MATH 1B03/1ZC3 Winter 2019

Lecture 15: Complex numbers II

Instructor: Dr Rushworth February 26th

Matrices with complex entries

(from Chapter 10.2 of Anton-Rorres 9th edition))

We have spent a significant part of this course introducing concepts related to matrices with real entries. All of these concepts, including transpose, inverse, determinant, extend to matrices with complex entries also. We must take care to treat the complex entries properly, however.

Example 15.1

Let

$$A = \begin{bmatrix} 4 & 0 & 1 \\ i & 8 & 3 \\ -7 & -2i & 5 \end{bmatrix}$$

then

$$2iA = \begin{bmatrix} 8i & 0 & 2i \\ -2 & 16i & 6i \\ -14i & 4 & 10i \end{bmatrix}$$

If

$$A = \begin{bmatrix} 4 & 0 & 1 \\ i & 8 & 3 \\ -7 & -2i & 5 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 0 & 2i & 1 \\ 3i & 0 & 2 \\ 9 & i & -1 \end{bmatrix}$$

then

$$A - B = \begin{bmatrix} 4 & -2i & 0 \\ -2i & 8 & 1 \\ -16 & -3i & 6 \end{bmatrix}$$

Let

$$M = \begin{bmatrix} 1+i & 2-i \\ 4+2i & 3 \end{bmatrix}$$

then

$$\begin{aligned} det(M) &= 3(1+i) - (2-i)(4+2i) \\ &= 3 + 3i - (8 + 4i - 4i + 2) \\ &= -7 + 3i \end{aligned}$$

This shows that it is possible for a matrix with complex entries to have complex determinant.
Further, we can compute M^{-1} be applying the formula we saw in previous lectures:

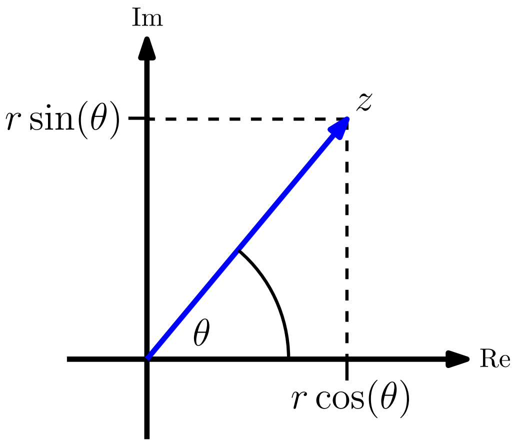
$$M^{-1} = \frac{1}{-7+3i} \begin{bmatrix} 3 & -2+i \\ -4-2i & 1+i \end{bmatrix}$$

The polar form of a complex number

(from Chapter 10.3 of Anton-Rorres 9th edition))

We have seen that complex numbers can be represented as vectors in the Argand plane. There is another way to express these vectors, which can make many aspects of working with complex numbers easier.

Let $z = a + ib$ be a complex number with $|z| = r$, then



where θ is the angle z makes to the positive real axis. Using trigonometric identities we can determine that

$$\begin{aligned} a &= r\cos(\theta) \\ b &= r\sin(\theta) \end{aligned}$$

and we may write

$$z = r\left(\cos(\theta) + i\sin(\theta)\right)$$

this is the polar form of the complex number z.

The angle θ is the argument of z. Arguments of complex numbers are **always** given in radians.

Recall that the functions sine and cosine are 2π-periodic. That means

$$\begin{aligned} \sin(x + 2\pi n) &= \sin(x) \\ \cos(x + 2\pi n) &= \cos(x) \end{aligned}$$

for any real number x and integer n. As a consequence, if we add an integer multiple of 2π to the argument of a complex number, we will not change the complex number described.

Specifically, if $z = r\left(\cos(\theta) + i\sin(\theta)\right)$, then

$$\begin{aligned} r\left(\cos(\theta + 2\pi n) + i\sin(\theta + 2\pi n)\right) &= r\left(\cos(\theta) + i\sin(\theta)\right) \\ &= z \end{aligned}$$

for any integer n.

To eliminate this indeterminacy we define the principle argument of a complex number to be the argument θ such that $-\pi < \theta \leqq \pi$. The principle argument of z is denoted $Arg(z)$.

Question 15.2

If z is a real number with $|z| > 0$ what is $Arg(z)$?
How about if z is a real number with $|z| < 0$?

To compute the arguments of general complex numbers we need to use trigonometry.

Example 15.3

Let $z = 2\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right)$.
Then

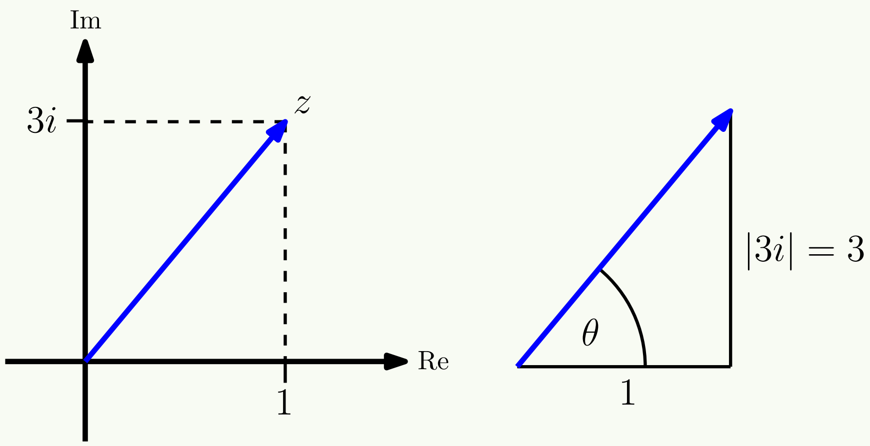
$$\theta = \frac{\pi}{2} + 2\pi = \frac{5\pi}{2}$$
$$\theta = \frac{\pi}{2} - 6\pi = -\frac{11\pi}{2}$$

are both possible arguments of z, but $Arg(z) = \frac{\pi}{2}$, as $-\pi < \frac{\pi}{2} \leqq \pi$.

Question: Express the complex number

$$z = 1 + 3i$$

in polar form.
Answer: Sketch the vector form of z:



and form a triangle with side lengths equal to $Re(z)$ and $Im(z)$ (given on the right). Using the trigonometric identity

$$\tan = \frac{opposite}{adjacent}$$

we see that

$$\tan(\theta) = \frac{3}{1}$$

and

$$\theta = \arctan(3) \approx 1.25\ldots$$

Next find the modulus of z:

$$|z| = \sqrt{1^2 + 3^2} = \sqrt{10}$$

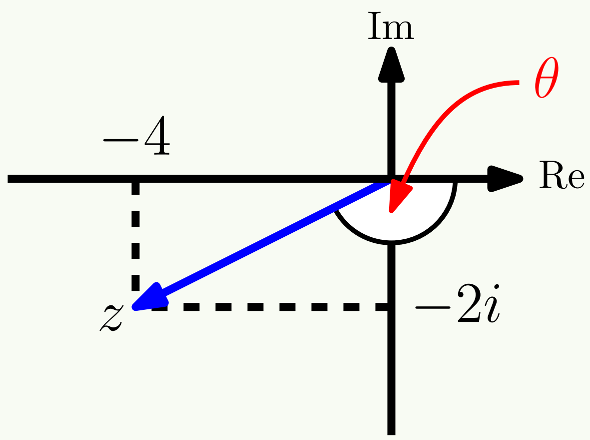
Therefore the polar form of z is

$$z = \sqrt{10}(\cos(1.25\ldots) + i\sin(1.25\ldots))$$

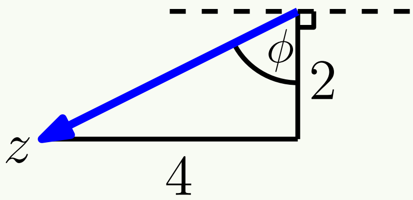
Question: Express the complex number

$$z = -4 - 2i$$

in polar form.
Answer: Sketch the vector form of z:



The relevant triangle is



Notice that we will compute the angle ϕ, and to recover the argument θ we must use

$$\theta = -\left(\phi + \frac{\pi}{2}\right)$$

The minus sign is required as we define the angle 0 to be lying on the real axis.
We see that

$$\tan(\phi) = \frac{4}{2}$$

so that $\phi = \arctan(2) \approx 1.11\ldots$
Then

$$\begin{aligned} \theta &= -\left(\phi + \frac{\pi}{2}\right) \\ &= -2.68\ldots \end{aligned}$$

Also

$$|z| = \sqrt{16+4} = 2\sqrt{5}$$

so that the polar form of z is

$$z = 2\sqrt{5}\,(\cos(-2.68\ldots) + i\sin(-2.68\ldots))$$

Multiplication and division in polar form

Writing complex numbers in polar form allows us to geometrically understand multiplication and division.

To do this, we need to recall the double angle formulae:

$$\begin{aligned} \cos(\theta_1 + \theta_2) &= \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) \\ \sin(\theta_1 + \theta_2) &= \sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2) \end{aligned}$$

Now let $z = r_1\,(\cos(\theta_1) + i\sin(\theta_1))$, and $w = r_2\,(\cos(\theta_2) + i\sin(\theta_2))$.

Then

$$\begin{aligned}
zw &= r_1\left(\cos(\theta_1) + i\sin(\theta_1)\right) r_2\left(\cos(\theta_2) + i\sin(\theta_2)\right)\\
&= r_1 r_2\left(\cos(\theta_1)\cos(\theta_2) + i\cos(\theta_1)\sin(\theta_2)\right.\\
&\quad \left.+ i\cos(\theta_2)\sin(\theta_1) - \sin(\theta_1)\sin(\theta_2)\right)\\
&= r_1 r_2\left(\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)\right.\\
&\quad \left.+ i\left(\cos(\theta_1)\sin(\theta_2) + \cos(\theta_2)\sin(\theta_1)\right)\right)
\end{aligned}$$

applying the double angle formula we obtain

$$zw = r_1 r_2\left(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right).$$

We have just proved the following facts.

Fact 15.4: Multiplication in polar form

Let $z = r_1\left(\cos(\theta_1) + i\sin(\theta_1)\right)$, and $w = r_2\left(\cos(\theta_2) + i\sin(\theta_2)\right)$ be complex numbers.
Then

$$zw = r_1 r_2\left(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right).$$

Fact 15.5

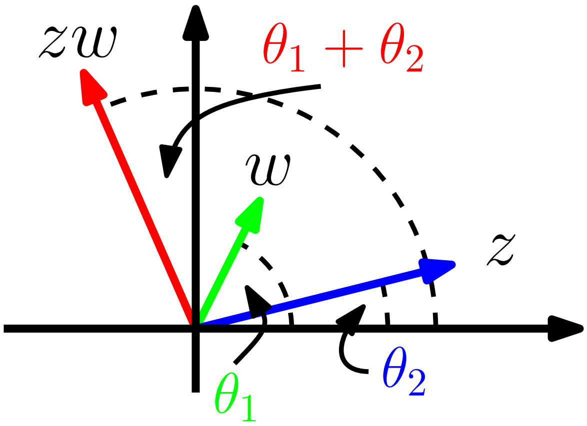
Let z and w be complex numbers. Then

$$|zw| = |z||w|$$

and

$$Arg(zw) = Arg(z) + Arg(w)$$

This is represented the Argand plane in the following manner:



Using the double angle formulae we can also prove similar facts about division.

Fact 15.6: Division in polar form

Let $z = r_1(\cos(\theta_1) + i\sin(\theta_1))$, and $w = r_2(\cos(\theta_2) + i\sin(\theta_2))$ be complex numbers.
Then

$$\frac{z}{w} = \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)).$$

Fact 15.7

Let z and w be complex numbers. Then

$$\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$$

and

$$Arg\left(\frac{z}{w}\right) = Arg(z) - Arg(w)$$

Notice that $Arg(i) = \frac{\pi}{2}$ and $|i| = 1$. Therefore multiplying a complex number by i, does not change its modulus, and rotates the associated vector anticlockwise by $\frac{\pi}{2}$ radians (or 90 degrees).

Question 15.8

Let $z = \cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)$. What is the argument of iz?
Sketch z and iz in the Argand plane.

De Moivre's Formula

Writing complex numbers in polar form can speed up certain calcluations, owning to the behaviour of the argument under multiplication, as described above.

Fact 15.9: De Moivre's Formula

Let z be a complex number with $|z| = 1$. Therefore its polar form is

$$z = \cos(\theta) + i\sin(\theta)$$

For any positive integer n, we have

$$z^n = \cos(n\theta) + i\sin(n\theta)$$

This equation is known as De Moivre's Formula.

This can be proved using Fact 15.4.

If $|z| \neq 1$, we can still use De Moivre's Formula to compute powers of z, using the equation

$$z^n = r^n\left(\cos(n\theta) + i\sin(n\theta)\right)$$

for $|z| = r$.

Example 15.10:

Given $z = 3\left(\cos(\frac{\pi}{3}) + i\sin(\frac{\pi}{3})\right)$ compute z^3.

Answer: We have

$$\begin{aligned} z^3 &= 3^3 \left(\cos(3\frac{\pi}{3}) + i\sin(3\frac{\pi}{3})\right) \\ &= 27\left(\cos(\pi) + i\sin(\pi)\right) \\ &= -27 \end{aligned}$$

Finding roots of complex numbers

We can use De Moivre's Formula to compute the roots of complex numbers.

Definition 15.11: Complex roots

Let z be a complex number and n a positive integer. An n-th root of z is a complex number w such that

$$w^n = z$$

We write $z^{\frac{1}{n}} = w$.

Given a complex number, the following fact tells us how we can compute its n-th roots.

Fact 15.12

Let $z = r\left(\cos(\theta) + i\sin(\theta)\right)$ be a complex number, and n a positive integer.
There are exactly n n-th roots of z, and they are given by

$$z^{\frac{1}{n}} = r^{\frac{1}{n}} \left(\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right)$$

for $k = 0, 1, 2, \ldots, n-1$.

This fact can be proved using De Moivre's Formula. An important detail to notice is that there are n n-th roots of a complex number. For example, there are 3 cube roots.

Example 15.13

Question: Given $z = 27$, find all of the complex cube roots of z.

Answer: We know that $(27)^{\frac{1}{3}} = 3$, if we are restricting to real numbers. However, Fact 15.12 tells us that z must have 2 more roots.
To find them we need to put z into polar form. As z is real this is easy, as its principle argument is 0

$$z = 27(\cos(0) + i\sin(0))$$

Using Fact 15.12 we obtain
When $k = 0$

$$\begin{aligned} z^{\frac{1}{3}} &= 27^{\frac{1}{3}}\left(\cos\left(\frac{\theta + 2\pi 0}{3}\right) + i\sin\left(\frac{\theta + 2\pi 0}{3}\right)\right) \\ &= 3 \end{aligned}$$

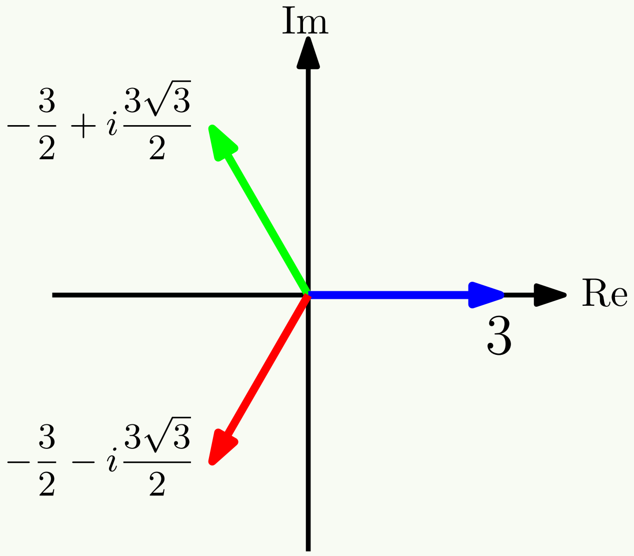
When $k = 1$

$$\begin{aligned} z^{\frac{1}{3}} &= 27^{\frac{1}{3}}\left(\cos\left(\frac{\theta + 2\pi}{3}\right) + i\sin\left(\frac{\theta + 2\pi}{3}\right)\right) \\ &= 3\left(\cos(\frac{2\pi}{3}) + i\sin(\frac{2\pi}{3})\right) \\ &= 3\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) \\ &= -\frac{3}{2} + i\frac{3\sqrt{3}}{2} \end{aligned}$$

When $k = 2$

$$
\begin{aligned}
z^{\frac{1}{3}} &= 27^{\frac{1}{3}}\left(\cos\left(\frac{\theta+4\pi}{3}\right)+i\sin\left(\frac{\theta+4\pi}{3}\right)\right)\\
&= 3\left(\cos\left(\frac{4\pi}{3}\right)+i\sin\left(\frac{4\pi}{3}\right)\right)\\
&= 3\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)\\
&= -\frac{3}{2}-i\frac{3\sqrt{3}}{2}
\end{aligned}
$$

Therefore 3, $-\frac{3}{2}+i\frac{3\sqrt{3}}{2}$, and $-\frac{3}{2}-i\frac{3\sqrt{3}}{2}$ are the complex cube roots of 27. In the Argand plane we have:



Question 15.14

Check that $-\dfrac{3}{2}+i\dfrac{3\sqrt{3}}{2}$ is a cube root of 27 by computing

$$\left(-\frac{3}{2}+i\frac{3\sqrt{3}}{2}\right)^3$$

Example 15.15

Question: Given $z = 1 - i$, find all of the complex square roots of z.
Answer: First, write z in polar form. Using the method of previous examples we obtain

$$z = \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$$

using Fact 15.12 we obtain the roots.
For $k = 0$:

$$\begin{aligned} z^{\frac{1}{2}} &= 2^{\frac{1}{4}}\left(\cos\left(-\frac{\frac{\pi}{4}}{2}\right) + i\sin\left(-\frac{\frac{\pi}{4}}{2}\right)\right) \\ &= 2^{\frac{1}{4}}\left(\cos\left(-\frac{\pi}{8}\right) + i\sin\left(-\frac{\pi}{8}\right)\right) \end{aligned}$$

For $k = 0$:

$$\begin{aligned} z^{\frac{1}{2}} &= 2^{\frac{1}{4}}\left(\cos\left(\frac{-\frac{\pi}{4}+2\pi}{2}\right) + i\sin\left(\frac{-\frac{\pi}{4}+2\pi}{2}\right)\right) \\ &= 2^{\frac{1}{4}}\left(\cos\left(\frac{7\pi}{8}\right) + i\sin\left(\frac{7\pi}{8}\right)\right) \end{aligned}$$

Therefore $2^{\frac{1}{4}}\left(\cos\left(-\frac{\pi}{8}\right) + i\sin\left(-\frac{\pi}{8}\right)\right)$ and $2^{\frac{1}{4}}\left(\cos\left(\frac{7\pi}{8}\right) + i\sin\left(\frac{7\pi}{8}\right)\right)$ are the square roots of $1 - i$.

Complex exponents

We conclude our introduction to complex numbers by demonstrating another way to express polar form.

Fact 15.16

Let $z = r(\cos(\theta) + i\sin(\theta))$ be a complex number. Then

$$z = re^{i\theta}$$

This fact can be proved using the Taylor series of the functions e^x, $\sin(x)$ and $\cos(x)$. It gives us yet another way of expressing complex numbers.

Example 15.17

We can write $z = 3\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)$ as

$$z = 3e^{i\frac{2\pi}{3}}$$

In Example 15.15 we showed that

$$2^{\frac{1}{4}}\left(\cos\left(-\frac{\pi}{8}\right) + i\sin\left(-\frac{\pi}{8}\right)\right)$$
$$2^{\frac{1}{4}}\left(\cos\left(\frac{7\pi}{8}\right) + i\sin\left(\frac{7\pi}{8}\right)\right)$$

are the square roots of $1 - i$. We can write this more succinctly in terms of exponentials

$$2^{\frac{1}{4}}\left(\cos\left(-\frac{\pi}{8}\right) + i\sin\left(-\frac{\pi}{8}\right)\right) = 2^{\frac{1}{4}}e^{-i\frac{\pi}{8}}$$
$$2^{\frac{1}{4}}\left(\cos\left(\frac{7\pi}{8}\right) + i\sin\left(\frac{7\pi}{8}\right)\right) = 2^{\frac{1}{4}}e^{i\frac{7\pi}{8}}$$

Fact 15.18

Let $z = re^{i\theta}$ be a complex number. Then

$$\text{Re}(z) = r\cos(\theta)$$
$$\text{Im}(z) = r\sin(\theta)$$

Suggested Problems

Practice the material covered in this lecture by attempting the following questions from Chapter 10.2 of Anton-Rorres 9th Edition (available on the coursepage)

- Questions 33, 5

and the questions from Chapter 10.3 of Anton-Rorres 9th Edition

- Questions 3, 5, $7(b)$, $7(c)$, 11, 14, 15