COMPSCI/SFWRENG 2FA3

Discrete Mathematics with Applications II

Winter 2020

Assignment 3

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Assignment 3 consists of four problems. You must write your solutions to the problems using LaTeX.

Please submit Assignment 3 as two files, `Assignment_3_`*`YourMacID`*`.tex` and `Assignment_3_`*`YourMacID`*`.pdf`, to the Assignment 3 folder on Avenue under Assessments/Assignments. *`YourMacID`* must be your personal MacID (written without capitalization). The `Assignment_3_`*`YourMacID`*`.tex` file is a copy of the LaTeX source file for this assignment (`Assignment_3.tex` found on Avenue under Contents/Assignments) with your solution entered after each problem. The `Assignment_3_`*`YourMacID`*`.pdf` is the PDF output produced by executing

```
pdflatex Assignment_3_YourMacID
```

This assignment is due **Sunday, February 9, 2020 before midnight.** You are allow to submit the assignment multiple times, but only the last submission will be marked. **Late submissions and files that are not named exactly as specified above will not be accepted!** It is suggested that you submit your preliminary `Assignment_3_`*`YourMacID`*`.tex` and `Assignment_3_`*`YourMacID`*`.pdf` files well before the deadline so that your mark is not zero if, e.g., your computer fails at 11:50 PM on February 9.

Although you are allowed to receive help from the instructional staff and other students, your submission must be your own work. Copying will be treated as academic dishonesty! If any of the ideas used in your submission were obtained from other students or sources outside of the lectures and tutorials, you must acknowledge where or from whom these ideas were obtained.

Background

Let $\Sigma = (\mathcal{B}, \mathcal{C}, \mathcal{F}, \mathcal{P}, \tau)$ be a finite signature of MSFOL, F_Σ be the set of Σ-formulas, and $A \in F_\Sigma$. Recall that the members of F_Σ are certain strings of symbols. A *subformula* of A is a $B \in F_\Sigma$ such that B is a substring of A. For example, let A be the formula $((0 = 2) \wedge (3 \mid 4))$, i.e., A is the string "$((0 = 2) \wedge (3 \mid 4))$". Then "$(0 = 2)$", "$(3 \mid 4)$", and "$((0 = 2) \wedge (3 \mid 4))$" are the subformulas of A, and "$(0 =$ " and "$\wedge$" are two substrings of A that are not subformulas of A.

A function $f : A \to B$ is *total* if it is defined on *all* members of A. A function $f : A \to B$ is a *partial* if it is be undefined on *some* members of A. For example, the square root function $\sqrt{\cdot} : \mathbb{R} \to \mathbb{R}$ is a partial function since $\sqrt{r}$ is undefined for all $r \in \mathbb{R}$ with $r < 0$. If $f, g : A \to B$ are partial or total functions, then f is a *subfunction* of g, written $f \sqsubseteq g$, if the domain D_f of f is a subset of the domain of g and, for all $x \in D_f$, $f(x) = g(x)$. In other words, f is a subfunction of g if $g(a)$ is defined and $f(a) = g(a)$ whenever $f(a)$ is defined.

Problems

1. **[10 points]** Let $\mathsf{subformulas} : F_\Sigma \to \mathcal{P}(F_\Sigma)$ be the function that maps a formula $A \in F_\Sigma$ to the set of subformulas of A. Define $\mathsf{subformulas}$ by structural recursion using pattern matching.

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   Solution:

   Let $\mathrm{x_i} \in T_{\sum}$.

   Let M, N be formulas that are in our signature. So: M, N $\in F_{\sum}$.

   - Let *eq* represent Equality.
   - Let *predi* represent Predicate.
   - Let *nega* represent Negation.
   - Let *impli* represent Implication.
   - Let *quants* represent Quantifications. (i.e. $\forall$)

   Structural recursion will be defined for all inputs of the above type, and the base case, which is the empty set.

   subformulas({ }) = { }

   subformulas(eq($\mathrm{x_e}$, $\mathrm{x_r}$)) = { eq($\mathrm{x_e}$, $\mathrm{x_r}$) } $\cup$ subforumlas(M)

   subformulas(predi($\mathrm{x_e}$, $\mathrm{x_r}$)) = { predi($\mathrm{x_e}$, $\mathrm{x_r}$) }
   $\cup$ subforumlas(M)

   subforumlas(nega(M)) = { nega(M) } $\cup$ subforumlas(M) subformulas(impli(M, N)) = { impli(M, N) } $\cup$ subformulas(M) $\cup$ subformulas(N)

subformulas(quants(z, a, M)) = { quants(z, a, M) } $\cup$ subformulas(M)

2. **[10 points]** Suppose F is the set of partial and total functions $f : \mathbb{N} \to \mathbb{N}$.

   a. Show that $(F, \sqsubseteq)$ is a weak partial order but not a weak total order.

   $(F, \sqsubseteq)$ is a weak partial order because it is reflexive, anti-symmetric, and transitive.

   $(F, \sqsubseteq)$ is not a weak total order because it is a weak partial order, and the relation is only one sided. If something is a partial order, then it can't be a total order. But if it is a total order, than it is also a partial order. And $(F, \sqsubseteq)$ is not a total order because there are elements in F that are not directly comparable. There is more than one way to compare the elements in F, thus making F a partial order. Every element in F is not related with one another.

   b. Describe the set of minimal elements of $(F, \sqsubseteq)$.

   F does not have any minimal elements because it is a weak partial order with an infinite number of functions. The minimal elements are all $\sqsubseteq$. And because of that, there is no minimal element.

   c. Describe the set of maximal elements of $(F, \sqsubseteq)$.

   $(F, \sqsubseteq)$ does not have any maximal elements in it, because there is no maximum or maximal element/number in $\mathbb{N}$. Therefore, $(F, \sqsubseteq)$ does have any any max element.

   d. Does $(F, \sqsubseteq)$ have a minimum element? If so, what is it?

   No, there is no minimum element, because it is a weak partial order.

   e. Does $(F, \sqsubseteq)$ have a maximum element? If so, what is it?

   No, there is no maximum element because there is no number in $\mathbb{N}$ that is greater than every number. In other words, $\mathbb{N}$ is Noetherian and has no maximum element. There is NO upperbound.

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Solution is above in italics and red color.