MATHEMATICS 1LT3 TEST 3

Day Class
Duration of Test: 60 minutes
McMaster University

E. Clements

16 March 2016

FIRST NAME (please print): SOLNS

FAMILY NAME (please print): ______

Student No.: ______

THIS TEST HAS 8 PAGES AND 8 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE.

Total number of points is 40. Marks are indicated next to the problem number in square brackets. You may use the McMaster standard calculator, Casio fx991 MS+, on this test.

USE PEN TO WRITE YOUR TEST. IF YOU USE A PENCIL, YOUR TEST WILL NOT BE ACCEPTED FOR REMARKING (IF NEEDED).

You need to show work to receive full credit, except for Multiple Choice.

Problem	Points	Mark
1	9	
2	6	
3	7	
4	3	
5	4	
6	4	
7	4	
8	3	
TOTAL	40	

MATH 1LT3 * Test 3 * 16 March 2016

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1. Multiple Choice. Clearly circle the one correct answer.

(a) [3] A partial table of values for a function $f(x, y)$ is given below. Which of the following is / are positive?

(I) $f_{xx}(4,0)$ $\ominus$ (II) $f_{xx}(4,1)$ $\oplus$ (III) $f_{xx}(4,2)$ $\ominus$

	$x=3$	$x=4$	$x=5$	$x=6$
$y=0$	2.3	2.2 $\rightarrow$	2.0	1.7
$y=1$	2.4	2.5 $\rightarrow$	2.7 $\rightarrow$	3.0
$y=2$	2.3	2.7 $\rightarrow$	2.9 $\rightarrow$	3.0

(A) none (B) I only (C) II only (D) III only
(E) I and II (F) I and III (G) II and III (H) all three

(b) [3] Consider the random experiment of rolling a fair, 6-sided die. Let A be the event of rolling an odd number and let B be the event of rolling a number greater than 2. Which of the following statements are true?

(I) $A \cap B^C = \emptyset$ ✗ (II) $A^C \cup B = \overset{\{2\}}{2}$ ✗ (III) $P(A \cup B) = P(A) + P(B)$ ✗

(A) none (B) I only (C) II only (D) III only
(E) I and II (F) I and III (G) II and III (H) all three

$A \cap B^c = \{1\}$ $\qquad A^c \cup B = \{2,3,4,5,6\}$

$P(A \cup B) = P(A) + P(B) - \underbrace{P(A \cap B)}_{>0} \quad \because A \cap B \neq \phi$

(c) [3] The probability that a couple has a female child is 0.45 and the probability that they have a male child is 0.55. Suppose that a couple has four children. Assuming that births are independent events, what is the probability that at least two children are boys?

(A) 0.039 (B) 0.7585 (C) 0.3333 (D) 0.4051
(E) 0.1883 (F) 0.1286 (G) 0.8605 (H) 0.0421

$A = \geq 2$ boys
$A^c = < 2$ boys $= \{GGGG, GGGB, GGBG, GBGG, BGGG\}$

$P(A) = 1 - P(A^c)$
$= 1 - [(0.45)^4 + 4(0.45)^3(0.55)]$
≈ 0.7585

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2. State whether each statement is true or false. Explain your reasoning.

(a) [2] $f(x, y) = x^4 - y^4$ has an ~~absolute~~ local maximum at $(0, 0)$.

FALSE.

$f(0,0) = 0$

In order for 0 to be a local max. value, $f(x,y) \leq 0$ for all $(x,y) \in B_r(0,0)$ where $r > 0$.

Consider $f(x,0) = x^4$. In any $B_r(0,0)$ where $r>0$, $f(x,0) \geq 0$

$\therefore$ 0 cannot be a maximum value.

(b) [2] If A and B are disjoint sets such that $P(A) > 0$ and $P(B) > 0$, then A and B cannot be independent.

TRUE.

If A and B are disjoint, then $P(A \cap B) = 0$ (*)

If A and B are also independent, then $P(A \cap B) = P(A)P(B)$.

But since $P(A) > 0$ and $P(B) > 0 \Rightarrow P(A \cap B) > 0$, which contradicts (*), A and B cannot be independent

(c) [2] It is possible that the function $F(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ 0.85 & \text{if} \quad 0 \leq x < 1 \\ 0.15 & \text{if} \quad 1 \leq x < 2 \\ 1 & \text{if} \quad x \geq 2 \end{cases}$

is the cumulative distribution function of a random variable X.

FALSE.

A CDF must be non-decreasing, which this function is not.

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3. Consider the function $f(x, y) = x^3 - 3x + 3xy^2$.

(a) [3] Find all four critical points of f.

$f_x = 3x^2 - 3 + 3y^2 \qquad f_y = 6xy$

$$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \Rightarrow \begin{cases} 3(x^2 - 1 + y^2) = 0 \\ 6xy = 0 \end{cases} \Longrightarrow \begin{cases} x^2 + y^2 = 1 \quad ① \\ x = 0 \text{ or } y = 0 \end{cases}$$

sub $x=0$ into ①: $0^2 + y^2 = 1 \Rightarrow y = \pm 1$ $\therefore$ $(0,1)$ and $(0,-1)$ are critical point

sub $y=0$ into ①: $x^2 + 0^2 = 1 \Rightarrow x = \pm 1$ $\therefore$ $(1,0)$ and $(-1,0)$ are critical points

4 critical points: $(-1,0), (1,0), (0,-1), (0,1)$

(b) [4] Use the second derivatives test to classify each point from (a) as either the location of a local maximum, local minimum, or saddle point of f.

$f_{xx} = 6x \qquad f_{xy} = 6y \qquad f_{yy} = 6x$

$D = 6x \cdot 6x - (6y)^2 = 36x^2 - 36y^2$

$D(\pm 1, 0) = 36$

$f_{xx}(-1,0) = -6 \Rightarrow f$ has a local max. at $(-1,0)$

$f_{xx}(1,0) = 6 \Rightarrow f$ has a local min. at $(1,0)$

$D(0, \pm 1) = -36 \Rightarrow f$ has saddle points at $(0,-1), (0,+1)$

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4. [3] Consider the sample space $S = \{1, 2, 3, 4, 5\}$ and the subsets $A = \{1, 2, 4, 5\}$ and $B = \{4, 5\}$. Suppose that $P(1) = 0.1$, $P(2) = 0.3$, $P(3) = 0.2$, $P(4) = 0.3$, and $P(5) = 0.1$. Find (i) $P(A \cap B)$, (ii) $P(A|B)$, and (iii) $P(B|A)$.

(i) $A \cap B = \{4,5\} = B$

$$P(A \cap B) = P(4) + P(5) = 0.3 + 0.1 = 0.4$$

(ii) $$P(A|B) = \frac{P(A \cap B)}{P(B)} = 1$$

(since $B \subseteq A$)

(iii) $$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{0.4}{1 - P(3)} = \frac{0.4}{1 - 0.2} = 0.5$$

5. A certain population consists of 20% children, 30% adolescents, and 50% adults. The probabilities that a certain member of this population catches the flu are 0.45 for a child, 0.2 for an adolescent, and 0.15 for an adult.

(a) [2] What is the probability that a randomly selected member of this population has the flu?

$$P(F) = P(F|C)P(C) + P(F|T)P(T) + P(F|A)P(A)$$
$$= (0.45)(.2) + (.2)(.3) + (.15)(.5)$$
$$= 0.225$$

(b) [2] What is the probability that a randomly selected person with the flu is a child?

$$P(C|F) = \frac{P(F|C)P(C)}{P(F)} = \frac{(.45)(.2)}{0.225} = 0.4$$

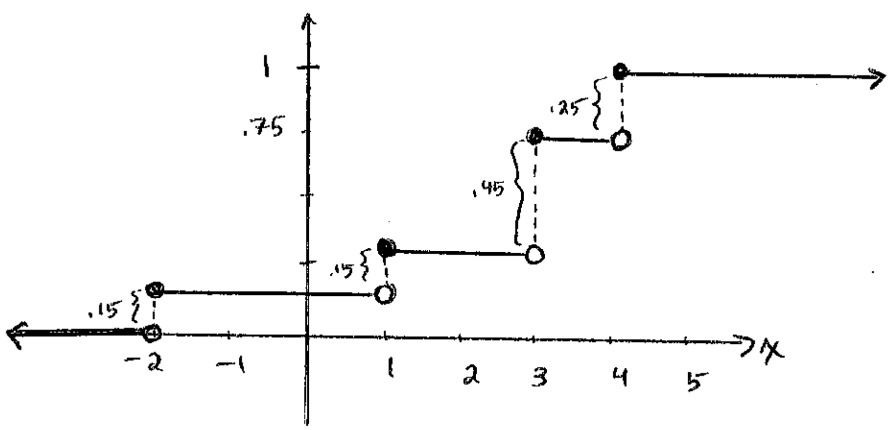
6. The probability mass function for a random variable X is given below:

$$p(-2) = 0.15,\ p(1) = 0.15,\ p(3) = 0.45,\ p(4) = 0.25$$

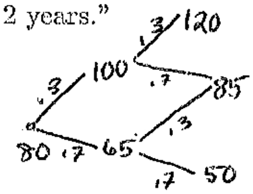
(a) [2] Find the cumulative distribution function, $F(x)$.

$$F(x) = \begin{cases} 0 & x < -2 \\ 0.15 & -2 \le x < 1 \\ 0.30 & 1 \le x < 3 \\ 0.75 & 3 \le x < 4 \\ 1 & x \geq 4 \end{cases}$$

(b) [2] Sketch the graph of $F(x)$.



7. [4] Consider a population of 80 elephants. Suppose that within any given year, there is a 30% chance the population will increase by 20 and a 70% chance it will decrease by 15. Assume that immigration from year to year is independent. Find the probability mass function for the random variable X = "population size of elephants after 2 years."

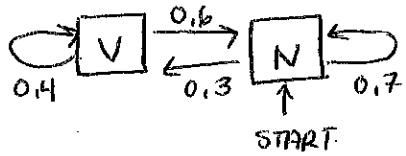


range of $X = \{50, 85, 120\}$

pmf:

x	$p(x) = P(X=x)$
50	$(.7)^2 = 0.49$
85	$2(.7)(.3) = 0.42$
120	$(.3)^2 = 0.09$

8. [3] If, at some time, a virus is present in a population, then it will be present the following month with probability 0.4. If the virus is absent from the population, then it will be absent the following month with a probability 0.7. Assume that at this moment the virus is absent from the population. Find the probability mass function for the random variable X = "number of virus-free months in the 2-month period from now."



x	$p(x) = P(X=x)$
0	$P(VV) = (0.3)(0.4) = 0.12$
1	$P(VN) + P(NV) = (0.3)(0.6) + (0.7)(0.3) = 0.39$
2	$P(NN) = (.7)^2 = 0.49$