Short proof of Feynman parametrizasion formula.

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Feynmans denominator formula , or simply feynman parametrizasion, is quite usefull and a well known result in high energy physics for evaluating certain tyles of Feynman diagrams, but i have not explicitly found (even though it of course might be out there on the internet somewhere) a detailed proof of this formula, so here is my attempt at such a detailed proof[1]. This is as well an excercise for me to make this derivation, and I hope you will find it usefull:

Theorem 1 (Feynman parametrizasion or Feynmans denominator formula)**.**

$$\frac{1}{\prod_{i=1}^{n} A_i^{\alpha_i}} = \frac{\Gamma\left(\sum_{i=1}^{n}\alpha_i\right)}{\prod_{i=1}^{n}\Gamma(\alpha_i)} \int_0^1 dx_1 \dots dx_n \, \delta\left(1 - \sum_{i=1}^{n} x_i\right) \frac{\left(\prod_{i=1}^{n} x_i^{\alpha_i - 1}\right)}{\left(\sum_{i=1}^{n} A_i x_i\right)^{\left(\sum_{i=1}^{n}\alpha_i\right)}}$$

Proof. First we note that the definition of the Euler-Γ function is:

$$\Gamma(\alpha) = \int_0^\infty dt \left(t^{\alpha-1} e^{-t}\right)$$

and that for the case where $\alpha \in \mathbb{N}$, we have that:

$$\Gamma(\alpha) = (\alpha - 1)!$$

It is not hard to generalize this formula a tiny amount to make another family of functions:

$$\Gamma_A(\alpha) = \int_0^\infty dt \left(t^{\alpha-1} e^{-At}\right) = \frac{\Gamma(\alpha)}{A^\alpha}$$

Now, imagine that we want the product metween n of of such functions, all with different A_i and α_i, where the subscript i is introduced only to distinguish between different A_i and α_i.

$$\prod_{i=1}^{n} \Gamma_{A_i}(\alpha_i) = \prod_{i=1}^{n} \int_0^\infty dt_i \left(t_i^{\alpha_i - 1} e^{-A_i t_i}\right) = \frac{\prod_{i=1}^{n}\Gamma(\alpha_i)}{\prod_{i=1}^{n} A_i^{\alpha_i}}$$

It is important to note here as well that this identity is not necessarily true for infinite values of n, thus we restrict ourselves to the case of finite n. To proceed from here, let the sum of all the integraiton variable be s, such that:

$$s = \sum_{i=1}^{n} t_i$$

To clarify, the ranges of each of the $t_i = [0, \infty]$, $\forall i \in \{1, 2, ..., n\}$, and thus we have that $s = [0, \infty]$ we can expand the integral part of the previous equation, by using a dirac-δ function such that:

$$\prod_{i=1}^{n} \Gamma_{A_i}(\alpha_i) = \int_0^\infty ds \left(\prod_{i=1}^{n} \int_0^\infty dt_i \left(t_i^{\alpha_i - 1} e^{-A_i t_i}\right)\right) \delta\left(s - \sum_{i=1}^{n} t_i\right) = \frac{\prod_{i=1}^{n}\Gamma(\alpha_i)}{\prod_{i=1}^{n} A_i^{\alpha_i}}$$

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[1] It should be noted though, that indirect proofs are plentifull and some quite good as well. Look no further than the wikipedia page for Feynman parametizasion, and you will see derivations of some simplified cases, but not the full machinery.

From here, new normalized variables x_i are introdused

$$x_i = \frac{t_i}{s} \quad , \quad x_i \in [0,1]$$

We can now make the following substitution[2]

$$\prod_{i=1}^{n} \Gamma_{A_i}(\alpha_i) = \int_0^\infty ds \left(\prod_{i=1}^{n} \int_0^1 dx_i \, s \left((sx_i)^{\alpha_i - 1} e^{-A_i s x_i} \right) \right) \delta \left(s \left(1 - \sum_{i=1}^{n} x_i \right) \right)$$

and using the deltafunction identity that:

$$\delta \left(s \left(1 - \sum_{i=1}^{n} x_i \right) \right) = \frac{1}{s} \delta \left(1 - \sum_{i=1}^{n} x_i \right)$$

we can write the above equation as:

$$\begin{aligned}
\prod_{i=1}^{n} \Gamma_{A_i}(\alpha_i) &= \int_0^\infty ds \left(\prod_{i=1}^{n} \int_0^1 dx_i \, s \left((sx_i)^{\alpha_i - 1} e^{-A_i s x_i} \right) \right) \frac{1}{s} \delta \left(1 - \sum_{i=1}^{n} x_i \right) \\
&= \int_0^\infty ds \left(\int_0^1 dx_1 \ldots dx_n \, s^n \left(\prod_{i=1}^{n} s^{\alpha_i - 1} x_i^{\alpha_i - 1} e^{-A_i s x_i} \right) \right) \frac{1}{s} \delta \left(1 - \sum_{i=1}^{n} x_i \right) \\
&= \int_0^1 dx_1 \ldots dx_n \left(\prod_{i=1}^{n} x_i^{\alpha_i - 1} \right) \delta \left(1 - \sum_{i=1}^{n} x_i \right) \int_0^\infty ds \left(\frac{1}{s} s^n s^{\sum_{i=1}^{n} (\alpha_i - 1)} e^{-s \sum_{i=1}^{n} (A_i x_i)} \right) \\
&= \int_0^1 dx_1 \ldots dx_n \left(\prod_{i=1}^{n} x_i^{\alpha_i - 1} \right) \delta \left(1 - \sum_{i=1}^{n} x_i \right) \int_0^\infty ds \left(s^{\sum_{i=1}^{n} (\alpha_i) - 1} e^{-s \sum_{i=1}^{n} (A_i x_i)} \right)
\end{aligned}$$

and noting that the s-integral has the exact form of the $\Gamma_A(\alpha)$ function, we can make the identification,

$$\int_0^\infty ds \left(s^{\sum_{i=1}^{n} (\alpha_i) - 1} e^{-s \sum_{i=1}^{n} (A_i x_i)} \right) = \Gamma_{\sum_{i=1}^{n} A_i x_i} \left(\sum_{i=1}^{n} \alpha_i \right) = \frac{\Gamma \left(\sum_{i=1}^{n} \alpha_i \right)}{\left(\sum_{i=1}^{n} A_i x_i \right)^{\left(\sum_{i=1}^{n} \alpha_i \right)}}$$

and thus:

$$\begin{aligned}
\prod_{i=1}^{n} \Gamma_{A_i}(\alpha_i) &= \int_0^1 dx_1 \ldots dx_n \left(\prod_{i=1}^{n} x_i^{\alpha_i - 1} \right) \delta \left(1 - \sum_{i=1}^{n} x_i \right) \int_0^\infty ds \left(s^{\sum_{i=1}^{n} (\alpha_i) - 1} e^{-s \sum_{i=1}^{n} (A_i x_i)} \right) \\
&= \int_0^1 dx_1 \ldots dx_n \left(\prod_{i=1}^{n} x_i^{\alpha_i - 1} \right) \delta \left(1 - \sum_{i=1}^{n} x_i \right) \frac{\Gamma \left(\sum_{i=1}^{n} \alpha_i \right)}{\left(\sum_{i=1}^{n} A_i x_i \right)^{\left(\sum_{i=1}^{n} \alpha_i \right)}} \\
&= \Gamma \left(\sum_{i=1}^{n} \alpha_i \right) \int_0^1 dx_1 \ldots dx_n \, \delta \left(1 - \sum_{i=1}^{n} x_i \right) \frac{\left(\prod_{i=1}^{n} x_i^{\alpha_i - 1} \right)}{\left(\sum_{i=1}^{n} A_i x_i \right)^{\left(\sum_{i=1}^{n} \alpha_i \right)}}
\end{aligned}$$

since the $\Gamma \left(\sum_{i=1}^{n} \alpha_i \right)$ factor is independent of the x_i Now, using what we have found, remember what the goal was; to find a integral-parametrized form of $\frac{1}{\prod_{i=1}^{n} A_i^{\alpha_i}}$, we can also remember that the initial representation we found for the $\Gamma_A(\alpha)$ function, we have actually (almost) arrivedat the end result:

$$\prod_{i=1}^{n} \Gamma_{A_i}(\alpha_i) = \frac{\prod_{i=1}^{n} \Gamma(\alpha_i)}{\prod_{i=1}^{n} A_i^{\alpha_i}} = \Gamma \left(\sum_{i=1}^{n} \alpha_i \right) \int_0^1 dx_1 \ldots dx_n \, \delta \left(1 - \sum_{i=1}^{n} x_i \right) \frac{\left(\prod_{i=1}^{n} x_i^{\alpha_i - 1} \right)}{\left(\sum_{i=1}^{n} A_i x_i \right)^{\left(\sum_{i=1}^{n} \alpha_i \right)}}$$

[2] I will ask you to forgive my notation; when I write $\int_0^1 dx_1 \ldots dx_n$, I implicitly refer to the nested integrals $\int_0^1 dx_1 \ldots \int_0^1 dx_n$ it is simply a notation that in the attempt of minimizing the clutter in writing, abuse the fact that $x_i \in [0,1] \quad , \forall i \in \{1, \ldots, n\}$

whereby we can conclude with the formula:

$$\frac{1}{\prod_{i=1}^{n} A_i^{\alpha_i}} = \frac{\Gamma\left(\sum_{i=1}^{n}\alpha_i\right)}{\prod_{i=1}^{n}\Gamma(\alpha_i)} \int_0^1 dx_1 \ldots dx_n\, \delta\left(1-\sum_{i=1}^{n} x_i\right) \frac{\left(\prod_{i=1}^{n} x_i^{\alpha_i - 1}\right)}{\left(\sum_{i=1}^{n} A_i x_i\right)^{\left(\sum_{i=1}^{n}\alpha_i\right)}}$$

$\square$