Partial Fraction Integration Examples

eg. $\displaystyle\int \frac{1}{x^2(x^2+4)}\,dx$

$$\frac{1}{x^2(x^2+4)} = \underbrace{\frac{A}{x} + \frac{B}{x^2}}_{} + \underbrace{\frac{Cx+D}{x^2+4}}_{\text{Irred. quadratic term}}$$

$\to$ 2 powers of "x"

$$1 = \underbrace{Ax(x^2+4)}_{Ax^3+4Ax} + B(x^2+4) + Cx^3 + Dx^2$$

Sorting by order:

$x^3 \to \quad 0 = A + C \qquad x \to 0 = 4A$

$x^2 \to \quad 0 = B + D \qquad \text{const} \to 1 = 4B$

So $A = 0 \Rightarrow C = -A = 0$, $B = \frac{1}{4} \Rightarrow D = -B = -\frac{1}{4}$

So $$\frac{1}{x^2(x^2+4)} = \cancel{\frac{0}{x}} + \frac{(1/4)}{x^2} + \frac{0x - \frac{1}{4}}{x^2+4}$$

$$\int \frac{1}{x^2(x^2+4)}\,dx = \frac{1}{4}\int \frac{1}{x^2}\,dx - \frac{1}{4}\int \frac{1}{x^2+4}\,dx$$

$$= -\frac{1}{4x} - \frac{1}{4} \cdot \frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right) + C$$

$$= -\frac{1}{4x} - \frac{1}{8} \tan^{-1}\left(\frac{x}{2}\right) + C$$

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$$

Extra Examples (as mentioned in class)

eg. $$\int \frac{x^4+7}{x^2(x^2+2x+5)}\,dx$$

Order of top (numerator) = 4

order of bottom (denominator) = 4

order top $\geq$ bottom, so divide

$$\begin{array}{r|l} & \quad 1 \\ \hline x^4+2x^3+5x^2 & x^4+0x^3+0x^2+0x+7 \\ & x^4+2x^3+5x^2 \\ \hline & -2x^3-5x^2+7 \end{array}$$

$-2x^3-5x^2+7 \}$ remainder

so
$$\frac{x^4+7}{x^2(x^2+2x+5)} = 1 + \frac{7-5x^2-2x^3}{x^2(x^2+2x+5)}$$

order top < order bottom so

break up with Partial Fractions.

$$\frac{-2x^3 - 5x^2 + 7}{x^2(x^2+2x+5)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+2x+5}$$

$$-2x^3 - 5x^2 + 7 = Ax(x^2+2x+5) + B(x^2+2x+5) + Cx^3 + Dx^2$$

$$\begin{aligned}-2x^3 - 5x^2 + 7 = {} & Ax^3 + 2Ax^2 + 5Ax \\ & + Bx^2 + 2Bx + 5B \\ & + Cx^3 + Dx^2\end{aligned}$$

note: I wrote them this way for easy collection!

Separated by <u>order</u>

x^3: $-2 = A + C$, x^2: $-5 = 2A + B + D$, x: $0 = 5A + 2B$

const: $5B = 7$

$\underline{\text{so}}$ $B = 7/5$ & $A = -\frac{2}{5}B = -14/25$

$C = -2 - A = -36/25$

$D = -5 - 2A - B = \frac{-125}{25} + \frac{28}{25} - \frac{35}{25} = \frac{-132}{25}$

$\underline{\text{so}}$

$$\frac{x^4+7}{x^2(x^2+2x+5)} = 1 + \frac{(-14/25)}{x} + \frac{(7/5)}{x^2} + \frac{1}{25}\left[\frac{-36x-132}{x^2+2x+5}\right]$$

b)

$$\int \frac{x^4+7}{x^2(x^2+2x+5)}\,dx = \overset{x}{\cancel{\int 1\,dx}} - \frac{14}{25}\overset{\ln|x|}{\cancel{\int \frac{1}{x}\,dx}} + \frac{7}{5}\overset{-\frac{1}{x}}{\cancel{\int \frac{1}{x^2}\,dx}}$$

$$-\frac{1}{25}\int \frac{36x+132}{x^2+2x+5}\,dx.$$

but $\int \frac{36x+132}{x^2+2x+5}\,dx$, $\quad x^2+2x+5$

$\quad = (x+1)^2+4$

$= \int \frac{36x+132}{(x+1)^2+4}\,dx$, let $u = x+1$ $\quad dx = du$

$x = u-1$

$= \int \frac{36u + \cancel{132-36}\ 96}{u^2+4}\,du = 18\int \frac{2u}{u^2+4}\,du + 31\int \frac{1}{u^2+2^2}\,du$

$= 18\ln|\cancel{u^2}+4| + 48\tan^{-1}\left(\frac{u}{2}\right) + C$

$(x+1)^2$

$= 18\ln|x^2+2x+5| + 48\tan^{-1}\left(\frac{x+1}{2}\right) + C$

So $\int \frac{x^4+7}{x^2(x^2+2x+5)}\,dx = x - \frac{14}{25}\ln|x| - \frac{7}{5x} - \frac{18}{25}\ln|x^2+2x+5|$

$- \frac{48}{25}\tan^{-1}\left(\frac{x+1}{2}\right) + C$