MATHEMATICS 1LS3 TEST 4

Day Class
Duration of Test: 60 minutes
McMaster University

E. Clements

25 March 2013

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FIRST NAME (please print) : Sol^NS

FAMILY NAME (please print) : ______________________

Student No.: ______________________

THIS TEST HAS 8 PAGES AND 9 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE.

Total number of points is 40. Marks are indicated next to the problem number in square brackets. Any Casio fx991 (or lower, non-graphing) calculator is allowed.

USE PEN TO WRITE YOUR TEST. IF YOU USE A PENCIL, YOUR TEST WILL NOT BE ACCEPTED FOR REMARKING (IF NEEDED).

You need to show work to receive full credit, except for Multiple Choice.

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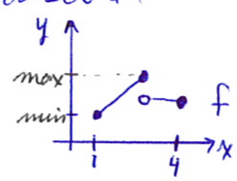
1. State whether each statement is **true or false** and then **explain** your reasoning.

(a) [2] If $f(x)$ is not continuous on $[a,b]$, then f cannot have both an absolute maximum and an absolute minimum on $[a,b]$.

FALSE.

If f is not continuous on $[a,b]$, then it is not guaranteed by the EVT to have both an absolute max and min, but it is still possible.

For example,



discontinuous on $[1,4]$

(b) [2] $P(t) = \dfrac{1}{1-\ln t}$ is a solution of the differential equation $\dfrac{dP}{dt} = \dfrac{P^2}{t}$ for $t > 0$.

Check: $P' = \dfrac{0\cdot(1-\ln t) - 1\cdot\left(-\frac{1}{t}\right)}{(1-\ln t)^2} = \dfrac{1}{t}\cdot\dfrac{1}{(1-\ln t)^2} = \dfrac{1}{t}\cdot P^2 = \dfrac{P^2}{t}$ ✓

TRUE.

2. **Multiple Choice**. Clearly **circle** the one correct answer.

(a) [3] It is known that $f'(a) = 0$ and $f''(a) < 0$. Which of the following is/are true?

(I) The graph of f is concave down at $x = a$. ✓

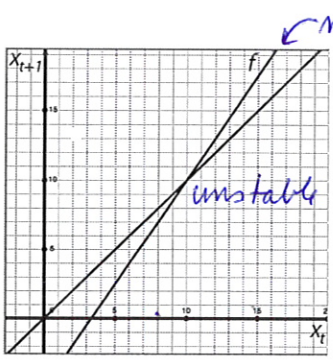
(II) $f(x)$ has a relative minimum at $x = a$. X (max)

(III) The graph of f has an inflection point at $x = a$. X (need $f''(a)=0$ + more information)

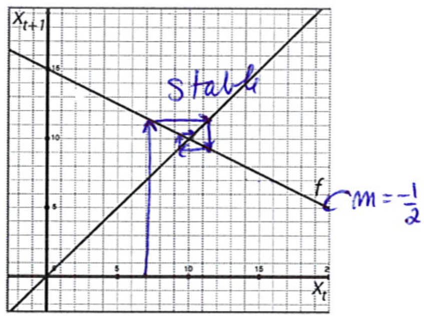
(A) none (B) I only (C) II only (D) III only

(E) I and II (F) I and III (G) II and III (H) all three

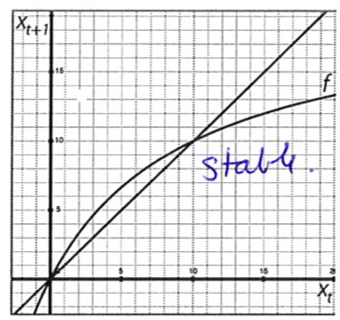
(b) [3] Below are the graphs of updating functions and the diagonal for three discrete-time dynamical systems $x_{t+1} = f(x_t)$. For which dynamical system(s) is $x^* = 10$ a stable equilibrium?



(I)



(II)



(III)

(A) none (B) I only (C) II only (D) III only

(E) I and II (F) I and III (G) II and III (H) all three

(c) [3] The function x^{-2} is the leading behaviour at ∞ (i.e., as $x \to \infty$) of which function(s)?

(I) $f(x) = x^{-2} + \underbrace{x^{-1}}_{LB}$ (II) $g(x) = \underbrace{3}_{LB} - x^{-2} - e^{-x}$ (III) $h(x) = \underbrace{x^{-2}}_{LB} + e^{-x}$

(A) none (B) I only (C) II only (D) III only

(E) I and II (F) I and III (G) II and III (H) all three

3. (a) [1] Write the formula for the Taylor polynomial $T_2(x)$ of a function $f(x)$ based at $a = 1$.

$$T_2(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2$$

(b) [2] Find the Taylor polynomial $T_2(x)$ for the function $f(x) = \ln x$ near $x = 1$.

$f(x) = \ln x \quad \ldots \quad f(1) = \ln 1 = 0$

$f'(x) = \frac{1}{x} \quad \ldots \quad f'(1) = \frac{1}{1} = 1$

$f''(x) = \frac{-1}{x^2} \quad \ldots \quad f''(1) = \frac{-1}{(1)^2} = -1$

$$T_2(x) = 1 \cdot (x-1) - \frac{1}{2}(x-1)^2$$

(c) [1] Use your answer in part (b) to find an approximation of $\ln 0.9$.

$\ln 0.9 \approx 1 \cdot (0.9 - 1) - \frac{1}{2}(0.9-1)^2$

$\approx (-0.1) - 0.5(-0.1)^2$

≈ -0.105

4. Suppose that a patient is given a dosage x of some medication where the probability of a cure is $P(x) = \dfrac{\sqrt{x}}{1+x}$ for $x > 0$.

(a) [2] Show that $P'(x) = \dfrac{1-x}{2\sqrt{x}(1+x)^2}$.

$$P' = \frac{\frac{1}{2\sqrt{x}} \cdot (1+x) - \sqrt{x} \cdot (1)}{(1+x)^2} \cdot \frac{2\sqrt{x}}{2\sqrt{x}}$$

$$= \frac{1 \cdot (1+x) - 2x}{2\sqrt{x}} \cdot \frac{1}{(1+x)^2}$$

$$= \frac{1-x}{2\sqrt{x}(1+x)^2}$$

(b) [2] What dosage maximizes the probabilitiy of a cure? What is the probability of a cure in this case?

$P' = 0$ when $1 - x = 0 \Rightarrow x = 1$

	0		1		$\to x$
P'		$+$		$-$	
P		$\nearrow$	max	$\searrow$	

$$P(1) = \frac{\sqrt{1}}{1+1} = \frac{1}{2}$$

$\therefore$ The probability of a cure is maximized when dosage is 1 unit. In this case, the chance of a cure is 50%.

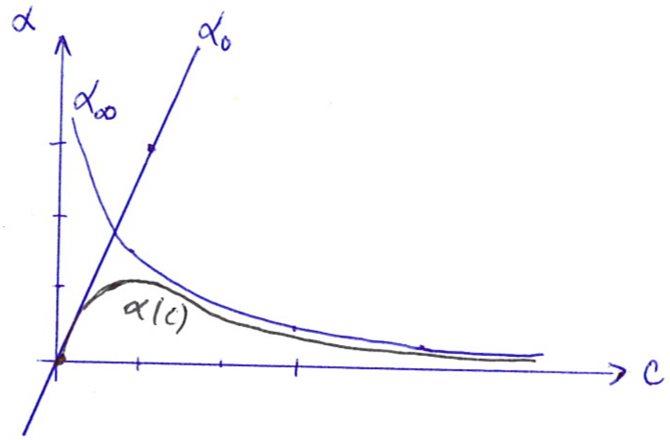
5. Consider the absorption function $\alpha(c) = \dfrac{3c}{1+2c^2}$, where c represents the concentration of a substance and α represents the amount absorbed.

(a) [2] Determine the leading behaviour of ~~$f(x)$~~ $\alpha(c)$ at both 0 and ∞.

$$\alpha_0(c) = \frac{3c}{1} = 3c$$

$$\alpha_\infty(c) = \frac{3c}{2c^2} = \frac{3}{2c}$$

(b) [2] Use your results in part (a) to sketch an approximate graph of $\alpha(c) = \dfrac{3c}{1+2c^2}$.



(c) [1] What happens to the absorption of the chemical as the concentration increases?

At first, absorption increases as concentration increases but then absorption will decrease towards 0 as concentration approaches infinity.

6. [3] Determine the absolute extrema of $f(x) = \arctan(x^2)$ on $[-1, 1]$.

$$f' = \frac{1}{1+(x^2)^2} \cdot (2x) = \frac{2x}{1+x^4} \Big\} \oplus$$

$f' = 0$ when $2x = 0 \Rightarrow x = 0$

x	$f(x)$
-1	$\arctan((-1)^2) = \arctan 1 = \frac{\pi}{4}$
0	$\arctan(0^2) = 0$ ABS. MIN.
1	$\arctan(1^2) = \frac{\pi}{4}$

ABS. MAX VALUE on $[-1,1]$.

7. [2] Evaluate $\lim_{x\to 0} \frac{x\cos x}{\sin x}$ using L'Hopital's rule.

$$\lim_{x \to 0} \frac{x \cdot \cos x}{\sin x}$$

$$\overset{LH}{=} \lim_{x \to 0} \frac{1 \cdot \cos x + x(-\sin x)}{\cos x}$$

$$= \frac{\overbrace{\cos 0}^{=1} - \overbrace{0 \cdot \sin 0}^{=0}}{\underbrace{\cos 0}_{=1}}$$

$$= 1$$

8. Consider the discrete-time dynamical system $p_{t+1} = 0.25p_t(1-p_t)^2$ for a population of birds, where p_t is the number of birds in thousands.

(a) [2] Find all three equilibria.

$$p^* = 0.25p^*(1-p^*)^2$$

$p^*=0$ is an eqm.

when $p^* \neq 0$, we can $\div p^*$:

$$1 = \frac{1}{4}(1-p^*)^2$$

$$4 = (1-p^*)^2$$

$$\pm 2 = 1-p^*$$

$$p^* = 1 \mp 2$$

$p^* = -1$ or $p^* = 3$

$\therefore$ equilibria are $-1, 0,$ and 3

(b) [3] Determine the stability of the biologically reasonable equilibria from part (a).

biologically reasonable equilibria: 0 and 3

$$f(p_t) = 0.25p_t(1-2p_t+p_t^2) = 0.25[p_t - 2p_t^2 + p_t^3]$$

$$f'(p_t) = 0.25[1-4p_t+3p_t^2]$$

$f'(0) = 0.25 \Rightarrow p^*=0$ is a STABLE eqm.

$f'(3) = 4 \Rightarrow p^*=3$ is an UNSTABLE eqm.

9. For each of the following, (i) describe the event as an initial value problem, i.e., write a differential equation and an initial condition, and (ii) state whether the differential equation is autonomous or pure-time.

(a) [2] A cell starts at a volume of $900\mu m^3$ and loses volume at a rate of $2t$ micrometres cubed per second.

$\ominus$

Let $V(t)$ be the volume of the cell at time t in seconds.

$V(0) = 900 \; (\mu m^3)$

$$\frac{dV}{dt} = -2t \; \left(\frac{\mu m^3}{s}\right)$$

PURE-TIME

(b) [2] The rate of change of the thickness of the ice on a lake is inversely proportional to the square root of the thickness. Initially, the ice is 3 mm thick.

Let $T(t)$ be the thickness of the ice at time t.

$$\frac{dT}{dt} \propto \frac{1}{\sqrt{T}}$$

$$\Rightarrow \frac{dT}{dt} = \frac{a}{\sqrt{T}} \quad \text{where } a > 0.$$

$T(0) = 3 \; (mm)$

AUTONOMOUS

THE END