MATHEMATICS 1LS3 TEST 2

Day Class
Duration of Examination: 60 minutes
McMaster University, 28 October 2015

E. Clements, G. Dragomir, M. Lovrić

First name (PLEASE PRINT): SOLUTIONS

Family name (PLEASE PRINT): ________

Student No.: ________

THIS TEST HAS 8 PAGES AND 7 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE.

Total number of points is 40. Marks are indicated next to the problem number. Any non-graphing calculator is allowed.

USE PEN TO WRITE YOUR TEST. IF YOU USE A PENCIL YOUR TEST WILL NOT BE ACCEPTED FOR REMARKING (IF NEEDED).

You must show work to receive full credit.

Problem	Points	Mark
1	4	
2	6	
3	6	
4	7	
5	6	
6	6	
7	5	
TOTAL	40	

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Name:______________
Student No.: ______________

1. Multiple choice questions: circle ONE answer. No justification is needed.

(a)[2] If $f(x) = Ax\ln(B+x)$, then $f'(0)$ is equal to

(A) A	(B) B	(C) AB	(D) $B\ln B$
(E) $AB\ln B$	(F) $AB\ln A$	(G) $A\ln B$	(H) $B\ln A$

$$f'(x) = A\left(\ln(B+x) + \underbrace{x \cdot \frac{1}{B+x}}_{0}\right)$$

$$f'(0) = A \cdot \ln B$$

(b)[2] If $f(x) = \arctan\left(\frac{x}{3}+1\right)$, then $f'(1)$ is equal to

(A) 9/75	(B) 9/15	(C) 1/75	(D) 1/25
(E) 1/15	(F) 3/25	(G) 9/5	(H) 3/17

$$f'(x) = \frac{1}{1+\left(\frac{x}{3}+1\right)^2} \cdot \frac{1}{3}$$

$$f'(1) = \frac{1}{1+\left(\frac{4}{3}\right)^2} \cdot \frac{1}{3} = \frac{1}{\frac{25}{9}} \cdot \frac{1}{3} = \frac{3}{25}$$

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2. Identify each statement as true or false, or yes or no (circle your choice). You do not need to justify your answer.

(a)[2] Let $m(t)$ represent the mass of melting snow in kilograms, where t is time in days. The units of $m'(t)$ are kilograms.

TRUE FALSE

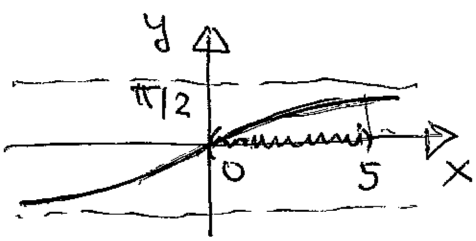
kg/day because $m'(t)$ is the rate of change of $m(t)$

(b)[2] Knowing that $g''(x) = (x-5)\arctan x$, we conclude that the function $g(x)$ is concave up on $(0, 5)$.

$\oplus$ on $(0,5)$

negative (from graph)

TRUE FALSE



(c)[2] The function $g(x) = x\sin(\pi x)$ has a horizontal tangent at $x = 1$.

TRUE FALSE

$g'(x) = \sin(\pi x) + x \cdot \cos(\pi x) \cdot \pi$

$g'(1) = \underbrace{\sin \pi}_{0} + \cos \pi \cdot \pi = -\pi \neq 0$

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Name:__________________
Student No.: __________________

Questions 3-7: You must show work to receive full credit.

3. (a)[3] Find $f'(1)$, if $f(x) = 3^{\ln x} + \sqrt{1 + \ln x} + 3^5$.

$$f'(x) = 3^{\ln x} \cdot \ln 3 \cdot \frac{1}{x} + \frac{1}{2}(1+\ln x)^{-1/2} \cdot \frac{1}{x} + 0$$

$$f'(1) = 3^0 \cdot \ln 3 \cdot 1 + \frac{1}{2} \cdot 1^{-1/2} \cdot 1 = \ln 3 + \frac{1}{2}$$

$$\approx 1.5986$$

(b)[3] Find $y'(x)$, if $\cos(x^2 y) = \sin y + \tan x$.

$$-\sin(x^2y) \cdot (2xy + x^2y') = \cos y \cdot y' + \sec^2 x$$

$$y'\left(-x^2\sin(x^2y) - \cos y\right) = \sec^2 x + \sin(x^2y) \cdot 2xy$$

$$y' = \frac{\sec^2 x + 2xy\sin(x^2y)}{-x^2\sin(x^2y) - \cos y}$$

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Name:

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4. (a)[3] In the article *Migration behaviour of grizzly bears in Northern British Columbia: contribution to a modelling approach.* G. Brown et al, Bear Science 4 (June 2012), we find the formula

$$P(t) = \arctan(1.7t) + 4.7$$

where t represents time.

Next, we read "initially, $P(t) \approx 1.7t + 4.7$, which gives a linear relationship." Explain why this statement is correct. [Hint: Think in terms of the linear approximation at $t = 0$.]

$$P(0) = \arctan(0) + 4.7 = 4.7$$

$$P'(0) = \frac{1}{1+(1.7t)^2} \cdot 1.7 \Big|_{t=0} = 1.7$$

linear approx. $= P(0) + P'(0)(t-0) = 4.7 + 1.7t$

near $t=0$, $P(t) \approx 4.7 + 1.7t$

(b)[4] A simple model of diffusion states that the concentration of a substance diffusing in air is given by

$$c(x) = e^{-ax^2+b}$$

where a and b are constants and x is the distance from the source. This formula is sometimes simplified using a quadratic approximation near $x = 0$. Find that approximation.

$$Q(x) = T_2(x) = \underline{c(0) + c'(0)x + \tfrac{1}{2}c''(0)x^2}$$

$$c(x) = e^{-ax^2+b} \to c(0) = e^b$$

$$c'(x) = e^{-ax^2+b}(-2ax) \to c'(0) = 0$$

$$c''(x) = e^{-ax^2+b}(-2ax)(-2ax) + e^{-ax^2+b}(-2a)$$

$$\to c''(0) = -2ae^b$$

thus $\underline{T_2(x) = e^b - ae^b x^2}$

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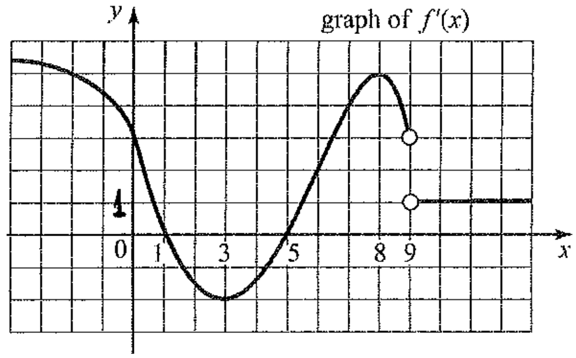
5. Drawn below is the derivative of a function $f(x)$.



(a)[2] State all intervals where $f(x)$ is increasing. Justify your answer.

$f(x)$ increasing $\leftrightarrow$ $f'(x) > 0$

$(-\infty, 1)$, $(5, 9)$, $(9, \infty)$

(b)[2] Find all intervals where $f(x)$ is concave down. Justify your answer.

$f(x)$ is concave down $\leftrightarrow$ $f'(x)$ is decreasing

$(-\infty, 3)$, $(8, 9)$

(c)[2] Describe in words the graph of $f(x)$ on the interval $(9, \infty)$.

$f'(x) = 1$ (constant function)

so $f(x)$ is a line of slope 1

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6. The quadratic model for the percent S of cancer cells surviving radiation treatment states that

$$S(d) = e^{-d^2-0.2}$$

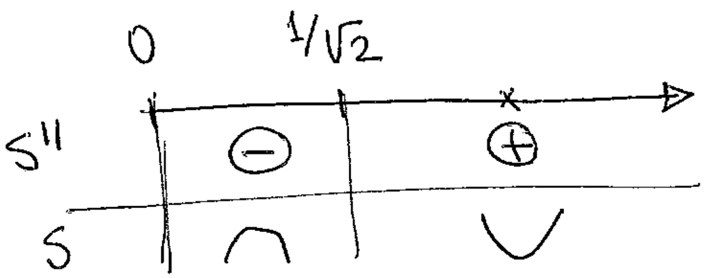
where $d \geqslant 0$ is the dose (in Gray) per treatment of radiation.

(a)[1] Show that S is a decreasing function of d when $d \geqslant 0$.

$$S'(d) = \underbrace{e^{-d^2-0.2}}_{\oplus}\underbrace{(-2d)}_{\ominus} < 0 \longrightarrow S \text{ is decreasing}$$

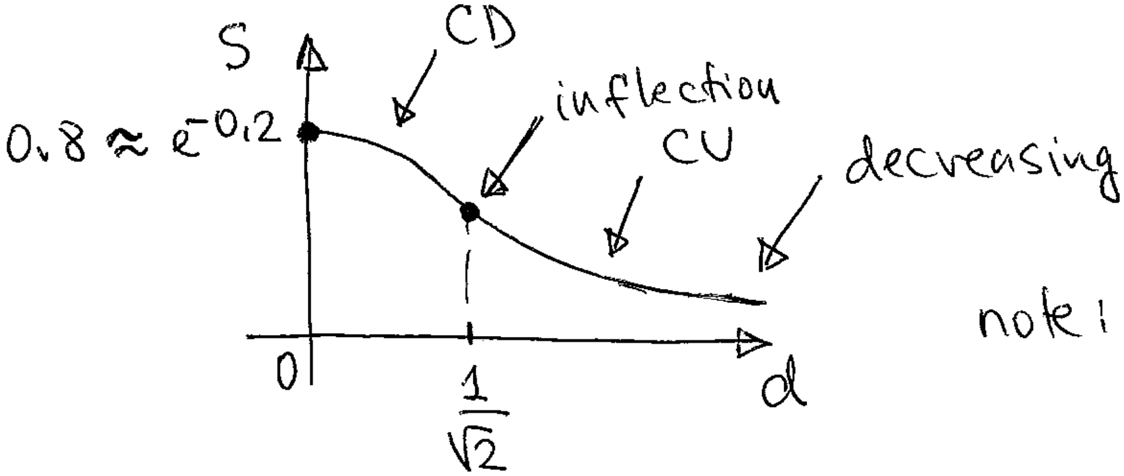
(b)[2] Find (if any) inflection points of $S(d)$. for $d \geqslant 0$

$$S''(d) = e^{-d^2-0.2}(-2d)(-2d) + e^{-d^2-0.2}(-2)$$
$$= 4d^2 e^{-d^2-0.2} - 2e^{-d^2-0.2}$$
$$= 2e^{-d^2-0.2}(2d^2-1) = 0 \rightarrow d^2 = \frac{1}{2}$$
$$d = \pm\frac{1}{\sqrt{2}}$$



$d = \frac{1}{\sqrt{2}}$ is an inflection point

(c)[3] Based on information in (a) and (b), make a sketch of the function $S(d)$ for $d \geqslant 0$. Label intercepts, if any.



note: $\underline{S(d) > 0}$

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7. The following excerpt is taken from *The laminar cortex model: a new continuum cortex model incorporating laminar architecture.* J. Du, V. Vegh, and D.C. Reutens. PLoS Computational Biology. 8.10 (Oct. 2012).

the average of membrane potentials of neurons in the element, that is

$$V = \frac{N_e V_e + N_i V_i}{N_e + N_i}$$

where N_e, N_i are the numbers of excitatory and inhibitory neurons and V_e and V_i are the (average) membrane potentials of excitatory and inhibitory neuron populations respectively.

You know that the numbers N_e and N_i are positive, and the membrane potentials V_e and V_i are negative.

(a)[3] Assume that V is a function of N_e. Find the derivative of V and simplify.

$$V' = \frac{V_e(N_e+N_i) - (N_eV_e + N_iV_i)\cdot 1}{(N_e+N_i)^2}$$

$$V' = \frac{V_eN_i - N_iV_i}{(N_e+N_i)^2} = \frac{N_i(V_e-V_i)}{(N_e+N_i)^2}$$

(b)[2] Assume that $V_e > V_i$. What does your answer in (a) say about the dependence of V on N_e? Justify your answer.

$$V' = \frac{N_i(V_e-V_i)}{(N_e+N_i)^2}$$

* $N_i > 0$
* $(N_e+N_i)^2 > 0$
* $V_e - V_i > 0$ by assumption

so $V' > 0 \rightarrow V$ is an increasing function of N_e

THE END