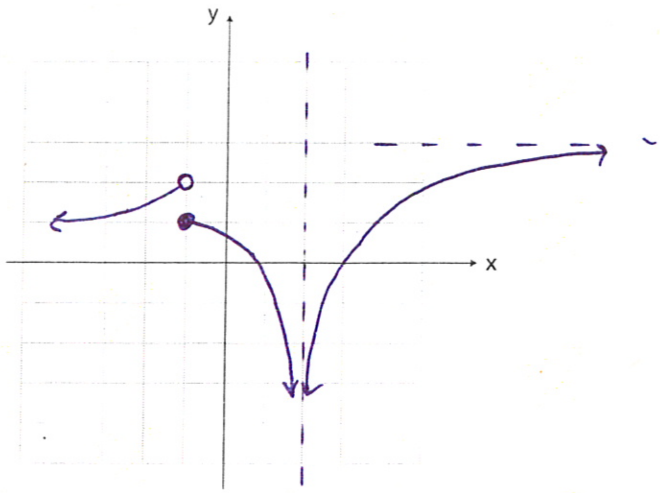
7. [3] Sketch a possible graph of a function $f(x)$ that satisfies all of the following conditions:

$\lim_{x\to -1^-} f(x) = 2$, $\lim_{x\to -1^+} f(x) = 1$, $\lim_{x\to 2} f(x) = -\infty$, and $\lim_{x\to\infty} f(x) = 3$.



8. [2] Suppose that the absorption of a toxic chemical by the body is modelled by

$$\alpha(c) = \frac{10c^2}{1+e^c}$$

where α represents the amount absorbed (in mg) and c represents the concentration of the chemical (in mmol/L).

What will happen long-term to the amount your body absorbs if the concentration of this chemical continues to increase steadily?

$$\lim_{c\to\infty} \alpha(c) = \lim_{c\to\infty} \frac{10c^2}{1+e^c} = \frac{\infty}{\infty}$$

but $e^c \to \infty$ faster than $10c^2 \to \infty$

$$\therefore \frac{10c^2}{1+e^c} \to \frac{\text{large \#}}{\infty} \to 0$$

Long-term, the amount you body absorbs decreases to 0 as the concentration continues to increase (saturation).

THE END