Complex Analysis

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Chapter 0

Introduction

0.3 Basic notations

Definition 1. We say that $U \subset C$ is *open* if for every $z \in U$ there is some $\varepsilon > 0$ such that an open ball $B(z, \varepsilon)\{w \in \mathbb{C} : |z - w| < \varepsilon\} \subset U$. Any open set U containing z is called a *neghborhood* of z.

Definition 2. A connected open subset $D \subseteq \mathbb{C}$ of the complex plane will be called a *domain*.

Chapter 1

Complex Differentiability

1.1 Complex Differentiability

Definition 3. (Complex Differentiability)

Example. Suppose we have a functor. If $G_X \ncong G_Y$, then X and Y are not homeomorphic. If 'shadows' are different, then objects themselves are different too.

Proof. Suppose X and Y are homeomorphic. Then $\exists f : X \to Y$ and $g : Y \to X$, maps (maps are always continuous in this course), such that $g \circ f = 1_X$ and $f \circ g = 1_Y$. Then $f_* : G_X \to G_Y$ and $g_* : G_Y \to G_X$ such that $(g \circ f)_* = (1_X)_*$ and $(f \circ g)_* = (1_Y)_*$. Using the rules discussed previously, we get

$$g_* \circ f_* = 1_{G_X} \quad f_* \circ g_* = 1_{G_Y},$$

which means that $f_* : G_X \to G_Y$ is an isomorphism.

Chapter 3

Cauchy's Formula and its Applications

3.5 Isolated Singularities

Definition 4. Let z_0 be an isolated singularity of f. Then the *residue* of f at z_0 is defined as the coefficient c_{-1} of the Laurent expansion and denoted by $Res_{z_0} f$ or $Res(f, z_0)$.

Theorem 1. Let z_0 be an isolated singularity of f. Let $\sum_{-\infty}^{\infty} c_n(z - z_0)^n$ be its Laurent expansion. Then z_0 is

- A removable singularity if $c_n = 0$ for all $n < 0$. Equivalently, the principal part vanishes.
- A pole of order n if $c_{-n} \neq 0$ and $c_k = 0$ for all $k < -n$. Equivalently, the principal part is non-trivial but contains only a finite number of non-zero terms.
- An essential singularity if there are arbitrarily large n such that $c_{-n} \neq 0$. Equivalently, the principal part contains finitely main non-zero terms.

Theorem 2. (Riemann's removable singularity theorem) Suppose that U is an open subset of $\mathbb{C}$ and $z_0 \in U$ and suppose that $f : U \setminus \{z_0\} \to \mathbb{C}$ is holomorphic. Then z_0 is a removable singularity if and only if f is bounded near z_0.

Lemma 1. Let f be a holomorphic function in a neighborhood of z_0. Then z_0 is a pole if and only if $|f(z)| \to \infty$ as $z \to z_0$. Moreover, in this case, the function

$$h(z) = \begin{cases} \frac{1}{f(z)}, & z \neq z_0; \\ 0, & z = z_0; \end{cases}.$$

is holomorphic in a neighborhood of z_0 and the multiplicty of its zero at z_0 is equal to the order of the pole of f.

Theorem 3. (Casorati-Weierstrass) Let U be an open subset of $\mathbb{C}$ and let $a \in U$. Suppose that $f : U \setminus \{a\} \to \mathbb{C}$ is a holomorphic function with an isolated essential singularity at a. Then for all $\rho > 0$ with $B(a, \rho) \subseteq U$, the set $f(B(a, a\rho) \setminus \{a\}$ is dense in $\mathbb{C}$, that is, the closure of $f(B(a, \rho) \setminus \{a\})$ is all of $\mathbb{C}$.

Theorem 4. (Residue Theorem) Suppose that U is an open set in $\mathbb{C}$ and γ is a closed curve that is contained in U together with its inside. Suppose that f is holomorphic on $U \setminus S$ where S is a finite set of isolated singularities of f. We also assume that f has no singularities on $\gamma\star$, that is $S \cap \gamma\star = \emptyset$. Then

$$\frac{1}{2\pi i} \int_\gamma f(z)dz = \sum_{a \in S} I(\gamma, a) Res_a(f).$$