

MATHEMATICS 1LS3 TEST 1

Day Class

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Duration of Examination: 60 minutes
McMaster University, 1 October 2012

FIRST NAME (please print): SOLUTIONS

FAMILY NAME (please print): _____

Student No.: _____

THIS TEST HAS 8 PAGES AND 7 QUESTIONS. YOU ARE RESPONSIBLE FORENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE.

Total number of points is 40. Marks are indicated next to the problem number. Any non-graphing calculator is allowed.

USE PEN TO WRITE YOUR TEST. IF YOU USE A PENCIL YOUR TEST WILL NOT BE ACCEPTED FOR REMARKING (IF NEEDED).

You need to show work to receive full credit.

Problem	Points	Mark
1	6	
2	6	
3	5	
4	7	
5	6	
6	4	
7	6	
TOTAL	40	

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1. Multiple choice questions: circle ONE answer. No justification is needed.

(a)[3] The wingspan W (in centimetres) of certain tropical birds is related to their body mass M (in grams) by $W = 2.37M^{2/3}$. Which of the following statements is/are true?

- (I) If M increases, so does W → W is an increasing function ✓
 (II) If M triples, so does W → NO because W is not proportional to M
 (III) W is proportional to $M^{2/3}$ ✓ by definition
- (A) none (B) I only (C) II only (D) III only
 (E) I and II (F) I and III (G) II and III (H) all three

(b)[3] Which of the following statements is/are true for $f(x) = 1.4 + 2.8 \sin(4x - \pi)$?

- (I) The period is π → NO; period is $\frac{2\pi}{4} = \frac{\pi}{2}$
 (II) The amplitude is 4.2 → NO; ampl. is 2.8
 (III) The average of is 2.8 → NO; it is 1.4
- (A) none (B) I only (C) II only (D) III only
 (E) I and II (F) I and III (G) II and III (H) all three

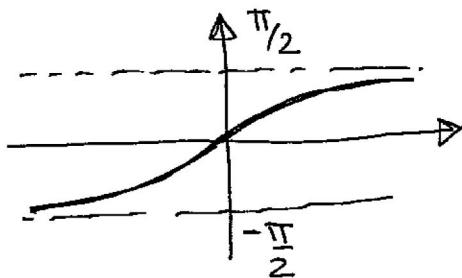
2. Identify each statement as true or false (circle your choice). No justification is needed.

(a)[2] If $m_{t+1} = 0.75m_t$ and $m_0 = 3$, then $m_t = 3e^{0.75t}$.

$$m_t = \underbrace{m_0 \cdot 0.75^t}_3$$

TRUE FALSE

(b)[2] The range of the function $f(x) = 2 \arctan x$ is $(0, \pi)$.



TRUE FALSE

range of $\arctan x$ is $(-\frac{\pi}{2}, \frac{\pi}{2})$
stretch vertically, so
range of $2 \arctan x$ is $(-\pi, \pi)$

(c)[2] The dynamical systems $b_{t+1} = b_t^2$ and $b_{t+1} = b_t^3$ have the same equilibrium points.

$$\begin{aligned} b^* &= b^{*2} \\ b^* - b^{*2} &= 0 \\ b^*(1-b^*) &= 0 \\ b^* &= 0, 1 \end{aligned}$$

$$\begin{aligned} b^* &= b^{*3} \\ b^* - b^{*3} &= 0 \\ b^*(1-b^{*2}) &= 0 \\ b^* &= 0, \pm 1 \end{aligned}$$

TRUE FALSE

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3. It has been determined that an average human body absorbs benzoylmethylecgonine (cocaine) according to $d(t) = 80e^{-0.71t}$, where $d(t)$ is in milligrams and t is time in hours.

(a)[1] Define the term: half-life of a drug d . (Avoid long explanations; one sentence suffices.)

half-life is the time needed for a drug d to
be reduced to $\frac{1}{2}$ of its original amount

KEY WORDS ARE UNDERLINED

(b)[2] Compute the half-life of benzoylmethylecgonine.

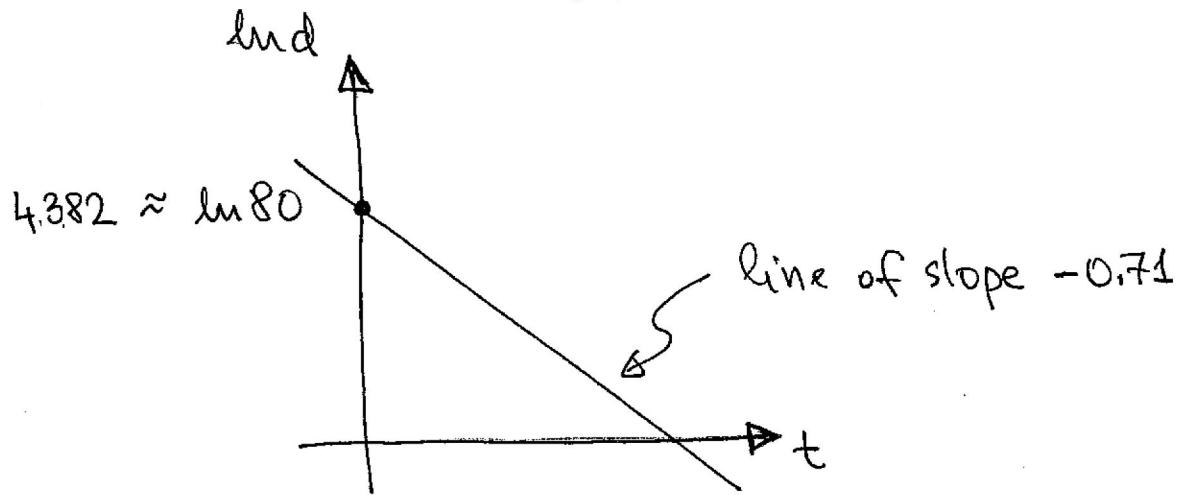
$$40 = 80 e^{-0.71t}$$

$$e^{-0.71t} = 0.5$$

$$t = \frac{\ln 0.5}{-0.71} \approx 0.976 \text{ hours}$$

(c)[2] Draw the semilog graph of $d(t)$. Label the axes.

$$\ln d(t) = \ln 80 - 0.71t$$



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4. Consider the dynamical system $M_{t+1} = 0.5M_t + 2.5$, where M_t represents the amount of drug (in mg) in patient's body at time t (time t is measured in days). It is given that $M_0 = 8$ mg.

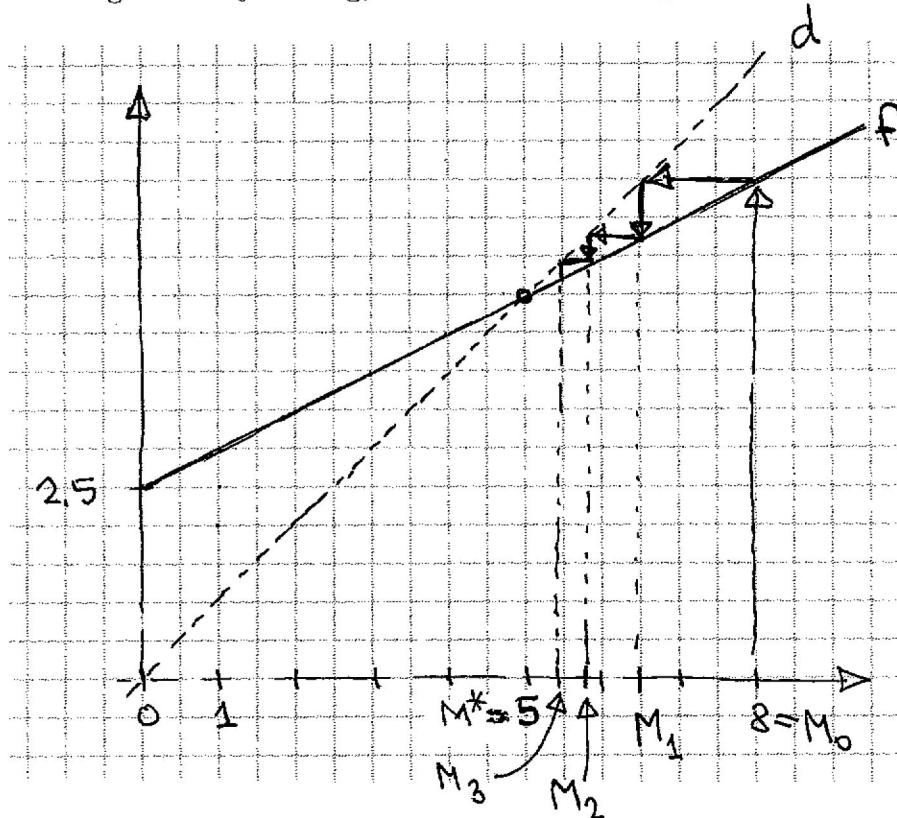
- (a)[2] Explain in words the dynamics described by this system.

Initial dose is 8 mg ; each day, the amount of the drug decreases to the half (so 1 day is half-life !)
and 2.5 units (mg) of drug are added

- (b)[2] Find all equilibrium points of the system.

$$M^* = 0.5M^* + 2.5 \rightarrow M^* = 5$$

- (c)[2] Starting with $M_0 = 8$ mg, cobweb for three steps.



- (d)[1] What does the cobwebbing in (c) suggest in terms of the amount of the drug in the patient's body?

the amount approaches the equilibrium $M^* = 5$

5. (a)[3] Find the domain of the function $f(x) = \frac{13}{2.1 + 1.4 \ln(x+1)}$.

$$\ln(x+1) \rightarrow x+1 > 0, \quad x > -1$$

$$\text{fraction} \rightarrow 2.1 + 1.4 \ln(x+1) \neq 0, \text{ so } \underline{x \neq e^{-\frac{3}{2}} - 1} \approx -0.777$$

$$2.1 + 1.4 \ln(x+1) = 0$$

$$\ln(x+1) = -\frac{2.1}{1.4} = -\frac{3}{2}$$

$$x+1 = e^{-\frac{3}{2}}$$

$$x = e^{-\frac{3}{2}} - 1$$

domain: x in $(-1, \infty)$ and $x \neq e^{-\frac{3}{2}} - 1$

- (b)[3] Find the range of the function $y = 3 + 2 \arcsin(12x)$.

≈ -0.777

range of $\arcsin x$ is $[-\frac{\pi}{2}, \frac{\pi}{2}]$

1. $\arcsin x \dots [-\pi, \pi]$

↑
irrelevant for range

2. $3 + 2 \arcsin x \dots [-\pi + 3, \pi + 3]$

6. The density of soil forming a forest floor is given by $d(x) = \frac{0.7}{0.8 + e^{-0.2x}}$, where x is the depth in metres (so $x = 0$ labels the surface, and $x = 3$ is 3 m below the surface).

(a)[1] State (in one sentence) what question is answered by finding the inverse function.

if we know the density of a soil sample;
underground was it taken from?

or: Given density, what depth?

- (b)[3] Find the inverse function of $d(x)$.

$$d = \frac{0.7}{0.8 + e^{-0.2x}}$$

$$0.8 + e^{-0.2x} = \frac{0.7}{d}$$

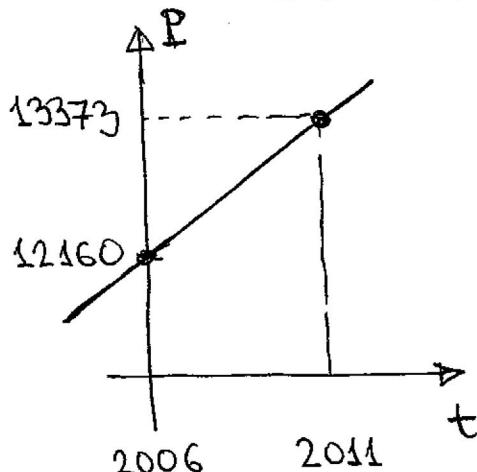
$$e^{-0.2x} = \frac{0.7}{d} - 0.8$$

$$x = \frac{\ln\left(\frac{0.7}{d} - 0.8\right)}{-0.2}$$

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7. According to the censuses conducted by Statistics Canada, the population of Ontario (in thousands) was 12,160 in 2006 and 13,373 in 2011. Let $P(t)$ denote the population of Ontario at time t , where $t = 0$ represents the year 2006.

(a)[3] Assuming that $P(t)$ changes linearly, find the formula that expresses $P(t)$ as a function of t . Sketch the graph of $P(t)$ and label given data on it.



$$\text{slope} = \frac{13373 - 12160}{5} = 242.6$$

$$P(t) = 242.6t + 12,160$$

(b)[1] What does the slope of the line in (a) say about the population of Ontario? (Saying that the slope is the rate of change does not suffice; you need to be more specific).

Yearly change (increase) in population of Ontario is 242,600 (i.e. 242.6 thousands)

(b)[2] In what year will the population of Ontario reach 20 million?

$$242.6t + 12,160 = 20,000$$

$$t = \frac{20,000 - 12,160}{242.6} = 32.3 \text{ years from 2006}$$

in year 2038