COMPSCI/SFWRENG 2FA3

Discrete Mathematics with Applications II

Winter 2020

Week 04 Exercises

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Background Definitions

Let $(S, <)$ be a strict partial order. $(S, <)$ is *dense* if, for all $x, y \in S$ with $x < y$, there is some $z \in S$ such that $x < z < y$. The strict total order $(\mathbb{Q}, <_{\text{rat}})$ of the rationals and the strict total order $(\mathbb{R}, <_{\text{real}})$ of the real numbers are both dense.

Exercises

1. Prove that $(\mathcal{P}(S), \subset)$ is a strict partial order where S is a nonempty set and $\mathcal{P}(S)$ is the power set of S.

2. Consider the weak partial order

$$P = (\{\{1\}, \{2\}, \{4\}, \{1, 2\}, \{1, 4\}, \{2, 4\}, \{3, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}, \subseteq).$$

   a. Find the maximal elements in P.
   b. Find the minimal elements in P.
   c. Find the maximum element in P if it exists.
   d. Find the minimum element in P if it exists.
   e. Find all the upper bounds of $\{\{2\}, \{4\}\}$ in P.
   f. Find the least upper bound of $\{\{2\}, \{4\}\}$ in P if it exists.
   g. Find all the lower bounds of $\{\{1, 3, 4\}, \{2, 3, 4\}\}$ in P.
   h. Find the greater lower bound of $\{\{1, 3, 4\}, \{2, 3, 4\}\}$ in P if it exists.

3. Let (U, I) where I is the *identity relation*, i.e., the binary relation such that $a \; I \; b$ iff $a = b$. Show that (U, I) is a weak partial order and not a weak total order.

4. Let $(\mathbb{Q} \cup \{-\infty, +\infty\}, <)$ be the strict total order such that $<$ is the same as $<_{\text{rat}}$ on $\mathbb{Q}$ and $-\infty$ and $+\infty$ are minimum and maximum elements, respectively, of $(\mathbb{Q} \cup \{-\infty, +\infty\}, <)$. Prove that

   $$(\mathbb{Q} \cup \{-\infty, +\infty\}, <)$$

   is dense. (Do not assume that $(\mathbb{Q}, <_{\text{rat}})$ is dense.)

5. Let $(S, <)$ be a strict total order such that there exist $a, b \in S$ with $a < b$ (i.e., S has at least two members). Show that, if $(S, <)$ is dense, then $(S, <)$ is not a well-order.

6. Consider the mathematical structure $(L, <_L)$ where L is a list of integers and $<_L$ is the binary relation on L defined by:

   $$[a_0, a_1, \ldots, a_n] <_L [b_0, b_1, \ldots, b_n] \text{ iff } \left(\sum_{i=0}^{n} a_i\right) < \left(\sum_{i=0}^{n} b_i\right).$$

   Prove that $(L, <_L)$ is a strict partial order that is not a strict total order.

7. Construct a strict partial order $(U, <)$ such that U is infinite, $<$ is well founded, and $(U, <)$ is not a total order (and thus $(L, <_L)$ is not a well-order).

8. Let Type be the inductive set (representing $\mathcal{B}$-types) defined in the lectures. Define $a(\alpha)$ be the number of $\mathbb{B}$ and Base constructors occurring in α and $b(\alpha)$ be the number of $\mathsf{Function}$ and $\mathsf{Product}$ constructors occurring in α. Prove by structural induction that, for all $\alpha \in \mathsf{Type}$,

   $$a(\alpha) \leq b(\alpha) + 1.$$

9. Construct a signature of MSFOL that is suitable for formalizing real number arithmetic.