MATHEMATICS 1LT3 TEST 2

Day Class
Duration of Test: 60 minutes
McMaster University

E. Clements

24 February 2016

FIRST NAME (please print): Sol^{N}S

FAMILY NAME (please print): ____________

Student No.: ____________

THIS TEST HAS 8 PAGES AND 9 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE.

Total number of points is 40. Marks are indicated next to the problem number in square brackets. You may use the McMaster standard calculator, Casio fx991 MS+, on this test.

USE PEN TO WRITE YOUR TEST. IF YOU USE A PENCIL, YOUR TEST WILL NOT BE ACCEPTED FOR REMARKING (IF NEEDED).

You need to show work to receive full credit, except for Multiple Choice.

Problem	Points	Mark
1	6	
2	6	
3	5	
4	3	
5	2	
6	3	
7	4	
8	5	
9	6	
TOTAL	40	

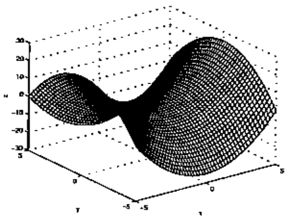
MATH 1LT3 * Test 2 * 24 February 2016

Name: ________________
Student No.: ________________

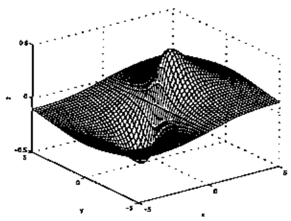
1. For part (a), write the letter corresponding to the graph of the function next to the equation in the space provided. For part (b), clearly circle the one correct answer.

(a) [3] Match the equation of each function with its graph below.

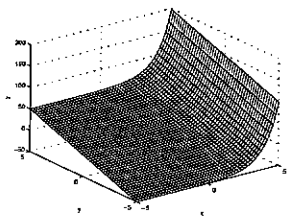
$f(x,y) = e^x + 10y$ C $g(x,y) = x^2 - y^2$ A $h(x,y) = \dfrac{x}{x^2+y^2+1}$ B



(A)



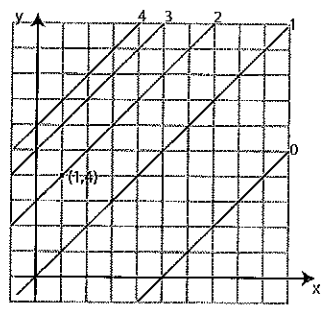
(B)



(C)

(b) [3] The contour map of a function $f(x,y)$ is given below. Which of the following are positive?

(I) $f_x(1,4)$ − (II) $f_y(1,4)$ + (III) $D_{\mathbf{v}} f(1,4)$ when $\mathbf{v} = \mathbf{i} + \mathbf{j}$
$= 0$



(A) none (B) I only (C) II only (D) III only
(E) I and II (F) I and III (G) II and III (H) all three

MATH 1LT3 * Test 2 * 24 February 2016

Name: ________________

Student No.: ________________

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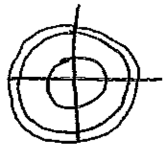
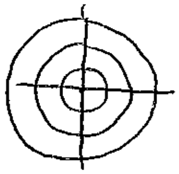
2. State whether each statement is true or false. Explain your reasoning.

(a) [2] The contour maps for $f(x,y) = x^2 + y^2$ and $g(x,y) = \sqrt{x^2+y^2}$ are identical.



for f: $x^2+y^2=k$ ... level curves are concentric circles with centre $(0,0)$ and radius $\sqrt{k}$



for g: $\sqrt{x^2+y^2}=k$ ... level curves are concentric circles with centre $(0,0)$ and radius k

$\Rightarrow x^2+y^2=k^2$

$\therefore$ FALSE.

(b) [2] If $f(x,y) \to L$ as $(x,y) \to (a,b)$ along every straight line through (a,b), then $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$.

FALSE.

In order for $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$, $f(x,y)$ must approach L along <u>every path</u> to (a,b) in the domain of f, not just along all linear paths to (a,b).

(c) [2] If $f(x,y) = \sin x + \sin y$, then $-\sqrt{2} \le D_{\mathbf{u}} f(x,y) \le \sqrt{2}$.

$f_x = \cos x \qquad f_y = \cos y$

$\nabla f = \cos x\,\hat{i} + \cos y\,\hat{j}$

$\|\nabla f\| = \sqrt{\underbrace{\cos^2 x}_{max=1} + \underbrace{\cos^2 y}_{max=1}}$

max. directional derivative $=$ max of $\|\nabla f\| = \sqrt{2}$

min. " " $= -(\text{max of } \|\nabla f\|) = -\sqrt{2}$

$\therefore$ TRUE

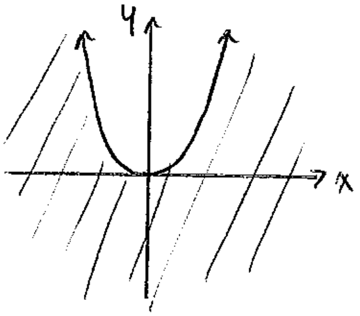
Name:

Student No.:

3. Consider the function $f(x, y) = e^{\sqrt{x^2-y}}$.

(a) [2] Find and sketch the domain of f.

$x^2 - y \geq 0 \Rightarrow y \leq x^2$



(b) [1] Determine the range of f. [You do not need to prove your result formally as we did in class.]

$\sqrt{x^2-y} \geq 0$

$\Rightarrow e^{\sqrt{x^2-y}} \geq e^0$

$\Rightarrow f(x,y) \geq 1$

$\therefore$ the range is $z \geq 1$.

(c) [2] Create a contour map for f. Include level curves corresponding to $k = 2$, $k = 6$, and $k = 10$.

level curves: $e^{\sqrt{x^2-y}} = k$ where $k \geq 1$.

$\sqrt{x^2-y} = \ln k$

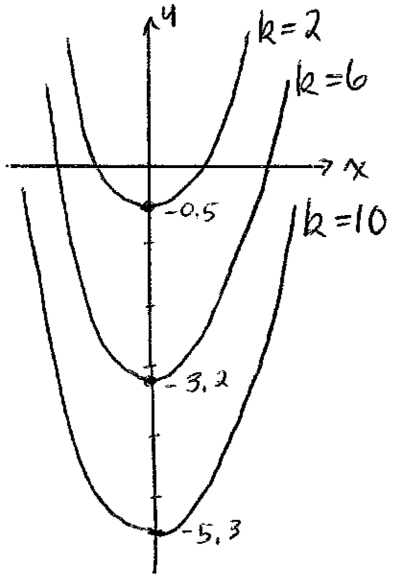
$x^2 - y = (\ln k)^2$

$y = x^2 - (\ln k)^2$

$k = 2 \Rightarrow y = x^2 - (\ln 2)^2 \approx x^2 - 0.5$

$k = 6 \Rightarrow y = x^2 - (\ln 6)^2 \approx x^2 - 3.2$

$k = 10 \Rightarrow y = x^2 - (\ln 10)^2 \approx x^2 - 5.3$



Name:

Student No.:

4. [3] Show that $\lim_{(x,y)\to(0,0)} \frac{y\sin x}{x^2+y^2}$ does not exist.

$f(0,y) = \frac{0}{y^2} = 0$

so, $z \to 0$ as $(x,y)\to(0,0)$ along $x=0$

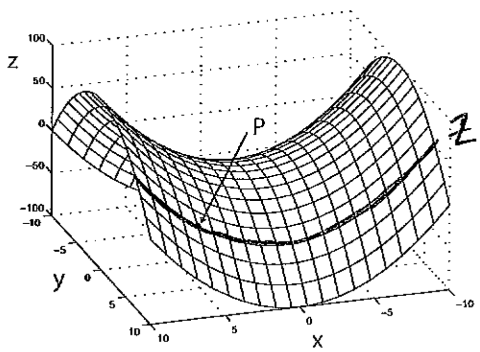
$f(x,x) = \frac{x\sin x}{x^2+x^2} = \frac{\sin x}{2x}$

$\lim_{x\to 0} \frac{\sin x}{2x} \overset{LH}{=} \lim_{x\to 0} \frac{\cos x}{2} = \frac{\cos 0}{2} = \frac{1}{2}$

so, $z \to \frac{1}{2}$ as $(x,y)\to(0,0)$ along $y=x$

$\therefore \lim_{(x,y)\to(0,0)} \frac{y\sin x}{x^2+y^2}$ DNE.

5. [2] Consider the graph of $z = x^2 - y^2$ and the point $P(6,7,-13)$ given below.



$z = f(x,7)$

Draw the curve $z = f(x,7)$ on the surface. What is the sign of $f_x(6,7)$?

y is held constant

$f_x(6,7) > 0$

Name:________________
Student No.: ________________

6. [3] A certain amount of PCBs (polychlorinated biphenyls, widely used as engine coolants) was released into Lake Ontario near Pickering. The function

$$c(x,t) = \frac{120}{t\sqrt{4\pi}} e^{-x^2/t}$$

models the concentration of PCBs (measured in milligrams of PCBs per litre of lake water) at a location x kilometres from Pickering, t days after the contamination occurred.

Find the partial derivative $c_x(1,4)$. Explain what your answer implies about the concentration of PCBs.

$$C_x = \frac{120}{t\sqrt{4\pi}} e^{-x^2/t}\left(-\frac{2x}{t}\right) = -\frac{240x}{t^2\sqrt{4\pi}} e^{-x^2/t}$$

$$C_x(1,4) = -\frac{240}{4^2\sqrt{4\pi}} e^{-1/4} = \frac{-15}{e^{1/4}\sqrt{4\pi}} \approx -3.3 \, \frac{mg/L}{km}$$

On day 4, the concentration at a location 1 km from the source decreases at a rate of approximately 3.3 mg/L per kilometre.

7. [4] Approximate the value of $\sin 0.75 \cos 3$ using the linearization of an appropriate function $f(x,y)$ at a suitable base point (a,b). Round your answer to two decimal places.

Let $f(x,y) = \sin x \cdot \cos y$.

Choose $(a,b) = (\frac{\pi}{4}, \pi)$ since $(\frac{\pi}{4}, \pi)$ is close to $(\frac{3}{4}, 3)$.

$f(\frac{\pi}{4}, \pi) = \sin\frac{\pi}{4} \cdot \cos\pi = -\frac{1}{\sqrt{2}}$

$f_x = \cos x \cdot \cos y \quad \cdots \quad f_x(\frac{\pi}{4}, \pi) = \cos\frac{\pi}{4} \cdot \cos\pi = -\frac{1}{\sqrt{2}}$

$f_y = -\sin x \cdot \sin y \quad . \quad f_y(\frac{\pi}{4}, \pi) = -\sin\frac{\pi}{4} \cdot \sin\pi = 0$

$\therefore L_{(\frac{\pi}{4},\pi)}(x,y) = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}(x - \frac{\pi}{4}) + 0(y - \pi)$

so, $\sin 0.75 \cos 3 \approx -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}(0.75 - \frac{\pi}{4}) \approx -0.68$

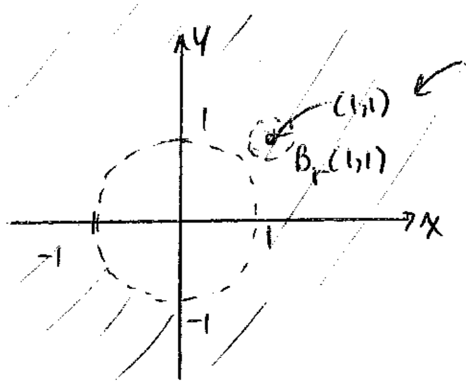
Name:
Student No.:

8. Let $f(x,y) = \ln(x^2 + y^2 - 1)$. domain : $x^2+y^2>1$

(a) [3] Compute f_x and f_y. Find and sketch the domain of f_x and f_y. Recall: The domain of a directional derivative of a function f must be a subset of the domain of f.

$$f_x = \frac{2x}{x^2+y^2-1} \qquad f_y = \frac{2y}{x^2+y^2-1}$$

$$x^2+y^2-1>0 \implies x^2+y^2>1$$



domain of f_x and f_y

(b) [2] Explain why f is differentiable at $(1,1)$.

f_x and f_y are rational functions and so they are continuous on their domains

We can find an open disk $B_r(1,1)$ centred at $(1,1)$ such that $B_r(1,1)$ is a subset of the domain of f_x and f_y for some $r>0$

since f_x and f_y are continuous on $B_r(1,1)$, f is differentiable at $(1,1)$.

Name:________________
Student No.: ________________

9. (a) [4] Find the directional derivative of the function $f(x, y) = \arctan\left(\frac{y}{x}\right)$ at the point (2,4) in the direction of the vector $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j}$.

$\|\vec{v}\| = \sqrt{3^2+4^2} = 5$ so $\hat{u} = \underbrace{\frac{3}{5}}_{u_1}\hat{i} + \underbrace{\frac{4}{5}}_{u_2}\hat{j}$

$f_x = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \left(-\frac{y}{x^2}\right) = \frac{-y}{x^2+y^2} \quad \ldots \quad f_x(2,4) = \frac{-4}{2^2+4^2} = -\frac{1}{5}$

$f_y = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \left(\frac{1}{x}\right) \cdot \frac{x}{x} = \frac{x}{x^2+y^2} \quad \ldots \quad f_y(2,4) = \frac{2}{2^2+4^2} = \frac{1}{10}$

$$
\begin{aligned}
D_{\vec{v}} f(2,4) &= f_x(2,4)u_1 + f_y(2,4)u_2 \\
&= -\frac{1}{5} \cdot \frac{3}{5} + \frac{1}{10} \cdot \frac{4}{5} \\
&= -\frac{1}{25}
\end{aligned}
$$

(b) [2] What is the maximum rate of change of $f(x, y) = \arctan\left(\frac{y}{x}\right)$ at the point (2,4)? In which direction does this occur?

$$
\begin{aligned}
\nabla f(2,4) &= f_x(2,4)\hat{i} + f_y(2,4)\hat{j} \\
&= -\frac{1}{5}\hat{i} + \frac{1}{10}\hat{j}
\end{aligned}
$$

$$
\begin{aligned}
\|\nabla f(2,4)\| &= \sqrt{\left(-\frac{1}{5}\right)^2 + \left(\frac{1}{10}\right)^2} \\
&= \frac{1}{\sqrt{20}}
\end{aligned}
$$

$\therefore$ The max. rate of change is $\frac{1}{\sqrt{20}}$ and occurs in the direction $\vec{v} = -\frac{1}{5}\hat{i} + \frac{1}{10}\hat{j}$.

THE END