1ZC3

Last Day adj & det

If A is invertible, then $\boxed{A^{-1} = \operatorname{adj} A \cdot \frac{1}{\det A}}$

& remember: $\operatorname{adj} A = ($transpose of matrix of cofactors of $A)$

$= [\, C_{ji} \,]$

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eg. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, use above to find A^{-1} (if exists!)

Solution $\det(A) = ad - bc$

$$\operatorname{adj} A = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^T = \begin{bmatrix} C_{11} & C_{21} \\ C_{12} & C_{22} \end{bmatrix}$$

$$= \begin{bmatrix} (+1)M_{11} & (-1)M_{21} \\ (-1)M_{12} & (+1)M_{22} \end{bmatrix} = \begin{bmatrix} M_{11} & -M_{21} \\ -M_{12} & M_{22} \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

so $A^{-1} = \operatorname{adj} A \cdot \frac{1}{\det A} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \cdot \frac{1}{ad-bc}$

Megatheorem Recap (The Theorem so far!)

If A is an $n \times n$ matrix, the following are all equivalent

1) A invertible
2) A row-equivalent to I (RREF form)
3) A is a product of elementary matrices.
4) $A\vec{x} = \vec{b}$ has a unique solution.
5) $A\vec{x} = \vec{b}$ always has a solution.
6) $A\vec{x} = \vec{0}$ has only $\vec{x} = 0$ (trivial) solution

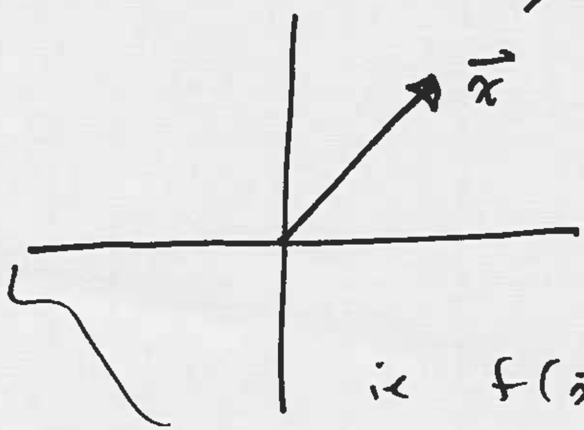
New! 7) $\det A \neq 0$

Eigenvectors & Eigenvalues

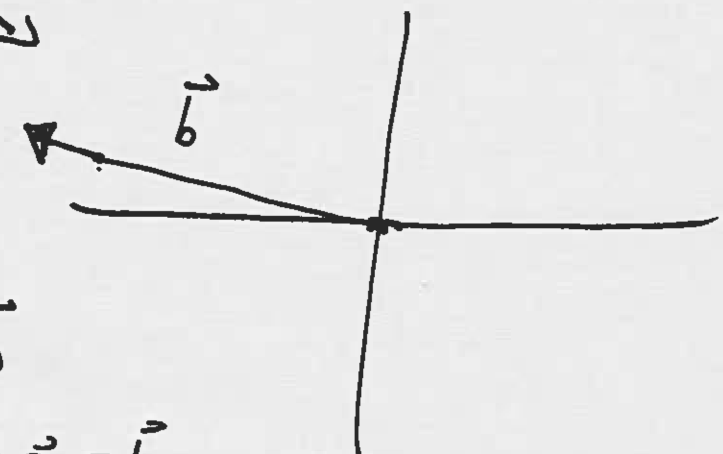
We've seen linear systems expressed as $\overset{\text{coeff}}{A}\,\overset{\text{variables}}{\vec{x}} = \overset{\text{const.}}{\vec{b}}$ } System as a matrix product

eg 2x2 matrix A

multiply by A



2D



ie $f(\vec{x}) = \vec{b}$

ie $f(\vec{x}) = A\vec{x} = \vec{b}$

In words: An eigenvector of a square matrix A is a vector for which mult. by A acts like mult. by a single real number

This real number is the eigenvalue

In symbols $\vec{x}$ is a λ-eigenvector of a square matrix A means $\boxed{A\vec{x} = \lambda \vec{x}}$ $\mid \vec{x} \neq 0$

λ is our eigenvalue!

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eg. If I tell you that $\begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$ is the $\lambda = 3$ eigenvector of a matrix A then:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -6 \\ 3 \\ 9 \end{bmatrix}$$

$$A \quad \vec{x} \quad = \quad \lambda x \qquad \lambda = 3$$

Notice Say $A\vec{x} = \lambda \vec{x}$ for some λ, $x \neq 0$

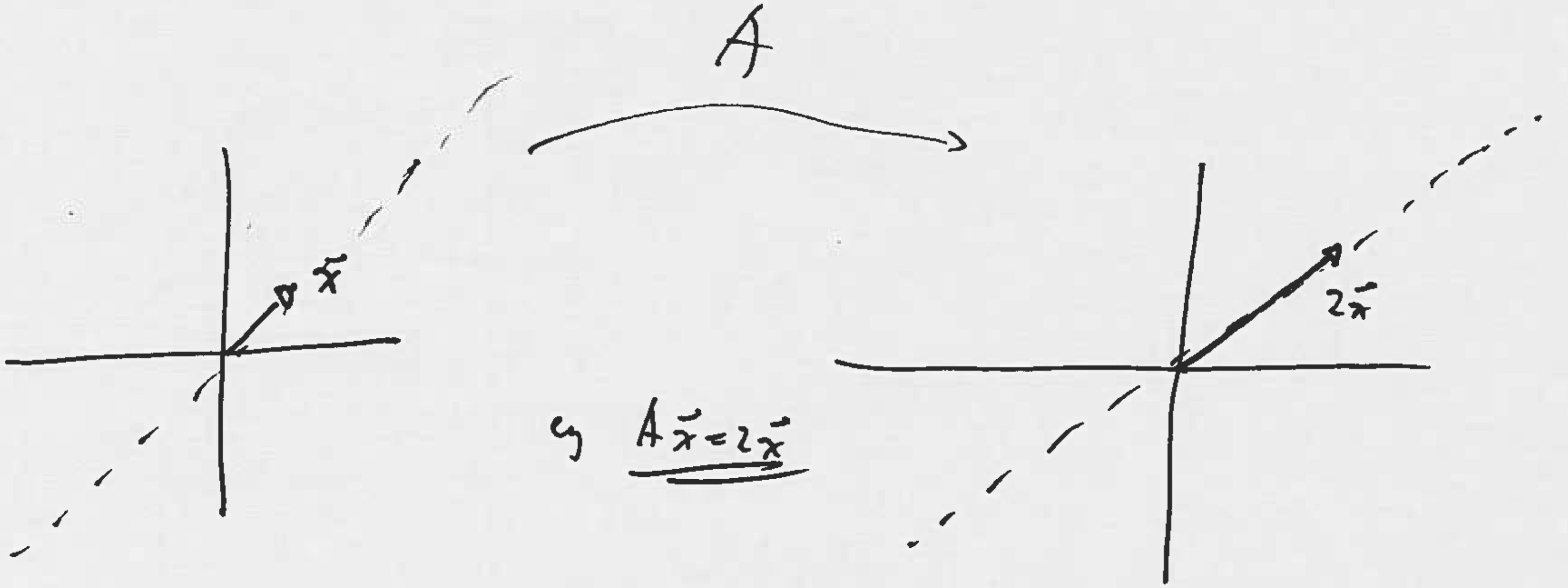
$$\underline{A(k\vec{x})} = k A\vec{x} = k \cdot \lambda \vec{x} = \underline{\lambda \cdot (k\vec{x})}$$

If $\vec{x}$ is an eigenvector, any non-zero multiple is an eigenvector

Also Say $\vec{x}, \vec{y} \neq 0$ & $A\vec{x} = \lambda\vec{x}$, $A\vec{y} = \lambda\vec{y}$ for the same λ $\Rightarrow$

$$\begin{aligned} A(\vec{x}+\vec{y}) &= A(\vec{x}) + A(\vec{y}) \\ &= \lambda\vec{x} + \lambda\vec{y} \\ &= \lambda(\vec{x}+\vec{y}) \end{aligned}$$

So, if $\vec{x} \neq -\vec{y}$, $\vec{x}+\vec{y}$ is <u>still</u> a <u>λ-eigenvector</u>!



eg $\underline{A\vec{x} = 2\vec{x}}$

The set of all $\vec{x}$ such that $A\vec{x} = \lambda\vec{x}$ (all λ-eigenvectors & $\vec{0}$) is called our <u>λ-eigenspace</u> & is <u>preserved</u> by action of <u>A</u>.

Hunting Eigenvectors & Eigenvalues

If $\vec{x} \neq 0$ & $A\vec{x} = \lambda \vec{x}$ & A square

then $A\vec{x} - \lambda \vec{x} = \vec{0}$ $\Leftrightarrow$

$(A - \lambda I)\vec{x} = \vec{0}$ for $\vec{x} \neq 0$

$\Updownarrow$

$(A - \lambda I)\vec{x} = \vec{0}$ has non-trivial soln!

$\Leftrightarrow$ $A - \lambda I$ is non-invertible!

$\Leftrightarrow$ $\det(A - \lambda I) = 0$

Reversible argument!

so If $\boxed{\det(A - \lambda I) = 0}$ $\Rightarrow$ λ is an eigenvalue of A

& $\vec{x}$ eigenvectors exist

$$C_A(\lambda) = |A - \lambda I| = \text{characteristic polynomial}$$

roots of $C_A(\lambda)$ are eigenvalues of A

eg. Let $A = \begin{bmatrix} 2 & 1 \\ 6 & 1 \end{bmatrix}$

$$C_A(\lambda) = |A - \lambda I| = \left| \begin{bmatrix} 2 & 1 \\ 6 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right|$$

$$= \begin{vmatrix} 2-\lambda & 1 \\ 6 & 1-\lambda \end{vmatrix}$$

$$= (2-\lambda)(1-\lambda) - 6$$

$$= \lambda^2 - 3\lambda - 4 \rightarrow$$

$$(\lambda - 4)(\lambda + 1) = 0$$

$$\lambda = 4, -1$$