Facts

Fact 3.1: Addition and subtraction

Let A and B be matrices of the same size. Then

1. $A + B = B + A$
2. $(A + B) + C = A + (B + C)$.
   That is, it doesn't matter whether we compute $A+B$ first, or $B + C$ first.
3. $A + 0 = 0 + A = A$
4. $A - A = A + (-1)A = 0$

Let λ and μ (the Greek letter mu, pronounced "mew") be scalars. Then

5. $\lambda(A + B) = \lambda A + \lambda B$
6. $(\lambda + \mu)A = \lambda A + \mu A$
7. $\lambda(\mu A) = \lambda\mu A$
8. $1A = A$

Fact 4.1: Transpose

Let A and B be matrices, and λ a scalar. Then

1. $(A^T)^T = A$
2. $(\lambda A)^T = \lambda A^T$
3. $(AB)^T = B^T A^T$, when AB and $A^T B^T$ are defined. Notice the order has swapped.
4. $(A + B)^T = A^T + B^T$, when $A + B$ is defined.

Fact 4.2: Trace

Let A and B be a square matrices of the same size, and λ a scalar. Then

1. $tr(\lambda A + B) = \lambda tr(A) + tr(B)$
2. $tr(AB) = tr(BA)$, even if $AB \neq BA$
3. $tr(A^T) = tr(A)$

Fact 4.3: Matrix mulitiplication

Let A be an $m \times n$ matrix, B and C be $n \times k$ matrices, and D a $k \times l$ matrix. Then

1. $A(B + C) = AB + AC$
2. $(B + C)D = BD + CD$ (notice the order)
3. $A(\lambda B) = (\lambda A)B = \lambda AB$, for λ a scalar
4. $A0 = 0A = 0$, for 0 the appropriate size 0 matrix

Fact 5.1: Inverse and product

Let A and B be invertible matrices of the same size. Then the product AB is invertible also, and

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Note the change of order.

Fact 5.2: Inverse and transpose

If A is an invertible matrix, then A^T is invertible also, and

$$(A^T)^{-1} = (A^{-1})^T$$

Fact 5.3: Laws of powers

Let A be a square matrix and k, $r > 0$ numbers. Then

1. $A^r A^s = A^{r+s}$
2. $(A^r)^s = A^{rs}$

If A is invertible then the above identities hold for negative powers also.

Fact 6.1

If E is the elementary matrix obtained by applying the elementary row operation O to the identity. Let A be another matrix. The product EA is the matrix obtained from A by applying the elementary row operation O.

Fact 6.2

Every elementary matrix is invertible, and the inverse is also an elementary matrix.

Fact 7.1

Let A and B be square matrices of the same size. Then

1. if $AB = I$, then $B = A^{-1}$
2. if $BA = I$, then $B = A^{-1}$
3. if $B = A^{-1}$, then $A = B^{-1}$ also
4. if AB is invertible then A and B are invertible also

Fact 7.2

Let A be an $n \times n$ matrix. The following are equivalent (that is, either all of them are true, or all of them are false):

1. A is invertible.
2. The matrix equation $A\mathbf{x} = 0$ has the unique solution $\mathbf{x} = 0$ (where 0 is the $n \times 1$ zero matrix).
3. The RREF of A is I_n.
4. A may be represented as a product of elementary matrices. That is $A = E_1 E_2 \cdots E_n$, where E_i is an elementary matrix.
5. The matrix equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$ (and is therefore consistent).

Fact 7.3

Let

$$D = \begin{bmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{bmatrix}$$

be a diagonal matrix. Then D is invertible if and only if every entry $d_i \neq 0$.
If D is invertible then

$$D^{-1} = \begin{bmatrix} \frac{1}{d_1} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{d_2} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{d_n} \end{bmatrix}$$

Fact 7.4

Let

$$D = \begin{bmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{bmatrix}$$

be a diagonal matrix. Then

$$D^k = \begin{bmatrix} d_1^k & 0 & 0 & \cdots & 0 \\ 0 & d_2^k & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & d_n^k \end{bmatrix}$$

for any integer k.

Fact 7.5: Properties of triangular matrices

1. The transpose of an upper triangular matrix is lower triangular, and vice versa.
2. The product of two upper triangular matrices is upper triangular. The product of two lower triangular matrices is lower triangular.
3. A triangular matrix is invertible if and only if its diagonal entries are non-zero.
4. The inverse of an upper triangular matrix is lower triangular. The inverse of a lower triangular matrix is upper triangular.

Fact 7.6: Products of triangular matrices

Let A and B be upper triangular matrices (or both lower triangular matrices). Then

$$(AB)_{ii} = (BA)_{ii} = (A)_{ii}\,(B)_{ii}$$

Fact 7.7: Properties of symmetric matrices

Let A and B be symmetric matrices of the same size, and λ a scalar. Then

1. A^T is symmetric
2. $A + B$ and $A - B$ are symmetric
3. λA is symmetric

Fact 7.8

If A is invertible and symmetric, then A^{-1} is symmetric also.

Fact 7.9

Let M be a matrix of any size.

1. MM^T and $M^T M$ are symmetric
2. If M is invertible, then MM^T and $M^T M$ are invertible also

Fact 8.1: Determinant of a triangular matrix

Let $A = \left[a_{ij}\right]$ be a triangular square matrix (either upper or lower). Then

$$det(A) = a_{11}a_{22}\cdots a_{nn}$$

i.e. it is the product of the diagonal entries.

Fact 8.2: Determinant of the transpose

Let A be a square matrix. Then

$$det(A) = det(A^T)$$

Fact 8.3

Let A be a square matrix. If A has a row or column of 0's, then $det(A) = 0$.

Fact 8.4: Determinants via row reduction

Let A and B be square matrices of the same size, related by exactly one elementary row operation.

1. If B is obtained from A by swapping two rows, then
$$det(B) = -det(A)$$
E.g.
$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = - \begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

2. If B is obtained from A by multiplying a row by the scalar λ, then
$$det(B) = \lambda det(A)$$
E.g.
$$\begin{vmatrix} \lambda a_{11} & \lambda a_{12} & \lambda a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \lambda \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

3. If B is obtained from A by adding a multiple of one row to another row, then
$$det(B) = det(A)$$
E.g.
$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} + a_{21} & a_{12} + a_{22} & a_{13} + a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Fact 8.5

Let A be a square matrix. If A has a column which is a scalar multiple of another column, or a row which is a scalar multiple of another row. Then $det(A) = 0$.

Fact 9.1: Determinants of elementary matrices

Let E be an elementary $n \times n$ matrix. Then

1. If E is obtained from I_n by swapping a row, then $det(E) = -1$.
2. If E is obtained from I_n by multiplying a row by a scalar λ, then $det(E) = \lambda$.
3. If E is obtained from I_n by adding a multiple of one row to another, then $det(E) = 1$.

Fact 9.2: Determinant and scalar multiplication

Let A be an $n \times n$ matrix, and λ a scalar. Then $det(\lambda A) = \lambda^n \det(A)$.

Fact 9.3

Let A and B be square matrices of the same size. Assume that A is equal to B except in the k-th row.
Let C be the matrix with rows equal to those of A (and B), except in the k-th row: let the k-th row of C be equal to the sum of the k-th row of A and the k-th row of B.
Then $det(C) = det(A) + det(B)$.

Fact 9.4: Determinant of a product

Let A and B be square matrices of the same size. Then
$$det(AB) = det(A)det(B).$$

Fact 9.5

Let A be a square matrix. Then A is invertible if and only if $det(A) \neq 0$.

Fact 9.6

Let A be an invertible matrix. Then
$$det(A^{-1}) = \frac{1}{det A}.$$

Fact 9.7: The inverse via the adjoint

Let A be a square matrix. If $det(A) \neq 0$, then
$$A^{-1} = \frac{1}{det(A)} adj(A).$$

Fact 10.1: Characteristic equation

Let A be an $n \times n$ matrix. The scalar λ is an eigenvalue of A if and only if it is a solution to the equation
$$det(\lambda I - A) = 0$$
(for I the identity matrix).
The equation
$$det(\lambda I - A) = 0$$
is known as the characteristic equation of A.

Fact 10.2: Eigenvalues of a triangular matrix

Let A be a triangular matrix. Then the eigenvalues of A are listed on its main diagonal.

Fact 10.3

A square matrix A is invertible if and only if 0 is not an eigenvalue of A.

Fact 11.1

Let A be a diagonalizable matrix, with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Let $\mathbf{x}_i$ be the eigenvector associated to the eigenvalue λ_i. Let

$$P = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

be the matrix of eigenvectors.
Then P diagonalizes A so that $A = PDP^{-1}$, where the diagonal matrix D has the form

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

That is, it is the matrix with the eigenvalues of A on the diagonal, and 0's elsewhere.

Fact 11.2

Let A be a square matrix and k a positive integer. If λ is an eigenvalue of A with associated eigenvector $\mathbf{x}$, then λ^k is an eigenvalue of A^k, with associated eigenvector $\mathbf{x}$.

Fact 12.1: Trivial solution

Let

$$\mathbf{y}' = A\mathbf{y}$$

be a system of differential equations. Then the vector $\mathbf{y} = 0$ is solution, known as the trivial solution.

Fact 13.1

Let z be a complex number and $\overline{z}$ its conjugate. Then

$$z\overline{z} = \overline{z}z$$

is a real number.

Fact 13.2

Let z be a complex number. Then

$$|z|^2 = z\overline{z}$$

Fact 13.3: Properties of the complex conjugate

Let z and w be complex numbers. Then

- $\overline{z+w} = \overline{z} + \overline{w}$
- $\overline{z-w} = \overline{z} - \overline{w}$
- $\overline{zw} = (\overline{z})(\overline{w})$
- $\overline{\dfrac{z}{w}} = \dfrac{\overline{z}}{\overline{w}}$
- $\overline{\overline{z}} = z$

Fact 14.1: Multiplication in polar form

Let $z = r_1(\cos(\theta_1) + i\sin(\theta_1))$, and $w = r_2(\cos(\theta_2) + i\sin(\theta_2))$ be complex numbers. Then

$$zw = r_1 r_2 (\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)).$$

Fact 14.2

Let z and w be complex numbers. Then

$$|zw| = |z||w|$$

and

$$Arg(zw) = Arg(z) + Arg(w)$$

Fact 14.3: Division in polar form

Let $z = r_1(\cos(\theta_1) + i\sin(\theta_1))$, and $w = r_2(\cos(\theta_2) + i\sin(\theta_2))$ be complex numbers. Then

$$\frac{z}{w} = \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)).$$

Fact 14.4

Let z and w be complex numbers. Then

$$\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$$

and

$$Arg\left(\frac{z}{w}\right) = Arg(z) - Arg(w)$$

Fact 14.5: De Moivre's Formula

Let z be a complex number with $|z| = 1$. Therefore its polar form is

$$z = \cos(\theta) + i\sin(\theta)$$

For any positive integer n, we have

$$z^n = \cos(n\theta) + i\sin(n\theta)$$

This equation is known as De Moivre's Formula.

Fact 14.6

Let $z = r(\cos(\theta) + i\sin(\theta))$ be a complex number, and n a positive integer.
There are exactly n n-th roots of z, and they are given by

$$z^{\frac{1}{n}} = r^{\frac{1}{n}}\left(\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right)$$

for $k = 0, 1, 2, \ldots, n-1$.

Fact 14.7

Let $z = r(\cos(\theta) + i\sin(\theta))$ be a complex number. Then

$$z = re^{i\theta}$$

Fact 14.8

Let $z = re^{i\theta}$ be a complex number. Then

$$\text{Re}(z) = r\cos(\theta)$$
$$\text{Im}(z) = r\sin(\theta)$$

Fact 15.1

Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ be vectors in $\mathbb{R}^n$, and k, m scalars. Then

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
3. $\mathbf{u} + \mathbf{0} = \mathbf{u}$
4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
5. $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$
6. $(k + m)\mathbf{u} = k\mathbf{u} + m\mathbf{u}$
7. $k(m\mathbf{u}) = (km)\mathbf{u}$
8. $1\mathbf{u} = \mathbf{u}$

Fact 15.2: Properties of the norm

Let $\mathbf{u}$ be a vector in $\mathbb{R}^n$, and k a scalar. Then

- $||\mathbf{u}|| \geq 0$
- $||\mathbf{u}|| = 0$ if and only if $\mathbf{u} = \mathbf{0}$
- $||k\mathbf{u}|| = k||\mathbf{u}||$

Fact 15.3: Distance between two points

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}n$. The distance between the endpoints of $\mathbf{u}$ and $\mathbf{v}$ is given by

$$||\mathbf{u} - \mathbf{v}|| \tag{1}$$

Fact 15.4

Let $\mathbf{u}$ be a vector in $\mathbb{R}^n$. Then

$$\mathbf{u} \bullet \mathbf{u} = ||\mathbf{u}||^2$$

Fact 15.5

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors and θ the angle between them. Then

$$\mathbf{u} \bullet \mathbf{v} = ||\mathbf{u}||||\mathbf{v}||\cos(\theta)$$

Fact 16.1: Properties of the dot product

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in $\mathbb{R}^n$, and k a scalar. Then

- $\mathbf{u} \bullet \mathbf{v} = \mathbf{v} \bullet \mathbf{u}$
- $\mathbf{u} \bullet (\mathbf{v} + \mathbf{w}) = \mathbf{u} \bullet \mathbf{v} + \mathbf{u} \bullet \mathbf{w}$
- $k(\mathbf{u} \bullet \mathbf{v}) = (k\mathbf{u}) \bullet \mathbf{v}$
- $\mathbf{u} \bullet \mathbf{u} \geq 0$, and $\mathbf{u} \bullet \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$

Fact 16.2: The Cauchy-Schwarz Inequality

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$. Then

$$|\mathbf{u} \bullet \mathbf{v}| \leq ||\mathbf{u}||||\mathbf{v}||$$

Fact 16.3: The Triangle Inequality

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$. Then

$$||\mathbf{u} + \mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||$$

Fact 16.4: Parallelogram Rule

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$. Then

$$||\mathbf{u} + \mathbf{v}||^2 + ||\mathbf{u} - \mathbf{v}||^2 = 2\left(||\mathbf{u}||^2 + ||\mathbf{v}||^2\right)$$

Fact 16.5

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$. Then

$$\mathbf{u} \bullet \mathbf{v} = \frac{1}{4}||\mathbf{u} + \mathbf{v}||^2 - \frac{1}{4}||\mathbf{u} - \mathbf{v}||^2$$

Fact 16.6

Let $\mathbf{u}$ and $\mathbf{v}$ be orthogonal vectors in $\mathbb{R}^n$. Then

$$||\mathbf{u} + \mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2$$

Fact 16.7: Calculating distance

Let $ax + by + c = 0$ be a line in $\mathbb{R}^2$, and (x_0, y_0) a point. The shortest distance from (x_0, y_0) to the line is given by

$$D = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}} \tag{2}$$

If $x + by + cz + d = 0$ is a plane in $\mathbb{R}^3$, and (x_0, y_0, z_0) is a point, then he shortest distance from (x_0, y_0, z_0) to the plane is given by

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}} \tag{3}$$

Fact 17.1

Let A be a an $m \times n$ matrix. An $n \times 1$ vector $\mathbf{x}$ is a solution to the equation

$$A\mathbf{x} = \mathbf{0}$$

if and only if $\mathbf{x}$ is orthogonal to every row vector of A.

Fact 17.2: Properties of the cross product

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in $\mathbb{R}^3$, and k a scalar. Then

1. $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$
2. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$
3. $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w}$
4. $k(\mathbf{u} \times \mathbf{v}) = (k\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (k\mathbf{v})$
5. $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$
6. $\mathbf{u} \times \mathbf{u} = \mathbf{0}$

Fact 17.3: Relation to other operations

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in $\mathbb{R}^3$. Then

1. $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. That is

$$\mathbf{u} \bullet (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \bullet (\mathbf{u} \times \mathbf{v}) = 0$$

2. $||\mathbf{u} \times \mathbf{v}||^2 = ||\mathbf{u}||^2 ||\mathbf{v}||^2 - (\mathbf{u} \bullet \mathbf{v})^2$
3. $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \bullet \mathbf{w})\mathbf{v} - (\mathbf{u} \bullet \mathbf{v})\mathbf{w}$

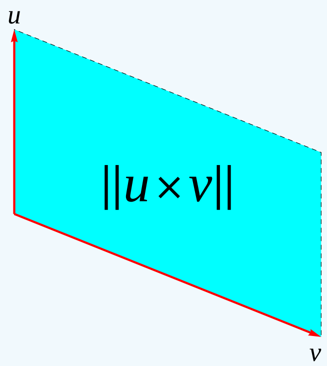
Fact 17.4

Let $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$ be vectors in $\mathbb{R}^3$. Their cross product may be computed

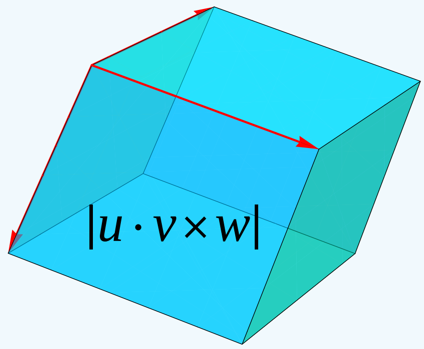
$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Fact 17.5

1. Let $\mathbf{u}$, $\mathbf{v}$ be vectors in $\mathbb{R}^3$. Then $||\mathbf{u} \times \mathbf{v}||$ is the area of the parallelogram defined by $\mathbf{u}$ and $\mathbf{v}$.



2. Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ be vectors in $\mathbb{R}^3$. Then $|\mathbf{u} \bullet (\mathbf{v} \times \mathbf{w})|$ is the area of the parallelepiped defined by $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.



Fact 17.6

Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ be vectors in $\mathbb{R}^3$. If $|\mathbf{u} \bullet (\mathbf{v} \times \mathbf{w})| = 0$ then $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ lie in a plane.

Fact 18.1

Let V be a vector space, $\mathbf{u} \in V$ and k a scalar. Then

1. $0\mathbf{u} = \mathbf{0}$
2. $k\mathbf{0} = \mathbf{0}$
3. $(-1)\mathbf{u} = -\mathbf{u}$
4. If $k\mathbf{u} = \mathbf{0}$ then either $k = 0$ or $\mathbf{u} = \mathbf{0}$.

Fact 18.2: When is a subset a subspace?

Let V be a vector space. The subset $W \subset V$ is a subspace if and only if

1. W is closed under addition: if $\mathbf{u} \in W$ and $\mathbf{v} \in W$, then $\mathbf{u} + \mathbf{v} \in W$
2. W is closed under scalar multiplication: if $\mathbf{u} \in W$ then $k\mathbf{u} \in W$ for all scalars k

Fact 18.3: Spans are subspaces

Let V be a vector space and $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}\}$. Then span$(S)$ is a subspace of V.

Fact 19.1

Let A be an $m \times n$ matrix. The set of solutions to the equation

$$A\mathbf{x} = \mathbf{0}$$

is a subspace of $\mathbb{R}^n$.

Fact 19.2

Let V be a vector space and W_1, W_2 subspaces. Then $W_1 \cap W_2$ is a subspace.

Fact 19.3

If S is a set of n linearly independent vectors in $\mathbb{R}^n$ then $\text{span}(S) = \mathbb{R}^n$.

Fact 20.1: A basis of P_n

Let P_n denote the vector space of polynomials in x of degree at most n. The set

$$S = \{1, x, x^2, \ldots, x^n\}$$

is a basis for P_n. The set S is known as the standard basis of P_n.

Fact 20.2

Let V be a vector space and $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ a basis. Every vector $\mathbf{v} \in V$ may be expressed as a linear combination of the basis vectors in exactly one way

$$\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 + \ldots + \mathbf{v}_n$$

Fact 21.1: Orthogonal (and non-zero) implies linearly independent

Let V be a vector space and S a set of vectors in $\mathbb{R}^n$, which **does not** contain the zero vector. If S is orthogonal then it is linearly independent.

Fact 21.2: A useful property of an orthonormal basis

Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis of $\mathbb{R}^n$. Given a vector $\mathbf{u}$

1. if S is orthogonal then

$$\mathbf{u} = \frac{\mathbf{u} \bullet \mathbf{v}_1}{||\mathbf{v}_1||^2}\mathbf{v}_1 + \ldots + \frac{\mathbf{u} \bullet \mathbf{v}_n}{||\mathbf{v}_n||^2}\mathbf{v}_n$$

   and the co-ordinate vector of $\mathbf{u}$ in terms of S is

$$\mathbf{u} = \left(\frac{\mathbf{u} \bullet \mathbf{v}_1}{||\mathbf{v}_1||^2}, \ldots \frac{\mathbf{u} \bullet \mathbf{v}_n}{||\mathbf{v}_n||^2}\right)_S$$

2. if S is orthonormal then

$$\mathbf{u} = (\mathbf{u} \bullet \mathbf{v}_1)\,\mathbf{v}_1 + \ldots + (\mathbf{u} \bullet \mathbf{v}_n)\,\mathbf{v}_n$$

   and the co-ordinate vector of $\mathbf{u}$ in terms of S is

$$\mathbf{u} = (\mathbf{u} \bullet \mathbf{v}_1, \ldots, \mathbf{u} \bullet \mathbf{v}_n)_S$$

Fact 21.3: Orthogonal decomposition

Let W be a subspace of $\mathbb{R}^n$. Given a vector $\mathbf{v} \in \mathbb{R}^n$, we have

$$\mathbf{v} = \text{proj}_W(\mathbf{v}) + \text{proj}_{W^\perp}(\mathbf{v})$$

where $\text{proj}_W(\mathbf{v}) \in W$ and $\text{proj}_{W^\perp}(\mathbf{v})$ is orthogonal to W.

Fact 21.4: Finding the orthogonal decomposition

Let W be a subspace of $\mathbb{R}^n$ with orthogonal basis

$$S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$$

Given a vector $\mathbf{v} \in \mathbb{R}^n$, we have

$$\text{proj}_W(\mathbf{v}) = \frac{\mathbf{v} \bullet \mathbf{v}_1}{||\mathbf{v}_1||^2}\mathbf{v}_1 + \frac{\mathbf{v} \bullet \mathbf{v}_2}{||\mathbf{v}_2||^2}\mathbf{v}_2 + \cdots + \frac{\mathbf{v} \bullet \mathbf{v}_n}{||\mathbf{v}_n||^2}\mathbf{v}_n$$

If the basis S is orthonormal then

$$\text{proj}_W(\mathbf{v}) = (\mathbf{v} \bullet \mathbf{v}_1)\,\mathbf{v}_1 + (\mathbf{v} \bullet \mathbf{v}_2)\,\mathbf{v}_2 + \cdots (\mathbf{v} \bullet \mathbf{v}_n)\,\mathbf{v}_n$$

The component of $\mathbf{v}$ orthogonal to W is given by

$$\text{proj}_{W^\perp}(\mathbf{v}) = \mathbf{v} - \text{proj}_W(\mathbf{v})$$

Fact 22.1

Let V be a vector space and S_1 and S_2 be bases for V. Then S_1 contains the same number of vectors as S_2.

Fact 22.2

Let V be a vector space and W a subspace of V. Then

$$\dim(W) \leq \dim(V)$$

Fact 22.3: Adding and removing vectors

Let V be a vector space. Then

1. let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a linearly independent set of vectors in V. If $\mathbf{u} \notin \text{span}(S)$, then the set

$$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k, \mathbf{u}\}$$

   is linearly independent also.

   That is, adding a vector which does lie in $\text{span}(S)$ does not break linear independence.

2. let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a linearly dependent set of vectors in V. If $\mathbf{v}_k$ may be written as a linear combination of the other vectors in S, then

$$\text{span}(\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{k-1}\}) = \text{span}(S)$$

   That is, removing vectors which can be written as linear combinations of the other vectors does not change the span.

3. let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a linearly independent set of vectors in V. Then the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{k-1}\}$ is linearly independent also.

   That is, removing any number of vectors from a linearly independent set does not break linear independence.

Fact 22.4

Let V be a finite dimensional vector space, and S a finite set of vectors in V. Then

1. if S is not linearly independent but span$(S) = V$, then S can upgraded to a basis of V by removing vectors.
2. if S is linearly independent but span$(S) \neq V$, then S can be upgraded to a basis of V by adding vectors which are not in span(S).