We will wait 10 minutes until 10:40 AM for all students to join into the meeting.

We will start the tutorial at **10:40 AM**.

CS 3SD3 - Concurrent Systems Tutorial 8

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Outline

- Announcement
- Ambiguous slides from lecture 12 (full review)
  - Incidence Matrix
  - Invariants
  - Concession

Announcements

- Midterm is marked. solution will not be posted.
- The deadline for assignment 2 has been extended until Thursday, November 11, 23:59 PM.

Lecture 12

The order of the slides are incorrect.

- Lecture 12
  Slides 32 -34 (invariants) Should be before slide 14.

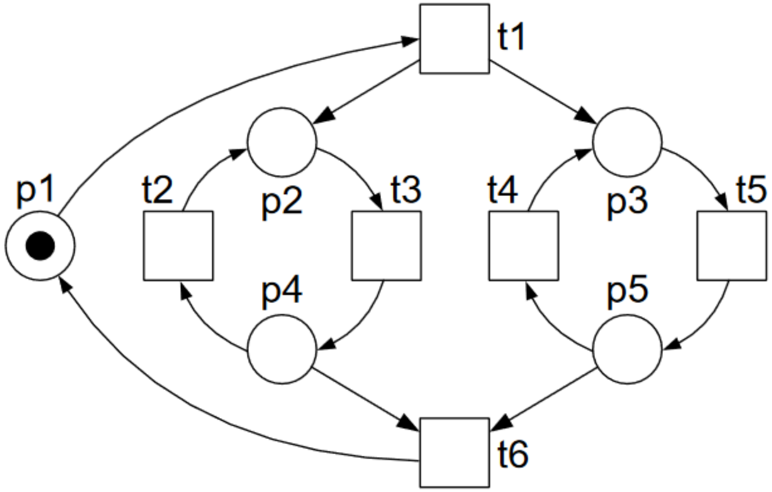
Two ways of calculating invariants

- "Intuitive way": Formulate the property that you think holds and verify it.
- "Linear-algebraic way": Solve a system of linear equations.

Humans tend to do it the intuitive way and computers do it the linear-algebraic way.

Exercise

Give the incidence matrix!



$$\begin{array}{c} \\ p1 \\ p2 \\ p3 \\ p4 \\ p5 \end{array} \begin{array}{c} \begin{array}{cccccc} \underline{t1} & \underline{t2} & \underline{t3} & \underline{t4} & \underline{t5} & \underline{t6} \end{array} \\ \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 1 & -1 \end{pmatrix} \end{array}$$

Invariants: Example

Example

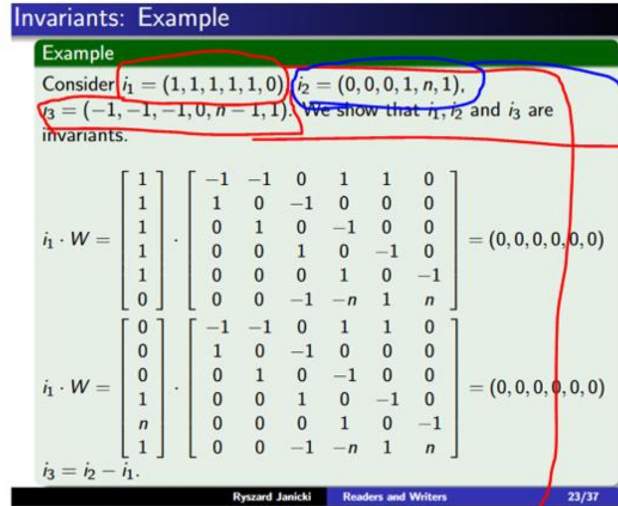
Consider $i_1 = (1,1,1,1,1,0)$, $i_2 = (0,0,0,1,n,1)$, $i_3 = (-1,-1,-1,0,n-1,1)$. We show that i_1, i_2 and i_3 are invariants.

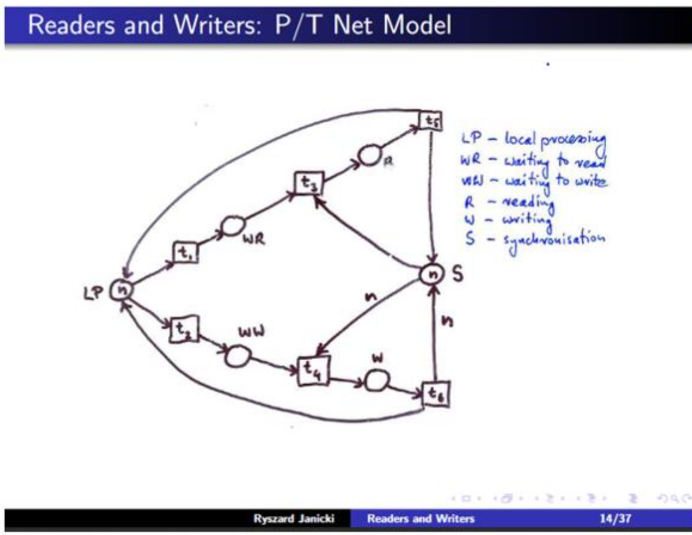
Incidence Matrix
each Ts are transitions

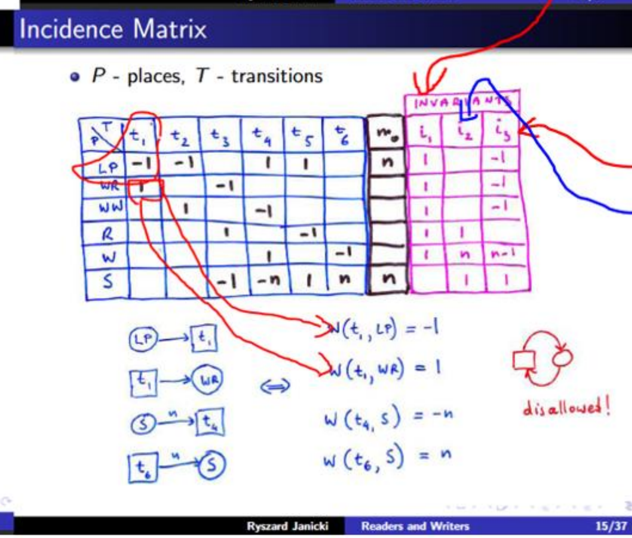
$$i_1 \cdot W = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -1 & -1 & 0 & 1 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & -n & 1 & n \end{bmatrix} = (0,0,0,0,0,0)$$

$$i_1 \cdot W = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ n \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & -1 & 0 & 1 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & -n & 1 & n \end{bmatrix} = (0,0,0,0,0,0)$$

$i_3 = i_2 - i_1$.

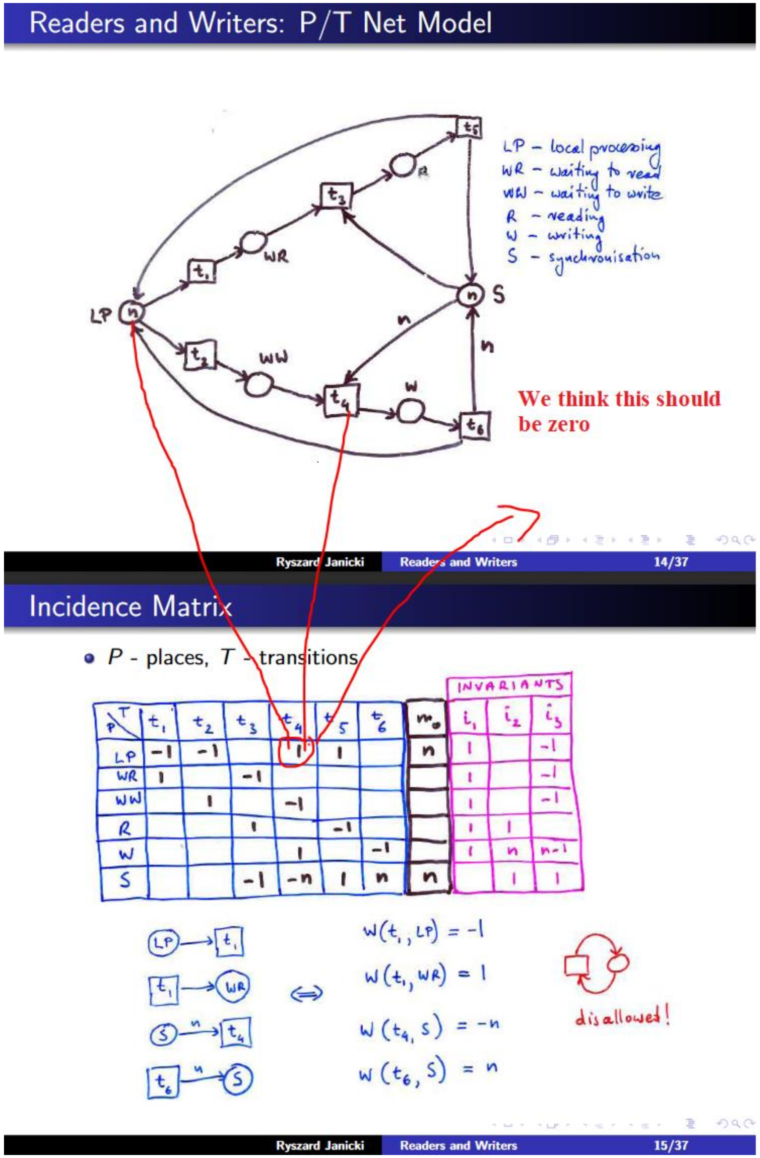






The distance from T_1 to LP is -1
The distance from T_1 to WR is 1
The distance from t_4 to S is -n
The Distance from t_6 to S is n

One Error in Slide



Basic Definitions

- Let **x** be a **multiset (weighted set) of transitions**, i.e.
  $\mathbf{x} : T \to \mathbb{N}$
- **x** is **positive** iff $\mathbf{x}(t) > 0$ for at least one $t \in T$, i.e. $\mathbf{x} \neq \emptyset$
- Marking: $m : P \to \mathbb{N}$.
  Marking **is not** interpreted as a multiset!
- $m \geq m' \iff \forall p \in P.\ m(p) \geq m'(p)$
- **Assumption:** Each place can hold an arbitrary number of tokens.
- Let W^- be the following matrix:

any negative number becomes positive
and any positive number becomes zero

$$\forall (p,t) \in P \times T.\ W^-(p,t) = \begin{cases} -W(p,t) & \text{if } W(p,t) < 0 \\ 0 & \text{if } W(p,t) \geq 0 \end{cases}$$

- A positive multiset of transitions **x** has **concession** in a marking m iff $m \geq W^- \cdot \mathbf{x}$

↑
matrix multiplication

- When **x** has concession, it may **fire**.

Example ($n = 15$)

- $\mathbf{x} = 10t_1 + 3t_2$ **has** a concession in $m_0 = (15,0,0,0,0,15)$, since

given

$$W^- \cdot \mathbf{x} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \cancel{0} & 15 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 10 \\ 3 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = (13,0,0,0,0,0),$$

and $m_0 > (13,0,0,0,0,0)$.

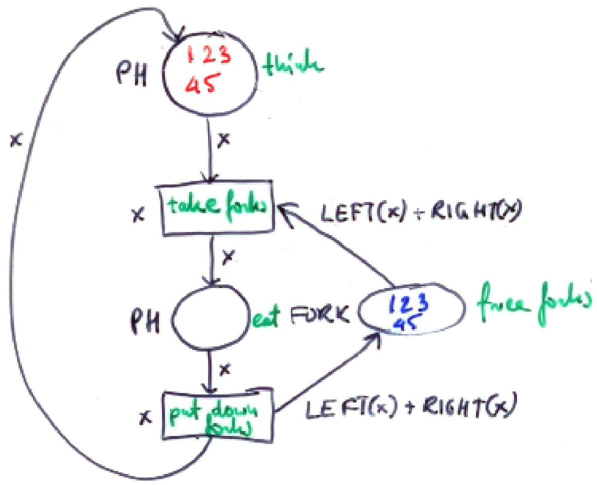
- $\mathbf{x} = t_4$ **does not have** a concession in $m = (8,3,1,2,0,13)$, since

$$W^- \cdot \mathbf{x} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 15 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = (0,0,1,0,0,15),$$

and m and $(0,0,1,0,0,15)$ are incomparable.

Invariants

- Invariants are equations that characterize all reachable markings.



- $M(think) + M(eat) = ph1 + ph2 + ph3 + ph4 + ph5$
  Each philosopher is either thinking or eating but not both. Also philosophers do not disappear and no new is born.
- $LEFT(M(eat)) + RIGHT(M(eat)) + M(free\ forks) =$
  $f_1 + f_2 + f_3 + f_4 + f_5$
  where $LEFT(X) = \sum_{x \in X} LEFT(x)$,
  $RIGHT(X) = \sum_{x \in X} RIGHT(x)$
  No philosopher can be eating at the same time as on of his

Multisets (or Bags)

- A multiset m, over a non-empty and finite set S is a function $m : S \to \mathbb{N} = \{0, 1, 2, \ldots\}$
- $m(s)$ is the number of appearances of s in m.
- notation: M is usually represented by:

$$\sum_{s \in S} m(s)s$$

$S = \{a, b, c, d, e\}$,
$m(a) = 3, m(b) = 1, m(c) = 0, m(d) = 183, m(e) = 4$

$$m = 3a + b + 183d$$

- $s \in m \iff m(s) \neq 0$
- $m(s)$ is a *coefficient*
- the *empty multiset* $m = \emptyset \iff m(s)$ for each $s \in S$.

Invariants

- Let **v** be a **multiset of places**, i.e. $\mathbf{v} : P \to \mathbb{N}$.

Note that $m : P \to \mathbb{N}$ and $\mathbf{v} : P \to \mathbb{N}$,but the interpretation is different, marking is not interpreted as a multiset!

Theorem (Lautenbach 1979)

Let **v** *be a multiset of places. If* $\mathbf{v} \cdot W = 0$ *and* $m \Rightarrow^* m'$ *then*

$$v \cdot m' = v \cdot m.$$

Proof.

It suffices to show it for $m \Rightarrow m'$. $v \cdot m' = v \cdot (m + W \cdot \mathbf{x}) = v \cdot m + v \cdot (W \cdot \mathbf{x}) = v \cdot m + (v \cdot W) \cdot \mathbf{x} = v \cdot m + 0 \cdot \mathbf{x} = v \cdot m.$ □

Definition

A multiset of places **v** is said to be an **invariant** iff $v \cdot W = 0$.

- Each linear combination of invariants is itself an invariant.

How to Find Invariants?

- Finding invariants can be reduced to finding non-negative integer solutions of some matrix equation:

$$W \cdot X = \mathbf{0}$$

  where $\mathbf{0}$ is a vector of zeros,
  W represents the structure of a net (incidence matrix),
  X represents an invariant.
- The number of invariants is infinite, but there is a finite number of linearly independent invariants
- Proper invariants are part of specification goals.
- Checking if an equation is an invariant is easy!

Invariant As an Expression

Definition

An invariant can also be defined as a **formula** obtained from $\mathbf{v} \cdot m_0 = \mathbf{v} \cdot m$, where $\mathbf{v}$ is an invariant, as defined previously, m_0 is the initial marking, and m is a marking variable.

Example

$i_1 = (1,1,1,1,1,0),\ m_0 = (n,0,0,0,0,n).$
$i_1 \cdot m_0 = (1,1,1,1,1,0) \cdot (n,0,0,0,0,n) = n$
$m = (m(LP), m(WR), m(WW), m(R), m(W), m(S))$
$i_1 \cdot m_0 =$
$(1,1,1,1,1,0) \cdot (m(LP), m(WR), m(WW), m(R), m(W), m(S)) =$
$m(LP) + m(WR) + m(WW) + m(R) + m(W).$
$i_1 \cdot m_0 = i_1 \cdot m \implies$
$$m(LP) + m(WR) + m(WW) + m(R) + m(W) = n$$

- **The number of processes is an invariant.**

Any Questions?