COMPSCI/SFWRENG 2FA3
Discrete Mathematics with Applications II
Winter 2020

Week 04 Exercises

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1. Prove that $(\mathcal{P}(S), \subset)$ is a strict partial order where S is a nonempty set and $\mathcal{P}(S)$ is the power set of S.

   SOLUTION:

   Proof Recall that a strict partial order $(S, <)$ has the following definitions:

   Irreflexivity: $\forall\, x \in S \,.\, \neg(x < x)$
   Asymmetry: $\forall\, x, y \in S \,.\, x < y \Rightarrow \neg(y < x)$
   Transitivity: $\forall\, x, y, z \in S \,.\, x < y \wedge y < z \Rightarrow x < z$

   We'll prove all three of these properties for $(\mathcal{P}(S), \subset)$.

   By definition $\forall s_1, s_2 \in \mathcal{P}(S) \,.\, s_1 \subset s_2 \iff (\forall y \in s_1 \,.\, y \in s_2) \wedge (\exists x \in s_2 \,.\, x \notin s_1)$

   Irreflexivity: $\forall s \in \mathcal{P}(S) \,.\, \neg(s \subset s)$. We proceed by contradiction:

$$\begin{aligned}
& s \subset s \\
\iff & (\forall y \in s \,.\, y \in s) \wedge (\exists x \in s \,.\, x \notin s) && \langle \text{definition of } \subset \rangle \\
\Rightarrow & \exists x \in s \,.\, x \notin s && \langle A \wedge B \Rightarrow B \rangle \\
\Rightarrow & False && \langle \text{obvious} \rangle
\end{aligned}$$

   We have a contradiction therefore $\neg(s \subset s)$. Alternatively, it is clearly irreflexive because no set is a proper subset of itself.

Asymmetry: $\forall s_1, s_2 \in \mathcal{P}(S) \;.\; s_1 \subset s_2 \Rightarrow \neg(s_2 \subset s_1)$.

$$
\begin{aligned}
& s_1 \subset s_2 \\
\iff & (\forall x \in s_1 \;.\; x \in s_2) \wedge (\exists y \in s_2 \;.\; y \notin s_1) && \langle \text{definition of } \subset \rangle \\
\iff & \neg(\exists x \in s_1 \;.\; x \notin s_2) \wedge (\exists y \in s_2 \;.\; y \notin s_1) && \langle \text{definition of } \exists \rangle \\
\iff & \neg(\exists x \in s_1 \;.\; x \notin s_2) \wedge \neg(\forall y \in s_2 \;.\; y \in s_1) && \langle \text{definition of } \exists \rangle \\
\iff & \neg((\exists x \in s_1 \;.\; x \notin s_2) \vee (\forall y \in s_2 \;.\; y \in s_1)) && \langle \text{De Morgan's} \rangle \\
\Rightarrow & \neg((\exists x \in s_1 \;.\; x \notin s_2) \wedge (\forall y \in s_2 \;.\; y \in s_1)) && \langle \neg(A \vee B) \Rightarrow \neg(A \wedge B) \rangle \\
\iff & \neg(s_2 \subset s_1) && \langle \text{definition of } \subset \rangle
\end{aligned}
$$

Therefore $s_1 \subset s_2 \Rightarrow \neg(s_2 \subset s_1)$. Alternatively, it is clearly asymmetric because no set can be both a proper subset and a proper superset of another set.

Transitivity: $\forall s_1, s_2, s_3 \in \mathcal{P}(S) \;.\; s_1 \subset s_2 \wedge s_2 \subset s_3 \Rightarrow s_1 \subset s_3$. First we'll show $s_1 \subset s_2 \wedge s_2 \subset s_3 \Rightarrow \forall x \in s_1 \;.\; x \in s_3$:

$$
\begin{aligned}
& s_1 \subset s_2 \wedge s_2 \subset s_3 \\
\iff & (\forall w \in s_1 \;.\; w \in s_2) \wedge (\exists x \in s_2 \;.\; x \notin s_1) \\
& \quad \wedge (\forall y \in s_2 \;.\; y \in s_3) \wedge (\exists z \in s_3 \;.\; z \notin s_2) && \langle \text{definition of } \subset \rangle \\
\Rightarrow & (\forall w \in s_1 \;.\; w \in s_2) \wedge (\forall y \in s_2 \;.\; y \in s_3) && \langle A \wedge B \Rightarrow A \rangle \\
\iff & (\forall x \;.\; x \in s_1 \Rightarrow x \in s_2) \wedge (\forall x \;.\; x \in s_2 \Rightarrow x \in s_3) \\
\iff & \forall x \;.\; (x \in s_1 \Rightarrow x \in s_2) \wedge (x \in s_2 \Rightarrow x \in s_3) \\
\Rightarrow & \forall x \;.\; x \in s_1 \Rightarrow x \in s_3 && \langle \text{transitivity of } \Rightarrow \rangle
\end{aligned}
$$

Next we'll show $s_1 \subset s_2 \wedge s_2 \subset s_3 \Rightarrow \exists x \in s_3 \;.\; x \notin s_1$:

$$
\begin{aligned}
& s_1 \subset s_2 \wedge s_2 \subset s_3 \\
\iff & (\forall w \in s_1 \;.\; w \in s_2) \wedge (\exists x \in s_2 \;.\; x \notin s_1) \\
& \quad \wedge (\forall y \in s_2 \;.\; y \in s_3) \wedge (\exists z \in s_3 \;.\; z \notin s_2) && \langle \text{definition of } \subset \rangle \\
\Rightarrow & (\exists x \in s_2 \;.\; x \notin s_1) \wedge (\forall y \in s_2 \;.\; y \in s_3) && \langle A \wedge B \Rightarrow A \rangle \\
\Rightarrow & \exists x \in s_2 \;.\; x \notin s_1 \wedge x \in s_3 \\
\Rightarrow & \exists x \in s_3 \;.\; x \notin s_1
\end{aligned}
$$

Therefore

$$s_1 \subset s_2 \wedge s_2 \subset s_3 \Rightarrow (\forall x \;.\; x \in s_1 \Rightarrow x \in s_3)$$

and

$$s_1 \subset s_2 \wedge s_2 \subset s_3 \Rightarrow (\exists x \in s_3 \;.\; x \notin s_1).$$

Using the definition of $\subset$ we can trivially see that these two implications mean that

$$s_1 \subset s_2 \wedge s_2 \subset s_3 \Rightarrow s_1 \subset s_3.$$

Alternatively, it is clearly transitive because $s_1 \subset s_2$ and $s_2 \subset s_3$ means that each element of s_1 must also be in s_2 and so also in s_3, and also that there is some element in s_2 that is not in s_1, but is in s_3.

$\square$

2. Consider the weak partial order

$$P = (\{\{1\}, \{2\}, \{4\}, \{1,2\}, \{1,4\}, \{2,4\}, \{3,4\}, \{1,3,4\}, \{2,3,4\}\}, \subseteq).$$

   a. Find the maximal elements in P.
   SOLUTION:
   The maximal elements in P are $\{1,2\}$, $\{1,3,4\}$, and $\{2,3,4\}$ as all other elements in P are "smaller" than at least one of these with respect to $\subseteq$ and each of these are only "smaller" than themselves (e.g. $\{1,2\} \subseteq \{1,2\}$, but for all other elements, x: $\{1,2\} \not\subseteq x$).

   b. Find the minimal elements in P.
   SOLUTION:
   The minimal elements in P are $\{1\}$, $\{2\}$, $\{4\}$ as all other elements in P are "greater" with respect to $\subseteq$ and these are only "greater" than themselves.

   c. Find the maximum element in P if it exists.
   SOLUTION:
   There is no maximum element in P, as the two maximal elements of P are not unique in P.

   d. Find the minimum element in P if it exists.
   SOLUTION:
   There is no minimum element in P, as the three minimal elements of P are not unique in P.

   e. Find all the upper bounds of $\{\{2\}, \{4\}\}$ in P.
   SOLUTION:
   The upper bounds of $\{\{2\}, \{4\}\}$ in P, are $\{2,4\}$, $\{2,3,4\}$ (*Note:* $\{1,2\}, \{1,4\}$ are **NOT** upper bounds on $\{\{2\}, \{4\}\}$, as an upper bound has to be larger with respect to $\subseteq$ for **BOTH** $\{2\}$ and $\{4\}$.)

f. Find the least upper bound of $\{\{2\}, \{4\}\}$ in P if it exists.

SOLUTION:

Out of the two upper bounds $\{2, 4\}, \{2, 3, 4\}$, $\{2, 4\} \subseteq \{2, 3, 4\}$, so $\{2, 4\}$ is the least upper bound.

g. Find all the lower bounds of $\{\{1, 3, 4\}, \{2, 3, 4\}\}$ in P.

SOLUTION:

The lower bounds of $\{\{1, 3, 4\}, \{2, 3, 4\}\}$ are $\{4\}, \{3, 4\}$.

h. Find the greater lower bound of $\{\{1, 3, 4\}, \{2, 3, 4\}\}$ in P if it exists.

SOLUTION:

Out of the two lower bounds, $\{4\} \subseteq \{3, 4\}$, so $\{3, 4\}$ is the greatest lower bound.

3. Let (U, I) where I is the binary relation such that $a\ I\ b$ iff $a = b$. Show that (U, I) is a weak partial order and not a weak total order.

   SOLUTION:

   The relation of a weak partial order is reflexive, antisymmetric and transitive. So we must show that I has these properties.

   - $\forall x \in U \,.\, (xIx)$, since $x = x$.
   - $\forall x, y \in U \,.\, xIy \wedge yIx \Rightarrow x = y$, I is equality.
   - $\forall x, y, z \in U \,.\, xIy \wedge yIz \Rightarrow xIz$, since equality is transitive.

   We have shown that (U, I) is a weak partial order. To show that (U, I) is *not* a weak *total* order, we observe that I is not total, since total means: $\forall x, y \in U \,.\, xIy \vee yIx$ but if $x \neq y$, we have neither xIy nor yIx.

   (Note: we have implicitly assumed that there are at least two individuals of sort U! Otherwise (U, I) would be a weak total order)

4. Let $(\mathbb{Q} \cup \{-\infty, +\infty\}, <)$ be the strict total order such that $<$ is the same as $<_{\text{rat}}$ on $\mathbb{Q}$ and $-\infty$ and $+\infty$ are minimum and maximum elements, respectively, of $(\mathbb{Q} \cup \{-\infty, +\infty\}, <)$. Prove that

   $$(\mathbb{Q} \cup \{-\infty, +\infty\}, <)$$

   is dense without assuming that $(\mathbb{Q}, <_{\text{rat}})$ is dense.

   SOLUTION: We recall the definition of dense:

   A set S is *dense* if and only if, for any interval $(a, b) \subseteq S$ where $a < b$, there exists a $c \in S$ such that $a < c < b$

We can express this as a predicate:

$$\text{dense}(S) \equiv \forall\, a, b \in S \,.\, a < b \Rightarrow (\exists\, c \in S \,.\, a < c \land c < b)$$

Now, we must show the predicate: $\text{dense}(\mathbb{Q} \cup \{-\infty, +\infty\})$ holds.

We observe 4 cases:

a. $a = -\infty, b = +\infty$:
Let c = 2. Therefore $a < c < b$.

b. $a = -\infty, b = \frac{x}{y}$ for $x, y \in \mathbb{N}$ (therefore $b \in \mathbb{Q}$):
Let $c = \frac{x}{2y}$. Therefore $a < c = \frac{x}{2y} = \frac{b}{2} < b$.

c. $a = \frac{x}{y}, b = +\infty$ for $x, y \in \mathbb{N}$ (therefore $a \in \mathbb{Q}$):
Let $c = \frac{2x}{y}$. Therefore $a < 2a = \frac{2x}{y} = c < b$.

d. $a = \frac{x}{y}, b = \frac{m}{n}$ for $x, y, m, n \in \mathbb{N}$ (therefore $a, b \in \mathbb{Q}$):
Let $c = \frac{2xn+1}{2yn}$. Note $a = \frac{xn}{yn}, b = \frac{my}{yn}$ and therefore: $xn < my$ and $xn < xn + 1 \leq my$ since $my, xn \in \mathbb{N}$. Therefore $a = \frac{x}{y} = \frac{2xn}{2yn} < \frac{2xn+1}{2yn} < \frac{2xn+2}{2yn} = \frac{2(xn+1)}{2yn} \leq \frac{2my}{2yn} = \frac{m}{n}$.

5. Let $(S, <)$ be a strict total order such that there exist $a, b \in S$ with $a < b$ (i.e., S has at least two members). Show that, if $(S, <)$ is dense, then $(S, <)$ is not a well-order.

   Proof By assumption, there are $a, b \in S$ with $a < b$. Assume $(S, <)$ is dense. We will show that $(S, <)$ is not a well-order. Define the infinite sequence $c_0, c_1, c_2, \ldots$ of members of S by natural number recursion as follows:

   a. c_0 is some member of S such that $a < c_0 < b$. We know c_0 exists since $(S, <)$ is dense.

   b. If $n > 0$, then c_n is some member of S such that $a < c_n < c_{n-1}$. We know c_n exists since $(S, <)$ is dense.

   By construction,

   $$c_0 > c_1 > c_2 > \cdots .$$

   $(S, <)$ is thus not Noetherian since there is an infinite descending sequence of members of S. $(S, <)$ is not Noetherian implies $(S, <)$ is not a well-order. □

6. Consider the mathematical structure $(L, <_L)$ where L is a list of integers and $<_L$ is the binary relation on L defined by:

   $$[a_0, a_1, \ldots, a_n] <^* [b_0, b_1, \ldots, b_n] \text{ iff } \left(\sum_{i=0}^{n} a_i\right) < \left(\sum_{i=0}^{n} b_i\right).$$

Prove that $(L, <_L)$ is a strict partial order that is not a strict total order.

SOLUTION:

Proof We need to prove the following statements about $(L, <_L)$:

$$\begin{aligned}
\text{Irreflexivity: } & \forall\, \ell \in L \,.\, \neg(\ell <_L \ell) \\
\text{Asymmetry: } & \forall\, \ell, k \in L \,.\, \ell <_L k \Rightarrow \neg(k <_L \ell) \\
\text{Transitivity: } & \forall\, \ell, k, j \in L \,.\, \ell <_L k \wedge k <_L j \Rightarrow \ell <_L j \\
\text{Anti-Trichotomous: } & \exists\, \ell, k \in L \,.\, \neg(\ell <_L k) \wedge \neg(\ell = k) \wedge \neg(k <_L \ell)^{\dagger}
\end{aligned}$$

†: Logically negate the definition of Trichotomy, then apply De Morgan's Law to reach the above definition.

Irreflexivity:

For any $\ell \in L$, we have:

$$\begin{aligned}
& \neg(\ell <_L \ell) && \langle \text{Definition of Irreflexivity} \rangle \\
\equiv\, & \neg\left(\left(\sum_{i=0}^{|\ell|} \ell_i\right) < \left(\sum_{i=0}^{|\ell|} \ell_i\right)\right) && \langle \text{Definition of } <_L \rangle \\
\equiv\, & \mathit{true} && \langle \text{Irreflexivity of } < \rangle
\end{aligned}$$

Asymmetry:

For any $\ell, k \in L$ satisfying $\ell <_L k$, we have:

$$\begin{aligned}
& \neg(k <_L \ell) && \langle \text{Definition of Asymmetry} \rangle \\
\equiv\, & \neg\left(\left(\sum_{i=0}^{|k|} k_i\right) < \left(\sum_{i=0}^{|\ell|} \ell_i\right)\right) && \langle \text{Definition of } <_L \rangle \\
\equiv\, & \mathit{true} && \langle \text{Asymmetry of } < \text{ with Assumption } \ell <_L k \text{, Def. of } <_L \rangle
\end{aligned}$$

Transitivity:

For any $\ell, k, j \in L$ satisfying $\ell <_L k \wedge k <_L j$, we have:

$$\begin{aligned}
& \ell <_L j && \langle \text{Definition of Transitivity} \rangle \\
\equiv\, & \left(\sum_{i=0}^{|\ell|} \ell_i\right) < \left(\sum_{i=0}^{|j|} j_i\right) && \langle \text{Definition of } <_L \rangle \\
\equiv\, & \mathit{true} && \langle \text{Transitivity of } < \text{ with Assumption, Def. of } <_L \rangle
\end{aligned}$$

Anti-Trichotomy:

$$
\begin{aligned}
&\exists\, \ell, k \in L \,.\, \neg(\ell <_L k) \wedge \neg(\ell = k) \wedge \neg(k <_L \ell) && \langle\text{Def. of Anti-Trichotomy}\rangle\\
\Leftarrow\, &(\neg(\ell <_L k) \wedge \neg(\ell = k) \wedge \neg(k <_L \ell))[\ell, k \mapsto [1,2],[2,1]] && \langle\exists \text{ witness}\rangle\\
\equiv\, &\neg([1,2] <_L [2,1]) \wedge \neg([1,2] = [2,1]) \wedge \neg([2,1] <_L [1,2]) && \langle\text{Substitution}\rangle\\
\equiv\, &\neg([1,2] <_L [2,1]) \wedge \mathit{true} \wedge \neg([2,1] <_L [1,2]) && \langle\text{Inequality of lists}\rangle\\
\equiv\, &\neg(1+2 < 2+1) \wedge \neg(2+1 < 1+2) && \langle\text{Def. of } <_L\rangle\\
\equiv\, &\mathit{true} && \langle\text{Asymmetry of } <\rangle
\end{aligned}
$$

□

7. Construct a strict partial order $(U, <)$ such that U is infinite, $<$ is well founded, and $(U, <)$ is not a total order (and thus $(L, <_L)$ is not a well-order).

   SOLUTION:

   Let $U \equiv \mathbb{N} \times \mathbb{N}$.
   For clarity we rename $<$ as $<_2$ and use $<$ as normal.
   Let $(x_1, x_2) <_2 (y_1, y_2) \iff x_1 < y_1 \wedge x_2 < y_2$.

   a. $(\mathbb{N} \times \mathbb{N}, <_2)$ is a strict partial order:
      Strict partial orders are irreflexive, asymmetric, and transitive.

      - Irreflexive:
        We proceed by contradiction.

$$
\begin{aligned}
&(x_1, x_2) <_2 (x_1, x_2) && \langle\text{negated definition of irreflexive}\rangle\\
\Rightarrow &x_1 < x_1 && \langle\text{ partial application of definition of } <_2\rangle\\
\Rightarrow &\mathit{False} && \langle(\mathbb{N}, <) \text{ is irreflexive}\rangle
\end{aligned}
$$

      - Asymmetric:

$$
\begin{aligned}
&(x_1, x_2) <_2 (y_1, y_2) &&\\
\Rightarrow &x_1 < y_1 && \langle\text{ partial application of definition of } <_2\rangle\\
\Rightarrow &\neg(y_1 < x_1) && \langle(\mathbb{N}, <) \text{ is asymmetric}\rangle\\
\Rightarrow &\neg((y_1, y_2) <_2 (x_1, x_2)) && \langle\text{trivial with } <_2 \text{ definition}\rangle
\end{aligned}
$$

        Therefore $(x_1, x_2) <_2 (y_1, y_2) \Rightarrow \neg((y_1, y_2) <_2 (x_1, x_2))$.

      - Transitive:

$$
\begin{aligned}
&(x_1, x_2) <_2 (y_1, y_2) \wedge (y_1, y_2) <_2 (z_1, z_2) &&\\
\Rightarrow &x_1 < y_1 \wedge x_2 < y_2 \wedge y_1 < z_1 \wedge y_2 < z_2 && \langle\text{definition of } <_2\rangle\\
\Rightarrow &x_1 < y_1 < z_1 \wedge x_2 < y_2 < z_2 && \langle\text{rearranging}\rangle\\
\Rightarrow &x_1 < z_1 \wedge x_2 < z_2 && \langle(\mathbb{N}, <) \text{ is transitive}\rangle\\
\Rightarrow &(x_1, x_2) <_2 (z_1, z_2) && \langle\text{definition of } <_2\rangle
\end{aligned}
$$

Therefore $(x_1, x_2) <_2 (y_1, y_2) \wedge (y_1, y_2) <_2 (z_1, z_2) \Rightarrow (x_1, x_2) <_2 (z_1, z_2)$.

Therefore our order is a strict partial order.

b. $(\mathbb{N} \times \mathbb{N}, <_2)$ is infinite:
We know $\mathbb{N}$ is infinite so clearly $\mathbb{N} \times \mathbb{N}$ is infinite (we can construct an infinite list of some of the members of $\mathbb{N} \times \mathbb{N}$ by pairing $x \in \mathbb{N}$ with itself for all such x).

c. $(\mathbb{N} \times \mathbb{N}, <_2)$ is well founded:
It is well founded if every nonempty subset of $\mathbb{N} \times \mathbb{N}$ has a $<_2$-minimal element. In other words,

$$\forall S \subseteq (\mathbb{N}\times\mathbb{N})\backslash\emptyset \,.\, \exists (y_1, y_2) \in S \,.\, \forall (x_1, x_2) \in \mathbb{N}\times\mathbb{N} \,.\, (x_1, x_2) \in S \Rightarrow \neg((x_1, x_2) <_2 (y_1, y_2))$$

We proceed by weak induction on the size of the nonempty subset.
Base Case:
The nonempty subset has 1 element. The 1 element is the minimal element and it is "less" than all 0 other elements.

Inductive Step:
Assume that subsets of size n have a minimal element and that the minimal element is "less" than all other elements.
Prove that subsets of size $n+1$ have a minimal element and that the minimal element is "less" than all other elements:
Split a subset of size $n+1$ into a set of size n and a set of size 1.
Let x be the minimal element of the set of size n which we know exists by the induction hypothesis.
Let y be the single element in the set of size 1.
In the original subset of size $n+1$, either x or y is a minimal element.
If $x <_2 y$, then x is a minimal element because no element is "less" than x, x is also "less" than all other elements.
If $y <_2 x$, then y is a minimal element because we have shown that $<_2$ is transitive and therefore y is "less" than all other elements because it is "less" than x which is less than all others excluding y and this makes y minimal because $<_2$ is asymmetric.
Therefore our order is well founded.

(Note: If we had not included in the induction hypothesis that the minimal element was "less" than all others, we would not have known that in the case of $y <_2 x$ that $\neg(z <_2 y)$ for all $z \in \mathbb{N} \times \mathbb{N}$, instead we would only know that $\neg(z <_2 x)$ for all $z \in \mathbb{N} \times \mathbb{N}$ where $z \neq y$ and that $\neg(x <_2 y)$. I.e. it would be plausible that x was not "less" than y, but that other things were "less" than y despite not being "less" than x)

d. $(\mathbb{N} \times \mathbb{N}, <_2)$ is not a total order:
Strict total orders have the properties of strict partial orders, but are also trichotomous so we must show $(\mathbb{N} \times \mathbb{N}, <_2)$ is not trichotomous:

$$\forall (x_1, x_2), (y_1, y_2) \in \mathbb{N} \times \mathbb{N} \; . \\ \neg((x_1, x_2) <_2 (y_1, y_2) \vee (y_1, y_2) <_2 (x_1, x_2) \vee (x_1, x_2) = (y_1, y_2))$$

Try $(x_1, x_2) = (3, 2),\ (y_1, y_2) = (2, 3)$:

$$\begin{aligned} & \neg((3,2) <_2 (2,3) \vee (2,3) <_2 (3,2) \vee (3,2) = (2,3)) \\ \iff & \neg(False \vee False \vee False) \\ \iff & True \end{aligned}$$

Therefore our order is not a strict total order.

Therefore $(\mathbb{N} \times \mathbb{N}, <_2)$ meets the given criteria.

8. Let Type be the inductive set (representing $\mathcal{B}$-types) defined in the lectures. Define $a(\alpha)$ be the number of $\mathbb{B}$ and Base constructors occurring in α and $b(\alpha)$ be the number of $\mathsf{Function}$ and $\mathsf{Product}$ constructors occurring in α. Prove by structural induction that, for all $\alpha \in \mathsf{Type}$,

$$a(\alpha) \leq b(\alpha) + 1.$$

SOLUTION:

Proof Let $P(\alpha) \equiv a(\alpha) \leq b(\alpha) + 1$. We will prove $P(\alpha)$ for all $\alpha \in \mathsf{Type}$ by structural induction.

Base case: $\alpha = \mathbb{B}$ or $\alpha \in \mathcal{B}$. We need to prove $P(\alpha)$.

$$\begin{aligned} & a(\alpha) && \langle \text{LHS of } P(\alpha) \rangle \\ \leq\ & 1 && \langle \text{definition of } a \rangle \\ =\ & 0 + 1 && \langle \text{arithmetic} \rangle \\ =\ & b(\alpha) + 1 && \langle \text{definition of } b\text{; RHS of } P(\alpha) \rangle \end{aligned}$$

So $P(\alpha)$ holds.

Induction step: $\alpha = C(\beta_1, \beta_2)$ where C is $\mathsf{Function}$ or $\mathsf{Product}$ and $\beta_1, \beta_2 \in \mathsf{Type}$. Assume $P(\beta_1)$ and $P(\beta_2)$. We need to prove $P(\alpha)$.

$$\begin{aligned} & a(C(\beta_1, \beta_2)) && \langle \text{LHS of } P(\alpha) \rangle \\ =\ & a(\beta_1) + a(\beta_2) && \langle \text{definition of } a \rangle \\ \leq\ & (b(\beta_1) + 1) + (b(\beta_2) + 1) && \langle \text{induction hypothesis} \rangle \\ =\ & (1 + b(\beta_1) + b(\beta_2)) + 1 && \langle \text{arithmetic} \rangle \\ =\ & b(C(\beta_1, \beta_2)) + 1 && \langle \text{definition of } b\text{; RHS of } P(\alpha) \rangle \end{aligned}$$

So $P(\alpha)$ holds.

Therefore, $P(\alpha)$ holds for all $\alpha \in \mathsf{Type}$ by structural induction. $\square$

9. Construct a signature of MSFOL that is suitable for formalizing real number arithmetic.

   SOLUTION:

   Let $\Sigma = (\mathcal{B}, \mathcal{C}, \mathcal{F}, \mathcal{P}, \tau)$ be the MSFOL signature that is suitable for arithmetic over $\mathbb{R}$. Then:

$$\begin{aligned}
\mathcal{B} &= \{\mathbb{R}\} \\
\mathcal{C} &= \{0, 1\} \\
\mathcal{F} &= \{+, *, -, \div\} \\
\mathcal{P} &= \{=, <\}
\end{aligned}$$

   Where τ maps the set $\mathcal{C} \cup \mathcal{F} \cup \mathcal{P}$ to $\mathcal{B}$ as follows:

$$\begin{aligned}
\tau(0) &= \mathbb{R} \\
\tau(1) &= \mathbb{R} \\
\tau(+) &= \mathbb{R} \times \mathbb{R} \to \mathbb{R} \\
\tau(-) &= \mathbb{R} \times \mathbb{R} \to \mathbb{R} \\
\tau(*) &= \mathbb{R} \times \mathbb{R} \to \mathbb{R} \\
\tau(\div) &= \mathbb{R} \times \mathbb{R} \to \mathbb{R} \\
\tau(=) &= \mathbb{R} \times \mathbb{R} \to \mathbb{B} \\
\tau(<) &= \mathbb{R} \times \mathbb{R} \to \mathbb{B}
\end{aligned}$$