MATHEMATICS 1LS3 TEST 3

Day Class
Duration of Test: 60 minutes
McMaster University

E. Clements

6 March 2013

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40

FIRST NAME (please print) : Sol^{NS}
FAMILY NAME (please print) : ____________
Student No.: ____________

THIS TEST HAS 8 PAGES AND 10 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE.

Total number of points is 40. Marks are indicated next to the problem number in square brackets. Any Casio fx991 (or lower, non-graphing) calculator is allowed.

USE PEN TO WRITE YOUR TEST. IF YOU USE A PENCIL, YOUR TEST WILL NOT BE ACCEPTED FOR REMARKING (IF NEEDED).

You need to show work to receive full credit, except for Multiple Choice.

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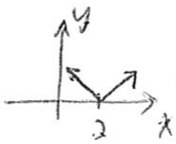
1. State whether each statement is **true or false** and then **explain** your reasoning.

(a) [2] If $\lim_{x\to 2} f(x) = f(2)$, then $f(x)$ is differentiable at $x = 2$. True or false? Explain.

FALSE!

$\lim_{x\to 2} f(x) = f(2) \Rightarrow f$ is continuous at $x=2$, but not necessarily differentiable at $x=2$



ex. $f(x) = |x-2|$ is continuous everywhere but not differentiable at $x=2$ (graph has a corner)

(b) [2] The largest slope of the graph of $y = \cos x$ is 1. True or false? Explain.

TRUE!

$y' = -\sin x \leftarrow$ determines slopes of graph of $y = \cos x$

$-1 \le -\sin x \le 1 \Rightarrow$ largest slope can be is 1

2. **Multiple Choice.** Clearly **circle** the one correct answer.

(a) [3] Which of the following statements is/are true, given that x approaches ∞?

(I) x^4 approaches ∞ faster than $100x^3$ ✓
(II) $\ln x$ approaches ∞ faster than $\sqrt{x}$ ✗
(III) x^5 approaches ∞ faster than $e^{0.1x}$ ✗

(A) none (B) I only (C) II only (D) III only
(E) I and II (F) I and III (G) II and III (H) all three

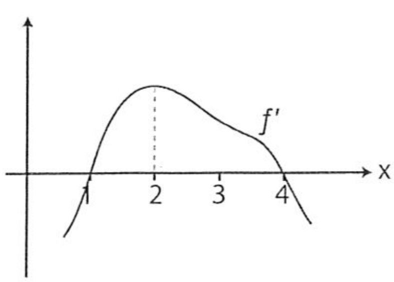
(b) [3] If $g(x) = \arctan(x^3) - e^{x^2-x}$, then $g'(1)$ is

(A) 0 (B) 1 (C) 1/2 (D) 0
(E) 2/3 (F) 3 (G) $\pi/2$ (H) 2

$$g' = \frac{3x^2}{1+(x^3)^2} - e^{x^2-x}\cdot(2x-1)$$

$$g'(1) = \frac{3}{2} - \underbrace{e^0}_{=1}\cdot(1) = \frac{1}{2}$$

(c) [3] The graph of the **derivative** $f'(x)$ of a function $f(x)$ is given. Which statements is/are true for $f(x)$?



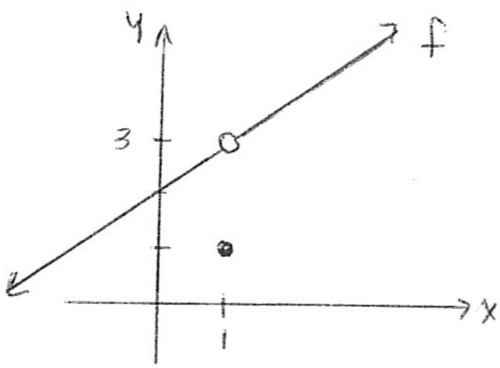
(I) $x = 1$ is a critical point of $f(x)$ ✓ ($x=4$ is too!)
(II) $x = 2$ is a critical point of $f(x)$ ✗
(III) $f(x)$ is increasing on $(1, 4)$ ✓

(A) none (B) I only (C) II only (D) III only
(E) I and II (F) I and III (G) II and III (H) all three

3. (a) [1] Write an equation that expresses the fact that a function $f(x)$ is continuous at $x = 1$.

$$\lim_{x \to 1} f(x) = f(1)$$

(b) [1] Sketch the graph of a function that satisfies $\lim_{x \to 1} f(x) = 3$, and that is **not** continuous at $x = 1$.



← one possible example

(c) [2] Using the definition in (a), show that the function $f(x) = \begin{cases} \dfrac{x^2 - 3x}{|x-3|} & \text{when} \quad x \neq 1 \\ -1 & \text{when} \quad x = 1 \end{cases}$ is continuous at $x = 1$.

① $\lim_{x \to 1} f(x) = \lim_{x \to 1} \dfrac{x^2 - 3x}{|x-3|} = \lim_{x \to 1} \dfrac{x(x-3)}{-(x-3)} = \lim_{x \to 1} (-x) = -1$

② $f(1) = -1$

③ $\because \lim_{x \to 1} f(x) = f(1)$ $\therefore$ f is continuous at $x = 1$ (by defn)

4. (a) [4] Using the **limit definition**, find the derivative of $f(x) = \dfrac{x-1}{x+3}$.

$$f'(x) = \lim_{h \to 0} \frac{\frac{(x+h)-1}{(x+h)+3} - \frac{x-1}{x+3}}{h}$$

$$= \lim_{h \to 0} \frac{(x+h-1)(x+3) - (x-1)(x+h+3)}{(x+h+3)(x+3)} \cdot \frac{1}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + \cancel{3x} + \cancel{xh} + 3h - \cancel{x} - \cancel{3} - \cancel{x^2} - \cancel{xh} - \cancel{3x} + \cancel{x} + h + \cancel{3}}{(x+h+3)(x+3) \cdot h}$$

$$= \lim_{h \to 0} \frac{4\cancel{h}}{(x+h+3)(x+3)\cancel{h}}$$

$$= \frac{4}{(x+3)^2}$$

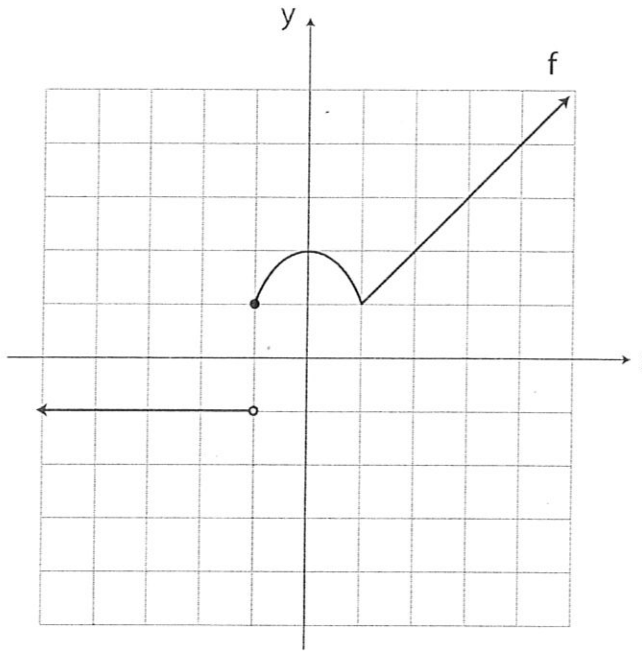
(b) [2] Determine the equation of the tangent line to the graph of $f(x) = \dfrac{x-1}{x+3}$ at $x = 1$.

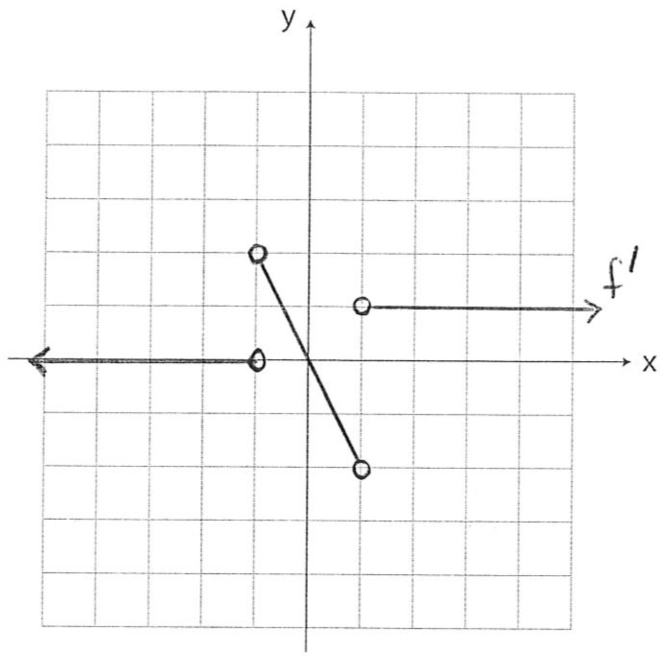
$f(1) = 0 \Rightarrow (x_1, y_1) = (1, 0)$

$f'(1) = \frac{4}{4^2} = \frac{1}{4}$

$y - 0 = \frac{1}{4}(x - 1) \Rightarrow y = \frac{1}{4}x - \frac{1}{4}$

5. [4] Looking at the graph of $f(x)$, sketch the graph of $f'(x)$.





6. [2] Use the table of values below to compute $(f \circ g)'(1)$. Chain Rule

x	f	g	f'	g'
0	3	6	4	8
1	6	0	2	5

$$(f \circ g)'(1) = f'(g(1)) \cdot g'(1)$$
$$= f'(0) \cdot g'(1)$$
$$= 4 \cdot 5$$
$$= 20$$

7. [2] Using the Quotient Rule, prove that $\frac{d}{dx}(\tan x) = \sec^2 x$.

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$$

$$= \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

$$= \sec^2 x$$

8. [3] Determine the points on the graph of $f(x) = \frac{e^{3x}}{x^2}$ where the tangent is horizontal.

$$f' = \frac{e^{3x} \cdot 3 \cdot x^2 - e^{3x} \cdot 2x}{x^4} = \frac{e^{3x} \cdot \cancel{x} \cdot [3x-2]}{x^{\cancel{4}\,3}} = \frac{e^{3x}(3x-2)}{x^3}$$

$f' = 0$ when $3x - 2 = 0$

$$x = \frac{2}{3}$$

$$f\left(\tfrac{2}{3}\right) = \frac{e^2}{\left(\frac{2}{3}\right)^2} = \frac{9}{4}e^2$$

$\therefore$ At the point $\left(\frac{2}{3}, \frac{9}{4}e^2\right)$ the tangent is horizontal.

9. [3] Determine all critical numbers of the function $g(x) = x^{\frac{1}{3}}(x-1)$.

$$g' = \frac{1}{3}x^{-2/3}\cdot(x-1) + x^{1/3}\cdot(1) = \frac{x-1}{3x^{2/3}} + \frac{x^{1/3}\cdot 3x^{2/3}}{3x^{2/3}} = \frac{4x-1}{3x^{2/3}}$$

$g' = 0$ when $4x-1=0$
$x = \frac{1}{4}$

g' dne when $3x^{2/3} = 0$
$x = 0$

Since 0 and $\frac{1}{4}$ are both in the domain of g, they are both critical #s of g.

10. [3] Determine the interval on which the function $f(x) = x \ln x$ is increasing.

$f' = 1\ln x + x\cdot\frac{1}{x} = \ln x + 1$

$f' = 0$ when $\ln x + 1 = 0$
$\ln x = -1$
$x = e^{-1}$

f' dne when $x \leq 0$

	e^{-2}	$\frac{1}{e}$	1	x
f'	$-$		$+$	
f	$\searrow$		$\nearrow$	

$\therefore$ f is increasing on $(\frac{1}{e}, \infty)$.