1ZA3

Areas & Sigmas: Riemann Sums

If $f(x) \geq 0$ & cont. on $[a,b]$

$\Rightarrow$ Area under $f(x) \approx \sum_{i=1}^{n} f(x_i)\Delta x$

sub-int. width $= \Delta x = \frac{b-a}{n}$

[x_i = ith sample point!

Define The Definite Integral: If $f(x)$ is cont. on $[a,b]$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

If it exists, $f(x)$ integrable on $[a,b]$

Definite Integral Properties

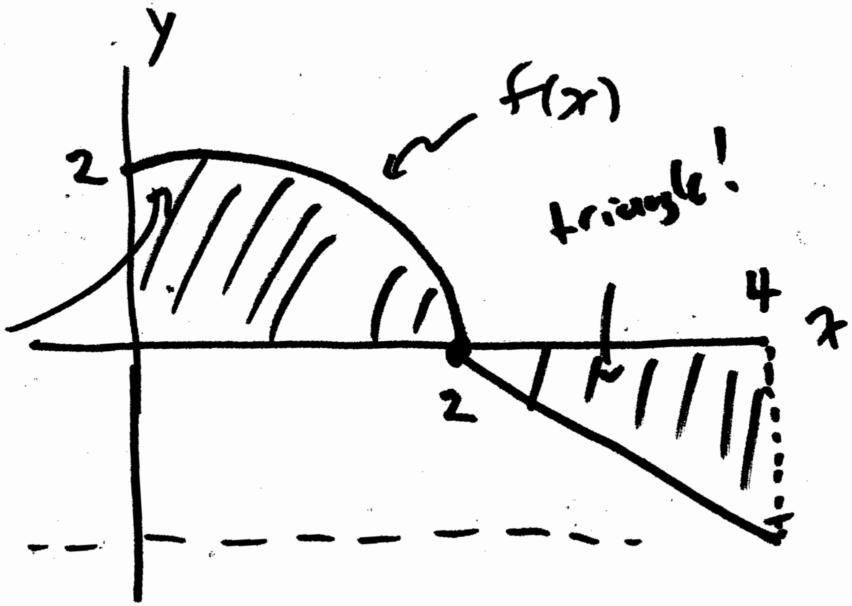
1) If $f(x) \geq 0 \Rightarrow \int_a^b f(x)\,dx \geq 0$ (and is area under the curve!)

2) If $f(x) \leq 0 \Rightarrow \int_a^b f(x) \leq 0$, (and is negative of area ~~below~~ above the graph.

3) In general

$$\int_a^b f(x)\,dx = (\text{Area above}) - (\text{Area below})$$

eg.



Assume part of circle!

Evaluate

$$\int_0^4 f(x)\,dx$$

$= \text{Area above} - \text{Area belo}$

$= \frac{1}{4}\pi 2^2 - \frac{1}{2}(4-2)(3)$

$= \underline{\pi - 3}$

More Properties

4) $$\int_a^b f(x) \pm g(x)\, dx = \int_a^b f(x)\, dx \pm \int_a^b g(x)\, dx$$

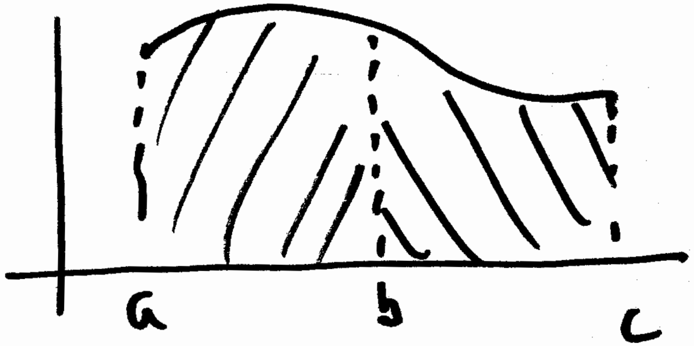
5) $$\int_a^b k f(x)\, dx = k \int_a^b f(x)\, dx$$

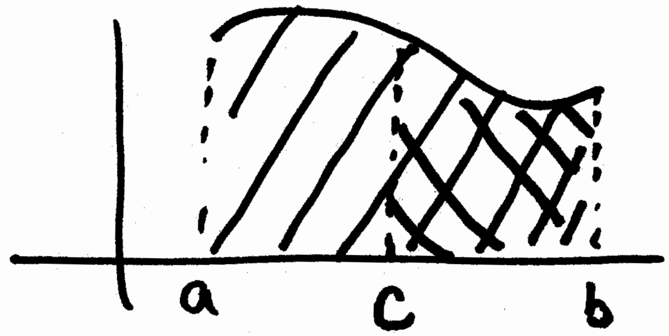
k constant only

6) $$\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$$

7) $$\int_a^a f(x)\, dx = 0$$

8) $$\int_a^c f(x)\, dx = \int_a^b f(x)\, dx + \int_b^c f(x)\, dx$$





$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$\Rightarrow \int_a^c f(x)\,dx = \int_a^b f(x)\,dx - \int_c^b f(x)\,dx$$

$$= \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

eg. Given $\int_1^7 g(x)\,dx = 2$, $\int_5^7 g(x)\,dx = -1$

$\int_1^3 f(x)\,dx = 4$ $\int_3^5 f(x)\,dx = -2$

Find $\int_1^5 2f(x) - 3g(x)\,dx$

Solution

$$= 2\underbrace{\int_1^5 f(x)\,dx}_{2} - 3\underbrace{\int_1^5 g(x)\,dx}_{3}$$

$$\int_1^5 f(x)\,dx = \int_1^3 f(x)\,dx + \int_3^5 f(x)\,dx = 4 + (-2) = 2$$

$$\int_1^5 g(x)\,dx = \int_1^7 g(x)\,dx - \int_5^7 g(x)\,dx = 2 - (-1) = 3$$

$$= 2(2) - 3(3) = \boxed{-5}$$

More Properties

9) From previous: $\int_a^b f(x)\,dx \geq 0$ if $f(x) \geq 0$ $(a \leq b)$

So If $f(x) \geq g(x) \Rightarrow f(x) - g(x) \geq 0$

$$\Rightarrow \int_a^b f(x) - g(x)\,dx \geq 0$$

$(a \leq b)$

9) $\Rightarrow$ If $f(x) \geq g(x)$, $a \leq b$

$$\Rightarrow \int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$$

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10) If $m \leq f(x) \leq M$ m, M constants

then $$\int_a^b m\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b M\,dx$$

so $$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

eg. Use above ↑ to approx. $\int_0^{\ln 4} e^x \, dx$

Solution

$$1 \le e^x \le 4 \quad \text{on } [0, \ln 4]$$

$\ln 4 > 0$

$$\Rightarrow 1(\ln 4 - 0) \le \int_0^{\ln 4} e^x \, dx \le 4(\ln 4 - 0)$$

$$\ln 4 \le \int_0^{\ln 4} e^x \, dx \le \ln(256)$$

$$= 8 \ln 2$$

The Fundamental Theorem of Calculus Part 1

If $f(x)$ is continuous on $[a, b]$

Then define $g(x) = \int_a^x f(t) \, dt$

then $\frac{d}{dx} g(x) = f(x)$