MATHEMATICS 1LT3 TEST 1

Evening Class
Duration of Examination: 60 minutes
McMaster University, 30 January 2014

Dr. M. Lovrić

FIRST NAME (please print): SOLUTIONS

FAMILY NAME (please print): ____________

Student No.: ____________

THIS TEST HAS 8 PAGES AND 6 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE.

Total number of points is 40. Marks are indicated next to the problem number. Any non-graphing calculator is allowed.

USE PEN TO WRITE YOUR TEST. IF YOU USE A PENCIL YOUR TEST WILL NOT BE ACCEPTED FOR REMARKING (IF NEEDED).

You must show work to receive full credit.

Problem	Points	Mark
1	6	
2	6	
3	4	
4	6	
5	8	
6	10	
TOTAL	40	

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1. **Multiple choice questions: circle ONE answer. No justification is needed.**

(a)[3] Consider the population model

$$P'(t) = 12P(t)\left(1 - \frac{P(t)}{600}\right)\left(1 - \frac{120}{P(t)}\right)$$

Which of the following statements is/are true?

(I) If $P(0) = 100$, then the population $P(t)$ decreases toward extinction. ✓

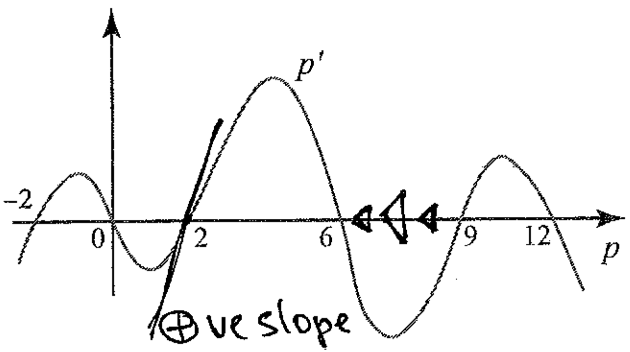
(II) The carrying capacity of the population is 120. X

(III) If $P(0) = 140$, then the population $P(t)$ increases toward the carrying capacity. ✓

(A) none (B) I only (C) II only (D) III only

(E) I and II (F) I and III (G) II and III (H) all three

(b)[3] Which of the following statements is/are true for the phase-line diagram of the differential equation $p' = f(p)$ given below



(I) For $6 < p < 9$, the arrows point right. X

(II) $p^* = 2$ is a stable equilibrium. X

(III) The largest right-pointing arrow is drawn somewhere between $p = 2$ and $p = 6$. ✓

(A) none (B) I only (C) II only (D) III only

(E) I and II (F) I and III (G) II and III (H) all three

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2. Identify each statement as true or false (circle your choice). You do not need to justify your answer.

(a)[2] $y' = x - 4y^2 - 4 + xy^2$ is a separable differential equation.

TRUE FALSE

$= x - 4 + xy^2 - 4y^2$

$= (x-4) + y^2(x-4)$

$y' = (1+y^2)(x-4)$

$\frac{1}{1+y^2} y' = x - 4$

(b)[2] The equilibrium solution of the differential equation $y' = \overbrace{2.3(50 - 3y)}^{f(y)}$ is unstable.

TRUE FALSE

$f'(y) = 2.3(-3) < 0$

so stable by the Stability Theorem

(c)[2] The solution of $P'(t) = 1.2P(t)\left(1 - \frac{P(t)}{14}\right)$ with $P(0) = 0.5$ looks initially (i.e., for small values of t) like $y = 0.5e^{1.2t}$.

TRUE FALSE

initial exponential growth!

$P'(t) = 1.2P(t)$ → solution is

$P(0)e^{1.2t}$

$= 0.5e^{1.2t}$

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Questions 3-7: You must show work to receive full credit.

3. [4] Find the explicit solution of the initial value problem

$$\frac{dy}{dt} = \frac{t\sqrt{1-y^2}}{\sqrt{1-t^2}}, \qquad y(0) = 0.$$

$$\frac{1}{\sqrt{1-y^2}} \cdot \frac{dy}{dt} = \frac{t}{\sqrt{1-t^2}}$$

$$\rightarrow \int \frac{1}{\sqrt{1-y^2}}\,dy = \underbrace{\int \frac{t}{\sqrt{1-t^2}}\,dt}$$

$$= \left\{ \begin{array}{l} u = 1-t^2 \\ \frac{du}{dt} = -2t \rightarrow t\,dt = -\frac{1}{2}du \end{array} \right\}$$

$$= \int \frac{1}{\sqrt{u}}\left(-\frac{1}{2}\right)du = -\frac{1}{2}u^{-1/2}du$$

$$= -u^{1/2} = -\sqrt{1-t^2}$$

$$\arcsin y = -\sqrt{1-t^2} + C$$

$y(0) = 0$ $\quad \arcsin 0 = -1 + C \Rightarrow C = 1$

$$\arcsin y = -\sqrt{1-t^2} + 1$$

$$y = \sin\left(1 - \sqrt{1-t^2}\right)$$

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4. Consider the differential equation $T'(t) = -2(35 - T(t))$.

(a)[3] Solve the equation given that $T(0) = 10$. $\longrightarrow$ separable

$$\frac{T'}{35-T} = -2 \longrightarrow \int \frac{T'}{35-T}\,dt = \int(-2)\,dt$$

ie $$\int \frac{1}{35-T}\,dT = \int(-2)\,dt$$

$$-\ln|35-T| = -2t + C$$

$$T(0)=10 \longrightarrow C = -\ln 25$$

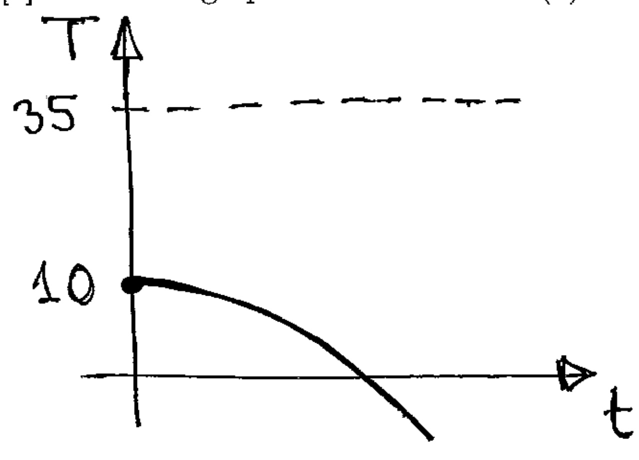
$$-\ln|35-T| = -2t - \ln 25$$

$$\underbrace{|35-T|}_{=\,35-T \text{ because } T<35} = e^{2t+\ln 25} = e^{2t}e^{\ln 25} = 25e^{2t}$$

$$T = 35 - 25e^{2t}$$

T is a decreasing function!

(b)[2] Sketch the graph of the solution in (a).



(c)[1] Does this equation describe Newton's Law of Cooling? Why, or why not?

NO. In NLC the solution below 35 increases toward 35 (colder object warms up).

In this case, colder object becomes even more cold.

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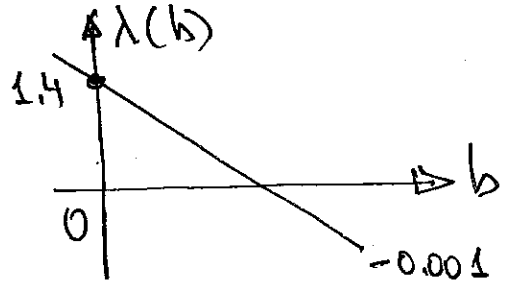
5. To model the competition within a population of bacteria, researchers use the differential equation $\frac{db}{dt} = \lambda(b)b$, where $\lambda(b)$ is the per capita production rate.

(a)[1] Should $\lambda(b)$ be an increasing or a decreasing function? Explain why.

competition means that with an increase in population, the per capita production must decrease (ie, fewer bacteria per bacterium)

(b)[2] Suppose that $\lambda(b)$ is a linear function of the population size b with maximum $\lambda(0) = 1.4$ and a slope of -0.001. Find $\lambda(b)$, and write the differential equation for b.



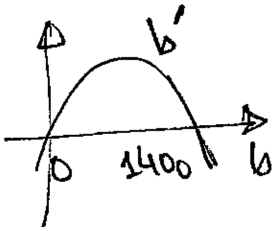
$$\lambda(b) = 1.4 - 0.001b$$

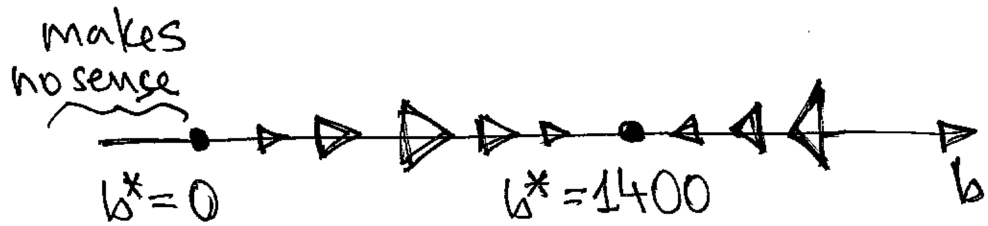
$$\frac{db}{dt} = (1.4 - 0.001b)\,b$$

(c)[2] Find all equilibrium solutions of this differential equation.

$$b^* = \frac{1.4}{0.001} = 1400 \qquad b^* = 0$$

(d)[3] Draw a phase-line diagram for this differential equation, clearly indicating the equilibria and intervals where the solutions are increasing and where they are decreasing.





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6. Consider the predator-prey model

$$\frac{da}{dt} = -0.1a + 0.0002ab$$

$$\frac{db}{dt} = 0.4b - 0.02ab$$

where $a(t)$ and $b(t)$ count the number of individuals in populations a and b.

(a)[1] Identify the prey equation and justify your choice.

SECOND equation $\boxed{b = \text{prey}}$

$0.4b$... on its own, prey pop. increases

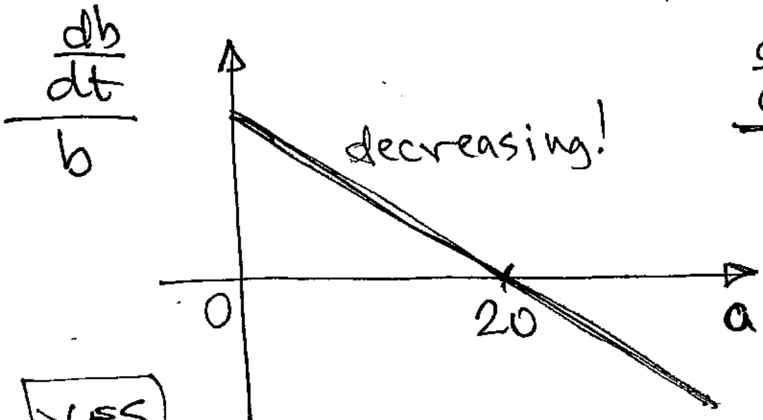
OR: $-0.02ab$... interactions decrease the population

(b)[2] What is the meaning of the term $0.0002ab$ in the first equation?

predator (a) eats prey (b) so benefits from interactions

OR: contribution to the increase in the population a (predator) due to the availability of food (population b)

(c)[2] Sketch the graph of the relative rate of change (i.e., the per capita change) of population b. Does it make sense? Explain why or why not.



$$\frac{\frac{db}{dt}}{b} = 0.4 - 0.02a$$

$$0.4 - 0.02a = 0$$

$$a = \frac{0.4}{0.02} = 20$$

$\boxed{\text{YES}}$

more predators force the decline in population b (prey)

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(d)[2] Find all biologically meaningful equilibria (i.e., those with $a > 0$ and $b > 0$).

$$\frac{da}{dt} = a(-0.1 + 0.0002b) = 0 \rightarrow b = \frac{0.1}{0.0002} = \underline{500}$$

$$\frac{db}{dt} = b(0.4 - 0.02a) = 0 \rightarrow a = \frac{0.4}{0.02} = \underline{\underline{20}}$$

(e)[3] Assume that $a_0 = 10$ and $b_0 = 100$, and the time is measured in months. Using the step size of 2 months (so $\Delta t = 2$), estimate the population sizes of a and b after 4 months.

$$a_0 = 10$$
$$b_0 = 100$$

$$a_1 = \underset{\substack{\shortparallel \\ 10}}{a_0} + \Big(\underbrace{-0.1(10)}_{-1} + \underbrace{0.0002(10)(100)}_{0.2}\Big)\cdot 2 =$$

$$= 10 + (-0.8)2 = 8.4$$

$$b_1 = b_0 + \Big(\underbrace{0.4(100)}_{40} - \underbrace{0.02(10)(100)}_{20}\Big)\cdot 2$$

$$= 100 + (20)\cdot 2 = 140$$

$$a_2 = 8.4 + \Big(-0.1(8.4) + 0.0002(8.4)(140)\Big)\cdot 2$$

$$\approx \underline{\underline{7.19}}$$

$$b_2 = 140 + \Big(0.4(140) - 0.02(8.4)(140)\Big)\cdot 2$$

$$\approx \underline{\underline{204.96}}$$

THE END