COMPSCI/SFWRENG 2FA3
Discrete Mathematics with Applications II
Winter 2020

Week 06 Exercises

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Background Definitions

Consider the following definitions:

1. $\Sigma_{\text{stack}} = (\mathcal{B}, \mathcal{C}, \mathcal{F}, \mathcal{P}, \tau)$ where:

   a. $\mathcal{B} = \{\mathsf{Element}, \mathsf{Stack}\}$.
   b. $\mathcal{C} = \{\mathsf{error}, \mathsf{bottom}\}$.
   c. $\mathcal{F} = \{\mathsf{push}, \mathsf{pop}, \mathsf{top}\}$.
   d. $\mathcal{P} = \emptyset$.
   e. $\tau(\mathsf{error}) = \mathsf{Element}$.
   f. $\tau(\mathsf{bottom}) = \mathsf{Stack}$.
   g. $\tau(\mathsf{push}) = \mathsf{Element} \times \mathsf{Stack} \rightarrow \mathsf{Stack}$.
   h. $\tau(\mathsf{pop}) = \mathsf{Stack} \rightarrow \mathsf{Stack}$.
   i. $\tau(\mathsf{top}) = \mathsf{Stack} \rightarrow \mathsf{Element}$.

2. $\Sigma_{\text{grp}} = (\{G\}, \{e\}, \{*, \mathsf{inv}\}, \emptyset, \tau)$ where $\tau(e) = G$, $\tau(*) = G \times G \rightarrow G$, and $\tau(\mathsf{inv}) = G \rightarrow G$.

3. Let Γ_{grp} be the following set of Σ-sentences:

   Assoc $\forall\, x, y, z : G \,.\, (x * y) * z = x * (y * z)$.
   IdLeft $\forall\, x : G \,.\, e * x = x$.
   IdRight $\forall\, x : G \,.\, x * e = x$.
   InvLeft $\forall\, x : G \,.\, \mathsf{inv}(x) * x = e$.
   InvRight $\forall\, x : G \,.\, x * \mathsf{inv}(x) = e$.

4. A *partition* of a set S is a nonempty set U of subsets of S such that, for all $x \in S$, x is a member of exactly one member of U. Hence (1) the members of U are disjoint and (2) their union equals S.

5. A *lattice* is a weak partial order $(U, \leq)$ such that each pair of elements of U has both a least upper bound and a greatest lower bound.

6. Let $M_1 = (D_1, e_1, *_1)$ and $M_2 = (D_2, e_2, *_2)$ be two monoids. A *monoid homomorphism* from M_1 to M_2 is a function $h : D_1 \rightarrow D_2$ such that:

   a. $h(x *_1 y) = h(x) *_2 h(y)$ for all $x, y \in D_1$.

   b. $h(e_1) = e_2$.

Exercises

1. Construct in MSFOL a theory $T = (\Sigma_{\text{stack}}, \Gamma_{\mathsf{stack}})$ of stacks. Γ_{stack} should contain axioms that say:

   a. The top of the bottom stack is the error element.

   b. Let s be a stack obtained by pushing an element e onto a stack s'. The top of s is e.

   c. Pop of the bottom stack is the bottom stack.

   d. Let s be a stack obtained by pushing an element e onto a stack s'. The pop of s is s'.

2. A *group* is a monoid with an inverse operation. $T_{\text{grp}} = (\Sigma_{\text{grp}}, \Gamma_{\text{grp}})$ is a theory of groups. Show that models of T_{grp} can be directly derived from $(\mathbb{Z}, 0, +)$ and $(\mathbb{Q}, 1, *)$ but not from $(\mathbb{N}, 0, +)$ and $(\mathbb{Z}, 1, *)$.

3. Let $\Sigma = (\alpha, p : \alpha \rightarrow \mathbb{B}, q : \alpha \rightarrow \mathbb{B})$ be a signature of MSFOL. What should Γ be so that each model for the theory $T = (\Sigma, \Gamma)$ is a set of values partitioned into two components defined by p and q.

4. Let Σ_{pairs} be the signature $(\mathcal{B}, \emptyset, \mathcal{F}, \emptyset, \tau)$ where:

   a. $\mathcal{B} = \{\alpha, \beta, \gamma\}$.

   b. $\mathcal{F} = \{\mathsf{mkPair}, \mathsf{left}, \mathsf{right}\}$ where $\tau(\mathsf{mkPair}) = \alpha \times \beta \rightarrow \gamma$, $\tau(\mathsf{left}) = \gamma \rightarrow \alpha$, and $\tau(\mathsf{right}) = \gamma \rightarrow \beta$.

   What should Γ_{pairs} be so that $T_{\text{pairs}} = (\Sigma_{\text{pairs}}, \Gamma_{\text{pairs}})$ is a theory of mathematical structures that contain (1) sets A, B, and C where C is a set of values that have the same structure as ordered pairs of members of A and B and (2) functions to construct and destruct the pairs in C?

5. Let $\Sigma_{\mathsf{lattice}} = (\{U\}, \emptyset, \emptyset, \{\leq\}, \tau)$ where $\tau(\leq) = U \times U \rightarrow \mathbb{B}$. Construct in MSFOL a theory $T = (\Sigma_{\mathsf{lattice}}, \Gamma_{\mathsf{lattice}})$ of lattices.

6. Explain why it is not possible to construct a theory of well-founded relations in MSFOL.

7. Construct in MSFOL a theory of vector spaces.

8. Construct in MSFOL a theory T of monoid homomorphisms where each model for T contains a monoid homomorphism.