MATHEMATICS 1LS3 TEST 3

Day Class
Duration of Examination: 60 minutes
McMaster University, 6 March 2012

E. Clements

FIRST NAME (please print): SOL^NS

FAMILY NAME (please print): GRADING

Student No.: GUIDE

THIS TEST HAS 8 PAGES AND 8 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE.

Total number of points is 40. Marks are indicated next to the problem number. Any non-graphing calculator is allowed.

USE PEN TO WRITE YOUR TEST. IF YOU USE A PENCIL YOUR TEST WILL NOT BE ACCEPTED FOR REMARKING (IF NEEDED).

You need to show work to receive full credit.

Problem	Points	Mark
1	6	
2	8	
3	5	
4	2	
5	7	
6	4	
7	3	
8	5	
TOTAL	40	

Name:______________
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1. (a) [2] Find $f'(1)$ if $f(x) = 2^{\ln x} + (\ln x)^5 + (\ln 5)^2$.

$$f'(x) = \underbrace{2^{\ln x} \cdot \ln 2 \cdot \frac{1}{x}}_{1} + \underbrace{5(\ln x)^4 \cdot \frac{1}{x}}_{1}$$

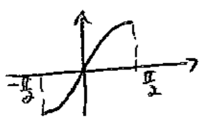
$$f'(1) = 2^{\ln 1} \cdot \ln 2 \cdot \frac{1}{1} + 5(\underbrace{\ln 1}_{=0})^4 \cdot \frac{1}{1}$$

$$= \ln 2$$

2 marks

(b) [2] Given $f(x) = \tan(\arcsin x)$, find $f'(0)$.

$$f'(x) = \underbrace{\sec^2(\arcsin x)}_{1} \cdot \frac{1}{\sqrt{1-x^2}} \Big\} \frac{1}{2}$$



$\sin 0 = 0$
$\Rightarrow \arcsin 0 = 0$

$$f'(0) = \sec^2(\arcsin 0) \cdot \frac{1}{\sqrt{1-0^2}}$$

$$= \sec^2 0$$

$$= \frac{1}{\cos^2 0}$$

$$= 1$$

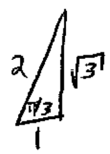
$\frac{1}{2}$

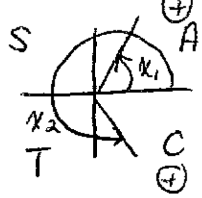
(c) [2] For what value(s) of x does the graph of $y = x - 2\sin x$ have a horizontal tangent?

$y' = 1 - 2\cos x$

$y' = 0 \Rightarrow \cos x = \frac{1}{2}$ 1

$x' = \frac{\pi}{3}$



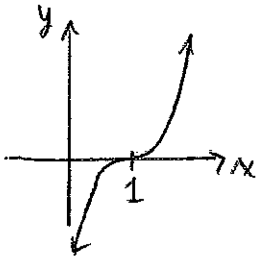


when $x = \frac{\pi}{3} + 2\pi k$
or $x = \frac{5\pi}{3} + 2\pi k$ 1

where $k \in \mathbb{Z}$

$x_2 = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$

2. (a) [2] Give an example (sketch a graph, or write down a formula) of a continuous function f such that $f'(1) = 0$, but f does not have an extreme value at $x = 1$.



$f(x) = (x-1)^3$

(b) [4] Find the critical numbers of $f(x) = x + \dfrac{1}{x}$ and use the second derivative test to classify them as local maxima, local minima, or state that the test is inconclusive.

$f' = 1 - \dfrac{1}{x^2} = \dfrac{x^2-1}{x^2}$

$f' = 0$ when $x^2 - 1 = 0 \Rightarrow x = \pm 1$

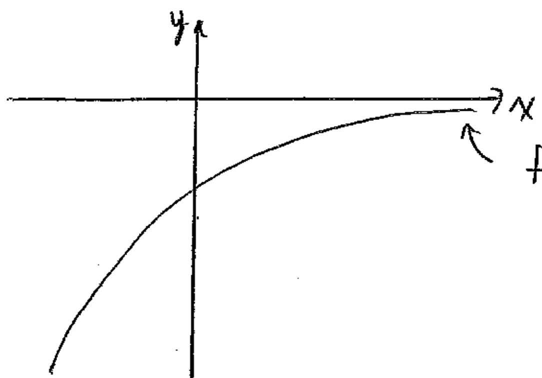
f' dne when $x^2 = 0 \Rightarrow x = 0$ but not a crit # since not in domain of f

$f'' = \dfrac{2}{x^3}$

$f''(1) = \dfrac{2}{(1)^3} = 2 > 0 \Rightarrow f \cup \Rightarrow f$ has a local min at $x = 1$

$f''(-1) = \dfrac{2}{(-1)^3} = -2 < 0 \Rightarrow f \cap \Rightarrow f$ has a local max at $x = -1$.

(c) [2] Draw the graph of a function that is defined for all x, negative for all x, but whose derivative is positive for all x.



$f(x) = -e^{-x}$, for example.

$f' = e^{-x} > 0 \quad \forall x$

Name:________________
Student No.: ________________

3. (a) [3] Determine the intervals on which the function $f(x) = xe^{-x^2}$ is increasing and the intervals on which it is decreasing.

$f' = 1 \cdot e^{-x^2} + x \cdot e^{-x^2}(-2x) = e^{-x^2}(1-2x^2)$

$f' = 0$ when $1 - 2x^2 = 0 \Rightarrow x = \pm \frac{1}{\sqrt{2}}$

f' dne? no x

	-1	$-\frac{1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$	1
f'	$-$		$+$		$-$
f	↘		↗		↘

(b) [2] Where does f have extreme values? What are the extreme values of the f?

f has a local min. at $x = \frac{-1}{\sqrt{2}}$. Local min. value: $f(-\frac{1}{\sqrt{2}}) = \frac{-1}{\sqrt{2e}}$

f has a local max. at $x = +\frac{1}{\sqrt{2}}$. Local max. value: $f(\frac{1}{\sqrt{2}}) = \frac{1}{\sqrt{2e}}$

4. [2] Given that $f'(5) = 0$, which of the following statements about f are true?

(I) 5 is a critical number of f T
(II) f has either a local maximum or minimum at $x = 5$ F (see #2(a))
(III) f is continuous at $x = 5$ T (diff $\Rightarrow$ cont)

(I) none (II) I and II (III) I and III (IV) I, II, and III

5. (a) [2] Let $f(x) = x^2 \ln x$. Show that $f''(x) = 2\ln x + 3$.

$f' = 2x\ln x + x^2 \cdot \frac{1}{x} = \underline{2x \ln x} + x$

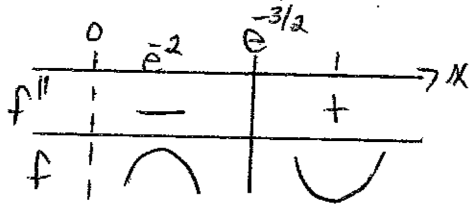
$f'' = 2 \cdot \ln x + 2x \cdot \frac{1}{x} + 1$

$= 2\ln x + 3$

(b) [2] Determine where the graph of the function $f(x) = x^2 \ln x$ is concave up and where it is concave down.

$f'' = 0$ when $2\ln x + 3 = 0 \Rightarrow \ln x = -\frac{3}{2} \Rightarrow x = e^{-3/2}$

f'' dne when $x \le 0$



$2\ln e^{-2} + 3 = -1 < 0$

$2\ln 1 + 3 = 3 > 0$

(c) [1] State the coordinates of any inflection points.

Inflection point when $x = e^{-3/2}$

$f(e^{-3/2}) = (e^{-3/2})^2 \cdot \ln e^{-3/2} = e^{-3} \cdot \left(-\frac{3}{2}\right) = \frac{-3}{2e^3}$

(d) [2] If $f''(x) = 0$, does it mean that x is an inflection point? Explain.

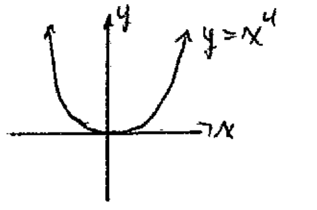
Not necessarily!

For example, $f(x) = x^4$.

$f' = 4x^3$

$f'' = 12x^2$

$f''(0) = 0$ but 0 is not an I.P. since the graph of $f(x)$ is concave up everywhere. (except at 0).



Name:

Student No.:

6. (a) [1] State the Extreme Value Theorem. Make sure to state all assumption(s) and conclusion(s).

If $f(x)$ is continuous on a closed interval $[a,b]$, then f has both an abs. max. and an abs. min on $[a,b]$.

(b) [3] Find the absolute extreme values of the function $f(x) = x^{\frac{2}{3}}$ on the interval $[-1, 8]$.

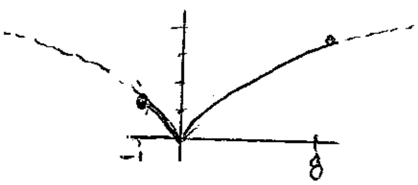
← domain is $\mathbb{R}$

$f(x) = (\sqrt[3]{x})^2$ or $= \sqrt[3]{x^2}$

$f' = \frac{2}{3}x^{-1/3} = \frac{2}{3x^{1/3}}$

$f' = 0$ no where

f' dne when $3x^{1/3} = 0 \Rightarrow \boxed{x=0}$ ← critical #



x	$f(x)$
-1	$(-1)^{2/3} = (\sqrt[3]{-1})^2 = (-1)^2 = 1$
0	$(0)^{2/3} = 0$ ← ABS. MIN
8	$(8)^{2/3} = (\sqrt[3]{8})^2 = (2)^2 = 4$ ← ABS. MAX

7. [3] The size of a population of bacteria introduced to a nutrient grows according to

$$N(t) = 5000 + \frac{30,000}{100 + t^2} = 5000 + 30\,000\,(100+t^2)^{-1}$$

where N represents the number of bacteria at time t, in hours. Find the maximum size of this population for $t \geq 0$.

$$N' = \frac{0 \cdot (100+t^2) - 30000(2t)}{(100+t^2)^2} = \frac{-60\,000t}{(100+t^2)^2}$$

$N' = 0$ when $-60\,000t = 0 \Rightarrow t = 0$

N' dne when $(100+t^2)^2 = 0$ (never happens)

	0	1
N'		$-$
N		↘

Since N is decreasing for all $t > 0$, N is at a max. when $t = 0$.

$$N(0) = 5000 + \frac{30\,000}{100+0^2} = 5300.$$

$\therefore$ The max. size of this pop^n is 5300 bacteria.

Name:________________
Student No.: ____________

8. (a) [1] Give a formula for the Taylor polynomial $T_3(x)$ of a function $f(x)$ near $x = a$.

$$T_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

(b) [2] Find the Taylor polynomial $T_3(x)$ for the function $f(x) = \ln x$ centred at $x = 1$.

$f' = \frac{1}{x}$ $\quad f'(1) = 1$ $\quad f(1) = 0$

$f'' = -\frac{1}{x^2}$ $\quad f''(1) = -1$

$f''' = \frac{2}{x^3}$ $\quad f'''(1) = 2$

$$\therefore T_3(x) = 1(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$$

(c) [2] Use the polynomial found in part (b) to estimate ln 0.9 and compare this to the actual value. How could we obtain a more accurate approximation?

$$\begin{aligned} \ln 0.9 &\approx 1(0.9-1) - \tfrac{1}{2}(0.9-1)^2 + \tfrac{1}{3}(0.9-1)^3 \\ &\approx (-0.1) - \tfrac{1}{2}(-0.1)^2 + \tfrac{1}{3}(-0.1)^3 \\ &\approx -0.1053\underset{\sim}{3} \end{aligned}$$

calculator value: $\ln 0.9 \approx -0.1053\underset{\sim}{6}$

We could obtain a more accurate approximation by using a higher degree Taylor polynomial.

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ROUGH WORK

THE END