MATHEMATICS 1LS3 TEST 3

Day Class
Duration of Examination: 60 minutes
McMaster University, 5 November 2014

E. Clements, M. Lovrić, O. Sanchez

FIRST NAME (please print): SOLUTIONS

FAMILY NAME (please print): ______________

Student No.: ______________

THIS TEST HAS 8 PAGES AND 7 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE.

Total number of points is 40. Marks are indicated next to the problem number. Any non-graphing calculator is allowed.

USE PEN TO WRITE YOUR TEST. IF YOU USE A PENCIL YOUR TEST WILL NOT BE ACCEPTED FOR REMARKING (IF NEEDED).

You must show work to receive full credit.

Problem	Points	Mark
1	6	
2	6	
3	7	
4	6	
5	6	
6	6	
7	3	
TOTAL	40	

Continued on next page

Name:__________________
Student No.: __________________

1. Multiple choice questions: circle ONE answer. No justification is needed.

(a)[3] The slope of the tangent to the curve given implicitly by $xy^4 + \cos(\pi x) = 2$ at the point $(1, 1)$ is

(A) $\pi/4$ (B) $\pi/2$ (C) 1 (D) -1

(E) $-1/4$ (F) $-1/2$ (G) $1/2$ (H) $1/4$

$$y^4 + x \cdot 4y^3 y' - \pi \sin(\pi x) = 0$$

$$y' = \frac{\pi \sin(\pi x) - y^4}{4xy^3}$$

$$x = 1,\ y = 1 \rightarrow y' = \frac{0-1}{4} = -\frac{1}{4}$$

(b)[3] It is known that $f(4) = 0$ and $f'(4) = 0$. Which statements is/are true for all functions $f(x)$ which satisfy these two conditions?

(I) $f(4) = 0$ is a local (relative) minimum of $f(x)$ $\times$

(II) the tangent line to the graph of $f(x)$ at $x = 4$ is $y = 0$

(III) $f(4) = 0$ is a global (absolute) minimum of $f(x)$ on the interval $[-2, 0]$ $\times$

(A) none (B) I only (C) II only (D) III only

(E) I and II (F) I and III (G) II and III (H) all three

$$\rightarrow \text{tangent} = \underbrace{f(4)}_{0} + \underbrace{f'(4)}_{0}(x-4) = 0$$

Continued on next page

Name: ____________________
Student No.: ____________________

2. **Identify each statement as true or false, or yes or no (circle your choice). You do not need to justify your answer.**

(a)[2] $x = 0$ is a critical point (critical number) of the function $f(x) = \sqrt[3]{x}$.

TRUE FALSE

$f(x) = x^{1/3}$

$\Rightarrow f'(x) = \frac{1}{3} x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}}$

f' dne when $x = 0$

and $x = 0$ is in the domain of f

(b)[2] If $f(x) = \arcsin(e^x - x - 1)$, then $f'(0) = 1$.

TRUE FALSE

$f'(x) = \frac{1}{\sqrt{1-(e^x-x-1)^2}} \cdot (e^x - 1)$

$f'(0) = \frac{1}{\sqrt{1}}(1-1) = 0$

(c)[2] The formula $\lim_{h\to 0} \frac{e^{2+h} - e^2}{h} = e^2$ is correct.

YES NO

$f(x) = e^x$

$f'(2) = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0} \frac{e^{2+h} - e^2}{h}$

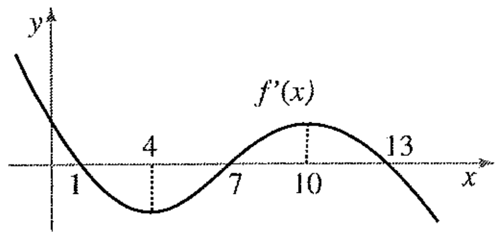
so this is the expression for the derivative of e^x at $x = 2$

Continued on next page

Name:________________
Student No.: ________________

Questions 3-7: You must show work to receive full credit.

3. The graph of the **derivative** f' of a function f is given below.



Answer the following questions. In order to receive credit, you need to justify your answers.

(a)[2] On which interval(s) is f decreasing? $\to$ $f' < 0$ (so above graph must be below the x-axis)

$(1,7)$ and $(13,\infty)$

(b)[2] On which interval(s) is f concave up? $\to$ $f'' > 0$ ie, $(f')' > 0$ (so above graph needs to be increasing)

$(4,10)$

(c)[3] At which value(s) of x does f have a maximum?

cp's are 1, 7 and 13

x		1		7		13	
f'	+		−		+		−
f	↗		↘		↗		↘
		max.				max.	

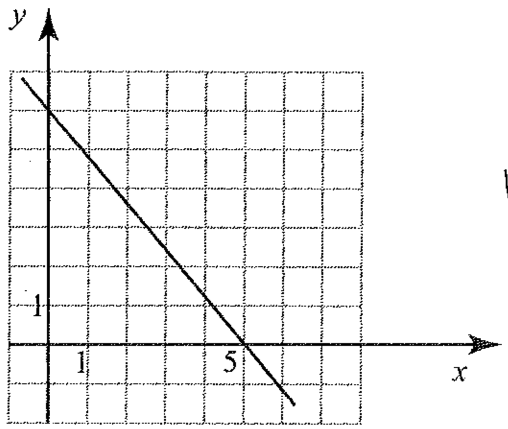
Continued on next page

4. (a)[2] Find $f'(1)$, if $f(x) = 2^{\ln x} + (\ln 5)^2$.

$$\hookrightarrow f'(x) = 2^{\ln x} \cdot \ln 2 \cdot \frac{1}{x} + 0$$

$$\text{so} \quad f'(1) = \underbrace{2^{\ln 1}}_{1} \cdot \ln 2 \cdot \frac{1}{1} = \underline{\underline{\ln 2}}$$

(b)[3] Let $h(x) = x \sin(f(x))$. The graph of $f(x)$ is a line shown below. Find $h'(5)$.



PRODUCT RULE

$$h'(x) = \sin(f(x)) + x \cdot \cos(f(x)) \cdot f'(x)$$

$$h'(5) = \underbrace{\sin(f(5))}_{\sin 0 = 0} + 5 \cdot \underbrace{\cos(f(5))}_{\cos 0 = 1} \cdot f'(5)$$

= slope of the line = $-6/5$

$$h'(5) = -6$$

(c)[2] Find $f'(0)$ if $f(x) = \ln \dfrac{ae^x + b}{ce^{-x} + d}$. $= \ln(ae^x + b) - \ln(ce^{-x} + d)$

$$f'(x) = \frac{1}{ae^x + b} \cdot ae^x - \frac{1}{ce^{-x} + d}\, ce^{-x}(-1)$$

$$f'(0) = \frac{a}{a+b} + \frac{c}{c+d}$$

Continued on next page

Name:

Student No.:

5. (a)[2] In the article *Migration behaviour of grizzly bears in Northern British Columbia: contribution to a modelling approach.* G. Brown et al, Bear Science 4 (June 2012), we find the formula

$$P(t) = \arctan(1.67t) + 4.71$$

where t represents time.

Next, we read "initially, $P(t) \approx 1.67t + 4.71$, which gives a linear relationship." Explain why this statement is correct. [Hint: Think in terms of the linear approximation at $t = 0$.]

$L_0(t) = P(0) + P'(0)(t-0)$

$P(0) = \underbrace{\cancel{\arctan(0)}}_{0} + 4.71 = 4.71$

$P'(x) = \dfrac{1}{1+(1.67t)^2} \cdot 1.67 \longrightarrow P'(0) = 1.67$

so $\boxed{L_0(t) = 4.71 + 1.67t}$

thus $P(t)$ is approximated by its linear approximation at $t=0$

(b)[3] The linear model for the ratio S of cancer cells surviving radiation treatment states that

$$S(x) = e^{-ax+b}$$

where a and b are constants and x is a radiation dose. This formula is sometimes simplified using a quadratic approximation near $x = 0$. Find that approximation.

$Q(x) = T_2(x) = S(0) + S'(0)x + \dfrac{S''(0)}{2}x^2$

$S = e^{-ax+b} \longrightarrow S(0) = e^b$

$S' = (-a)e^{-ax+b} \longrightarrow S'(0) = -ae^b$

$S'' = (-a)(-a)e^{-ax+b} \longrightarrow S''(0) = a^2e^b$

so $T_2(x) = e^b - ae^b x + \dfrac{a^2e^b}{2}x^2$

Continued on next page

Name:________________
Student No.: ________________

6. (a)[2] The function $f(x) = x^2e^{4x}$ has two critical points. Find them.

$$f'(x) = 2x\,e^{4x} + x^2 e^{4x} \cdot 4$$
$$= 2xe^{4x}(1+2x) = 0$$
$$x = 0 \qquad x = -1/2$$

(b)[2] Give a statement of the Extreme Value Theorem. Make sure to clearly identify assumption(s) and conclusion(s).

IF $f(x)$ is continuous on closed interval $[a,b]$

THEN $f(x)$ has an absolute max. and an absolute min. in $[a,b]$

(c)[2] Find the absolute maximum and the absolute minimum of the function $f(x) = x^2e^{4x}$ on the interval $[0, 1]$.

closed

continuous

x	$f(x)=x^2e^{4x}$	
0	0	$\longrightarrow$ abs. min. $=0$, at $x=0$
1	e^4	$\longrightarrow$ abs. max. $= e^4$, at $x=1$
$-1/2$ IGNORE		

Continued on next page

Name:

Student No.:

7. [3] The interstitial fluid pressure p at a location r mm from the centre of a tumour is given by

$$p(r) = 0.267p_i + \frac{2\sinh(0.4r)}{r}$$

where p_i is the atmospheric pressure (assumed constant). Searching Wikipedia, you found that $\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$.

Researchers claim that p is an increasing function of r when $r > 2.5$. Justify their claim.

$$p(r) = 0.267p_i + \frac{e^{0.4r} - e^{-0.4r}}{r}$$

$$p'(r) = \frac{\left(e^{0.4r}(0.4) - e^{-0.4r}(-0.4)\right)r - \left(e^{0.4r} - e^{-0.4r}\right)}{r^2}$$

$$= \frac{1}{r^2}\left(\underbrace{e^{0.4r}}_{\oplus}(0.4r-1) + \underbrace{e^{-0.4r}}_{\oplus}\underbrace{(0.4r+1)}_{\oplus}\right)$$

$$0.4r - 1 > 0.4(2.5) - 1 = 0$$

so when $r > 2.5$, then $0.4r-1$ is $\oplus$

thus $p'(r) > 0$ and $p(r)$ is increasing