Announcements

Topics:

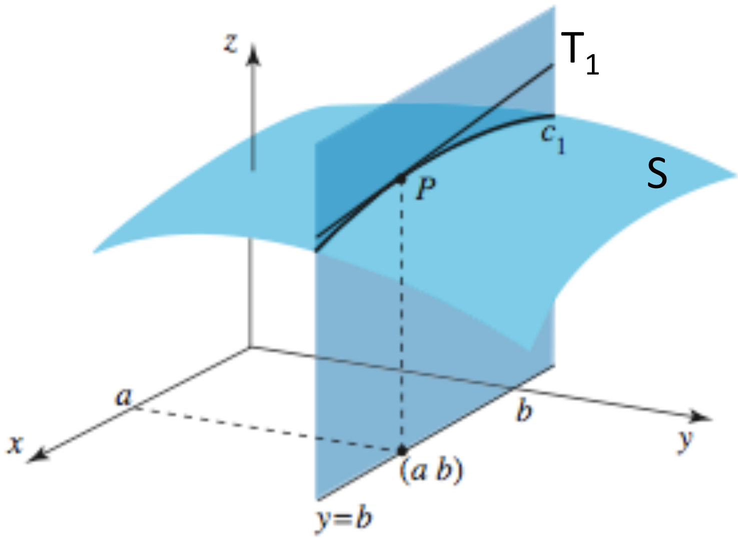
In the Functions of Several Variables module:

- **Section 7:** Second-Order Partial Derivatives
- **Section 9:** Directional Derivative and Gradient
- **Section 10:** Local Extreme Values (note, we will not study absolute extreme values in this course)

To Do:

- Read sections 7, 9, and 10 in the "Functions of Several Variables" module
- Work on Assignments and Suggested Practice Problems assigned on the webpage under the SCHEDULE + HOMEWORK link

Partial Derivatives

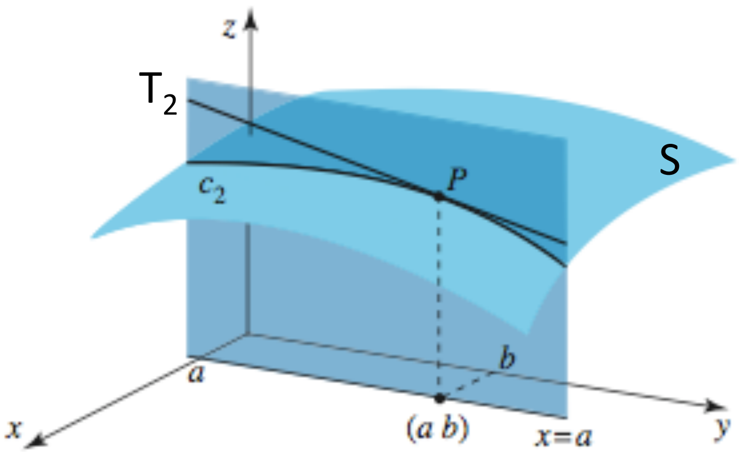


Recall:

$$f_x(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

f_x represents the rate of change of *f* in the *x*-direction at (*a*,*b*)

Partial Derivatives



Recall:

$$f_y(a,b) = \lim_{h \to 0} \frac{f(a, b+h) - f(a,b)}{h}$$

f_y represents the rate of change of *f* in the *y*-direction at (*a*,*b*)

The Directional Derivative

These two partial derivatives are special cases of a **directional derivative**.

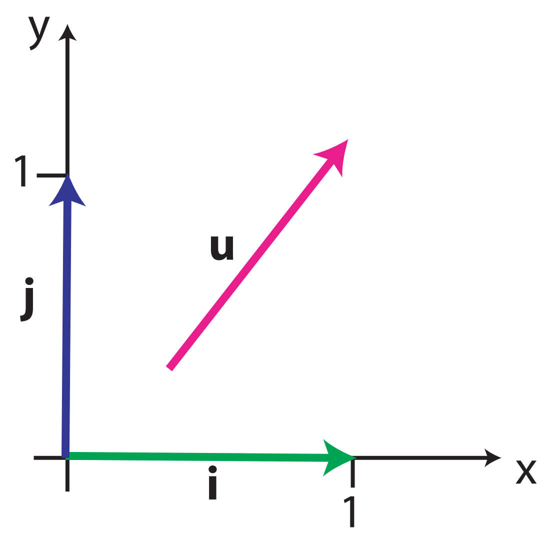
A directional derivative allows us to find the rate of change of a function of two variables, $f(x,y)$, in an arbitrary direction.

The Directional Derivative

Vectors

To describe a direction, we use a unit vector **u**.

A **unit vector** is any vector with magnitude (length) one unit, i.e. $\|u\| = 1$.



The Directional Derivative

Vectors

Vector **v** expressed as a linear combination of the standard unit vectors **i** and **j**:

$$\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j}$$

Magnitude of vector **v**:

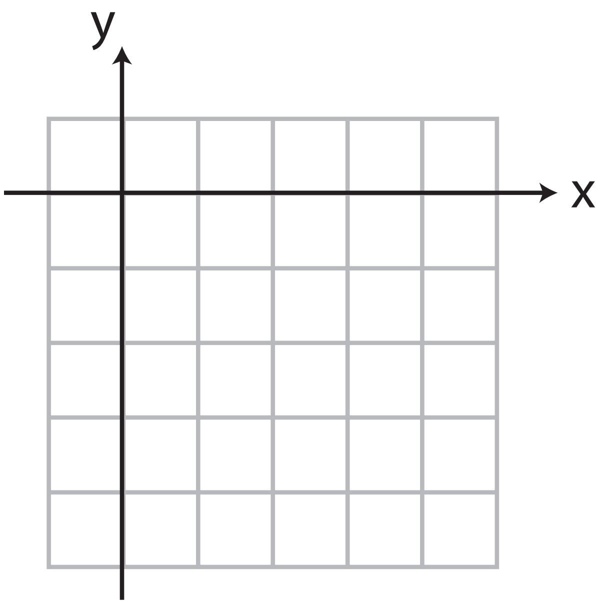
$$\|\mathbf{v}\| = \sqrt{(v_1)^2 + (v_2)^2}$$

Unit vector in the same direction as **v**:

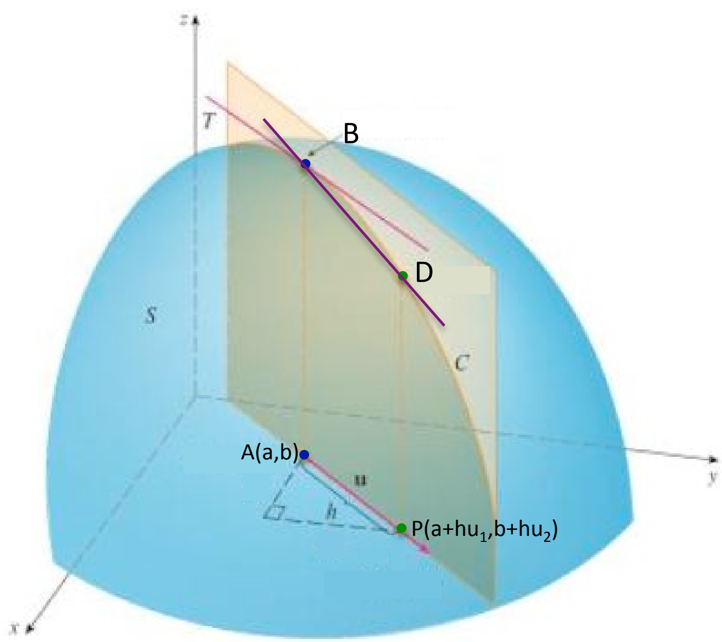
$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{v_1\mathbf{i} + v_2\mathbf{j}}{\sqrt{(v_1)^2 + (v_2)^2}}$$

Example:

Find the magnitude and determine the corresponding unit vector of the vector $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$.



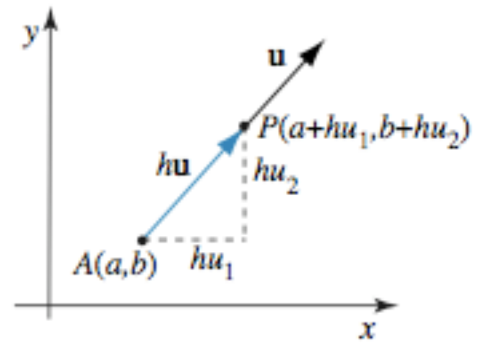
Geometric Interpretation of the Directional Derivative



The average rate of change of $f(x,y)$ in the direction of **u** from the point A to P is the slope of the secant line connecting points B and D on the curve C.

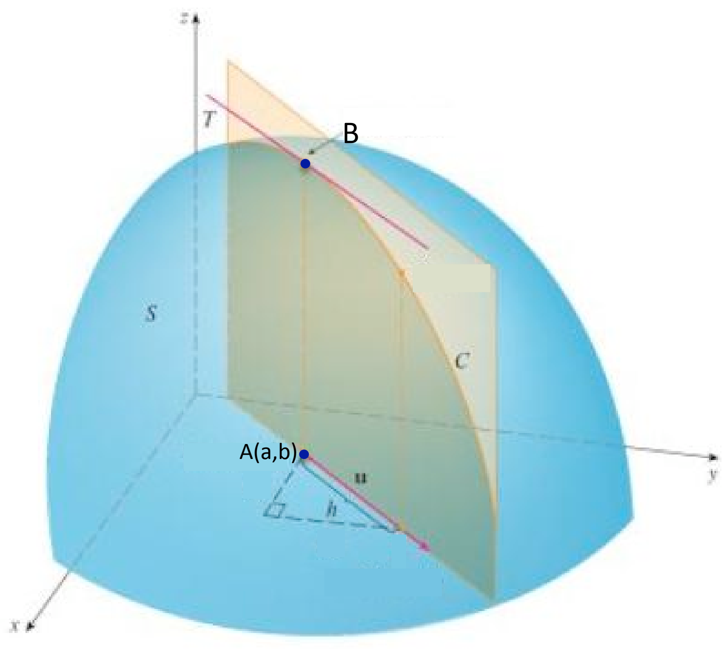
Geometric Interpretation of the Directional Derivative

Average rate of change of f(x,y) in the direction of $\mathbf{u}=u_1\mathbf{i}+u_2\mathbf{j}$ from the point A to P:



$$\frac{\text{change in } f \text{ from } A \text{ to } P}{\text{distance from } A \text{ to } P} = \frac{f(a+hu_1, b+hu_2) - f(a,b)}{h}$$

Geometric Interpretation of the Directional Derivative



The slope of the tangent T to the curve C at the point B is called the **directional derivative of *f* at (*a*,*b*) in the direction of the unit vector u**=u_1**i**+u_2**j** and is denoted by $D_{\mathbf{u}}f(a,b)$.

The Directional Derivative

<u>**Definition**</u>:
The directional derivative of a function *f*(x,y) at a point (*a*,*b*) in the direction of a <u>*unit vector*</u> **u**=u_1**i**+u_2**j** is given by

$$D_u f(a,b) = \lim_{h \to 0} \frac{f(a + hu_1, b + hu_2) - f(a,b)}{h}$$

provided that the limit exists.

The Directional Derivative

$D_u f(a,b)$ represents the rate of change of $f(x,y)$ in the direction of the vector $\mathbf{u}=u_1\mathbf{i}+u_2\mathbf{j}$.

<u>*Note*</u>:
when $\mathbf{u}=1\mathbf{i}+0\mathbf{j}$,

$$D_u f(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h} = f_x(a,b)$$

when $\mathbf{u}=0\mathbf{i}+1\mathbf{j}$,

$$D_u f(a,b) = \lim_{h \to 0} \frac{f(a,b+h) - f(a,b)}{h} = f_y(a,b)$$

i.e. the partial derivatives of *f* with respect to *x* and *y* are just special cases of the directional derivative.

The Directional Derivative

Example:

Find the directional derivative of $f(x,y) = x^2 - y$ at the point A(1,1) in the direction of the vector **v**=2**i**-5**j.**

The Directional Derivative

Theorem:

Assume that f is a differentiable function and $\mathbf{u}=u_1\mathbf{i}+u_2\mathbf{j}$ is a unit vector. The directional derivative of f at a point (a,b) in the direction $\mathbf{u}$ is given by

$$D_{\mathbf{u}}f(a,b) = f_x(a,b)u_1 + f_y(a,b)u_2$$

The Directional Derivative

Example (revisited):

Find the directional derivative of $f(x,y) = x^2 - y$ at the point A(1,1) in the direction of the vector **v**=2**i**-5**j.**

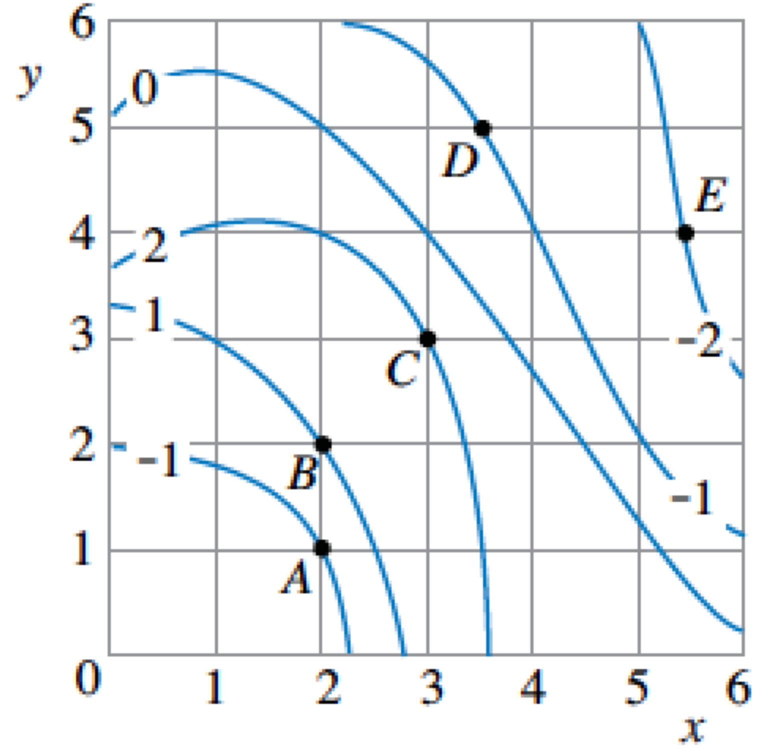
The Directional Derivative

Example :

#20. Find the directional derivative of $f(x,y) = e^{-x^2-y^2}$ at the point A(0,1) in the direction of the vector **v=i+j.**

The Directional Derivative

Example :
Consider the contour diagram of a function f(x,y). Estimate the value of the directional derivative at the given point in the given direction.



(a) At B, in the direction **v=i+j**
(b) At D, in the direction **v=-i**

The Gradient Vector

<u>Definition</u>:

Assume that f is a differentiable function. The <u>*gradient*</u> of f at (x,y) is the **vector** $\nabla f(x,y)$ defined by

$$\nabla f(x,y) = f_x(x,y)\mathbf{i} + f_y(x,y)\mathbf{j}$$

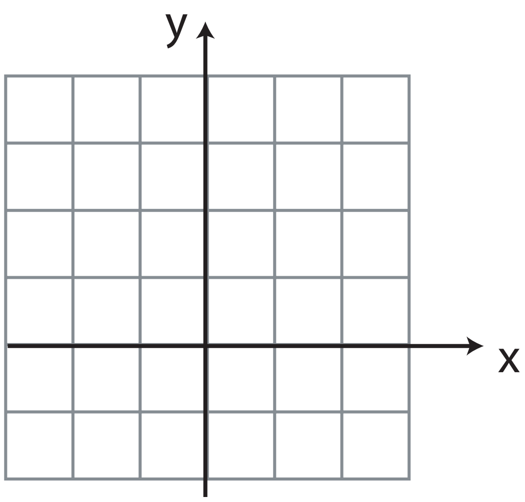
The Gradient Vector

Example:

Determine the gradient vector of

$$f(x,y) = x^2 - y$$

and use this to compute the gradient at several points in the domain of f.



The Gradient Vector

The gradient of *f* associates a vector to each point in the domain of *f*, where the partial derivatives are defined.

What is special about these vectors? What information do they provide about *f*?

The Gradient Vector and the Directional Derivative

If $f(x,y)$ is a differentiable function and $\mathbf{u}=u_1\mathbf{i}+u_2\mathbf{j}$ is a unit vector, then

$$D_{\mathbf{u}} f(a,b) = \left\|\nabla f(a,b)\right\| \cos\theta$$

where θ is the angle between $\mathbf{u}$ and $\nabla f(a,b)$.

The Gradient Vector

Theorem:

Extreme Values of the Directional Derivative

Assume that *f* is a differentiable function, and (*a*,*b*) is a point in its domain where $\nabla f(a,b) \neq 0$.

(a) The ***maximum*** rate of change of *f* at (*a*,*b*), is equal to $\|\nabla f(a,b)\|$ and occurs in the direction of the gradient $\nabla f(a,b)$.

(b) The ***minimum*** rate of change of *f* at (*a*,*b*) is equal to $-\|\nabla f(a,b)\|$ and occurs in the direction opposite to the gradient $\nabla f(a,b)$.

The Gradient Vector

Example:

Find the maximum rate of change of the function $f(x,y) = \arctan\left(\frac{3y}{x}\right)$ at (1,1) and the direction in which it occurs.

Geometric Interpretations of the Gradient Vector

The gradient vector is perpendicular to the level curves and points in the direction of largest increase.

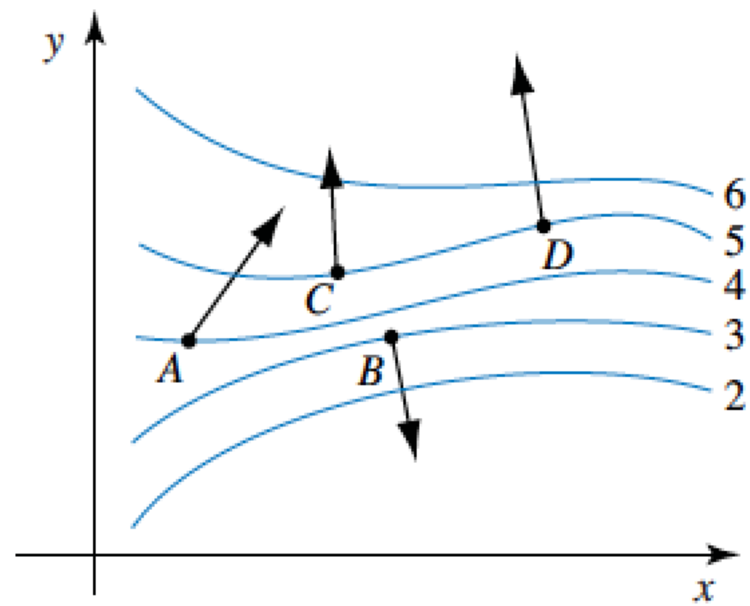
The magnitude of the vector is equal to the largest rate of change.

Geometric Interpretations of the Gradient Vector

Example 9.8:

(a) Explain why the vectors at A and B cannot represent the gradient of *f*.

(b) Explain why the vectors at C and D can represent the gradient of *f*.

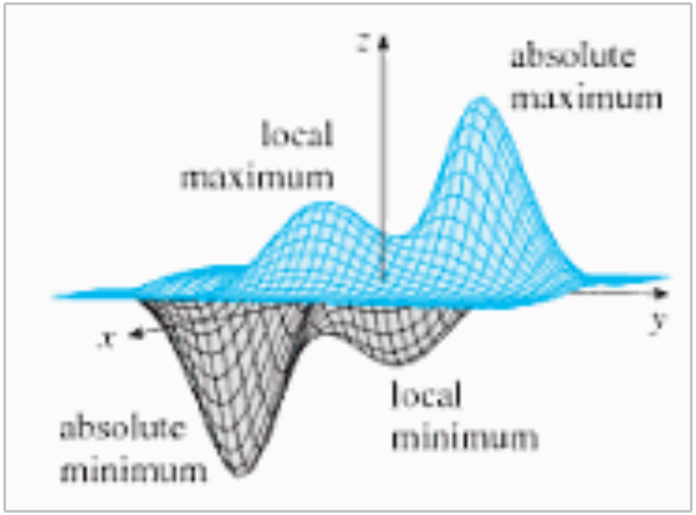


Maximum and Minimum Values

<u>Definition</u>:

A function $f(x,y)$ has a **local maximum** at (a,b) if $f(a,b) \geq f(x,y)$ when (x,y) is near (a,b).

The number $f(a,b)$ is called a **local maximum value.**

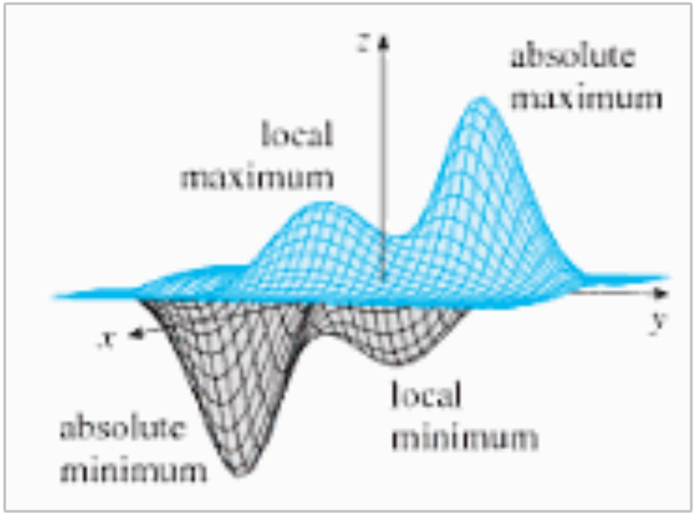


Maximum and Minimum Values

Definition:

A function $f(x,y)$ has a **local minimum** at (a,b) if $f(a,b) \le f(x,y)$ when (x,y) is near (a,b).

The number f(a,b) is called a **local minimum value.**

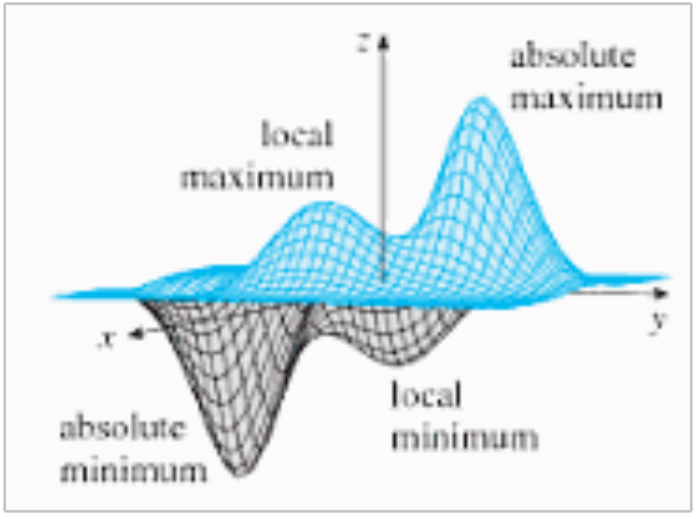


Maximum and Minimum Values

<u>Definition</u>:

A function $f(x,y)$ has an **absolute maximum** at (a,b) if $f(a,b) \geq f(x,y)$ for all (x,y) in the domain of f.

The number $f(a,b)$ is called the **absolute maximum value.**

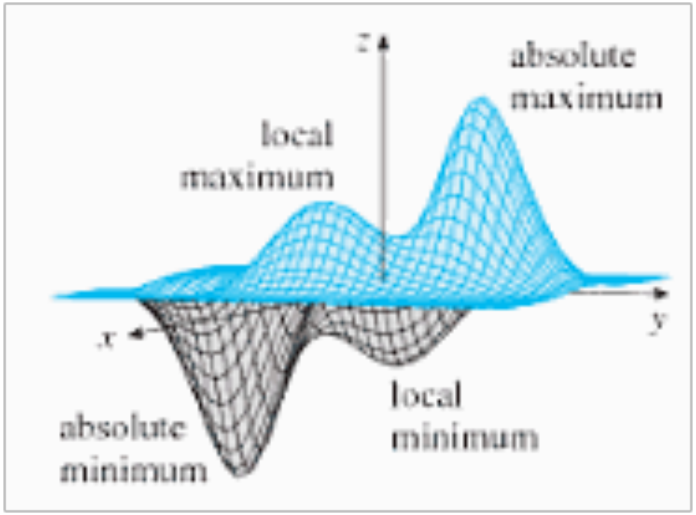


Maximum and Minimum Values

Definition:

A function $f(x,y)$ has an **absolute minimum** at (a,b) if $f(a,b) \leq f(x,y)$ for all (x,y) in the domain of f.

The number $f(a,b)$ is called the **absolute minimum value.**



Fermat's Theorem

If a function $f(x,y)$ has a local minimum or a local maximum at (a,b), then (a,b) is a <u>*critical point*</u> of f.

Critical Points

Definition:

A point (a,b) in the domain of a function $f(x,y)$ is called a **critical point** if either

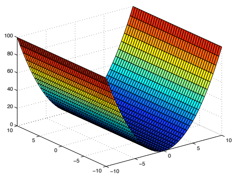
(a) $f_x(a,b)=0$ <u>and</u> $f_y(a,b)=0$, or

(b) at least one of $f_x(a,b)$ or $f_y(a,b)$ does not exist.

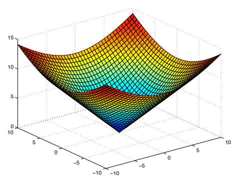
Critical Points

Some interesting cases:

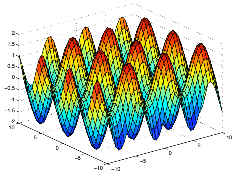
(a) $f(x,y) = x^2$



(b) $f(x,y) = \sqrt{x^2 + y^2}$



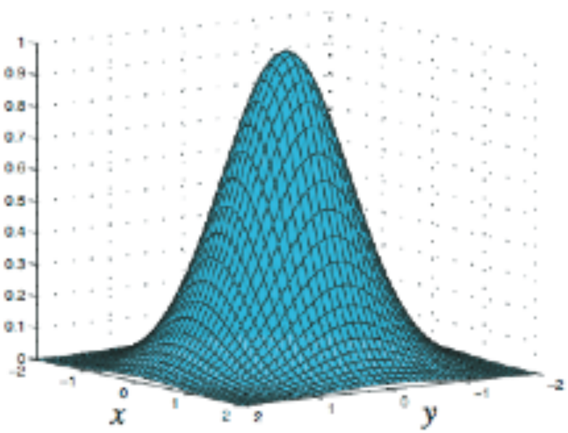
(c) $f(x,y) = \sin x - \sin y$

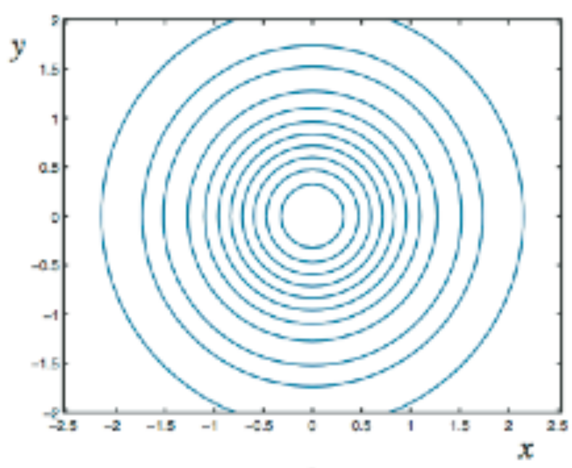


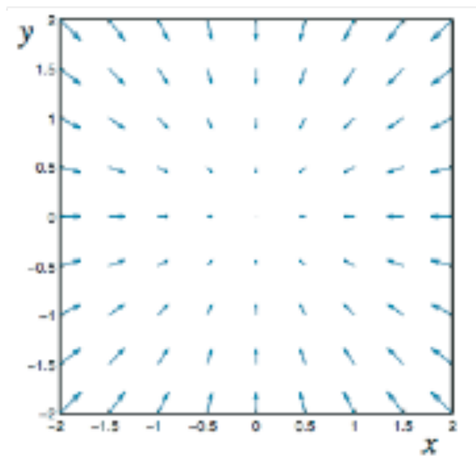
Critical Points

Some interesting cases:

(d) $f(x,y) = e^{-x^2-y^2}$

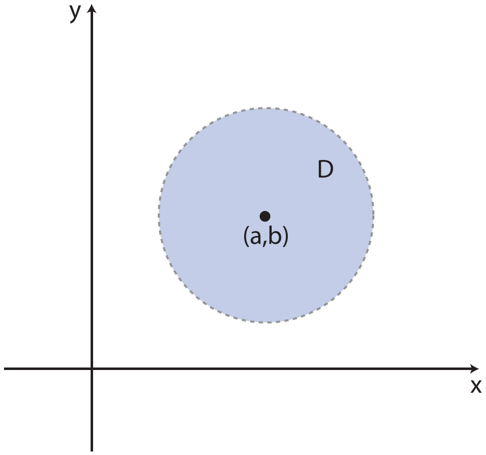






Second Derivatives Test

Suppose the second partial derivatives of f are continuous on a disk with centre (a,b) and suppose that $f_x(a,b)=0$ and $f_y(a,b)=0$.

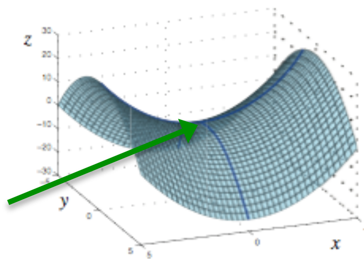


Second Derivatives Test

Let $D = D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - \left[f_{xy}(a,b)\right]^2$.

(a) If $D > 0$ and $f_{xx}(a,b) > 0$, then $f(a,b)$ is a local min.
(b) If $D > 0$ and $f_{xx}(a,b) < 0$, then $f(a,b)$ is a local max.
(c) If $D < 0$, then $f(a,b)$ is not a local max or min and we say (a,b) is a saddle point of *f*.

(0,0) is a saddle point



__Note__:
If *D*=0, the test gives no information: *f* could have a local max or min at (*a*,*b*) or (*a*,*b*) could be a saddle point of *f*.

Second Derivatives Test

Example:

Find the local minimum and maximum values and saddle points (if any) of each function.

#12. $f(x,y) = x^2 + y^2 + 2xy^2$

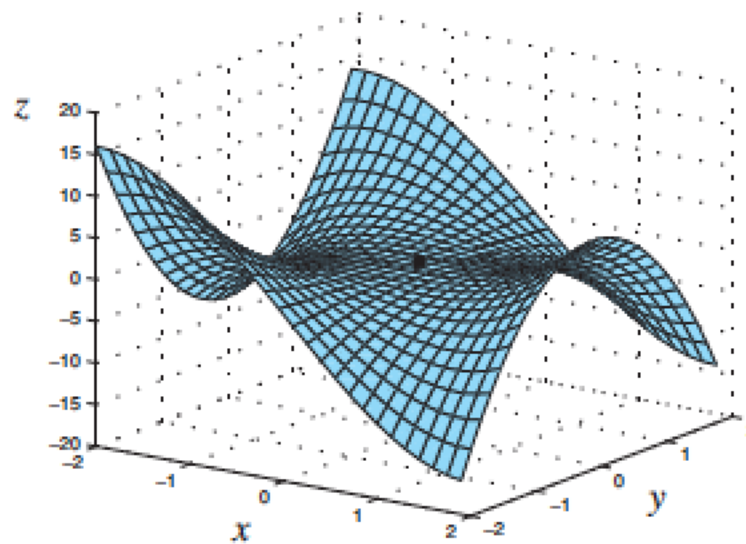
#16. $f(x,y) = xye^{-x-y}$

When The Second Derivatives Test Does Not Apply

Example:

For the function to the right, it can be shown that (0,0) is the only critical point of *f* and that D(0,0)=0 and so the second derivatives test is inconclusive.

$$f(x, y) = x^3 - 3xy^2$$



What is (0,0)?