1ZC3 Last Day — Span & Independence

Remember: A "L.C." or Linear Combination of vectors $\{\vec{v}_1, \dots \vec{v}_n\}$

has the form $a_1\vec{v}_1 + a_2\vec{v}_2 + \dots a_n\vec{v}_n = \sum a_i\vec{v}_i$

where a_i are real constant coefficients

The Span of a set of vectors $\{\vec{v}_1, \dots \vec{v}_n\}$ is $\text{Span}(\{\vec{v}_1 \dots \vec{v}_n\})$

is the set of all possible L.C. of $\{\vec{v}_1, \dots \vec{v}_n\}$

- Spans are always subspaces.

- If $S = \text{Span}(\{\vec{v}_1 \dots \vec{v}_n\})$, then $\{\vec{v}_1 \dots \vec{v}_n\}$ is a spanning set of S. (& very non-unique!)

A set of vectors $\{\vec{v}_1, \dots, \vec{v}_n\}$ is "L.I.", Linearly Independent

if $\quad a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n = \vec{0}$

iff $\quad a_1 = a_2 = a_3 = \cdots = a_n = 0$

equivalently: no $\vec{v}_i$ is a L.C. of the others!

Conversely if there is any $\vec{v}_i$ which is a L.C. of rest $\Rightarrow$ L.D.

ie Linearly Dependent

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eg. Are x^2-1, x^2+x, $x-2$ L.I. functions in $\mathbb{P}_2$?

Solution

$$a(x^2-1) + b(x^2+x) + c(x-2) = 0$$

LI iff $a=b=c=0$ only.

So solve?

$$\underbrace{(a+b)}_{\overset{\shortparallel}{0}}x^2 + \underbrace{(b+c)}_{\overset{\shortparallel}{0}}x + \underbrace{(-a-2c)}_{\overset{\shortparallel}{0}} = 0$$

"Two poly. equal iff coeff. equal!"

$$\left.\begin{aligned} a+b+0c &= 0 \\ 0a+b+c &= 0 \\ -1a+0b-2c &= 0 \end{aligned}\right\} \Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Yeah! Square coeff. matrix $\Rightarrow$ can shortcut!

If det (coeff matrix) $\neq 0 \Rightarrow$ invertible

$\Rightarrow$ only trivial soln!

(ie $a=b=c=0$ only!)

$$\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & -2 \end{vmatrix} \overset{R_3+R_1}{=} \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -2 \end{vmatrix} \overset{R_3-R_2}{=} \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -3 \end{vmatrix}$$

$= -3 \neq 0 \Rightarrow$ Independent.

Why LI?

We saw last day if $S = \mathrm{Span}(\{\vec{v}_1, \ldots \vec{v}_n\})$

& $\{\vec{v}_1 \ldots \vec{v}_n\}$ is L.D.

$\Rightarrow$ some vector is a L.C. of rest! say $\vec{v}_n$.

$$\Rightarrow \quad \mathrm{Span}(\{\vec{v}_1, \ldots \vec{v}_{n-1}\}) = \mathrm{Span}(\{\vec{v}_1 \ldots \vec{v}_n\})!$$

$\Rightarrow$ Can "throw out" redundant vector, ie ones that are L.C. of rest! Won't change span.

If $\{\vec{v}_1 \ldots \vec{v}_n\}$ finite! $\Rightarrow$ will get a L.I. span!

eg Let $S = \operatorname{Span}\left(\left\{ \begin{bmatrix}1\\1\\2\end{bmatrix} \begin{bmatrix}1\\0\\1\end{bmatrix} \begin{bmatrix}0\\1\\0\end{bmatrix} \begin{bmatrix}1\\0\\0\end{bmatrix} \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}\right)$

Find a L.I. set with same span!

Solution

$$\begin{bmatrix}1\\1\\2\end{bmatrix} = \begin{bmatrix}0\\1\\0\end{bmatrix} + \begin{bmatrix}1\\0\\0\end{bmatrix} + 2\begin{bmatrix}0\\0\\1\end{bmatrix} \Rightarrow \text{drop!}$$

$$\left\{ \begin{bmatrix}1\\0\\1\end{bmatrix} \begin{bmatrix}1\\0\\0\end{bmatrix} \begin{bmatrix}0\\1\\0\end{bmatrix} \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}$$

$$\begin{bmatrix}0\\0\\1\end{bmatrix} = -\begin{bmatrix}1\\0\\0\end{bmatrix} + \begin{bmatrix}1\\0\\1\end{bmatrix} \Rightarrow \text{drop!}$$

$$\left\{ \begin{bmatrix}1\\0\\1\end{bmatrix} \begin{bmatrix}1\\0\\0\end{bmatrix} \begin{bmatrix}0\\1\\0\end{bmatrix} \right\}$$ LI set! Can't get smaller.

L.I Spans are Special: Called Bases

If you are a basis every vector in span has a unique representation as a L.C.

Why? Say $\vec{u} = a_1\vec{v}_1 + \dots + a_n\vec{v}_n$

and $\vec{u} = b_1\vec{v}_1 + \dots + b_n\vec{v}_n$ $\Big\}$ not all $a_i = b_i$

Two distinct LC

by subtraction $\vec{0} = (a_1 - b_1)\vec{v}_1 + \dots + (a_n - b_n)\vec{v}_n$

$\Rightarrow$ non-zero coeff gives $\vec{0}$

contradiction! not possible if $\vec{v}$'s L.I

So all bases give unique L.C. for each vector in their span! (ie Unique co-ordinates!)

Say you have a finite element basis of a v space, V (ie a finite set of vectors span V and are L.I.) then no L.I. in V can have more vectors.

ie If basis has n vectors $\Rightarrow$ (number in any L.I set) $\leq n$

eg. Say $\{\vec{u}_1, \vec{u}_2\}$ is L.I. & Span V.

(basis of V)

Say $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} \subseteq V$ is <u>L.I.</u>

$$\left.\begin{aligned} \vec{v}_1 &= \overset{\#?}{v_{11}}\,\vec{u}_1 + \overset{\#?}{v_{12}}\,\vec{u}_2 \\ \vec{v}_2 &= \cdots\cdots \\ \vec{v}_3 &= \cdots\cdots \end{aligned}\right\} \text{ unique L.C. in term of } \vec{u}_1 \text{ \& } \vec{u}_2$$

$$a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_2 = \vec{0}$$

$$a(v_{11}\vec{u}_1 + v_{12}\vec{v}_2) + b(\quad) + c(\quad) = \vec{0}$$

$$(av_{11} + bv_{21} + cv_{31})\vec{u}_1 + (\quad)\vec{u}_2 = 0$$

<u>L.I. u's</u>

$$\Rightarrow \quad (a v_{11} + b v_{21} + c v_{31}) = 0 \leftarrow \vec{u}_1 \text{ coeff}$$
$$(a v_{12} + b v_{22} + \cdots) = 0 \leftarrow \vec{u}_2 \text{ coeff}$$
$$\cancel{(\qquad\qquad\qquad)} = 0$$

$$\text{2 rows}\begin{array}{c} \vec{u}_1 \\ u_2 \end{array}\left[\begin{array}{ccc} & & \\ & & \end{array}\right]\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\underbrace{v_1 \qquad v_2 \qquad v_3}_{\text{3 colm}}$$

If # col > # rows ⇒ soln. has a param!

⇒ ∞ of (a, b, c) values!

⇒ $\underline{V}$'s can't be LI ever!

Same happens if # $\underline{v\text{'s}}$ > # u's } ⇒ can't have

LI set   basis!   L.I sets bigger than basis!

EVER

⇒ all bases same size!   dim (V)   Our dimension!