MATH 1B03/1ZC3 Winter 2019

# Lecture 5: More about the inverse

Instructor: Dr Rushworth January 15th

## Covered in last lecture:

- Transpose
- Trace
- More properties of matrix multiplication
- The identity matrix
- The inverse of a matrix

## More on the inverse

(from Chapter 1.4 of Anton-Rorres)

Recall that a square matrix $A$ is invertible if there exists another square matrix $A^{-1}$ such that

$$AA^{-1} = A^{-1}A = I$$

where $I$ is the identity matrix.

**Example 5.1**

Let

$$A = \begin{bmatrix} 3 & 8 \\ 1 & 3 \end{bmatrix}$$

Then

$$A^{-1} = \begin{bmatrix} 3 & -8 \\ -1 & 3 \end{bmatrix}$$

Let's check:

$$\left(AA^{-1}\right)_{11} = (3 \cdot 3) + (8 \cdot -1) = 1$$
$$\left(AA^{-1}\right)_{12} = (3 \cdot -8) + (8 \cdot 3) = 0$$
$$\left(AA^{-1}\right)_{21} = (1 \cdot 3) + (3 \cdot -1) = 0$$
$$\left(AA^{-1}\right)_{22} = (1 \cdot -8) + (3 \cdot 3) = 1$$

so

$$\begin{bmatrix} 3 & 8 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 3 & -8 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

If a matrix is not square then it does not have an inverse. Even if a matrix is square, it does not necessarily have an inverse.

**Definition 5.2: Singular matrix**

Let $A$ be a square matrix which does not have an inverse. We say that $A$ is a singular matrix, or that A is singular.

For example, the zero matrix $0$ is singular because

$$M0 = 0M = 0 \neq I$$

for any matrix $M$ (recall Fact 4.9 from Lecture 4).

We will develop methods for determining whether or not a matrix is invertible, and how to find the inverse, if one exists. In the case of $2 \times 2$ matrices, there is a simple formula to do this.

## Recipe 5.3: Inverse of a $2 \times 2$ matrix

Let

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

be a $2 \times 2$ matrix.

**Step 1:** Compute the quantity $ad - bc$.
If $ad - bc = 0$ then $A$ is singular.
If $ad - bc \neq 0$ then $A$ is invertible.

**Step 2:** If $ad - bc \neq 0$ then the inverse $A^{-1}$ is given by

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The quantity $ad - bc$ is an example of a determinant of a matrix. Determinants are very important in linear algebra, and we'll learn more about them later on in the course. For now we will use the quantity $ad - bc$ to find $2 \times 2$ determinants.

## Example 5.4

Lets use the above formula to confirm the working in Example 5.1. Let

$$A = \begin{bmatrix} 3 & 8 \\ 1 & 3 \end{bmatrix}$$

Then $a = 3,\ b = 8,\ c = 1,\ d = 3$ and

$$\begin{aligned} ad - bc &= 3 \cdot 3 - 8 \cdot 1 \\ &= 9 - 8 \\ &= 1 \\ &\neq 0 \end{aligned}$$

so that $A$ is invertible. The formula in Recipe 5.3 gives

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$
$$= \frac{1}{1}\begin{bmatrix} 3 & -8 \\ -1 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} 3 & -8 \\ -1 & 3 \end{bmatrix}$$

as in Example 5.1.

**Fact 5.5**

Let $A$ and $B$ be invertible matrices of the same size. Then the product $AB$ is invertible also, and

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Note the change of order.

In general, if $M_1, M_2, \ldots, M_{k-1}, M_k$ are invertible matrices of the same size, then the product $M_1M_2 \cdots M_{k-1}M_k$ is invertible and

$$(M_1M_2 \cdots M_k)^{-1} = M_k^{-1}M_{k-1}^{-1} \cdots M_2^{-1}M_1^{-1}.$$

Notice that the order of the product has been reversed.

The inverse and the transpose interact in a nice way.

**Fact 5.6**

If $A$ is an invertible matrix, then $A^T$ is invertible also, and

$$(A^T)^{-1} = (A^{-1})^T$$

### Powers of matrices

As we have seen, we can do many things with matrices that we could do with numbers (with some differences). Just as we can raise numbers to powers, we can also raise matrices to powers, provided the matrices are square.

**Definition 5.7: Powers of matrices**

Let $A$ be a square matrix, and $k > 0$ a postive number. Define the $k$-th power of $A$ as

$$A^k = \underbrace{AA\cdots A}_{A \text{ multiplied by itself } k \text{ times}}$$

where $A^0 = I$.
We call $A^k$ "$A$ to the power of $k$".
If $A$ is invertible, then define

$$A^{-k} = \underbrace{A^{-1}A^{-1}\cdots A^{-1}}_{A^{-1} \text{ multiplied by itself } k \text{ times}}$$

**Fact 5.8: Laws of powers**

Let $A$ be a square matrix and $k, r > 0$ numbers. Then

1. $A^r A^s = A^{r+s}$
2. $(A^r)^s = A^{rs}$

If $A$ is invertible then the above identities hold for negative powers also.

## Finding the inverse of larger matrices

(from Chapter 1.5 of Anton-Rorres)

In Recipe 5.3 we saw a quick way of finding the inverse of a $2 \times 2$ matrix. Finding the inverse of larger matrices is harder, but there is still a recipe to do so.

Recall the elementary row operations:

1. swap two rows
2. multiply a row by a non-zero number
3. add a non-zero multiple of a row to another row

We can use the elementary row operations and Gauss-Jordan elimination to determine if a matrix of any size is invertible, and find the inverse, if it exists.

**Recipe 5.9: Inverse of an $n \times n$ matrix**

Let $A$ be an $n \times n$ matrix.

**Step 1:** Place $A$ on the left of $I_n$

$$\left[\begin{array}{c|c} \quad A \quad & \quad I_n \quad \end{array}\right]$$

**Step 2:** Do Gauss-Jordan elmination on $A$, while doing the same elementary row operations on $I_n$.

**Step 3:** There are two possibilities.
If you encounter a row of $0$'s during Gauss-Jordan elmination, then $A$ is singular, and $A^{-1}$ does not exist.
If you <u>do not</u> encounter a row of $0$'s, you will be able to put $A$ into RREF. In fact, the RREF of $A$ will be $I_n$. What you obtain on the right is $A^{-1}$:

$$\left[\begin{array}{c|c} \quad I_n \quad & \quad A^{-1} \quad \end{array}\right]$$

In Lecture 1 we saw that there are three possibilities for the number of solutions of a SLE, and that this was related to the RREF of its augmented matrix. In general, the RREF of a matrix is not the identity matrix i.e. when the associated SLE has an infinite number of solutions.

Square matrices are a special case, however, and invertible matrices are even more special. As we shall see later, a square matrix is invertible if and only if its RREF is the identity matrix. Further into the course we shall see how to interpret this in a geometric way.

**In summary:** to find the inverse of an $n \times n$ matrix $A$, place $A$ on the left of $I_n$, and attempt convert $A$ into $I_n$. If you encounter a row of $0$'s, then $A$ is singular. If you are able to put $A$ into RREF, then what appears on the right hand side is $A^{-1}$.

**Example 5.10: Using the recipe**

**Question:** Find the inverse of the following matrix, if it exists.

$$A = \begin{bmatrix} 2 & 6 & 6 \\ 2 & 7 & 6 \\ 2 & 7 & 7 \end{bmatrix}$$

**Answer:** Put $A$ next to $I_3$ and apply Gauss-Jordan elimination. Remember to apply the elementary row operations to both $A$ and $I_3$.

$$\left[\begin{array}{ccc|ccc} 2 & 6 & 6 & 1 & 0 & 0 \\ 2 & 7 & 6 & 0 & 1 & 0 \\ 2 & 7 & 7 & 0 & 0 & 1 \end{array}\right]$$

$$\wr \qquad \tfrac{1}{2}R1$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 3 & \frac{1}{2} & 0 & 0 \\ 2 & 7 & 6 & 0 & 1 & 0 \\ 2 & 7 & 7 & 0 & 0 & 1 \end{array}\right]$$

$$\wr \qquad -2R1 + R2$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 3 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 2 & 7 & 7 & 0 & 0 & 1 \end{array}\right]$$

$$\wr \qquad -2R1 + R3$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 3 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\wr \qquad -R2 + R3$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 3 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

$$\wr \qquad -3R2 + R1$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & \frac{7}{2} & -3 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

$$\wr \qquad -3R3 + R1$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{7}{2} & 0 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

We have placed $A$ into RREF, which in fact is $I_3$! Therefore, what appears on the right hand side is the inverse of $A$

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & -3 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

**Question:** Find the inverse of the following matrix, if it exists.

$$B = \begin{bmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & -9 \end{bmatrix}$$

**Answer:** Repeat the method of the previous example.

$$\left[\begin{array}{ccc|ccc} -1 & 3 & -4 & 1 & 0 & 0 \\ 2 & 4 & 1 & 0 & 1 & 0 \\ -4 & 2 & -9 & 0 & 0 & 1 \end{array}\right]$$

$$\rightsquigarrow \qquad 2R1 + R2, \text{ and } -4R1 + R3$$

$$\left[\begin{array}{ccc|ccc} -1 & 3 & -4 & 1 & 0 & 0 \\ 0 & 10 & -7 & 2 & 1 & 0 \\ 0 & -10 & 7 & -4 & 0 & 1 \end{array}\right]$$

$$\rightsquigarrow \qquad R2 + R3$$

$$\left[\begin{array}{ccc|ccc} -1 & 3 & -4 & 1 & 0 & 0 \\ 0 & 10 & -7 & 2 & 1 & 0 \\ 0 & 0 & 0 & -2 & 1 & 1 \end{array}\right]$$

as we have hit a row of $0$'s $A$ is singular and $A^{-1}$ does not exist.

## Elementary matrices

In mathematics it is often useful to be able to express the same thing in different ways. We're going to do this with the inverse of a matrix, and give a list of different ways to tell if a matrix is invertible or not. For this we need to define elementary matrices.

**Definition 5.11: Elementary matrix**

An elementary matrix is a matrix which can be obtained by applying exactly one elementary row operation to the identity matrix.

Examples of elementary matrices are

$$\begin{bmatrix} 1 & 0 \\ 0 & 17 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

**Question 5.12: (A bit harder)**

Is the follow matrix an elementary matrix?

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In the next lecture we'll see a Fact that will make the above question easy.

## Suggested problems

Practice the material in this lecture by attempting the following problems in **Chapter 1.3** of Anton-Rorres, starting on page $58$

- Questions $1$, $3$, $5$, $7$, $11$, $29$

Not all of these questions can be completed using material from this lecture - some require material from next lecture.