1ZC3

Elementory Matrices

- They are the result of a single elementary row op. applied to I_n, an Identity matrix

- If E is elementory, EA, if defined, is the matrix A after the application of the corresponding row op.

eg "Swap two rows" $\rightsquigarrow$ $E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ ← op. on I_2.

$$E \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix}$$

Swap on new matrix

- All such E are invertible & inverse is also elementory corresponding to the inverse row op.

Say I have a ~~n×n~~ matrix A which is row-equivalent to I

ie row ops. can turn A into I (RREF of A is I)

$$\Rightarrow \underbrace{E_n \cdots E_2 E_1}_{\text{elementary}} A = I \quad \text{for some element. matrices.}$$

$\Rightarrow$ by mult. by inverses of E's $\Rightarrow A = \underbrace{E_1^{-1} E_2^{-1} E_3^{-1} \cdots}_{\text{product of elem. matrices!}}$

$\boxed{\text{So (A row-equivalent to I)} \Rightarrow \text{(A is a product of } E\text{'s)}}$

Elementary Matrix Decomposition

I claim $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is row-equivalent to I_2.

Let's find an elementary matrix decomposition for this!

Solution

$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Row2 − 3 Row1 $\xrightarrow{\text{act on } I_2}$ $E_1 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \Rightarrow E_1^{-1} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$ (Row2 + 3 Row1)

$$E_1 A = \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$$

Row1 + Row2 $\xrightarrow{\text{act on } I_2}$ $E_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \Rightarrow E_2^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ (Row1 − Row2)

$$E_2 E_1 A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$$

Row $2 \cdot \left(-\frac{1}{2}\right)$ $\xrightarrow{\text{act on } I_2}$ $E_3 = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} \Rightarrow E_3^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$

Row $2 \cdot (-2)$

$$E_3 E_2 E_1 A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ "}I$$

$$A = E_1^{-1} E_2^{-1} E_3^{-1}$$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \underline{\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}}$$

A (non-unique!) <u>Elem. matrix decomposition!</u>

<u>Remember</u> if A, B <u>invertible</u> last day argued that

$$A B B^{-1} A^{-1} = I = B^{-1} A^{-1} A B$$

$\Rightarrow$ AB <u>invertible</u> & $(AB)^{-1} = B^{-1} A^{-1}$

So if A is a product of elem matrices (all invertible!)

$\Rightarrow$ A^{-1} exists! $\quad (A^{-1} = E_n \cdots E_3 E_2 E_1$

ie $(E_1^{-1} \cdots E_n^{-1})^{-1}$)

$\boxed{\text{So } A \text{ is a product of } E\text{'s} \Rightarrow A^{-1} \text{ exists}}$

Say A is invertible

Look at lin. system $A\vec{x} = \vec{b}$

$\Rightarrow \quad A^{-1}A\vec{x} = I\vec{x} = \boxed{\vec{x} = A^{-1}\vec{b}}$

$\Rightarrow$ only 1 solution to any $A\vec{x} = \vec{b}$

So $\boxed{A^{-1} \text{ exists} \Rightarrow A\vec{x} = \vec{b} \text{ has a unique solution}}$

If $A\vec{x}=\vec{b}$ has a unique solution: Row reduce!

Assume A = square matrix

At first: Aug. matrix $[A \mid \vec{b}]$

Reduced: $[RREF \mid \vdots]$

$\Rightarrow$ $\left.\begin{array}{l} x = \# \\ y = \# \\ z = \# \end{array}\right\}$ no parameter! no free variables!

$\Rightarrow$ 1 leading 1 for each var!

n leading 1s & A is $n \times n$ $\Rightarrow$ RREF of A is I

ie $\boxed{A\vec{x}=\vec{b} \text{ has a unique soln} \Rightarrow A \text{ row-equivalent to } I}$

Beginnings of The Mega Theorem

If A is an $n \times n$ matrix the following are equivalent!

1) A has RREF of I (row equivalent to I)
2) A is a product of E's (elementary matrices!)
3) A^{-1} exists (A invertible)
4) $A\vec{x} = \vec{b}$ has a unique solution.
5)

(More will be added!)

Now! an Ap! A^{-1} exists $\Leftrightarrow$ A is a product of E's

$\Rightarrow$ $A^{-1} = E_n \cdots E_3 E_2 E_1$

So Combine A with I

$$[A \mid I]$$

Reduce A to RREF

$$[\underbrace{E_n \cdots E_1 A}_{I} \mid \underbrace{E_n \cdots E_1 \cancel{I}}_{A^{-1}}]$$

$\Rightarrow$ A^{-1} exists

eg. Find the inverse of $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 0 & 5 \\ 2 & 1 & 4 \end{bmatrix}$

Solution $[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 2 & 0 & 5 & 0 & 1 & 0 \\ 2 & 1 & 4 & 0 & 0 & 1 \end{array}\right]$

Row 2 − 2 Row 1, then Row 3 − 2 Row 1

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & -2 & 0 & 1 \end{array}\right]$$

Swap Row 2 ⟷ Row 3

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 1 & -2 & 1 & 0 \end{array}\right]$$

Row 1 − 2 Row 3

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 5 & -2 & 0 \\ 0 & 1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 1 & -2 & 1 & 0 \end{array}\right]$$

$$\underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I} \qquad \underbrace{\begin{matrix} 5 & -2 & 0 \\ -2 & 0 & 1 \\ -2 & 1 & 0 \end{matrix}}_{A^{-1}}$$

Can check $A^{-1}A = AA^{-1} = I$

If reducing does not get I as RREF on left $\Rightarrow$ no A^{-1}

A not invertible

Why?

Megatheorem says for A square ($n \times n$)

1) A RREF is I_n
2) A is a product of E's
3) A^{-1} exists
4) $A\vec{x} = \vec{b}$ has a unique solution

These are all equivalent

ie If any 1 thing is true, all of these must be true

$\Longleftrightarrow$

"Dark" Version

1) A does not reduce to I_n
2) A is not a product of E's
3) A is singular (A^{-1} DNE)
4) $A\vec{x} = \vec{b}$ does not have a unique solution

These are also all equivalent!