Series Convergence & Divergence Summary:

Special Series with Known Convergence:

"p"-series:

$$\sum \frac{1}{n^p} \qquad \text{converges if } p > 1 \quad \text{and} \quad \text{diverges if } p \le 1$$

The Only Two Types ***of (non-power) series, in this course, for which we that we can compute exact values:***

Geometric Series:

$$\sum_{n=0}^{\infty} ar^{n-1} = \sum_{n=1}^{\infty} ar^n = \frac{a}{1-r} \quad \text{if } |r| < 1, \ \text{ diverges if } |r| > 1.$$

Telescoping Series:

Typical case:

$$\sum_{n=0}^{\infty} \big(f(n+1) - f(n)\big) \quad \Rightarrow \quad \begin{aligned} S_m = {} & f(1) - f(0) \\ & + f(2) - f(1) \\ & + f(3) - f(2) \\ & \quad \ddots \\ & + f(m-1) - f(m-2) \\ & + f(m) - f(m-1) \\ & + f(m+1) - f(m) \end{aligned} \quad \Rightarrow \quad \begin{aligned} \sum_{n=0}^{\infty} \big(f(n+1) - f(n)\big) &= \lim_{m\to\infty} S_m \\ &= \Big(\lim_{m\to\infty} f(m+1)\Big) - f(0) \end{aligned}$$

In General:

$$\sum_{n=n_0}^{\infty} \big(f(n+p) - f(n)\big), \text{ where } n_0, p \text{ are positive integers}$$

$$\begin{aligned} = \sum_{n=n_0}^{\infty} \big(f(n+p) - f(n)\big) &= \lim_{m\to\infty}\big(f(m+1) + f(m+2) + \ldots + f(m+p)\big) \\ &\quad - \big(f(n_0) + f(n_0+1) + \ldots + f(n_0+p-1)\big) \\ &= \Big(p \lim_{m\to\infty} f(m)\Big) - \big(f(n_0) + f(n_0+1) + \ldots + f(n_0+p-1)\big) \end{aligned}$$

Convergence Tests:

Divergence Test:

$\lim_{n\to\infty} a_n \neq 0$ then $\sum a_n$ diverges (Notice, this test CANNOT show $\sum a_n$ converges!)

Alternating Series Test (AST):

An alternating series $\sum (-1)^n b_n$ converges if:

1) $b_n \geq b_{n+1}$ (monotonically decreasing)

2) $\lim_{n\to\infty} b_n = 0$

(Notice, this test CANNOT show $\sum a_n$ diverges! If either of the two criteria fail, the test fails)

(***But:*** *If the limit is non-zero, this is the condition for the* *Divergence Test*, *and the series will* *diverge*)

The Following Four Tests *are for use on Non-Negative series.*

Each can be used on the positive form of the series, $\sum |a_n|$, to demonstrate Absolute Convergence.

1) Comparison Test:

$\underline{\textit{Given } \sum a_n \text{ and } \sum b_n \textit{ non-negative series}}$

$$a_n \leq b_n \quad \Rightarrow \quad \begin{cases} \sum b_n \text{ converges} \Rightarrow \sum a_n \text{ converges} \\ \sum b_n \text{ diverges} \Rightarrow \text{no information} \end{cases}$$

$$a_n \geq b_n \quad \Rightarrow \quad \begin{cases} \sum b_n \text{ diverges} \Rightarrow \sum a_n \text{ diverges} \\ \sum b_n \text{ converges} \Rightarrow \text{no information} \end{cases}$$

2) Limit Comparison Test:

$\underline{\textit{Given } \sum a_n \text{ and } \sum b_n \textit{ non-negative series}}$

$$\text{if } \lim_{n\to\infty}\frac{a_n}{b_n} \text{ exits, then } \lim_{n\to\infty}\frac{a_n}{b_n} = \begin{cases} \infty \Rightarrow \sum a_n \text{ diverges if } \sum b_n \text{ diverges} \\ \# \Rightarrow \begin{cases} \sum a_n \text{ diverges if } \sum b_n \text{ diverges, and} \\ \sum a_n \text{ converges if } \sum b_n \text{ converges} \end{cases} \\ 0 \Rightarrow \sum a_n \text{ converges if } \sum b_n \text{ converges} \end{cases}$$

3) Ratio Test:

$\underline{\textit{Given } \sum a_n, \textit{ a non-negative series}}:$

$$\text{If } \left(\lim_{n\to\infty}\frac{a_{n+1}}{a_n}\right) \text{ exists, then } \quad \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \begin{cases} > 1 \Rightarrow \sum a_n \text{ Diverges} \\ < 1 \Rightarrow \sum a_n \text{ Converges} \\ = 1 \Rightarrow \text{Test Fails, No Information} \end{cases}$$

4) Root Test:

$\underline{\textit{Given } \sum a_n, \textit{ a non-negative series}}:$

$$\text{If } \left(\lim_{n\to\infty}\sqrt[n]{a_n}\right) \text{ exists, then } \quad \lim_{n\to\infty}\sqrt[n]{a_n} = \begin{cases} > 1 \Rightarrow \sum a_n \text{ Diverges} \\ < 1 \Rightarrow \sum a_n \text{ Converges} \\ = 1 \Rightarrow \text{Test Fails, No Information} \end{cases}$$

(*Note: Usually, the ratio test is easier, use this only when we have a_n's with n exponents.*)

Error Estimates:

Alternate Series Error Estimate:

Given an alternating series $\sum (-1)^n b_n$ which converges: $|S - S_m| \le b_{n+1}$

Strategies to Test Convergence:

There's no hard and fast algorithm to checking for convergence, the best way to learn which tests to use when, is to practice, practice, practice. Nothing beats experience.

With that said, here are some tips:

- Make sure you are working with a series. Often people try to use series convergence rules on Sequences. This of course is horrible. The tests of series convergence are usable only on series.

- If someone asks for the EXACT value of an infinite series, the series in question is usually a variant of telescoping, or geometric series, or some combination thereof.

- Always at least consider the Divergence test first. If the series is ugly, taking the limit may be too much hard work, but if you can see, with quick inspection, that the terms do not go to zero, the series diverges, and you've saved yourself some time and trouble.

- Always look for the usual suspects. Identify telescoping, geometric or p-series first.

- If the series *ALMOST* is one of the special cases, try the comparison/limit comparison tests.

- Alternating series test is one of the easier tests. If you get an alternating series, try this test first.

- If doing the AST, make sure you always check the limit before the monotonicity. If the limit doesn't go to zero, AST not only fails, but the ***Divergence test*** says that the series *must diverge*. This often saves work.

- If the series has factorials (!), it's often nice to use ratio test.

- Ratio test ALWAYS fails on rational functions. If your series terms are polynomials, or ratios of polynomials, the test will always yield a limit of 1, and thus give no information.

  (Also, although not of issue here, Ratio test on power series, BY DEFINITION, never work on the endpoints of an interval of convergence! You'll have to try something else!)

- Avoid using the root test unless your series terms are variants of the form: $a_n = (f(n))^n$. In cases such as these, the nth power and your nth root in the test nicely cancel.

- Sometimes negatives get in the way. It may be easier to look at the Absolute convergence of a series, to show convergence, than looking at the behaviour of the series itself. Remember, though, the divergence of the absolute value series does NOT mean the divergence of the original.

Strategies to Test Absolute or Conditional Convergence:

- If your series is already strictly non-negative, absolute convergence and convergence are the same thing. A positive series cannot be conditionally convergent.

- Absolute convergence is, of course a "stronger" condition than conditional. Always test for the abs. convergence first.

- If the absolute value terms produce a finite sum, then we have absolute convergence, (and thus convergence)

  Remember, *the divergence of the absolute value series does NOT mean the divergence of the original series. There is no absolute divergence.*

- If the absolute value series diverges, we still must test for conditional convergence. This often means using AST.

  Remember, *A conditionally convergent series is still convergent. Just not absolutely convergent. There is no such thing as conditional divergence.*

- Ratio, and to a lesser extent, Root tests are popular for testing absolute convergence.

  This is primarily because they tend to be relatively easy to compute, and perhaps more relevantly, they require no insight to use.

  Unfortunately, in many cases these tests do fail to give you any information, so be ready to use one of the comparison tests when needs be.