MATHEMATICS 1LS3 TEST 1

Day Class
Duration of Examination: 60 minutes
McMaster University, 24 January 2012

E. Clements

FIRST NAME (please print): SOL^{NS}

FAMILY NAME (please print): __________

Student No.: __________

THIS TEST HAS 8 PAGES AND 10 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE.

Total number of points is 40. Marks are indicated next to the problem number. Any non-graphing calculator is allowed.

USE PEN TO WRITE YOUR TEST. IF YOU USE A PENCIL YOUR TEST WILL NOT BE ACCEPTED FOR REMARKING (IF NEEDED).

You need to show work to receive full credit.

Problem	Points	Mark
1	6	
2	4	
3	2	
4	3	
5	3	
6	6	
7	6	
8	3	
9	4	
10	3	
TOTAL	40	

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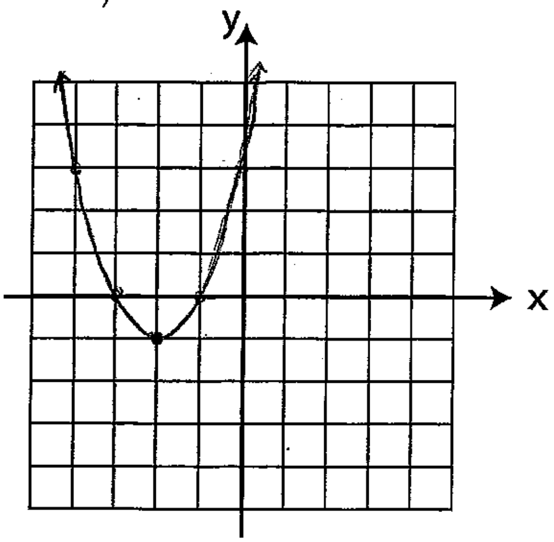
1. (a) [2] Sketch the graph of $f(x) = x^2 + 4x + 3$. (Hint: Complete the square to find the vertex.)

$\hookrightarrow (\frac{4}{2})^2 = (2)^2 = 4$



$f(x) = \underbrace{x^2 + 4x + 4} - 4 + 3$

$= (x+2)^2 - 1$

Vertex: $V(-2,-1)$

(b) [2] Determine the domain of $h(x) = \dfrac{3x-2}{\sqrt{x^2+4x+3}}$.

$x^2 + 4x + 3 > 0$

from graph in part (a):

$x < -3$ or $x > -1$

$(x \in (-\infty, -3) \cup (-1, \infty))$

(c) [2] Explain why the set of points $R = \{(1,-2), (3,4), (1,7)\}$ do **not** describe a function.

the input 1 corresponds to 2 output values: -2 or 7

A function is a rule that assigns to each input value exactly one output value.

So R does <u>not</u> describe a f^N.

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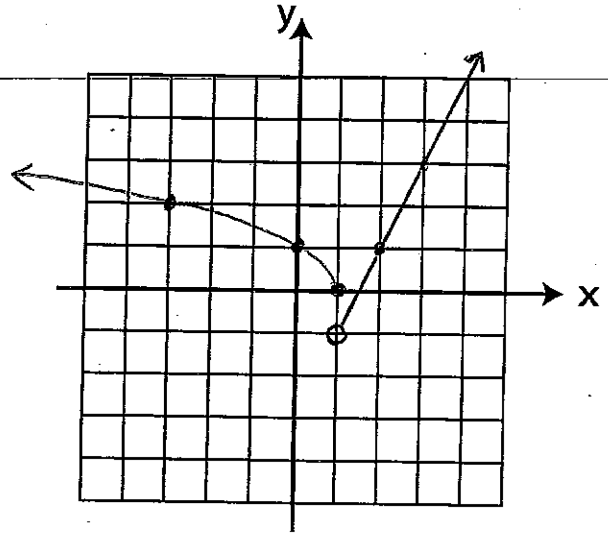
2. (a) [3] Sketch the graph of $f(x) = \begin{cases} \sqrt{1-x} & \text{if } x \le 1 \\ 2x-3 & \text{if } x > 1 \end{cases}$. $\sqrt{-(x-1)}$



(b) [1] Determine the range of f.

$y > -1$ or $y \in (-1, \infty)$

3. [2] Show that the function $f(x) = x^2 + 5x + 6$ is **not** one-to-one.

$= (x+2)(x+3)$

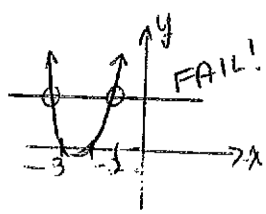
$f(-2) = 0$ AND $f(-3) = 0$

so when $y = 0$, $x = -2$ OR $x = -3$

$\Rightarrow f$ is NOT 1-1

OR

f is not 1-1 $\because$ it's graph fails the HLT



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4. [3] By referring to the graphs of two functions f and g below, compute the following: (Hint: You do **not** need to find the equations of the lines to do this!)

(a) $(f+g)(0)$

$= f(0) + g(0)$

$= 5 + (-1)$

$= 4$

(b) $(fg)(2)$

$= f(2) \cdot g(2)$

$= (3)(5)$

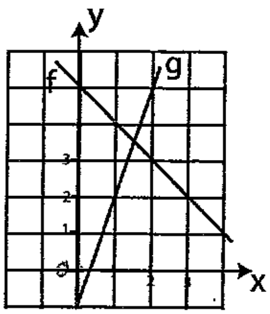
$= 15$

(c) $(g \circ f)(3)$

$= g(f(3))$

$= g(2)$

$= 5$



5. [3] The function $f(x) = 2(x+3)^3 - 1$ is one-to-one. Find f^{-1}.

$$y = 2(x+3)^3 - 1$$

$$y + 1 = 2(x+3)^3$$

$$\frac{y+1}{2} = (x+3)^3$$

$$\sqrt[3]{\frac{y+1}{2}} = x + 3$$

$$\sqrt[3]{\frac{y+1}{2}} - 3 = x$$

$$\therefore \quad f^{-1}(x) = \sqrt[3]{\frac{x+1}{2}} - 3$$

6. Insulin and blood glucose in the human body are inversely proportional. Suppose that the level of insulin is inversely proportional to the square of the blood glucose level.

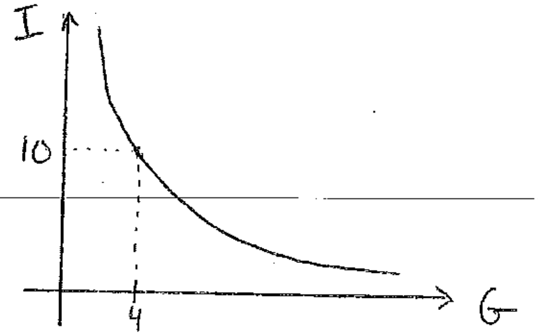
(a) [2] Express insulin level, I, as a function of blood glucose level, G. At a certain time during the day, the level of insulin in a healthy person was found to be 10 IU/mL while the level of glucose was measured to be 4 mmol/L. Solve for the proportionality constant.

$$I \propto \frac{1}{G^2} \Rightarrow I = a \cdot \frac{1}{G^2}$$

$$\left.\begin{matrix} I = 10 \\ G = 4 \end{matrix}\right\} \Rightarrow 10 = a \cdot \frac{1}{4^2} \Rightarrow a = 160$$

$$\therefore I(G) = \frac{160}{G^2}$$

(b) [2] Sketch the relationship in part (a).



(c) [2] After a meal, blood sugar levels increase. If the glucose level in the blood increases by 20%, how will this affect the insulin level in the body?

$$G_{new} = 1.20\, G_{old}$$

$$I_{new} = \frac{160}{G_{new}^2} = \frac{160}{(1.2 G_{old})^2} = \frac{160}{1.2^2 \cdot G_{old}^2} = \frac{1}{1.2^2} \cdot I_{old}$$

$\Rightarrow I_{new} \approx 0.69\, I_{old}$ (70% of old insulin level)

So, when glucose levels increase by 20%, insulin levels will drop by about 30%.

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7. The half-life of caffeine is approximately 5 hours. Suppose at 8 a.m. you drink a tall coffee from Second Cup (approximately 240 mg of caffeine).

(a) [2] Write a formula for $C(t)$, the amount of caffeine in your body at time t.

$$C(t) = C_0 e^{kt}$$

Let $t=0$ correspond to 8 am.

Then $C_0 = C(0) = 240$.

Also, $C(5) = \frac{1}{2} C_0 = \frac{1}{2} 240 = 120$.

$$\Rightarrow \quad \frac{1}{2} C_0 = C_0 e^{k(5)} \qquad \text{So,} \quad C(t) = 240 e^{\frac{\ln\frac{1}{2}}{5} t}$$

$$\ln \tfrac{1}{2} = k(5)$$

$$\Rightarrow k = \ln \tfrac{1}{2} / 5$$

(b) [2] At what time will there only be 10 mg left in your body?

Set $C(t) = 10$: $\quad 240 e^{\ln\frac{1}{2}/5\, t} = 10$

$$e^{(\ln\frac{1}{2}/5)t} = \frac{1}{24}$$

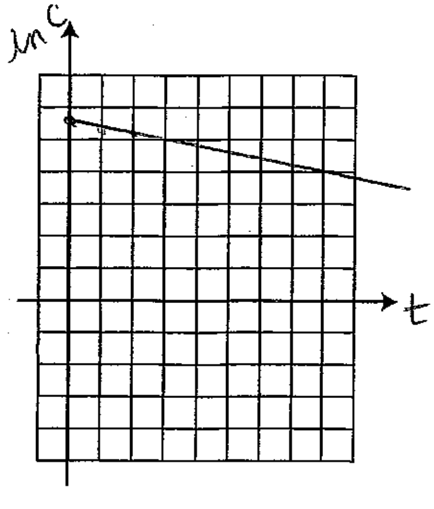
$$\frac{\ln\frac{1}{2}}{5} t = \ln \frac{1}{24}$$

$$\Rightarrow t = \frac{5 \ln\frac{1}{24}}{\ln\frac{1}{2}}$$

≈ 23 hours $\approx$ 7am

almost time for another coffee!

(c) [2] Plot the semilog graph for $C(t)$ from part (a). Label the axes.



$$\ln C = \ln 240 e^{\frac{\ln\frac{1}{2}}{5} t}$$

$$= \ln 240 + \ln e^{\frac{\ln\frac{1}{2}}{5} t}$$

$$= \underbrace{\ln 240}_{\approx 5.5} + \underbrace{\frac{\ln\frac{1}{2}}{5}}_{\approx -0.1386} t$$

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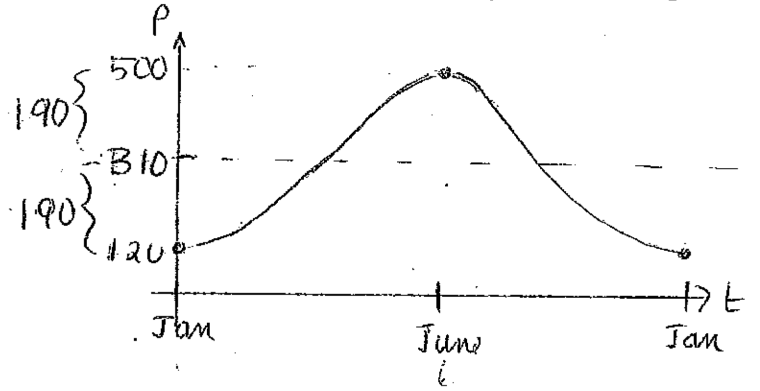
8. A population of birds changes periodically throughout the course of a year. In January, the population reaches a low of 120 birds. Between January and June, the population increases and reaches a maximum of 500 birds in June. From June to January, the population will decline again.

[3] Using a trigonometric function, find a formula which describes how the population of birds changes over time.
(Hint: It might help to sketch the graph first and then determine the transformations required to obtain this graph starting from the graph of $y = \cos x$).



period = 12 months

$\frac{2\pi}{k} = 12 \Rightarrow k = \frac{2\pi}{12} = \frac{\pi}{6}$

avg. value $= \frac{500+120}{2} = 310$

amplitude $= \frac{500-120}{2} = 190$

So, $P(t) = -190\cos\frac{\pi}{6}t + 310$

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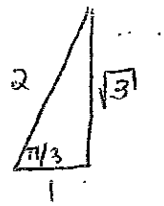
9. (a) [2] Compute the exact value of $\arcsin(-\frac{\sqrt{3}}{2})$. **Show your work!**

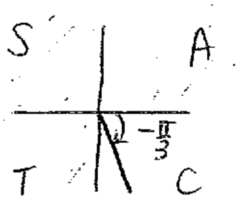
$\arcsin\left(-\frac{\sqrt{3}}{2}\right) = \theta$

$\Updownarrow$

$\sin\theta = -\frac{\sqrt{3}}{2}$ for $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



$\theta' = \frac{\pi}{3}$



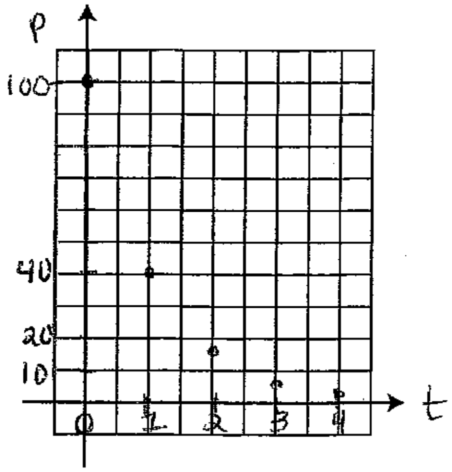
$\theta = -\frac{\pi}{3}$ so $\arcsin\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$

9. (b) [2] Is it true that $\arctan(\tan x) = x$ for all x? Explain.

NO! $\arctan(\tan x) = x$ only for $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$.
$\tan x$ is not naturally a 1-1 f^N so its domain must be restricted first.

10. Consider the discrete-time dynamical system $P_{t+1} = 0.4P_t$, which describes the dynamics of a certain population with an initial size of 100 ($P_0 = 100$).

(a) [2] Find and graph the size of the population over the next four years. Label the axes and scale appropriately.



$P_1 = 0.4P_0 = 0.4(100) = 40$
$P_2 = 0.4P_1 = 0.4(40) = 16$
$P_3 = 0.4P_2 = 0.4(16) = 6.4$
$P_4 = 0.4P_3 = 0.4(6.4) = 2.56$

(b) [1] Write a formula for the solution to this dynamical system and substitute in $t = 4$ to determine the size of this population after 4 years (and to confirm your answer in part (a)).

$$P_t = P_0 \cdot (0.4)^t$$
$$= 100(0.4)^t$$
$$P_4 = 100(0.4)^4 = 2.56 \checkmark$$