MATHEMATICS 1LS3 TEST 4

Day Class
Duration of Examination: 60 minutes
McMaster University, 20 March 2012

E. Clements

FIRST NAME (please print): SOL^{NS}

FAMILY NAME (please print):

Student No.:

THIS TEST HAS 8 PAGES AND 8 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE.

Total number of points is 40. Marks are indicated next to the problem number. Any non-graphing calculator is allowed.

USE PEN TO WRITE YOUR TEST. IF YOU USE A PENCIL YOUR TEST WILL NOT BE ACCEPTED FOR REMARKING (IF NEEDED).

You need to show work to receive full credit.

Problem	Points	Mark
1	7	
2	7	
3	6	
4	3	
5	4	
6	4	
7	3	
8	6	
TOTAL	40	

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1. Consider the function $f(x) = \dfrac{x^2}{2x^2+1}$.

(a) [2] Find the leading behaviour of $f(x)$ at 0 and at ∞.

$$f_0(x) = \frac{x^2}{1} = x^2$$

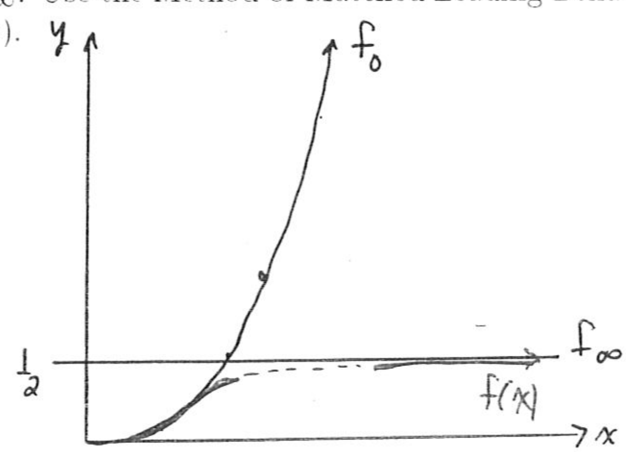
$$f_\infty(x) = \frac{x^2}{2x^2} = \frac{1}{2}$$

(b) [2] Find the limit of $f(x)$ as x approaches 0 and as x approaches ∞.

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} f_0(x) = \lim_{x\to 0} x^2 = 0$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} f_\infty(x) = \lim_{x\to\infty} \frac{1}{2} = \frac{1}{2}$$

(c) [3] Sketch f_0 and f_∞. Use the Method of Matched Leading Behaviours to sketch a rough graph of $f(x)$ on $[0, \infty)$.



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2. Find each limit using L'Hopital's rule.

(a) [2] $\lim_{x\to 0^+} \dfrac{1-\sqrt{1-x}}{x}$

$$\overset{L\hat{H}}{=} \lim_{x\to 0^+} \frac{\frac{1}{2\sqrt{1-x}}}{1}$$

$$= \frac{1}{2\sqrt{1-0}}$$

$$= \frac{1}{2}$$

(b) [3] $\lim_{x\to\infty} x\sin\left(\dfrac{1}{x}\right)$ $\quad (= \infty \cdot 0\ ?)$

$$= \lim_{x\to\infty} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} \quad \left(= \frac{0}{0} \rightarrow \text{use L'H}\right)$$

$$\overset{LH}{=} \lim_{x\to\infty} \frac{\cos\left(\frac{1}{x}\right)\cdot\cancel{\left(-\frac{1}{x^2}\right)}}{\cancel{-\frac{1}{x^2}}}$$

$$= \cos\left(\cancelto{0}{\frac{1}{\infty}}\right)$$

$$= 1$$

(c) [2] For $f(x) = 2\sqrt{x}$ and $g(x) = 25\ln x$, decide which of the functions approaches ∞ faster as x approaches ∞. Verify your answer using L'Hopital's Rule.

$f(x) \to \infty$ faster as $x \to \infty$.

To show this, show that $\lim_{x\to\infty} \dfrac{f(x)}{g(x)} = \infty$

$$\lim_{x\to\infty} \frac{2\sqrt{x}}{25\ln x} \overset{LH}{=} \lim_{x\to\infty} \frac{\frac{1}{\sqrt{x}}}{\frac{25}{x}} = \lim_{x\to\infty} \frac{1}{x^{1/2}}\cdot\frac{x}{25} = \lim_{x\to\infty} \frac{\sqrt{x}}{25} = \infty$$

$\therefore$ top f^n, $f(x) = 2\sqrt{x}$, approaches ∞ faster.

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3. A population is changing according to the dynamical system $p_{t+1} = 2.5p_t(1 - p_t)$.

(a) [2] Find all equilibrium points.

$$p^* = 2.5\,p^*(1-p^*)$$
$$p^* = 2.5p^* - 2.5(p^*)^2$$
$$2.5(p^*)^2 - 2.5p^* + p^* = 0$$
$$p^*\left[2.5p^* - 1.5\right] = 0$$
$$\Rightarrow p^* = 0 \quad \text{or} \quad p^* = \frac{1.5}{2.5} = 0.6$$

(b) [2] Describe how we could determine the stability of equilibrium points by cobwebbing.

Starting both to the left and to the right of the equilibrium (but close by), cobweb a few steps. If sol'ns from both sides approach equilibrium value, then eq'm is stable. If both move away, it is unstable. If one moves towards + the other moves away, it is half-stable.

(c) [2] Use the Stability Theorem to determine whether your equilibrium points from (a) are stable or unstable.

$$f(x) = 2.5x - 2.5x^2$$
$$f'(x) = 2.5 - 5x$$
$$f'(0) = 2.5 > 1 \Rightarrow p^* = 0 \text{ is UNSTABLE}$$
$$f'(0.6) = 2.5 - 5(0.6) = -0.5$$
$$|f'(0.6)| = |-0.5| = 0.5 < 1 \Rightarrow p^* = 0.6 \text{ is STABLE}$$

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4. [3] Verify that $y = \dfrac{\ln x + 5}{x}$ is a solution of the differential equation $x^2 y' + xy = 1$.

$$y' = \frac{\frac{1}{x} \cdot x - (\ln x + 5) \cdot 1}{x^2} = \frac{-\ln x - 4}{x^2}$$

$$\begin{aligned} LS &= x^2 y' + xy \\ &= x^2 \left[\frac{-\ln x - 4}{x^2} \right] + x \left[\frac{\ln x + 5}{x} \right] \\ &= -\ln x - 4 + \ln x + 5 \\ &= 1 \\ &= RS \checkmark \end{aligned}$$

$\therefore$ $y = \dfrac{\ln x + 5}{x}$ satisfies the D.E.

5. Describe the following events as initial value problems (i.e., in each case, write down a differential equation and an initial condition). Do not solve the equation.

(a) [2] The relative rate of change of a population of monkeys is 0.35 baby monkeys per month per monkey. Initially, there are 78 monkeys.

Let M rep. the # of monkeys @ time t.

$$\frac{\frac{dM}{dt}}{M} = 0.35 \Rightarrow \frac{dM}{dt} = 0.35M \qquad M(0) = 78$$

(b) [2] A pie, initially at the temperature of 20^oC, is put into a 300^oC oven. Let $T(t)$ be the temperature of the pie at time t. The temperature of the pie changes proportionally to the difference between the temperature of the oven and the temperature of the pie.

$$T(0) = 20^\circ C$$

$$\frac{dT}{dt} = k(300 - T(t))$$

6. A population P(t) of frogs in a pond is modelled by the autonomous differential equation

$$P'(t) = 0.2P(t)\left(1 - \frac{P(t)}{2400}\right)\left(1 - \frac{45}{P(t)}\right), \quad P(t) > 0.$$

[1] (a) For which value(s) of $P(t)$ will the population remain unchanged over time?

$P' = 0$ when $P = 2400$ or $P = 45$

[3] (b) For what values of $P(t)$ is the population increasing? For what values of $P(t)$ is the population decreasing? Interpret your results.

	0	45	2400 → P
P'	−	+	−
P	↘	↗	↘

∴ P is increasing when P is between 45 and 2400 frogs ($45 < P < 2400$) and decreasing otherwise.

If pop^n is less than 45 frogs, the pop^n will decrease (not enough variation genetically to have healthy offspring). If pop^n is greater than 2400, pop^n will decline as resources become limited. When $45 < P < 2400$, we have enough frogs to reproduce well + enough resources for all frogs ⇒ pop^n will increase.

7. [3] The rate at which new influenza cases occured in 2007 in Greater Toronto Area follows the formula $125.4e^{0.3t}$, where t is time in ~~days~~ years measured from 1 December 2007 (so $t = 0$ represents 1 December 2007.) It is known that there were 56 cases of influenza on 1 December 2007. Find the formula for the total number of influenza cases on ~~day~~ year t and use this to determine the total number of influenza cases on 1 December 2011.

Let I = # of influenza cases at time t.

$\frac{dI}{dt} = 125.4e^{0.3t}$ $\qquad I(0) = 56$ $\qquad$ t = 0 ... Dec. 1/07; t = 1460 ... Dec. 1/11.

$I(t) = \int 125.4e^{0.3t}\,dt$

$= \frac{125.4e^{0.3t}}{0.3} + C$

$I(0) = 56 \Rightarrow 56 = 418e^0 + C$

$\Rightarrow C = -362$

$\therefore I(t) = 418e^{0.3t} - 362$

$I(4) = 418e^{0.3(4)} - 362$

$\simeq 1026$ cases

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8. (a) [2] Compute $\int \left(\frac{1}{2\sqrt{x}} - 5\sqrt{x}\right) dx$ and $\int \left(2^x + x^2\right) dx$.

$$\int \left(\frac{1}{2\sqrt{x}} - 5\sqrt{x}\right) dx = \sqrt{x} - 5\,\frac{x^{3/2}}{3/2} + C$$

$$\int (2^x + x^2)\, dx = \frac{2^x}{\ln 2} + \frac{x^3}{3} + C$$

(b) [2] Compute $\int (1+7x)^4 dx$ and $\int \frac{1}{1+7x} dx$.

$$\int (1+7x)^4\, dx = \frac{(1+7x)^5}{5} \cdot \frac{1}{7} + C$$

$$\int \frac{1}{1+7x}\, dx = \frac{\ln|1+7x|}{7} + C$$

(c) [2] Is it true that $\int \underbrace{\frac{2x}{(1-x^2)^2}}_{f} dx = \underbrace{\frac{1}{1-x^2}}_{F(x)} + C$? Explain.

$$F' = \frac{0(1-x^2) - 1(-2x)}{(1-x^2)^2}$$

$$= \frac{2x}{(1-x^2)^2}$$

F(x) is the antiderivative of f(x).

YES!