Summary of Identities and Integration Techniques
From Math 1A03/1ZA3

1. Identities

Ratios & Definitions

$$\sin(\theta)=\frac{opp}{hyp}=\frac{y}{r},\quad \cos(\theta)=\frac{adj}{hyp}=\frac{x}{r},\quad \tan(\theta)=\frac{opp}{adj}=\frac{y}{x}$$

$$\sec(\theta)=\frac{1}{\cos(\theta)},\quad \csc(\theta)=\frac{1}{\sin(\theta)},\quad \tan(\theta)=\frac{\sin(\theta)}{\cos(\theta)},\quad \cot(\theta)=\frac{1}{\tan(\theta)}=\frac{\cos(\theta)}{\sin(\theta)}$$

Pythagorean Identities

$$\sin^2(\theta)+\cos^2(\theta)=1,\quad \tan^2(\theta)+1=\sec^2(\theta)$$

Double Angle & Half Angle Identities

$$\sin(2\theta)=2\sin(\theta)\cos(\theta),\quad \cos(2\theta)=\cos^2(\theta)-\sin^2(\theta)=1-2\sin^2(\theta)=2\cos^2(\theta)-1$$

$$\cos^2(\theta)=\frac{1}{2}(1+\cos(2\theta)),\quad \sin^2(\theta)=\frac{1}{2}(1-\cos(2\theta))$$

Hyperbolic Identities & Properties

$$\cosh(t)=\frac{e^t+e^{-t}}{2},\quad \sinh(t)=\frac{e^t-e^{-t}}{2},\quad \tanh(t)=\frac{e^t-e^{-t}}{e^t+e^{-t}}=\frac{\sinh(t)}{\cosh(t)}$$

$$\cosh^2(t)-\sinh^2(t)=1,\quad \cosh(t)+\sinh^2(t)=e^t$$

$$\frac{d}{dt}\cosh(t)=\sinh(t),\quad \frac{d}{dt}\sinh(t)=\cosh(t)$$

2. Basic Integral Properties

$$\int kf(x)\,dx=k\int f(x)\,dx \qquad \int f(x)\pm g(x)\,dx=\int f(x)\,dx\pm\int g(x)\,dx$$

$$\int_a^b f(x)dx=F(b)-F(a)=F(x)\Big|_a^b,\ \text{if } F(x) \text{ is continuous on } [a,b]$$

3. Common Basic Integrals

$$\int 1\ dx=x+C \quad \int x\ dx=\frac{1}{2}x^2+C \quad \int\frac{1}{x}dx=\ln|x|+C \quad \int x^p dx=\frac{1}{p+1}x^{p+1}+C,\ \text{if } p\neq 1$$

$$\int e^x\,dx=e^x+C \qquad \int a^x\,dx=\frac{1}{\ln(a)}a^x+C$$

$$\int\sinh(ax)dx=\frac{1}{a}\cosh(ax)+C \qquad \int\cosh(ax)dx=\frac{1}{a}\sinh(ax)+C$$

$$\int\sin(ax)dx=-\frac{1}{a}\cos(ax)+C \qquad \int\cos(ax)dx=\frac{1}{a}\sin(ax)+C$$

$$\int \sec^2(ax)dx = \frac{1}{a}\tan(ax) + C \qquad \int \tan(ax)\sec(ax)dx = \frac{1}{a}\sec(ax) + C$$

$$\int \frac{1}{\sqrt{1-x^2}}dx = \arcsin(x) + C \qquad \int \frac{1}{1+x^2}dx = \arctan(x) + C \qquad \int \frac{1}{a^2+x^2}dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$$

4. Substitutions

If $\int_a^b f(g(x))\,dx$, let $u = g(x)$, $du = g'(x)dx$, Replace $a \rightarrow g(a)$, $\rightarrow g(b)$ as endpoints, and re-express the entire integral in terms of u. (And hope it is an improvement.)

5. Integration by Parts

$\int u\,dv = uv - \int v\,du$, or equivalently, $\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,du$

- The *ENTIRE* original integral must correspond to $\int u\,dv$ to use this identity
- Note that this is an integral identity, and "bad" choices of u,v will still give true statements, but may make the result harder to integrate!
- Choose the u to simplify the integral. General rule of thumb (not a hard-and-fast rule)
  - Best u's: $\ln(x)$, $\arctan(x)$, $\arcsin(x)$
    (ie. *Things with nice derivatives, bad integrals*)
  - Middling u's: x, x^2, x^p
  - Poor u's e^x: $\cos(x)$, $\sin(x)$, $\cosh(x)$, $\sinh(x)$
    (ie. *Things where derivative makes next to no change in the expression*)
- $a^x\sin(bx)$, or $a^x\cosh(bx)$, or any case where both terms are one of the "bad" u choices, integrate by parts twice to get the same integral type on both sides of the equation and solve for the integral.
- Don't be afraid to use $dv = 1dx$ if needed.

6. Special Trigonometric Integrals

Integrating $\sin^n(x)\cos^m(x)$

- If n odd, $u = \cos(x)$, $du = -\sin(x)\,dx$, use $\sin^2 x = 1 - \cos^2 x = 1 - u^2$ to simplify.
- If m odd, $u = \sin(x)$, $du = \cos(x)dx$, use $\cos^2 x = 1 - \sin^2 x = 1 - u^2$ to simplify.
- Both powers are even, use half-angle identities to reduce the powers
  $\cos^2 x = (1+\cos(2x))/2$, $\sin^2 x = (1-\cos(2x))/2$
  and try to integrate again.

Integrating $\sec^m x \tan^n x$

- If m even, use substitution, $u = \tan x$, $du = \sec^2 x\,dx$.and $\sec^2 x = 1 + \tan^2 x = 1 + u^2$
- If n odd, use substitution, $u = \sec x$, $du = \sec x \tan x\,dx$. $\tan^2 x = \sec^2 x - 1 = u^2 - 1$
- Otherwise get creative! Often required: $\int \sec(x)dx = \ln|\sec(x) + \tan(x)| + C$

7. Trig. Substitutions

Usually only used if you have fractional powers of squares of x.

If You Have	Substitute	Conversion	Differential	Domain
$\sqrt{a^2-x^2}$	$x=a\sin(t)$	$\sqrt{a^2-x^2}=\sqrt{a^2-a^2\sin^2(t)}=\sqrt{a^2\cos^2(t)}=a\cos(t)$	$dx=a\cos(t)$	$t\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$
$\sqrt{a^2+x^2}$	$x=a\tan(t)$	$\sqrt{a^2+x^2}=\sqrt{a^2+a^2\tan^2(t)}=\sqrt{a^2\sec^2(t)}=a\sec(t)$	$dx=a\sec^2(t)$	$t\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$
$\sqrt{x^2-a^2}$	$x=a\sec(t)$	$\sqrt{x^2-a^2}=\sqrt{a^2\sec^2(t)-a^2}=\sqrt{a^2\tan^2(t)}=a\tan(t)$	$dx=a\sec(t)\tan(t)$	$t\in\left[0,\frac{\pi}{2}\right]\cup\left[\pi,\frac{3\pi}{2}\right]$

8. Partial Fractions

Given $f(x)=\dfrac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials, we want to write $f(x)$ as the sum of simpler fractions.

- If we have order $P(x)\geq$ order $Q(x)$, first do ("synthetic") division.
- If $Q(x)$ includes the factor $(x-a)$, then we have a term $\dfrac{A}{x-a}$
- If $Q(x)$ includes $(x-a)^n$, then we have terms $\dfrac{A_1}{x-a}+\dfrac{A_2}{(x-a)^2}+\ldots\dfrac{A_n}{(x-a)^n}$
- If $Q(x)$ includes an irreducible quadratic factor, (ax^2+bx+c), ie. where $b^2-2ac\leq 0$

  then we have terms $\dfrac{Ax+B}{ax^2+bx+c}$
- If $Q(x)$ includes a repeated irreducible quadratic factor, $(ax^2+bx+c)^n$,

  then we have terms $\dfrac{A_1x+B_1}{ax^2+bx+c}+\dfrac{A_2x+B_2}{(ax^2+bx+c)^2}++\dfrac{A_nx+B_n}{(ax^2+bx+c)^n}$

Once you have the form, solve for the constants by multiplying both sides of the equation by the original denominator, Q(x), to eliminate fractions. Then solve for the constants by setting x to convenient values, or by comparing and equating coefficients of powers of x.