ASSIGNMENT 1

Review of Chapter 6 + Section 7.1

1. Verify that $y(x) = e^{1-\sqrt{1+x^2}} - 1$ is a solution of the differential equation $\underbrace{\frac{dy}{dx}}_{LS} = \underbrace{-\frac{x(1+y)}{\sqrt{1+x^2}}}_{RS}$.

$$LS = \frac{dy}{dx}$$

$$= e^{1-\sqrt{1+x^2}} \cdot -\frac{2x}{2\sqrt{1+x^2}}$$

$$= -\frac{x \cdot e^{1-\sqrt{1+x^2}}}{\sqrt{1+x^2}}$$

$$= -\frac{x \cdot \left(e^{1-\sqrt{1+x^2}} \overbrace{-1+1}^{=0}\right)}{\sqrt{1+x^2}}$$

$$= -\frac{x(y+1)}{\sqrt{1+x^2}}$$

$= RS$ $\therefore$ $y(x)$ is a sol^n to the D.E.

2. Sketch the graph of $f'(x) = 10e^{-2x} - 40$. Use this to sketch the graph of $f(x)$ given that $f(0) = 0$.

$f'(0) = 10e^{-2(0)} - 40 = 10 - 40 = -30$

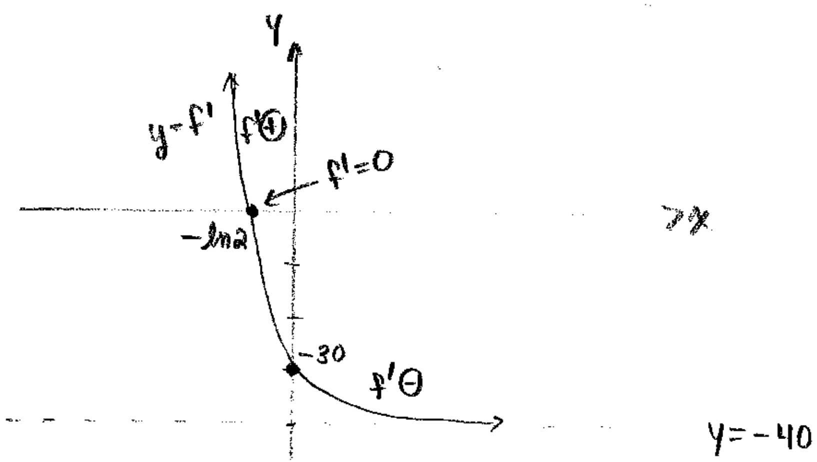
$f'(x) = 0 \Rightarrow 10e^{-2x} - 40 = 0$

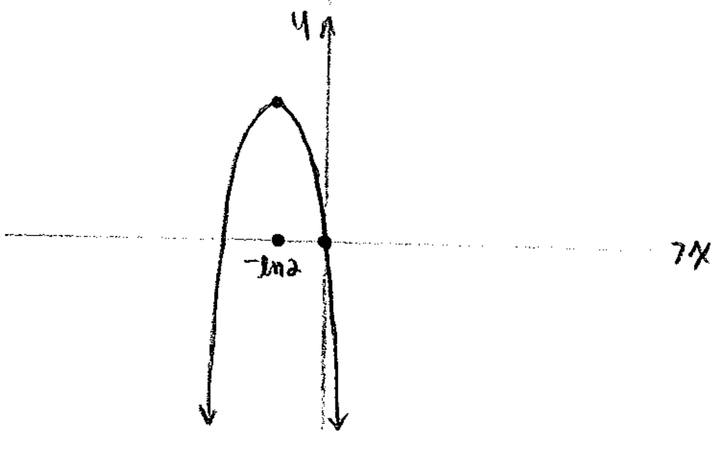
$e^{-2x} = 4$

$x = \frac{\ln 4}{-2}$

$= -\ln 2$

critical # of $f(x)$





3. Find the general solution of the following pure-time differential equations.

(a) $\frac{dy}{dx} = \frac{x}{\sqrt{1-x^2}}$ * requires SUBSTITUTION

$$y(x) = \int \frac{x}{\sqrt{1-x^2}}\,dx$$

Let $u = 1-x^2$. Then $\frac{du}{dx} = -2x \Rightarrow dx = \frac{du}{-2x}$.

$$\int \frac{x}{\sqrt{1-x^2}}\,dx = \int \frac{x}{\sqrt{u}}\,\frac{du}{-2x}$$

$$= -\frac{1}{2}\int u^{-\frac{1}{2}}\,du$$

$$= -\frac{1}{2}\cdot 2u^{\frac{1}{2}} + C$$

$$= -\sqrt{1-x^2} + C$$

$\therefore\ y(x) = -\sqrt{1-x^2} + C$

(b) $y' = xe^{2x}$ * requires Integration by Parts : $\int u\,dv = uv - \int v\,du$

$$y(x) = \int xe^{2x}\,dx$$

Let $u = x$ and $dv = e^{2x}\,dx$.

Then $du = dx$ and $v = \frac{e^{2x}}{2}$

$$\therefore \int xe^{2x}\,dx = \frac{xe^{2x}}{2} - \int \frac{e^{2x}}{2}\,dx$$

$$= \frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + C$$

$\therefore\ y(x) = \left(\frac{2x-1}{4}\right)e^{2x} + C$

4. Consider the pure-time differential equation $\frac{dw}{dt} = \frac{2}{1+t}$ with initial condition $w(0) = 3$.

(a) Solve the IVP. What is the value of $w(1)$?

$$w(t) = \int \frac{2}{1+t}\, dt$$

$$= 2\ln|1+t| + C$$

$$w(0) = 3 \Rightarrow 3 = 2\underbrace{\ln 1}_{=0} + C \Rightarrow C = 3$$

$\therefore$ $w(t) = 2\ln|1+t| + 3$ is the sol^n to the IVP.

$$w(1) = 2\ln 2 + 3 \approx 4.39$$

(b) Apply Euler's method using a step size of $h = 0.25$ and starting from the initial condition $w(0) = 3$ to estimate $w(1)$.

$t_0 = 0$
$w_0 = 3$

$t_1 = t_0 + h = 0 + 0.25 = 0.25$
$w_1 = w_0 + \left.\frac{dw}{dt}\right|_{t=t_0} \cdot h = 3 + \frac{2}{1+0} \cdot 0.25 = 3.5$

$t_2 = 0.5$
$w_2 = 3.5 + \frac{2}{1+0.25} \cdot 0.25 = 3.9$

$t_3 = 0.75$
$w_3 = 3.9 + \frac{2}{1+0.5} \cdot 0.25 \approx 4.23$

$t_4 = 1$
$w_4 = 4.23 + \frac{2}{1+0.75} \cdot 0.25 \approx 4.52$ $\therefore$ $w(1) \approx 4.52$

$\checkmark\ \frac{dT}{dt} = \alpha(A-T)$

5. In the textbook, read example 7.1.4 **Newtons Law of Cooling** (pages 520-521) and answer the following on p. 527:

(a) Question 50. $\alpha = 0.2/\text{min}$ $A = 10°C$ $T(0) = 40°C$

$$\text{(a) } T(t) = 10 + (40-10)e^{-0.2t} = 10 + 30e^{-0.2t}$$

check $T(t)$ is a sol^n to $\frac{dT}{dt} = 0.2(10-T)$ (*):

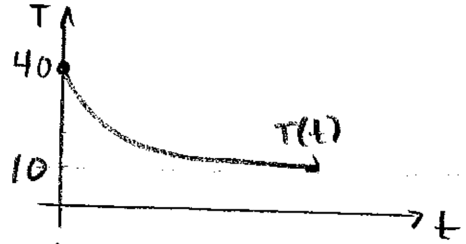
$$LS = \frac{dT}{dt} = 30e^{-0.2t}(-0.2) = -6e^{-0.2t}$$

$$RS = 0.2(10-T) = 0.2(10-(10+30e^{-0.2t})) = 0.2(-30e^{-0.2t}) = -6e^{-0.2t}$$

$LS = RS$ $\therefore$ $T(t)$ is the sol^n to (*).

(b) $T(1) = 10 + 30e^{-0.2} \approx 34.56°C$
$T(2) = 10 + 30e^{-0.4} \approx 30.12°C$

(c)



$$\lim_{t\to\infty} T(t) = 10 + 30\underbrace{e^{-\infty}}_{\to 0} = 10°C$$

(b) Question 52.
$\Delta t = 1$

$t_0 = 0$
$T_0 = 40$

$t_1 = t_0 + \Delta t = 0 + 1 = 1$
$T_1 = T_0 + \left.\frac{dT}{dt}\right|_{T=T_0} \cdot \Delta t = 40 + 0.2(10-40)\cdot 1 = 34\ °C$

$T(1) \approx 34°C$
(actual temp: $T(1) \approx 34.56°C$)

$t_2 = 2$
$T_2 = 34 + 0.2(10-34)\cdot 1 = 29.2\ °C$
$T(2) \approx 29.2\ °C$ (actual temp: $T(2) \approx 30.12°C$)

6. Psychologists interested in learning theory study **learning curves**. A learning curve is the graph of a function $P(t)$, the performance of someone learning a skill as a function of the training time t. The derivative dP/dt represents the rate at which performance improves.

(a) When do you think P increases most rapidly? What happens to $dP/d\overset{t}{\cancel{P}}$ as t increases? Explain.

We would expect P to increase most rapidly at the beginning. As time goes on and performance reaches a maximum, the rate at which performance improves slows down.

For example, consider the process of learning a language. In the beginning, your performance improves dramatically with training but as you master the language, the rate at which you are improving slows down.

(b) If M is the maximum level of performance of which the learner is capable, explain why the differential equation

$$\frac{dP}{dt} = k(M - P), \quad k \text{ a positive constant}$$

is a reasonable model for learning.

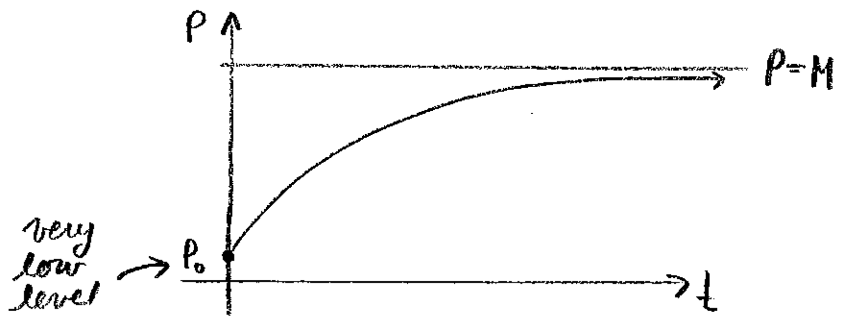
For small P (low performance level),

$\frac{dP}{dt} \approx kM \Rightarrow P(t)$ increases exponentially

As $P \to M$ (as performance reaches max. level),

$\frac{dP}{dt} \to 0$ (rate of improvement approaches 0).

(c) Make a rough sketch of a possible solution of this differential equation.



7. Consider the ~~modified logistic~~ autonomous differential equation $P' = 2P(1 - \frac{120}{P})$.

(a) For which values of P is the population increasing? For which values of P is the population decreasing?

	0		120		$\rightarrow P$
P'		$-$		$+$	
P		$\searrow$		$\nearrow$	

The popⁿ is decreasing when $P \in (0, 120)$ and it is increasing when $P \in (120, \infty)$.

(b) Check that the constant function $P(t) = 120$ is a solution of the equation. What is special about it?

$P = 120$
$P' = 0$

$LS = P'$
$= 0$

$RS = 2P\left(1 - \frac{120}{P}\right)$
$= 2(120)\left(1 - \frac{120}{120}\right)$
$= 0$

$\because LS = RS \therefore P = 120$ is a solⁿ of the DE.

Since the popⁿ dies out if it falls below 120 individuals, 120 is called the "existential threshold".

7. continued....

(c) Apply Euler's method using a step size of $h = 5$ and starting from the initial condition $P(0) = 200$ to estimate $P(15)$.

$t_0 = 0$ $\quad h = 5$

$P_0 = 200$

$t_1 = t_0 + h = 5$

$P_1 = P_0 + P'(P_0) \cdot h = 200 + 2(200)\left(1 - \frac{120}{200}\right) \cdot 5 = 1000$

$t_2 = t_1 + h = 10$

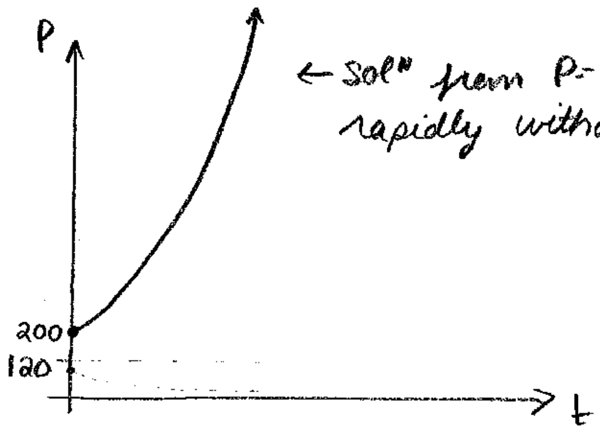
$P_2 = P_1 + P'(P_1) \cdot h = 1000 + 2(1000)\left(1 - \frac{120}{1000}\right) \cdot 5 = 9800$

$t_3 = t_2 + h = 15$

$P_3 = P_2 + P'(P_2) \cdot h = 9800 + 2(9800)\left(1 - \frac{120}{9800}\right) \cdot 5 = 106\,600$

$\therefore P(15) \approx 106\,600$

(d) Make a rough sketch of the solution.



← sol^n from $P = 200$ increases rapidly without bound.

THE END