Complex Analysis

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Chapter 1

Vector Spaces and Linear Maps

1.1 Vector Spaces

Definition 1. A set $\mathbb{F}$ with two binary operations $+$ and $\times$ is a **field** if both $(\mathbb{F}, +, 0)$ and $(\mathbb{F} \setminus \{0\}, \times, 1)$ are abelian groups and the distribution law holds:

(a+b)c = ac + bc, for all a, b, c, $\in \mathbb{F}$

Definition 2. The smallest integer p such that

$$1 + 1 + \cdots + 1 \quad (p \text{ times}) = 0$$

is called the **characteristic** of $\mathbb{F}$. If no such p exists, the charecteristic of $\mathbb{F}$ is defined to be 0. If such a p exists, it is necessarily prime.

Definition 3. A **vector space** V over a field $\mathbb{F}$ is an abelian group $(V, +, 0)$ together with a scalar multiplication $\times\ \mathbb{F} \times V \to V$ such that for all $a, b \in F, v, w, \in V$:

1. $a(v + w) = av + aw$
2. $(a + b)v = av + bv$
3. $(ab)v = a(bv)$
4. $1 \cdot v = v$

Definition 4. Let V be a vector space over $\mathbb{F}$.

1. A set $S \subseteq V$ is **linearly independent** if whenever $a_1, \cdots, a_n \in \mathbb{F}$,

and $s_1, \cdots, s_n \in S$,

$$a_1 s_1 + \cdots + a_n s_n = 0 \Rightarrow a_1 = \cdots = a_n = 0.$$

2. A set $S \subseteq V$ is **spanning** if for all $v \in V$ there exists $a_1, \cdots, a_n \in \mathbb{F}$ and $s_1, \ldots, s_n \in S$ with $v = a_1 s_1 + \cdot a_n s_n$.

3. A set $\mathcal{B} \subseteq V$ is a **basis** of V if $\mathcal{B}$ is spanning and linearly independent. The size of $\mathcal{B}$ is the **dimension** of V.

Example. Let $V = \mathbb{R}^{\mathbb{N}} = \{(a_1, a_2, a_3, \cdots) \mid a_i \in \mathbb{R}\}$. Then $S = \{e_1, e_2, \cdots\}$ where $e_1 = (1, 0, 0, \cdots), \cdots$ is linearly independent but its span W is a proper subset of V.

Proof. It is important that n is finite.

1.2 Linear Maps

We consider linear maps and their relation to matrices.

Definition 5. Suppose V and W are vector spaces over $\mathbb{F}$. A map $T : V \to W$ is a **linear map** if for all $a \in \mathbb{F}, v, v' \in V$,

$$T(av + v') = aT(v) + T(v').$$

A bijective linear map is called an **isomorphism** of vector spaces.