12C2 Last Day Determinants

If A is 2×2 $\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

If A is $n \times n$ $\det A = \sum_{i=1}^{n} a_{ij} C_{ij}$ or $\sum_{j=1}^{n} a_{ij} C_{ij}$ } Cofactor expansion along any 1 row or 1 col

Don't forget {

C_{ij} = cofactor $= \underbrace{(-1)^{i+j}}_{\text{pos. sign}} \cdot \underbrace{M_{ij}}_{\text{Minor}}$

M_{ij} = det. of A, with row i, col. j removed.

or Row-reduce A using elem. ops. to a triangular matrix

- Swapping rows changes sign of det.
- mult. by k on 1 row $\Rightarrow$ $k \cdot$ det.
- adding a row to another changes nothing
- det. of triangular (or diag. matrix) = (product on principal diagonal)

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eg. Say $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ & $\det(A) = 7$

& $B = \begin{bmatrix} 3c & 3d \\ 2a-c & 2b-d \end{bmatrix}$, find $\det(B)$

Solution

$$\det B = \begin{vmatrix} 3c & 3d \\ 2a-c & 2b-d \end{vmatrix}$$

Row1 $\cdot \frac{1}{3}$

$$= 3 \begin{vmatrix} c & d \\ 2a-c & 2b-d \end{vmatrix} \overset{\text{Row2 + Row1}}{=} 3 \begin{vmatrix} c & d \\ 2a & 2b \end{vmatrix}$$

Row2 $\cdot \frac{1}{2}$

$$= 3 \cdot 2 \begin{vmatrix} c & d \\ a & b \end{vmatrix} \overset{\text{Row1} \leftrightarrow \text{Row2}}{=} 3 \cdot 2(-1) \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$= 3 \cdot 2(-1) \cdot \underbrace{\det(A)}_{7} = -6 \cdot 7 = \underline{\underline{-42}}$$

Determinant Properties

1) $|A^T| = |A|$

2) Row or col. of zeros $\Rightarrow |A| = 0$

3) Two rows are [redacted] same or multiples?

$\Rightarrow |A| = 0$

4) $|AB| = |A||B|$ (if A, B are $n \times n$)

(shown last day)

$$\Rightarrow |ABBAACCACB| = |A|^4 |B|^3 |C|^3$$

breaks into product of det $\Rightarrow$ numbers! $\Rightarrow$ order no longer matters!

5) $|A^k| = |A|^k$

6) $|A^{-1}| = |A|^{-1}$

$AA^{-1} = I$

$|AA^{-1}| = |I| = 1$

$|A||A^{-1}| = 1$

$|A^{-1}| = 1/|A|$

7) $|kA|$, $k \in \mathbb{R}$.

$= k^n |A|$

$|kA|$

$= |kIA|$

$= |kI||A| = \begin{vmatrix} k & & 0 \\ & \ddots & \\ 0 & & k \end{vmatrix} |A|$

8) $|A+B|$ = <u>Trouble!</u>

(no good formula except in special cases)

<u>Adjoints</u> or <u>Adjugates</u>

The adjoint (adjugate?) of A = $\operatorname{adj}(A)$

$\operatorname{adj} A$ = <u>Transpose of matrix of Cofactors of A</u>

$= [C_{ij}]^T$

eg. If A is $\underline{\underline{3\times 3}}$ $\Rightarrow$ $\operatorname{adj} A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$

$$= \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

<u>Note</u> $A \cdot \operatorname{adj} A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & - & - \\ C_{13} & - & - \end{bmatrix}$

along row 1

$|A|$

$= [\; a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$

along row 2

$|A|$

$a_{21}C_{11} + a_{22}C_{12} + a_{23}C_{13}$

$a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}$

along row 3

$\# \;]$

$= \det A.$

note

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$= \det A$$

but

$$\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{21}C_{11} + a_{22}C_{12} + a_{23}C_{13}$$

copy

$= \boxed{0}$ by parallel rows!

(b/c rows are identical!)

so $A \cdot \operatorname{adj} A = \begin{bmatrix} |A| & & 0 \\ & |A| & \\ 0 & & |A| \end{bmatrix} = |A|\,I$

so if $\det A \neq 0$ $\qquad A \cdot \left(\dfrac{\operatorname{adj} A}{\det A}\right) = I$

$$\boxed{A^{-1} = \operatorname{adj} A / \det A}$$

eg. Let $B = \begin{bmatrix} 3 & 5 & 1 \\ 0 & 0 & 2 \\ 1 & 2 & -1 \end{bmatrix}$ Find entry in row 2, col 3 of B^{-1}

Solution: $B^{-1} = \operatorname{adj} B / \det B$

$$\det B = 0 + 0 + 2 \cdot C_{23} = 2 \cdot (-1)^{2+3} \begin{vmatrix} 3 & 5 \\ 1 & 2 \end{vmatrix}.$$

-1

$$= -2(6-5) = -2$$

Now I want row 2, col 3 of B^{-1}

$$= (\text{row } 2, \text{col } 3 \text{ of adj } B) / \det B$$

$$= C_{32} / (\det B) = \frac{(-1)^{3+2}}{-2} M_{32} = \frac{1}{2} \begin{vmatrix} 3 & 1 \\ 0 & 2 \end{vmatrix}$$

$$= \frac{1}{2} \cdot 6 = \boxed{3}.$$