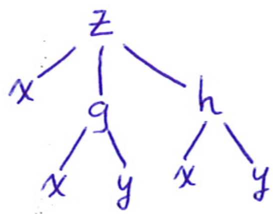
McMaster University
Math 1LT3E

ASSIGNMENT 6

Sections 6, 7, 9, and 10 in the Red Module

1. Suppose that $z = F(x, g(x, y), h(x, y))$. Sketch a tree diagram and find formulas for $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.



$$\frac{\partial z}{\partial x} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial g}\cdot\frac{\partial g}{\partial x} + \frac{\partial F}{\partial h}\cdot\frac{\partial h}{\partial x}$$

$$\frac{\partial z}{\partial y} = \frac{\partial F}{\partial g}\cdot\frac{\partial g}{\partial y} + \frac{\partial F}{\partial h}\cdot\frac{\partial h}{\partial y}$$

2. Wheat production W in a given year depends on the average temperature T and the annual rainfall R. Scientists estimate that the average temperature is rising at a rate of 0.15^oC/year and rainfall is decreasing at a rate of 0.1 cm/year. They also estimate that, at current production levels, $\frac{\partial W}{\partial T} = -2$ and $\frac{\partial W}{\partial R} = 8$.

(a) What is the significance of the signs of these partial derivatives?

$\frac{\partial W}{\partial T} = -2 \Rightarrow$ wheat production DECREASES as average temperature increases.

$\frac{\partial W}{\partial R} = 8 \Rightarrow$ wheat production INCREASES as annual rainfall increases.

(b) Estimate the current rate of change of wheat production, $\frac{dW}{dt}$.

$$\frac{dW}{dt} = \frac{\partial W}{\partial T}\cdot\frac{dT}{dt} + \frac{\partial W}{\partial R}\cdot\frac{dR}{dt}$$

$$= (-2)(+0.15) + (8)(-0.1)$$

$$= -1.1 \text{ units of wheat/year}$$

So, wheat production is decreasing at about 1.1 units per year.

3. Use the chain rule to find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

(a) $z = y^2 e^{-x}$, $x = 2s - 5t$, $y = -s - 4t$

$$\frac{dz}{ds} = \frac{dz}{dy}\cdot\frac{dy}{ds} + \frac{dz}{dx}\cdot\frac{dx}{ds}$$
$$= (2ye^{-x})(-1) + (-y^2e^{-x})(2) = -2ye^{-x}(1+y)$$
$$= -2(-s-4t)e^{-(2s-5t)}(1 \; -s-4t) = 2(s+4t)(1-s-4t)e^{5t-2s}$$

$$\frac{dz}{dt} = \frac{dz}{dy}\cdot\frac{dy}{dt} + \frac{dz}{dx}\cdot\frac{dx}{dt}$$
$$= (2ye^{-x})(-4) + (-y^2e^{-x})(-5) = ye^{-x}(-8+5y)$$
$$= (-s-4t)e^{-(2s-5t)}(-8+5(-s-4t)) = (s+4t)(8+5s+20t)e^{5t-2s}$$

(b) $z = \dfrac{ab-1}{b^2+1}$, $a = 3s$, $b = st$

$$\frac{dz}{ds} = \frac{dz}{da}\cdot\frac{da}{ds} + \frac{dz}{db}\cdot\frac{db}{ds}$$
$$= \frac{b}{b^2+1}\cdot 3 + \frac{a(b^2+1)-(ab-1)(2b)}{(b^2+1)^2}\cdot t = \frac{3b(b^2+1)+(a-ab^2+2b)t}{(b^2+1)^2}$$
$$= \frac{3st(s^2t^2+1)+(3s-3s^3t^2+2st)t}{(b^2+1)^2}$$
$$=$$

$$\frac{dz}{dt} = \frac{dz}{da}\cdot\frac{da}{dt} + \frac{dz}{db}\cdot\frac{db}{dt}$$
$$= \frac{b}{b^2+1}\cdot 0 + \frac{a-ab^2+2b}{(b^2+1)^2}\cdot s$$
$$= \frac{3s^2-3s^4t^2+2s^2t}{(s^2t^2+1)^2}$$
$$= \frac{s^2\cdot(3-3s^2t^2+2t)}{(s^2t^2+1)^2}$$

4. Find all second-order partial derivatives of $f(x, y) = \dfrac{xy}{x^2+1}$.

$$f_x = \frac{y(x^2+1) - xy(2x)}{(x^2+1)^2} = \frac{y(1-x^2)}{(x^2+1)^2}$$

$$f_y = \frac{x}{x^2+1}$$

$$f_{xx} = \frac{-2xy(x^2+1)^{\cancel{2}} - y(1-x^2)2\cancel{(x^2+1)}(2x)}{(x^2+1)^{\cancel{4}3}} = \frac{-2xy[x^2+1+2(1-x^2)]}{(x^2+1)^3}$$

$$= -\frac{2xy(3-x^2)}{(x^2+1)^3}$$

$$f_{xy} = \frac{1-x^2}{(x^2+1)^2} \quad (= f_{yx})$$

$$f_{yy} = 0$$

5. (a) Compute the quadratic approximation of the function $f(x, y) = x^2 \arctan(y)$ at $(1, 0)$.

$f_x = 2x \arctan y \quad \ldots \quad f_x(1,0) = 0$

$f_y = x^2 \cdot \frac{1}{1+y^2} \quad \ldots \quad f_y(1,0) = 1$

$f_{xx} = 2\arctan y \quad \ldots \quad f_{xx}(1,0) = 0$

$f_{xy} = \frac{2x}{1+y^2} \quad \ldots \quad f_{xy}(1,0) = 2$

$f_{yy} = \frac{-x^2(2y)}{(1+y^2)^2} \quad \ldots \quad f_{yy}(1,0) = 0$

$$T_2(x,y) = f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y-0) + \frac{f_{xx}(1,0)}{2}(x-1)^2 + \frac{f_{yy}(1,0)}{2}y^2 + f_{xy}(1,0)(x-1)y$$

$$= 0 + 0 + 1 \cdot y + 0 + 0 + 2(x-1)y$$

$$= y + 2(x-1)y$$

(b) Use your formula in part (a) to approximate the value of the function at $(1.05, 0.05)$ and compare this to the actual value of $f(1.05, 0.05)$.

$$T_2(1.05, 0.05) = 0.05 + 2(1.05-1)(0.05) = 0.055$$

$$f(1.05, 0.05) = (1.05)^2 \arctan(0.05) \approx 0.055079$$

$$\text{so, } T_2(1.05, 0.05) \approx f(1.05, 0.05)$$

$= 2x \ln y + \frac{x}{y}$

6. (a) Find the directional derivative of the function $f(x, y) = x \ln y^2 + \frac{x}{y}$ at the point (2,1) in the direction of the vector $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j}$.

$\|\vec{v}\| = \sqrt{3^2+4^2} = 5 \qquad \vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{5}\vec{v} = \underbrace{\frac{3}{5}}_{u_1}\vec{i} + \underbrace{\frac{4}{5}}_{u_2}\vec{j}$

$f_x = \ln y^2 + \frac{1}{y} \quad \ldots \quad f_x(2,1) = 1$

$f_y = \frac{2x}{y} - \frac{x}{y^2} \quad \ldots \quad f_y(2,1) = 2$

$D_{\vec{u}} f(2,1) = f_x(2,1) \cdot u_1 + f_y(2,1) \cdot u_2$

$= (1)(\frac{3}{5}) + (2)(\frac{4}{5})$

$= \frac{11}{5}$

(b) What does this number tell us about the function f at the point (2,1)?

This tells us that f is increasing in the direction indicated by $\vec{u}$ at the point (2,1).

(c) Is it possible that (in some direction other than that specified by the vector $\mathbf{v}$ in part (a)) the directional derivative of f at (2,1) is equal to 3? Explain.

The maximum rate of change at (2,1) is $\|\nabla f(2,1)\|$.

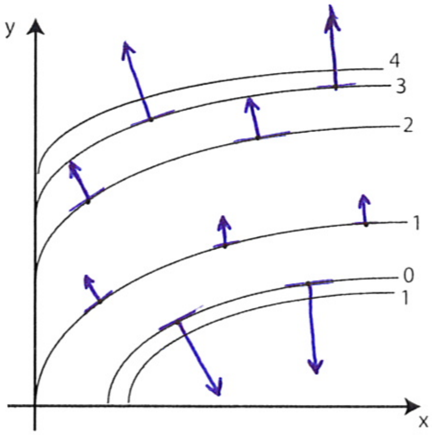
$\nabla f(2,1) = f_x(2,1)\vec{i} + f_y(2,1)\vec{j}$

$= 1\vec{i} + 2\vec{j}$

$\|\nabla f(2,1)\| = \sqrt{(1)^2 + (2)^2} \approx 2.2 < 3$

$\therefore$ No! The max. rate of change at (2,1) is about 2.2 So it is impossible that a directional derivative equals 3 at (2,1). (All directional derivatives must be $\leq \sqrt{5}$ at (2,1).)

7. On the contour diagram for $f(x, y)$ below, draw gradient vectors at the indicated points.



8. Find the maximum rate of change of the function $f(x, y) = 2ye^x + e^{-x}$ at the point $(0, 0)$ and the direction in which it occurs.

$f_x = 2ye^x - e^{-x} \ldots f_x(0,0) = -1$

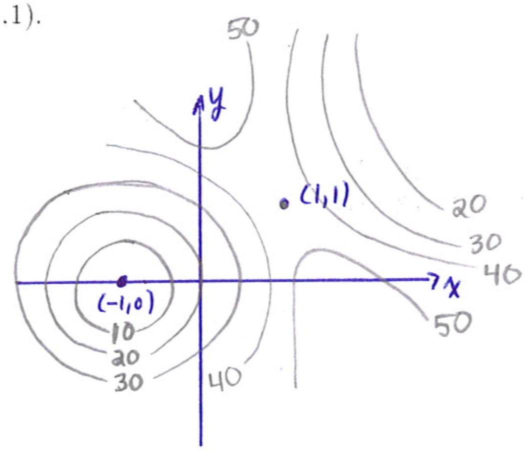
$f_y = 2e^x \ldots f_y(0,0) = 2$

$\nabla f(0,0) = -1\vec{i} + 2\vec{j}$

$\|\nabla f(0,0)\| = \sqrt{(-1)^2 + 2^2} = \sqrt{5} \approx 2.2$

$\therefore$ The max. rate of change of f at $(0,0)$ is $\sqrt{5} \approx 2.2$ and occurs in the direction of $\nabla f(0,0) = -1\vec{i} + 2\vec{j}$

9. Draw a contour diagram of a function that has a minimum at (-1,0) and a saddle point at (1,1).



10. Reason geometrically (i.e., without the second derivatives test) to show that the function $f(x, y) = y^3 - 4x^2y$ has a saddle point at (0,0).

$f_x = -8xy$

$f_y = 3y^2 - 4x^2$

$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \Rightarrow \begin{cases} -8xy = 0 & ① \\ 3y^2 - 4x^2 = 0 & ② \end{cases}$

② $\Rightarrow y^2 = \frac{4}{3}x^2 \Rightarrow y = \pm \frac{2}{\sqrt{3}}x$

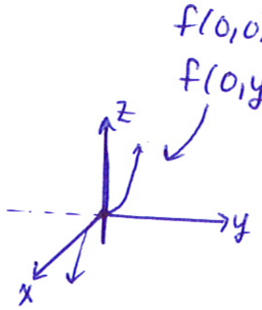
sub into ① $\Rightarrow -8x\left(\pm\frac{2}{\sqrt{3}}x\right) = 0 \Rightarrow \pm\frac{16}{\sqrt{3}}x^2 = 0 \Rightarrow \boxed{x = 0}$

sub $x = 0$ into ② : $\boxed{y = 0}$

so, (0,0) is the only critical point of f

$f(0,0) = 0 \leftarrow$ max or min ?

$f(0,y) = y^3$



0 cannot be a minimum value since $f(0,y) < 0$ when $y < 0$, 0 cannot be a maximum value since $f(0,y) > 0$ when $y > 0$.

$\therefore$ f has a saddle point at (0,0).

11. Find the local minimum and maximum values and saddle points (if any) of each function.

(a) $f(x, y) = x^3 - 2y^2 + 3xy + 4$

$f_x = 3x^2 + 3y$ $\quad f_{xx} = 6x$ $\quad f_{xy} = 3$

$f_y = -4y + 3x$ $\quad f_{yy} = -4$

$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \Rightarrow \begin{cases} 3(x^2+y) = 0 \\ -4y + 3x = 0 \end{cases} \Rightarrow \begin{cases} y = -x^2 \ ① \\ y = \frac{3}{4}x \ ② \end{cases}$

sub ② into ①: $\frac{3}{4}x = -x^2 \Rightarrow x^2 + \frac{3}{4}x = 0 \Rightarrow x(x+\frac{3}{4}) = 0 \Rightarrow x = 0$ or $x = -\frac{3}{4}$

sub $x=0$ into ②: $y = 0$

sub $x = -\frac{3}{4}$ into ②: $y = -\frac{9}{16}$

$\Rightarrow (0,0)$ and $(-\frac{3}{4}, -\frac{9}{16})$ are critical points of f

$D(x,y) = f_{xx} \cdot f_{yy} - (f_{xy})^2 = (6x)(-4) - (3)^2 = -24x - 9$

$D(0,0) = -9 < 0 \Rightarrow f$ has a saddle point at $(0,0)$

$D(-\frac{3}{4}, -\frac{9}{16}) = 9 > 0$

$f_{xx}(-\frac{3}{4}, -\frac{9}{16}) = -\frac{9}{2} < 0$

$\Rightarrow f$ has a local maximum at $(-\frac{3}{4}, -\frac{9}{16})$

local max value: $f(-\frac{3}{4}, -\frac{9}{16}) = \frac{539}{128}$

(b) $f(x, y) = xye^{-x-y}$

$f_x = y \cdot e^{-x-y} + xy \cdot e^{-x-y}(-1) = e^{-x-y} \cdot y(1-x)$

$f_y = e^{-x-y} \cdot x(1-y)$

$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \Rightarrow \begin{cases} y(1-x) = 0 \\ x(1-y) = 0 \end{cases} \Rightarrow \begin{cases} y = 0 \text{ or } x = 1 \\ x = 0 \text{ or } y = 1 \end{cases} \Rightarrow$ critical points are $(0,0)$ and $(1,1)$.

$f_{xx} = -e^{-x-y} \cdot y(1-x) + e^{-x-y} \cdot y(-1) = -e^{-x-y} \cdot y(2-x) = e^{-x-y} \cdot y(x-2)$

$f_{yy} = e^{-x-y} x(y-2)$

$f_{xy} = -e^{-x-y} \cdot y(1-x) + e^{-x-y} \cdot (1-x) = e^{-x-y}(1-x)(1-y)$

$D(0,0) = -1 \Rightarrow f$ has a saddle point at $(0,0)$.

$D(1,1) = e^{-2} (-1) \cdot e^{-2} (-1) - [\ 0]^2 = e^{-4} > 0$

$f_{xx}(1,1) = e^{-2} (-1) < 0$

$\Rightarrow f$ has a local max. at $(1,1)$

local max. value: $f(1,1) = e^{-2}$.

THE END