12C3

Last Day: Orthogonal Sets & Basis

A set $\{\vec{v}_1, \dots, \vec{v}_n\}$ is orthogonal if $\langle \vec{v}_i, \vec{v}_j \rangle = 0$ $i \neq j$

$\neq 0$ if $i = j$

i.e. all pairwise orthogonal

A set is orthonormal if $\langle \vec{v}_i, \vec{v}_j \rangle = 0$ $i \neq j$

$= 1$ $i = j$

i.e. orthogonal & unit length

If $\vec{u} \in \mathrm{Span}(\{\vec{v}_i\text{'s}\})$ & $\vec{v}_i$'s form an orthogonal set

$$\Rightarrow \vec{u} = a_1 \vec{v}_1 + a_2 \vec{v}_2 + \cdots a_n \vec{v}_n, \quad a_i = \frac{\langle \vec{u}, \vec{v}_i \rangle}{\|\vec{v}_i\|^2}$$

$$= \text{proj}_{\vec{v}_1} \vec{u} + \text{proj}_{\vec{v}_2} \vec{u} + \cdots$$

$$\vec{u} = \sum_{i=1}^{n} \text{proj}_{\vec{v}_n} \vec{u}$$

and if $\vec{v}$'s are orthonormal $\Rightarrow \|\vec{v}_i\| = 1$

$$\Rightarrow \vec{u} = \sum_{i=1}^{n} \langle \vec{u}, \vec{v}_i \rangle \vec{v}_i$$

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eg. In $\mathbb{R}^2$, $\left\{ \begin{bmatrix} 4 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \end{bmatrix} \right\}$ is an orthogonal set

& $n = 2 = \dim \mathbb{R}^2$

$\Rightarrow$ LI & $n = \dim V \Rightarrow$ basis

Express $\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ in the basis

<u>Solution</u> $\vec{u} = a_1 \vec{v}_1 + a_2 \vec{v}_2 \Rightarrow \begin{bmatrix} 1 \\ 2 \end{bmatrix} = a_1 \begin{bmatrix} 4 \\ 3 \end{bmatrix} + a_2 \begin{bmatrix} -3 \\ 4 \end{bmatrix}$

by <u>orthogonal</u> $a_1 = \dfrac{\langle \vec{u}, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}$, $a_2 = \dfrac{\langle \vec{u}, \vec{v}_2 \rangle}{\|v_2\|^2}$

$$= \frac{1(4) + 2(3)}{4^2 + 3^2} \qquad = \frac{1(-3) + 2(4)}{3^2 + 4^2}$$

$$= 10/5^2 = 2/5 \qquad = 1/5$$

<u>so</u> $\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{2}{5} \begin{bmatrix} 4 \\ 3 \end{bmatrix} + \frac{1}{5} \begin{bmatrix} -3 \\ 4 \end{bmatrix}$

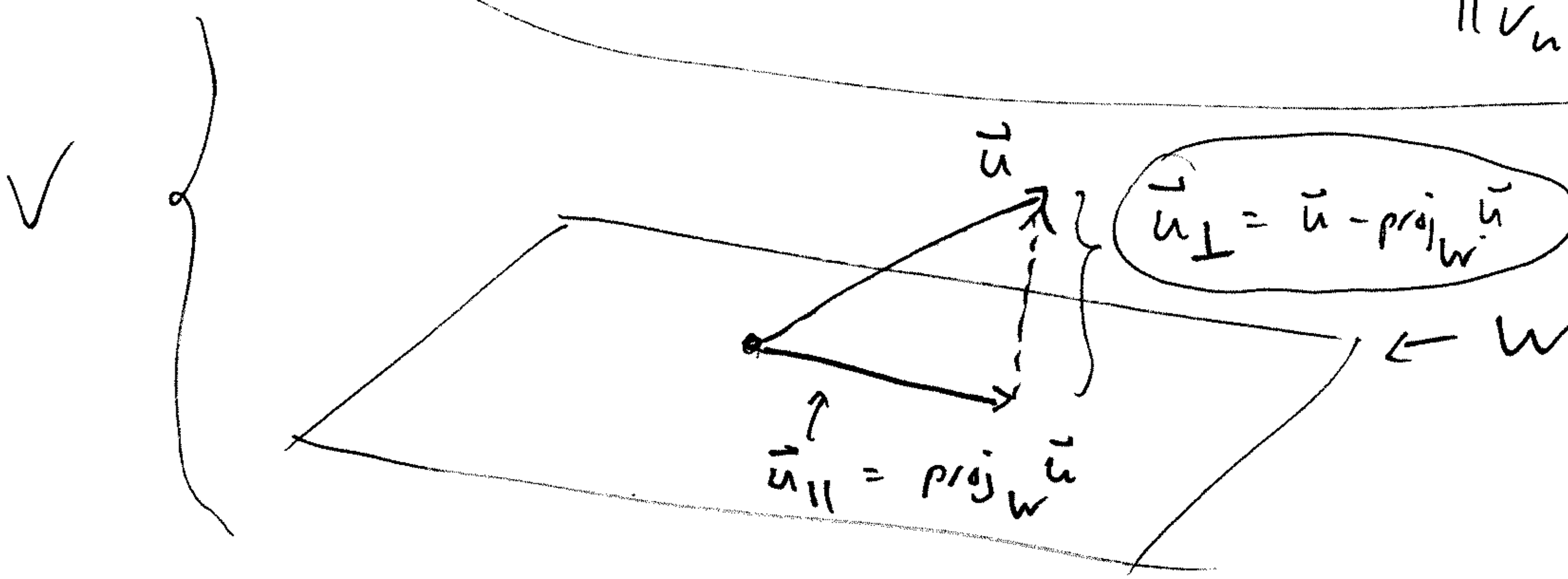
<u>or</u> $\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \left(\frac{2}{5}, \frac{1}{5} \right)$ in new basis

Say $W = \mathrm{Span}(\{\vec{v}_1, \ldots \vec{v}_n\}) \subseteq V$, our vspace

& $\{\vec{v}_1, \ldots \vec{v}_n\}$ orthogonal & $W \neq V$

Claim If $\vec{u} \in V$ then $\mathrm{proj}_W \vec{u} = \sum_{i=1}^{n} \mathrm{proj}_{\vec{v}_i} \vec{u}$

or $\mathrm{proj}_W \vec{u} = \frac{\langle \vec{u}, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 + \cdots + \frac{\langle \vec{u}, \vec{v}_n \rangle}{\|v_n\|^2} \vec{v}_n$



So **check!** is $(\vec{u} - \text{proj}_W \vec{u})$ orthogonal to W?

i.e. orthogonal to any vector in W

i.e. orthog. to all $\vec{v}_i$

Check $(\vec{u} - \text{proj}_W \vec{u}) \cdot \vec{v}_i$

$$= \vec{u} \cdot \vec{v}_i - \sum_{j=1}^{n} \left(\text{proj}_{\vec{v}_j} \vec{u}\right) \cdot \vec{v}_i$$

$$= \vec{u} \cdot \vec{v}_i - 0 - 0 - \cdots - \left(\text{proj}_{\vec{v}_i} \vec{u}\right) \cdot \vec{v}_i - 0 \cdots 0$$

$$= \vec{u} \cdot \vec{v}_i - \frac{\vec{u} \cdot \vec{v}_i}{\cancel{\|\vec{v}_i\|^2}} \left(\cancel{\vec{v}_i \cdot \vec{v}_i}\right) = \boxed{0} \checkmark$$

The Gram - Schmidt Process "Pimp my basis"

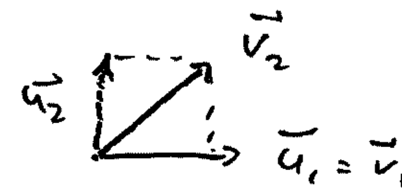
If I have a basis $\{\vec{v}_1 \ldots \vec{v}_n\}$ of V

G.S. process turns it into a new orthogonal basis!

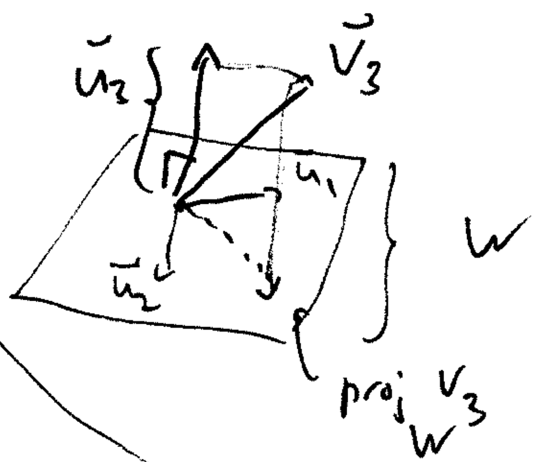
(& then orthonormal)

The Process If $\{\vec{v}_1, \ldots \vec{v}_n\}$ is our basis

Let $\vec{u}_1 = \vec{v}_1$



$$\vec{u}_2 = \vec{v}_2 - \text{proj}_{\vec{u}_1} \vec{v}_2$$



$$\vec{u}_3 = \vec{v}_3 - \text{proj}_{\vec{u}_1} \vec{v}_3 - \text{proj}_{\vec{u}_2} \vec{v}_3$$

$$= \vec{v}_3 - \text{proj}_{\text{span}\{\vec{u}_1, \vec{u}_2\}} \vec{v}_3$$

etc.

eg.

Say I have an $\mathbb{R}^3$ basis: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

Use G. S. process on this basis in this order

to get an orthog. basis of $\mathbb{R}^3$.

note result depends on order used

Solution: $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$$\vec{u}_1 = \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{u}_2 = \vec{v}_2 - \text{proj}_{\vec{u}_1}\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - \frac{\langle \vec{v}_2, \vec{u}_1 \rangle}{\|\vec{u}_1\|^2}\vec{u}_1$$

$$= \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - \underbrace{\frac{(1^2 + 0(-1) + 0^2)}{1^2 + 0^2 + 1^2}}_{1/2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} +1/2 \\ -1 \\ -1/2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$$

$$= \vec{u}_2$$

$$\vec{u}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \text{proj}_{\vec{u}_1} \vec{v}_3 - \text{proj}_{\vec{u}_2} \vec{v}_3$$

$$\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \frac{\vec{u}_1 \cdot \vec{v}_3}{\|\vec{u}_1\|^2} \vec{u}_1 - \frac{\vec{u}_2 \cdot \vec{v}_3}{\|\vec{u}_2\|^2} \vec{u}_2$$

$$= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - 0 - \frac{(0 + (-1) + 0)}{\frac{1}{4}(1^2 + (-2)^2 + (-1)^2)} \left(\frac{1}{2}\right) \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \underbrace{\frac{1 \cdot 2}{6}}_{1/3} \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/3 \\ -1/3 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

<u>Orthogonal</u> basis: $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$

$$= \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{2}\begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}, \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}$$

<u>Orthonormal basis</u> $\vec{w}_i = \vec{u}_i / \|\vec{u}_i\|$

$$\vec{w}_1 = \frac{1}{\sqrt{1^2+1^2+0^2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

$$\vec{w}_2 = \frac{1}{\sqrt{1^2+2^2+1^2}} \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix}$$

(feel free to <u>drop</u> any <u>+ve</u> <u>multiple</u> when finding a unit vector!

$$\frac{k\vec{v}}{\|k\vec{v}\|} = \frac{k\vec{v}}{|k|\,\|\vec{v}\|} = \cancel{\frac{k}{|k|}}^{\,1} \frac{\vec{v}}{\|\vec{v}\|} \quad \text{if } k>0 \;)$$

$$\vec{w}_3 = \begin{bmatrix}1\\1\\-1\end{bmatrix} \cdot \frac{1}{\sqrt{1^2+1^2+1^2}} = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\-1\end{bmatrix} = \begin{bmatrix}1/\sqrt{3}\\1/\sqrt{3}\\-1/\sqrt{3}\end{bmatrix}$$

$\{\vec{w}_1, \vec{w}_2, \vec{v}_3\}$ is our orthonormal basis.