12/13

Graph Sketching! The Final Compilation!

1) **Symmetry**

- "even" $\iff$ $f(-x) = f(x)$ 
- "odd" $\iff$ $f(-x) = -f(x)$ 
- "periodic" $\iff$ $f(x+\lambda) = f(x)$ λ-periodic!

practically,
just look to see if cos/sin/sec/ etc. periodic functions.

2) **Domain** (& Range, but only if easy!)

3) **Intercepts** - x-intercepts $(y=0)$ (potentially ∞)

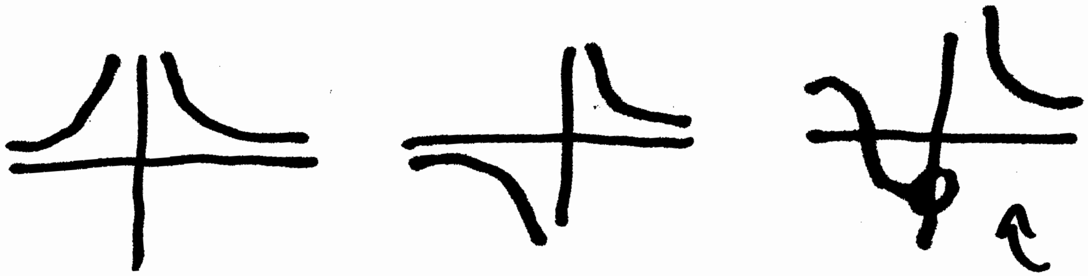
- y-intercept $(x=0)$ (≤ 1 possible)

4) **Asymptotes** - Vertical Asymptotes (VA)

- Horizontal Asymptotes (HA)

VA at $x=a \iff \lim_{x \to a} f(x) = \pm\infty$

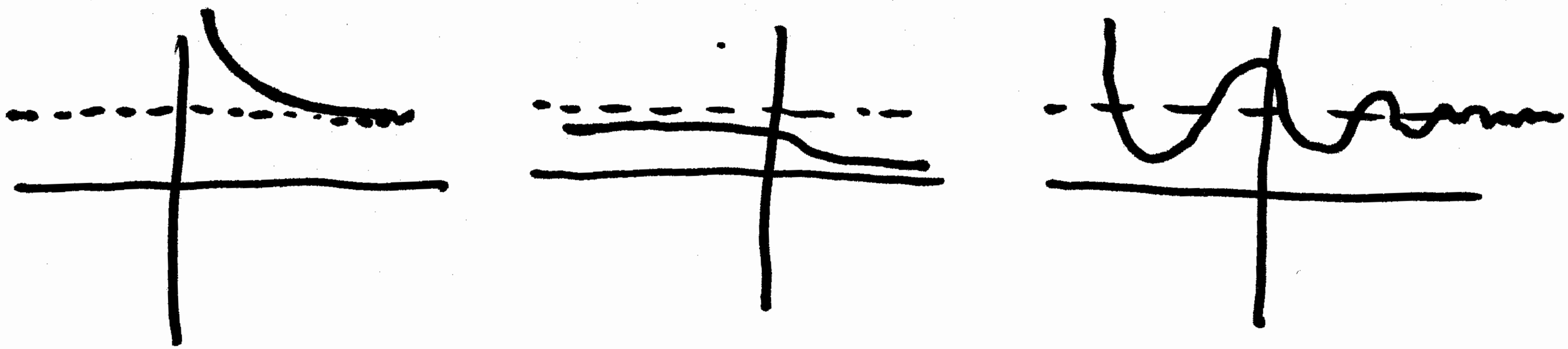
or $\lim_{x \to a^+} f(x) = \pm\infty$ **or** $\lim_{x \to a^-} f(x) = \pm\infty$



Isolated **VA**
It's **still** a **VA**.

HA $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$

Note at **most** 2 **H.A**, one at **$+\infty$**, one at **$-\infty$**.



5) $f'(x)$, the **First Derivative**

- **c.n.** where $f'(x)=0$, $f'(x)$ DNE in $f(x)$ **domain**.
- Intervals of inc/dec. (do a **chart**)
- **1st** derivative test: **Local max/min points!**

6) $f''(x)$, the Second Derivative

- check where $f''(x)=0$, $f''(x)$ DNE, get inflection points
- intervals of concavity (C.U. $\cup$, C.D. $\cap$)

7) Sketch it!

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eg. Sketch $y = \dfrac{x^2}{1-x^2}$

Solution

1) Symmetry: Periodic? Clearly no

Even/odd? Check $f(-x)$

$$f(-x) = \frac{(-x)^2}{1-(-x)^2} = \frac{x^2}{1-x^2} = f(x)$$

$\Rightarrow$ $f(x)$ is <u>even</u>

2) <u>Domain</u> $f(x)$ DNE if $\frac{x^2}{1-x^2}$ DNE

$\Rightarrow 1 - x^2 = 0 \Rightarrow x = \pm 1$

so <u>$f(x)$ defined for</u> <u>$x \neq \pm 1$</u>

3) <u>Intercepts</u>

<u>y-int</u>: if $x = 0$ $y = f(0) = \frac{0^2}{1-0^2} = \underline{0}$

<u>x-int</u>: if $y = 0 \Rightarrow 0 = \frac{x^2}{1-x^2} \Rightarrow x^2 = 0$

$x = 0$

So $(0,0)$ is only intercept

4) HA

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{x^2}{1-x^2} \sim \lim_{x \to \infty} \frac{x^2}{-x^2} = -1$$

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{x^2}{1-x^2} = -1$$

Same by even symmetry.

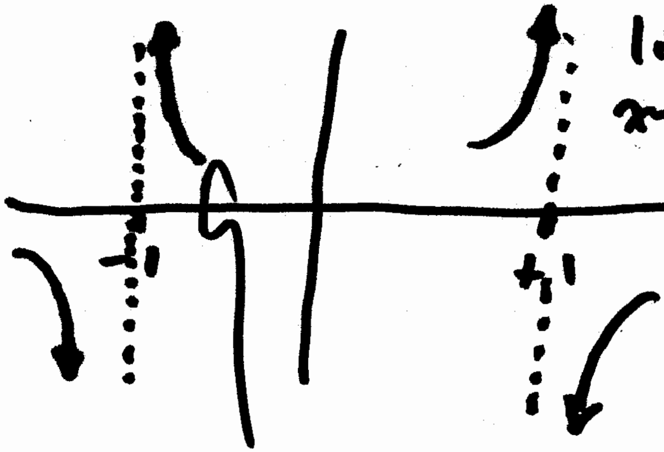
VA check our discont. at $x = \pm 1$

$$\lim_{x \to +1} f(x) = \lim_{x \to +1} \frac{x^2}{1-x^2} = \frac{1}{0} = ?$$

$$\lim_{\substack{x \to 1^+ \\ x>1}} \frac{x^2}{1-x^2} = \frac{1}{1-1^+} = \frac{1}{0^-} = -\infty$$

$$\lim_{x\to 1^-} \frac{x^2}{1-x^2} = \frac{1}{1-1^-} = \frac{1}{0^+} = \underline{\underline{+\infty}}\,.$$

Graphically



$\lim_{x\to 1^-} f(x) = +\infty$

$\lim_{x\to 1^+} f(x) = -\infty$

$\lim_{x\to -1^-} f(x) = -\infty$ $\Rightarrow$

$\lim_{x\to -1^+} f(x) = \infty$

by (even) symmetry!

5. Check $f'(x)$

$$f(x) = \frac{x^2}{1-x^2} \Rightarrow f'(x) = \frac{2x(1\cancel{-x^2}) - (\cancel{-2x})x^2}{(1-x^2)^2}$$

$$= \frac{2x}{(1-x^2)^2}$$

c.n. Case #1 $f'(x) = 0 = \frac{2x}{(1-x^2)^2}$

$\Rightarrow 2x = 0 \Rightarrow x = 0 \Rightarrow (x,y) = (0,0)$

Case #2 $f'(x)$ DNE $\Rightarrow \frac{2x}{(1-x^2)^2}$ DNE

$\hookrightarrow 1 - x^2 = 0 \rightarrow x = \pm 1$

not in domain of $f(x) \Rightarrow$ not c.n.

but f' can change sign! Use in chart!

c.n.

	$(-\infty,-1)$	$(-1,0)$	$(0,1)$	$(1,\infty)$
$2x$	$-$	$-$	$+$	$+$
$(1-x^2)^2$	$+$	$+$	$+$	$+$
$f'(x)$	$-$	$-$	$+$	$+$
$f(x)$	dec $\searrow$	dec $\searrow$	inc $\nearrow$	inc $\nearrow$

local min at $x=0$ by 1st deriv test.

6) $f''(x) = \frac{d}{dx} \frac{2x}{(1-x^2)^2}$

$$= \frac{2(1-x^2)^2 - (2x)\cdot 2(1-x^2)\cdot(-2x)}{(1-x^2)^4}$$

$$= \frac{2(1-x^2)}{(1-x^2)^4}\left[1 - \cancel{x^2} + 4\cancel{x^2}\right] = \frac{2(1+3x^2)}{(1-x^2)^3}$$

$f''(x) = 0$? none $\qquad$ $f''(x)$ DNE? $\quad x = \pm 1$

not I.P. since not in domain of $f(x)$.

	$(-\infty, -1)$	$(-1, 1)$	$(1, \infty)$
f''	$-$	$+$	$-$
$f(x)$	CD	CU	CD