Examples of Proof by Induction

Introduction:

"A Journey of a thousand miles begins with a single step"

This phrase rather nicely sums up the core idea of proof by induction where we attempt to demonstrate that a property holds in an infinite, but countable, number of cases, by extrapolating from the first few.

More specifically, we have the following:

1) ***The "base" case***: Our property holds for some initial case(s).

2) ***The induction step:*** Show if the property holds for some n^{th} case, it must hold for the next, the "n+1" case.

3) ***The conclusion:*** Once we have the first case holds, from the induction step we know if any step holds, the next step holds. So the first implies the 2^{nd}, implies the third etc., etc. Thus our property holds for all cases.

(*Note*: *technically, this is what is known as* <u>*weak induction*</u>*. This differs in step 2 from the other version, called* <u>*strong induction.*</u> *With strong, we assume, in the induction step that all earlier cases have the requisite property, whereas here we only required the n^{th} step to work.*)

Generally it's much easier to see, if we work through some examples.

Example #1

Show, using induction, that $\sum_{i=1}^{n} (i+1) = \frac{n(n+3)}{2}$

Solution:

Well, we can see this is obviously true, even without induction, by deconstructing it into the expression: $\Sigma(i + 1) = \Sigma\, i + \Sigma\, 1$, and using the relations from Appendix E. But let's work this through now with induction.

1) ***Base case*****:** The simplest possible sum occurs at $n = 1$.

$$\sum_{i=1}^{1} (i+1) = (1+1) = 2 = \frac{1(1+3)}{2} = \left.\frac{n(n+3)}{2}\right|_{n=1}$$

Which fits the equation as predicted.

From here, we could check $n = 1$, $n = 2$, $n = 3$ etc. and each of these would fit our formula. But how do we know that $n = 100$, $n = 10000$, $n = 10^{100}$ works? That's why we need step #2.

2) ***Induction Step:*** Here we assume our relation holds for the sum up to $n = k$:

$$\sum_{i=1}^{k}(i+1) = \frac{k(k+3)}{2}$$

If it's true for $n = k$, is it true for the "next" expression, $n = k + 1$? That is, can we say that:

$$\sum_{i=1}^{k+1}(i+1) = \frac{(k+1)((k+1)+3)}{2} = \frac{(k+1)(k+4)}{2}$$

Well, let's see. We can split up our new $k+1^{\text{th}}$ sum in terms of the previous sum, which we know worked (by our assumption).

$$\sum_{i=1}^{k+1}(i+1) = \sum_{i=1}^{k}(i+1) + ((k+1)+1)$$

We now use the property on the k^{th} sum, yielding:

$$\sum_{i=1}^{k+1}(i+1) = \frac{k(k+3)}{2} + ((k+1)+1) = k+2+\frac{k^2+3k}{2}$$
$$= \frac{2k+4+k^2+3k}{2} = \frac{k^2+5k+4}{2} = \frac{(k+1)(k+4)}{2}$$

Just as we had hoped.

3) ***Conclusion:*** We know at the start that:

$$\sum_{i=1}^{1}(i+1) = (1+1) = 2 = \frac{1(1+3)}{2}$$

Our induction step shows then that:

$$\sum_{i=1}^{2}(i+1) = \frac{2(2+3)}{2} \Rightarrow \sum_{i=1}^{3}(i+1) = \frac{3(3+3)}{2} \Rightarrow \ldots \Rightarrow \sum_{i=1}^{n}(i+1) = \frac{n(n+3)}{2} \Rightarrow \ldots$$

That is we can continue on as far as we like, up to any n-value. So our relation holds for all n.

Let's move on, now, to a second example using sequences.

Example #2

Show, using induction, that the sequence defined by $a_{n+1} = \left(a_n - 2\right) \quad a_1 = 5$ is decreasing.

(*Note: this is a recursive definition of a sequence of numbers, so each step in the sequence depends on the previous ones.*)

Solution:

Again, that this is true it trivially obvious, but let's belabour the point and use the new technique.

Before we start, we have to understand what decreasing means as a mathematical statement. That is, we say:

$$a_1 \geq a_2 \geq a_3 \geq a_4 \geq a_5 \geq \ldots$$

Which of course means if we look at any two successive numbers in the sequence, we have a nice inequality relation. For example:

$$a_3 \geq a_4, \text{ or } a_7 \geq a_8 \ \ldots \text{ or in general, } a_n \geq a_{n+1}$$

So if our inequality expression, $a_n \geq a_{n+1}$ holds for all *n,* we'll have a decreasing sequence.

Now let's prove it.

1) *Base case*:

Does our first step have the correct inequality?

$$a_1 = 5 \ (Given) \quad a_2 = 5 - 2 = 3 \quad \Rightarrow \quad \text{Yes, in fact } a_1 \geq a_2$$

2) *Induction Step:*

If we know that our k^{th} inequality holds, that is, if we know $a_k \geqslant a_{k+1}$, then can we show the next inequality, $a_{k+1} \geqslant a_{k+2}$, also holds? Well, we'll see if we can construct the required expression, using the assumed expression.

$$a_k \geq a_{k+1} \quad \Rightarrow \quad a_k - 2 \geq a_{k+1} - 2 \quad \Rightarrow \quad a_{k+1} \geq a_{k+2}$$

as required.

Notice here we've used the often handy method of using our recursive relation to rewrite our assumption into a relation involving the required *a*-terms.

3) ***Conclusion:***

From our base case, we know $a_1 \geq a_2$, and using our induction we can conclude:

$$a_1 \geq a_2 \Rightarrow a_2 \geq a_3 \Rightarrow a_3 \geq a_4 \Rightarrow a_4 \geq a_5 \Rightarrow a_5 \geq a_6 \ \ldots$$

As before, then, this progresses logically to cover any n value we choose, so we get

$$a_n \geq a_{n+1}$$

for all possible n.

Now, let's look at a typical textbook problem from Stewart's *Calculus: Early Transcendentals* where we apply this induction logic to a larger problem.

(***Notice:*** *This uses some concepts related to limits, taken from Chapter 11.1, Sequences.*)

Question (#68, *Chap.* 11.1)

A sequence $\{a_n\}$ is given by $a_1 = 2,\ a_{n+1} = \sqrt{2 + a_n}$

a) By induction or otherwise show that $\{a_n\}$ is increasing and bounded above by 3.
Hint: Apply "Theorem 12" to show that $\lim_{n\to\infty} a_n$ exits.

b) Find $\lim_{n\to\infty} a_n$.

Solution

a) We want to show that $\{a_n\}$ is **both** *increasing* and *bounded above by 3*.
We can prove this by the process of "induction on n" discussed above.

We could show the "increasing" aspect, and then the "bounded" aspect by separate inductions, but why bother? Both follow from the same logic, using the same equations, so why not do them at the same time.

To show that $\{a_n\}$ is increasing, and always less than 3, we must show:

$$a_n \leq a_{n+1} \leq 3 \ \text{ for all values of } n$$

Base Case

It's clear, at least initially this relation holds.

$$a_1 = \sqrt{2},\ \ a_2 = \sqrt{2 + a_1} = \sqrt{2 + \sqrt{2}}\text{ , so clearly } a_1 \le a_2 \le 3,$$

So we can see that, at least for $n = 1$, our inequalities hold.

Induction Step

Now, let's say we know that $a_n \le a_{n+1} \le 3$ for some particular value of n. We can use our series-defining relationship $\left(a_{n+1} = \sqrt{2 + a_n}\right)$ to show one step leads to the next.

$$\begin{aligned} & a_n \le a_{n+1} \le 3 \\ \Rightarrow\ & 2 + a_n \le 2 + a_{n+1} \le 2 + 3 = 5 \\ \Rightarrow\ & \sqrt{2 + a_n} \le \sqrt{2 + a_{n+1}} \le \sqrt{5} \le 3 \\ \Rightarrow\ & a_{n+1} \le a_{n+2} \le 3 \end{aligned}$$

Since we know already that $a_1 \le a_2 \le 3$, we can, as before, apply the result of the induction step to conclude that $a_n \le a_{n+1} \le 3$ for all possible n.

Theorem 12 from the text says that if a sequence is *bounded* and *monotonic*, it <u>must</u> converge. Here we have shown that $\{a_n\}$ is increasing, and whatever else, each a_n is always less than 3. So our sequence $\{a_n\}$ must converge to some value.

For the purpose of the next section, let's call this limit, L.

b) This actually is quite straight forward: We just take the limit of both sides of our original expression. We know that $\lim_{n\to\infty} a_n = L$. So if we take the limit of both sides of our general expression?

$$\begin{aligned} & \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty}\sqrt{2 + a_n} \\ \Rightarrow\ & L = \sqrt{\lim_{n\to\infty}(2 + a_n)} = \sqrt{2 + \lim_{n\to\infty} a_n} = \sqrt{2 + L} \\ \Rightarrow\ & L = \sqrt{2 + L} \\ \Rightarrow\ & L^2 - L - 2 = 0 \\ \Rightarrow\ & L = 2, -1 \end{aligned}$$

Clearly, $L = -1$ doesn't work as a solution, so our limit is $L = 2$.

Just a Little Bit More:

There's a couple of important things to note about our solution to part ***b)*** here.

First, students are often perplexed by the idea that if $\lim_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} a_{n+1} = L$ as well. The sequence $\{a_n\}$ is, in essence, just a list of values, a_1, a_2, a_3.... If this path of values leads to a number, L, then in our limits, a_n and a_{n+1} travel this same path, one slightly ahead of the other, and so ultimately get to the same destination.

Secondly, this limit calculation is *not* sufficient to show a limit exists, only that if there is a limit, it gives a list of possible values. The entire calculation is predicated on the existence of the value L. If L doesn't exist, its results are pure fantasy.

Lastly, notice that, by the construction of our limit calculation, this list of possible limits we calculate is also a list of the fixed point of our mapping that creates a_{n+1} from a_n.

Although not directly addressed in this course, one finds that typically, either these fixed points are stable (so a starting a_1 value near them means we approach, or are “attracted” to that value) or are unstable, (so a starting value near them move away, or are “repelled” from that value). We get all possible fixed points as potential limits because our process of taking the limit has no information as to our possible starting point, and, without a starting point for context, our process has no means to distinguish between them.

Stability and fixed points of discrete mappings have some very fun properties, and even lead into the study of chaos! You can see more of this in Math 3DC3, our Chaotic Dynamics course.