Math 1LS3 Week 3: Discrete Time Dynamical Systems

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This week covers Chapter 2 of the textbook (2.1-2.3; we're skipping 2.4,2.5). Next week is 3.1-3.4. Warning for pacing: 3.3 is relatively long. We'll then finish Chapter 3 the following week.

Overview

You studied discrete-time dynamical systems (DTDS) in grade school!

- Start at 0 and repeatedly add 1 – **counting**
- Start at s and repeatedly add 1 (t times) – **addition** ($s + t$)
- Start at 0 and repeatedly add d (t times) – **multiplication** ($d \cdot t$)
- Start at 1 and multiply by r (t times) – **exponentiation** (r^t)

A **DTDS** consists of:

- Starting value, m_0: "**intial value**"
- Operation to repeat, f: "**rule**"/"**updating function**"

Main Question: What happens as we repeatedly apply f?
This week, we'll start to answer this question. After we learn some calculus, we will return to it in week 9.

A discrete time dynamical system yields a **sequence** by *iteration*.

$$\boxed{\begin{array}{c}\text{DTDS:}\\ m_0 \ \&\ f\end{array}} \Longrightarrow \boxed{\begin{array}{c}\text{Sequence:}\\ m_0, m_1, m_2, \ldots\end{array}}$$

$$\begin{aligned} m_1 &= f(m_0) \\ m_2 &= f(m_1) = f^2(m_0) = f(f(m_0)) \\ m_3 &= f(m_2) = f^3(m_0) = f(f(f(m_0))) \\ m_4 &= f(m_3) = f^4(m_0) = f(f(f(f(m_0)))), \quad \text{etc.} \end{aligned}$$

Example

Initial Value: $m_0 = 1$. Updating Function: $f(m) = 2 \cdot m$.

1	2	4	8	16	32	64	128
m_0	m_1	m_2	m_3	m_4	m_5	m_6	m_7

Rules vs. Updating Functions

A **rule** is pretty much the same information as an *updating function*.

Example

Initial Value: $m_0 = 3$. Rule: $m_{t+1} = m_t - 1$.

3	2	1	0	-1	-2	-3	-4
m_0	m_1	m_2	m_3	m_4	m_5	m_6	m_7

rule $m_{t+1} = m_t - 1$ corresponds to **updating function** $f(m) = m - 1$

Example

What is the rule corresponding to $f(m) = m^2$? Answer: $m_{t+1} = m_t^2$

$$\boxed{\text{DTDS}} = \boxed{\begin{array}{c}\text{Initial value } m_0 \text{ \&} \\ \text{(Rule } \textit{or} \text{ Updating Function)}\end{array}}$$

Updating Functions vs. Sequences

- The updating function f is a function.

$$\text{current entry in sequence} \mapsto \text{next entry in sequence}$$

- The sequence m_t is also a function.

$$t \mapsto m_t$$

Warning: these are **not** the same functions!

- The updating function tells how to do *one step*.
- We will graph both functions later to study the DTDS.

Iteration

Repeatedly doing something is called **iteration**.

Initial Value: m_0
Updating Function: f

$$\begin{aligned}
m_1 &= f(m_0)\\
m_2 &= f(m_1) = f(f(m_0))\\
m_3 &= f(m_2) = f(f(f(m_0)))\\
m_4 &= f(m_3) = f(f(f(f(m_0))))\\
&\vdots\\
m_{t+1} &= f(m_t) = f^{t+1}(m_0)
\end{aligned}$$

Warning: $f^n(x)$ does not mean $(f(x))^n$, but $\sin^2(\varphi)$ means $(\sin(\varphi))^2$.
"$\sin^2(\varphi)$ is odious to me"–C.F. Gauss

Solving a DTDS means finding a direct formula for m_t.

Solution to a DTDS

Solving a DTDS means finding a direct formula for m_t.

- Start at $m_0 = 0$.
- $f(x) = x + 6$.
- Repeatedly add 6: $m_{t+1} = m_t + 6$.

What is m_7? $m_7 = 6 \cdot 7 = 42$.
What is m_t? $m_t = 6 \cdot t$.

$\boxed{m_t = 6 \cdot t}$ is a **solution** to this DTDS.
$6 \cdot t$ is a direct function of t: no reference to m_{t-1}.

Q: Why did you learn how to multiply?
A: To repeatedly add without having to repeatedly add.
Moral: solutions are nice – they fully answer our main question.

Using a DTDS as a mathematical model

In a DTDS model:

- t is usually **time** (independent variable).
- m is some physical quantity (dependent variable).
- m_t is the value of m at time t.

When constructing a model:

1. First identify the *updating function* (one step: from m_t to m_{t+1}).
2. *After* that, look for a *solution* (from t to m_t).

In addition to the updating function, the model should specify:

- How long is one time step?
- What does m represent?
- In what units is m measured?

Updating Functions

In a DTDS model, t is usually **time**.

An *updating function* f tells you how to compute the next value (m_{t+1}) from the current value (m_t).

$$m_{t+1} = f(m_t)$$

Problem

Find updating functions for the follwing scenarios.

1. *A tree grows 0.8m every year. Its height at time t is h_t, with t in years, h_t in meters.*
2. *A bacteria colony doubles in size every hour. Its population at time t is P_t, with t in hours.*

Solution

1. $h_{t+1} = h_t + 0.8$, *so the updating function is* $f(h_t) = h_t + 0.8$.
2. $P_{t+1} = 2P_t$, *so the updating function is* $g(P_t) = 2P_t$.

Solving a discrete-time dynamical system

Solving a DTDS means expressing m_t as a function of just t.

Problem

A tree starts at 1m and grows 0.8m each year. Write and solve the dynamical system for the height h_t.

Solution

The dynamical system consists of:

- *Initial height:* $h_0 = 1$.
- *Interval length: 1 year.*
- *Updating function:* $h_{t+1} = h_t + 0.8$.

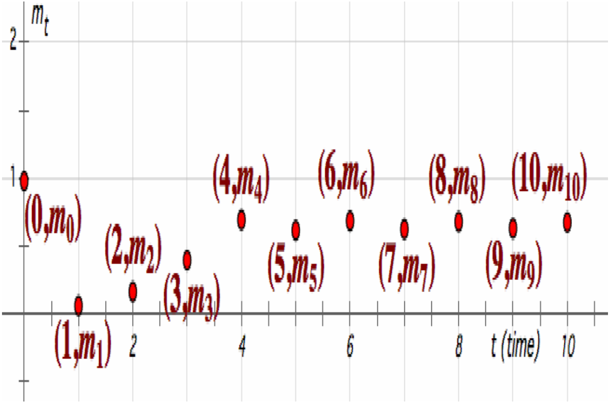
The solution is $h_t = 1 + 0.8t$.

Note: The solution expresses h_t in terms of t. The updating function expresses h_{t+1} in terms of h_t.

Visual Representation 1: *Graphing a DTDS*

Just a "line graph" without the lines.

t	m_t
0	0.978897
1	0.059908
2	0.163326
3	0.396286
4	0.693806
5	0.616074
6	0.685928
7	0.624749
8	0.679869
9	0.631177
10	0.675099



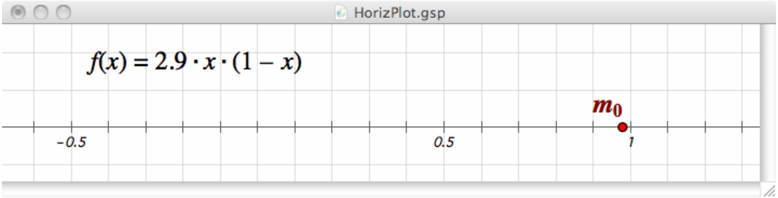
- What does graph suggest about the "Main Question" for this DTDS?
- Do we know for sure?
- The graph consists of discrete, isolated points. (Why?)

Graphing a Sequence along a Line

Visual Representation 2: *Graphing a DTDS along a line*

Same DTDS as previous slide.

- Plot the values on a horizontal or vertical line.
- Label the points



What does the image suggest about the "Main Question" for this DTDS?

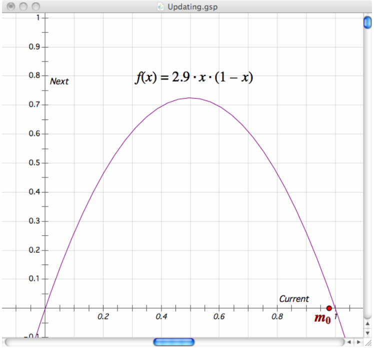
Do we know for sure?

Graphing the Updating Function

Visual Representation 3: *Graphing the Updating Function*

Just the graph of $y = f(x)$. Same DTDS as previous slides.

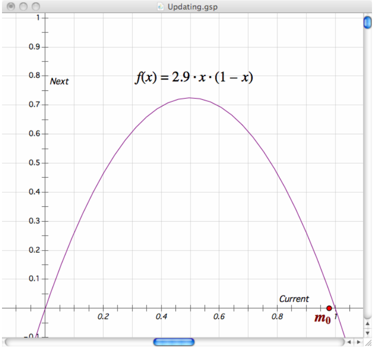
- Textbook labels the axes: "initial" (x-axis) and "final" (y-axis).
- Better to say: "current" (x-axis) and "next" (y-axis).



Note: f is typically continuous but m_t is discrete.

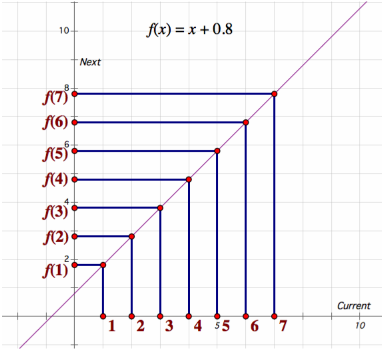
Graphing the Updating Function

What can we tell about *this particular* DTDS from the graph of $y = f(x)$?



- If current value is small, next value is small (but a little bigger).
- Medium (near .5) $\mapsto$ big, while big $\mapsto$ small.
- What happens upon many iterations? Not easy to say (yet).

But some updating functions *do* answer the main question.
DTDS with $h_{t+1} = f(h_t) = h_t + 0.8$. What happens?

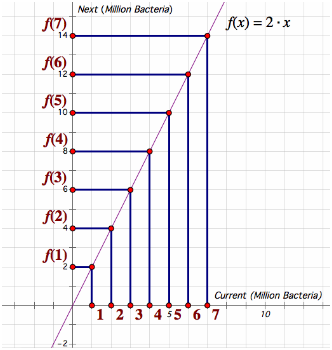


(Some equally spaced inputs)

- Current value increases by fixed amount to get new value.
- What happens upon many iterations? Height goes off to ∞.

Graphing the Updating Function: Bacteria Growth

DTDS with $P_{t+1} = f(P_t) = 2P_t$. What happens?



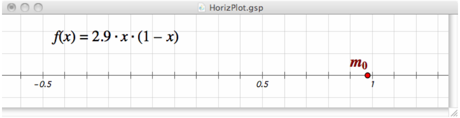
(Some equally spaced inputs)

- Current value doubles to get new value.
- Larger current values experience even larger growth.
- What happens upon many iterations? Population explodes exponentially to ∞.

Cobwebbing: Combining the Best of (2) and (3)

Visual Representation 4: *Cobwebbing* Same DTDS as in (1),(2),(3)

We want to take this sequence from before...

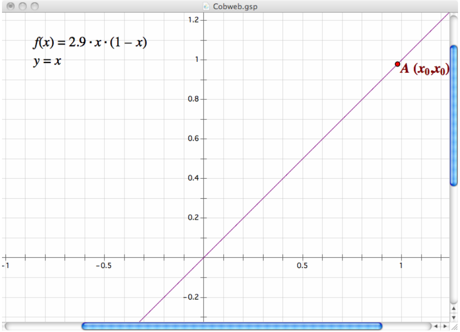


and plot instead along the *diagonal* line ($y = x$).

Cobwebbing: Combining the Best of (2) and (3)

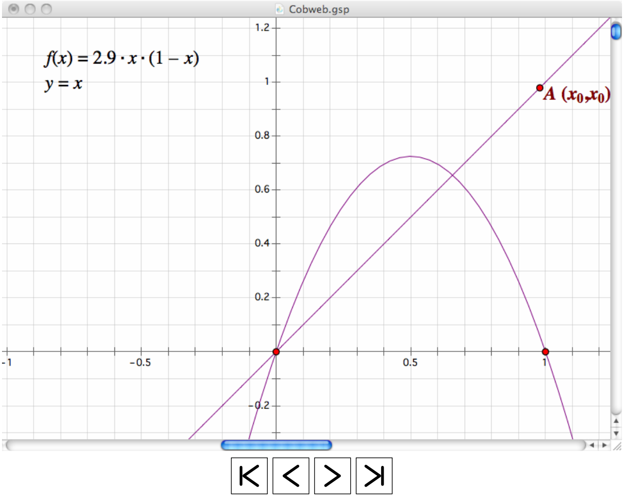
Visual Representation 4: *Cobwebbing* Same DTDS as in (1),(2),(3)

Instead of plotting x_0, plot (x_0, x_0).
Instead of plotting x_1, plot (x_1, x_1), etc.



Now we'll *connect* the points to visualize the sequence order.

Cobwebbing



$$x_1 = f(x_0)$$

$$x_2 = f(x_1)$$

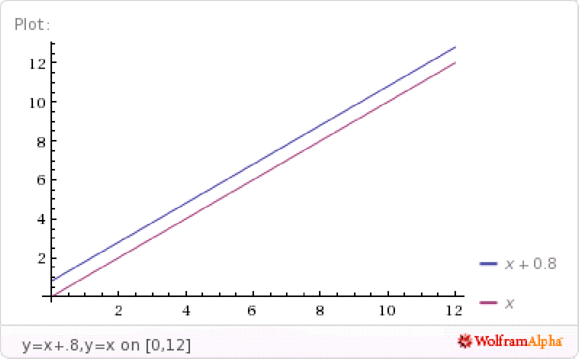
$$x_3 = f(x_2)$$

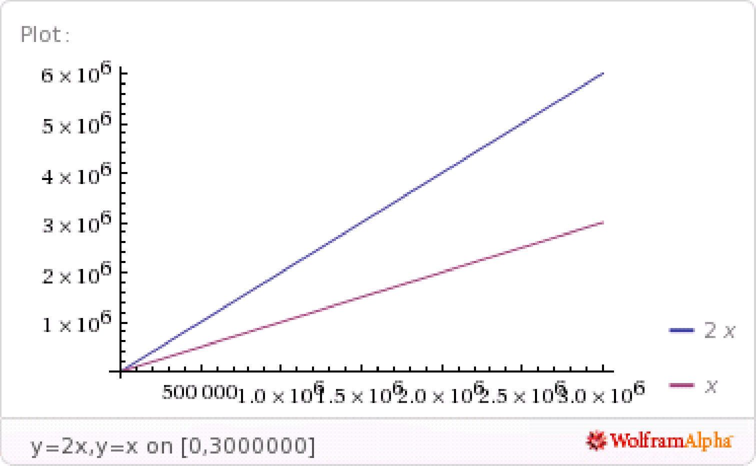
- One step is a vertical move followed by a horizontal move.
- What do we see about the Main Question?

Cobwebbing (Tree Growth Example)

Using the graph of the updating function, *cobwebbing* lets you see lots of iterates.

- (Current, next) pair (m_t, m_{t+1}) on graph of f.
- Move horizontally to line $y = x$: get point (m_{t+1}, m_{t+1}).
- Move vertically to f: get point (m_{t+1}, m_{t+2}).
- Repeat.





Click: Cobweb Plot Applet. Try 2*x with initial value 0.001.

Bacteria Population Example

Problem

A bacteria population P_t doubles every hour, starting at 1 million. It doubles every hour. Write and solve a dynamical system.

Solution

The dynamical system consists of:

- *Initial population:* $P_0 = 10^6$.
- *Interval length: 1 hour.*
- *Updating function:* $P_{t+1} = 2P_t$.

The solution is $P_t = 10^6 * 2^t$.

Repeated Addition, Repeated Multiplication

Basic additive DTDS:
If

- Initial value is A, and
- updating function is $f(m_t) = m_t + B$,

then the solution is $m_t = A + t * B$.

Basic exponential DTDS:
If

- Initial value is A, and
- updating function is $f(m_t) = B * m_t$,

then the solution is $m_t = A * B^t$.

Memorize this slide. (Better yet, understand it as the *definition* of multiplication, exponentiation!)

Dynamics of Absorption of Pain Meds (p.95)

Problem

A patient takes one dose of the pain drug methadone each day. The half-life of the drug in this patient is 24 hours. Describe an updating function for the amount of drug in the body and solve the DTDS.

Solution

If m_t is the current amount (in doses), then tomorrow the patient has

- *half of today's m_t left, plus*
- *the 1 new dose they take tomorrow.*

So the update function is $f(m_t) = \frac{m_t}{2} + 1$.

Examples 2.1.8, 2.1.9 (p.100-101) solve the dynamical system for different initial values of m. Let's try the general case...

It's hard – so just focus on the main ideas.

Solving a linear DTDS: $f(m_t) = (m_t)/2 + 1$

Solution

Observe: if $m_t = 2$ doses, then $m_{t+1} = 2$. So $m_{t+2} = 2$, etc.
Idea: introduce a new variable $e = m - 2$, so $m_t = 2 + e_t$.
e represents excess dosage – dosage beyond 2.
Find a DTDS for e:

$$e_{t+1} = m_{t+1} - 2 = \left(\frac{m_t}{2} + 1\right) - 2 = \frac{2 + e_t}{2} - 1 = \frac{e_t}{2}$$

So e_t is an exponential DTDS:

$$e_t = e_0 \cdot \left(\frac{1}{2}\right)^t = (m_0 - 2)\left(\frac{1}{2}\right)^t$$

Solving the original DTDS:

$$\boxed{m_t =} 2 + e_t = \boxed{2 + (m_0 - 2) * (1/2)^t}.$$

Running a DTDS Backwards

How can we run a discrete-time dynamical system backwards?
f does one step forward, so f^{-1} does one step backward.
The "backwards DTDS" has updating function f^{-1}.

Problem

Bacteria pop. doubles each hr. Now it's 10^6. *What was it 3 hrs ago?*

Solution

Update function $f(P_t) = 2P_t$, *so* $f^{-1}(P_t) = \frac{1}{2}P_t$. *Current value is* 10^6.

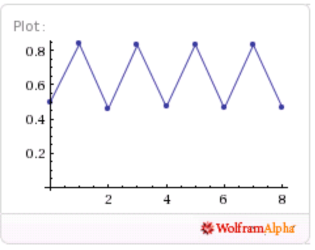
time	*population*
now	10^6
1 hour ago	$f^{-1}(10^6) = 500000$
2 hours ago	$f^{-1}(500,000) = 250000$
3 hours ago	$f^{-1}(250,000) = 125000$

Self-Composition

$f(m_t) = m_{t+1}$. In words, f computes the next value from the current value. What does $f \circ f$ compute? Two steps into the future.

Example

The DTDS: $f(x_t) = 3.35x_t(1 - x_t), x_0 = 0.5$ *oscillates*.



Study an *auxilliary DTDS* with:

- updating function $f \circ f$
- initial value 0.5

Iterate the *auxilliary DTDS*. What happens?
m_{even}'s only. Stabilizes!

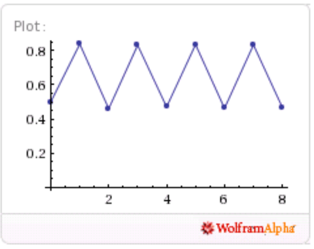
Click: Cobweb Plot Applet. Compare $f(x) =$ `3.35*x*(1-x)` and
$f \circ f(x) =$ `11.2225*x-48.8179*x^2+75.1908*x^3-37.5954*x^4`.

Self-Composition

$f(m_t) = m_{t+1}$. In words, f computes the next value from the current value. What does $f \circ f$ compute? Two steps into the future.

Example

The DTDS: $f(x_t) = 3.35x_t(1 - x_t), x_0 = 0.5$ *oscillates*.



Study an *auxilliary DTDS* with:

- updating function $f \circ f$
- initial value $f(0.5) = 0.8375$

Iterate the *auxilliary DTDS*. What happens?

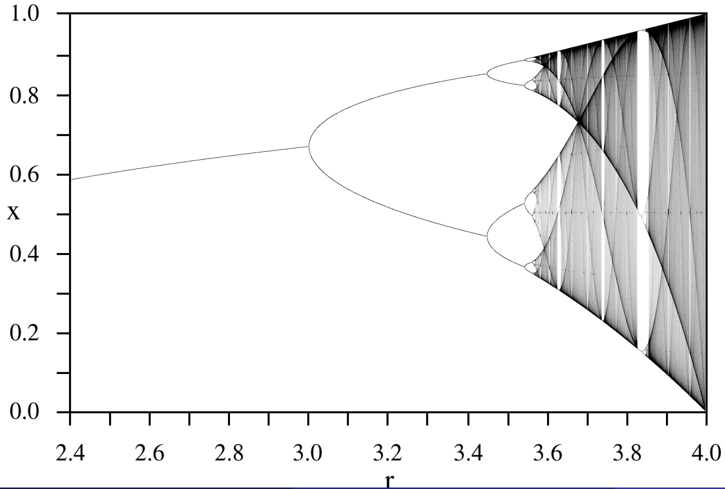
m_{odd}'s only. Stabilizes!

Click: Cobweb Plot Applet. Compare $f(x) =$ `3.35*x*(1-x)` and $f \circ f(x) =$ `11.2225*x-48.8179*x^2+75.1908*x^3-37.5954*x^4`.

Play with the parameter: change 3.35

Click: Cobweb Plot Applet

We looked at $f(x) =$ `3.35*x*(1-x)`. Change $r = 3.35$ to interesting r-values below. See what happens. Challenge: compute bifurcation points.



Equilibrium (Algebraic Description)

An *equilibrium point* m^* for a dynamical system is where $f(m^*) = m^*$.
If m^* is equilibrium point and $m_t = m^*$, then $m_{t+1} = m^*$, $m_{t+2} = m^*$, etc.

Problem

Find the equilibrium points for the pain medication DTDS: $f(m_t) = \frac{m_t}{2} + 1$.

Solution

Solve $f(m^*) = m^*$.

$$m^* = \frac{m^*}{2} + 1$$

$$\frac{m^*}{2} = 1$$

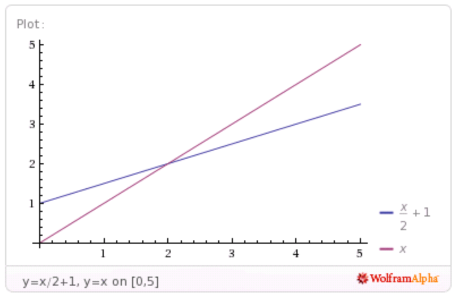
$$m^* = 2$$

There is just the one equilibrium point: $m^* = 2$.

Equilibrium: Geometric Description

Find the equilibrium points for $f(m_t) = 1 + \frac{m_t}{2}$ geometrically. Then cobweb starting with $m_0 = 1$ and with $m_0 = 4$ to see what happens.

Solution



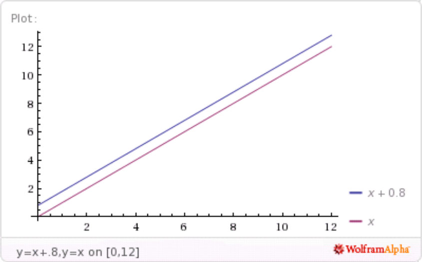
The equilibrium point is the intersection point: $m^* = 2$.
Starting at $m_0 = 1$ *or* $m_0 = 4$, *cobwebbing draws us in to this equilibrium.*

An equilibrium point is **stable** if nearby values are drawn in.
An equilibrium point is **unstable** if nearby values are pushed away.

Tree Growth, Bacteria Colony Equilibria

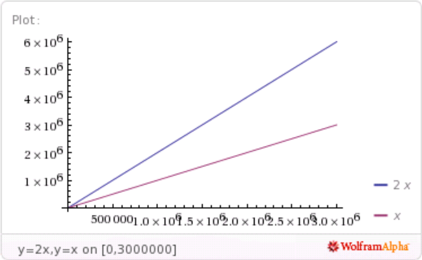
Find the equilibria. Are they stable?

Tree growth: $f(m_t) = m_t + 0.8$.



No equilibrium value!

Bacteria colony: $g(P_t) = 2P_t$.



Equilibrium only at $P^* = 0$.

Is it stable? No, unstable.

Stable/Unstable Equilibria Summary

- An equilibrium point is where $f(m^*) = m^*$.
- An equilibrium point is where $y = f(x)$ intersects $y = x$.
- The equilibrium m^* is *stable* if nearby points are drawn in by cobwebbing.
- The equilibirum m^* is *unstable* if nearby points are pushed away.

Note: if you can't

- graph f, or
- directly solve $f(m^*) = m^*$,

you can computer search for *stable* equilibria.

How? By iterating! Unstable equilibria remain hidden. $\ddot\frown$

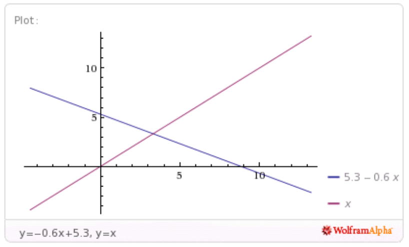
Example 2.2.6: p.117

Codfish population is given by the updating function $n_{t+1} = -0.6n_t + 5.3$ (million codfish). Find equilibria and classify as stable/unstable.

Solution

Solve $-0.6n^* + 5.3 = n^*$ *to find unique equilibrium* $n^* \approx 3.3125$

Cobweb to determine stability:



Spirals in towards equilibrium: stable!

One more problem (time permitting): go back to $f(t) = 3.35 * x * (1 - x)$ and find stable/unstable fixed points. What about $f \circ f$?

Per Capita Production: p.123-4

Problem

Consider the bacteria population model $P_{t+1} = r * P_t$.
*The parameter $r \geq 0$ is called **per capita production**.*
Find equilibria and classify as stable/unstable.

Note: this is the **basic exponential DTDS**.

Solution

For equilibrium P^*, solve $rP^* = P^*$.
If $r \neq 1$, the unique solution is $P^* = 0$.
If $r = 1$, then every point is an equilibrium value!
If $r > 1$, the equilibrium is unstable. (Exponential growth!)
If $r < 1$, the equilibrium is stable. (Exponential decay!)
Homework: verify stability by Cobwebbing.

Memorize (or understand) this result! We'll use it later.
Note: r is still called **per capita production** if it's a *function* of P_t.

Limited Population Model: p.126-7

In *limited population models*, the per capita production is a variable rate. It decreases as the population gets bigger. The resulting population doesn't grow as quickly as in the exponential case.

Problem

Consider the limited population model

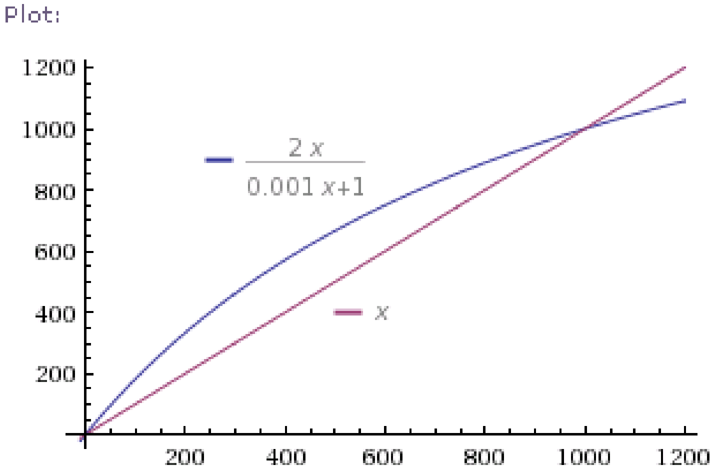
$$b_{t+1} = \left(\frac{2}{1 + 0.001 b_t} \right) b_t.$$

Find equilibria and classify as stable/unstable. (Cobweb.)

Solution

Solve $b^* = \frac{2}{1+0.001b^*} b^*$.
Either $b^* = 0$ *or* $1 = \frac{2}{1+0.001b^*} \implies 1 + 0.001b^* = 2 \implies b^* = 1000$.
There are two equilibria ($b^* = 0$ *is unstable;* $b^* = 1000$ *is stable).*

Limited Population Model: p.126-7 Cobweb Plot



Classify the stability of equilibria.

Caffeine Absorption Model

Problem

The body eliminates caffeine at a constant rate of 13% per hour. Find the updating function if d extra mg of caffeine are consumed each hour.

- c_t is caffeine present in body in mg.
- One time step = 1 hour

Solution

$f(c_t) = .87c_t + d$

- The half-life is ≈ 5 hours.
- Caffeine taken by 2PM should be about quartered by midnight.

Alcohol Dynamics

Unlike caffeine, which is eliminated at a constant proportion, the liver removes alcohol at a rate that decreases with amount present:

$$r(a_t) = \frac{10.1}{4.2 + a_t}.$$

Problem

A student drinks d grams of alcohol at the end of each hour. Write an updating function to describe the amount a_t.

Solution

$$a_{t+1} = a_t - r(a_t)a_t + d = a_t - \frac{10.1a_t}{4.2 + a_t} + d$$

Alcohol Equilibrium

Problem

A student drinks one standard drink (14g) and keeps consuming a drink on the hour. What happens?

Solution

To find equilibrium:

$$a^* = a^* - \frac{10.1a^*}{4.2 + a^*} + 14$$

$$10.1a^* = 14(4.2 + a^*) = 58.8 + 14a^*$$

$$a^* = -15.1$$

The only equilibrium is biologically meaningless, so a_t *can't approach an equilibrium.*

Cobwebbing shows alcohol level increases without end...

Alcohol Equilibrium Continued

Solution