Boosting

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What you will know

→ The idea behind boosting

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→ How to create a strong and efficient learning algorithm

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→ What is AdaBoost and why is it so successful?

Let’s talk about training a model

How to train a machine learning model

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But there is one problem...

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...so what can we do?

A first idea...

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Let's call ERM on a simple class a **weak learner**. We will formally define it later...

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But first, let's get back to weak learning.

Weak Learnability

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In weak learning, we only want the error to be less than 50%.

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...but how does this help us?

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Lets look at an example (Decision Stumps)

Spam detection with decision stumps

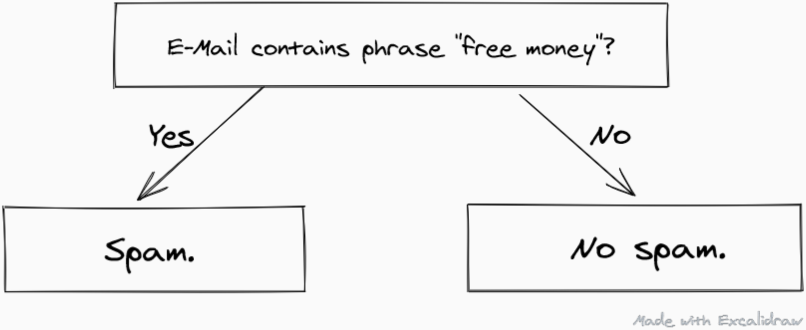

Figure 1: This is a Decision Stump.

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$$\min_{j \in [d]} \min_{\theta \in \mathbb{R}} \left(\sum_{i: y_i = 1}^{m} D_i \mathbb{1}_{[x_{i,j} > \theta]} + \sum_{i: y_i = -1}^{m} D_i \mathbb{1}_{[x_{i,j} \leq \theta]} \right)$$

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$\boxed{\text{This can be solved in } \mathcal{O}(dm)\text{!}}$

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But how to do that? The AdaBoost algorithm will tell us...

AdaBoost

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Let's look at this in more detail...

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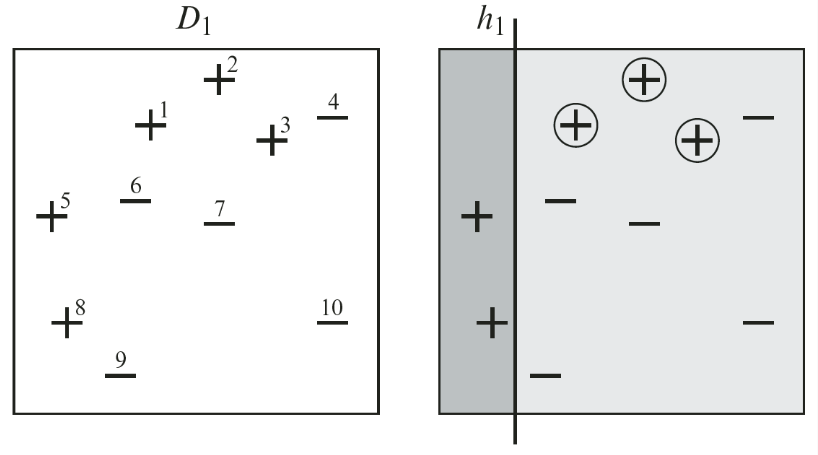
3. Update the weights $D_i^{(t)}$ like this

$$D_i^{(t+1)} = \frac{D_i^{(t)} \exp\left(-w_t y_i h_t(\mathbf{x}_i)\right)}{\sum_{j=1}^{m} D_j^{(t)} \exp\left(-w_t y_j h_t(\mathbf{x}_j)\right)}$$

A step by step example[2]



[2]Taken from the book *Boosting: Foundations and Algorithms* written by Freund and Schapire [3]. You can read it for free at `https://mitpress.mit.edu/books/boosting`

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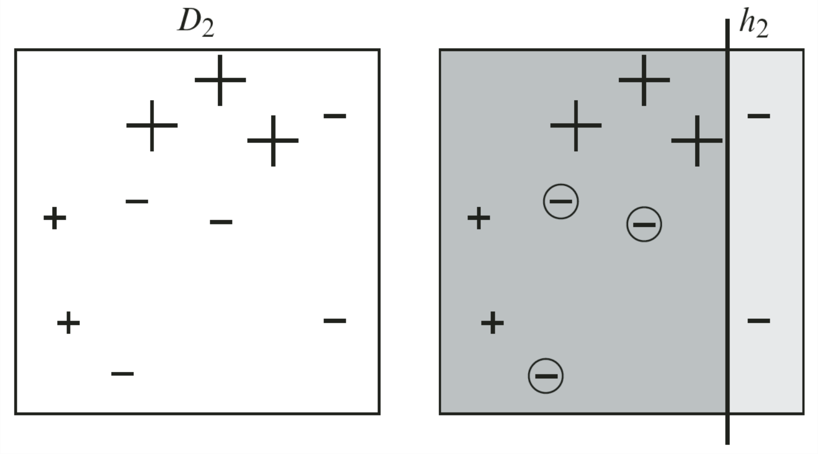

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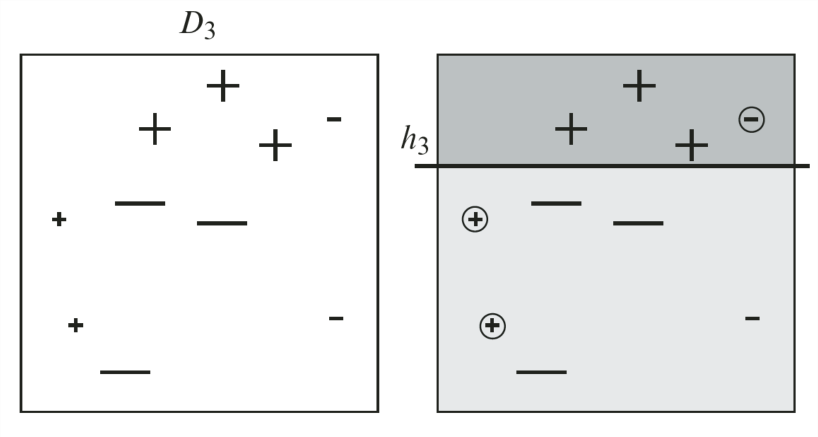

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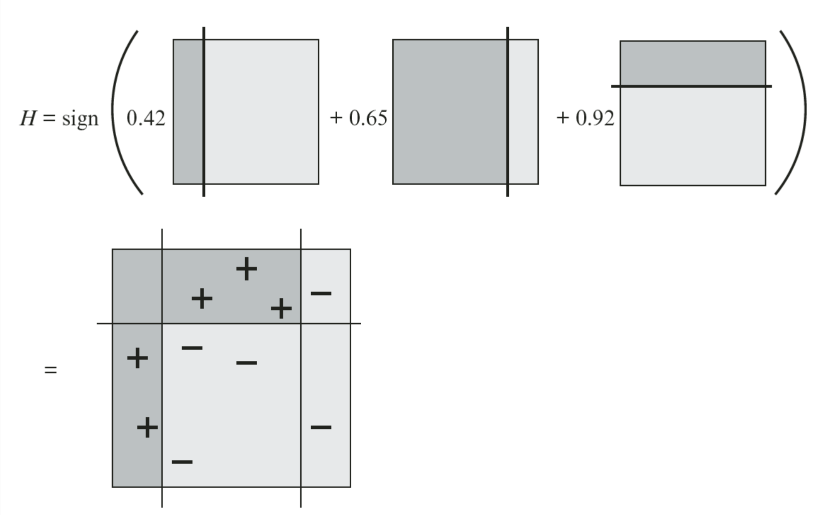

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...but what about the out of sample error?

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The End.

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