Discrete Mathematics with Applications I

COMPSCI&SFWRENG 2DM3

McMaster University, Fall 2019

Wolfram Kahl

2019-09-10

Ladies or Tigers

Raymond Smullyan provides, in **The Lady or the Tiger?**, the following context for a number of puzzles to follow:

> [...] the king explained to the prisoner that each of the two rooms contained either a lady or a tiger, but it *could* be that there were tigers in both rooms, or ladies in both rooms, or then again, maybe one room contained a lady and the other room a tiger.

In the first case, the following signs are on the doors of the rooms:

1	2
In this room there is a lady, and in the other room there is a tiger.	In one of these rooms there is a lady, and in one of these rooms there is a tiger.

We are told that one of the signs is true, and the other one is false.

"Which door would you open (assuming, of course, that you preferred the lady to the tiger)?"

Plan for Today

- Meaning of Boolean Operators
- Modeling English Propositions
- Proving Theorems by Calculational Reasoning

Plan for Tomorrow

- Substitution
- Leibniz

Truth Values

Boolean constants/values: *false*, *true*

The type of Boolean values: $\mathbb{B}$

— This is the type of propositions, for example: $(x = 1) : \mathbb{B}$

— For any type t, equality $\_{=}\_$ can be used on expressions of that type: $\_{=}\_ : t \to t \to \mathbb{B}$

Boolean operators:

- $\neg\_ : \mathbb{B} \to \mathbb{B}$ — negation, complement, "logical not"
- $\_\wedge\_ : \mathbb{B} \to \mathbb{B} \to \mathbb{B}$ — conjunction, "logical and"
- $\_\vee\_ : \mathbb{B} \to \mathbb{B} \to \mathbb{B}$ — disjunction, "logical or"
- $\_\Rightarrow\_ : \mathbb{B} \to \mathbb{B} \to \mathbb{B}$ — implication, "implies", "if ... then ..."
- $\_\equiv\_ : \mathbb{B} \to \mathbb{B} \to \mathbb{B}$ — equivalence, "if and only if", "iff"
- $\_\not\equiv\_ : \mathbb{B} \to \mathbb{B} \to \mathbb{B}$ — inequivalence, "exclusive or"

Some Laws for the Boolean Operators

(3.12) **Double negation**: $\neg\neg p \equiv p$
(3.36) **Symmetry of $\wedge$**: $p \wedge q \equiv q \wedge p$
(3.37) **Associativity of $\wedge$**: $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
(3.38) **Idempotency of $\wedge$**: $p \wedge p \equiv p$
(3.39) **Identity of $\wedge$**: $p \wedge true \equiv p$
(3.40) **Zero of $\wedge$**: $p \wedge false \equiv false$
(3.42) **Contradiction**: $p \wedge \neg p \equiv false$

(3.24) **Symmetry of $\vee$**: $p \vee q \equiv q \vee p$
(3.25) **Associativity of $\vee$**: $(p \vee q) \vee r \equiv p \vee (q \vee r)$
(3.26) **Idempotency of $\vee$**: $p \vee p \equiv p$
(3.29) **Zero of $\vee$**: $p \vee true \equiv true$
(3.30) **Identity of $\vee$**: $p \vee false \equiv p$
(3.28) **Excluded Middle**: $p \vee \neg p$

(3.45) **Distributivity of $\vee$ over $\wedge$**: $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
(3.46) **Distributivity of $\wedge$ over $\vee$**: $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
(3.47) **De Morgan**: $\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\qquad \neg(p \vee q) \equiv \neg p \wedge \neg q$

Truth Values and Equivalence

Boolean constants/values: *false*, *true*

The set/type of Boolean values: $\mathbb{B}$

Equality of Boolean values is also called **equivalence** and written $\equiv$

$p \equiv q$ can be read as: p is equivalent to q
or: p exactly when q
or: p if-and-only-if q
or: p iff q

p	q	$p \equiv q$	
false	*false*	*true*	The moon is green **iff** $2 + 2 = 7$.
false	*true*	*false*	The moon is green **iff** $1 + 1 = 2$.
true	*false*	*false*	$1 + 1 = 2$ **iff** the moon is green.
true	*true*	*true*	$1 + 1 = 2$ **iff** the sun is a star.

Table of Precedences

- $[x := e]$ (textual substitution) **(highest precedence)**
- $.$ (function application)
- unary prefix operators $+, -, \neg, \#, \sim, \mathcal{P}$
- $**$
- $\cdot \quad / \quad \div \quad \textbf{mod} \quad \textbf{gcd}$
- $+ \quad - \quad \cup \quad \cap \quad \times \quad \circ \quad \bullet$
- $\downarrow \quad \uparrow$
- $\#$
- $\lhd \quad \rhd \quad \hat{}$
- $= \quad \neq \quad < \quad > \quad \in \quad \subset \quad \subseteq \quad \supset \quad \supseteq \quad |$ (conjunctional)
- $\vee \quad \wedge$
- $\Rightarrow \quad \not\Rightarrow \quad \Leftarrow \quad \not\Leftarrow$
- $\equiv \quad \not\equiv$ **(lowest precedence)**

All non-associative binary infix operators associate to the left, except $**$, $\lhd$, $\Rightarrow$, $\rightarrow$, which associate to the right.

Conjunctional Operators

Chains can involve different conjunctional operators:

$$
\begin{array}{ll}
 & 1 < i \leq j < 5 = k \\
\equiv & \langle \text{ conjunctional operators } \rangle \\
 & 1 < i \quad \wedge \quad i \leq j \quad \wedge \quad j < 5 \quad \wedge \quad 5 = k \\
\equiv & \langle \quad \wedge \quad \text{has lower precedence } \rangle \\
 & (1 < i) \quad \wedge \quad (i \leq j) \quad \wedge \quad (j < 5) \quad \wedge \quad (5 = k)
\end{array}
$$

$$
\begin{array}{ll}
 & x < 5 \in S \subseteq T \\
\equiv & \langle \text{ conjunctional operators } \rangle \\
 & x < 5 \quad \wedge \quad 5 \in S \quad \wedge \quad S \subseteq T \\
\equiv & \langle \quad \wedge \quad \text{has lower precedence } \rangle \\
 & (x < 5) \quad \wedge \quad (5 \in S) \quad \wedge \quad (S \subseteq T)
\end{array}
$$

Equality versus Equivalence

The operators $=$ (as Boolean operator) and $\equiv$

- have the **same meaning** (represent the same function),
- but **are used with different notational conventions:**
  - different precedences ($\equiv$ has lowest)
  - different **chaining behaviour**:
    - $\equiv$ is associative:

      $$\boxed{(p \equiv q \equiv r) \quad = \quad ((p \equiv q) \equiv r) \quad = \quad (p \equiv (q \equiv r))}$$

    - $=$ is **conjunctional**:

      $$\boxed{(p = q = r) \quad = \quad ((p = q) \quad \wedge \quad (q = r))}$$

Binary Boolean Operators: Equivalence

Args.		$\equiv$	
F	F	T	The moon is green **iff** $2 + 2 = 7$.
F	T	F	The moon is green **iff** $1 + 1 = 2$.
T	F	F	$1 + 1 = 2$ **iff** the moon is green.
T	T	T	$1 + 1 = 2$ **iff** the sun is a star.

Binary Boolean Op.: Inequivalence ("exclusive or")

Args.		$\not\equiv$	
F	F	F	Either the moon is green, or $2 + 2 = 7$.
F	T	T	Either the moon is green, or $1 + 1 = 2$.
T	F	T	Either $1 + 1 = 2$, or the moon is green.
T	T	F	Either $1 + 1 = 2$, or the sun is a star.

Binary Boolean Operators: Implication

Args.		$\Rightarrow$	
F	F	T	If the moon is green, then $2 + 2 = 7$.
F	T	T	If the moon is green, then $1 + 1 = 2$.
T	F	F	If $1 + 1 = 2$, then the moon is green.
T	T	T	If $1 + 1 = 2$, then the sun is a star.

$$p \Rightarrow q \qquad \equiv \qquad \neg p \vee q$$

If you don't eat your spinach, I'll spank you.	$\equiv$	You eat your spinach, or I'll spank you.

Binary Boolean Operators: Consequence

Args.		$\Leftarrow$	
F	F	T	The moon is green **if** $2+2=7$.
F	T	F	The moon is green **if** $1+1=2$.
T	F	T	$1+1=2$ **if** the moon is green.
T	T	T	$1+1=2$ **if** the sun is a star.

$$p \Leftarrow q \quad \equiv \quad p \vee \neg q$$

Selected Laws for Implication

Consequence: $p \Leftarrow q \;\equiv\; q \Rightarrow p$

ex falso quodlibet: $\mathit{false} \Rightarrow p \;\equiv\; \mathit{true}$

Left-identity of $\Rightarrow$: $\mathit{true} \Rightarrow p \;\equiv\; p$

Right-zero of $\Rightarrow$: $p \Rightarrow \mathit{true} \;\equiv\; \mathit{true}$

Definition of $\neg$: $p \Rightarrow \mathit{false} \;\equiv\; \neg p$ $\qquad \neg p \;\equiv\; p \Rightarrow \mathit{false}$

Evaluation of Boolean Expressions Using Truth Tables

p	q	$\neg p$	$q \wedge \neg p$	$p \vee (q \wedge \neg p)$
F	F	T	F	F
F	T	T	T	T
T	F	F	F	T
T	T	F	F	T

- Identify variables
- Identify subexpressions
- Enumerate possible states (of the variables)
- Evaluate (sub-)expressions in all states

Evaluation of Boolean Expressions Using Truth Tables

p	q	r	$\neg r$	$q \wedge \neg r$	$p \vee (q \wedge \neg r)$
F	F	F	T	F	F
F	F	T	F	F	F
F	T	F	T	T	T
F	T	T	F	F	F
T	F	F	T	F	T
T	F	T	F	F	T
T	T	F	T	T	T
T	T	T	F	F	T

			$\wedge$					$\not\equiv$ $\neq$	$\vee$	nor	$\equiv$ $=$		$\Leftarrow$		$\Rightarrow$	nand	
F	F	F	F	F	F	F	F	F	F	T	T	T	T	T	T	T	T
F	T	F	F	F	F	T	T	T	T	F	F	F	F	T	T	T	T
T	F	F	F	T	T	F	F	T	T	F	F	T	T	F	F	T	T
T	T	F	T	F	T	F	T	F	T	F	T	F	T	F	T	F	T

Modeling English Propositions 1

- Henry VIII had one son and Cleopatra had two.

  Henry VIII had one son and Cleopatra had two sons.

  Declarations:

  h :≡ Henry VIII had one son

  c :≡ Cleopatra had two sons

  Formalisation:

  $h \wedge c$

Modeling English Propositions — Recipe

- Transform into shape with clear subpropositions
- Introduce Boolean variables to denote subpropositions
- Replace these subpropositions by their corresponding Boolean variables
- Translate the result into a Boolean expression, using (no perfect translation rules are possible!) **for example**:

and, but	becomes	$\wedge$
or	becomes	$\vee$
not	becomes	$\neg$
it is not the case that	becomes	$\neg$
if p then q	becomes	$p \Rightarrow q$

Ladies or Tigers — The First Case

Raymond Smullyan provides, in **The Lady or the Tiger?**, the following context for a number of puzzles to follow:

> [...] the king explained to the prisoner that each of the two rooms contained either a lady or a tiger, but it *could* be that there were tigers in both rooms, or ladies in both rooms, or then again, maybe one room contained a lady and the other room a tiger.

In the first case, the following signs are on the doors of the rooms:

1	2
In this room there is a lady, and in the other room there is a tiger.	In one of these rooms there is a lady, and in one of these rooms there is a tiger.

We are told that one of the signs is true, and the other one is false.

"Which door would you open (assuming, of course, that you preferred the lady to the tiger)?"

Ladies or Tigers — The First Case — Formalisation 1

Raymond Smullyan provides, in **The Lady or the Tiger?**, the following context for a number of puzzles to follow:

> [...] the king explained to the prisoner that each of the two rooms contained either a lady or a tiger, but it *could* be that there were tigers in both rooms, or ladies in both rooms, or then again, maybe one room contained a lady and the other room a tiger.

$R1L$:= There is a lady in room 1

$R1T$:= There is a tiger in room 1

$R2L$:= There is a lady in room 2

$R2T$:= There is a tiger in room 2

"Each of the two rooms contained either a lady or a tiger":

Axiom (R1): $R1L \not\equiv R1T$

Axiom (R2): $R2L \not\equiv R2T$

Ladies or Tigers — The First Case — Formalisation 2

"Each of the two rooms contained either a lady or a tiger": **Axiom** (R1): $R1L \not\equiv R1T$ **Axiom** (R2): $R2L \not\equiv R2T$

In the first case, the following signs are on the doors of the rooms:

1	2
In this room there is a lady, and in the other room there is a tiger.	In one of these rooms there is a lady, and in one of these rooms there is a tiger.

We are told that one of the signs is true, and the other one is false.

S_1 := Sign 1 is true	$S_1 := R1L \wedge R2T$
S_2 := Sign 2 is true	$S_2 := (R1L \vee R2L) \wedge (R1T \vee R2T)$

What does "in one of these rooms" mean?

The second sign could more concisely be formalised as: $R1L \equiv R2T$
— but that is quite a large step...

"one of the signs is true, and the other one is false": $S_1 \not\equiv S_2$

Ladies or Tigers — The First Case — Truth table

In the first case, the following signs are on the doors of the rooms:

1	2
In this room there is a lady, and in the other room there is a tiger.	In one of these rooms there is a lady, and in one of these rooms there is a tiger.

We are told that one of the signs is true, and the other one is false.

$R1L$:= There is a lady in room 1
$R2T$:= There is a tiger in room 2
S_1 := $R1L \wedge R2T$
S_2 := $R1L \equiv R2T$

$$\begin{array}{ll} & S_1 \not\equiv S_2 \\ = & \langle \text{ Def. } S_1, S_2 \rangle \\ & (R1L \wedge R2T) \not\equiv (R1L \equiv R2T) \end{array}$$

$R1L$	$R2T$	$R1L \wedge R2T$	$R1L \equiv R2T$	$(R1L \wedge R2T) \not\equiv (R1L \equiv R2T)$
F	F	F	T	T
F	T	F	F	F
T	F	F	F	F
T	T	T	T	F

Ladies or Tigers: First Case, Formalisation, Long S_2

In the first case, the following signs are on the doors of the rooms:

1	2
In this room there is a lady, and in the other room there is a tiger.	In one of these rooms there is a lady, and in one of these rooms there is a tiger.

We are told that one of the signs is true, and the other one is false.

$R1L$:= There is a lady in room 1
$R2T$:= There is a tiger in room 2

$S_1 \equiv R1L \wedge R2T$
$S_2 \equiv (R1L \vee \neg R2T) \wedge (\neg R1L \vee R2T)$

$$S_1 \not\equiv S_2$$

Ladies or Tigers: First Case, Long S_2, Solution

$R1L$:= There is a lady in room 1
$R2T$:= There is a tiger in room 2

$S_1 \equiv R1L \wedge R2T$
$S_2 \equiv (R1L \vee \neg R2T) \wedge (\neg R1L \vee R2T)$

$$\begin{array}{ll} & S_1 \not\equiv S_2 \\ = & \langle \text{ Def. } S_1, S_2 \rangle \\ & (R1L \wedge R2T) \not\equiv ((R1L \vee \neg R2T) \wedge (\neg R1L \vee R2T)) \\ = & \langle \text{ (3.14) } p \not\equiv q \equiv \neg p \equiv q, \text{ (3.35) Golden Rule } \rangle \\ & \neg(R1L \wedge R2T) \equiv R1L \vee \neg R2T \equiv \neg R1L \vee R2T \equiv R1L \vee \neg R2T \vee \neg R1L \vee R2T \\ = & \langle \text{ (3.28) Excluded Middle, (3.29) Zero of } \vee \rangle \\ & \neg(R1L \wedge R2T) \equiv R1L \vee \neg R2T \equiv \neg R1L \vee R2T \equiv true \\ = & \langle \text{ (3.47) De Morgan, (3.3) Identity of } \equiv \rangle \\ & \neg R1L \vee \neg R2T \equiv R1L \vee \neg R2T \equiv \neg R1L \vee R2T \\ = & \langle \text{ (3.32) } p \vee q \quad \equiv \quad p \vee \neg q \quad \equiv \quad p \rangle \\ & \neg R2T \equiv \neg R1L \vee R2T \\ = & \langle \text{ (3.32) } p \vee q \quad \equiv \quad p \vee \neg q \quad \equiv \quad p \rangle \\ & \neg R2T \equiv \neg R1L \vee \neg R2T \equiv \neg R1L \\ = & \langle \text{ (3.35) Golden Rule } \rangle \\ & \neg R1L \wedge \neg R2T \\ = & \langle R1T = \neg R1L \text{ and } R2L = \neg R2T \rangle \\ & R1T \wedge R2L \end{array}$$

Calculational Proof Format

$$
\begin{array}{ll}
& E_0 \\
= & \langle \text{ Explanation of why } E_0 = E_1 \ \rangle \\
& E_1 \\
= & \langle \text{ Explanation of why } E_1 = E_2 \ \rangle \\
& E_2 \\
= & \langle \text{ Explanation of why } E_2 = E_3 \ \rangle \\
& E_3
\end{array}
$$

This is a proof for:

$$E_0 = E_3$$

Calculational Proof Format

$$
\begin{array}{ll}
& E_0 \\
= & \langle \text{ Explanation of why } E_0 = E_1 \ \rangle \\
& E_1 \\
= & \langle \text{ Explanation of why } E_1 = E_2 \ \rangle \\
& E_2 \\
= & \langle \text{ Explanation of why } E_2 = E_3 \ \rangle \\
& E_3
\end{array}
$$

Details (will be revisited): This reads as:

$$E_0 = E_1 \qquad \wedge \qquad E_1 = E_2 \qquad \wedge \qquad E_2 = E_3$$

Because $=$ is **transitive**, this justifies:

$$E_0 = E_3$$

Calculational Proofs of Theorems — (15.17) $-(-a) = a$

(15.3) **Identity of +** $0 + a = a$	(15.13) **Unary minus** $a + (-a) = 0$

Theorem (15.17): $-(-a) = a$
Proof:

$$
\begin{array}{ll}
& -(-a) \\
= & \langle \text{ Identity of + (15.3) } \rangle \\
& 0 + -(-a) \\
= & \langle \text{ Unary minus (15.13) } \rangle \\
& a + (-a) + -(-a) \\
= & \langle \text{ Unary minus (15.13) } \rangle \\
& a + 0 \\
= & \langle \text{ Identity of + (15.3) } \rangle \\
& a
\end{array}
$$