MATH 1B03/1ZC3 Winter 2019

Lecture 23: Dimension of a vector space

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Dimension of a vector space

(from Chapter 4.5 of Anton-Rorres)

The notion of a basis allows us to describe the ‘size’ of a vector space. This generalized notion of size is known as dimension.

Definition 23.1: Dimension of a vector space

Let V be a vector space. The dimension of V, denoted $\dim(V)$, is the number of non-zero vectors in a basis for V.
If $\dim(V) = k$ we say that V is k-dimensional.

That is, the dimension of a vector space is the number of linearly independent vectors needed to span the entire space.

There are two important things to say immediately:

1. Recall that the trivial vector space is the vector space with a single element. If $V = \{\mathbf{0}\}$ is a trivial vector space, then

$$\{\mathbf{0}\}$$

is a basis, but it does not contain any non-zero vectors. Therefore the dimension of the trivial vector space is 0.

2. It is possible for the dimension of a vector space to be infinite. That is, if V is infinite dimensional then every basis of V contains an infinite number of vectors.

The following fact shows that Definition 23.1 is correct.

Fact 23.2

Let V be a vector space and S_1 and S_2 be bases for V. Then S_1 contains the same number of vectors as S_2.

Proof: This fact can be proved in general by contradiction, but we will suffice ourselves with the following instructive example.

Let $V = \mathbb{R}^n$. Suppose that S_1 and S_2 are bases for $\mathbb{R}^n$ and

$$S_1 = \{\mathbf{v}_1\}$$
$$S_2 = \{\mathbf{v}_2, \mathbf{v}_3\}$$

But $\operatorname{span}(S_1)$ is a line (as S_1 contains only one vector), and $\operatorname{span}(S_1)$ is a plane (as S_2 contains two vectors). A line is certainly not equal to a plane, so S_1 and S_2 cannot have a different number of vectors. ■

This explains why Fact 20.9 is true. Recall that the standard basis of $\mathbb{R}^n$ has exactly n vectors. Therefore $\dim(\mathbb{R}^n) = n$. If S is a set of n linearly independent vectors in $\mathbb{R}^n$, then $\operatorname{span}(S)$ is n-dimensional. But the only n-dimensional subspace of $\mathbb{R}^n$ is $\mathbb{R}^n$ itself! Therefore $\operatorname{span}(S) = \mathbb{R}^n$.

Example 23.3

The vector space $\mathbb{R}^n$ has the standard basis

$$\{(1, 0, \ldots, 0), (0, 1, \ldots, 0), \ldots, (0, 0, \ldots, 1)\}$$

containing n vectors, so $\dim(\mathbb{R}^n) = n$

The vector space P_n has the standard basis

$$\{1, x, x^2, \ldots, x^n\}$$

containing $n+1$ vectors, so $\dim(\mathbb{R}^n) = n+1$.

The vector space of upper triangular 2×2 matrices has the basis

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

containing 3 vectors, it is 3-dimensional.

Definition 23.4: Dimension of a subspace

Let V be a vector space and W a subspace of V. The dimension of W is defined as the number of non-zero vectors in a basis for W, and is denoted $\dim(W)$.

Fact 23.5

Let V be a vector space and W a subspace of V. Then

$$\dim(W) \leq \dim(V)$$

Proof: Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a basis for V, so that $\dim(V) = k$. If $\mathbf{w} \in W$, then $\mathbf{w} \in V$ also as $W \subset V$, so

$$\mathbf{w} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k$$

and we observe that any $\mathbf{w} \in W$ can be written as a linear combination of at most k linearly independent vectors. Therefore $\dim(W) \leq k$. ■

Fact 23.6: Adding and removing vectors

Let V be a vector space. Then

1. let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a linearly independent set of vectors in V. If $\mathbf{u} \notin \text{span}(S)$, then the set

$$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k, \mathbf{u}\}$$

is linearly independent also.

That is, adding a vector which does lie in span(S) does not break linear independence.

2. let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a linearly dependent set of vectors in V. If $\mathbf{v}_k$ may be written as a linear combination of the other vectors in S, then

$$\text{span}(\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{k-1}\}) = \text{span}(S)$$

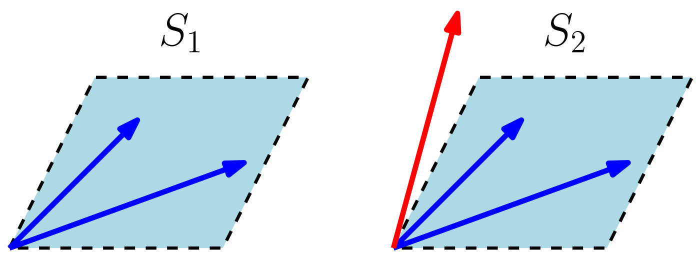
That is, removing vectors which can be written as linear combinations of the other vectors does not change the span.

3. let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a linearly independent set of vectors in V. Then the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{k-1}\}$ is linearly independent also.

That is, removing any number of vectors from a linearly independent set does not break linear independence.

We can understand these facts diagrammatically as follows:

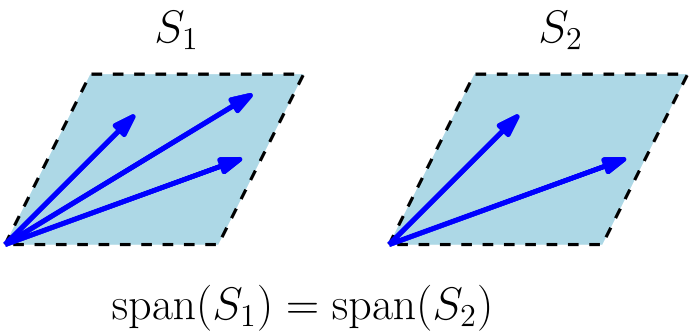
1.



S_1 linearly independent then
S_2 linearly independent also

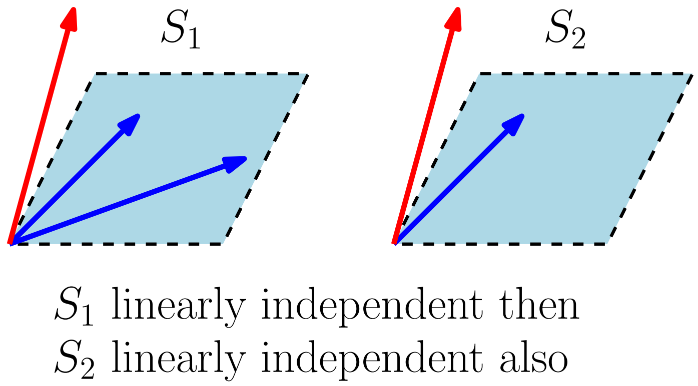
Then dim(span(S_2)) = dim(span(S_1)) + 1

2.



Then $\dim(\text{span}(S_2)) = \dim(\text{span}(S_1))$

3.



Then $\dim(\text{span}(S_2)) = \dim(\text{span}(S_1)) - 1$

Example 23.7

Solutions to an homogeneous system: Find a basis for the set of solutions to the SLE

$$\begin{aligned} 3x_1 + 2x_3 &= 0 \\ x_1 - 2x_2 + x_3 &= 0 \\ 4x_1 + x_2 + x_3 &= 0 \end{aligned}$$

and compute the dimension of the space of solutions.
Answer: Recall that if A is an $m \times n$ matrix then the set of solutions to an equation $A\mathbf{x} = \mathbf{0}$ forms a subspace of $\mathbb{R}^n$, so it must have a basis.
Solve the SLE using its augmented matrix:

$$\begin{bmatrix} 3 & 0 & 2 & 0 \\ 1 & -2 & 1 & 0 \\ 4 & 1 & 1 & 0 \end{bmatrix}$$

Gauss-Jordan elimination yields

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

this yields the unique solution

$$\begin{aligned} x_1 &= 0 \\ x_2 &= 0 \\ x_3 &= 0 \end{aligned}$$

so that the space of solutions contains exactly one point. The only vector space which contains a single point is the trivial vector space, which is 0 dimensional. Therefore the space of solutions is 0 dimensional also.

Question: Find a basis for the set of solutions to the SLE

$$\begin{aligned} 8x_1 - 2x_2 + x_3 &= 0 \\ 2x_1 + x_2 + 5x_3 &= 0 \end{aligned}$$

and compute the dimension of the space of solutions.
Answer: Form the augmented matrix

$$\begin{bmatrix} 8 & -2 & 1 & 0 \\ 2 & 1 & 5 & 0 \end{bmatrix}$$

and apply Gauss-Jordan elimination to obtain

$$\begin{bmatrix} 1 & 0 & \frac{11}{12} & 0 \\ 0 & 1 & \frac{19}{6} & 0 \end{bmatrix}$$

Then we have

$$\begin{aligned} x_1 + \frac{11}{12}x_3 &= 0 \\ x_2 + \frac{19}{6}x_3 &= 0 \end{aligned}$$

The variable x_3 is free, so let $x_3 = s$, then

$$\begin{aligned} x_1 &= -\frac{11}{12}s \\ x_2 &= -\frac{6}{9}s \end{aligned}$$

and the solutions to the SLE are of the form

$$\left(-\frac{11}{12}s,\ -\frac{6}{9}s,\ s\right) = s\left(-\frac{11}{12},\ -\frac{6}{9},\ 1\right)$$

and we see that a basis for the space of solutions is

$$\left\{\left(-\frac{11}{12},\ -\frac{6}{9},\ 1\right)\right\}$$

so that the dimension of the space of solutions is 1.

Dimension in terms of degrees of freedom

We can often predict the dimension of vector spaces by considering the number of degrees of freedom. That is, the number of free variables needed to describe the vector space.

In the example above we considered vectors in $\mathbb{R}^3$ which satisfied two equations. The equations were of a particular form: they are linear combinations of vectors in $\mathbb{R}^3$ set to zero. That is

$$\begin{aligned} 8x_1 - 2x_2 + x_3 &= 0 \\ 2x_1 + x_2 + 5x_3 &= 0 \end{aligned}$$

are both examples of equations of the form

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k = 0$$

for a_i scalars and $\mathbf{v}_i$ vectors.

As $\mathbb{R}^3$ is 3-dimensional and we have imposed 2 equations, the space of solutions will be 1 dimensional (or 0 dimensional if no solutions exist). This argument can be used in general.

Let V be an n-dimensional vector space, and W a subspace defined by specifying k equations of the form

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k = 0$$

Then W is of dimension $n - k$.

Example 23.8

Question: Find a basis for the plane in $\mathbb{R}^3$ defined by the equation

$$3x + 7y = 0$$

Answer: The subspace is defined by 1 equation, so we expect it to be $3-1 = 2$ dimensional. A plane is 2 dimensional, so this makes sense.
To find a basis, consider the equation

$$3x + 7y + 0z = 0$$

then $3x = -7y$ with z a free variable. Setting $y = t$ and $z = s$, we have

$$x = -\frac{7}{3}t$$

and any vector in the plane can be written

$$\begin{aligned} \left(-\frac{7}{3}t,\, t,\, s\right) &= \left(-\frac{7}{3}t,\, t,\, 0\right) + (0,\, 0,\, s) \\ &= t\left(-\frac{7}{3},\, 1,\, 0\right) + s\,(0,\, 0,\, 1) \end{aligned}$$

from which we observe that

$$\left\{\left(-\frac{7}{3},\, 1,\, 0\right),\, (0,\, 0,\, 1)\right\}$$

forms a basis for the plane. It contains 2 non-zero vectors, so the plane is 2-dimensional, as we expected!

Question: Find a basis and the dimension of the subspace of vectors $(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5$ which satisfy

$$\begin{aligned} x_1 &= x_3 + x_4 \\ x_2 &= x_4 + x_5 \end{aligned}$$

Answer: As $\mathbb{R}^5$ is 5 dimensional and we have 2 equations, we expect the subspace to be $5 - 2 = 3$ dimensional.
Rearrange the equations to obtain

$$\begin{aligned} x_1 - x_3 - x_4 &= 0 \\ x_2 - x_4 - x_5 &= 0 \end{aligned}$$

and produce the associated augmented matrix:

$$\begin{bmatrix} 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 \end{bmatrix}$$

This matrix is already in reduced row echelon form. The variables x_3, x_4 and x_5 are free variables. Let $x_3 = s$, $x_4 = t$ and $x_5 = r$, then

$$\begin{aligned} x_1 - s - t &= 0 \\ x_2 - t - r &= 0 \end{aligned}$$

which yields

$$\begin{aligned} x_1 &= s + t \\ x_2 &= t + r \end{aligned}$$

Therefore solutions to the equations are of the form

$$\begin{aligned} (s + t, t + r, s, t, r) &= (s, 0, s, 0, 0) + (t, t, 0, t, 0) + (0, r, 0, 0, r) \\ &= s\,(1, 0, 1, 0, 0) + t\,(1, 1, 0, 1, 0) + r\,(0, 1, 0, 0, 1) \end{aligned}$$

and the set

$$\{(1, 0, 1, 0, 0), (1, 1, 0, 1, 0), (0, 1, 0, 0, 1)\}$$

is a basis for the subspace. It follows that the subspace is 3-dimensional, as expected.

In vector spaces which are not $\mathbb{R}^n$ we need to investigate further to determine dimension. The principle of degrees of freedom holds, and it allows us to make informed predictions about the dimension of vector spaces.

Example 23.9

Question: Let V be the vector space of $n \times n$ matrices with 0 along the main diagonal. Find dim(V).
Answer: An $n \times n$ matrix A is in V if it has the form

$$\begin{bmatrix} 0 & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & 0 & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & 0 \end{bmatrix}$$

Notice that there are no conditions on the other entries a_{ij} of A: they are free variables.
In each row there are $n-1$ free variables, and there are n rows. Therefore there are $n(n-1)$ free variables, and dim$(V) = n(n-1)$.

Question: Let W be the subspace of polynomials $\mathbf{p} \in P_3$ defined by the equations

$$\begin{aligned} \mathbf{p}(1) &= 0 \\ \mathbf{p}(-1) &= 0 \end{aligned}$$

Find a basis for W and dim(W).
Answer: A polynomial $\mathbf{p} \in P_3$ has the form

$$\mathbf{p} = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

Then

$$\begin{aligned} \mathbf{p}(1) &= a_0 + a_1 + a_2 + a_3 = 0 \\ \mathbf{p}(-1) &= a_0 - a_1 + a_2 - a_3 = 0 \end{aligned}$$

Notice that these equations are of the form

$$b_0 + b_1 x + b_2 x^2 + b_3 x^3 = 0$$

that is, a linear combination set to 0. As dim$(P_3) = 4$ and we have 2 equations, we expect W to be of $4 - 2 = 2$ dimensional.
To find a basis, consider the equations

$$\begin{aligned} a_0 + a_1 + a_2 + a_3 &= 0 \\ a_0 - a_1 + a_2 - a_3 &= 0 \end{aligned}$$

This is an SLE with augmented matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & -1 & 1 & -1 & 0 \end{bmatrix}$$

Applying Gauss-Jordan elimination we obtain

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Set $a_2 = s$ and $a_3 = t$, then

$$\begin{aligned} a_0 &= -s \\ a_1 &= -t \end{aligned}$$

and

$$\begin{aligned} \mathbf{p} &= -s - tx + sx^2 + tx^3 \\ &= s(x^2 - 1) + t(x^3 - x) \end{aligned}$$

and a basis for W is given by

$$\{x^2 - 1, x^3 - x\}$$

Building bases

When working with finite dimensional vector spaces we can produce bases by adding or removing certain vectors.

Fact 23.10

Let V be a finite dimensional vector space, and S a finite set of vectors in V. Then

1. if S is not linearly independent but $\text{span}(S) = V$, then S can upgraded to a basis of V by removing vectors.
2. if S is linearly independent but $\text{span}(S) \neq V$, then S can be upgraded to a basis of V by adding vectors which are not in $\text{span}(S)$.

Example 23.11

Question: Enlarge the set $S = \{(1, -3, 4), (7, 0, 5)\}$ to a basis of $\mathbb{R}^3$.
Answer: As $\mathbb{R}^3$ is finite dimensional Fact 23.10 applies. We must add a vector $\mathbf{v} \notin \text{span}(S)$ to produce a basis of $\mathbb{R}^3$.
One possibility is a vector which is orthogonal to both $(1, -3, 4)$ and $(7, 0, 5)$. Let $\mathbf{v} = (v_1, v_2, v_3)$ and set

$$(v_1, v_2, v_3) \bullet (1, -3, 4) = v_1 - 3v_2 + 4v_3 = 0$$

and

$$(v_1, v_2, v_3) \bullet (7, 0, 5) = 7v_1 - 5v_3 = 0$$

This yields the system

$$\begin{aligned} v_1 - 3v_2 + 4v_3 &= 0 \\ 7v_1 - 5v_3 &= 0 \end{aligned}$$

which can be reduced to

$$\begin{aligned} v_1 &= \frac{5}{7}v_3 \\ v_2 &= \frac{11}{7}v_3 \end{aligned}$$

Picking $v_3 = 7$ we obtain

$$\begin{aligned} v_1 &= 5 \\ v_2 &= 11 \end{aligned}$$

so that $\mathbf{v} = (5, 11, 7)$.
Then the set

$$\{(1, -3, 4), (7, 0, 5), (5, 11, 7)\}$$

is a basis for $\mathbb{R}^3$.

Suggested Problems

Practice the material covered in this lecture by attempting the following questions from Chapter 4.4 of Anton-Rorres, starting on page 228

- Questions 4, 7, 8, 9, 11, 15
- True/False (b), (c), (d), (e)