Priority Queues and Applications

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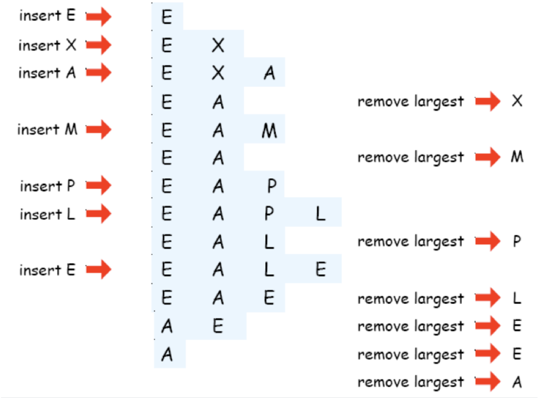
Priority Queue

- A collection is a data type that stores a group of items.
- **Priority queue** is a collection of objects which can be compared. It supports inserting an item, and removing the largest (smallest) item.



Priority queue: applications

- Event-driven simulation – customers in a line
- Numerical computation – reducing roundoff error
- Discrete optimization – scheduling
- Operating systems – load balancing, interrupt handling
- Data compression – Huffman codes
- Graph searching – Dijkstra's algorithm, Prim's algorithm
- and many more!

Priority Queue API

Key must be Comparable
(bounded type parameter)

```
public class MaxPQ<Key extends Comparable<Key>>
```

	MaxPQ()	*create an empty priority queue*
	MaxPQ(Key[] a)	*create a priority queue with given keys*
void	insert(Key v)	*insert a key into the priority queue*
Key	delMax()	*return and remove a largest key*
boolean	isEmpty()	*is the priority queue empty?*
Key	max()	*return a largest key*
int	size()	*number of entries in the priority queue*

Priority queue: implementations costs

Challenge: Implement all operations efficiently.

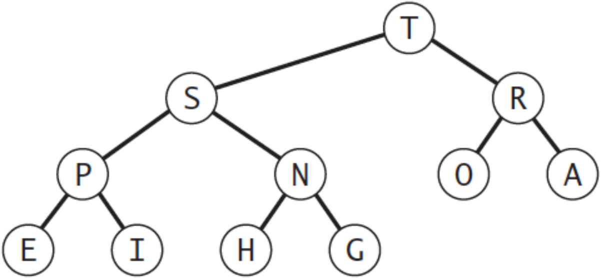
implementation	insert	del max
unordered array	1	n
ordered array	n	1
goal	$\boxed{\log n}$	$\boxed{\log n}$

Solution: Heap

Binary Heap Ordered Tree

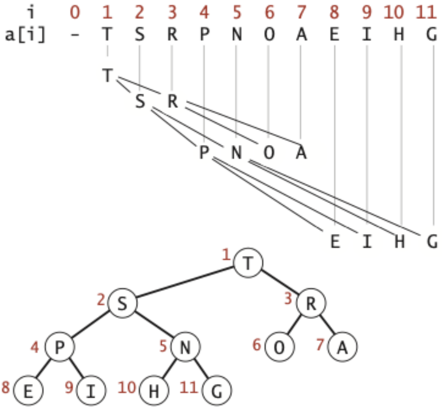
Binary Heap ordered tree: is a complete binary tree where

- the keys are in nodes, and
- every parent's key $\geq$ children's keys (Max. heap property).



Binary Heap - array representation

- Indices start at 1.
- Take nodes in level order; that is, with the root at position 1, its children at positions 2 and 3, their children in positions 4, 5, 6, and 7, and so on.
- In a heap, the parent of the node in position k is in position $\lfloor k/2 \rfloor$.
- Conversely, the two children of the node in position k are in positions 2k and 2k + 1.



Heap representations

Binary heap: Operations (see demo)

- Insert. Add node at end, then swim it up.
- Remove the maximum. Exchange root with node at end, then sink it down.

See Demo - `https://algs4.cs.princeton.edu/lectures/`

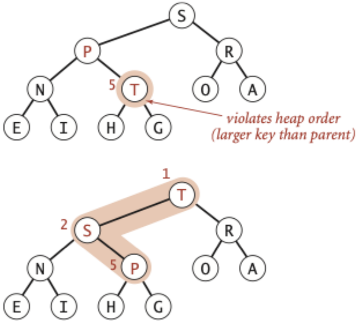
Binary heap: swim operation

Scenario. A key becomes larger than its parent's key. To eliminate the violation:

- Exchange key in child with key in parent.
- Repeat until heap order restored.

```
private void swim(int k)
{
   while (k > 1 && less(k/2, k))
   {
      exch(k, k/2);
      k = k/2;
   }
}
```

parent of node at k is at k/2

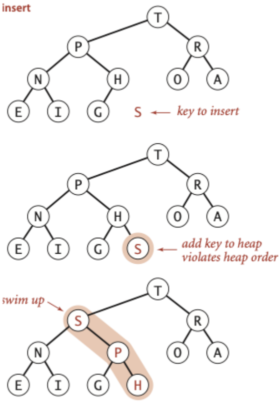


Binary heap: Insert

Insert.

- Add node at end, then swim it up.
- Cost. At most $1 + \log_2 n$ compares.

```
public void insert(Key x)
{
   pq[++n] = x;
   swim(n);
}
```



Binary heap: sink operation

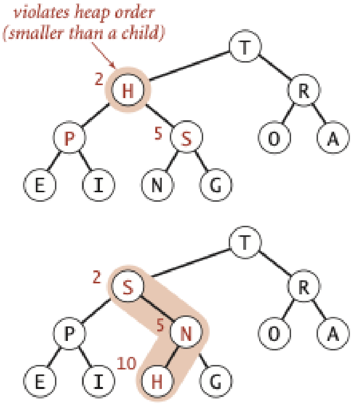
Scenario. A key becomes smaller than one (or both) of its children's.

To eliminate the violation:

- Exchange key in parent with key in larger child.
- Repeat until heap order restored.

```
private void sink(int k)
{
   while (2*k <= n)
   {
      int j = 2*k;
      if (j < n && less(j, j+1)) j++;
      if (!less(k, j)) break;
      exch(k, j);
      k = j;
   }
}
```

children of node at k are 2*k and 2*k+1



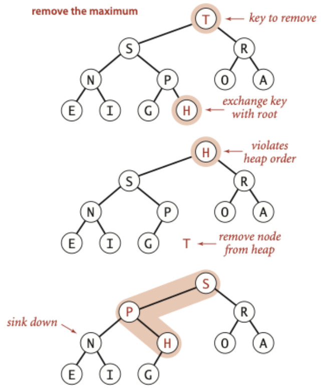
Top-down reheapify (sink)

Binary heap: delete maximum

- Delete max: Exchange root with node at end, then sink it down.
- Cost: At most $2\log_2 n$ compares.

```
public Key delMax()
{
   Key max = pq[1];
   exch(1, n--);
   sink(1);
   pq[n+1] = null;
   return max;
}
```

prevent loitering



Max. Priority Queue

```
public class MaxPQ<Key extends Comparable<Key>>
{
   private Key[] pq;
   private int n;

   public MaxPQ(int capacity)
   {  pq = (Key[]) new Comparable[capacity+1];  }

   public boolean isEmpty()
   {  return n == 0;  }
   public void insert(Key key)  // see previous code
   public Key delMax()          // see previous code

   private void swim(int k)     // see previous code
   private void sink(int k)     // see previous code

   private boolean less(int i, int j)
   {  return pq[i].compareTo(pq[j]) < 0;  }
   private void exch(int i, int j)
   {  Key t = pq[i]; pq[i] = pq[j]; pq[j] = t;  }
}
```

fixed capacity (for simplicity)

PQ ops

heap helper functions

array helper functions

Priority Queue: implementations cost summary

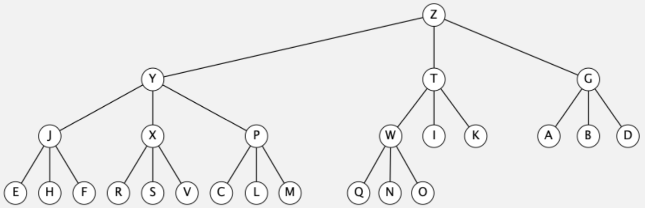
implementation	insert	del max	max
unordered array	1	n	n
ordered array	n	1	1
binary heap	$\log n$	$\log n$	1

order-of-growth of running time for priority queue with n items

Binary heap: practical improvements

Multiway heaps.

- Complete d-way tree.
- Parent's key no smaller than its children's keys.
- Fact. Height of complete d-way tree on n nodes is $\log_d n$.

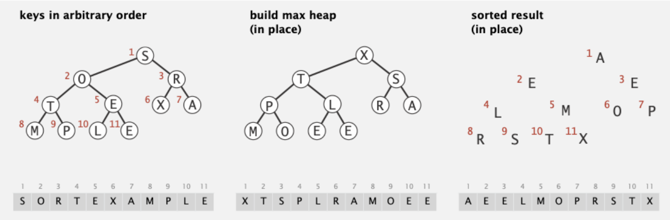


3-way heap

Heapsort: Idea

Basic plan for in-place sort.

- View input array as a complete binary tree.
- Heap construction: build a max-heap with all n keys.
- Sortdown: repeatedly remove the maximum key.



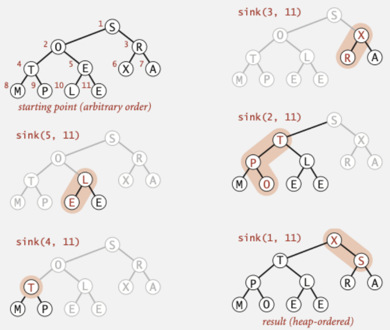
See Demo - `https://algs4.cs.princeton.edu/lectures/`

Heapsort: Construction

First pass: Build max heap using bottom-up (sink()) method.

We assume array entries are from 1 to n.

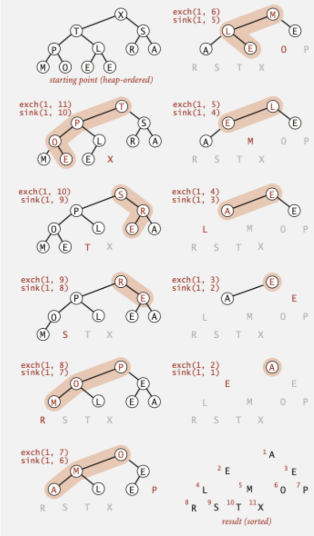
```
for (int k = n/2; k >= 1; k--)
   sink(a, k, n);
```



Heapsort: Sortdown

- Second pass: Remove the maximum, one at a time.
- Repeatedly delete the largest remaining item.

```
while (n > 1)
{
   exch(a, 1, n--);
   sink(a, 1, n);
}
```



Heapsort: Java implementation

```
public class Heap
{
   public static void sort(Comparable[] a)
   {
      int n = a.length;
      for (int k = n/2; k >= 1; k--)
         sink(a, k, n);
      while (n > 1)
      {
         exch(a, 1, n);
         sink(a, 1, --n);
      }
   }
                        but make static (and pass arguments)
   private static void sink(Comparable[] a, int k, int n)
   {  /* as before */  }

   private static boolean less(Comparable[] a, int i, int j)
   {  /* as before */  }

   private static void exch(Object[] a, int i, int j)
   {  /* as before */  }

}                                    but convert from 1-based
                                     indexing to 0-base indexing
```

Heapsort: Trace

a[i]

N	k	0	1	2	3	4	5	6	7	8	9	10	11
initial values			S	O	R	T	E	X	A	M	P	L	E
11	5		S	O	R	T	L	X	A	M	P	E	E
11	4		S	O	R	T	L	X	A	M	P	E	E
11	3		S	O	X	T	L	R	A	M	P	E	E
11	2		S	T	X	P	L	R	A	M	O	E	E
11	1		X	T	S	P	L	R	A	M	O	E	E
heap-ordered			X	T	S	P	L	R	A	M	O	E	E
10	1		T	P	S	O	L	R	A	M	E	E	X
9	1		S	P	R	O	L	E	A	M	E	T	X
8	1		R	P	E	O	L	E	A	M	S	T	X
7	1		P	O	E	M	L	E	A	R	S	T	X
6	1		O	M	E	A	L	E	P	R	S	T	X
5	1		M	L	E	A	E	O	P	R	S	T	X
4	1		L	E	E	A	M	O	P	R	S	T	X
3	1		E	A	E	L	M	O	P	R	S	T	X
2	1		E	A	E	L	M	O	P	R	S	T	X
1	1		A	E	E	L	M	O	P	R	S	T	X
sorted result			A	E	E	L	M	O	P	R	S	T	X

Heapsort trace (array contents just after each sink)

Heapsort: Analysis

Proposition. Heap construction uses $\leq 2n$ compares and $\leq n$ exchanges.

Proposition. Heapsort uses $\leq 2n \log_2 n$ compares and exchanges. - The algorithm can be improved to $\approx 1n \log_2 n$, but no such variant is known to be practical.

Significance. In-place sorting algorithm with $n \log n$ worst-case.

- Mergesort: no, linear extra space. [in-place merge possible, not practical]
- Quicksort: no, quadratic time in worst case. [$n \log n$ worst-case quicksort possible, not practical]
- Heapsort: yes!

Bottom line. Heapsort is optimal for both time and space, but:

- Inner loop longer than quicksort's.
- Makes poor use of cache: array entries are rarely compared with nearby array entries, so the number of cache misses is far higher than for quicksort, mergesort, where most compares are with nearby entries.

Sorting and Stability

A sorting method is **stable** if it preserves the relative order of equal keys in the array.

sorted by time		sorted by location (not stable)		sorted by location (stable)	
Chicago	09:00:00	Chicago	09:25:52	Chicago	09:00:00
Phoenix	09:00:03	Chicago	09:03:13	Chicago	09:00:59
Houston	09:00:13	Chicago	09:21:05	Chicago	09:03:13
Chicago	09:00:59	Chicago	09:19:46	Chicago	09:19:32
Houston	09:01:10	Chicago	09:19:32	Chicago	09:19:46
Chicago	09:03:13	Chicago	09:00:00	Chicago	09:21:05
Seattle	09:10:11	Chicago	09:35:21	Chicago	09:25:52
Seattle	09:10:25	Chicago	09:00:59	Chicago	09:35:21
Phoenix	09:14:25	Houston	09:01:10	Houston	09:00:13
Chicago	09:19:32	Houston	09:00:13	Houston	09:01:10
Chicago	09:19:46	Phoenix	09:37:44	Phoenix	09:00:03
Chicago	09:21:05	Phoenix	09:00:03	Phoenix	09:14:25
Seattle	09:22:43	Phoenix	09:14:25	Phoenix	09:37:44
Seattle	09:22:54	Seattle	09:10:25	Seattle	09:10:11
Chicago	09:25:52	Seattle	09:36:14	Seattle	09:10:25
Chicago	09:35:21	Seattle	09:22:43	Seattle	09:22:43
Seattle	09:36:14	Seattle	09:10:11	Seattle	09:22:54
Phoenix	09:37:44	Seattle	09:22:54	Seattle	09:36:14

no longer sorted by time

still sorted by time

Stability when sorting on a second key

Sorting and Stability

Stable sorts

- Insertion sort
- Mergesort

Below are NOT stable sorting algorithms

- Selection sort
- Shellsort
- Quicksort
- Heapsort

Sorting Summary

algorithm	stable?	in place?	*order of growth to sort N items* running time	extra space	notes
selection sort	no	yes	N^2	1	
insertion sort	yes	yes	between N and N^2	1	depends on order of items
shellsort	no	yes	$N \log N$? $N^{6/5}$?	1	
quicksort	no	yes	$N \log N$	$\lg N$	probabilistic guarantee
mergesort	yes	no	$N \log N$	N	
heapsort	no	yes	$N \log N$	1	

Performance characteristics of sorting algorithms

Sorting Applications

- Commercial computing
- Searching
- Operations research
- Event-driven simulation
- String processing
- And many more!