Math 1B03 Term 2/1ZC3

Sample Exam

Name:_______________________________________
(Last Name) (First Name)

Student Number:____________________ **Tutorial Number:**____________________

This exam consists of 44 multiple choice questions worth 1 mark each (no part marks). All questions must be answered on the COMPUTER CARD with an HB PENCIL. Marks will not be deducted for wrong answers (i.e., there is no penalty for guessing). You are responsible for ensuring that your copy of the test is complete. Bring any discrepancy to the attention of the invigilator. Calculators are NOT allowed.

1. Let **u**, **v**, and **w** be independent vectors in $\mathbb{R}^3$. Which of the following sets are also independent?
(i) $\{\mathbf{u}-\mathbf{v}, \mathbf{v}-\mathbf{w}, \mathbf{w}-\mathbf{u}\}$
(ii) $\{\mathbf{u}, \mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{v}+\mathbf{w}\}$
(iii) $\{\mathbf{u}-\mathbf{v}, \mathbf{v}-\mathbf{w}, \mathbf{u}+\mathbf{v}+\mathbf{w}, \mathbf{w}\}$
(a) (i) only **(b)** (ii) only **(c)** (i) and (ii) only **(d)** all of them **(e)** (ii) and (iii) only

2. The following set of vectors $\{(1,-1,1,-1),(2,0,1,0),(0,-2,1,-2)\}$
(a) Spans $\mathbb{R}^4$ but is not independent.
(b) Is independent, but does not span $\mathbb{R}^4$
(c) Is independent and spans $\mathbb{R}^4$
(d) Is not independent, and does not span $\mathbb{R}^4$
(e) None of the above.

3. Suppose that $W = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ where each $\mathbf{v}_i$ is in $\mathbb{R}^3$. Consider the following statements.
(i) If **x** is in W then $\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$ for some scalars c_1, c_2, c_3.
(ii) $3\mathbf{v}_1 - 2\mathbf{v}_2$ is in W.
(iii) W is a subspace of $\mathbb{R}^3$.

Which of the above statements are always true?
(a) (i) only
(b) (ii) only
(c) (ii) and (iii) only
(d) (i) and (ii) only
(e) (i), (ii), and (iii)

4. Let $\mathbf{u} = (0, -2, 2)$ and $\mathbf{v} = (1, 3, -1)$. Which of the following vectors are in span$\{\mathbf{u}, \mathbf{v}\}$?
(i) $(1, -1, 2)$
(ii) $(1, 1, 1)$
(iii) $(5, 3, 7)$
(a) all of them **(b)** (ii) only **(c)** (iii) only **(d)** (i) and (iii) only **(e)** (ii) and (iii) only

5. If $\{\mathbf{v}, \mathbf{w}\}$ is independent, find conditions on the scalars k_1 and k_2 so that the set $\{k_1\mathbf{v} + \mathbf{w}, \mathbf{v} + k_2\mathbf{w}\}$ is also independent.
(a) $k_1 + k_2 = 1$ **(b)** $k_1 + k_2 \neq 1$ **(c)** $k_1 \neq k_2$ **(d)** $k_1k_2 = 1$ **(e)** $k_1k_2 \neq 1$

6. Let $\mathbf{p} = 2 - x + x^2$. Find the coordinates of $\mathbf{p}$ with respect to the following basis of P_2 $\{1 + x, 1 + x^2, x + x^2\}$.
(a) $(1, -1, 2)$ **(b)** $(0, 2, -1)$ **(c)** $(0, 2, 2)$ **(d)** $(2, -1, 0)$ **(e)** $(-1, 1, 3)$

7. Let V be a vector space with dimension n. Consider the following statements.
(i) Every independent set in V is a basis for V
(ii) Every set in V that spans V must be independent
(iii) Every set in V with less than n vectors must be independent.
Which of the above statements is always true?
(a) (ii) and (iii) only **(b)** (iii) only **(c)** (ii) only **(d)** none of them **(e)** (i) only

8. Find the dimension of the following vector spaces.
(i) The set of all 2×2 skew-symmetric matrices
(ii) The set of all polynomials $a + bx + cx^2$ where $a = b + c$.
(a) 1 and 2 **(b)** 2 and 3 **(c)** 1 and 3 **(d)** 3 and 3 **(e)** 4 and 2

9. If A is a 4×4 matrix and the columns of A are linearly dependent then,
(a) every vector $\mathbf{b}$ in $\mathbb{R}^4$ is in the column space of A
(b) no vector $\mathbf{b}$ is in the column space of A
(c) The column vectors of A form a basis for $\mathbb{R}^4$
(d) None of the above

10. Let $\mathbf{u} = (1, -2, 1, 6)$ in $\mathbb{R}^4$, and let $W = \text{span}\{(1, 1, -1, 0), (1, 1, 0, 0)\}$. Compute $\text{proj}_W\,\mathbf{u}$.
(a) $(-\frac{1}{2}, 0, 1, \frac{1}{2})$ **(b)** $(-1, -\frac{1}{2}, \frac{1}{2}, 0)$ **(c)** $(-\frac{1}{2}, -1, 1, 0)$
(d) $(-\frac{1}{2}, -\frac{1}{2}, 1, 0)$ **(e)** $(\frac{1}{2}, -1, -\frac{1}{2}, 0)$

11. Find a basis of the following subspace of $\mathbb{R}^4$.
$W =$ all vectors of the form (a, b, c, d) where $a + b - c + d = 0$.
(a) $\{(1, 0, 0, -1), (0, 1, 0, -1), (0, 0, 1, 1)\}$
(b) $\{(1, 0, 0, -1), (0, 1, 0, -1)\}$
(c) $\{(1, 0, 0, -1), (0, 1, 0, -1), (0, 0, 1, -1), (0, 1, -1, 0)\}$
(d) $\{(1, 0, 0, -1), (0, 1, 0, -1), (0, 1, -1, 0)\}$
(e) $\{(1, 0, -1, 0), (0, 1, 0, -1), (0, 0, 1, -1)\}$

12. Find the dimension of the subspace of $\mathbb{R}^3$ spanned by the following set of vectors.
$\{(1, 5, 6), (2, 6, 8), (3, 7, -1), (4, 8, 12)]$
(a) 1 **(b)** 2 **(c)** 3 **(d)** 4 **(e)** 0

13. Decode the message AOJX given that it is a Hill cipher with enciphering matrix

$$\begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$$

(a) MATE **(b)** HILL **(c)** HELP **(d)** GOOD **(e)** MATH

14. Consider the following statements.
(i) Suppose that $W = \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ and that $A\mathbf{u}_i = \mathbf{b}$ for each i. If the vector $\mathbf{u}$ is in W then $A\mathbf{u} = \mathbf{b}$.
(ii) Let W be the set of all vectors $\mathbf{x}$ in $\mathbb{R}^n$ that are solutions to the equation $A\mathbf{x} = 0$. W is a subspace of $\mathbb{R}^n$.
Which of the above statements is always true?
(a) (i) only **(b)** (ii) only **(c)** (i) and (ii) **(d)** neither

15. Find a basis for the null space of A. $A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & -2 & 3 \\ -1 & 0 & -3 & 2 \end{bmatrix}$

(a) $\{(1, 0, -1), (2, 1, 0)\}$
(b) $\{(0, 2, -3, 1), (1, 2, -3, 0)\}$
(c) $\{(1, 2, -1, 4), (0, 1, -2, 3)\}$
(d) $\{(-1, 2, 0, 3), (2, 1, 0, -3)\}$
(e) $\{(2, -3, 0, 1), (-3, 2, 1, 0)\}$

16. Let A be a matrix with 4 rows and 7 columns. Then the column space of A
(a) is a subspace of $\mathbb{R}^4$
(b) has dimension 4
(c) is equal to the column space of A^T
(d) none of the above

For Questions 17-19, determine which of the following answers is correct for the given subset W of $\mathbb{R}^3$.
(a) W is a subspace
(b) W is closed under addition, but not closed under scalar multiplication
(c) W is closed under scalar multiplication, but not closed under addition
(d) W is not closed under scalar multiplication, and not closed under addition

17. $W =$ all vectors of the form $(a, 3b, c)$ where $a = c + 1$.
(a) **(b)** **(c)** **(d)**

18. $W =$ all vectors of the form $(2a, -b^2, -c)$
(a) **(b)** **(c)** **(d)**

19. Let **b** be a nonzero vector in $\mathbb{R}^4$ and let A be a 4×4 matrix.
Let $W =$ all vectors **x** in $\mathbb{R}^4$ that are solutions to the equation $A\mathbf{x} = \mathbf{b}$.
(a) **(b)** **(c)** **(d)**

20. Suppose that W is a subspace of a vector space V. Consider the following statements.
(i) If **u** is in W and $a\mathbf{u} - b\mathbf{v}$ is in W (where $b \neq 0$) then **v** is in W.
(ii) If **u** is in W and **v** is in W then $a\mathbf{u} - b\mathbf{v}$ is in W.

Which of the above statements are always true?
(a) (i) only
(b) (i) only
(c) (i) and (ii)
(d) neither of them

21. Let $W = \text{span}\{(1,1,1,1),\ (3,1,3,1),\ (6,2,4,0)\}$. Find an orthonormal basis of W using the Gram-Schmidt process.
(a) $\{(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}), (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})\}$
(b) $\{(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), (0, -\frac{1}{2}, 1, -\frac{1}{2}), (\frac{1}{\sqrt{6}}, 0, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}})\}$
(c) $\{(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), (0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0), (\frac{1}{\sqrt{2}}, 0, 0, -\frac{1}{\sqrt{2}})\}$
(d) $\{(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0, 0), (0, 0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})\}$
(e) $\{(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), (-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0), (0, \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}})\}$

22. Consider the following set of orthogonal vectors,

$$\mathbf{v}_1 = (1, -1, 2, -1),\ \mathbf{v}_2 = (-2, 2, 3, 2),\ \mathbf{v}_3 = (1, 2, 0, -1),\ \mathbf{v}_4 = (1, 0, 0, 1).$$

Let $\mathbf{u} = (3, 1, -2, 4)$. Find c such that $\mathbf{u} = a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3 + d\mathbf{v}_4$
(a) $\frac{5}{6}$ **(b)** $\frac{1}{6}$ **(c)** $\frac{1}{3}$ **(d)** $\frac{1}{2}$ **(e)** $\frac{2}{3}$

23. If A and B are both $n \times n$ invertible matrices, which of the following matrices is the inverse of $(A^{-1}B)^T$?
(a) $(B^{-1}A)^T$ **(b)** $(AB^{-1})^T$ **(c)** $B^T(A^T)^{-1}$ **(d)** $(A^T)^{-1}B^T$ **(e)** $(B^TA^T)^{-1}$

24. Consider the following statements.
(i) If **u** and **v** are orthogonal in $\mathbb{R}^3$ then $\|\mathbf{u}+\mathbf{v}\| = \|\mathbf{u}-\mathbf{v}\|$
(ii) $\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 = \frac{1}{4}\|\mathbf{u}+\mathbf{v}\|^2 + \frac{1}{4}\|\mathbf{u}-\mathbf{v}\|^2$ for all **u**, **v** in $\mathbb{R}^3$.
Which of the above statements is always true?
(a) neither **(b)** (i) only **(c)** (ii) only **(d)** (i) and (ii)

25. Recall that B is *similar* to A if there is an invertible matrix P such that $B = P^{-1}AP$.
Suppose that B is similar to A. Consider the following statements.
(i) A and B have the same determinant
(ii) B^{-1} is similar to A^{-1}

Which of the above statements are always true?
(a) (i) only
(b) (ii) only
(c) (i) and (ii)
(d) neither of them

26. A matrix P is called **orthogonal** if $PP^T = I$. Consider the following statements.
(i) If P is an orthogonal matrix then $2P$ is also orthogonal.
(ii) If P is an orthogonal matrix then $\det P = \pm 1$
Which of the above statements is always true?
(a) (i) ony **(b)** (ii) only **(c)** (i) and (ii) **(d)** neither

27. Let $A = \begin{bmatrix} 0 & 0 & a \\ 0 & b & 0 \\ a & 0 & 0 \end{bmatrix}$. Find the characteristic polynomial $p(\lambda)$ of A.
(a) $(\lambda + b)(\lambda + a)^2$ **(b)** $(\lambda - b)(\lambda + a)^2$ **(c)** $(\lambda - b)(\lambda - a)^2$
(d) $(\lambda - b)(\lambda - a)(\lambda + a)$ **(e)** $(\lambda + b)(\lambda - a)(\lambda + a)$

28. Consider the following statements.
(i) $\{(1,-1,2,3),\ (2,1,-1,1),(1,\ 8,\ -13,-12)\}$ is an independent set.
(ii) $\{(1,2,-1),\ (-1,1,2),(-5,-1,8)\}$ spans $\mathbb{R}^3$.
Which of the above statements is true?
(a) (i) only **(b)** (ii) only **(c)** (i) and (ii) **(d)** neither

29. Consider the triangle with vertices P, Q, and R. Which of the following is a right-angled triangle?
(a) $P(1,1,0), Q(1,0,1), R(1,-1,2)$ **(b)** $P(1,1,0), Q(1,0,1), R(1,2,2)$
(c) $P(1,1,0), Q(1,0,1), R(1,0,2)$ **(d)** $P(1,1,0), Q(1,0,1), R(1,1,3)$
(e) $P(1,1,0), Q(1,0,1), R(1,3,2)$

30. Find the shortest distance from the point $P(0,1,-1)$ to the line
$(x,y,z) = (1,1,0) + t(1,-1,-2)$.
(a) $\frac{1}{6}\sqrt{66}$ **(b)** $\frac{1}{6}\sqrt{65}$ **(c)** $\frac{4}{3}$ **(d)** $\frac{1}{6}\sqrt{62}$ **(e)** $\frac{1}{6}\sqrt{61}$

31. Find the equation of the plane containing the point $P(3,0,-1)$ and the line
$(x,y,z) = (2,1,3) + t(3,-1,-2)$.
(a) $2x - 6y + 2z = 4$ **(b)** $x + 5y - z = 4$ **(c)** $x + 6y - z = 4$
(d) $3x - 17y + 5z = 4$ **(e)** $16y - 4z = 4$

32. Consider the following matrix (where only the first row is given): $A = \begin{bmatrix} 3 & -2 \\ * & * \end{bmatrix}$.
If $\begin{bmatrix} 1+i \\ 2 \end{bmatrix}$ is an eigenvector of A, what is the corresponding eigenvalue?
(a) $2-2i$ **(b)** $2-i$ **(c)** $1+2i$ **(d)** $1-i$ **(e)** $3+i$

33. Consider the line through $P(1,2,3)$ that is parallel to $\mathbf{v} = (1,0,1)$. Which of the following planes does the line lie in?
(a) $x + 2y + 2z + 1 = 0$ **(b)** $3x + 2y - 3z + 2 = 0$ **(c)** $-2y - z + 1 = 0$
(d) $3x - y + z + 2 = 0$ **(e)** $2x + 2y + z - 3 = 0$

34. If A and B are $n \times n$ symmetric matrices, which of the following matrices are always symmetric?
(i) $A - B^T$
(ii) $A^T B - B^T A$
(a) (i) only **(b)** (ii) only **(c)** (i) and (ii) **(d)** neither

35. Consider the following matrices.

$$A = \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

B can be obtained from A by the following sequence of row operations on A:
1. Switch row 1 and row 2
2. Replace row 2 by (row 2 $-$ 2 $\times$ row 1)
Using the above sequence of row operations (in the above order), find an invertible matrix U such that $UA = B$.
(a) $\begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix}$ **(b)** $\begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$ **(c)** $\begin{bmatrix} 0 & 1 \\ 2 & -4 \end{bmatrix}$ **(d)** $\begin{bmatrix} -1 & 3 \\ 2 & -4 \end{bmatrix}$ **(e)** $\begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}$

36. Compute the determinant of the following matrix.

$$\begin{bmatrix} 0 & 1 & -1 & 0 \\ 3 & 0 & 0 & 2 \\ 0 & 1 & 2 & 1 \\ 5 & 0 & 0 & 7 \end{bmatrix}$$

(a) 0 **(b)** 5 **(c)** -33 **(d)** -17 **(e)** 8

37. Let A be a 2×2 matrix, with $\det A = 2$. Evaluate $\det(2\,\mathrm{adj}(A))$.
(a) 2 **(b)** 4 **(c)** 8 **(d)** 16 **(e)** 32

38. A square matrix P is called **idempotent** if $P^2 = P$. Let A and B be $n \times n$ idempotent matrices. Which of the following matrices are always idempotent?

(i) $A - B$
(ii) AB

(a) (i) only **(b)** (ii) only **(c)** (i) and (ii) **(d)** neither

39. Suppose that a matrix A has eigenvectors $\mathbf{x}_1 = (1, 2, 1)$, $\mathbf{x}_2 = (1, 0, -1)$, and $\mathbf{x}_3 = (1, -2, 1)$, with corresonding eigenvalues $\lambda_1 = 1, \lambda_2 = \frac{1}{2}$, and $\lambda_3 = 0$, respectively. Let $\mathbf{v} = \frac{1}{4}\mathbf{x}_1 - \frac{1}{4}\mathbf{x}_2 + \frac{9}{50}\mathbf{x}_3$. Find the constant b so that $A^5\mathbf{v} = a\mathbf{x}_1 + b\mathbf{x}_2 + c\mathbf{x}_3$.

(a) $-\frac{1}{1024}$ **(b)** $-\frac{1}{32}$ **(c)** $\frac{1}{32}$ **(d)** $-\frac{1}{128}$ **(e)** $\frac{1}{1024}$

40. Let z be a complex number. Which of the following statements is correct?
(a) $\overline{z} + z$, $(\overline{z} - z)i$, $\overline{z}z$ are all real numbers
(b) $\overline{z} + z$, $(\overline{z} - z)i$, $\overline{z}z$ all have modulus 1
(c) $\overline{z} + z$ and $\overline{z}z$ are real numbers, but $(\overline{z} - z)i$ is not a real number.
(d) If z is a complex number and $|z| = 1$, then $z = 1$ or $z = -1$.
(e) none of the above

41. Find all complex numbers z so that $z^3 = -8i$.
(a) $\sqrt{3} + i, -\sqrt{3} + i, -2i$ **(b)** $\sqrt{2} - i, -\sqrt{2} - i, 2i$ **(c)** $\sqrt{3} - i, -\sqrt{3} + i, -2i$
(d) $\sqrt{2} + i, -\sqrt{2} + i, 2i$ **(e)** $\sqrt{3} - i, -\sqrt{3} - i, 2i$

42. Find a matrix P which diagonalizes

$$A = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}.$$

(a) $\begin{bmatrix} -\frac{1}{2} & 1 \\ 1 & \frac{1}{2} \end{bmatrix}$ **(b)** $\begin{bmatrix} 3 & 1 \\ 1 & -3 \end{bmatrix}$ **(c)** $\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$ **(d)** $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ **(e)** $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

43. Let A be an $n \times n$ matrix. Suppose that there exists an invertible matrix P such that $P^{-1}AP = D$, where D is a diagonal matrix. Consider the following statements.
(i) $A^2 = P^2 D^2 (P^{-1})^2$
(ii) $A^2 = P^{-1} D^2 P$
Which of the above statements is always true?
(a) (i) only **(b)** (ii) only **(c)** (i) and (ii) **(d)** neither

44. Solve the following initial value problem,

$$\begin{aligned} y_1' &= 5y_1 - 2y_2 \\ y_2' &= 12y_1 - 5y_2 \end{aligned}, \qquad y_1(0) = 1,\ y_2(0) = 4$$

(a) $y_1 = -e^{-x} + 2e^{x}$, $y_2 = -2e^{-x} + 6e^{x}$ **(b)** $y_1 = -e^{x} + 2e^{-x}$, $y_2 = -2e^{x} + 6e^{-x}$ **(c)** $y_1 = -2e^{x} + 6e^{-x}$, $y_2 = -e^{x} + 2e^{-x}$
(d) $y_1 = -2e^{x} + 6e^{-x}$, $y_2 = -e^{x} + 2e^{-x}$ **(e)** $y_1 = 2e^{x} - e^{-x}$, $y_2 = -2e^{x} + 6e^{-x}$

Answers

1. b **2.** d **3.** e **4.** e **5.** e **6.** b **7.** d **8.** a **9.** d **10.** d
11. a **12.** c **13.** e **14.** b **15.** e **16.** a **17.** d **18.** b **19.** d **20.** c
21. a **22.** b **23.** a **24.** b **25.** c **26.** b **27.** d **28.** d **29.** e **30.** a
31. b **32.** c **33.** b **34.** a **35.** b **36.** c **37.** c **38.** d **39.** d **40.** a
41. e **42.** c **43.** d **44.** b