

Complex Analysis

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Chapter 0

Introduction

0.3 Basic notations

Definition 1. We say that $U \subset \mathbb{C}$ is *open* if for every $z \in U$ there is some $\varepsilon > 0$ such that an open ball $B(z, \varepsilon) = \{w \in \mathbb{C} : |z - w| < \varepsilon\} \subset U$. Any open set U containing z is called a *neighborhood* of z .

Definition 2. A connected open subset $D \subseteq \mathbb{C}$ of the complex plane will be called a *domain*.

Chapter 1

Complex Differentiability

1.1 Complex Differentiability

Definition 3. (Complex Differentiability)

Example. Suppose we have a functor. If $G_X \not\cong G_Y$, then X and Y are not homeomorphic. If ‘shadows’ are different, then objects themselves are different too.

Proof. Suppose X and Y are homeomorphic. Then $\exists f : X \rightarrow Y$ and $g : Y \rightarrow X$, maps (maps are always continuous in this course), such that $g \circ f = 1_X$ and $f \circ g = 1_Y$. Then $f_* : G_X \rightarrow G_Y$ and $g_* : G_Y \rightarrow G_X$ such that $(g \circ f)_* = (1_X)_*$ and $(f \circ g)_* = (1_Y)_*$. Using the rules discussed previously, we get

$$g_* \circ f_* = 1_{G_X} \quad f_* \circ g_* = 1_{G_Y},$$

which means that $f_* : G_X \rightarrow G_Y$ is an isomorphism.

Chapter 3

Cauchy's Formula and its Applications

3.1 Cauchy's Integral Formula

Theorem 1 (Cauchy's Integral Formula). Suppose that $f : U \rightarrow \mathbb{C}$ is a holomorphic function on an open set U , $w \in U$ and γ is a simple positively oriented closed curve such that γ^* and the interior of γ are inside of U . Then for all w that are inside of γ we have

$$f(w) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - w} dz.$$

Remark. Same result holds for any oriented curve γ once we weigh the LHS by the winding number of a path around the point $w \notin \gamma^*$, provided that f is holomorphic on the inside of γ .

Corollary (Cauchy Formula for multiple curves). Let U be a bounded domain with piecewise C^1 boundary which has finitely many components and f be a function holomorphic in the closure of U (this means that it is holomorphic in some open domain that contains the closure of U). We parametrize each boundary component of U by a contour γ_i in such way that $i\gamma_i'(t)$ is an inward normal. This means that the 'outer' boundary is positively oriented (i.e. counter-clockwise) and all 'inner' components are negatively oriented (i.e. clockwise). Denoting $\int_{\delta U} = \sum \int_{\gamma_i}$ we have

$$\int_{\delta U} f(z) dz = 0$$

and

$$\frac{1}{2\pi i} \int_{\delta U} \frac{f(z)}{z - w} dz = f(w), w \in U.$$

3.2 Homotopy Version of Cauchy's Theorem

NON-EXAMINABLE

3.3 Applications of the Integral Formula

Corollary. If $f : U \rightarrow \mathbb{C}$ is holomorphic on an open set U , then for any $z_0 \in U$, $f(z)$ is equal to its Taylor series at z_0 and the Taylor series converges on any open disk centered at z_0 lying in U . Moreover the derivatives of f at z_0 are given by

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{\gamma(z_0, r)} \frac{f(z)}{(z - z_0)^{n+1}} dz$$

Definition 4. A function which is locally given by a power series is said to be *analytic*. We have thus shown that any holomorphic function is actually analytic, and from now on we may use the terms interchangeably.

Definition 5. A function $f : \mathbb{C} \rightarrow \mathbb{C}$ is *entire* if it is complex differentiable on the whole complex plane.

Theorem 2 (Liouville). Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be an entire function. If f is bounded then it is constant.

Remark. Liouville's theorem is another manifestation of the unique properties of holomorphic functions. In real analysis, $f(x) = \frac{1}{1+x^2}$ is real-analytic in the entire \mathbb{R} , but it is bounded. **This is also one of the first examples of a dichotomy which often appears in complex analysis.** In many cases objects are as good or as bad as they could be but nothing in between.

Theorem 3. Suppose that $p(z) = \sum_{k=0}^n a_k z^k$ is a non-constant polynomial where $a_k \in \mathbb{C}$ and $a_n \neq 0$. Then there is a $z_0 \in \mathbb{C}$ for which $p(z_0) = 0$.

Remark. The crucial point of the above proof is that on term of the polynomial dominates the behavior for large z . All proofs of the fundamental theorem hinge on essentially this point.

Theorem 4. Suppose $f : U \rightarrow \mathbb{C}$ is a continuous function on an open subset $U \subseteq \mathbb{C}$. If for any closed path $\gamma : [a, b] \rightarrow oU$ we have $\int_{\gamma} f(z) dz = 0$, then f is holomorphic.

3.4 The Identity Theorem

3.5 Isolated Singularities

Definition 6. Let $f : U \rightarrow \mathbb{C}$ be a function, where U is open. We say that $z_0 \in \overline{U}$ is a *regular* point of f if f is holomorphic at z_0 . Otherwise we say that z_0 is *singular*.

We say that z_0 is an *isolated singularity* if f is holomorphic on $B(z_0, r) \setminus \{z_0\}$ for some $r > 0$.

Definition 7. A function on an open set U which has only isolated singularities all of which are poles is called a *meromorphic* function on U . (Strictly, it is a function only defined on the complement of the poles in U .)

Definition 8. Let z_0 be an isolated singularity of function f . We say that z_0 is

- A *removable singularity* if there is a function g holomorphic in $B(z_0, r)$ for some $r > 0$ such that $f(z) = g(z)$ in $B(z_0, r) \setminus \{z_0\}$.
- A *pole of order n* if there is a function g holomorphic in $B(z_0, r)$ for some $r > 0$ such that $g(z_0) \neq 0$ and $f(z) = (z - z_0)^{-n}g(z)$ in $B(z_0, r) \setminus \{z_0\}$.

Example. Let $f(z) = \frac{\sin z}{z}$. $z_0 = 0$ is a removable singularity.

Proof. The function

$$g(z) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots$$

is entire and coincides with f outside of z_0 .

Example. Let $f(z) = \frac{\sin z}{z^{n+1}}$. This function has a pole of order n at z_0 .

Proof. It is easy to see for $z \neq 0$ we have $f(z) = \frac{g(z)}{z^n}$ where g is the entire function from the previous example. Obviously $g(0) = 1 \neq 0$.

Example. Let $f(z) = \sin(\frac{1}{z})$. This function is holomorphic in $C \setminus \{0\}$. It can be shown 0 is neither a removable singularity nor a pole, so it must be an essential singularity.

Theorem 5. (Laurent's Theorem) Suppose that $0 < r < R$ and

$$A = A(z_0, r, R) = \{z : r < |z - z_0| < R\}$$

is an annulus centered at z_0 . If $f : U \rightarrow \mathbb{C}$ is holomorphic on an open set U which contains \overline{A} , then there exist $c_n \in \mathbb{C}$ such that

$$\sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$$

The series converges for all $z \in A$ and it converges uniformly for all $z \in A(z_0, r', R')$ where $r < r' < R' < R$. The series is called the *Laurent series* of f . Moreover, the c_n are unique and are given by the following formulae:

$$c_n = \frac{1}{2\pi i} \int_{\gamma_s} \frac{f(z)}{(z - z_0)^{n+1}} dz,$$

where $s \in [r, R]$ and for any $s > 0$ we set $\gamma_s(t) = z_0 + se^{2\pi i t}$.

Remark. Given a formula for c_n in terms of f , the Laurent expansion is unique. If two series converge to the same function then they coincide term-by-term.

Remark. If f is holomorphic in $B(z_0, R)$, then for $n < 0$ the integrand in the formula for c_n is holomorphic, hence $c_n = 0$ for all $n < 0$. For $n \geq 0$ formulas for c_n are exactly the same as in Taylor's theorem so in this case the Laurent series is the same as the Taylor series.

Corollary. If $f : U \rightarrow \mathbb{C}$ is a holomorphic function and z_0 is an isolated singularity, then f has a Laurent expansion on punctured disc $B(z_0, R) \setminus \{z_0\}$ for any R such that $B(z_0, R) \setminus \{z_0\} \subset U$.

Definition 9. Let z_0 be an isolated singularity of f and $\sum c_n (z - z_0)^n$ be its Laurent expansion. Its *principal part* of f at z_0 is the sum of terms with negative powers and denoted $P_{z_0} f$. Namely,

$$P_{z_0} f = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n = \sum_{n=1}^{\infty} c_{-n} (z - z_0)^n$$

The principal part of f at z_0 converges on $\mathbb{C} \setminus \{z_0\}$ and converges uniformly on $\mathbb{C} \setminus B(z_0, r)$.

Definition 10. Let z_0 be an isolated singularity of f . Then the *residue* of f at z_0 is defined as the coefficient c_{-1} of the Laurent expansion and denoted by $\text{Res}_{z_0} f$ or $\text{Res}(f, z_0)$.

Theorem 6. Let z_0 be an isolated singularity of f . Let $\sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$ be its Laurent expansion. Then z_0 is

- A removable singularity if $c_n = 0$ for all $n < 0$. Equivalently, the principal part vanishes.
- A pole of order n if $c_{-n} \neq 0$ and $c_k = 0$ for all $k < -n$. Equivalently, the principal part is non-trivial but contains only a finite number of non-zero terms.
- An essential singularity if there are arbitrarily large n such that $c_{-n} \neq 0$. Equivalently, the principal part contains finitely many non-zero terms.

Theorem 7. (Riemann's removable singularity theorem) Suppose that U is an open subset of \mathbb{C} and $z_0 \in U$ and suppose that $f : U \setminus \{z_0\} \rightarrow \mathbb{C}$ is holomorphic. Then z_0 is a removable singularity if and only if f is bounded near z_0 .

Lemma 1. Let f be a holomorphic function in a neighborhood of z_0 . Then z_0 is a pole if and only if $|f(z)| \rightarrow \infty$ as $z \rightarrow z_0$. Moreover, in this case, the function

$$h(z) = \begin{cases} \frac{1}{f(z)}, & z \neq z_0; \\ 0, & z = z_0; \end{cases}$$

is holomorphic in a neighborhood of z_0 and the multiplicity of its zero at z_0 is equal to the order of the pole of f .

Theorem 8. (Casorati-Weierstrass) Let U be an open subset of \mathbb{C} and let $a \in U$. Suppose that $f : U \setminus \{a\} \rightarrow \mathbb{C}$ is a holomorphic function with an isolated essential singularity at a . Then for all $\rho > 0$ with $B(a, \rho) \subseteq U$, the set $f(B(a, \rho) \setminus \{a\})$ is dense in \mathbb{C} , that is, the closure of $f(B(a, \rho) \setminus \{a\})$ is all of \mathbb{C} .

Theorem 9. (Residue Theorem) Suppose that U is an open set in \mathbb{C} and γ is a closed curve that is contained in U together with its inside. Suppose that f is holomorphic on $U \setminus S$ where S is a finite set of isolated singularities of f . We also assume that f has no singularities on γ^* , that is $S \cap \gamma^* = \emptyset$. Then

$$\frac{1}{2\pi i} \int_{\gamma} f(z) dz = \sum_{a \in S} I(\gamma, a) \text{Res}_a(f).$$