Logistic Regression Explanation

We start with a random variable Y following a Bernoulli distribution

$$Y \mid X = x \sim Bernoulli(p(x)), \quad Y \in \{0, 1\} \tag{1}$$

Alternatively,

$$\begin{aligned} P(Y = 1 \mid X = x) &= p(x) \\ P(Y = 0 \mid X = x) &= 1 - p(x) \end{aligned}$$

Now compute the conditional expectation,

$$\begin{gathered} E(Y \mid X = x) = \sum y \cdot P(Y = y \mid X = x) \\ E(Y \mid X = x) = 1 \cdot P(Y = 1 \mid X = x) + 0 \cdot P(Y = 0 \mid X = x) \\ E(Y \mid X = x) = P(Y = 1 \mid X = x) \end{gathered}$$

Now we define:

$$P(x) = P(Y = 1 \mid X = x)$$

$$\boxed{P(x) = E(Y \mid X = x)} \tag{2}$$

Intuitively, because $Y \in \{0, 1\}$, average is equivalent to probability. $E(Y \mid X = x)$ is telling us if we look at all patients with feature x, what is the average value of Y among them. Suppose 70 are malignant ($Y = 1$) and 30 are benign ($Y = 0$), the average value of $Y = 0.7$, but that is equivalent to $P(Y = 1 \mid X = x)$ because 70 out of 100 were malignant. Because Y only takes values 0 and 1, When taking the average of 0's and 1's: The average equals the fraction of 1's. And the fraction of 1's is the probability of 1.

Linear Regression

Suppose we have random variables $Y, X_1, X_2, ..., X_p$. We define 2 criteria for a linear model, namely: (note, this idea is taken from math.stackexchange)

$$1.\quad Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + ... + \beta_p X_p + \epsilon$$

$$2.\quad E(Y \mid X_1 = x_1, X_2 = x_2, ..., X_p = x_p) = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + ... + \beta_p X_p$$

Where $\beta_0, \beta_1, ..., \beta_p$ are constants.

For criteria 2, we enforce this linear relationship on average because real-world data contains randomness. We can imagine if it were exact in the sense $Y = \beta_0 + \beta_1 X$ with no error term, it will not make sense of what we observe in reality. Two people studying the same amount of hours, say 5 hours may not score the same in an exam.

Take the conditional expectation of 1:

$$E(Y \mid X_1 = x_1, ..., X_p = x_p) = E(\beta_0 + \beta_1 X_1 + \beta_2 X_2 + ... + \beta_p X_p + \epsilon \mid X_1 = x_1, ..., X_p = x_p) \tag{3}$$

Since we condition on $X_1 = x_1, ..., X_p = x_p$, the random variables X_i are "fixed" at those values.

$$E(Y \mid X_1 = x_1, ..., X_p = x_p) = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + ... + \beta_p X_p + E(\epsilon \mid X_1 = x_1, ..., X_p = x_p)$$

Plugging in criteria 2:

$$\beta_0 + \beta_1 X_1 + \beta_2 X_2 + ... + \beta_p X_p = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + ... + \beta_p X_p + E(\epsilon \mid X_1 = x_1, ..., X_p = x_p)$$

$$E(\epsilon \mid X_1 = x_1, ..., X_p = x_p) - 0 \tag{4}$$

The statistical definition of a linear regression model is defined as:

$$\boxed{E(Y \mid X_1 = x_1, ..., X_p = x_p) = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + ... + \beta_p X_p} \tag{5}$$

Remember we did not derive this, because that line of reasoning is circular from criteria 2, we however did derive the ”Zero Conditional Mean Assumption” in (4).

Back to Logistic Regression

In Linear Regression, the random variable Y is continuous, in logistic regression, it is binary. Most importantly, notice (2) and (5) are both modelling the conditional expectation. Let us now try to define the logistic regression, keeping in mind a constraint:

$$0 < P(x) < 1 \tag{6}$$

Looking at (2) and (5), a natural first attempt is:

$$P(x) = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + ... + \beta_p X_p$$

However, this linear function is unbounded and contradicts (6). It's very important to understand we want some probability related quantity to equal a linear function, so we attempt to do a transformation on $P(x)$ to match the unboundedness of the linear function. In other words we want some transformation g such that:

$$g : (0, 1) \to (-\infty, \infty)$$

We shall define the odds to remove the upper bound of 1.

$$odds = \frac{P}{1-P}, \quad bound\, is\, now (0, \infty)$$

$odds > 1$ if $Y = 1$ is more likely, $odds < 1$ if $Y = 0$ is more likely. Now take the ln to remove the lower bound.

$$ln\left(\frac{P(x)}{1-P(x)}\right), \quad bound\, is\, now (\infty, \infty)$$

Therefore, the logistic regression model is defined as the Logit:

$$\boxed{Logit = ln(odds) = ln\left(\frac{P(x)}{1-P(x)}\right) = \beta_0 + \beta_1 X_1 + ... + \beta_p X_p} \tag{7}$$

Note in lecture notes, $P(x)$ is referred to as $P(Y = 1)$. The bigger the $Logit$ is, the bigger $P(Y = 1)$ is.

Because we want to predict probability which is $P(x)$, we now solve for $P(x)$ by multiplying by exponent on both sides.

$$e^{ln\left(\frac{P}{1-P}\right)} = e^{\beta_0 + \beta_1 X_1 + ... + \beta_p X_p}$$

$$\frac{P}{1-P} = e^{\beta_0 + \beta_1 X_1 + ... + \beta_p X_p}$$

$$\begin{aligned} P &= e^{\beta_0 + \beta_1 X_1 + ... + \beta_p X_p} - Pe^{\beta_0 + \beta_1 X_1 + ... + \beta_p X_p} \\ &= P\left[\frac{e^{\beta_0 + \beta_1 X_1 + ... + \beta_p X_p}}{P} - e^{\beta_0 + \beta_1 X_1 + ... + \beta_p X_p}\right] \\ 1 &= \frac{e^{\beta_0 + \beta_1 X_1 + ... + \beta_p X_p}}{P} - e^{\beta_0 + \beta_1 X_1 + ... + \beta_p X_p} \end{aligned}$$

$$P\left(1+e^{\beta_0+\beta_1 X_1+\ldots+\beta_p X_p}\right)=e^{\beta_0+\beta_1 X_1+\ldots+\beta_p X_p}$$

$$P=\frac{e^{\beta_0+\beta_1 X_1+\ldots+\beta_p X_p}}{1+e^{\beta_0+\beta_1 X_1+\ldots+\beta_p X_p}}$$

Therefore, we arrive at the Sigmoid function:

$$\boxed{P=\frac{1}{1+e^{-(\beta_0+\beta_1 X_1+\ldots+\beta_p X_p)}}} \tag{8}$$

In R, the function predict() has the following syntax:

```
predict(object from "glm", newdata = NULL,
        type = c("link", "response", "terms"),
        se.fit = FALSE, dispersion = NULL, ...)
```

and returns P, the Sigmoid function when `type = "response"`

Application (Graduate Admissions Logistic Regression supplementary notes)

In R, when we performed:

```
grad$rank <- factor(grad$rank, levels = c(1, 2, 3, 4))
logit_model <- glm(admit ~ gre + gpa + rank,
                   data = train_data,
                   family = binomial)
```

R now understands `rank` as categorical, not numeric, so R creates a dummy variable where `rank 1` is the baseline. The logistic regression model now becomes:

$$ln(odds) = \beta_0 + \beta_1 gre + \beta_2 gpa + (-0.89) rank2 + (-1.48) rank3 + (-1.80) rank4$$

Suppose we want to compare the difference in odds between rank 1.

For rank 1 (all dummy variables = 0):

$$ln(odds_1) = \beta_0 + \beta_1 gre + \beta_2 gpa$$

For rank 2 (only rank 2 = 1):

$$ln(odds_2) = \beta_0 + \beta_1 gre + \beta_2 gpa + (-0.89)$$

$$\begin{gathered} ln(odds_2) - ln(odds_1) = -0.89 \\ ln\left(\frac{odds_2}{odds_1}\right) - = 0.89 \\ \frac{odds_2}{odds_1} = e^{-0.89} \approx 0.41 \\ odds_2 = 0.41 odds_1 \\ 1 - 0.41 = 0.59 \end{gathered}$$

Therefore, rank 2 odds are 59 percent lower than rank 1, meaning applicants from rank 2 institutions have a 59 percent lower odds of admission than those from rank 1 institutions.