MATH 1AA3/1ZB3 Test #1, Seating #1 Full Solutions
Versions #1-4, Alternate & SAS #5
(*Questions sorted by course topic order*)

1. Evaluate the improper integral: $\displaystyle\int_2^\infty \frac{\ln(x)}{x^2}\,dx$ or state it is divergent.

Solution:

Integrating by parts, we get that:

$$\int \frac{\ln(x)}{x^2}\,dx = \int x^{-2}\ln(x)\,dx = \ln(x)(-1)x^{-1} - \int(-1)(x^{-1})(x^{-1})\,dx = -\frac{\ln(x)}{x} + \int x^{-2}\,dx = -\frac{(\ln(x)+1)}{x} + C$$

So our Type I improper integral becomes:

$$\int_2^\infty \frac{\ln(x)}{x^2}\,dx = \lim_{b\to\infty} -\left.\frac{(\ln(x)+1)}{x}\right|_2^b = \frac{(\ln(2)+1)}{2} - \lim_{b\to\infty}\frac{(\ln(b)+1)}{b} = \frac{(\ln(2)+1)}{2}$$ since $\ln(x)$ grows slower than x,

or explicitly, since $\displaystyle\lim_{b\to\infty}\frac{(\ln(b)+1)}{b} \overset{H}{=} \lim_{b\to\infty}\frac{1/b}{1} = \lim_{b\to\infty}\frac{1}{b} = 0$

Answer: $(\ln(2)+1)/2$

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2. For what values of k does the integral $\displaystyle\int_{-1}^5 (5-x)^{k/2}\,dx$ converge?

Solution:

Let's do a substitution: $u = 5 - x$ so we get: $\displaystyle\int_{-1}^5 (5-x)^{k/2}\,dx = \int_6^0 u^{k/2}(-1)\,du = -\int_0^6 \frac{1}{u^{-k/2}}\,du$.

Notice that this is a multiple of a classic Type II improper p-integral, so it converges if $-k/2 < 1$, or $k > -2$.

Answer: $k > -2$

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3. Which of the following improper integrals converge? I) $\displaystyle\int_1^\infty \arctan(x)e^{-x}\,dx$ II) $\displaystyle\int_1^\infty \frac{1+e^{-x}}{\sqrt{x}}\,dx$

Solution:

I) $0 \le \arctan(x)e^{-x} \le \frac{\pi}{2}e^{-x}$, and $\displaystyle\int_1^\infty e^{-x}\,dx = -\lim_{b\to\infty} e^{-x}\Big|_1^b = \frac{1}{e}$, ie. convergent.

So $\displaystyle\int_1^\infty \arctan(x)e^{-x}\,dx$ is **convergent** by Integral Comparison

II) $0 \le \frac{1}{\sqrt{x}} \le \frac{1+e^{-x}}{\sqrt{x}}$, and $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ is a divergent Type I p-integral, $p \le 1$, so it is divergent.

So $\int_1^{\infty} \frac{1+e^{-x}}{\sqrt{x}}\,dx$ is **divergent** by Integral Comparison

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4. Which of the following sequences converge? I) $a_n = \frac{3}{n}$ II) $b_n = (-1)^n \left(\frac{4n}{\sqrt{n^2+3}}\right)$

Solution:

I) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3}{n} = 0$ that is, $a_n = \frac{3}{n}$ is a **convergent sequence**

II) $\lim_{n\to\infty} |\, b_n \,| = \lim_{n\to\infty} \frac{4n}{\sqrt{n^2+3}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{3}{n}}} = 1 \neq 0$ Since $b_n = (-1)^n \left(\frac{4n}{\sqrt{n^2+3}}\right)$ is an alternating sequence,

it is a **divergent sequence.**
(Specifically values alternate between almost 1 and almost –1 as n goes to infinity.)

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5. If the series defined by $a_{n+1} = \frac{a_n^2+3}{4}$, $a_1 = 2$ is convergent, what is the limit?

Solution:

If the sequence converges, then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = L$ for some value of L.
Taking the limit of both sides of the recursion relation, we get:
$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{a_n^2+3}{4}$ becomes $L = \frac{L^2+3}{4}$ or $L^2 - 4L + 3 = 0 = (L-3)(L-1)$

So L = 1 or 3. But which is it?
$a_1 = 2$, $a_2 = (4+3)/4 = 7/4$, etc., so it appears to be decreasing. (Prove by induction, perhaps!)

So if the limit exists, it's lower than $a_1 = 2$.

Answer: $L = 1$

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6. Given the sequence: $b_{n+1} = \frac{2}{5-b_n}$, $b_1 = 3$, we wish to show that it is monotonic using mathematical induction. Which of the following statements corresponds to a possible induction step?

Solution:

To show it's monotonic, we first need to know if we're to show it's monotonic increasing or monotonic decreasing.
$b_1 = 3$, $b_2 = \frac{2}{5-b_1} = \frac{2}{5-3} = 1 < 3$, so $b_1 > b_2$. At least initially we're decreasing.

To show it keeps decreasing, we need to assume that if it decreases from b_k to b_{k+1}, that it also decreases from b_{k+1} to b_{k+2}. And the only way we have to do this is using the recursion relation, so:

$b_k > b_{k+1}$ implies $\dfrac{2}{5-b_k} > \dfrac{2}{5-b_{k+1}}$ which implies $b_{k+1} > b_{k+2}$ Or in other words:

Answer: Assume $b_k > b_{k+1}$, and show $\dfrac{2}{5-b_k} > \dfrac{2}{5-b_{k+1}}$

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7. For the series $\sum_{n=1}^{\infty} b_n$ the m^{th} partial sum is given by: $S_m = \dfrac{m+2}{3m}$. Find the value of $b_2 + b_3$.

Solution:

$S_m = \sum_{n=1}^{m} b_n = \dfrac{m+2}{3m}$, so $S_3 = \dfrac{3+2}{3(3)} = \dfrac{5}{9} = b_1 + b_2 + b_3$, and $S_1 = \dfrac{1+2}{3(1)} = 1 = b_1$

Then $b_2 + b_3 = S_3 - S_1 = 5/9 - 1 = -4/9$

Answer: $-4/9$

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8. Find the sum of the series, $\sum_{n=1}^{\infty} \dfrac{2^{2n}}{5^{n+1}}$ or state it is divergent.

Solution:

This is a geometric series: $\sum_{n=1}^{\infty} \dfrac{2^{2n}}{5^{n+1}} = \sum_{n=1}^{\infty} \dfrac{4^n}{5^{n+1}} = \sum_{n=1}^{\infty} \dfrac{4}{5^2}\left(\dfrac{4}{5}\right)^{n-1}$. So $a = \dfrac{4}{25}, r = \dfrac{4}{5}$. Since $|r| < 1$ it converges to

$$\frac{a}{1-r} = \frac{4/25}{1-4/5} = \frac{4/25}{1/5} = \frac{4}{5}.$$

Answer: 4/5

Equivalently, the series is geometric, so $\sum_{n=1}^{\infty} \dfrac{2^{2n}}{5^{n+1}} = \dfrac{4}{25} + \dfrac{16}{125} + \ldots = a + ar + \ldots$ So $a = \dfrac{4}{25}, r = \dfrac{4}{5}$, etc.

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9. Which of the following series converge? I) $\sum_{n=2}^{\infty} \dfrac{1}{n\ln(n)}$ II) $\sum_{n=2}^{\infty} \left(\dfrac{1}{2n} - \dfrac{1}{2n-2}\right)$

Solution:

I) Let $f(x) = \dfrac{1}{x\ln(x)}$. The function $f(x)$ is positive, continuous and decreasing for $n > 1$.

If we let $u = \ln(x)$, then $\int_2^{\infty} \dfrac{1}{x\ln(x)}dx = \int_{\ln(2)}^{\infty} \dfrac{1}{u}\,du$, a divergent Type I p-integral ($p = 1 \le 1$)

Thus $\sum_{n=2}^{\infty} \dfrac{1}{n\ln(n)}$ is **divergent** by the Integral comparison test.

(Note, the ratio/root tests fail here, as do most obvious uses of the comparison and limit comparison tests. And since the terms go to zero, the divergence test does not apply.)

II) $\sum_{n=2}^{\infty}\left(\frac{1}{2n}-\frac{1}{2n-2}\right)$ is a telescopic series, with terms which approach 0 at infinity. So it is **convergent**.

Explicitly,

$$S_m=\sum_{n=2}^{m}\left(\frac{1}{2n}-\frac{1}{2n-2}\right)=\cancel{\frac{1}{4}}-\frac{1}{2}+\cancel{\frac{1}{6}}-\cancel{\frac{1}{4}}+\cancel{\frac{1}{8}}-\cancel{\frac{1}{6}}+\ldots+\cancel{\frac{1}{2m-2}}-\cancel{\frac{1}{2m-4}}+\frac{1}{2m}-\cancel{\frac{1}{2m-2}}$$

$$=-\frac{1}{2}+\frac{1}{m}\to-\frac{1}{2},\text{ as } m \text{ goes to infinity.}$$

Alternatively, $\sum_{n=2}^{\infty}\left(\frac{1}{2n}-\frac{1}{2n-2}\right)=-\frac{1}{2}\sum_{n=2}^{\infty}\frac{1}{n(n-1)}=-\frac{1}{2}\sum_{n=2}^{\infty}\frac{1}{n^2-n}$, which converges like $\sum 1/n^2$.

since $\frac{1}{n^2-n}\geq 0$, and $\lim_{n\to\infty}\frac{\frac{1}{n^2-n}}{\frac{1}{n^2}}=\lim_{n\to\infty}\frac{n^2}{n^2-n}=\lim_{n\to\infty}\frac{1}{1-\frac{1}{n}}=1$ (Limit comparison test)

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10. If the sum of the series, $\sum_{n=1}^{\infty}\frac{1}{4n^3}$ is approximated by $S_m=\sum_{n=1}^{m}\frac{1}{4n^3}$, find the smallest possible m such that the **integral error estimate** says $S-S_m<0.01$?

Solution:

We can use our integral estimate here since if $f(x)=1/4x^3$, $f(n)=a_n$, and $f(x)$ is a positive, continuous and decreasing function.

So by the **integral error estimate,** $S-S_m\leq\int_m^{\infty}\frac{1}{4x^3}dx<0.01$, so $\frac{1}{8m^2}<\frac{1}{100}$ or equivalently $m^2>12.5$

Answer: $m=4$

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11. Which of the following series converge? I) $\sum_{n=1}^{\infty}\cos\left(\frac{1}{\sqrt{n}}\right)$ II) $\sum_{n=1}^{\infty}\frac{\sin^4(n)}{n^{4/3}}$

Solution:

I) $\lim_{n\to\infty}\cos\left(\frac{1}{\sqrt{n}}\right)=\cos(0)=1\neq 0$, so by the Divergence test, $\sum_{n=1}^{\infty}\cos\left(\frac{1}{\sqrt{n}}\right)$ **diverges**

II) $0\leq\frac{\sin^4(n)}{n^{4/3}}\leq\frac{1}{n^{4/3}}$, and $\sum_{n=1}^{\infty}\frac{1}{n^{4/3}}$ is a convergent p-series (with $p=4/3\geq 1$), so $\sum_{n=1}^{\infty}\frac{\sin^4(n)}{n^{4/3}}$ **converges** by comparison test.

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12. Which of the following series converge? I) $\sum_{n=1}^{\infty} \frac{5}{n^{1.1}}$ II) $\sum_{n=1}^{\infty} \frac{3n+n^3}{n^4+n}$

Solution:

I) $\sum_{n=1}^{\infty} \frac{5}{n^{1.1}}$ is a multiple of a p-series, $p = 1.1 > 1$, so it is **convergent**.

II) Informally, $\sum_{n=1}^{\infty} \frac{3n+n^3}{n^4+n}$ has $a_n = \frac{3n+n^3}{n^4+n} \approx \frac{n^3}{n^4} = \frac{1}{n}$ for large n. So it behaves approximately like a **divergent** p-series with $p = 1 \le 1$.

Or, more formally using a Limit comparison test with the series $\sum 1/n$, we get:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(\frac{3n+n^3}{n^4+n}\right)}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n(3n+n^3)}{n^4+n} = \lim_{n\to\infty} \frac{\frac{3}{n^2}+1}{1+\frac{1}{n^3}} = 1$$

So $\sum_{n=1}^{\infty} \frac{3n+n^3}{n^4+n}$ diverges like our $p = 1$ p-series.

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13. If the sum of the series, $\sum_{n=1}^{\infty} \frac{(-1)^n}{4n^3}$ is approximated by $S_3 = \sum_{n=1}^{3} \frac{(-1)^n}{4n^3}$, which of the following numbers does the **alternating series error estimate** give as a bound of the magnitude of the remainder, $|S - S_3|$?

Solution:

For the alternating series error estimate, $|S - S_m| \le b_{m+1}$. Here $m = 3$, and $b_n = \frac{1}{4n^3}$ so we get:

$$|S - S_3| \le b_4 = \frac{1}{4(4)^3} = \frac{1}{4^4} = \frac{1}{2^8} = \frac{1}{256}$$

Answer: $\frac{1}{256}$

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14. Which of the following converges absolutely: I) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ II) $\sum_{n=1}^{\infty} \frac{(-1)^n}{5^n}$

Solution:

I) $\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent p-series ($p \le 1$) so $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is **not absolutely convergent**

(Specifically since it also converges by alternating series test, whereas the positive term version diverges, this series is in fact conditionally convergent.)

II) $\sum_{n=1}^{\infty} \frac{1}{5^n}$ is a geometric series, $|r| = 1/5 < 1$, so $\sum_{n=1}^{\infty} \frac{(-1)^n}{5^n}$ is **absolutely convergent.**

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15. The series, $\sum_{n=1}^{\infty} (-1)^n b_n$ converges and $b_n > 0$, consider:

I) The sequence given by the terms, b_n II) The series $\sum_{n=1}^{\infty} b_n$

Which of the following statements must be true?

a) I must converge, II may or may not converge.
b) II must converge, I may or may not converge.
c) Both I and II must converge
d) Both I and II may or may not converge
e) Both must diverge

Solution:

We're given $\sum_{n=1}^{\infty} (-1)^n b_n$ is an alternating series which converges. But it might only be conditionally convergent.

So the positive term version: $\sum_{n=1}^{\infty} b_n$ may not converge. But since our series converges, we know that the limit of our terms, $\lim_{n\to\infty} (-1)^n b_n = 0 = \lim_{n\to\infty} b_n$, so the sequence, b_n, is convergent.

Answer: The sequence, b_n, must converge, but the series $\sum_{n=1}^{\infty} b_n$ may or may not converge.

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16. Which of the following series converge? I) $\sum_{n=1}^{\infty} n^{-n/5}$ II) $\sum_{n=1}^{\infty} \frac{(2n)!}{3^n n!}$

I) For $\sum_{n=1}^{\infty} n^{-n/5}$ since it's a function if n to a power of n, we apply the ratio test.

$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{n^{-n/5}} = \lim_{n\to\infty} n^{-1/5} = \lim_{n\to\infty} \frac{1}{n^{1/5}} = 0 < 1$. So the series $\sum_{n=1}^{\infty} n^{-n/5}$ is **convergent**

II) For $\sum_{n=1}^{\infty} \frac{(2n)!}{3^n n!}$, since we have powers of constants, and factorials, we're best off using the ratio test.

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\left(\frac{(2(n+1))!}{3^{n+1}(n+1)!}\right)}{\left(\frac{(2n)!}{3^n n!}\right)} = \lim_{n\to\infty} \frac{(2n+2)!}{3^{n+1}(n+1)!} \frac{3^n n!}{(2n)!} = \lim_{n\to\infty} \frac{1}{3} \frac{(2n+2)!}{(2n)!} \frac{n!}{(n+1)!}$$

$$= \frac{1}{3} \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{n+1} = \frac{2}{3} \lim_{n\to\infty} (2n+1) = \infty > 1$$

So our series $\sum_{n=1}^{\infty} \frac{(2n)!}{3^n n!}$ is **divergent**

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17. Which of the following represents all real values of r such that the series $\sum_{n=0}^{\infty} 2^{nr}$ converges?

Solution:

We can re-write our series as $\sum_{n=0}^{\infty} 2^{nr} = \sum_{n=0}^{\infty} (2^r)^n$. So this is a geometric series with a ratio of 2^r, which converges if $2^r = |2^r| < 1 = 2^0$, so $2^r < 2^0$, so $r < 0$

Answer: $r < 0$

Equivalently, you could perform the ratio or root test. For instance the ratio test gives the result:

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2^{r(n+1)}}{2^{rn}} = \lim_{n\to\infty} \frac{2^{rn+r}}{2^{rn}} = 2^r < 1$, so it converges if $r < 0$ (and not if $r = 0$, since in that case we would have $\sum 1$, which clearly diverges.)

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18. Find the interval of convergence of the power series, $\sum_{n=1}^{\infty} \frac{x^n}{2^n \sqrt{n+1}}$.

Solution:

As usual we first do the ratio test:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{2^{n+1}\sqrt{n+2}}}{\frac{x^n}{2^n\sqrt{n+1}}} \right| = \lim_{n\to\infty} \frac{1}{2} \sqrt{\frac{n+1}{n+2}} |x| = \frac{|x|}{2} < 1 \quad \text{so} \quad -2 < x < 2$$

And now we just need to check the endpoints for convergence:

At $x = -2$ we get $\sum_{n=1}^{\infty} \frac{(-2)^n}{2^n \sqrt{n+1}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$. Notice this series is alternating, with decreasing magnitude terms which approach 0, so the series converges. (By Alternating series test)

At $x = +2$ we get $\sum_{n=1}^{\infty} \frac{(2)^n}{2^n \sqrt{n+1}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}$. And we can compare this to the p-series with $p = 1/2$:

$$\lim_{n\to\infty}\frac{\frac{1}{\sqrt{n+1}}}{\frac{1}{\sqrt{n}}}=\lim_{n\to\infty}\sqrt{\frac{n}{n+1}}=1$$ so our series and $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ both behave the same by Limit comparison test. and p = 1/2 < 1, so both diverge.

Thus our power series converges at x = –2 and diverges at x = 2, or the interval of convergence is $[-2,\ 2)$

Answer: $[-2,\ 2)$

(Also notice, since the ratio test always "loses" any terms that grow slower than exponential rate, by inspection we can see that the power series behaves like $a_n = |x|/2$ under the ratio test. So our radius will be R = 1/(1/2) = 2. To understand the endpoints, we look at the non-exponentially growing terms, and we have $\frac{1}{\sqrt{n+1}}$. For large n this the terms are essentially $\frac{1}{\sqrt{n}}=\frac{1}{n^{1/2}}$. Thus when positive it diverges like a positive term p-series, since p = 1/2 < 1, but since the terms are decreasing and approaching 0, the series will still converge at the left endpoint as an alternating series.)

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19. The power series, $\sum_{n=1}^{\infty} b_n(x-1)^n$ converges at x = 2 and diverges at x = – 3 .

What are the minimum and maximum possible values of its radius of convergence?

Solution:

The series $\sum_{n=1}^{\infty} b_n(x-1)^n$ has a centre of x = 1. If it converges also at x = 2, then the radius of convergence, $R \geq |1-2| = 1$. If it diverges at x = – 3 , then $R \leq |-3-1| = 4$.

Answer: min 1, max 4

MATH 1AA3/1ZB3 Test #1, Seating #2 Full Solutions
Versions 1– 4 (Version 5 – 9 on childsmath.ca)
(*Questions sorted by course topic order*)

1. Evaluate the improper integral: $\int_0^2 x^3 \ln(x)\,dx$ or state it is divergent.

Solution:

Integrating by parts, we get that:

$$\int x^3 \ln(x)\,dx = \ln(x)\left(\frac{1}{4}\right)x^4 - \int\left(\frac{1}{4}\right)(x^4)(x^{-1})\,dx = \frac{x^4\ln(x)}{4} - \frac{1}{4}\int x^3\,dx = \frac{x^4\ln(x)}{4} - \frac{x^4}{16} + C$$

So our Type II improper integral becomes:

$$\int_0^2 x^3 \ln(x)\,dx = \lim_{a\to 0^+} \left.\frac{x^4\ln(x)}{4} - \frac{x^4}{16}\right|_a^2 = 4\ln(2) - 1 - \lim_{a\to 0^+}\frac{a^4\ln(a)}{4} = 4\ln(2) - 1 \text{ since}$$

$$\lim_{a\to 0^+}\frac{a^4\ln(a)}{4} = \lim_{a\to 0^+}\frac{\ln(a)}{(4/a^4)} \overset{H}{=} \lim_{a\to 0^+}\frac{1/a}{-12\,/\,a^5} = -\lim_{a\to 0^+}\frac{a^4}{12} = 0$$

Answer: $4\ln(2) - 1$

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2. For what values of k does the integral $\int_{-1}^5 \left(\sqrt{x+1}\right)^k dx$ converge?

Solution:

Let's do a substitution: $u = x + 1$ so we get: $\int_{-1}^5 \left(\sqrt{x+1}\right)^k dx = \int_0^6 u^{k/2}\,du = \int_0^6 \frac{1}{u^{-k/2}}\,du\,.$

Notice that this is a multiple of a classic Type II improper p-integral, so it converges if $-k/2 < 1$, or $k > -2$.

Answer: $k > -2$

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3. Which of the following improper integrals converge? I) $\int_1^\infty \left(1+\frac{1}{x}\right)e^{-x}\,dx$ II) $\int_1^\infty \frac{\arctan(x)}{x^7}\,dx$

Solution:

I) $0 \le \left(1+\frac{1}{x}\right)e^{-x} \le 2e^{-x}$, and $\int_1^\infty e^{-x}\,dx = -\lim_{b\to\infty} \left. e^{-x}\right|_1^b = \frac{1}{e}$, ie. convergent.

So $\int_1^\infty \left(1+\frac{1}{x}\right)e^{-x}\,dx$ is **convergent** by Integral Comparison

II) $0 \le \dfrac{\pi/4}{x^7} \le \dfrac{\arctan(x)}{x^7}$, for $x \ge 1$, and $\displaystyle\int_1^\infty \frac{1}{x^7}\,dx$ is a convergent Type I p-integral, $p \ge 1$, so it is convergent.

So $\displaystyle\int_1^\infty \frac{\arctan(x)}{x^7}\,dx$ is **divergent** by Integral Comparison

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4. Which of the following sequences converge? I) $a_n = \dfrac{1}{\sqrt{n}}$ II) $b_n = (-1)^n\left(\dfrac{2+n}{3n}\right)$

I) $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$ that is, $a_n = \dfrac{1}{\sqrt{n}}$ is a **convergent sequence**

II) $\displaystyle\lim_{n\to\infty} |\, b_n \,| = \lim_{n\to\infty} \frac{2+n}{3n} = \lim_{n\to\infty} \frac{\frac{2}{n}+1}{3} = \frac{1}{3} \ne 0$ Since $b_n = (-1)^n\left(\dfrac{2+n}{3n}\right)$ is an alternating sequence, it is a **divergent sequence.**
(Specifically values alternate between almost 1/3 and almost –1/3 as n goes to infinity.)

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5. If the series defined by $a_{n+1} = \dfrac{10 + a_n^2}{7}$, $a_1 = 3$ is convergent, what is the limit?

Solution:

If the sequence converges, then $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = L$ for some value of L.
Taking the limit of both sides of the recursion relation, we get:

$\displaystyle\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{10 + a_n^2}{7}$ becomes $L = \dfrac{10 + L^2}{7}$ or $L^2 - 7L + 10 = 0 = (L-2)(L-5)$

So L = 2 or 5. But which is it?
$a_1 = 3$, $a_2 = (10+9)/7 = 19/7 < 3$, etc., so it appears to be decreasing. (Prove by induction, perhaps!)

So if the limit exists, it's lower than $a_1 = 3$.

Answer: $L = 2$

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6. Given the sequence: $a_{n+1} = \dfrac{2}{5 - a_n}$, $a_1 = 1$, we wish to show that it is monotonic using mathematical induction. Which of the following statements corresponds to a possible induction step?

Solution:

To show it's monotonic, we first need to know if we're to show it's monotonic increasing or monotonic decreasing.

$a_1 = 1,\ a_2 = \dfrac{2}{5-a_1} = \dfrac{2}{5-1} = \dfrac{1}{2} < 1$, so $a_1 > a_2$. At least initially we're decreasing.

To show it keeps decreasing, we need to assume that if it decreases from a_k to a_{k+1}, that it also decreases from a_{k+1} to a_{k+2}. And the only way we have to do this is using the recursion relation, so:
$a_k > a_{k+1}$ implies $\dfrac{2}{5-a_k} > \dfrac{2}{5-a_{k+1}}$ which implies $a_{k+1} > a_{k+2}$ Or in other words:

Answer: Assume $a_k > a_{k+1}$, and show $\dfrac{2}{5-a_k} > \dfrac{2}{5-a_{k+1}}$

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7. For the series $\sum_{n=1}^{\infty} c_n$ the m^{th} partial sum is given by: $S_m = \dfrac{5m}{m+1}$. Find the value of $c_3 + c_4$.

Solution:

$S_m = \sum_{n=1}^{m} c_n = \dfrac{5m}{m+1}$, so $S_4 = \dfrac{5(4)}{4+1} = 4 = c_1 + c_2 + c_3 + c_4$, and $S_2 = \dfrac{5(2)}{2+1} = \dfrac{10}{3} = c_1 + c_2$
Then $b_2 + b_3 = S_3 - S_1 = 4 - 10/3 = 2/3$

Answer: 2/3

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8. Find the sum of the series, $\sum_{n=1}^{\infty} \dfrac{4^{n+1}}{3^{2n-1}}$ or state it is divergent.

Solution:

This is a geometric series: $\sum_{n=1}^{\infty} \dfrac{4^{n+1}}{3^{2n-1}} = \sum_{n=1}^{\infty} \dfrac{4^{n+1}}{9^n(3^{-1})} = \sum_{n=1}^{\infty} \dfrac{4^2}{9(3^{-1})}\left(\dfrac{4}{9}\right)^{n-1} = \sum_{n=1}^{\infty} \dfrac{16}{3}\left(\dfrac{4}{9}\right)^{n-1}$. So $a = \dfrac{16}{3}, r = \dfrac{4}{9}$.

Since $|r| < 1$ it converges to

$$\frac{a}{1-r} = \frac{16/3}{1-4/9} = \frac{16/3}{5/9} = \frac{48}{5}.$$

Answer: 48/5

Equivalently, the series is geometric, so $\sum_{n=1}^{\infty} \dfrac{4^{n+1}}{3^{2n-1}} = \dfrac{16}{3} + \dfrac{64}{27} + \ldots = a + ar + \ldots$ So $a = \dfrac{16}{3}, r = \dfrac{4}{9}$, etc.

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9. Which of the following series converge? I) $\sum_{n=2}^{\infty} \dfrac{1}{n\ln(n)}$ II) $\sum_{n=1}^{\infty} \left(\dfrac{1}{n+1} - \dfrac{1}{n}\right)$

Solution:

I) Let $f(x) = \dfrac{1}{x\ln(x)}$. The function $f(x)$ is positive, continuous and decreasing for $n > 1$.

If we let $u = \ln(x)$, then $\displaystyle\int_2^{\infty} \frac{1}{x\ln(x)}dx = \int_{\ln(2)}^{\infty} \frac{1}{u}\,du$, a divergent Type I p-integral ($p = 1 \le 1$)

Thus $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$ is **divergent** by the Integral comparison test.

(Note, the ratio/root tests fail here, as do most obvious uses of the comparison and limit comparison tests. And since the terms go to zero, the divergence test does not apply.)

II) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n}\right)$ is a telescopic series, with terms which approach 0 at infinity. So it is **convergent**.

Explicitly, $\displaystyle S_m = \sum_{n=1}^{m}\left(\frac{1}{n+1} - \frac{1}{n}\right) = \cancel{\frac{1}{2}} - \frac{1}{1} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{2}} + \cancel{\frac{1}{4}} - \cancel{\frac{1}{3}} + \ldots + \cancel{\frac{1}{m}} - \cancel{\frac{1}{m-1}} + \frac{1}{m+1} - \cancel{\frac{1}{m}}$

$\displaystyle = -1 + \frac{1}{m+1} \to -$, as m goes to infinity.

Alternatively, $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n}\right) = -\sum_{n=1}^{\infty}\frac{1}{n(n+1)} = -\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^2+n}$, which converges like $\sum 1/n^2$.

since $\dfrac{1}{n^2+n} \ge 0$, and $\displaystyle\lim_{n\to\infty}\frac{\frac{1}{n^2+n}}{\frac{1}{n^2}} = \lim_{n\to\infty}\frac{n^2}{n^2+n} = \lim_{n\to\infty}\frac{1}{1+\frac{1}{n}} = 1$ (Limit comparison test)

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10. If the sum of the series, $\displaystyle\sum_{n=1}^{\infty}\frac{1}{4n^3}$ is approximated by $\displaystyle S_3 = \sum_{n=1}^{3}\frac{1}{4n^3}$, which of the following numbers does the **integral error estimate** give as the upper bound of the remainder, $S - S_3$?

Solution:

We can use our integral estimate here since if $f(x) = 1/4x^3$, $f(n) = a_n$, and $f(x)$ is a positive, continuous and decreasing function.

So by the **integral error estimate,** $\displaystyle S - S_3 \le \int_3^{\infty}\frac{1}{4x^3}dx = \lim_{b\to\infty} -\frac{1}{8x^2}\bigg|_3^b = \frac{1}{8(3)^2} - \lim_{b\to\infty}\frac{1}{8b^2} = \frac{1}{72}$

Answer: 1/72

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11. Which of the following series converge? I) $\displaystyle\sum_{n=1}^{\infty}\cos\left(\frac{1}{n^2}\right)$ II) $\displaystyle\sum_{n=2}^{\infty}\frac{n+n^3}{n^5-4n}$

Solution:

I) $\lim_{n\to\infty} \cos\left(\frac{1}{n^2}\right) = \cos(0) = 1 \neq 0$, so by the Divergence test, $\sum_{n=1}^{\infty} \cos\left(\frac{1}{n^2}\right)$ **diverges**

II) Informally, $\sum_{n=2}^{\infty} \frac{n+n^3}{n^5-4n}$ has $a_n = \frac{n+n^3}{n^5-4n} \approx \frac{n^3}{n^5} = \frac{1}{n^2}$ for large n. So it behaves approximately like a **convergent** p-series with $p = 2 \geq 1$.

Or, more formally using a Limit comparison test with the series $\sum 1/n^2$, we get:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(\frac{n+n^3}{n^5-4n}\right)}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2(n+n^3)}{n^5-4n} = \lim_{n\to\infty} \frac{\frac{1}{n^2}+1}{1-\frac{4}{n^4}} = 1$$ So $\sum_{n=2}^{\infty} \frac{n+n^3}{n^5-4n}$ converges like the $p = 2$ p-series.

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12. Which of the following series converge? I) $\sum_{n=1}^{\infty} \frac{\cos^2(n)}{n^{3/2}}$ II) $\sum_{n=1}^{\infty} \frac{2}{n^{0.8}}$

Solution:

I) $0 \leq \frac{\cos^2(n)}{n^{3/2}} \leq \frac{1}{n^{3/2}}$, and $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is a convergent p-series (with $p = 3/2 \geq 1$), so $\sum_{n=1}^{\infty} \frac{\cos^2(n)}{n^{3/2}}$ **converges** by comparison test.

II) $\sum_{n=1}^{\infty} \frac{2}{n^{0.8}}$ is a multiple of a p-series, $p = 0.8 < 1$, so it is **divergent**.

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13. If the sum of the series, $\sum_{n=1}^{\infty} \frac{(-1)^n}{4n^3}$ is approximated by $S_m = \sum_{n=1}^{m} \frac{(-1)^n}{4n^3}$, find the smallest possible m such that the **alternating series error estimate** says $|S - S_m| < 0.01$?

Solution:

For the alternating series error estimate, $|S - S_m| \leq b_{m+1}$. Here $b_n = \frac{1}{4n^3}$ so we get:

$$|S - S_m| \leq b_{m+1} = \frac{1}{4(m+1)^3} < 0.01, \text{ so } 4(m+1)^3 > 100 \text{ and } (m+1)^3 \geq 25$$

Answer: $m = 2$

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14. Which of the following converges absolutely: I) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$ II) $\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n}$

Solution:

I) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a divergent p-series ($p \le 1$) so $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$ is **not absolutely convergent**

(Specifically since it also converges by alternating series test, whereas the positive term version diverges, this series is in fact conditionally convergent.)

II) $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a geometric series, $|r| = 1/2 < 1$, so $\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n}$ is **absolutely convergent.**

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15. Given the series, $\sum_{n=1}^{\infty} b_n$ converges and all $b_n > 0$, consider:

I) The series $\sum_{n=1}^{\infty} (-1)^n b_n$ II) The sequence given by the terms, b_n

Which of the following statements must be true?

a) I must converge, II may or may not converge.
b) II must converge, I may or may not converge.
c) Both I and II must converge
d) Both I and II may or may not converge
e) Both must diverge

Solution:

We're given $\sum_{n=1}^{\infty} b_n$ is a positive term series which converges. So $\sum_{n=1}^{\infty} (-1)^n b_n$ is absolutely convergent, and so must converge. And since our series converges, we know that the limit of our terms, $\lim_{n\to\infty} b_n = 0$, so the sequence, b_n, is convergent.

Answer: The sequence, b_n, must converge, and the series $\sum_{n=1}^{\infty} (-1)^n b_n$ must converge as well.

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16. Which of the following series converge? I) $\sum_{n=1}^{\infty} n^{-n/3}$ II) $\sum_{n=1}^{\infty} \frac{3^n n!}{(2n)!}$

Solution:

I) For $\sum_{n=1}^{\infty} n^{-n/3}$ since it's a function if n to a power of n, we apply the ratio test.

$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{n^{-n/3}} = \lim_{n\to\infty} n^{-1/3} = \lim_{n\to\infty} \frac{1}{n^{1/3}} = 0 < 1$. So the series $\sum_{n=1}^{\infty} n^{-n/3}$ is **convergent**

II) For $\sum_{n=1}^{\infty} \frac{3^n n!}{(2n)!}$, since we have powers of constants, and factorials, we're best off using the ratio test.

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}=\lim_{n\to\infty}\frac{\left(\dfrac{3^{n+1}(n+1)!}{(2(n+1))!}\right)}{\left(\dfrac{3^n n!}{(2n)!}\right)}=\lim_{n\to\infty}\frac{3^{n+1}(n+1)!}{(2n+2)!}\frac{(2n)!}{3^n n!}=\lim_{n\to\infty}3\frac{(2n)!}{(2n+2)!}\frac{(n+1)!}{n!}$$

$$=3\lim_{n\to\infty}\frac{n+1}{(2n+2)(2n+1)}=\frac{3}{2}\lim_{n\to\infty}\frac{1}{2n+1}=0<1$$

So our series $\sum_{n=1}^{\infty}\frac{(2n)!}{3^n n!}$ is **convergent**

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17. Which of the following represents all real values of r such that the series $\sum_{n=0}^{\infty}\frac{1}{2^{nr}}$ converges?

Solution:

We can re-write our series as $\sum_{n=0}^{\infty}\frac{1}{2^{nr}}=\sum_{n=0}^{\infty}(2^{-r})^n$. So this is a geometric series with a ratio of 2^{-r}, which converges if $2^{-r}=|2^{-r}|<1=2^0$, so $2^{-r}<2^0$, so $-r<0$, and $r>0$

Answer: $r > 0$

Equivalently, you could perform the ratio or root test. For instance the ratio test gives the result:

$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{1/2^{r(n+1)}}{1/2^{rn}}=\lim_{n\to\infty}\frac{2^{rn}}{2^{rn+r}}=2^{-r}<1$, so it converges if $r > 0$ (and not if $r = 0$, since in that case we would have $\sum 1$, which clearly diverges.)

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18. The power series, $\sum_{n=1}^{\infty}c_n(x-2)^n$ converges at $x = 3$ and diverges at $x = -4$.

What are the minimum and maximum possible values of its radius of convergence?

Solution:

The series $\sum_{n=1}^{\infty}c_n(x-2)^n$ has a centre of $x = 2$. If it converges also at $x = 3$, then the radius of convergence, $R \geq |3-2| = 1$. If it diverges at $x = -4$, then $R \leq |-4-2| = 6$.

Answer: min 1, max 6

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19. Find the interval of convergence of the power series, $\sum_{n=1}^{\infty}\frac{(-2x)^n}{\sqrt{n+1}}$.

Solution:

As usual we first do the ratio test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{(-2)^{n+1}x^{n+1}}{\sqrt{n+2}}}{\dfrac{(-2)^n x^n}{\sqrt{n+1}}}\right| = \lim_{n\to\infty} 2\sqrt{\frac{n+1}{n+2}}\,|x| = 2|x|<1 \quad \text{so} \quad -\frac{1}{2}<x<\frac{1}{2}$$

And now we just need to check the endpoints for convergence:

At $x = +\frac{1}{2}$ we get $\sum_{n=1}^{\infty}\frac{(-2)^n(\frac{1}{2})^n}{\sqrt{n+1}} = \sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n+1}}$. Notice this series is alternating, with decreasing magnitude terms which approach 0, so the series converges. (By Alternating series test)

At $x = -\frac{1}{2}$ we get $\sum_{n=1}^{\infty}\frac{(-2)^n(-\frac{1}{2})^n}{\sqrt{n+1}} = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n+1}}$. And we can compare this to the p-series with $p = 1/2$:

$\lim_{n\to\infty}\dfrac{\frac{1}{\sqrt{n+1}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty}\sqrt{\frac{n}{n+1}} = 1$ so our series and $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ both behave the same by Limit comparison test.

And $p = 1/2 < 1$, so both diverge.

Thus our power series converges at $x = +\frac{1}{2}$ and diverges at $x = -\frac{1}{2}$, or the interval of convergence is $\left(-\frac{1}{2},\frac{1}{2}\right]$

Answer: $\left(-\frac{1}{2},\frac{1}{2}\right]$

(Also notice, since the ratio test always "loses" any terms that grow slower than exponential rate, by inspection we can see that the power series behaves like $a_n = 2|x|$ under the ratio test. So our radius will be $R = 1/2$.
To understand the endpoints, we look at the non-exponentially growing terms, and we have $\frac{1}{\sqrt{n+1}}$. For large n this the terms are essentially $\frac{1}{\sqrt{n}} = \frac{1}{n^{1/2}}$. Thus when the terms are positive at $x = -1/2$, it diverges like a positive term p-series, since $p = 1/2 < 1$, but since the terms are decreasing and approaching 0, the series will still converge at the right endpoint, $x = 1/2$, as an alternating series.)