MATHEMATICS 1LT3 TEST 1

Day Class
Duration of Test: 60 minutes
McMaster University, 1 February 2012

E. Clements

FIRST NAME (please print): SOL^{NS}

FAMILY NAME (please print): __________

Student No.: __________

THIS TEST HAS 8 PAGES AND 5 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE.

Total number of points is 40. Marks are indicated next to the problem number. Any non-graphing calculator is allowed.

USE PEN TO WRITE YOUR TEST. IF YOU USE A PENCIL YOUR TEST WILL NOT BE ACCEPTED FOR REMARKING (IF NEEDED).

You need to show work to receive full credit.

Problem	Points	Mark
1	8	
2	5	
3	7	
4	12	
5	8	
TOTAL	40	

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1. Use the separation of variables technique to solve each differential equation.

(a) [4] $P'(t) - 10P(t)t = 0, \;\; P(0) = 210$

$$\frac{dP}{dt} = 10P \cdot t$$

$$\int \frac{1}{P}\, dP = \int 10t\, dt$$

$$\ln|P| = 5t^2 + C$$

$$|P| = e^{5t^2 + C}$$

$$= e^C \cdot e^{5t^2}$$

$$P = \pm e^C \cdot e^{5t^2}$$

so, $P(t) = K \cdot e^{5t^2}$, where $K = \pm e^C$

$P(0) = 210 \Rightarrow \;\; 210 = Ke^{5(0)^2} \Rightarrow K = 210$

$\therefore P(t) = 210e^{5t^2}$

(b) (i) [3] $\dfrac{dy}{dx} = y^2 \cos x$ (*)

$$\int y^{-2} dy = \int \cos x\, dx$$

$$-y^{-1} = \sin x + C$$

$$\frac{1}{y} = -\sin x - C$$

$$\Rightarrow y = \frac{1}{-\sin x - C}$$

(ii) [1] Are there any other solutions to this differential equation **not** covered by the equation you found in part (i)?

YES! $y = 0$ is also a sol^N to (*)

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2. Assuming that there is competition within a population of bacteria for resources, a model for limited bacterial growth is given by

$$\frac{db}{dt} = \lambda(b)b$$

where the per capita production rate, λ, is a decreasing function of the population size, b.

Suppose that the per capita production rate is a linear function of population size with a maximum of $\lambda(0) = 1$ and a slope of -0.002.

(a) [2] Find $\lambda(b)$ and the differential equation for b.

$$\lambda(b) = -0.002b + 1$$

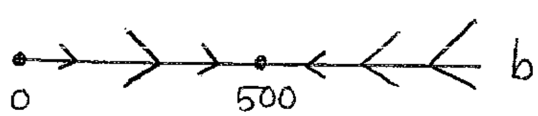
$$\frac{db}{dt} = (-0.002b + 1)b$$

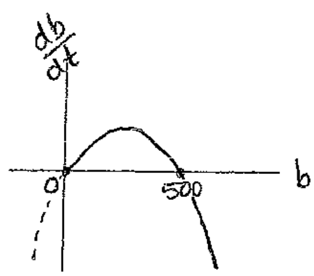
(b) [1] Find equilibrium solutions of this autonomous differential equation.

$$\frac{db}{dt} = 0 \quad \text{when} \quad -0.002b + 1 = 0 \quad \text{or} \quad \boxed{b^* = 0}$$

$$\Rightarrow \boxed{b^* = 500}$$

(c) [2] When is the population increasing? When is it decreasing? Display this information using a phase-line diagram.





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3. Consider the following autonomous differential equation, where α and β are parameters:

$$\frac{dW}{dt} = \alpha e^{\beta W} - 1$$

(a) [2] Find the equilibrium for this differential equation.

$\frac{dW}{dt} = 0$ when $\alpha e^{\beta W} - 1 = 0$

$e^{\beta W} = \frac{1}{\alpha}$ ← note $\alpha > 0$ $\because$ $e^{\beta W} > 0$

$\beta W = \ln \frac{1}{\alpha}$

$= -\ln \alpha$

$\Rightarrow \boxed{W^* = -\frac{\ln \alpha}{\beta}}$

(b) [3] Using the stability theorem, determine the values of the parameter β for which this equilibrium is stable and the values for which it is unstable.

$f(W) = \alpha e^{\beta W} - 1$

$f'(W) = \alpha e^{\beta W} \cdot \beta$

$f'\left(-\frac{\ln \alpha}{\beta}\right) = \alpha e^{\beta\left(\frac{\ln \alpha^{-1}}{\beta}\right)} \cdot \beta$

$= \alpha \cdot e^{\ln \alpha^{-1}} \cdot \beta$

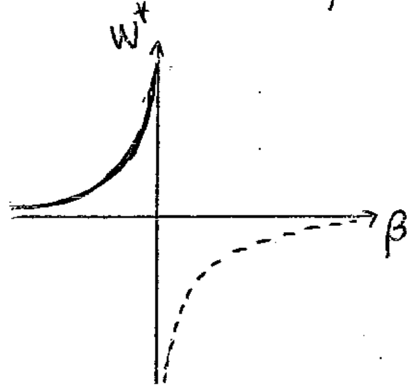
$= \beta$

when $\beta < 0$, $W^* = -\frac{\ln \alpha}{\beta}$ is stable.

when $\beta > 0$, $W^* = -\frac{\ln \alpha}{\beta}$ is unstable

(c) [2] Suppose that $\alpha = e$. Graph the equilibrium in part (a) as a function of the parameter value β. Use a solid line when the equilibrium is stable and a dashed line when the equilibrium is unstable.

$\alpha = e \Rightarrow W^* = -\frac{\ln e}{\beta} = -\frac{1}{\beta}$



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4. The logistic differental equation describing the growth of a population of monkeys is given by

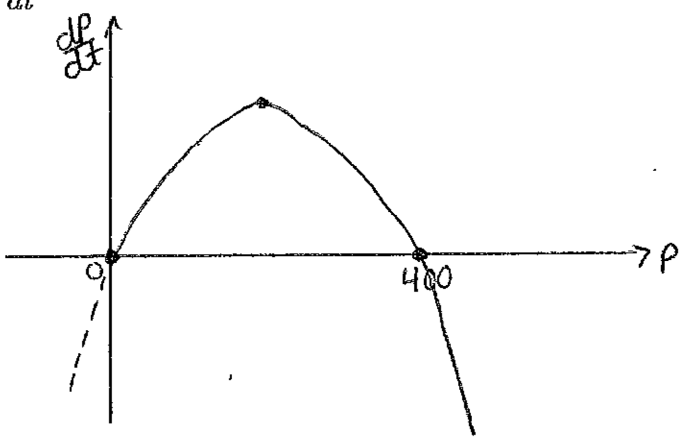
$$\frac{dP}{dt} = 0.8P\left(1 - \frac{P}{400}\right).$$

(a) [2] Determine the equilibria for this population. What do these numbers represent?

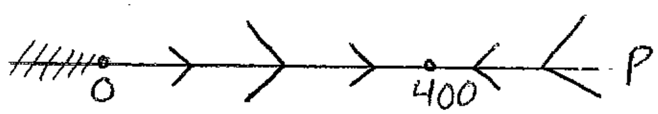
$\frac{dP}{dt} = 0$ when $P = 0$ or $P = 400$

$P=0$ is a trivial equilibrium since a pop^N of size zero will have a growth rate of zero.
$P=400$ represents the carrying capacity for this pop^N. This is the max. # of monkeys the environment can support in the long run.

(b) [2] Sketch $\frac{dP}{dt}$ as a function of P.

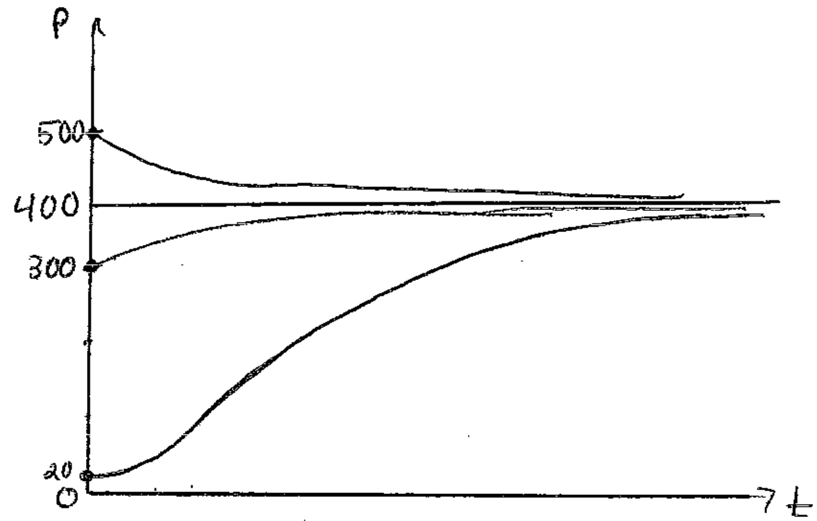


(c) [2] Draw a phase-line diagram for this differential equation.



4. continued

(d) [4] Sketch the equilibrium solutions and sketch typical solution curves corresponding to the initial conditions $P(0) = 20$, $P(0) = 300$, and $P(0) = 500$.



(e) [2] The solution of this equation is given by $P(t) = \dfrac{400}{1 + 19e^{-0.8t}}$.

When will the population reach 95% of its carrying capacity?

$P(t_?) = 0.95(400)$

$0.95(4\cancel{00}) = \dfrac{4\cancel{00}}{1+19e^{-0.8t}}$

$1 + 19e^{-0.8t} = \dfrac{1}{0.95}$

$\Rightarrow \quad 19e^{-0.8t} = \dfrac{100}{95} - 1$

$e^{-0.8t} = \dfrac{1}{361}$

$\Rightarrow \quad t = -\dfrac{\ln 361}{-0.8} \approx \underline{7.36}$ years (?)

5. The following pair of equations represent the population growth of two different species where one is the predator and the other is the prey.

$$\frac{dA}{dt} = 0.1A - 0.005AB$$
$$\frac{dB}{dt} = -0.05B + 0.0001AB$$

(a) [2] Determine the equilibrium solutions for this system of equations.

$\frac{dA}{dt} = 0$ AND $\frac{dB}{dt} = 0$

$\Rightarrow A(0.1 - 0.005B) = 0$ AND $B(-0.05 + 0.0001A) = 0$

$A = 0$ or $B = 20$ AND $B = 0$ or $A = 500$

$\therefore$ eq^n sol^{n}s are $\begin{cases} A=0 \\ B=0 \end{cases}$ or $\begin{cases} A=500 \\ B=20 \end{cases}$

(b) [2] Which of the variables represents the predator population and which represents the prey population? Explain.

A represents the prey pop^n since the coefficient of the interact^n term is negative (interactions have a negative effect on growth rate of prey). Also, when $B=0$, A's pop^n will grow exponentially (another characteristic of prey).

B represents the predator pop^n since it benefits from interactions ($+0.0001AB$ increases the growth rate of B) and without prey (ie when $A=0$) the pop^n of B will die out exponentially (typical of predators).

5.continued...

(c) [1] Why are the coefficients of the AB term not the same in both equations?

"-0.005" is a measure of how harmful interactions are for pop^N A and "$+0.0001$" measures how beneficial interactions are for type B.
The size of these numbers indicates that interactions will be very harmful for type A but only moderately beneficial for type B.

(d) [3] If $A_0 = 40$ and $B_0 = 15$, approximate the size of both populations after one year using Euler's method and a step size of 4 months. Here, t is measured in months.

$t_0 = 0 \qquad \Delta t = 4$

$A_0 = 40$

$B_0 = 15$

$t_1 = t_0 + \Delta t = 4$

$A_1 = 40 + [0.1(40) - 0.005(40)(15)]4 = 44$

$B_1 = 15 + [-0.05(15) + 0.0001(40)(15)]4 = 12.24$

$t_2 = t_1 + \Delta t = 8$

$A_2 = 44 + [0.1(44) - 0.005(44)(12.24)]4 = 50.8288 \approx 50.83$

$B_2 = 12.24 + [-0.05(12.24) + .0001(44)(12.24)]4 \doteq 10.01$

$t_3 = t_2 + \Delta t = 12$

$A_3 = 50.83 + [0.1(50.83) - 0.005(50.83)(10.01)]4 \approx 60.98$

$B_3 = 10.01 + [-0.05(10.01) + .0001(50.83)(10.01)]4 \approx 8.21$

$\therefore$ After 1 year, pop^N will have approximately 61 individuals and pop^N B will have approximately 8.

THE END