1ZC3

Assignment Due ~~Thurs~~ Extended!

Last Day Intro. to diagonalization

Remember: Diagonal Matrices:

the only non-zero entries are on diagonal.

$$\begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \lambda_3 \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

- each entry on principal diagonal is a $\underline{\underline{\lambda}}$ an eigenvalue!
- # ~~[illegible]~~ repeats = algebraic multiplicity.
- corresponding eigenvectors are $\vec{i}, \vec{j}, \vec{k}$ (in general $\vec{e_i} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

1 in ith spot, 0 elsewhere

- $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \Rightarrow D^k = \begin{bmatrix} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \lambda_n^k \end{bmatrix}$ } easy powers & mult.

Let's turn a matrix diagonal

eg $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ To diagonalize get λ (eigenvalues)

& corresponding $\vec{x}$ eigenvectors

characteristic polynomial

$$C_A(\lambda) = |A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix}$$

roots

$$= (1-\lambda)^2 - 2^2 = \lambda^2 - 2\lambda + 1 - 4$$

$$\left.\begin{aligned} &= \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1) \\ &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} \lambda &= 3 \\ \lambda &= -1 \end{aligned}$$

Side note $\lambda = 3$, $\lambda = -1$ have only $\underline{\underline{1}}$ factor each in $C_A(\lambda)$

$\Rightarrow$ alg. mult. $= 1$ for each!

& $1 \leq$ geo. mult. $\leq$ alg. mult. $\Rightarrow$ geo mult $= 1$ for each too!

Let's get eigenvectors Solve $(A - \lambda I)\vec{x} = 0$

$\left\{ A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \right.$

$\lambda = 3$ $(A - \lambda I)\vec{x} = \vec{0} \Rightarrow \left[\begin{array}{cc|c} 1-3 & 2 & 0 \\ 2 & 1-3 & 0 \end{array}\right]$

$\Rightarrow \left[\begin{array}{cc|c} -2 & 2 & 0 \\ 2 & -2 & 0 \end{array}\right]$ $Row_2 \to Row\,2 + Row\,1$ then $Row\,1 \cdot (-\frac{1}{2})$

$\Rightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $\leadsto x - y = 0 \Rightarrow x = y$; $\leadsto 0 = 0$ $\Big\}$ $x = t$, $y = t$

So $\begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \leadsto \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\lambda = 3$ basis eigenvector

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

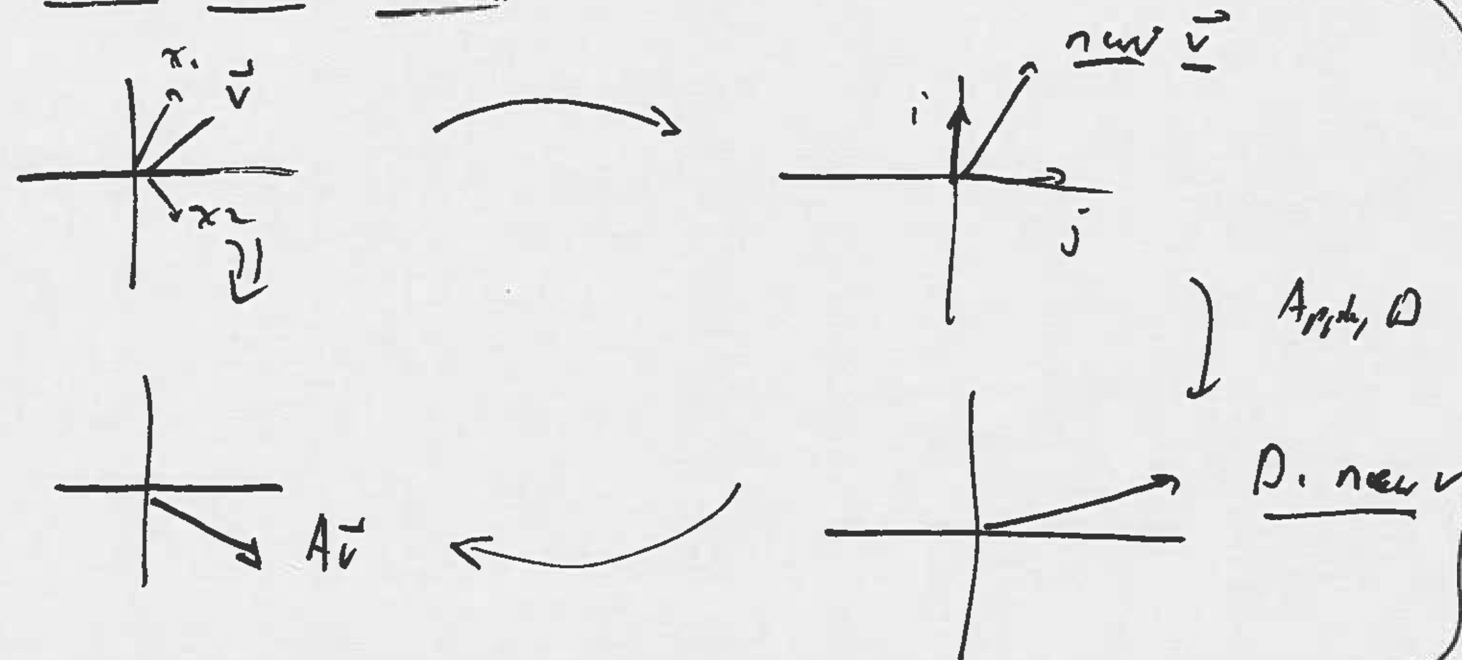
$\lambda = -1$

$$(A - \lambda I)\vec{x} = 0 \Rightarrow \left[\begin{array}{cc|c} 2 & 2 & 0 \\ 2 & 2 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$\Rightarrow \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $\lambda = -1$ basis eigenvector

Now Remember our dream!



Let P = matrix of basis eigenvectors of A

$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ $\vec{x}_{1,2}$

$\lambda_1 = 3 \Rightarrow \vec{x} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2 = -1 \Rightarrow \vec{x} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\vec{x}_2$

In general $P = \left[\vec{x}_1 \,|\, \vec{x}_2 \,|\, \vec{x}_3 \ldots | \vec{x}_n \right]$

note if geo < alg. $\Rightarrow$ not enough $\vec{x}$'s

$\Rightarrow$ Fail! Can't do it!

Here $P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$\vec{x}_1$ $\vec{x}_2$

note $P \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \vec{x}_1$

$P \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = x_2$

$P \vec{e}_i = \vec{x}_i$

Get $\underline{\underline{P^{-1}}}$ $= \begin{bmatrix} 1 & +1 \\ -1 & 1 \end{bmatrix} \cdot \dfrac{1}{(1)(1) - (-1)(1)} = \dfrac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}$

NB:

$P^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$P^{-1} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$P^{-1} \vec{x}_i = e_i$

$\underline{\underline{D}}$ is corresponding diagonal matrix

$=$ matrix of λ's $\Rightarrow$ λ's have to match P's order of eigenvectors

$P = \begin{bmatrix} \vec{x}_1 \mid \vec{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$

& $P^{-1} = \dfrac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ as above!

Claim $A = P D P^{-1}$

here $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$

& $D = P^{-1} A P$

Note One superpower of diagonalizability!

Eg Calculate $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}^{500}$

Solution If A diagonalizable (i.e. geo. = alg.

$\Rightarrow$ P has suff. columns to be square!

$$\Rightarrow \quad A = P D P^{-1}$$

$$A^{500} = (PDP^{-1})^{500}$$

$$= P D \underbrace{P^{-1} P}_{I} D \underbrace{P^{-1} P}_{I} D P^{-1} \cdots\cdots$$

← 500 repeats!

$$= P D^{500} P^{-1}$$

Here $P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ $\quad D^{500} = \begin{bmatrix} 3^{500} & 0 \\ 0 & +1 \end{bmatrix}$ $\quad P^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \frac{1}{2}$

$$A^{500} = P D^{500} P^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3^{500} & 3^{500} \\ -1 & 1 \end{bmatrix} \cdot \frac{1}{2}$$

$$= \begin{bmatrix} \frac{3^{500}+1}{2} & \frac{3^{500}-1}{2} \\ \frac{3^{500}-1}{2} & \frac{3^{500}+1}{2} \end{bmatrix}.$$