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10.3 POLAR FORM OF A COMPLEX NUMBER

In this section we shall discuss a way to represent complex numbers using trigonometric properties. Our work will lead to an important formula for powers of complex numbers and to a method for finding nth roots of complex numbers.

Polar Form

If $z = x + iy$ is a nonzero complex number, $r = |z|$, and θ measures the angle from the positive real axis to the vector z, then, as suggested by Figure 10.3.1,

$$x = r\cos\theta, \qquad y = r\sin\theta \tag{1}$$

so that $z = x + iy$ can be written as $z = r\cos\theta + ir\sin\theta$ or

$$z = r(\cos\theta + i\sin\theta) \tag{2}$$

This is called a ***polar form of z***.

Argument of a Complex Number

The angle θ is called an ***argument of z*** and is denoted by

$$\theta = \arg z$$

The argument of z is not uniquely determined because we can add or subtract any multiple of 2π from θ to produce another value of the argument. However, there is only one value of the argument in radians that satisfies

$$-\pi < \theta \le \pi$$

This is called the ***principal argument of z*** and is denoted by

$$\theta = \operatorname{Arg} z$$

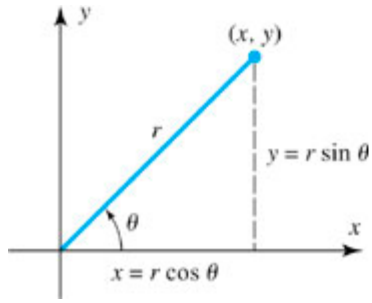

Figure 10.3.1

EXAMPLE 1 Polar Forms

Express the following complex numbers in polar form using their principal arguments:

(a) $z = 1 + \sqrt{3}i$

(b) $z = -1 - i$

Solution (a)

The value of r is

$$r = |z| = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{4} = 2$$

and since $x = 1$ and $y = \sqrt{3}$, it follows from 1 that

$$1 = 2\cos\theta \qquad \text{and} \qquad \sqrt{3} = 2\sin\theta$$

so $\cos\theta = 1/2$ and $\sin\theta = \sqrt{3}/2$. The only value of θ that satisfies these relations and meets the requirement $-\pi < \theta \leq \pi$ is $\theta = \pi/3(=60°)$ (see Figure 10.3.2*a*). Thus a polar form of z is

$$z = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

Solution (b)

The value of r is

$$r = |z| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$$

and since $x = -1$, $y = -1$, it follows from 1 that

$$-1 = \sqrt{2}\cos\theta \quad \text{and} \quad -1 = \sqrt{2}\sin\theta$$

so $\cos\theta = -1/\sqrt{2}$ and $\sin\theta = -1/\sqrt{2}$. The only value of θ that satisfies these relations and meets the requirement $-\pi < \theta \leq \pi$ is $\theta = -3\pi/4 (= -135°)$ (Figure 10.3.2*b*). Thus, a polar form of z is

$$z = \sqrt{2}\left(\cos\frac{-3\pi}{4} + i\sin\frac{-3\pi}{4}\right)$$

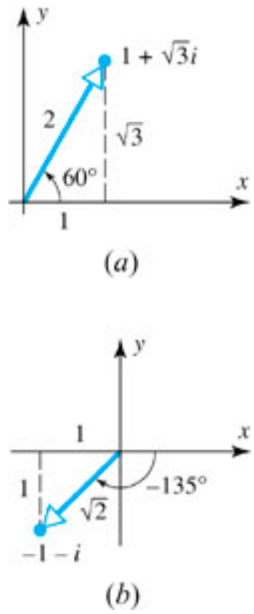

Figure 10.3.2

Multiplication and Division Interpreted Geometrically

We now show how polar forms can be used to give geometric interpretations of multiplication and division of complex numbers. Let

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \quad \text{and} \quad z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

Multiplying, we obtain

$$z_1 z_2 = r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)]$$

Recalling the trigonometric identities

$$\cos(\theta_1 + \theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2$$
$$\sin(\theta_1 + \theta_2) = \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2$$

we obtain

$$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] \tag{3}$$

which is a polar form of the complex number with modulus $r_1 r_2$ and argument $\theta_1 + \theta_2$. Thus we have shown that

$$|z_1 z_2| = |z_1||z_2| \tag{4}$$

and

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2$$

(Why?) In words, *the product of two complex numbers is obtained by multiplying their moduli and adding their arguments* (Figure 10.3.3).

We leave it as an exercise to show that if $z_2 \neq 0$, then

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)] \tag{5}$$

from which it follows that

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \qquad \text{if } z_2 \neq 0$$

and

$$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$$

In words, *the quotient of two complex numbers is obtained by dividing their moduli and subtracting their arguments* (*in the appropriate order*).

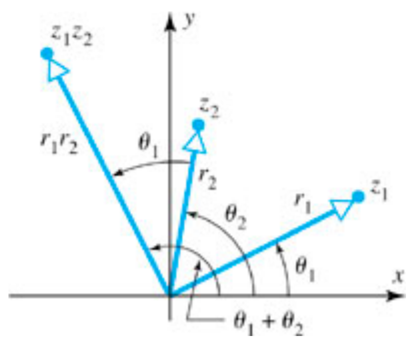

Figure 10.3.3
The product of two complex numbers.

EXAMPLE 2 A Quotient Using Polar Forms

Let

$$z_1 = 1 + \sqrt{3}i \qquad \text{and} \qquad z_2 = \sqrt{3} + i$$

Polar forms of these complex numbers are

$$z_1 = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) \qquad \text{and} \qquad z_2 = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$

(verify) so that from 3,

$$z_1z_2 = 4\left[\cos\left(\frac{\pi}{3}+\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{3}+\frac{\pi}{6}\right)\right]$$
$$= 4\left[\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right] = 4[0+i] = 4i$$

and from 5,

$$\frac{z_1}{z_2} = 1\cdot\left[\cos\left(\frac{\pi}{3}-\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{3}-\frac{\pi}{6}\right)\right]$$
$$= \cos\frac{\pi}{6}+i\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2}+\frac{1}{2}i$$

As a check, we calculate z_1z_2 and z_1 / z_2 directly without using polar forms for z_1 and z_2:

$$z_1z_2 = (1+\sqrt{3}i)(\sqrt{3}+i) = (\sqrt{3}-\sqrt{3})+(3+1)i = 4i$$
$$\frac{z_1}{z_2} = \frac{1+\sqrt{3}i}{\sqrt{3}+i}\cdot\frac{\sqrt{3}-i}{\sqrt{3}-i} = \frac{(\sqrt{3}+\sqrt{3})+(-i+3i)}{4} = \frac{\sqrt{3}}{2}+\frac{1}{2}i$$

which agrees with our previous results.

The complex number *i* has a modulus of 1 and an argument of $\pi / 2(=90°)$, so the product iz has the same modulus as *z*, but its argument is 90° greater than that of *z*. In short, *multiplying z by i rotates z counterclockwise by* 90° (Figure 10.3.4).

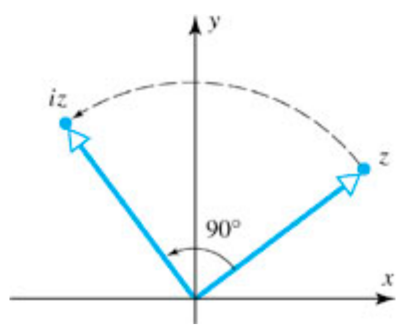

Figure 10.3.4
Multiplying by *i* rotates *z* counterclockwise by 90°.

DeMoivre's Formula

If *n* is a positive integer and $z = r(\cos\theta + i\sin\theta)$, then from Formula 3,

$$z^n = \underbrace{z\cdot z\cdot z\cdots z}_{n-\text{factors}} = r^n[\cos\underbrace{(\theta+\theta+\cdots+\theta)}_{n-\text{terms}} + i\sin\underbrace{(\theta+\theta+\cdots+\theta)}_{n-\text{terms}}]$$

or

$$z^n = r^n(\cos n\theta + i\sin n\theta) \tag{6}$$

Moreover, 6 also holds for negative integers if $z \neq 0$ (see Exercise 23).

In the special case where $r = 1$, we have $z = \cos\theta + i\sin\theta$, so 6 becomes

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \tag{7}$$

which is called ***DeMoivre's formula***. Although we derived 7 assuming *n* to be a positive integer, it will be shown in the exercises that this formula is valid for all integers *n*.

Finding *n*th Roots

We now show how DeMoivre's formula can be used to obtain roots of complex numbers. If *n* is a positive integer and *z* is any complex number, then we define an ***nth root of z*** to be any complex number *w* that satisfies the equation

$$w^n = z \tag{8}$$

We denote an *n*th root of *z* by $z^{1/n}$. If $z \neq 0$, then we can derive formulas for the *n*th roots of *z* as follows. Let

$$w = \rho(\cos\alpha + i\sin\alpha) \qquad \text{and} \qquad z = r(\cos\theta + i\sin\theta)$$

If we assume that *w* satisfies 8, then it follows from 6 that

$$\rho^n(\cos n\alpha + i\sin n\alpha) = r(\cos\theta + i\sin\theta) \tag{9}$$

Comparing the moduli of the two sides, we see that $\rho^n = r$ or

$$\rho = \sqrt[n]{r}$$

where $\sqrt[n]{r}$ denotes the real positive *n*th root of *r*. Moreover, in order to have the equalities $\cos n\alpha = \cos\theta$ and $\sin n\alpha = \sin\theta$ in 9, the angles $n\alpha$ and θ must either be equal or differ by a multiple of 2π. That is,

$$n\alpha = \theta + 2k\pi \qquad \text{or} \qquad \alpha = \frac{\theta}{n} + \frac{2k\pi}{n}, \qquad k = 0, \pm 1, \pm 2, \ldots$$

Thus the values of $w = \rho(\cos\alpha + i\sin\alpha)$ that satisfy 8 are given by

$$w = \sqrt[n]{r}\left[\cos\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)\right], \qquad k = 0, \pm 1, \pm 2, \ldots$$

Although there are infinitely many values of *k*, it can be shown (see Exercise 16) that $k = 0$, 1, 2, …, $n - 1$ produce distinct values of *w* satisfying 8 but all other choices of *k* yield duplicates of these. Therefore, there are exactly *n* different *n*th roots of $z = r(\cos\theta + i\sin\theta)$, and these are given by

$$z^{1/n} = \sqrt[n]{r}\left[\cos\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)\right], \qquad k = 0, 1, 2, \ldots, n-1 \tag{10}$$

Abraham DeMoivre *(1667–1754)* was a French mathematician who made important contributions to probability, statistics, and trigonometry. He developed the concept of statistically independent events, wrote a major and influential treatise on probability, and helped transform trigonometry from a branch of geometry into a branch of analysis through his use of complex numbers. In spite of his important work, he barely managed to eke out a living as a tutor and a consultant on gambling and insurance.

EXAMPLE 3 Cube Roots of a Complex Number

Find all cube roots of −8.

Solution

Since −8 lies on the negative real axis, we can use $\theta = \pi$ as an argument. Moreover, $r = |z| = |-8| = 8$, so a polar form of −8 is

$$-8 = 8(\cos \pi + i \sin \pi)$$

From 10 with $n = 3$, it follows that

$$(-8)^{1/3} = \sqrt[3]{8}\left[\cos\left(\frac{\pi}{3} + \frac{2k\pi}{3}\right) + i \sin\left(\frac{\pi}{3} + \frac{2k\pi}{3}\right)\right], \qquad k = 0, 1, 2$$

Thus the cube roots of −8 are

$$2\left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right) = 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right) = 1 + \sqrt{3}i$$

$$2(\cos \pi + i \sin \pi) = 2(-1) = -2$$

$$2 \cos\left(\frac{5\pi}{3} + i \sin \frac{5\pi}{3}\right) = 2\left(\frac{1}{2} - \frac{\sqrt{3}}{2} i\right) = 1 - \sqrt{3}i$$

As shown in Figure 10.3.5, the three cube roots of −8 obtained in Example 3 are equally spaced $\pi / 3$ radians $(= 120°)$ apart around the circle of radius 2 centered at the origin. This is not accidental. In general, it follows from Formula 10 that the *n*th roots of *z* lie on the circle of radius $\sqrt[n]{r} (= \sqrt[n]{|z|})$ and are equally spaced $2\pi / n$ radians apart. (Can you see why?) Thus, once one *n*th root of *z* is found, the remaining $n - 1$ roots can be generated by rotating this root successively through increments of $2\pi / n$ radians.

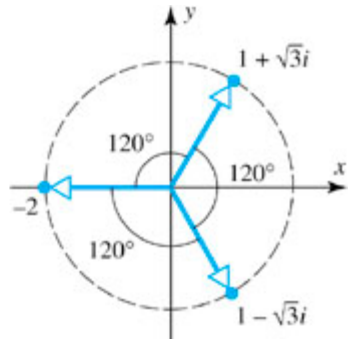

Figure 10.3.5

The cube roots of −8.

EXAMPLE 4 Fourth Roots of a Complex Number

Find all fourth roots of 1.

Solution

We could apply Formula 10. Instead, we observe that $w = 1$ is one fourth root of 1, so the remaining three roots can be generated by rotating this root through increments of $2\pi / 4 = \pi / 2$ radians $(= 90°)$. From Figure 10.3.6, we see that the fourth roots of 1 are

$$1, \quad i, \quad -1, \quad -i$$

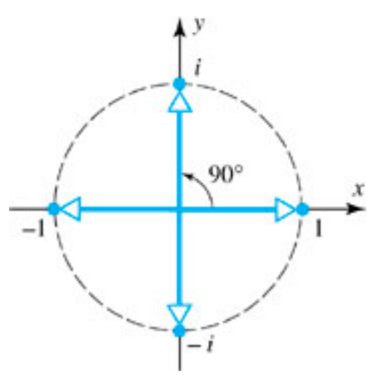

Figure 10.3.6

The fourth roots of 1.

Complex Exponents

We conclude this section with some comments on notation.

In more detailed studies of complex numbers, complex exponents are defined, and it is shown that

$$\cos\theta + i\sin\theta = e^{i\theta} \tag{11}$$

where *e* is an irrational real number given approximately by $e \approx 2.71828\ldots$. (For readers who have studied calculus, a proof of this result is given in Exercise 18.)

It follows from 11 that the polar form

$$z = r(\cos\theta + i\sin\theta)$$

can be written more briefly as

$$z = re^{i\theta} \tag{12}$$

EXAMPLE 5 Expressing a Complex Number in Form 12

In Example 1 it was shown that

$$1 + \sqrt{3}i = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

From 12 this can also be written as

$$1 + \sqrt{3}i = 2e^{i\pi/3}$$

It can be proved that complex exponents follow the same laws as real exponents, so if

$$z_1 = r_1e^{i\theta_1} \qquad \text{and} \qquad z_2 = r_2e^{i\theta_2}$$

are nonzero complex numbers, then

$$z_1z_2 = r_1r_2e^{i\theta_1 + i\theta_2} = r_1r_2e^{i(\theta_1+\theta_2)}$$
$$\frac{z_1}{z_2} = \frac{r_1}{r_2}e^{i\theta_1 - i\theta_2} = \frac{r_1}{r_2}e^{i(\theta_1-\theta_2)}$$

But these are just Formulas 3 and 5 in a different notation.

We conclude this section with a useful formula for $\bar{z}$ in polar notation. If

$$z = re^{i\theta} = r(\cos\theta + i\sin\theta)$$

then

$$\bar{z} = r(\cos\theta - i\sin\theta) \tag{13}$$

Recalling the trigonometric identities

$$\sin(-\theta) = -\sin\theta \qquad \text{and} \qquad \cos(-\theta) = \cos\theta$$

we can rewrite 13 as

$$\bar{z} = r[\cos(-\theta) + i\sin(-\theta)] = re^{i(-\theta)}$$

or, equivalently,

$$\bar{z} = re^{-i\theta} \tag{14}$$

In the special case where $r = 1$, the polar form of z is $z = e^{i\theta}$, and 14 yields the formula

$$\overline{e^{i\theta}} = e^{-i\theta} \tag{15}$$

Exercise Set 10.3

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1. In each part, find the principal argument of z.

(a) $z = 1$

(b) $z = i$

(c) $z = -i$

(d) $z = 1 + i$

(e) $z = -1 + \sqrt{3}i$

(f) $z = 1 - i$

2. In each part, find the value of $\theta = \arg(1 - \sqrt{3}i)$ that satisfies the given condition.

(a) $0 < \theta \leq 2\pi$

(b) $-\pi < \theta \leq \pi$

(c) $-\frac{\pi}{6} \leq \theta < \frac{11\pi}{6}$

3. In each part, express the complex number in polar form using its principal argument.

(a) $2i$

(b) -4

(c) $5 + 5i$

(d) $-6 + 6\sqrt{3}i$

(e) $-3 - 3i$

(f) $2\sqrt{3} - 2i$

4. Given that $z_1 = 2(\cos \pi / 4 + i \sin \pi / 4)$ and $z_2 = 3(\cos \pi / 6 + i \sin \pi / 6)$, find a polar form of

(a) $z_1 z_2$

5. Express $z_1 = i$, $z_2 = 1 - \sqrt{3}i$, and $z_3 = \sqrt{3} + i$ in polar form, and use your results to find $z_1 z_2 / z_3$. Check your results by performing the calculations without using polar forms.

6. Use Formula 6 to find

(a) $(1+i)^{12}$

(b) $\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)^{-6}$

(c) $(\sqrt{3}+i)^7$

(d) $(1 - i\sqrt{3})^{-10}$

7. In each part, find all the roots and sketch them as vectors in the complex plane.

(a) $(-i)^{1/2}$

(b) $(1+\sqrt{3}i)^{1/2}$

(c) $(-27)^{1/3}$

(d) $(i)^{1/3}$

(e) $(-1)^{1/4}$

(f) $(-8+8\sqrt{3}i)^{1/4}$

8. Use the method of Example 4 to find all cube roots of 1.

9. Use the method of Example 4 to find all sixth roots of 1.

10. Find all square roots of $1+i$ and express your results in polar form.

11. Find all solutions of the equation $z^4-16=0$.

12. Find all solutions of the equation $z^4+8=0$ and use your results to factor z^4+8 into two quadratic factors with real coefficients.

13. It was shown in the text that multiplying z by i rotates z counterclockwise by 90°. What is the geometric effect of dividing z by i?

14. In each part, use 6 to calculate the given power.

(a) $(1+i)^8$

(b) $(-2\sqrt{3}+2i)^{-9}$

15. In each part, find $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$.

(a) $z=3e^{i\pi}$

(b) $z=3e^{-i\pi}$

(c) $\bar{z}=\sqrt{2}e^{\pi i/2}$

(d) $\bar{z} = -3e^{-2\pi i}$

16.
(a) Show that the values of $z^{1/n}$ in Formula 10 are all different.

(b) Show that integer values of *k* other than $k = 0$, 1, 2, …, $n - 1$ produce values of $z^{1/n}$ that are duplicates of those in Formula 10.

17. Show that Formula 7 is valid if $n = 0$ or *n* is a negative integer.

18. **(For Readers Who Have Studied Calculus)** To prove Formula 11, recall that the Maclaurin series for e^x is

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$$

(a) By substituting $x = i\theta$ in this series and simplifying, show that

$$e^{i\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right)$$

(b) Use the result in part (a) to obtain Formula 11.

19. Derive Formula 5.

20. When $n = 2$ and $n = 3$, Equation 7 gives

$$(\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta$$
$$(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$$

Use these two equations to obtain trigonometric identities for $\cos 2\theta$, $\sin 2\theta$, $\cos 3\theta$, and $\sin 3\theta$.

21. Use Formula 11 to show that

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \text{and} \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

22. Show that if $(a + bi)^3 = 8$, then $a^2 + b^2 = 4$.

23. Show that Formula 6 is valid for negative integer exponents if $z \neq 0$.