

Perspectives: Equivalence between functions and vectors

Christian Berrig ^{*} ¹

¹*Institute of Science and Environment, RUC*

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1 Equivalence between functions and vectors

As is well known one way of describing a function is in terms of the functions domain and co-domain. This notion of sets, defining what a function maps from and to, is useful in describing very basic properties of the functions in question. Take as an example a function $f : \mathbb{R} \mapsto \mathbb{R}$. What such a function does is exactly, with this statement, to associate a value (in the real numbers, namely the image) to any real number (in the domain) given as an input. This of course is also true for extended number-systems, like the complex numbers, but for clarity, we stick to the reals.

Let us then see what a vector is; beside the geometric interpretation, and the specific parametrization a vector is given, what it does is that for any particular n -dim vector, there is for each of the entries given a (real) number, associated with the index. In other words, we can use the notion that we know from functions in a similar way such that: $v : i \rightarrow v_i$

Note here, that working in the abstract, we don't need the vector to be finite dimensional (although it can be), and as such we can consider vectors as functions, from the natural numbers (the domain of the indices) to the reals (the entries of the vectors) $v : \mathbb{N} \mapsto \mathbb{R}$

Thus vectors and functions are essentially the same type of object, let's just call them functions, but working on different domains.

1.1 Inner products and projections

From the considerations above, we suddenly can gain an intuition from the inner product between functions of continuous parameters. Let us consider the two forms (discrete vs continuous domain) side by side:

$$\begin{aligned}\langle f, g \rangle &= \int_{-\infty}^{\infty} dx [f^*(x)g(x)] \\ \langle f, g \rangle &= \sum_i f_i^* g_i\end{aligned}$$

where the $*$ denotes complex conjugation. In the first case f, g are considered functions and in the latter case they are vectors and the sum is over the relevant set of indices. But note here how naturally the integral formulation is interpreted as the inner product over vectors with a continuous index, being the independent variable.

Further note what the interpretation of the inner product can yield further insights to the integral interpretation; The inner product for vectors are the projection of vector f onto the direction of g scaled with length of g . Thus the integral over fg can be considered a projection operator, and the integral over f^2 be considered the Euclidean norm of f , squared. This can also be seen in the notation and naming convention of typical norms.

^{*}Electronic address: chrberri@protonmail.ch

The "usual" discrete Euclidean norm is noted ℓ_2 -norm and defined:

$$\ell_2(f) = \|f\|_2 = \left(\sum_i |f_i|^2 \right)^{1/2}$$

And is generalized to the ℓ_p norm as

$$\ell_p(f) = \|f\|_p = \left(\sum_i |f_i|^p \right)^{1/p}$$

But an equivalent for continuous variables exists as well, namely the L_2 -norm:

$$L_2(f) = \|f\|_2 = \left(\int_{-\infty}^{\infty} dx |f(x)|^2 \right)^{1/2}$$

which is generalized to the L_p norm in the similar way:

$$L_p(f) = \|f\|_p = \left(\int_{-\infty}^{\infty} dx |f(x)|^p \right)^{1/p}$$

1.2 Orthogonality

In the same sense that vectors can be mutual orthogonal, in the sense that the projection between the two vectors is zero, the same notion can be carried over to the continuous variable case. This means that orthogonality between functions suddenly makes sense, both in sense of functions and vectors being essentially the same type of object, but also as a geometric concept, in terms of projections. Canonical examples of such functions are e.g. legendre polynomials and complex exponentials.

Examining complex exponentials as a specific example, provides us with an alternative route to fourier transforms than the computational convenient one most people are probably introduced to.

INSERT EXAMPLE HERE!

When this notion of orthogonality between functions are understood, a very powerfull statement can be made, namely that functions can be decomposed into sets of orthogonal functions, jyst equivalent to how vectors can be decomposed into a set of orthogonal basis-vacors. This decomposition takes the form:

$$f(x) = \sum_i \langle b_i, f \rangle b_i(x)$$

Here i have chosen to write the representation out as a sum, but note here that we do nao neccessarily have a discrete set of basis-functions; these too can line in a continuous variable space, such that:

$$f(x) = \int_{-\infty}^{\infty} dy \langle b_y, f \rangle b_y(x)$$

Ideally the "basis-functions" are also normal, in the inner product sense, such that:

$$\langle b_i, b_j \rangle = \delta_{ij}$$

but as this is not always the case, the functions f_i can be "normalized", such that they fulfill this requirement.

Take the fourier transform as an example: The basis vectors/functions are $e^{i\omega t}$, such that the projection onto these vectors for any (sufficiently well behaved) function $f(t)$ is:

$$\tilde{f}(\omega) = \langle e^{i\omega t}, f(t) \rangle = \int_{-\infty}^{\infty} dt e^{-i\omega t} f(t)$$

and can thus be representet in the following way:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{i\omega t} \langle e^{i\omega t}, f(t) \rangle = \int_{-\infty}^{\infty} d\omega e^{i\omega t} \tilde{f}(\omega)$$

The factor of $\frac{1}{2\pi}$ in fromt is simply the normalilzasion required for orthonormal basis vectors, as mentioned earlier.

1.3 Integral transforms

Now an extra layer of nomenclature is introduced, that is never the less usefull as this is the guise of these subjects, as normally introduced. The concept of integral transforms, first seem very strange, due to the nature of the subject introduced, as a way of transforming one variable into another through an integral. Computationally, this is done in the following way, that we will soon understand in terms of abstracty vectors. say, we have a function of 2 variables $K(y, x)$ and multiplying this with the target function $f(x)$, we can "integrate out" the variable x , leaving us with a function:

$$\tilde{f}(y) = \int_{-\infty}^{\infty} dx K(y, x) f(x)$$

The function $K(y, x)$ is called the integration kernel, and is what defines the "type" of integral transform;

e^{iyx} is the kernel for the fourier transform

e^{yx} is the kernel for the laplace transform

etc. etc.

But we can now see what the Fourier transform is explicitly: It is the inner product between the target function $f(t)$ and set of basis-vectors, indexed by the continuous variable ω and t . And the inverse Fourier transform is simply the representation of the target-function as expansion/decomposition in terms of the basis vectors, using the coefficients given by the inner product.

1.4 Kernels

The kernel functions by themselves can therefore also be understood in the context of linear transforms, as an analogy to what is learned from linear algebra. In the same manner that a matrix is a linear transform that takes a vector from space V and returns a vector in space W , the matrix M is a linear mapping:

$$M : V \mapsto W$$

such that:

$$w_j = \sum_i M_{ji} v_i$$

we can consider an integration kernel as the transform from X (the space of x) to Y (space of variable y) in the following way:

$$K : X \mapsto Y$$

such that:

$$\tilde{f}(y) = \int_X dx K(y, x) f(x)$$

and seen in this way, when abstracting the target function away, we consider

$$\int_X dx K(y, x) : X \mapsto Y$$

the linear transform. The kernel is thus the continuous variable equivalent to a matrix and the integral transform the continuous variable analogue to matrix vector multiplication, resulting in a vector in the co-domain space.

1.5 Propagators and (abstract) algebra

1.5.1 composition

Finally layering on another layer of abstraction, we can now talk about composition

1.5.2 matrix-matrix multiplication: composition of linear transforms in discrete spaces

In the same manner that one can combine to linear transforms over discrete spaces as a multiplication of matrices, such that

1.5.3 kernel transforms: composition of linear transforms in continuous spaces

1.5.4 abstract algebra

1.5.5 propagators