Math 1LS3 Week 10: Antiderivatives and Integrals

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Week 10: sections 6.2 (one lecture), 6.3 (two lectures)

Antiderivatives

In the following, assume functions are defined on some open interval.

Definition

$F(x)$ is an antiderivative (or *indefinite integral*) for $f(x)$ if $F'(x) = f(x)$. Write:

$$\int f(x)dx = F(x).$$

Problem

Find two antiderivatives of $\cos(x)$.

Solution

One function whose derivative is $\cos(x)$ *is* $\sin(x)$. *Another is* $\sin(x) + 17$.

$$\int \cos(x)dx = \sin(x) \quad \textit{and also} \quad \int \cos(x)dx = \sin(x) + 17.$$

Antiderivatives

Problem

Find three indefinite integrals of $\frac{1}{1+x^2}$.

Solution

One function whose derivative is $\frac{1}{1+x^2}$ *is* $\arctan(x)$. *So:*

$$\int \frac{dx}{1+x^2} = \arctan(x).$$

$$\int \frac{dx}{1+x^2} = \arctan(x) + 17.$$

$$\int \frac{dx}{1+x^2} = \arctan(x) - \pi^2.$$

In fact, $\arctan(x) + C$ is an antiderivative for any constant C.

General Antiderivative

Theorem

If $F(x)$ is one antiderivative of $f(x)$, then any other antiderivative is $F(x) + C$ for some constant C.

$$\int f(x)dx = F(x) + C$$

Geometrically, this means all antiderivatives of the same function are vertical translates of each other.

It also means that a single initial condition is enough to nail down the solution.

Checking an Antiderivative

Problem (6.2.2)

Show that $\int x\cos(4x)dx = \frac{1}{16}\cos(4x) + \frac{1}{4}x\sin(4x) + C$.

Solution

We must show: $\left(\frac{1}{16}\cos(4x) + \frac{1}{4}x\sin(4x) + C\right)' = x\cos(4x)$.

$$\left(\frac{1}{16}\cos(4x) + \frac{1}{4}x\sin(4x) + C\right)' =$$

$$\frac{1}{16}(-4\sin(4x)) + \frac{1}{4}(1\cdot\sin(4x) + x\cdot 4\cos(4x)) + 0$$

$$= -\frac{1}{4}\sin(4x) + \frac{1}{4}\sin(4x) + \frac{4}{4}x\cos(4x) = x\cos(4x)\checkmark$$

Antiderivative Rules: Power Rule

Every derivative rule yields a corresponding integral rule.

Example (Power Rule)

$$(x^{n+1})' = (n+1)x^n$$

$$\left(\frac{x^{n+1}}{n+1}\right)' = x^n$$

Therefore:

$$\boxed{\int x^n dx = \frac{x^{n+1}}{n+1} + C}$$

Note: $\boxed{\int \frac{dx}{x} = \ln(x) + C}$ (not $\frac{x^0}{0} + C$.)

Go back and memorize all basic derivatives in the reverse direction!

Sum and Constant Multiple Rules

Every derivative rule yields a corresponding integral rule.

$$(F(x)+G(x))' = F'(x)+G'(x)$$

$$\implies \boxed{\int (f(x)+g(x))dx = \int f(x)dx + \int g(x)dx}$$

$$(F(x)-G(x))' = F'(x)-G'(x)$$

$$\implies \boxed{\int (f(x)-g(x))dx = \int f(x)dx - \int g(x)dx}$$

$$(cF(x))' = cF'(x) \implies \boxed{\int cf(x)dx = c\int f(x)dx}$$

Breaks an integral up into simpler sub-problems

Sum Rule: An Example

Problem

Evaluate:

$$\int \sec^2(\theta) + \sec(\theta)\tan(\theta)d\theta.$$

Solution

$$\int \sec^2(\theta) + \sec(\theta)\tan(\theta)d\theta = \int \sec^2(\theta)d\theta + \int \sec(\theta)\tan(\theta)d\theta$$

$$= \boxed{\tan(\theta) + \sec(\theta) + C}$$

Difference Rule: An Example

Problem

Evaluate:

$$\int (e^x - x^2)dx.$$

Solution

$$\int (e^x - x^2)dx = \int e^x dx - \int x^2 dx$$

$$\boxed{= e^x - \frac{1}{3}x^3 + C}$$

Constant Multiple Rule: An Example

Problem

Evaluate:

$$\int \left(\frac{1}{2x^3} - 4\frac{1}{x} + \sqrt{x} \right) dx.$$

Solution

$$\int \left(\frac{1}{2x^3} - 4\frac{1}{x} + \sqrt{x} \right) dx = \frac{1}{2} \int x^{-3} dx - 4 \int \frac{dx}{x} + \int x^{1/2} dx$$

$$= \frac{1}{2} \cdot \frac{1}{-2} x^{-2} - 4\ln(x) + \frac{1}{3/2} x^{3/2} + C$$

$$= \boxed{-\frac{1}{4x^2} - 4\ln(x) + \frac{2}{3} x\sqrt{x} + C}$$

Solving a DiffEq with Initial Value

Problem (6.2.13)

Early in the AIDS epidemic, the number $A(t)$ of cases was found to satisfy:

$$\frac{dA}{dt} = 523.8t^2$$

If 1981 corresponds to $t = 0$ with $A(0) = 340$ people, find $A(t)$.

Solution

$$A(t) = \int 523.8t^2 dt = \frac{523.8}{3}t^3 + C$$

When $t = 0$, $A = 340$, *so* $340 = \frac{523.8}{3}(0)^3 + C$. *So* $C = 340$. *Therefore:*

$$A(t) = \frac{523.8}{3}t^3 + 340 = \boxed{174.6t^3 + 340}.$$

Acceleration (Optional)

Near the surface of the Earth, acceleration due to gravity is a constant $-9.8m/s^2$. So $dv/dt = -9.8$ where velocity $v = dy/dt$. Find a formula for $y(t)$ in terms of initial position $y(0)$ and initial speed $v(0)$.

Area under a Curve

Suppose $f \geq 0$ is a function defined on $[a, b]$. How would you approximate the area between $[a, b]$ and $f(x)$?

Problem

Slice the region between $[1, 3]$ and $f(x) = x^2$ into 4 vertical rectangular strips. Approximate the area two ways.

Solution

Width of a rectangle $= (3 - 1)/4 = 1/2$.
Hint: $f(1) = 1, f(1.5) = 2.25, f(2) = 4, f(2.5) = 6.25, f(3) = 9$.

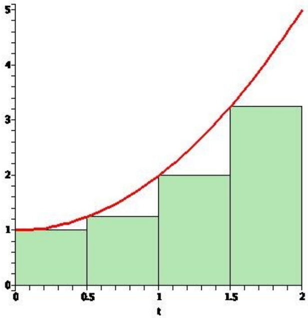
Left sum: $\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 2.25 + \frac{1}{2} \cdot 4 + \frac{1}{2} \cdot 6.25 = 6.75$
Right sum: $\frac{1}{2} \cdot 2.25 + \frac{1}{2} \cdot 4 + \frac{1}{2} \cdot 6.25 + \frac{1}{2} \cdot 9 = 10.75$

The actual area should be somewhere in between these **Riemann Sums**.

Riemann sums

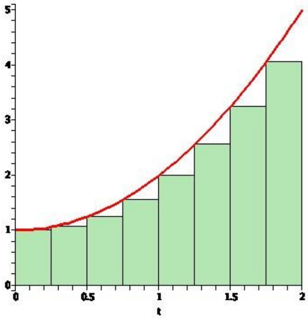
$$f(t) = t^2 + 1 \quad \text{on} \quad [0,2]$$

Riemann Sums



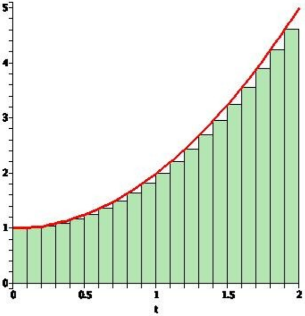
$L_4 = 3.75$

Riemann Sums



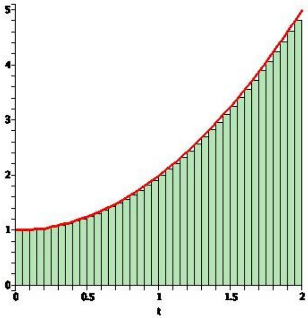
$L_8 = 4.1875$

Riemann Sums



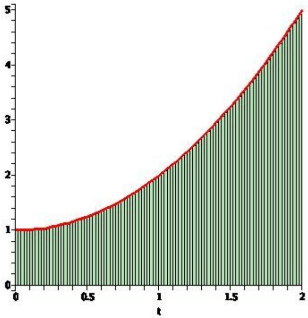
$L_{20} = 4.47$

Riemann Sums



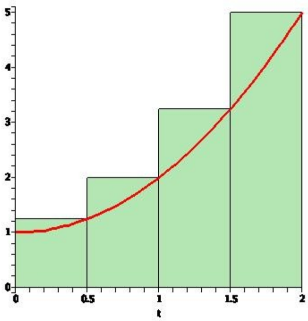
$L_{40} = 4.5675$

Riemann Sums



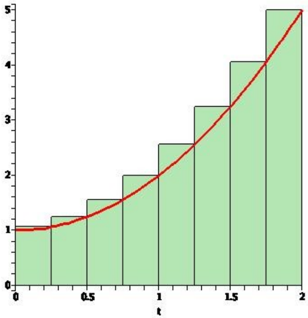
$L_{100} = 4.6268$

Riemann Sums



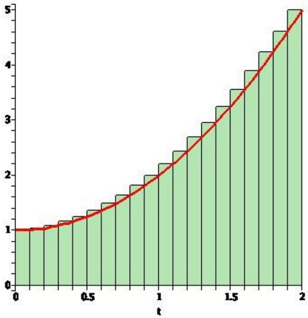
$R_4 = 5.75$

Riemann Sums



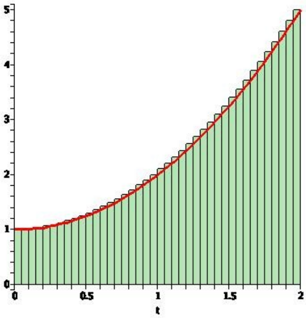
$R_8 = 5.1875$

Riemann Sums



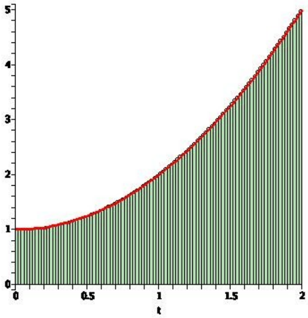
$R_{20} = 4.87$

Riemann Sums



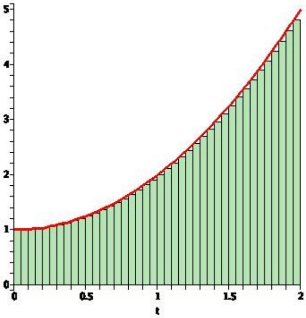
$R_{40} = 4.7675$

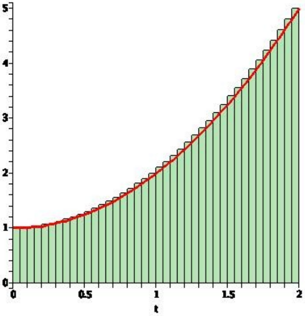
Riemann Sums



$R_{100} = 4.7068$

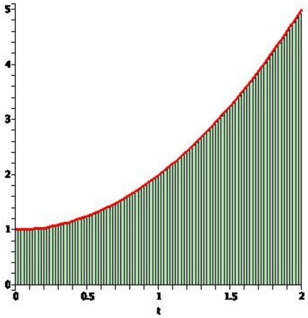
Riemann Sums



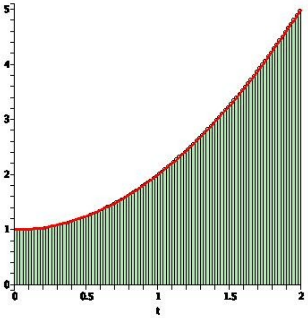


$L_{40} = 4.5675$ $\quad$ $R_{40} = 4.7675$

Riemann Sums

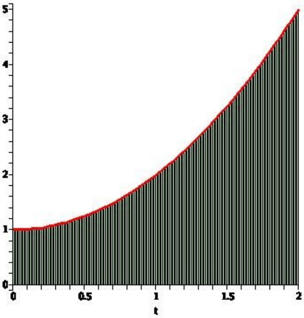


$L_{100} = 4.6268$

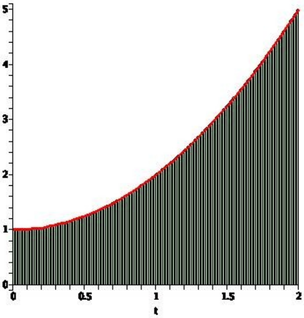


$R_{100} = 4.7068$

Riemann Sums



$L_{200} = 4.6467$



$R_{200} = 4.6867$

Riemann Sums

$L_{500} = 4.658672$

$L_{1000} = 4.662668$

$L_{10000} = 4.666266$

$R_{500} = 4.674672$

$R_{1000} = 4.670668$

$R_{10000} = 4.667066$

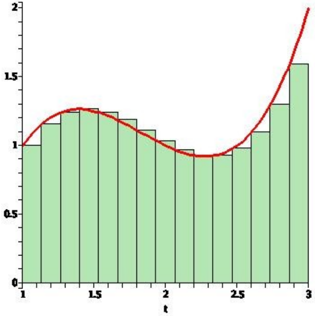
....

true value: $14/3 = 4.6666666\ldots$

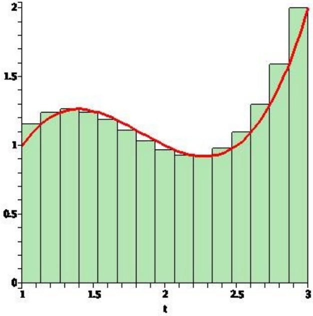
More complicated case …

Riemann Sums

$$f(t) = t^3 + 5.5t^2 + 9.5t - 4 + 1 \quad \text{on} \quad [1,3]$$



L_{15}



R_{15}

Riemann Sum

Area under $f(x) \geq 0$ above $[a, b]$ is:

$\approx$(area rect 1)+(area rect 2)+$\cdots$+(area rect n)

=(height 1)(width 1)+(height 2)(width 2)+$\cdots$+(height n)(width$n$)

$$= f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x$$

$$= \sum_{i=1}^{n} f(x_i^*)\Delta x \quad \text{“Sigma notation”}$$

Definition (Area, Definite Integral)

Suppose $f \geq 0$. The area under f over $[a, b]$ is:

$$\int_a^b f(x)dx := \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

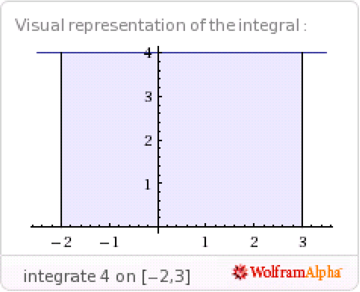
Definite Integrals using Geometric Reasoning

Problem (6.3.5)

Evaluate the definite integral

$$\int_{-2}^{3} 4dx$$

Solution



$$\int_{-2}^{3} 4dx = Area$$

$$= [Width][Height] = 5 \cdot 4 = \boxed{20}$$

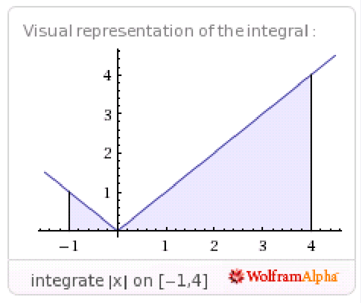
Definite Integrals using Geometric Reasoning

Problem (6.3.6)

Evaluate the definite integral

$$\int_{-1}^{4} |x| dx$$

Solution



$$\int_{-1}^{4} |x| dx = Area\ Left\ \Delta + Area\ Right\ \Delta$$

$$= \frac{1}{2} 1 \cdot 1 + \frac{1}{2} 4 \cdot 4 = \boxed{\frac{17}{2}}.$$

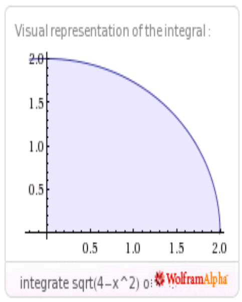
Definite Integrals using Geometric Reasoning

Problem (6.3.7)

Evaluate the definite integral

$$\int_0^2 \sqrt{4 - x^2}dx$$

Solution



$$\int_0^2 \sqrt{4 - x^2}dx = \frac{1}{4} Area\ of\ Circle\ of\ Radius\ 2$$

$$= \frac{1}{4} \cdot \pi(2)^2 = \boxed{\pi}.$$

Definite Integrals and Negative-Valued Functions

The limit in our definition:

Definition (Area, Definite Integral)

Suppose $f \geq 0$. The area under f over $[a, b]$ is:

$$\int_a^b f(x)dx := \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

makes sense even when f is allowed to take negative values. So:

Definition (Definite Integral)

Suppose f is defined on $[a, b]$. The *definite integral* is:

$$\int_a^b f(x)dx := \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

Theorem: all continuous functions are **integrable**.

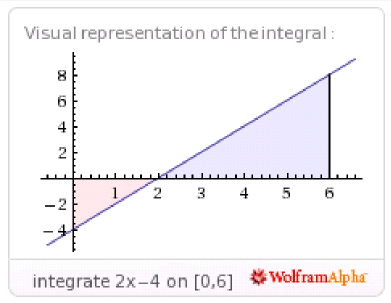
Area and Signed Area

Problem (6.3.8)

Evaluate the definite integral

$$\int_0^6 (2x-4)\,dx$$

Solution



$$\int_0^6 (2x-4)\,dx =$$

Area Above - Area Below

$$= \frac{1}{2}(4)(8) - \frac{1}{2}(2)(4) = 16 - 4 = \boxed{12}.$$

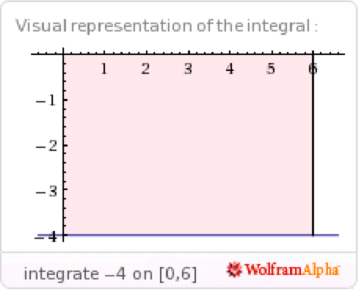
Area and Signed Area

Problem (6.3.9)

Evaluate the definite integral

$$\int_0^6 -4dx$$

Solution



$$\int_0^6 4dx = -\ Area\ Below$$

$$\text{=-}(width)(height) = -6 \cdot 4 = \boxed{-24}.$$

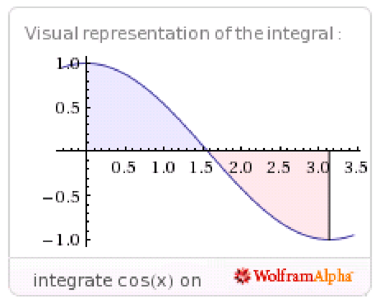
Area and Signed Area

Problem (6.3.10)

Evaluate the definite integral

$$\int_0^{\pi} \cos(x)dx$$

Solution



$$\int_0^{\pi} \cos(x)dx = \mathcal{Area\ Above} - \mathcal{Area\ Below}$$

$= \boxed{0}$ *(by symmetry).*

Integral Properties: p.447

Assuming $a < b$ and the integrals are defined:

$$\int_a^a f(x)dx = 0 \quad \text{(No width} \implies \text{0 area)}$$

$$\int_b^a f(x)dx := -\int_a^b f(x)dx \quad \text{(Definition)}$$

$$\int_a^b cf(x)dx = c\int_a^b f(x)dx \quad \text{(think vertical stretching)}$$

$$\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx \quad \text{(think translation)}$$

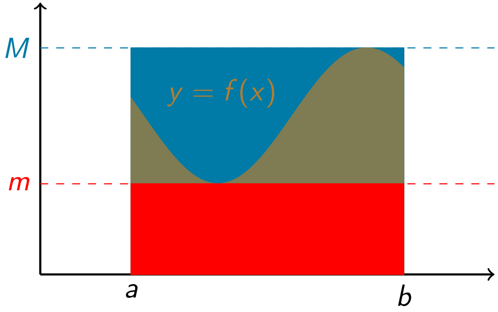
$$\int_a^b cdx = c(b-a) \quad \text{(area of rectangle)}$$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \quad \text{(Horizontal Subdivision)}$$

Bounding a Definite Integral

Assuming $a < b$ and the integrals are defined:

If $m \leq f(x) \leq M$ for all x in $[a, b]$, then:

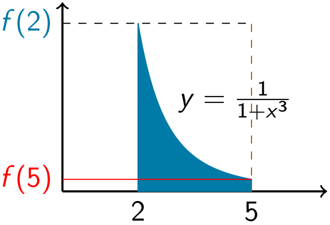


$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a).$$

Bounding a Definite Integral (6.3.13)

Problem

Approximate $\int_2^5 \frac{1}{1+x^3}dx$ *by bounding the integrand.*



Solution

Since $1/(1+x^3)$ *is* ***decreasing***, $\frac{1}{1+2^3} \geq f(x) \geq \frac{1}{1+5^3}$ *on* $[2,5]$. *So*

$$0.02381 \approx 3f(5) \leq \int_2^5 \frac{1}{1+x^3}dx \leq 3f(2) \approx 0.33333$$

NB: not all functions are monotone! What to do then?

The Fundamental Theorem of Calculus

- Pay particular attention to reading Example 6.3.14 in the text (p. 448)
- It reasons out why the *Fundamental Theorem* is true.
- At beginning of next lecture, I'll explain the same idea in a slightly different setting.