MATHEMATICS 1LT3 TEST 1

Evening Class
Duration of Test: 60 minutes
McMaster University

E. Clements

6 February 2017

FIRST NAME (please print): Sol^{N}s

FAMILY NAME (please print): ______

Student No.: ______

THIS TEST HAS 8 PAGES AND 6 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE.

Total number of points is 40. Marks are indicated next to the problem number in square brackets. You may use the McMaster standard calculator, Casio fx991 MS+, on this test.

USE PEN TO WRITE YOUR TEST. IF YOU USE A PENCIL, YOUR TEST WILL NOT BE ACCEPTED FOR REMARKING (IF NEEDED).

You need to show work to receive full credit, except for Question 1.

Problem	Points	Mark
1	6	
2	6	
3	7	
4	8	
5	6	
6	7	
TOTAL	40	

MATH 1LT3 * Test 1 * 6 February 2017

Name:____________________

Student No.: ____________________

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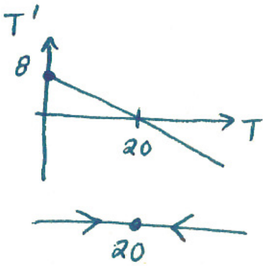
1. **Multiple Choice**. Clearly **circle** the one correct answer.

(a) [3] Suppose the temperature of an object changes according to $dT/dt = 0.4(20 - T)$, where $T(0) = 15$. Which of the following statements is/are true?



(I) dT/dt is a decreasing function of T. ✓

(II) $T = 20$ is a stable equilibrium. ✓

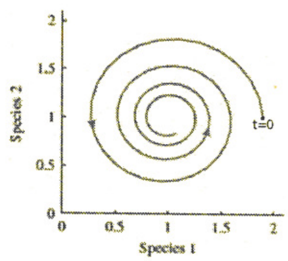
(III) $T(t) = 15e^{-0.4t}$ is the solution of the initial value problem. ✗

$T' = 15e^{-0.4t}(-0.4) \Rightarrow T' = -0.4T \neq 0.4(20-T)$

(A) none (B) I only (C) II only (D) III only

(E) I and II (F) I and III (G) II and III (H) all three

(b) [3] Consider the phase-plane trajectory given below.



Which of the following statements is/are true?

(I) The population of both species changes periodically. ✓

(II) The size of both populations approaches 1 as $t \to \infty$. ✓

(III) The maximum of both populations occurs when $t = 0$. ✗

NOTE: Answer (C) will also be accepted as correct

(A) none (B) I only (C) II only (D) III only

(E) I and II (F) I and III (G) II and III (H) all three

Name:

Student No.:

2. State whether each statement is **true or false**. **Explain** your reasoning.

(a) [2] Consider the selection model, $dp/dt = 0.4p(1-p)$, where $p(t)$ represents the proportion of abnormal cells within a tissue sample at time t and $p(0) = 0.25$. Using Euler's method with a step size $h = 1$, we find that $p(2) \approx 0.34$.

$t_0 = 0$
$p_0 = 0.25$

$t_1 = 1$
$p_1 = 0.25 + 0.4(0.25)(1-0.25)\cdot 1$
$= 0.325$

$t_2 = 2$
$p_2 = 0.325 + 0.4(0.325)(1-0.325)\cdot 1$
≈ 0.413

$\therefore$ FALSE

$p(2) \approx 0.41$

(b) [2] $y^* = 0$ is a stable equilibrium of $dy/dt = \underbrace{ye^{-y} - 0.5y}_{f(y)}$.

$f(0) = 0 \cdot e^0 - 0.5(0) = 0$
$\Rightarrow y^* = 0$ is an eq^{m}

$f'(y) = 1 \cdot e^{-y} + ye^{-y}(-1) - 0.5$
$f'(0) = e^{-0} + 0 - 0.5 = 0.5 \Rightarrow y^* = 0$ is unstable

$\therefore$ FALSE

(c) [2] The system $dx/dt = -x + xy$ and $dy/dt = -y + xy$, could describe a situation in which two predators need to eat each other in order to survive.

TRUE. In the absence of the other species, both predators will die out:

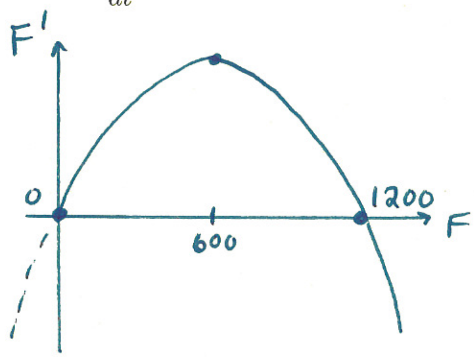
$$\frac{dx}{dt} = -x \qquad \frac{dy}{dt} = -y$$

coefficients of both interaction terms are $+$

Name:____________________
Student No.: ____________________

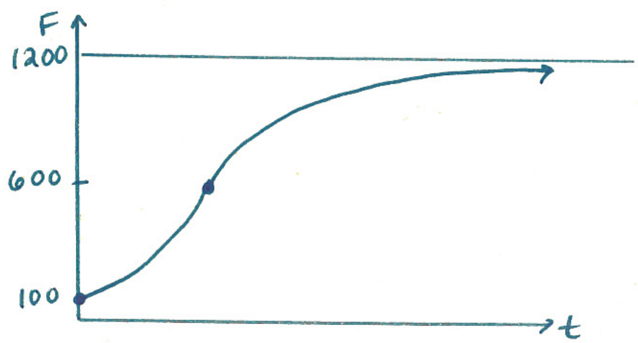
3. A population of frogs is modelled by $\dfrac{dF}{dt} = 0.2F\left(1 - \dfrac{F}{1200}\right)$.

(a) [2] Graph $\dfrac{dF}{dt}$ as a function of F.



(b) [2] Draw a phase-line diagram for $\dfrac{dF}{dt} = 0.2F\left(1 - \dfrac{F}{1200}\right)$.

(c) [2] Suppose that initially there are 100 frogs. Sketch the solution curve $F(t)$.



(d) [1] Suppose it is known that the population will die out if it falls below 80 frogs. Write a modified logistic differential equation to model this.

$$\frac{dF}{dt} = 0.2F\left(1 - \frac{F}{1200}\right)\left(1 - \frac{80}{F}\right)$$

Name:

Student No.:

4. Suppose that a virus is spreading within an isolated population according to

$$dI/dt = 0.3I(1 - I) - 0.2I$$

where $I(t)$ represents the fraction of the population infected at time t.

(a) [2] Find all biologically meaningful equilibria for this equation.

$f(I) = I[0.3(1-I) - 0.2]$

$f(I) = 0$ when $I^* = 0$ or $1 - I^* = \frac{0.2}{0.3}$

$\Rightarrow I^* = \frac{1}{3}$

(b) [3] Determine the stability of the equilibria found in part (a).

$f(I) = 0.3I - 0.3I^2 - 0.2I$

$f'(I) = 0.3 - 0.6I - 0.2 = 0.1 - 0.6I$

$f'(0) = 0.1 \Rightarrow I^* = 0$ is an unstable eq^{M}

$f'(\frac{1}{3}) = 0.1 - 0.6(\frac{1}{3}) = -0.1 \Rightarrow I^* = \frac{1}{3}$ is a stable eq^{M}

(c) [2] Suppose that initially 100% of the population was infected with the virus. Describe how this proportion will change over time.

This proportion of infected individuals will decrease and approach $\frac{1}{3}$ of the whole population.

(d) [1] Suppose that S represents the fraction of the population susceptible at time t such that $S = 1 - I$. Write an autonomous differential equation for dS/dt.

$$\frac{dS}{dt} = -0.3(1-S)S + 0.2(1-S)$$

Name:

Student No.:

5. Use the separation of variables technique to solve each initial value problem.

(a) [3] $\frac{dy}{dx} = \frac{\ln x}{xy}$, where $y(1) = 2$

Let $u = \ln x$. Then $du = \frac{1}{x}dx$

$\int \frac{\ln x}{x} dx = \int u\, du$

$= \frac{u^2}{2} + C$

$= \frac{(\ln x)^2}{2} + C$

$\int y\, dy = \int \frac{\ln x}{x} dx$

$\frac{y^2}{2} = \frac{(\ln x)^2}{2} + C$

$y(1) = 2 \Rightarrow \frac{2^2}{2} = \frac{(\ln 1)^2}{2} + C \Rightarrow C = 2$

$\therefore \frac{y^2}{2} = \frac{(\ln x)^2}{2} + 2$

$y^2 = (\ln x)^2 + 4$

$\therefore y = \sqrt{(\ln x)^2 + 4}$

(b) [3] $\frac{dx}{dt} = \frac{x+1}{1+t^2}$, where $x(0) = 1$

$\int \frac{1}{x+1} dx = \int \frac{1}{1+t^2} dt$

$\ln|x+1| = \arctan t + C$

$|x+1| = e^{\arctan t + C}$

$x+1 = \pm e^{C} \cdot e^{\arctan t}$

$x = Ke^{\arctan t} - 1$, where $K = \pm e^{C}$

$x(0) = 1 \Rightarrow 1 = Ke^{\arctan(0)} - 1 \Rightarrow 1 = K - 1 \Rightarrow K = 2$

$\therefore x = 2e^{\arctan t} - 1$

Name:__________________

Student No.: __________________

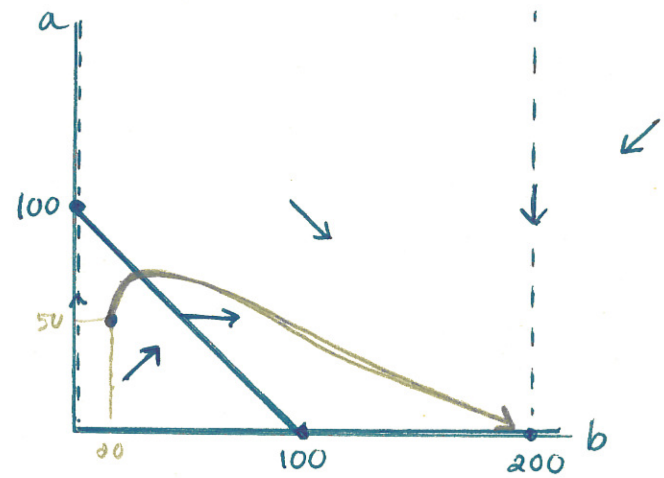
6. Consider the modified competition equations

$$\frac{da}{dt} = 2\left(1 - \frac{a+b}{100}\right)a, \quad \frac{db}{dt} = \left(1 - \frac{b}{200}\right)b$$

(a) [2] Find and graph the nullclines in the phase plane.

$a' = 0$ when $a = 0$ or $1 - \frac{a+b}{100} = 0 \Rightarrow a + b = 100 \Rightarrow a = 100 - b$

$b' = 0$ when $b = 0$ or $1 - \frac{b}{200} = 0 \Rightarrow b = 200$



(b) [1] Identify the equilibria.

$(200, 0)$, $(0, 100)$, $(0, 0)$

(c) [2] Add direction arrows to your phase-plane diagram in part (a).

(d) [2] Sketch a phase-plane trajectory starting from $a(0) = 20$ and $b(0) = 50$.