Arc Length in x: The 1/2 Trick

In general calculating the arc length of a function of *x* often becomes quite challenging. The square root often means resorting to trigonometric substitutions, and more often than not finding a nice form for the integral proves seemingly impossible.

But there is one common special case that appears in many of these calculations that allows a nice form for the resulting integral. We'll call it "The 1/2 trick".

The Trick in Action:

First let's remember the form of the arc length differential term. $ds = \sqrt{1 + (f'(x))^2}\, dx$

Say we wish to evaluate this for a very specific function: $f(x) = \frac{x^4}{4} + \frac{1}{8x^2}$

$$f(x) = \frac{x^4}{4} + \frac{1}{8x^2}, \quad \text{so} \quad f'(x) = x^3 - \frac{1}{4x^3}$$

We then can square this to get:

$$\left(f'(x)\right)^2 = \left(x^3 - \frac{1}{4x^3}\right)\left(x^3 - \frac{1}{4x^3}\right) = x^6 + \frac{1}{16x^6} - 2x^3\left(\frac{1}{4x^3}\right) = x^6 + \frac{1}{16x^6} - \frac{1}{2}$$

Now consider what happens if we add 1, as in the arc length differential:

$$1 + \left(f'(x)\right)^2 = x^6 + \frac{1}{16x^6} - \frac{1}{2} + 1 = x^6 + \frac{1}{16x^6} + \frac{1}{2}$$

Notice that by adding 1, we've switched the sign on the 1/2 without changing anything else. This looks almost exactly like the result of the square of the same polynomial, but with the opposite sign.

$$1 + \left(f'(x)\right)^2 = x^6 + \frac{1}{16x^6} + \frac{1}{2} = x^6 + \frac{1}{16x^6} + 2x^3\left(\frac{1}{4x^3}\right) = \left(x^3 + \frac{1}{4x^3}\right)^2$$

And our arc length differential then is:

$$ds = \sqrt{1 + (f'(x))^2}\, dx$$

$$= \sqrt{1 + \left(x^3 - \frac{1}{4x^3}\right)^2}\, dx = \sqrt{x^6 + \frac{1}{16x^6} + \frac{1}{2}}\, dx = \sqrt{\left(x^3 + \frac{1}{4x^3}\right)^2}\, dx = \left(x^3 + \frac{1}{4x^3}\right) dx$$

Note that this is, of course, much, much more reasonable form and trivial to integrate.

The "General" Case

The take-home idea for this is to notice that we should always be on the lookout for perfect squares when simplifying our square roots.

Of course generally this will prove difficult. But if our function's derivative has the form:

$$\boxed{f'(x) = ax^n - \frac{b}{x^n}, \quad \text{such that } \; 2ab = \frac{1}{2}}$$

Then

$$\left(f'(x)\right)^2 = \left(ax^n - \frac{b}{x^n}\right)^2 = a^2x^{2n} + \frac{b^2}{x^{2n}} - 2ax^n\left(\frac{b}{x^n}\right) = a^2x^{2n} + \frac{b^2}{x^{2n}} - \frac{1}{2}$$

and

$$ds = \sqrt{1 + \left(f'(x)\right)^2}\; dx = \sqrt{a^2x^{2n} + \frac{b^2}{x^{2n}} + \frac{1}{2}}\; dx = \sqrt{\left(ax^n + \frac{b}{x^n}\right)^2}\; dx = \left(ax^n + \frac{b}{x^n}\right)dx$$

A Slightly Irregular Case:

Make sure you watch out for situations where the correct form arises from non-polynomial functions. For instance if $\ln(x)$ gets involved:

$$f(x) = \frac{x^2}{8} + \ln(x), \quad \text{so} \quad f'(x) = \frac{x}{4} - \frac{1}{x}$$

$$\text{and} \quad 1 + \left(f'(x)\right)^2 = 1 + \left(\frac{x^2}{16} + \frac{1}{x^2} - 2\left(\frac{x}{4}\right)\left(\frac{1}{x}\right)\right) = \frac{x^2}{16} + \frac{1}{x^2} + \frac{1}{2} = \left(\frac{x}{4} + \frac{1}{x}\right)^2$$

So

$$ds = \sqrt{1 + \left(f'(x)\right)^2}\; dx = \left(\frac{x}{4} + \frac{1}{x}\right)dx$$