MATHEMATICS 1LT3 TEST 2

Evening Class
Duration of Examination: 60 minutes
McMaster University, 27 February 2014

Dr. M. Lovrić

FIRST NAME (please print): SOLUTIONS

FAMILY NAME (please print): ____________

Student No.: ____________

THIS TEST HAS 8 PAGES AND 7 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE.

Total number of points is 40. Marks are indicated next to the problem number. Any non-graphing calculator is allowed.

USE PEN TO WRITE YOUR TEST. IF YOU USE A PENCIL YOUR TEST WILL NOT BE ACCEPTED FOR REMARKING (IF NEEDED).

You must show work to receive full credit.

Problem	Points	Mark
1	6	
2	5	
3	6	
4	7	
5	5	
6	6	
7	5	
TOTAL	40	

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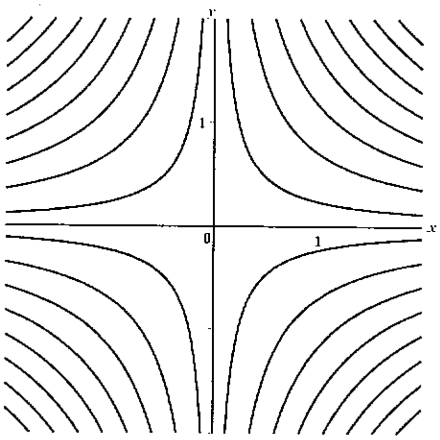
1. **Multiple choice questions: circle ONE answer. No justification is needed.**

(a)[3] Identify all functions (if any) whose contour diagram is shown below.

(I) $f(x,y) = e^{x^2+y^2} = C$

(II) $f(x,y) = e^{xy} = C$

(III) $f(x,y) = y - \frac{2}{x} = C$



I ... $x^2+y^2 = \ln C$ (circles)

II $xy = \ln C \to y = \frac{\ln C}{x}$

(A) none (B) I only (C) II only (D) III only

(E) I and II (F) I and III (G) II and III (H) all three

III $y = \frac{2}{x} + C$ (hyperbolas in 1st and 3rd quadrants)

(b)[3] The humidex $H(T,h)$ is a measure used by meteorologists to describe the combined effects of heat and humidity on an average person's feeling of hotness. In the table below we give values of humidex based on measurements of temperature (in degrees Celsius) and relative humidity h (given as a percent).

	$T = 22$	$T = 26$	$T = 30$	$T = 34$
$h = 70$	27	33	41	49
$h = 60$	25	32	38	46
$h = 50$	24	30	36	43

Which of the following statements is/are true?

(I) $H_T(30, 60)$ is negative X (II) $H_h(26, 60)$ is positive ✓ (III) $H_T(26, 60)$ is negative X

(A) none (B) I only (C) II only (D) III only

(E) I and II (F) I and III (G) II and III (H) all three

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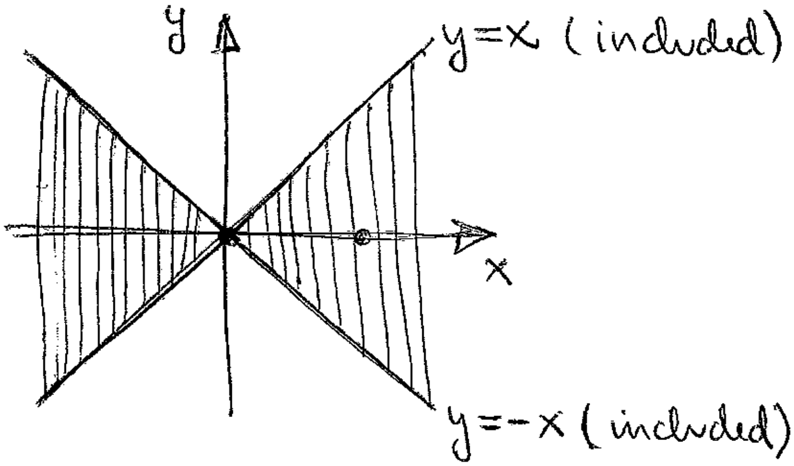
Questions 2-7: You must show work to receive full credit.

2. (a)[3] Find the domain of the function $g(x,y) = \sqrt{x^2 - y^2}$. Answer the question by shading the region(s) that form the domain. Clearly indicate whether or not the boundary lines/curves belong to the domain.

$x^2 - y^2 \geq 0$ .. $x^2 - y^2 = 0$, $y^2 = x^2$, $y = \pm x$

four regions, use test points



(b)[2] Find the range of the function $f(x,y) = \dfrac{4}{x^2 + y^2 + 1}$. You do not need to provide a formal proof as we did in class, but have to correctly explain your reasoning.

$f(x,y) \geq 0$ as a quotient of positive numbers

$f(x,y) > 0$ since the numerator $\neq 0$

as well, $x^2 + y^2 + 1 \geq 1$, so $\dfrac{4}{x^2+y^2+1} \leq 4$

(ie when we divide 4 by a number ≥ 1 we get a number ≤ 4)

range: $(0, 4]$

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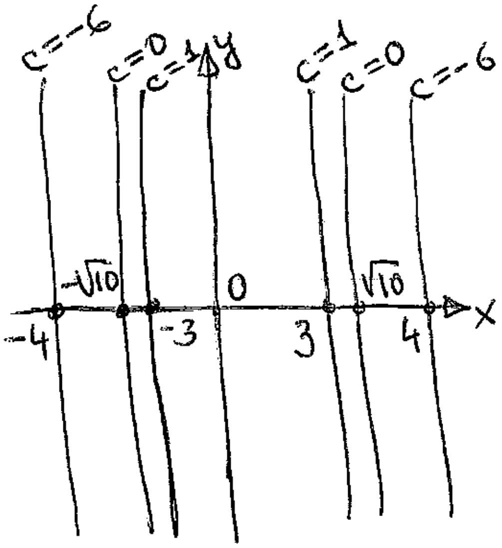
3. (a)[3] Sketch a contour diagram of the function $f(x, y) = 10 - x^2$ by drawing level curves which correspond to $z = 0$, $z = 1$ and $z = -6$. Label all curves in your diagram.



$f = 10 - x^2 = c \rightarrow x = \pm\sqrt{10-c}$

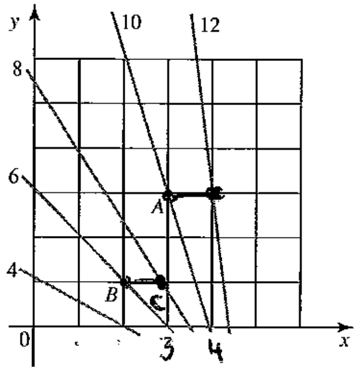
$c = 0 \ldots\ x = \pm\sqrt{10} \approx 3{,}16$

$c = 1 \ldots\ x = \pm 3$

$c = -6 \ldots\ x = \pm 4$

(vertical lines)

Given below is a contour diagram of the function $h(x, y)$.



(b)[1] Find an estimate for the partial derivative h_x at A. Explain your reasoning.

$$h_x(A) \approx \frac{12-10}{4-3} = 2$$

$(h_y)_x$

(c)[2] What is the sign of the second derivative h_{yx} at B? Explain your reasoning.

$$h_y(B) \approx \frac{2}{1{,}5}$$

$$h_y(C) \approx \frac{2}{2{,}5}$$

as we move away from B in the x-direction, hy decreases

so $(h_y)_x < 0$

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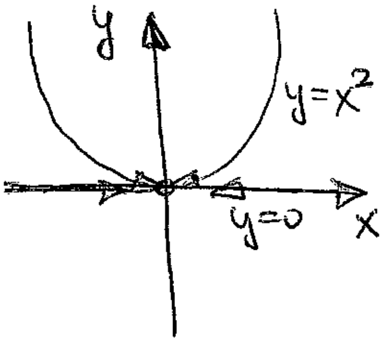
4. (a)[2] Somebody tells you that $\lim_{(x,y)\to(0,0)} f(x,y) = 4$. You try to check the answer by computing the limit along different paths. Along $y = x$ you compute the limit to be 4, but along $y = -x$ you obtain the value of -4 for the limit. Based on these answers, what can you conclude? Explain.

If somebody is right, then I made a mistake in calculating the limit along $y=-x$.
If my calculations are correct, then limit dne $\Rightarrow$ somebody is wrong

(b)[3] Show that $\lim_{(x,y)\to(0,0)} \frac{x^2y}{2x^4 + 3y^2}$ does not exist. Sketch the domain and the paths that you chose.



along $y=0$... $\lim = \lim_{x\to 0} \frac{0}{2x^4} = 0$

along $y=x$... $\lim = \lim_{x\to 0} \frac{x^3}{2x^4+3x^2}$

$= \lim_{x\to 0} \frac{x}{2x^2+3} = 0$

or

along $y=x^2$... $\lim = \lim_{x\to 0} \frac{x^4}{2x^4+3x^4} = \frac{1}{5}$

limit dne

(c)[2] Determine whether or not the function

$$f(x,y) = \begin{cases} \frac{2e^{x+y}}{xy-1} & \text{if} \quad (x,y) \neq (1,-1) \\ -1 & \text{if} \quad (x,y) = (1,-1) \end{cases}$$

is continuous at $(1, -1)$.

$$\lim_{(x,y)\to(1,-1)} f(x,y) = \lim_{(x,y)\to(1,-1)} \frac{2e^{x+y}}{xy-1} = \frac{2e^0}{-1-1} = -1$$

equal to $f(1,-1)$

So f $\underline{\underline{\text{is}}}$ cont. at $(1,-1)$

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5. Certain amount of PCBs (polychlorinated biphenyls, widely used as engine coolants) was released into Lake Ontario near Pickering. The function

$$c(x,t) = \frac{120}{t\sqrt{4\pi}} e^{-x^2/t}$$

models the concentration of PCBs (measured in milligrams of PCBs per litre of lake water) at a location x kilometres away from Pickering, t days after the contamination occurred.

(a)[3] Find the partial derivative $c_x(1,4)$. Say what your answer implies about the concentration of PCBs.

$$c_x = \frac{120}{t\sqrt{4\pi}} e^{-\frac{x^2}{t}} \left(-\frac{2x}{t}\right) = \frac{-240x}{t^2\sqrt{4\pi}} e^{-x^2/t}$$

$$= \frac{-120x}{t^2\sqrt{\pi}} e^{-x^2/t} < 0$$

$$c_x(1,4) \approx -3.30$$

On day 4 ($t=4$ is fixed), the concentration at a location $x=1$ km from the source decreases at a rate of approximately 3.30 mg per L per kilometre

(b)[2] You computed the second partial derivative and obtained $c_{xx}(1,4) \approx 1.56$. Interpret your answer.

$$\overset{\parallel}{(c_x)_x}$$

$(c_x)_x$ is the rate of change in the decrease c_x of the concentration

Since it is $\oplus$ve, the amounts of decrease are increasing with distance

so concentration decreases, but at a slower and slower rate

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6. Type-2 functional response in a predator-prey context is given by

$$c(a, N, T_h) = \frac{aN}{1 + aNT_h},$$

where $c(a, N, T_h)$ is the number of prey captured, a is the attack rate (number of atacks per day), N is the density of prey (number of prey per square kilometre) and T_h is the handling time (in days).

(a)[3] Compute $\partial c/\partial a$ and say what it means in the context of the predator-prey interactions.

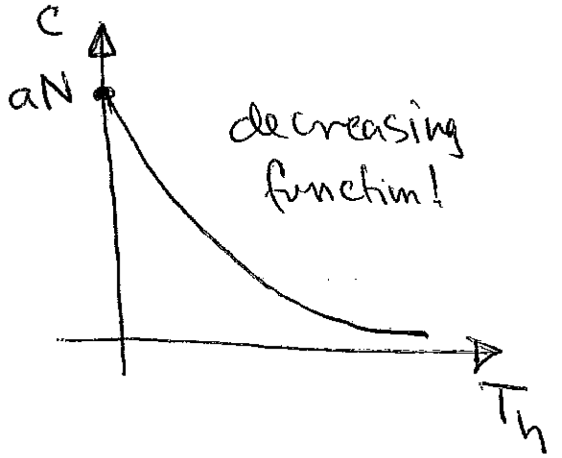
$$\frac{\partial c}{\partial a} = \frac{N(1+aNT_h) - aN(NT_h)}{(1+aNT_h)^2} = \frac{N}{(1+aNT_h)^2}$$

So $\frac{\partial c}{\partial a} > 0$ ie, the number of prey captured is an increasing function of the attack rate

or: as the attack rate increases, so does the number of prey captured

(b)[3] Consider $c(T_h)$, i.e., view c as a function of T_h. Sketch the graph of $c(T_h)$ (label intercepts, if any) and explain why it makes sense.

$$c(T_h) = \frac{aN}{1+aNT_h} = \frac{\text{constant}}{1 + \text{constant} \cdot T_h} \quad \left(\text{looks like } \frac{2}{1+3x} \right)$$



as the handling time increases (ie, the time between successive hunts increases) the number of prey captured with a and N unchanged! decreases

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7. The function $s(x,d) = e^{-x+0.25d^2}$ counts the number of seeds (in some units) that fell within the distance x from a tree; d is the density of the seeds.

(a)[2] Find the linear approximation $L_{(1,2)}(x,d)$ of $s(x,d)$ near $(1,2)$.

$$s(x,d) = e^{-x+0.25d^2} \quad \therefore \quad s(1,2) = e^0 = 1$$

$$s_x = -e^{-x+0.25d^2} \quad \therefore \quad s_x(1,2) = -1$$

$$s_d = e^{-x+0.25d^2}(0.5d) \quad \therefore \quad s_d(1,2) = 1$$

$$L_{(1,2)}(x,d) = \underbrace{s(1,2)}_{1} + \underbrace{s_x(1,2)}_{-1}(x-1) + \underbrace{s_d(1,2)}_{1}(d-2)$$

$$= 1 - (x-1) + (d-2) = -x + d$$

(b)[3] Find the quadratic approximation of $s(x,d)$ near $(1,2)$.

$$s_{xx} = e^{-x+0.25d^2} \quad \therefore \quad s_{xx}(1,2) = 1$$

$$s_{xd} = -e^{-x+0.25d^2}(0.5d) \quad \therefore \quad s_{xd}(1,2) = -1$$

$$s_{dd} = e^{-x+0.25d^2}(0.5d)(0.5d) + e^{-x+0.25d^2}(0.5)$$

$$\therefore \quad s_{dd}(1,2) = 1.5$$

$$T_2(x,d) = L_{(1,2)}(x,d) + \frac{1}{2}\Big(\overset{1}{s_{xx}}(x-1)^2 + 2\overset{-1}{s_{xd}}(x-1)(d-2) + \underset{1.5}{s_{dd}}(d-2)^2\Big)$$

$$= -x + d + \frac{1}{2}(x-1)^2 - (x-1)(d-2) + \underbrace{\frac{1.5}{2}}_{0.75}(d-2)^2$$

THE END