MATHEMATICS 1LS3 TEST 1

Day Class
Duration of Examination: 60 minutes
McMaster University, 1 October 2014

E. Clements, M. Lovrić, O. Sanchez

FIRST NAME (please print): SOLUTIONS

FAMILY NAME (please print): _____

Student No.: _____

THIS TEST HAS 8 PAGES AND 7 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE.

Total number of points is 40. Marks are indicated next to the problem number. Any non-graphing calculator is allowed.

USE PEN TO WRITE YOUR TEST. IF YOU USE A PENCIL YOUR TEST WILL NOT BE ACCEPTED FOR REMARKING (IF NEEDED).

You must show work to receive full credit.

Problem	Points	Mark
1	6	
2	6	
3	5	
4	6	
5	4	
6	6	
7	7	
TOTAL	40	

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 Name:_____
Student No.: _____

1. Multiple choice questions: circle ONE answer. No justification is needed.

(a)[3] Start with the graph of $y = \cos x$. Scale (expand) the graph horizontally by a factor of 3 and then shift right the graph you obtained by 6 units. Finally, expand this graph vertically by a factor of 4. The graph you obtained is

(A) $y = \frac{1}{4}\cos\left(\frac{x+2}{6}\right)$ (B) $y = \frac{1}{4}\cos\left(\frac{x-2}{6}\right)$ (C) $y = 4\cos\left(\frac{x+6}{3}\right)$

(D) $y = 4\cos\left(\frac{x}{3} - 2\right)$ (E) $y = 4\cos\left(\frac{x}{3} + 6\right)$ (F) $y = 4\cos\left(\frac{x}{3} - \frac{2}{3}\right)$

(G) $y = \frac{1}{4}\cos\left(\frac{x+2}{3}\right)$ (H) $y = \frac{1}{4}\cos\left(\frac{x-6}{3}\right)$

$\cos x \longrightarrow \cos\frac{x}{3} \longrightarrow \cos\frac{x-6}{3}$

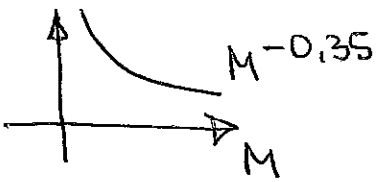
$\longrightarrow 4\cdot\cos\frac{x-6}{3} = 4\cos\left(\frac{x}{3}-2\right)$

(b)[3] The maximum flying speed S (in metres per second) of tropical birds living in rain forests of Cook Islands (South Pacific Ocean) is related to their body mass M (in grams) by $S = 44.5M^{-0.35}$. Which of the following statements is/are true?

(I) S is decreasing as a function of M ✓

(II) If M triples, so does S ✗

(III) S is inversely proportional to $M^{0.35}$



(A) none (B) I only (C) II only (D) III only

(E) I and II (F) I and III (G) II and III (H) all three

$S = 44.5 \cdot \frac{1}{M^{0.35}}$ ✓

true for linear functions $y = mx + b$ with $b = 0$ only

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2. **Identify each statement as true or false (circle your choice). You do not need to justify your answer.**

(a)[2] If $m_{t+1} = 0.8m_t$ and $m_0 = 1000$, then m_{10} is larger than 100.

(TRUE) FALSE

$m_t = m_0 \cdot 0.8^t$

$\rightarrow m_{10} = 1000 \cdot 0.8^{10} \approx 107.37$

(b)[2] For the linear function $y = 3x + 2$, the ratio of change in output over change in input is constant.

(TRUE) FALSE

this is the slope; and the slope of a line is constant

(c)[2] The label on the milk carton which you bought in Sydney, Australia, says "energy value of one cup of milk is 609 kJ." (Australians use kJ=kilo-Joules, whereas in North America we use calories.) Knowing that 1 calorie = 4.2 kJ, you calculated that that cup of milk contains more than 150 calories. Is your calculation correct?

YES (NO)

$609\ kJ = \frac{609}{4.2}$ calories

145

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Name:_____
Student No.: _____

Questions 3-7: You must show work to receive full credit.

3. It has been determined that an average human body absorbs benzoylmethylecgonine (cocaine) according to $d(t) = 80e^{-0.71t}$, where $d(t)$ is in milligrams and t is time in hours.

(a)[1] Define the term: half-life of a drug. (Avoid long explanations; one sentence suffices.)

time needed for a substance to decay / reduce to 1/2 of its original amount

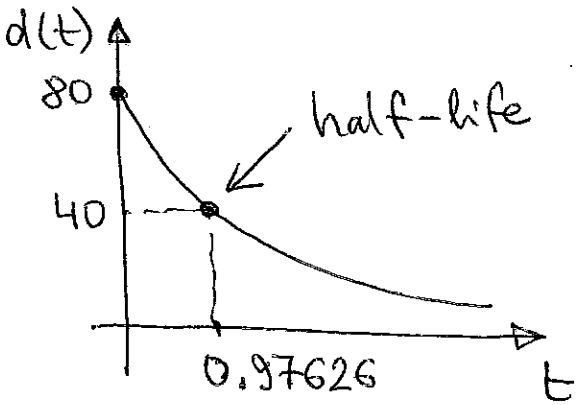
(b)[2] Compute the half-life of benzoylmethylecgonine.

$$40 = 80\,e^{-0.71t}$$

$$0.5 = e^{-0.71t} \longrightarrow \ln 0.5 = -0.71t$$

$$t = \frac{\ln 0.5}{-0.71} \approx 0.97626 \text{ hours}$$

(c)[2] Sketch the graph of $d(t)$ and identify the point on the graph which corresponds to the half-life. ("To identify the point" means to say what its coordinates are.)



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4. (a)[1] What is the phase of the oscillation $f(t) = 3 - 2.4\cos(2t - 7)$?

$\cos 2(t - \frac{7}{2})$

so phase $= \frac{7}{2}$ (right shift)

(b)[2] What is the domain of the function $f(x) = 3\arcsin(2x - 5)$?

irrelevant for domain

$-1 \le 2x - 5 \le 1 \mid +5$

$4 \le 2x \le 6 \mid \div 2$

$2 \le x \le 3$, ie, $[2,3]$

fact: arcsin * is defined when $-1 \le * \le 1$

(c)[3] What is the range of the function $P(t) = 4.4\left(\frac{\pi}{2} + \arctan\frac{t}{42}\right)$?

range of arctan t is $(-\frac{\pi}{2}, \frac{\pi}{2})$

arctan t/42 is a horizontal transformation, so does not affect the range

range of $\frac{\pi}{2} + \arctan\frac{t}{42}$ is $\frac{\pi}{2} - \frac{\pi}{2} = 0$, $\frac{\pi}{2} + \frac{\pi}{2} = \pi$

ie $(0, \pi)$

$\Rightarrow$ range of P(t) is $(0, 4.4\pi)$

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5. Based on the density of a soil sample taken from a forest floor, scientists can determine the depth it came from, by using the formula

$$d = -5 \ln\left(\frac{0.7}{\rho} - 0.8\right)$$

In this formula, ρ is the density of a soil sample and d is the depth in metres (so $d = 0$ labels the surface, and $d = 3$ is 3 m below the surface).

(a)[1] In the above formula, d is a function of ρ. State (in one sentence) what question is answered by finding the inverse function of d.

What is the density ρ of a soil sample at depth d?

(b)[3] Find a formula for the inverse function of d.

$$d = -5 \ln\left(\frac{0.7}{\rho} - 0.8\right)$$

$$-0.2d = \frac{d}{-5} = \ln\left(\frac{0.7}{\rho} - 0.8\right) \quad | \text{ apply } e$$

$$e^{-0.2d} = \frac{0.7}{\rho} - 0.8$$

$$\frac{0.7}{\rho} = 0.8 + e^{-0.2d} \quad | \text{ apply reciprocal}$$

$$\rightarrow \frac{\rho}{0.7} = \frac{1}{0.8 + e^{-0.2d}}$$

$$\rho = \frac{0.7}{0.8 + e^{-0.2d}}$$

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6. In *Linear quadratic and tumour control probability modelling in external beam radiotherapy,* S.F.C. O'Rourke, H. McAneney, T. Hillen. J. Math. Biol. (2009) 58:799–817, we read

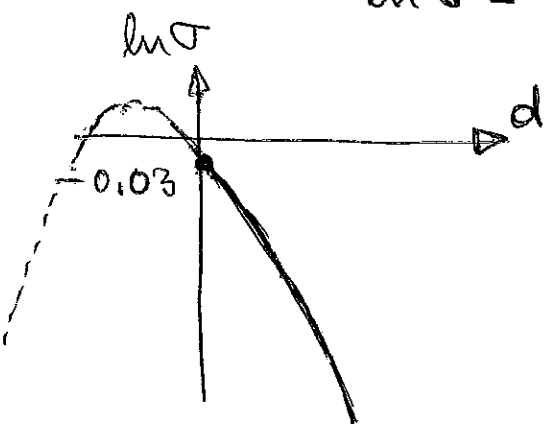
> predicted clonogenic survival [27,85,96]. Such a model is in popular use and may be written in the form.
>
> $$\ln \sigma = -n(\alpha d + \beta d^2) - \lambda T, \qquad (12)$$
>
> where T is the *overall* exposure time (i.e. the complete timescale of the treatment

In formula (12), σ is the survival rate of cancer cells; it is a function of the applied radiation dose d. Reading the paper, we learn that all parameters are positive.

(a)[3] Assume that $n = 1$, $\alpha = 1$, $\beta = 1$, $\lambda = 0.03$ and $T = 1$. For these values, sketch the semi-log graph of $\sigma(d)$ for $d \geq 0$.

$\ln \sigma = -d - d^2 - 0.03 = -d^2 - d - 0.03$

↑ parabola opens downward ∩



intercepts $\begin{cases} d^2 + d + 0.03 = 0 \\ d = \dfrac{-1 \pm \sqrt{1 - 0.12}}{2} \end{cases}$

= both negative so not relevant here

(b)[1] Explain why $\sigma(d) < 1$ for all values of $d > 0$.

because $\ln \sigma < 0$ for $d \geq 0$ → $e^{\ln \sigma} < e^0$ (≈1) → $\sigma < 1$

OR: if $\ln A < 0$ then $A < 1$

(c)[2] Given that $n = 1$, $\alpha = 1$, $\beta = 1$, $\lambda = 0.03$ and $T = 1$, find $\sigma(0)$ and interpret your answer.

$\ln \sigma(0) = -0.03$ → $\sigma(0) = e^{-0.03} \approx 0.97$

so even when no radiation is applied ($d=0$) some cancer cells (3%) will die, as the survival rate is $0.97 = 97\%$

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7. Consider the dynamical system $M_{t+1} = 0.5M_t + 2.5$, where M_t represents the amount of drug (in mg) in a patient's body at time t (time t is measured in days). It is given that $M_0 = 8$ mg.

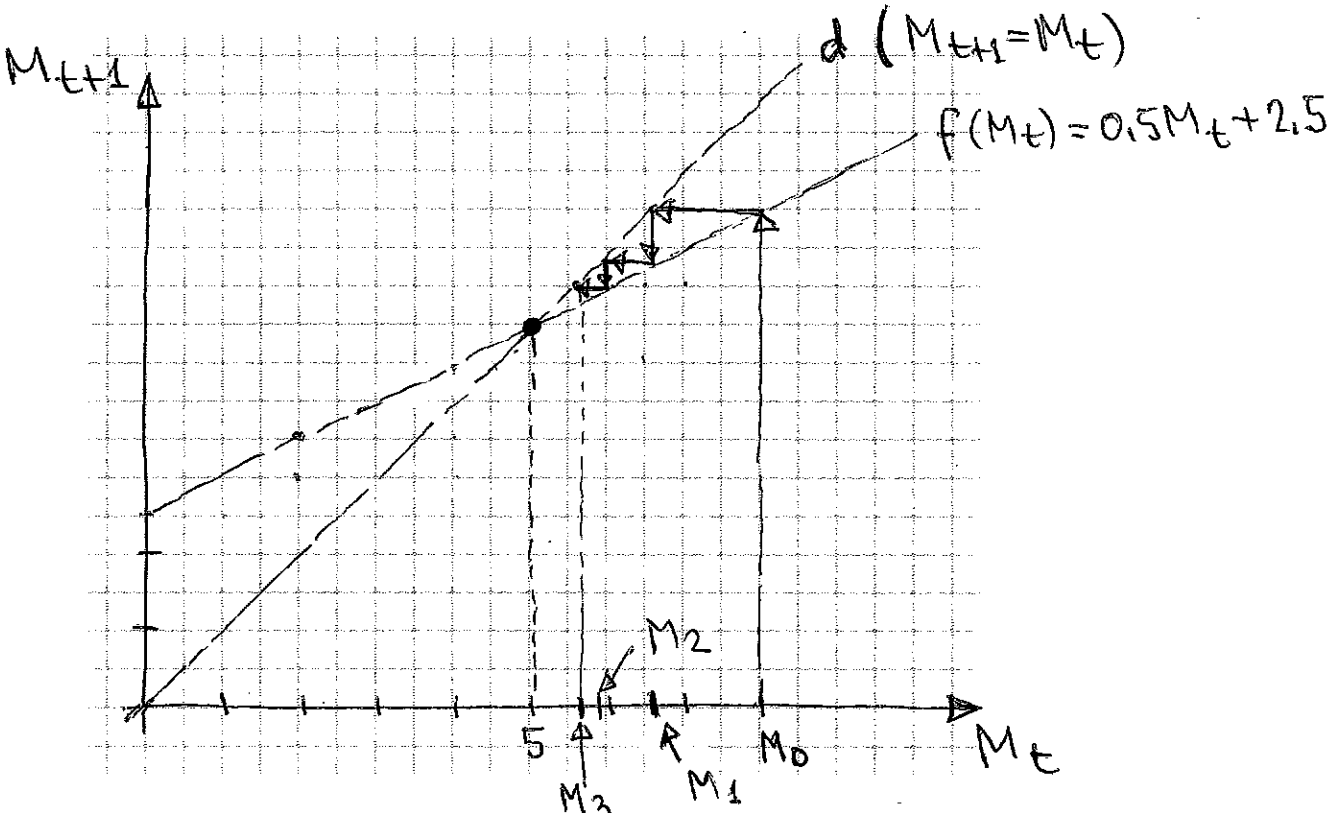
(a)[2] Explain in words the dynamics described by this system.

Initially, 8 mg of the drug are given
At each half-life, 2.5 mg are given
↳ which is 1 day in this case

(b)[2] Find all equilibrium points of the system.

$M^* = 0{,}5M^* + 2{,}5 \rightarrow M^* = 5$

(c)[2] Starting with $M_0 = 8$ mg, cobweb for three steps.



(d)[1] What does the cobwebbing in (c) suggest in terms of the amount of the drug in the patient's body?

It will decrease toward the equilibrium value of $M^* = 5$

THE END