Math 1LS3 Week 4: Limits

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This week covers most of 3.1-3.3. We will finish 3.3 next week and do 3.4,3.5.

Overview

- Goal: understand instantaneous rate of change (speed at one instant)
- Paradox? Don't we need two points in time to define speed?
- Geometric reasoning shows how to do it!
- (Geometric reasoning $\rightarrow$ concrete computations) requires **limits**
- Two weeks of limits. Then we can apply them to understand speed at one instant.
- We handle limits *informally/intuitively* in this class. [If you're not satisfied, look up the notorious δ-ϵ definition. It's not really so bad.]

Average Speed

Problem

If you travel 240km in 4 hours, traveling

- *80km the first hour,*
- *0km the second hour,*
- *100km the third hour, and*
- *60 km the last hour*

What was your average speed?

Solution

Your average speed is the ***total distance*** *divided by* ***total time****.*

$$(240km)/(4hr) = 60kph$$

Average speed is the same speed as if you were to do the same trip at a constant rate in the same time.

Average Rate of Change

Problem

*A population of bacteria saisfies $b(t) = 2^t$ million bacteria at hour t. What is the **average** rate of population change (a) in the first hour? (b) In the first two hours? (c) In the next two hours?*

Solution

$\Delta t =$ *change in* t

$\Delta b =$ *change in* $b(t)$ *corresponding to* Δt

(a) In the first hour:

$\Delta t = 1 - 0 = 1 hour\ and\ \Delta b = 2^1 - 2^0\ million = 1\ million\ bacteria.$

$$Average\ rate\ of\ change = \frac{\Delta b}{\Delta t} = \frac{1}{1} = \boxed{1\ million\ bacteria/hr.}$$

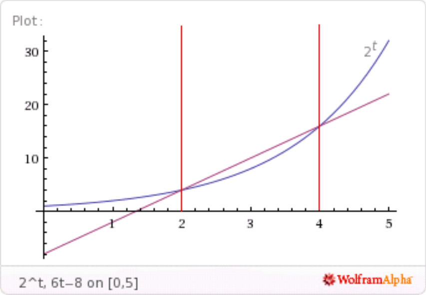
(b) $\Delta t = 2$ *and* $\Delta b = 2^2 - 2^0 = 3$. *So* $\frac{\Delta b}{\Delta t} = \boxed{1.5\ million\ bacteria/hr.}$

(c) $\frac{\Delta b}{\Delta t} = \frac{2^4 - 2^2}{4-2} = \boxed{6\ million\ bacteria\ per\ hour.}$

Average Rate of Change: Geometric Interpretation

The fractions $\frac{\Delta b}{\Delta t}$ on the previous slide should make you think of slope.

Indeed, the average rate of change over an interval $[t_1, t_2]$ is the slope of the *secant line* from (t_1, b_1) to (t_2, b_2).

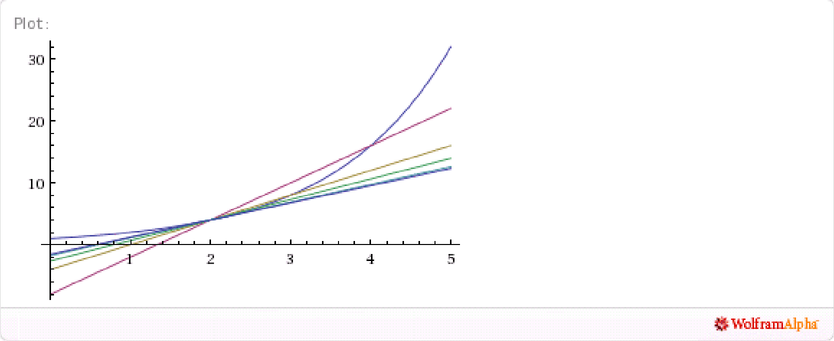


A secant line is the straight line through two given points on a curve.

Instantaneous Rates of Change

Instantaneous speed (not average speed) is what speedometers show.

Geometric interpretation:

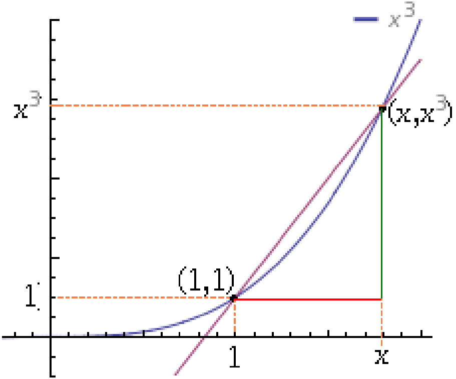


- Use shorter and shorter intervals to measure rate.
- The secant lines get close to a "tangent" line.
- The **instantaneous rate of change** is the *slope of the tangent line*.

The Derivative

Instantaneous Rate of Change = Slope of Tangent Line = **The Derivative**

Concrete example: compute slope of $y = x^3$ at $x = 1$:



$$\text{Secant Slope} = \frac{\Delta y}{\Delta x} = \frac{x^3 - 1}{x - 1}$$

This ratio stabilizes as x nears 1:

$$\frac{\Delta y}{\Delta x} \text{ approaches } \frac{dy}{dx}$$

Think of dx as *infinitessimal* – i.e., infinitely small.

Goal: How can we evaluate this **derivative** $\frac{dy}{dx}$?

What is a limit? (Numerical)

$$f(x) = \frac{x^3 - 1}{x - 1}$$

What happens to $f(x)$ as x gets close to 1?

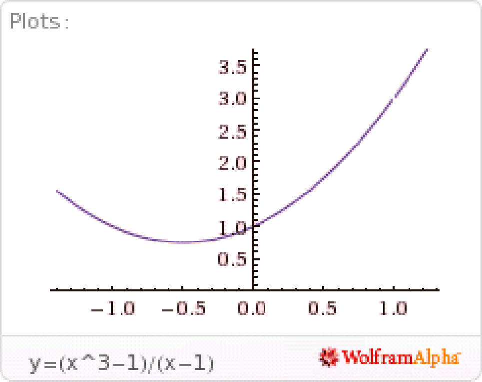
x	f(x)
0.5	1.75
0.9	2.71
0.95	2.8525
0.99	2.9701
1.0001	3.0003...
0.999999	2.9999969...
1.0000000000000001	3.0000000000000003...

We write $\lim_{x \to 1} f(x) = 3$ and say "as x approaches 1, $f(x)$ approaches 3."

What is a limit? (Geometric)

The function is not defined at $x = 1$.

$$f(x) = \frac{x^3 - 1}{x - 1}$$



Which point is "missing"?

$(1, 3)$

- We say "as x approaches 1, $f(x)$ approaches 3."
- We write $\lim_{x \to 1} f(x) = 3$.

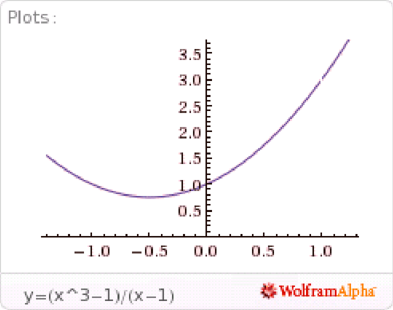
What is a limit?

Definition (Def 3.1, p.169)

We say that the *limit of $f(x)$ as x approaches a is L*, and we write

$$\lim_{x \to a} f(x) = L$$

if we can make the values of $f(x)$ as close to L as desired by taking x close enough to a (but not equal to a).



The derivative $\frac{dy}{dx}$ is a limit of secant slopes $\frac{\Delta y}{\Delta x}$.

Equations involving derivatives are called differential equations. Example:

$$\frac{dy}{dx} = ky(1 - y)$$

Solving Diff. Eqs. is a huge practical application of calculus in science.

They give a continuous analogue of Discrete Time Dynamical Systems.

- Diff. Eq. relates change in y to its current value (and to x).
- Solution to a Diff. Eq. expresses y directly in terms of x.

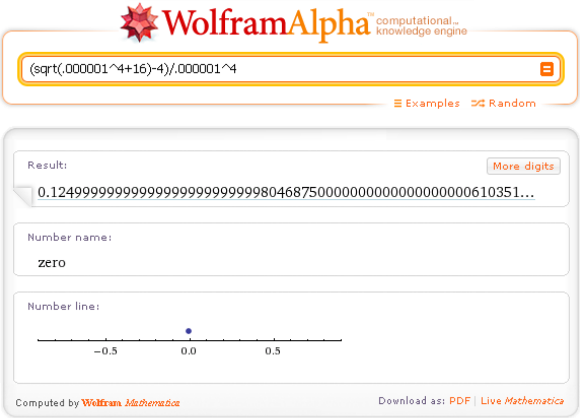
Slogan (not quite true): Initial Value + differential equation determines solution

Numerical Limits: A Quick Caution

Most calculators use rounding after each step in a computation.

Moral: take calculator/Wolfram results with a grain of salt.

See Example 3.2.3 of your text (p.172): $f(x) = \frac{\sqrt{x^4+16}-4}{x^4}$.



Sum Law (for Limits)

Example

- Your scale is never more than $0.01g$ off.
- Suppose you measure two items and get $4200.00g$ and $3800.00g$.
- The total mass should be around $4200.00 + 3800.00 = 8000.00g$.

How far can $8000.00g$ be from the actual total?

Answer: at most $0.02g$.

- e.g., the actual masses might be $42000.01g$ and $3800.01g$.

Moral: really good approximations for x and y lead to a good approximation for $x + y$.

Theorem

If $\lim_{x \to a} f(x) = L$ *and* $\lim_{x \to a} g(x) = M$ *then* $\lim_{x \to a} [f(x) + g(x)] = L + M$.

Other Limit Laws

Memorize the limit laws in your text (Thm 3.1, p.175).

Example

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$$

Fine print: whenever $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a} g(x)$ exists and $\lim_{x\to a} g(x) \neq 0$.

Problem

Find $\lim_{x\to 1} \frac{1+x}{x^2}$.

Solution

$$\lim_{x\to 1} \frac{1+x}{x^2} = \frac{\lim_{x\to 1}(1+x)}{\lim_{x\to 1} x^2} = \frac{\left(\lim_{x\to 1} 1\right) + \left(\lim_{x\to 1} x\right)}{\left(\lim_{x\to 1} x\right) \cdot \left(\lim_{x\to 1} x\right)} = \frac{1+1}{1\cdot 1} = 2$$

Direct Substitution Rule

For *decent* functions, just plug in to find limit!

$$\lim_{x \to a} f(x) = f(a)$$

"Direct substitution rule".

"*Decent*" includes all your faves:

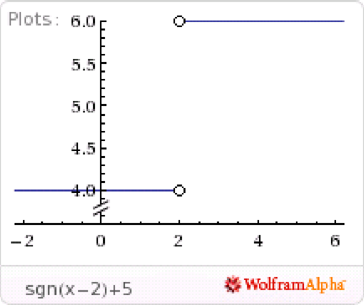
- algebraic
- exponential
- log
- trig
- inverse trig

Example:

$$\lim_{x \to e} \frac{\ln(x)}{\sin(x)} = \frac{\ln(e)}{\sin(e)} = \boxed{\frac{1}{\sin(e)}}$$

One–Sided Limits

Graph of $y = f(x)$:



Left-hand limit $\lim_{x \to 2^-} f(x)$.

$$\lim_{x \to 2^-} f(x) = \boxed{4}$$

"x is 2 minus a little bit."

Right-hand limit $\lim_{x \to 2^+} f(x)$.

$$\lim_{x \to 2^+} f(x) = \boxed{6}$$

"x is 2 plus a little bit."

If $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$ then $\lim_{x \to a} f(x)$ *does not exist.*

Two–Sided Limits

Theorem

$\lim_{x \to a} f(x)$ *exists* ***if and only if****:*

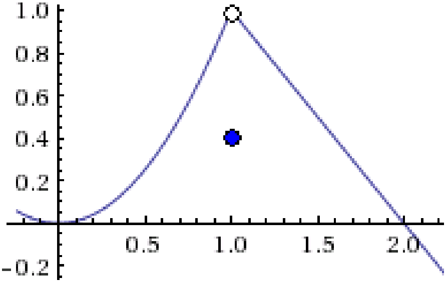
1. $\lim_{x \to a^+} f(x)$ *exists;*
2. $\lim_{x \to a^-} f(x)$ *exists; AND*
3. *Both these one–sided limits are equal.*

Limits are sometimes called **two–sided limits** for emphasis.

Example

$$f(x) = \begin{cases} x^2, & \text{if } x < 1 \\ 0.4, & \text{if } x = 1 \\ 2 - x, & \text{if } x > 1 \end{cases}$$

Does $\lim_{x \to 1} f(x)$ exist? What is its value?

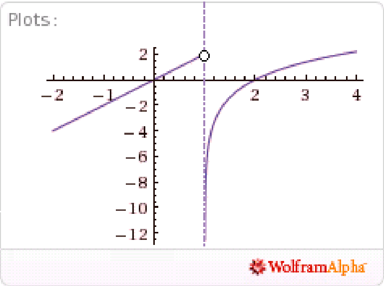


1. $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x^2 = 1$
2. $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 2 - x = 1$
3. $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x)$, so:

$$\boxed{\lim_{x \to 1} f(x) \text{ exists and } \lim_{x \to 1} f(x) = 1.}$$

Infinite Limits

$\boxed{\text{Infinite Limit as } x \to a \text{ corresponds to Vertical Asymptote at } x = a}$



Vertical asymptote at $x = 1$.

How can we express this as a limit?

$$\lim_{x \to 1^-} f(x) = 2 \qquad \lim_{x \to 1^+} f(x) = -\infty$$

If $f(x)$ gets (and stays) arbitrarily big when x gets close to a, we say:

$$\lim_{x \to a} f(x) = +\infty$$

$\lim_{x \to a^\pm} f(x) = \pm\infty$ defined analogously.

Graphing a Vertical Asymptote

Problem

Show the vertical asymptotes of $y = \frac{1}{(1-x)^3}$ *on a graph.*

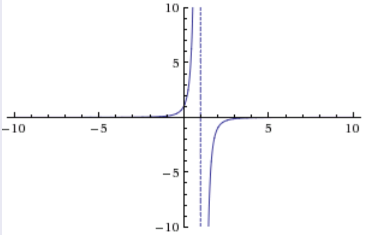
Solution

Q: Where does function "blow up"?

- *Ans: Near where* $1 - x = 0$.
- *So look at* $\lim\limits_{x\to 1^-}$ *and* $\lim\limits_{x\to 1^+}$.

$$\lim_{x\to 1^-} \frac{1}{(1-x)^3} = \lim \frac{1}{\textit{small positive}} = +\infty$$

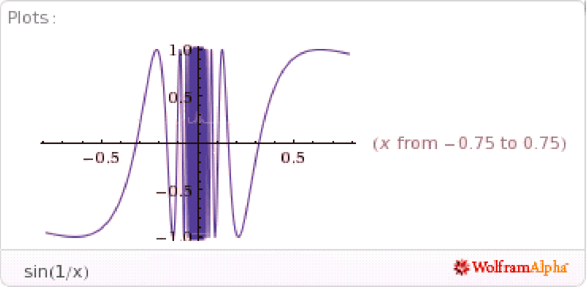
$$\lim_{x\to 1^+} \frac{1}{(1-x)^3} = \lim \frac{1}{\textit{small negative}} = -\infty$$



A Notational Warning about Infinite Limits

Please carefully read Table 3.3.4 on p.191 for the precise meaning of "limit does not exist". In particular, note:

- It is perfectly consistent to simultaneously say BOTH "$\lim_{x\to 0} f(x) = +\infty$" and "$\lim_{x\to 0} f(x)$ does not exist".
- On the other hand, there are far stranger ways for the limit to not exist.



Algebra Tricks 1: Factoring

When direct substitution doesn't work:

Problem

Find

$$\lim_{t \to 1} \frac{t^2 + t - 2}{t^2 - t}$$

Solution

If we plug in, we get $\frac{0}{0}$. *Not defined. Instead factor:*

$$\lim_{t \to 1} \frac{t^2 + t - 2}{t^2 - t} = \lim_{t \to 1} \frac{(t-1)(t+2)}{t(t-1)} = \lim_{t \to 1} \frac{(t+2)}{t} = \frac{1+2}{1} = \boxed{3}.$$

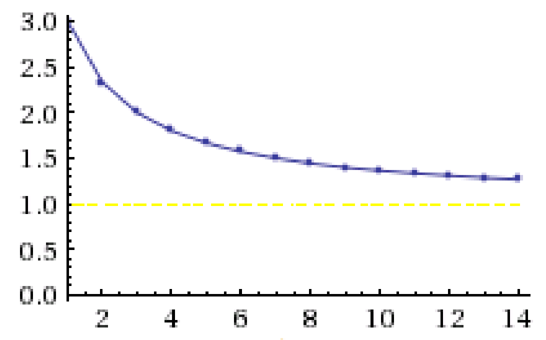
Limit of a Sequence

- DTDS m_t is only defined at discrete points.

Example

$$\lim_{x\to\infty} \frac{x+5}{x+1} = 1$$

$$\lim_{n\to\infty} \frac{n+5}{n+1} = 1$$



If $\lim_{x\to\infty} f(x)$ converges then $\lim_{n\to\infty} f(n)$ converges.

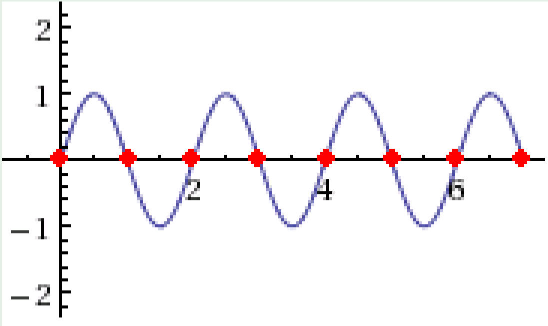
Limit of a Sequence

Example

$$\lim_{x\to\infty} \sin(\pi x) \text{ doesn't exist}$$

but

$$\lim_{n\to\infty} \sin(\pi n) = 0$$



If $\lim_{x\to\infty} f(x)$ diverges, $\lim_{n\to\infty} f(n)$ might still converge!

- For a DTDS, a limit at ∞ is a **stable equilibrium**.