Notes to Hands-on Intro to R

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Section 6: $Z = X + Y$

Let X and Y be random variables which are independent and have a uniform distribution in $[0, 1]$. Our goal is to find the cdf and pdf[1] for the random variable $Z = X + Y$.
Their joint pdf $f_{X,Y}(x, y)$ (due to the independence of X and Y) is given by:

$$\begin{aligned} f_{X,Y}(x,y) &= f_X(x)\, f_Y(y) \\ &= \begin{cases} 1, & 0 \le x \le 1 \text{ and } 0 \le y \le 1 \\ 0, & \text{else} \end{cases} \end{aligned}$$

Thus, the random variable Z can take on values in the interval $[0, 2]$. In order to derive the cdf and pdf for Z, we need to consider 2 cases:

1. $0 \le z \le 1$
2. $1 \le z \le 2$

Case 1

The cdf $F_Z(Z \le z)$, where $0 \le z \le 1$ requires the calculation of the following integral:

$$\begin{aligned} F_Z(Z \le z) &= \int_{x=0}^{x=z} \int_{y=0}^{y=z-x} f_{X,Y}(x,y)\, dx\, dy \\ &= \frac{z^2}{2} \end{aligned}$$

Due to the fact that the joint density function $f_{X,Y}(x, y)$ is 1 within the area $0 \le x \le 1$, $0 \le y \le 1$ the aforementioned integral is identical to the area of the triangle with the following vertices: $(0, 0)$ and $(0, z)$ and $(z, 0)$.

[1] cdf stands for the cumulative density function. pdf stands for the probability density function. If $f_X(x)$ is the pdf associated with the random variable X, then $F_X(X \le x) := \int_{-\infty}^{x} f_X(t)dt$ is the corresponding cdf.

Case 2

The cdf $F_Z(Z \leq z)$, where $1 \leq z \leq 2$ requires the calculation of the following integral:

$$
\begin{aligned}
F_Z(Z \leq z) &= 1 - \int_{x=z-1}^{x=1} \int_{y=z-x}^{y=1} f_{X,Y}(x,y)\, dx\, dy \\
&= 1 - \frac{(2-z)^2}{2}
\end{aligned}
$$

Using the same argument as in Case 1, the aforementioned integral is identical to the area of a triangle. The triangle's vertices are $(z-1,1)$ and $(1,z-1)$ and $(1,1)$. Its area is therefore $\dfrac{(2-z)^2}{2}$. The cdf $F_Z(Z \leq z)$ is the complementary area.

Calculation of the pdf

Above, we obtained the cdf $F_Z(z)$:

$$
F_Z(z) = \begin{cases} \frac{z^2}{2} & 0 \leq z \leq 1 \\ 1 - \frac{(2-z)^2}{2} & 1 \leq z \leq 2 \\ 0 & \text{else} \end{cases}
$$

From the cdf $F_Z(z)$ its pdf $f_Z(z)$ can be easily derived:

$$
f_Z(z) = \frac{dF_Z(z)}{dz}
$$

The pdf $f_Z(z)$ thus becomes:

$$
f_Z(z) = \begin{cases} z & 0 \leq z \leq 1 \\ 2 - z & 1 \leq z \leq 2 \\ 0 & \text{else} \end{cases}
$$