COMPSCI/SFWRENG 2FA3
Discrete Mathematics with Applications II
Winter 2020

Week 05 Exercises

Dr. William M. Farmer
McMaster University

Revised: February 6, 2020

Background Definitions

Consider the following definitions:

1. $\Sigma_{\text{stack}} = (\mathcal{B}, \mathcal{C}, \mathcal{F}, \mathcal{P}, \tau)$ where:

   a. $\mathcal{B} = \{\mathsf{Element}, \mathsf{Stack}\}$.
   b. $\mathcal{C} = \{\mathsf{error}, \mathsf{bottom}\}$.
   c. $\mathcal{F} = \{\mathsf{push}, \mathsf{pop}, \mathsf{top}\}$.
   d. $\mathcal{P} = \emptyset$.
   e. $\tau(\mathsf{error}) = \mathsf{Element}$.
   f. $\tau(\mathsf{bottom}) = \mathsf{Stack}$.
   g. $\tau(\mathsf{push}) = \mathsf{Element} \times \mathsf{Stack} \rightarrow \mathsf{Stack}$.
   h. $\tau(\mathsf{pop}) = \mathsf{Stack} \rightarrow \mathsf{Stack}$.
   i. $\tau(\mathsf{top}) = \mathsf{Stack} \rightarrow \mathsf{Element}$.

2. $\Sigma_{\text{mon}} = (\{M\}, \{e\}, \{*\}, \emptyset, \tau)$ where $\tau(e) = M$ and $\tau(*) = M \times M \rightarrow M$.

3. $\mathcal{M}_{\text{nat}}$ is the Σ_{mon}-structure derived from $(\mathbb{N}, 0, +)$.

4. $\mathcal{M}_{\text{triv}}$ is the Σ_{mon}-structure derived from the *trivial monoid* $(\{0\}, 0, +)$.

5. Let Γ_{mon} be the following set of Σ-sentences:

   Assoc $\forall\, x, y, z : M \,.\, (x * y) * z = x * (y * z)$.
   IdLeft $\forall\, x : M \,.\, e * x = x$.
   IdRight $\forall\, x : M \,.\, x * e = x$.

6. $T_{\text{mon}} = (\Sigma_{\text{mon}}, \Gamma_{\text{mon}})$.

Exercises

1. Let $\Sigma = (\alpha, a : \alpha, f : \alpha \times \alpha \to \alpha, p : \alpha \times \alpha \to \mathbb{B})$. Compute fvar and bvar for each of the following Σ-formulas:

   a. $\exists\, x : \alpha \,.\, \exists\, y : \alpha \,.\, p(z : \alpha)$.

   b. $f(x : \alpha) = a \wedge$
   $\forall\, y : \alpha \,.\, ((p\,(y : \alpha) \vee p\,(x : \alpha)) \Rightarrow \exists\, x : \alpha \,.\, p(f(x : \alpha)))$.

2. Compute the following substitutions:

   a. $f(x : \alpha) = a \wedge$
   $\forall\, y : \alpha \,.\, ((p(y : \alpha) \vee p(x : \alpha)) \Rightarrow \exists\, x : \alpha \,.\, p(f(x : \alpha)))[x \mapsto f(a)]$.

   b. $f(x : \alpha) = a \wedge$
   $\forall\, y : \alpha \,.\, ((p(y : \alpha) \vee p(x : \alpha)) \Rightarrow \exists\, x : \alpha \,.\, p(f(x : \alpha)))[y \mapsto f(a)]$.

   c. $f(x : \alpha) = a \wedge$
   $\forall\, y : \alpha \,.\, ((p(y : \alpha) \vee p\,(x : \alpha)) \Rightarrow \exists\, x : \alpha \,.\, p(f(x : \alpha)))[z \mapsto f(a)]$.

3. Construct a signature of MSFOL that is suitable for formalizing:

   a. A queue of abstract elements.

   b. An abstract field.

   c. An abstract vector space over an abstract field.

4. Let $\Sigma_{\text{ord}} = (\mathcal{B}, \mathcal{C}, \mathcal{F}, \mathcal{P}, \tau)$ be the signature defined in the lecture slides. Construct Σ_{ord}-structures that define the following mathematical structures: $(\mathbb{N}, \leq)$, $(\mathbb{Z}, <)$, $(\mathbb{Q}, >)$, and $(\mathbb{R}, \neq)$.

5. Let Σ_{stack} be the signature defined above. Construct a Σ_{stack}-structure such that $D_{\mathsf{Element}} = \mathbb{N}$, D_{Stack} is the set of finite sequences of members of $\mathbb{N}$, and the function symbols of Σ_{stack} manipulate the members of D_{Stack} as stacks.

6. Which of the following Σ_{mon}-formulas are satisfiable and which are universally valid?

   a. $e = e$.

   b. $e = e * e$.

   c. $\forall\, x : M \,.\, x = e$.

   d. $\forall\, x : M \,.\, x \neq e$.

7. Which of the following Σ_{mon}-formulas are valid in $\mathcal{M}_{\text{nat}}$ and which are valid in $\mathcal{M}_{\text{triv}}$?

   a. $e = e$.

   b. $e = e * e$.

c. $\forall x : M \,.\, x = e$.

d. $\forall x : M \,.\, x \neq e$.

8. Which are the following Σ_{mon}-formulas are valid in T_{mon}.

   a. $e = e$.

   b. $e = e * e$.

   c. $\forall x : M \,.\, x = e$.

   d. $\forall x : M \,.\, x \neq e$.