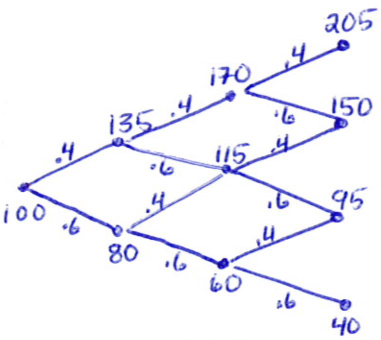
McMaster University
Math 1LT3E

ASSIGNMENT 8

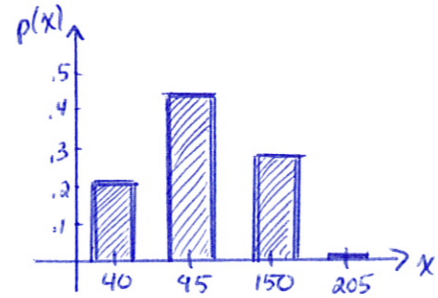
Sections 6, 7, 8, 13 and 14 in the Grey Module

1. Consider a population of tigers that grows according to $p_{t+1} = p_t + I_t$ where $I_t = 35$ with a 40% chance and $I_t = -20$ with a 60% chance, and t is measured in years. Suppose that initially there are 100 tigers. Define X to be the number of tigers after three years.

(a) Determine the probability mass function, $p(x)$, for X. Draw a histogram for $p(x)$.



x	$p(x) = P(X=x)$
40	$(.6)^3 = 0.216$
95	$3(.6)^2(.4) = 0.432$
150	$3(.4)^2(.6) = 0.288$
205	$(.4)^3 = 0.064$



(b) Compute the expected value and standard deviation for X.

$E(X) = \sum_x x\,p(x) = 40(.216) + 95(.432) + 150(.288) + 205(.064) = 106$ tigers

$Var(X) = \sum_x (x-106)^2 p(x) = (40-106)^2(.216) + (95-106)^2(.432) + (150-106)^2(.288) + (205-106)^2(.064)$

$= 2178$ tigers^2

Standard Deviation $= \sqrt{2178} \approx 47$ tigers

(c) What is the probability that the number of tigers will be within one standard deviation of the mean after three years?

$\mu \pm \sigma$

$P(106-47 \le X \le 106+47) = P(59 \le X \le 153)$

$= P(X=95) + P(X=150)$

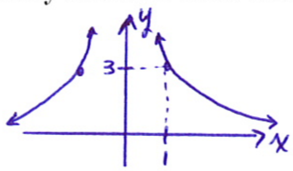
$= 0.432 + 0.288$

$= \underline{\underline{0.72}}$

2. Consider the function $f(x) = \frac{3}{x^4}$ on the interval $[1, \infty)$.

(a) Verify that $f(x)$ is a probability density function for a continuous random variable X.

(1) $f(x) > 0$ on $[1, \infty)$:

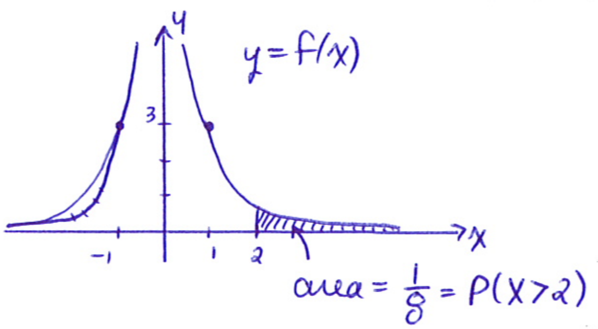


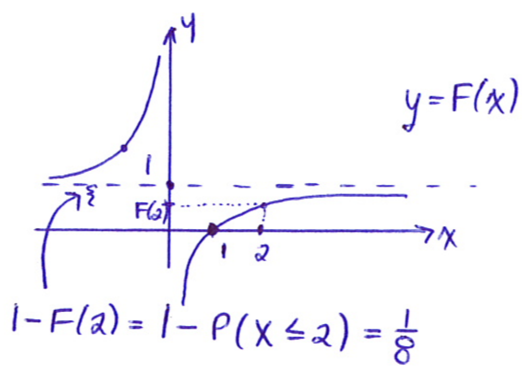
(2) $$\int_1^\infty f(x)\,dx = \lim_{T\to\infty} \int_1^T 3x^{-4}\,dx = \lim_{T\to\infty} \left[-\frac{1}{x^3}\right|_1^T = \lim_{T\to\infty}\left[1 - \frac{1}{T^3}\right] = 1 \checkmark$$

(b) Determine the corresponding cumulative distribution function, $F(x)$.

$$F(x) = \int_1^x f(t)\,dt = \left[-\frac{1}{t^3}\right|_1^x = 1 - \frac{1}{x^3}$$

(c) Graph both $f(x)$ and $F(x)$.





2. continued...

(d) Calculate $P(X > 2)$. Illustrate on both graphs in (c) what this number represents geometrically.

$$P(X>2) = 1-P(X\leq 2)$$
$$= 1-P(1\leq X\leq 2)$$
$$= 1-\int_1^2 f(x)\,dx$$
$$= 1-\left[-\frac{1}{x^3}\right|_1^2$$
$$= 1-\left[1-\frac{1}{2^3}\right]$$
$$= \frac{1}{8}$$

(e) Compute the expected value and standard deviation of X.

$$E(X) = \int_1^\infty x\cdot f(x)\,dx = \lim_{T\to\infty}\int_1^T 3x^{-3}\,dx = \lim_{T\to\infty}\left[-\frac{3}{2}\cdot\frac{1}{x^2}\right|_1^T = \dots$$
$$\dots = \lim_{T\to\infty}\left[\frac{3}{2}-\frac{3}{2T^2}\right] = \frac{3}{2}$$

$$E(X^2) = \int_1^\infty x^2 f(x)\,dx = \lim_{T\to\infty}\int_1^T 3x^{-2}\,dx = \lim_{T\to\infty}\left[-\frac{3}{x}\right|_1^T = \dots$$
$$\dots = \lim_{T\to\infty}\left[3-\frac{3}{T}\right] = 3$$

$$Var(X) = E(X^2)-[E(X)]^2$$
$$= 3-\left(\frac{3}{2}\right)^2$$
$$= 0.75$$

Standard Deviation of $X = \sqrt{0.75} \approx 0.87$

(f) What is the probability that a value X will be within one standard deviation of its mean?

$$\mu \pm \sigma = \frac{3}{2} \pm \sqrt{.75}$$

$$P\left(\frac{3}{2}-\sqrt{.75} \leq X \leq \frac{3}{2}+\sqrt{.75}\right) \approx P(0.63 \leq X \leq 2.37)$$
$$\approx P(1\leq X\leq 2.37) \quad \because P(X<1)=0$$
$$\approx \int_1^{2.37} f(x)\,dx$$
$$\approx \left[-\frac{1}{x^3}\right|_1^{2.37}$$
$$\approx \frac{1}{(1.)^3}-\frac{1}{(2.37)^3}$$
$$\approx \underline{\underline{0.92}}$$

3. Find the following using Table 14.4.

(a) Given that X is normally distributed with mean 3 and variance 4, find the probability that X is less than 4.1.

$$X \sim N(3, 2^2)$$

$$P(X < 4.1) = P\left(\frac{X-3}{2} < \frac{4.1-3}{2}\right)$$
$$= P(Z < 0.55)$$
$$= F(0.55)$$
$$= 0.708840$$

(b) Let $X \sim N(-2, 0)$; find $P(1 \leq X \leq 5)$.

typo... should be "9"

$$P(1 \leq X \leq 5) = P\left(\frac{1-(-2)}{3} \leq \frac{X-(-2)}{3} \leq \frac{5-(-2)}{3}\right)$$
$$\doteq P(1 \leq Z \leq 2.3)$$
$$= F(2.3) - F(1)$$
$$= 0.989276 - 0.841345$$
$$= 0.147931$$

(c) Suppose that $X \sim N(2, 144)$. Find an approximate value of x such that (i) $P(X \leq x) = 0.95$ and (ii) $P(X > x) = 0.3$.

(i) $P(Z \leq z) = 0.95 \Leftrightarrow F(z) = 0.95$ when $z \approx 1.65$

$$Z = \frac{X-2}{12} \Rightarrow X = 12Z + 2$$

so, $P(X \leq x) = 0.95$ when $x \approx 12(1.65) + 2 \approx 21.8$

(ii) $P(Z > z) = 1 - P(Z \leq z) = 1 - F(z) = 0.3 \Rightarrow F(z) = 0.7$ when $z \approx 0.55$

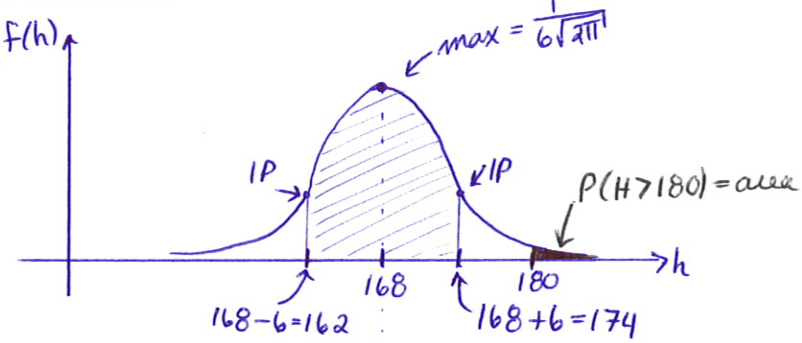
so, $P(X > x) = 0.3$ when $x \approx 12(0.55) + 2 \approx 8.6$

4. Let H represent the height of a university student. Assume that heights are approximately normally distributed with a mean of 168 cm and a standard deviation of 6 cm.

(a) Write the formula for the probability density function for H and sketch its graph, labelling the mean, maximum, and location of inflection points. Approximately what percentage of students are between 162 cm and 174 cm tall?

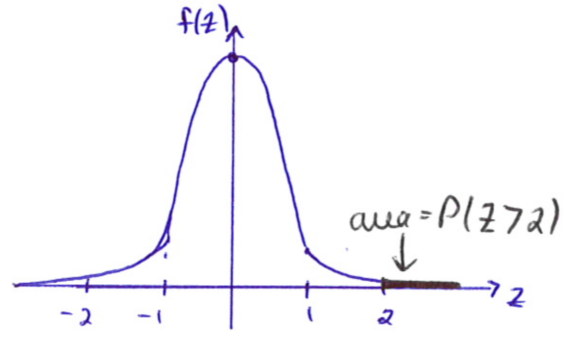
$H \sim N(168, 6^2)$

$$f(h) = \frac{1}{6\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{h-168}{6}\right)^2}$$



Approximately 68% of students are within one standard deviation of the mean.

(b) What is the probability that a randomly chosen student is taller than 180 cm? Shade the area representing this probability on your sketch in part (a). Sketch the probability density function for the standard normal random variable Z and shade in the area representing the equivalent probability.



$$\begin{aligned} P(H>180) &= 1 - P(H \le 180) \\ &= 1 - P\left(\frac{H-168}{6} \le \frac{180-168}{6}\right) \\ &= 1 - P(Z \le 2) \\ &= 1 - F(2) \\ &= 1 - 0.977250 \\ &= 0.02275 \end{aligned}$$

$\therefore$ The probability is about 2.3%.

5. (a) Determine the Taylor polynomial of degree 6 for $f(x) = e^{-\frac{x^2}{2}}$ near $x = 0$. (Hint: Find the Taylor polynomial of degree 3 for $g(x) = e^x$ then replace x by $-\frac{x^2}{2}$ in the formula).

$$T_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} \quad (\text{for } f(x) = e^x)$$

so, $e^x \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$ for x near 0

$$\Rightarrow e^{-\frac{x^2}{2}} \approx 1 + \left(-\frac{x^2}{2}\right) + \frac{\left(-\frac{x^2}{2}\right)^2}{2} + \frac{\left(-\frac{x^2}{2}\right)^3}{6} \approx \underbrace{1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{48}}_{T_6(x)} \text{ for } x \text{ near } 0$$

(b) Use your approximation in part (a) to estimate $P(0 \le Z \le 0.5)$. Compare this to the value obtained using Table 14.4.

$$P(0 \le Z \le 0.5) = \int_0^{0.5} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\, dz$$

$$\approx \frac{1}{\sqrt{2\pi}} \int_0^{0.5} \left(1 - \frac{z^2}{2} + \frac{z^4}{8} - \frac{z^6}{48}\right) dz$$

$$\approx \frac{1}{\sqrt{2\pi}} \left[z - \frac{z^3}{6} + \frac{z^5}{40} - \frac{z^7}{336} \right|_0^{0.5}$$

$$\approx 0.19146224$$

$$\left.\begin{aligned} P(0 \le Z \le 0.5) &= F(0.5) - F(0) \\ &= 0.691462 - 0.5 \\ &= 0.191462 \end{aligned}\right\} \text{using Table 14.4}$$

THE END