ANALYSIS OF AN ELLIPTIC EQUATION FROM POPULATION DYNAMICS

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Abstract. We study the equation

$$-\Delta u(x) = f(x, u(x)), \quad x \in \mathbb{R}^d,$$

where the reaction term f is periodic in x. We prove the existence of the principal eigenvalue λ_1 for the associated linearised problem with periodicity condition. Then, we prove that the sign of this eigenvalue is a criterium in the existence result for this equation. Finally, we study numerically the influence of the shape of f on λ_1.

Contents

1. Introduction

1.1. **Definitions and assumptions.** Berestycki et al. ([2]) were insterested in the existence of a stationary solution for the reaction-diffusion equation

$$u_t(x) - \nabla \cdot (A(x)\nabla u(x)) = f(x, u(x)) \qquad (t \in \mathbb{R}_+,\ x \in \mathbb{R}^d), \tag{1.1}$$

that is a positive solution of the following elliptic equation

$$-\nabla \cdot (A(x)\nabla u(x)) = f(x, u(x)) \qquad (x \in \mathbb{R}^d).$$

The equation (1.1) can be used to model the spatial propagation of biological species (bacteria, insects, plants, etc.). The nonlinear reaction term f considered here may be of the form

$$f(x, s) = s(\mu(x) - s), \tag{1.2}$$

where $\mu(x)$ represents the instinsic growth rate of the population. The model takes into account the possible heterogeneity of the environment : the diffusion coefficient $A(x)$ and the intrinsic growth rate of the population $\mu(x)$ depend on the space variable x. Regions where $\mu(x)$ is relatively high represents more favourable zones than low $\mu(x)$ regions. We first give definitions and make assumptions about the functions and quantities involved in this system.

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Let $d \in \mathbb{N}^*$ be fixed and let $L_1, L_2, \dots, L_d > 0$ be given real numbers. We will denote by $\mathcal{O}$ the following subset of $\mathbb{R}^d$

$$\mathcal{O} = (0, L_1) \times (0, L_2) \times \ldots \times (0, L_d).$$

Definition 1.3. We say that a function $g \colon \mathbb{R}^d \mapsto \mathbb{R}$ is periodic if

$$g(x_1, x_2, \dots, x_k + L_k, \dots, x_d) = g(x_1, x_2, \dots, x_d)$$

for all $k \in \{1, 2, \dots, d\}$.

The diffusion matrix $A(x) = (a_{ij}(x))_{1 \le i,j \le d}$ is periodic, of class $C^{1,\alpha}$, and is uniformly elliptic in the sense that

$$\exists \alpha_0 > 0,\ \forall x \in \mathbb{R}^d,\ \forall \xi \in \mathbb{R}^d, \quad \sum_{i=1}^{d} \sum_{j=1}^{d} a_{ij}(x) \xi_i \xi_j \ \ge\ \alpha_0 |\xi|^2.$$

Locally in s (i.e. for s in compact sets), the function

$$f \colon \begin{cases} \mathbb{R}^d \times \mathbb{R}_+ & \longrightarrow \mathbb{R} \\ \quad (x, s) & \longmapsto f(x, s) \end{cases}$$

is of class $C^{0,\alpha}$ with respect to x. Additionally, f is locally Lipschitz-continuous with respect to s and periodic with respect to x. We set

$$f(x, 0) = 0 \qquad (x \in \mathbb{R}^d).$$

Moreover, f is of class $C^1(\mathbb{R}^d \times \mathbb{R}_+)$, and the derivative of $s \longmapsto f(x, s)$ evaluated in 0 is written as

$$f_u(x, 0) := \lim_{s \to 0^+} \frac{f(x, s)}{s} \qquad (x \in \mathbb{R}^d).$$

For the existence result, we will also assume that

$$\forall x \in \mathbb{R}^d, \quad s \longmapsto \frac{f(x, s)}{s} \quad \text{is decreasing in } s, \tag{1.4}$$

and/or

$$\exists M \ge 0, \text{ such that } \sup_{s \ge M,\ x \in \mathbb{R}^d} f(x, s) \le 0. \tag{1.5}$$

For example, f could be as in (1.2) with μ and ν periodic. In this work, we choose for simplicity $A(x)$ to be the identity matrix I, and thus we study the equation

$$-\Delta u(x) = f(x, u(x)) \qquad (x \in \mathbb{R}^d). \tag{1.6}$$

1.2. **Theorems.** Here we state without proof some theorems used to prove the results that follow.

Theorem 1.7 (Krein-Rutman theorem)**.** *Let X be a Banach space, $Y \subset X$ a cone such that $\mathring{Y} \neq \emptyset$. Let $T \in \mathscr{L}(X)$ be strongly positive with respect to Y (i.e. $T(Y \setminus \{0\}) \subset \mathring{Y}$) and compact. Then, there exists a unique $\lambda \ge 0$ and a unique normalized vector $\phi \in \mathring{Y}$ such that*

$$T\phi = \lambda \phi.$$

λ is called the principal eigenvalue, and ϕ the principal eigenvector.

Theorem 1.8 (Strong maximum principle). *Assume that $\Omega \subset \mathbb{R}^d$ is open and bounded, $\varphi \in C^0(\overline{\Omega}) \cap C^2(\Omega)$, $\alpha \in C^0(\overline{\Omega})$ with $\alpha(x) \leq 0$, and $(\Delta + \alpha(x))\varphi \geq 0$. If there exists $x_0 \in \Omega$ such that*

$$\varphi(x_0) = \max_{\overline{\Omega}} \varphi$$

(i.e. φ atteins its maximum in the interior of Ω), then φ is constant equal to $\varphi(x_0)$.

Remark 1.9. Replacing φ with $-\varphi$ and using that the maximum of a function ψ is equal to $-\min(-\psi)$, we remark that under the same hypothesis on Ω, φ and α, the strong maximum principle can also be stated as follows :

If $(\Delta + \alpha)\varphi \leq 0$ and if there exists $x_0 \in \Omega$ such that

$$\varphi(x_0) = \min_{\overline{\Omega}} \varphi$$

(i.e. φ atteins its minimum in the interior of Ω), then φ is constant equal to $\varphi(x_0)$.

The results on Sobolev embeddings may be found in [1]. The results on elliptic regularity may be found in Gilbarg-Trudinger [3] (for example Theorem 9.11).

2. The principal eigenvalue

Theorem 2.1. *There is a unique real number λ_1 such that there exists a function $\phi > 0$ periodic with $\|\phi\|_\infty = 1$, which satisfies*

$$-\Delta\phi(x) - f_u(x,0)\phi(x) = \lambda_1\phi(x) \qquad (x \in \mathbb{R}^d). \tag{2.2}$$

λ_1 is called the principal eigenvalue of the operator $\mathscr{L}_0 \colon \phi \longmapsto -\Delta\phi - f_u(x,0)\phi$.

Proof. We want to prove the existence and uniqueness of λ_1. Let X be the Banach space

$$X = \{\varphi \in C^0(\mathcal{O}) \colon \varphi \text{ is periodic}\}$$

endowed with the norm $\|\varphi\|_\infty = \max_{\mathcal{O}} |\varphi|$. We choose the cone Y to be the closure of

$$\mathring{Y} = \{\varphi \in X \colon \varphi > 0\}$$

$\mathring{Y}$ is indeed open: if ψ lies in this set,

$$\|\psi\|_\infty = \sup_{\mathcal{O}} \psi > 0,$$

since ψ is positive and attains its maximum. Consequently, we can choose r real such that $0 < r < \|\psi\|_\infty$. Set

$$B(\psi, r) = \{\varphi \in X \colon \|\varphi\|_\infty < r\}.$$

For $\phi \in B(\psi, r)$,

$$\|\varphi\|_\infty \;\geq\; \big|\underbrace{\|\varphi - \psi\|_\infty}_{<r<\|\psi\|_\infty} - \|\psi\|_\infty\big| \;=\; \|\psi\|_\infty - \|\varphi - \psi\|_\infty \;\geq\; \|\psi\|_\infty - r \;>\; 0.$$

Thus $\phi \in B(\psi, r)$, and $B(\psi, r) \subset \mathring{Y}$.

Set $r(x) = f_u(x,0)$ and $c > 0$. Let $A\colon \mathscr{D}(A) \longmapsto L^2(\mathcal{O})$ be the operator defined by

$$\begin{aligned}&\mathscr{D}(A) = \{u \in H^2(\mathcal{O})\colon u \text{ is periodic, } \partial_k u \text{ is periodic, } k \in \{1 \dots d\}\}\\ &Au(x) = -\Delta u(x) + cu(x) - r(x)u(x).\end{aligned}$$

A is self-adjoint, and

$$\begin{aligned}\langle Au, u\rangle_{L^2} &= \langle -\Delta u, u\rangle_{L^2} + c\|u\|_{L^2}^2 - \langle r(x)u, u\rangle_{L^2}\\ &= -\sum_{k=1}^{d} \int_{\tilde{\mathcal{O}}} \Big(\int_0^{L_k} (\partial_{x_k}^2 u) u dx_k \Big) d\tilde{x} + c\|u\|_{L^2}^2 - \langle r(x)u, u\rangle_{L^2},\end{aligned}$$

where $\tilde{\mathcal{O}} = (0, L_1) \times (0, L_2) \times \dots \times (0, L_{k-1}) \times (0, L_{k+1}) \times \dots \times (0, L_d)$, and $\tilde{x} = (x_1, x_2, \dots, x_{k-1}, x_{k+1}, \dots, x_d)$. Hence,

$$\langle Au, u\rangle_{L^2} = -\sum_{k=1}^{d} \int_{\tilde{\mathcal{O}}} \Big(\int_0^{L_k} -(\partial_{x_k} u)^2 dx_k + 0 \Big) d\tilde{x} + c\|u\|_{L^2}^2 - \langle r(x)u, u\rangle_{L^2},$$

because $x \longmapsto (\partial_{x_k} u)(x)u(x)$ is periodic. Thus, we obtain

$$\begin{aligned}\langle Au, u\rangle_{L^2} &= -\sum_{k=1}^{d} \int_{\mathcal{O}} -(\partial_{x_k} u)^2 dx + c\|u\|_{L^2}^2 - \langle r(x)u, u\rangle_{L^2}\\ &= \|\nabla u\|_{L^2}^2 + c\|u\|_{L^2}^2 - \langle r(x)u, u\rangle_{L^2}\\ &\geq \underbrace{(c - \max_{\mathcal{O}} r)}_{>0} \|u\|_{L^2}^2,\end{aligned}$$

for c large enough. Consequently, 0 lies in the resolvent set of A, i.e. $A : \mathscr{D}(A) \longmapsto L^2(\mathcal{O})$ is bijective and A^{-1} lies in $\mathscr{L}(L^2(\mathcal{O}))$.

We now define the operator $T\colon X \longmapsto X$ by

$$T(\varphi) = J \circ A^{-1}\varphi,$$

where $J\colon W^{2,p} \longmapsto X$ is an embedding. We first show that $A^{-1} : X \longmapsto W^{2,p}$ (for any $1 < p < \infty$) is bounded. For $\varphi \in X$, $A^{-1}\varphi = u \in \mathscr{D}(A)$, which is equivalent to $Au = \varphi \in X$. Thus, we have

$$\Delta u = \varphi + (c - r(x))u.$$

For the next steps we use elliptic regularity and Sobolev embedding results. Being continuous, φ lies in any $L^p(\mathcal{O})$. Moreover, for $p \geq 2$, $W^{2,2}(\mathcal{O}) \subset L^p(\mathcal{O})$ continously

$$\|(c - r(x))u\|_{L^p} \lesssim \|u\|_{L^p} \lesssim \|u\|_{W^{2,2}(\mathcal{O})}$$

(where $\lesssim$ means inferior up to a strictly positive constant), in particular $x \longmapsto (c - r(x))u(x)$ lies in L^p. Thus $u \in W^{2,p}(\mathcal{O})$ and we have the following estimate (where $c_{\mathcal{O}} > 0$)

$$\begin{aligned}\|u\|_{W^{2,p}(\mathcal{O})} &\leq c_{\mathcal{O}}\big(\|u\|_{L^p} + \|\varphi + (c - r(x))u\|_{L^p}\big)\\ &\lesssim \|u\|_{L^p} + \|\varphi\|_{L^p}.\end{aligned} \tag{2.3}$$

Since A^{-1} is bounded in L^2,

$$\|u\|_{L^2(\mathcal{O})} = \|A^{-1}\varphi\|_{L^2(\mathcal{O})} \lesssim \|\varphi\|_{L^2(\mathcal{O})},$$

thus (2.3) (with $p = 2$) yields

$$\|u\|_{W^{2,2}(\mathcal{O})} \lesssim \|\varphi\|_{L^2} \lesssim \|\varphi\|_{L^\infty}$$

Hence, (2.3) and the continuous embedding $W^{2,2}(\mathcal{O}) \subset L^p(\mathcal{O})$ yield

$$\|u\|_{W^{2,p}(\mathcal{O})} \lesssim \|\varphi\|_{W^{2,2}} + \|\varphi\|_{L^p} \lesssim \|\varphi\|_{L^\infty}.$$

We have obtained that A^{-1} is bounded from $L^\infty(\mathcal{O})$ into $W^{2,p}(\mathcal{O})$. Thus A^{-1} is bounded from X into $W^{2,p}(\mathcal{O})$. Moreover, for p large enough $J : W^{2,p} \longmapsto X$ is compact. We conclude that T is compact from X into itself.

Next, we want to show that T goes from Y to itself. This means that if we take $\varphi \in Y$ then the continuous periodic function $u = \big(-\Delta + c - r(x)\big)^{-1}\varphi$ is nonnegative. Assume by contradiction that there exists $x_0 \in \mathbb{R}^d$ such that

$$u(x_0) = \min(u) < 0.$$

Then $\Delta u(x_0) \geq 0$. Moreover, we know that $0 \leq \varphi = (-\Delta + c - r(x))u$. This leads to the following contradiction,

$$0 \leq \varphi(x_0) = -\Delta u(x_0) + \underbrace{(c - r(x_0))}_{>0} u(x_0) < 0.$$

Consequently $u \geq 0$. It remains to show that T is strongly positive. We moreover assume that $\varphi \not\equiv 0$, and seek to show that $u > 0$. Suppose by contradiction that there exists $x_1 \in \mathbb{R}^d$ such that $u(x_1) = 0$. Since $u \geq 0$ and $(\Delta + c - r(x))u = -\varphi \leq 0$, it follows from the strong maximum principle (theorem 1.8) that $u \equiv 0$. But this is in contradiction with $\varphi \not\equiv 0$, so $u \in \mathring{Y}$.

We can conclude, from the Krein-Rutman theorem (theorem 1.7), that there exists a unique $\lambda > 0$ and a unique $\phi \in Y$ (with $\|\phi\|_\infty = 1$) such that

$$T\phi = \lambda\phi,$$

which is equivalent to

$$(-\Delta - r(x))\phi = \Big(\frac{1}{\lambda} - c\Big)\phi. \qquad \square$$

3. Existence of positive periodic solution

Theorem 3.1 (Existence 1)**.** *If f satifies* (1.5) *and $\lambda_1 < 0$, then there exists a positive and periodic solution p of* (1.6).

Proof. Assume that $\lambda_1 < 0$ and that there exists $M \geq 0$ such that

$$\sup_{s \geq M,\ x \in \mathbb{R}^d} f(x, s) \leq 0.$$

We denote by ϕ the unique solution of

$$\begin{cases} -\Delta\phi - f_u(x,0)\phi = \lambda_1\phi & (x \in \mathbb{R}^d) \\ \phi > 0,\ \phi \text{ periodic},\ \|\phi\|_\infty = 1 \end{cases}$$

The function f lies in $C^1(\mathbb{R}^d \times \mathbb{R}_+)$, consequently for fixed $x \in \mathbb{R}^d$ and for $K > 0$ we have (remembering that $\phi(x) > 0$)

$$\lim_{K \to 0^+} \frac{f(x, K\phi(x)) - f(x,0)}{K\phi(x) - 0} = f_u(x,0).$$

Using the definition of the limit and the fact that $-\frac{\lambda_1}{2} > 0$, we obtain

$$\Big| \frac{f(x, K\phi(x)) - f(x,0)}{K\phi(x) - 0} - f_u(x,0) \Big| < -\frac{\lambda_1}{2} \quad \Longrightarrow \quad \frac{f(x, K\phi(x))}{K\phi(x)} - f_u(x,0) > \frac{\lambda_1}{2}.$$

Thus, for $x \in \mathbb{R}^d$, for $K \leq K_x$ with K_x sufficiently small we have

$$f(x, K\phi(x)) > f_u(x,0)K\phi(x) + \frac{\lambda_1}{2} K\phi(x).$$

This inequality remains true in $B(x, r_x)$ for r_x sufficiently small (this is a consequence of the continuity of $f,\ f_u$ and ϕ). Since $f,\ f_u$ and ϕ are periodic, we consider $x \in \overline{C}$ (a compact set) which is contained in

$$\bigcup_{x \in \overline{C}} B(x, r_x),$$

and using compactness of $\overline{C}$, we deduce

$$\overline{C} \subset \bigcup_{k=1}^{N} B(x_k, r_{x_k}),$$

where $x_1,\ x_2,\ \ldots,\ x_N\ \in \overline{C}$. Choosing $\tilde{K} = \min\{K_{x_1},\ K_{x_2},\ \ldots,\ K_{x_N}\}$, for $K \leq \tilde{K}$

$$f(x, K\phi(x)) > f_u(x,0)K\phi(x) + \frac{\lambda_1}{2} K\phi(x), \qquad \forall x \in \mathbb{R}^d.$$

It follows that for every $x \in \mathbb{R}^d$

$$-\Delta K\phi(x) - f(x, K\phi(x)) \ \leq\ -\Delta K\phi(x) - f_u(x,0)K\phi(x) - \frac{\lambda_1}{2} \ =\ \frac{\lambda_1}{2}\ \leq 0, \tag{3.2}$$

where we used that ϕ is the principal eigenvector. In addition, for every $x \in \mathbb{R}^d$ $f(x, M) \leq 0$. Since M is constant,

$$-\Delta M - f(x, M) \geq 0 \qquad (x \in \mathbb{R}^d). \tag{3.3}$$

Consequently we have obtained (for K small enough)

$$K\phi(x) \leq M \qquad (x \in \mathbb{R}^d). \tag{3.4}$$

Set $\underline{u} = K\phi,\ \overline{u} = M$. From (3.3) and (3.2), we see that these two functions are such that

$$\begin{aligned} -\Delta \overline{u} + \kappa \overline{u} &\geq f(x, \overline{u}) + \kappa \overline{u} \\ -\Delta \underline{u} + \kappa \underline{u} &\leq f(x, \underline{u}) + \kappa \underline{u} \end{aligned}$$

We construct a periodic solution as the limit of the sequence $(u^n)_n$ defined as follows

$$\begin{cases} (u^n) \text{ sequence of periodic functions} \\ 0 = \Delta u^{n+1} - \kappa u^{n+1} + f(x, u^n) + \kappa u^n \\ \underline{u} \leq u^0 \leq \overline{u} \end{cases}$$

The constant $\kappa > 0$ has been fixed so that the function $s \longmapsto f(x, s) + \kappa s$ is increasing. We will show that (u^n) is an increasing sequence and that for every $n \in \mathbb{N}$,

$$\underline{u} \leq u^n \leq \overline{u}. \tag{3.5}$$

Moreover, one can study the regularity of the terms u^n using the relation $\Delta u^n = \kappa u^n - f(x, u^{n-1}) - \kappa u^{n-1}$ and elliptic estimates, and prove that up to the extraction of subsequences (u_n) converges to a periodic function.

First we show (3.5) by induction. Set $u^0 = \underline{u}$, (3.5) is true for u^0. Assuming that (3.5) is true for u^n, we have to show that $\underline{u} \leq u^{n+1} \leq \overline{u}$. We will use the relation

$$0 = \Delta u^{n+1} - \kappa u^{n+1} + f(x, u^n) + \kappa u^n \tag{3.6}$$

Since $u^n \leq \overline{u}$, we have $f(x, u^n) + \kappa u^n \leq f(x, \overline{u}) + \kappa \overline{u} \leq -\Delta \overline{u} + \kappa \overline{u}$. Thus $(u^{n+1} - \overline{u})$ satisfies

$$0 \leq \Delta(u^{n+1} - \overline{u}) - \kappa(u^{n+1} - \overline{u}). \tag{3.7}$$

We can deduce that $u^{n+1} - \overline{u} \leq 0$. Indeed, if we assume by contradiction that $\max(u^{n+1} - \overline{u}) > 0$, and we denote by x_0 the element of $\mathcal{O}$ such that

$$(u^{n+1} - \overline{u})(x_0) = \max_{\mathcal{O}}(u^{n+1} - \overline{u}).$$

Then $\Delta(u^{n+1} - \overline{u})(x_0) \leq 0$ and since κ and $(u^{n+1} - \overline{u})(x_0)$ are positive we get

$$\Delta(u^{n+1} - \overline{u})(x_0) - \kappa(u^{n+1} - \overline{u})(x_0) < 0,$$

which contradicts (3.7). On the other hand, since $\underline{u} \leq u^n$, we have $-\Delta \underline{u} + \kappa \underline{u} \leq f(x, \underline{u}) + \kappa \underline{u} \leq f(x, u^n) + \kappa u^n$. Thus $(u^{n+1} - \underline{u})$ satisfies

$$\Delta(u^{n+1} - \underline{u}) - \kappa(u^{n+1} - \underline{u}) \leq 0. \tag{3.8}$$

We can deduce that $0 \leq u^{n+1} - \underline{u}$. Indeed, if we assume by contradiction that $\min(u^{n+1} - \overline{u}) < 0$, and we denote by x_1 the element of $\mathcal{O}$ such that

$$(u^{n+1} - \underline{u})(x_1) = \min_{\mathcal{O}}(u^{n+1} - \underline{u}).$$

Then $\Delta(u^{n+1} - \underline{u})(x_1) \geq 0$ and since $\kappa < 0$ and $(u^{n+1} - \underline{u})(x_1) > 0$ we get

$$0 < \Delta(u^{n+1} - \underline{u})(x_1) - \kappa(u^{n+1} - \underline{u})(x_1),$$

which contradicts (3.8).

Secondly, we show by induction that (u^n) is increasing. We already showed that $u^0 \leq u^1$. Suppose that $u^n \leq u^{n+1}$, then $-\Delta u^{n+1} + \kappa u^{n+1} = f(x, u^n) + \kappa u^n \leq f(x, u^{n+1}) + \kappa u^{n+1}$ and

$$\begin{aligned} & \Delta u^{n+2} - \kappa u^{n+2} + f(x, u^{n+1}) + \kappa u^{n+1} = 0 \\ \Longrightarrow \quad & \Delta(u^{n+2} - u^{n+1}) - \kappa(u^{n+2} - u^{n+1}) \leq 0. \end{aligned}$$

By the same argument as before we can show, by contradiction, that $u^{n+1} \leq u^{n+2}$. $\square$

Theorem 3.9 (Existence 2)**.** *If f satisfies* (1.4) *and $\lambda_1 \geq 0$, then there is no positive bounded solution of* (1.6)*.*

Proof. We assume that there exists a bounded nonnegative solution p of (1.6). We denote by λ_1 and ϕ the principal eigenvalue and its associated (normalized) principal eigenvector. That is (λ_1, ϕ) satisfies

$$\begin{cases} -\Delta\phi - f_u(x,0)\phi = \lambda_1\phi \\ \phi \text{ periodic}, \ \phi > 0, \ \|\phi\|_\infty = 1. \end{cases} \tag{3.10}$$

We also assume that for every $x \in \mathbb{R}^d$ the function $s \longmapsto \frac{f(x,s)}{s}$ is deacreasing in $s > 0$. Using this fact, we can say that for every $x \in \mathbb{R}^d$ and every $\gamma > 0$, there exists $s_{x,\gamma} > 0$ such that $s_{x,\gamma} \leq \gamma\phi(x)$ and consequently

$$\frac{f(x,\gamma\phi(x))}{\gamma\phi(x)} \leq \frac{f(x,s)}{s}, \qquad \forall s \leq s_{x,\gamma}.$$

Thus, using the definition of $f_u(x,0)$, we deduce

$$f(x,\gamma\phi(x)) < f_u(x,0)\gamma\phi(x).$$

We can now bound $-\Delta\gamma\phi - f(x,\gamma\phi)$ from below and use (3.10) to obtain

$$-\Delta\gamma\phi - f(x,\gamma\phi) \ > \ -\Delta\gamma\phi - f_u(x,0)\gamma\phi \ = \ \gamma\lambda_1\phi \ \geq \ 0,$$

since λ_1 is assumed to be nonnegative. The eigenvector ϕ being periodic and the function p being bounded, the quantity $\frac{\|p\|_\infty}{\min\phi}$ is well defined. Since any $\gamma > \frac{\|p\|_\infty}{\min\phi} \geq 0$ satisfies $\gamma\phi > p$ in $\mathbb{R}^d$, the following quantity is nonnegative and well defined

$$\gamma^* = \inf\{\gamma > 0\colon \gamma\phi(x) > p(x), \ \forall x \in \mathbb{R}^d\}. \tag{3.11}$$

If γ^* were equal to zero, then there would exist a sequence $(\gamma_n) \subset \{\gamma > 0\colon \gamma\phi > p \text{ in } \mathbb{R}^d\}$ which converges to 0. Since for any $n \in \mathbb{N}$, $\gamma_n \geq \gamma_n\phi(x) > p(x) \geq 0$, the solution p would be the zero function, and the proof would be over.

That is why we are going to show by contradiction that $\gamma^* = 0$. Assume $\gamma^* > 0$ and set

$$z = \gamma^*\phi - p,$$

which is nonnegative. We prove by contradiction that

$$\alpha \overset{\text{def.}}{=} \inf\{\gamma^*\phi(x) - p(x)\colon x \in \mathbb{R}^d\} = 0.$$

Indeed, assume that $\alpha > 0$, then one could choose

$$\gamma = \gamma^* - \delta\ , \qquad \text{where } \ \delta = \min\Big(\frac{\alpha}{9}, \frac{\gamma^*}{7}\Big).$$

By construction $\gamma^* > \gamma > 0$. Moreover, for every $x \in \mathbb{R}^d$,

$$\gamma\phi(x) - p(x) \ = \ \underbrace{\gamma^*\phi(x) - p(x)}_{\geq\alpha} - \delta \ \geq \ \alpha - \alpha/7 \ > \ 0,$$

which contradicts (3.11). Thus $\alpha = 0$, and by the definition of the infimum one can infer that there exists a sequence $(x_n) \subset \mathbb{R}^d$ such that

$$z(x_n) \xrightarrow[n\to\infty]{} 0.$$

From (x_n), we define the sequence $(\overline{x}_n) \subset \overline{\mathcal{O}} = [0, L_1] \times [0, L_2] \times \ldots \times [0, L_d]$ by $x_n - \overline{x}_n \in (L_1\mathbb{Z}) \times (L_2\mathbb{Z}) \times \ldots \times (L_d\mathbb{Z})$. Since $\overline{\mathcal{O}}$ is compact, up to the extraction of a subsequence, there exists $\overline{x}_\infty \in \overline{\mathcal{O}}$ such that $\overline{x}_n \xrightarrow[n\to\infty]{} \overline{x}_\infty$. We define the following periodic functions

$$\begin{aligned} \phi_n(x) &= \phi(x + x_n) \\ p_n(x) &= p(x + x_n), \end{aligned}$$

for which we know that

$$\begin{aligned} &-\Delta\gamma^*\phi_n(x) - f(x + x_n, \gamma^*\phi_n(x)) > 0 \\ &-\Delta p_n(x) - f(x + x_n, p_n(x)) = 0. \end{aligned}$$

Because of the periodicity of f with respect to x, and the fact that $x + x_n$ can be rewriten as $x + \overline{x}_n + (x_n - \overline{x}_n)$ these equations are equivalent to

$$\begin{aligned} &-\Delta\gamma^*\phi_n(x) - f(x + \overline{x}_n, \gamma^*\phi_n(x)) > 0 \\ &-\Delta p_n(x) - f(x + \overline{x}_n, p_n(x)) = 0. \end{aligned}$$

One can show, using elliptic estimates, that up to the extraction of some subsequences, (p_n) converges in C^2_{loc} to a function p_∞ satisfying

$$-\Delta p_\infty - f(x + \overline{x}_\infty, p_\infty) = 0, \qquad (x \in \mathbb{R}^d) \tag{3.12}$$

and $(\gamma^*\phi_n)$ converges to the function $\gamma^*\phi(\,.\, + \overline{x}_\infty)$ (that we will denote by $\gamma^*\phi_\infty$) satisfying

$$-\Delta\gamma^*\phi_\infty - f(x + \overline{x}_\infty, \gamma^*\phi_\infty) > 0, \qquad (x \in \mathbb{R}^d). \tag{3.13}$$

Set $z_\infty(x) = \gamma^*\phi_\infty(x) - p_\infty(x)$, then

$$z_\infty(x) = \lim_{n\to+\infty} \big(\gamma^*\phi(x + x_n) - p(x + x_n)\big) = \lim_{n\to+\infty} z(x + x_n) \geq 0.$$

Remark that substracting (3.13) from (3.12) yields

$$\Delta(\gamma^*\phi_\infty - p_\infty) + f(x + \overline{x}_\infty, \gamma^*\phi_\infty) - f(x + \overline{x}_\infty, p_\infty) < 0.$$

Denote by α the continuous function on $\mathbb{R}^d$ defined by

$$\alpha(x) = \begin{cases} \frac{f(x+\overline{x}_\infty, \gamma^*\phi_\infty) - f(x+\overline{x}_\infty, p_\infty)}{p_\infty(x) - \gamma^*\phi_\infty(x)}, \text{ if } p_\infty(x) \neq \gamma^*\phi_\infty(x) \\ \partial_s f(x + \overline{x}_\infty, p_\infty(x)), \quad \text{if } p_\infty(x) = \gamma^*\phi_\infty(x) \end{cases}$$

We have $(\Delta + \alpha)z_\infty \leq 0$. Since $z_\infty(0) = 0$ and $z_\infty \geq 0$, it follows from the strong maximum principle (Theorem 1.8) that $z_\infty = 0$. Thus, $\gamma^*\phi_\infty = p_\infty$ and we reach a contradiction since

$$0 = -\Delta\gamma^*\phi_\infty - f(x + \overline{x}_\infty, \gamma^*\phi_\infty) > 0.$$

Consequently $\gamma^* = 0$, and there is no other nonnegative bounded solution of (1.6) besides $p \equiv 0$. $\square$

4. Influence of the shape of $f_u(x,0)$

For the numerical simulations, we set $d=1$ and choose the function f to be

$$f(x,s)=s(\mu(x)-s), \qquad (x\in\mathbb{R},\ s\in\mathbb{R}_+)$$

where $\mu\colon\mathbb{R}\longmapsto\mathbb{R}$ is a periodic function. The stationnary problem becomes

$$-\varepsilon\partial^2_{x^2}u(x)=u(x)(\mu(x)-u(x)), \qquad (x\in\mathbb{R}), \tag{4.1}$$

where we added the diffusion coefficient ε. The term $f_u(x,0)$ is equal to $\mu(x)$. Our main aim is to approximate the principal eigenvalue λ_1 of the operator

$$\mathscr{L}_0\colon\varphi\longmapsto-\varepsilon\partial^2_{x^2}\varphi-\mu(x)\varphi,$$

and observe the influence of the shape of μ on λ_1. For this purpose, we use an algorithm called the *power iteration method*. For $x\in\mathbb{R}$, $\mu(x)$ represents the growth rate of the population which is located at the position x. A positive $\mu(x)$ means that the "place" x is favorable for the reproduction, a negative $\mu(x)$ means the "place" x is hostile for the population. For example, μ could represents the distribution of food in the space $\mathbb{R}$. We know from theorem 3.1 that if λ_1 is negative, then the system has a positive periodic solution. We choose functions μ having different "profiles" in order to compare the corresponding eigenvalues, and for the comparison to be relevant the functions μ_i must satisfy

$$\int_{\mathcal{O}}\mu_i=\int_{\mathcal{O}}\mu_j.$$

We also approximate the corresponding solution u of (4.1).

Firstly, we choose four nonnegative functions μ for which the integral stays the same (around 12.55) but the shape goes from a constant line to a series of crenel (with a increasing frequency). The length of the interval $[0,L_1]$ stays the same throughout all the simulations ($L_1=\frac{5\pi}{2}$).

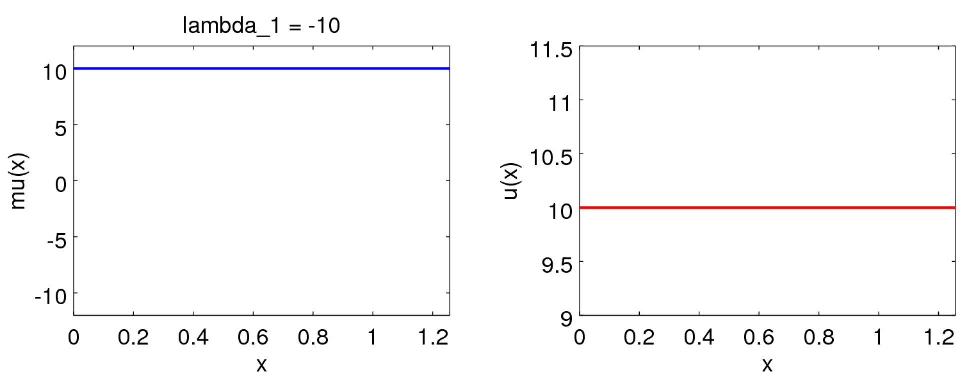


Figure 1: We set $\mu\equiv 10$, $\varepsilon=0.1$. The approximation of the principal eigenvalue is $\lambda_1=-10$.

According to Proposition 5.3 of [2], which compares the effect of μ and of its average, in the three following case λ_1 should be less than -10. The approximations of the principal eigenvalue satisfy this property (Figure 1). Moreover, we observe that the most negative λ_1 is obtained when the shape of μ contains a unique crenel (Figure 2). The case with two crenels follows, λ_1 being approximately the half of the previous one

(Figure 3). In the last case (five crenels), λ_1 is the closest to -10 (Figure 4). Thus, a favourable zone regrouped at the center of the periodicity cell, seems to maximize the (absolute value of) the principal eigenvalue.

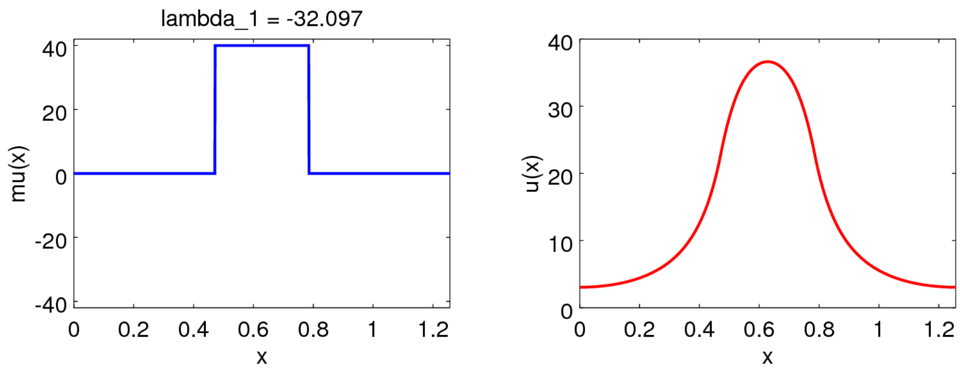


Figure 2: We set $\mu \equiv \mathbb{1}_{[\frac{3L_1}{8}, \frac{5L_1}{8}]}$ and $\varepsilon = 0.1$. The approximation of the principal eigenvalue is $\lambda_1 = -32.097$.

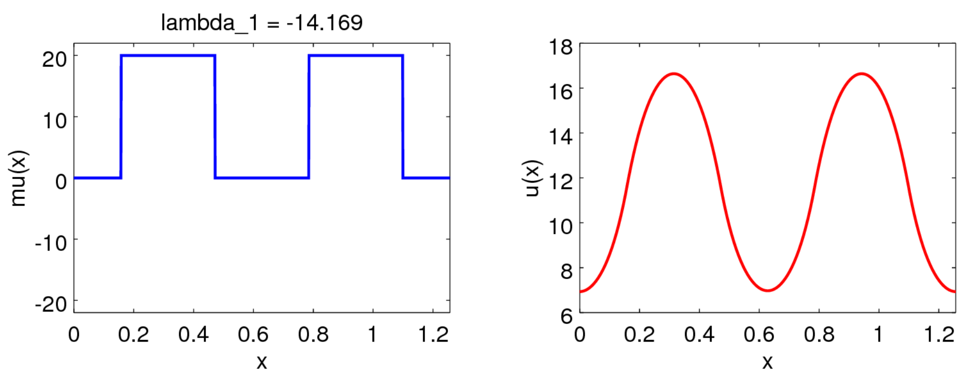


Figure 3: We set $\mu \equiv \mathbb{1}_{[\frac{L_1}{8}, \frac{3L_1}{8}] \cup [\frac{5L_1}{8}, \frac{7L_1}{8}]}$ and $\varepsilon = 0.1$. The approximation of the principal eigenvalue is $\lambda_1 = -14.169$.

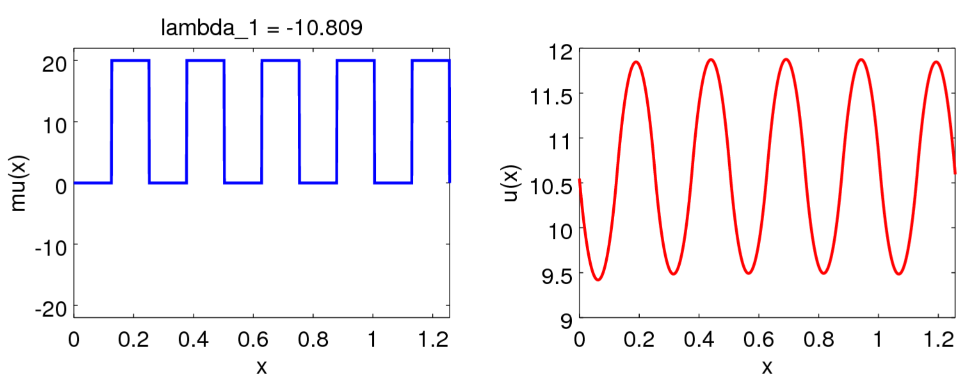


Figure 4: We set $\varepsilon = 0.1$. The approximation of the principal eigenvalue is $\lambda_1 = -10.809$.

We also compared the constant function to a sine function with two "hills" (both of average around 6.32). We also observe that λ_1 is more negative for the non-constant function : -7.7502 for the sine function, -5 for the constant function (Figure 5, Figure 6).

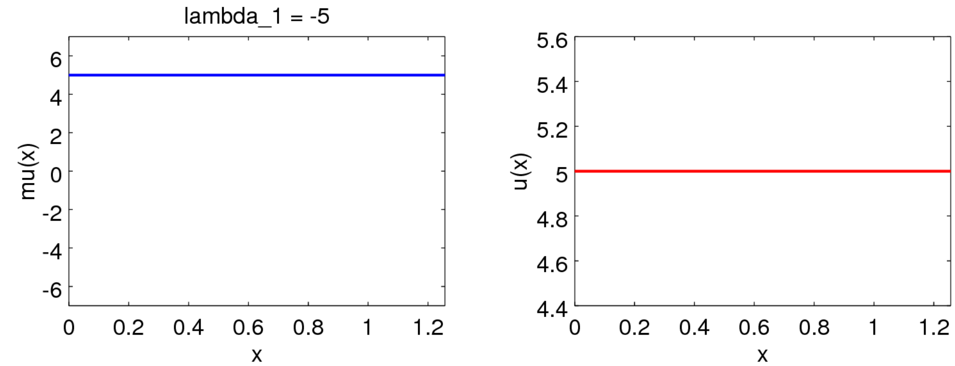


Figure 5: We set $\mu \equiv 5$, $\varepsilon = 0.1$. The approximation of the principal eigenvalue is $\lambda_1 = -5$.

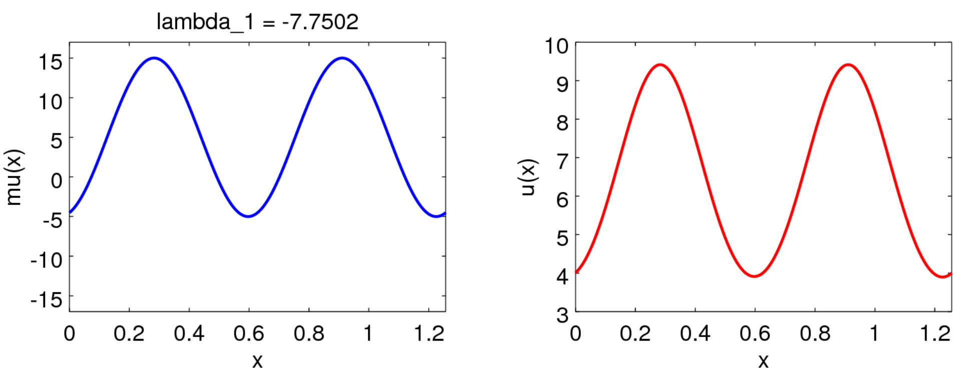


Figure 6: We set $\mu(x) = 10\sin(10x - L_1) + 5$, $\varepsilon = 0.1$. The approximation of the principal eigenvalue is $\lambda_1 = -7.7502$.

In order to better interpret these results, it would be necessary to study if taking step functions is relevant or if their discontinuity deforms the results (would the discretization methods be inadequate). If so, an alternative is to use well configured trigonometric functions (as in Figure 6), while keeping the same program. It would also be interesting to compare the results to theorical results of the litterature.

4.1. **Discretization.** The power iteration method allows to compute simultaneously the principal eigenvector and the principal eigenvalue of a given operator. We have seen that there exists a unique positive λ and a unique positive periodic ϕ with $\|\phi\|_\infty = 1$ satisfying

$$\big(-\varepsilon\partial^2_{x^2} + c - \mu(x)\big)^{-1}\phi = \lambda\phi.$$

Consequently we use the iteration method on the operator

$$\mathscr{L}\colon \phi \longmapsto \big(-\varepsilon\partial^2_{x^2} + c - \mu(x)\big)^{-1}\phi.$$

The eigenvalue λ_1 solution of (2.2) is equal to $\frac{1}{\lambda} - c$. At each iteration, for a given periodic function w, the power method computes the quantity

$$v = \big(-\varepsilon\partial^2_{x^2} + c - \mu(x)\big)^{-1} w.$$

Consequently, we have to discretize the problem

$$(P_{\lambda_1})\colon \begin{cases} \big(-\varepsilon\partial^2_{x^2} + c - \mu(x)\big)v(x) = w(x) \qquad (x \in [0, L_1]) \\ v \text{ periodic} \\ w \text{ periodic (given),} \end{cases}$$

in order to obtain the matrix system $Bv = w$ (and solve it by using a built-in Matlab function).

Secondly, we want to compute an approximation of the corresponding periodic positive solution u of (4.1), using a fixed point method. The method demands to rewrite (4.1) so that it is equivalent to an equation of the form $u = F(u)$, and compute at each iteration the quantity $u^{k+1} = F(u^k)$. The approximate solution u is the limit of the sequence (u^k). The equation (4.1) is equivalent to

$$u - \varepsilon\partial^2_{x^2}u + u^2 = \mu(x)u + u.$$

Then, we choose which term are implicit and which ones are explicit

$$u^{k+1} - \varepsilon\partial^2_{x^2}u^{k+1} + u^k u^{k+1} = \mu(x)u^k + u^k,$$

to finally obtain the system

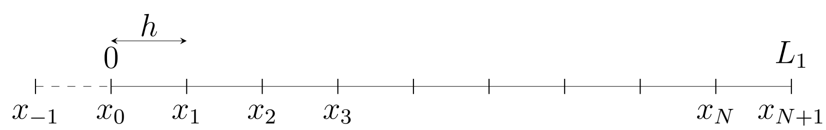
$$u^{k+1} = \big(I - \varepsilon\partial^2_{x^2} + u^k\big)^{-1}\big(\mu(x)u^k + u^k\big),$$

where I denotes the identity operator. Consequently, we have to discretize the problem

$$(P_u)\colon \begin{cases} \big(I - \varepsilon\partial^2_{x^2} + \tilde{w}(x)\big)v(x) = w(x) \qquad (x \in [0, L_1]) \\ v \text{ periodic} \\ w,\ \tilde{w} \text{ periodic (given),} \end{cases}$$

in order to obtain the matrix system $Bv = w$ (and solve it by using a built-in Matlab function). In both cases we must discretize $\partial^2_{x^2}$ with periodic boundary conditions.

The space $[0, L_1]$ is discretized the following way.



We set $h = \frac{L_1}{N+1}$, $x_i = ih$, $v_i = v(x_i)$ and $w_i = w(x_i)$. Consider the equation

$$-\partial^2_{x^2}v = w.$$

We must compute $v_0,\ v_1,\ \ldots,\ v_{N+1}$. There are $N+1$ unknowns $v_0,\ v_1,\ \ldots,\ v_N$ since $v_0 = v_{N+1}$. The centered finite difference method gives the following second order approximation

$$-\frac{v_{i-1} - 2v_i + v_{i+1}}{h^2} = w_i \qquad i \in \{0, \ldots, N\}$$

When $i = 0$, using $v_{-1} = v_N$ we obtain

$$-\frac{v_N - 2v_0 + v_1}{h^2} = w_0.$$

When $i = N$, using $v_0 = v_{N+1}$ we obtain

$$-\frac{v_{N-1} - 2v_N + v_0}{h^2} = w_N.$$

Theses equations are equivalent to the matrix system

$$\frac{1}{h^2}\underbrace{\begin{pmatrix} 2 & -1 & 0 & \dots & 0 & -1 \\ -1 & 2 & -1 & \ddots & & 0 \\ 0 & -1 & 2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & \ddots & \ddots & -1 \\ -1 & 0 & \dots & 0 & -1 & 2 \end{pmatrix}}_{=A} \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ \vdots \\ \vdots \\ v_N \end{pmatrix} = \begin{pmatrix} w_0 \\ w_1 \\ \vdots \\ \vdots \\ \vdots \\ w_N \end{pmatrix}$$

We can now write the discretizations of (P_{λ_1}) and (P_u) (setting $\mu_i = \mu(x_i)$, $v_i = v(x_i)$, $w_i = w(x_i)$ and $\tilde{w}_i = \tilde{w}(x_i)$), as

$$(P_{\lambda_1})\colon \left(\frac{\varepsilon}{h^2}A + cI - \begin{pmatrix} \mu_0 & & \\ & \ddots & \\ & & \mu_N \end{pmatrix}\right) \begin{pmatrix} v_0 \\ \vdots \\ v_N \end{pmatrix} = \begin{pmatrix} w_0 \\ \vdots \\ w_N \end{pmatrix}$$

$$(P_u)\colon \left(I + \frac{\varepsilon}{h^2}A + \begin{pmatrix} \tilde{w}_0 & & \\ & \ddots & \\ & & \tilde{w}_N \end{pmatrix}\right) \begin{pmatrix} v_0 \\ \vdots \\ v_N \end{pmatrix} = \begin{pmatrix} w_0 \\ \vdots \\ w_N \end{pmatrix},$$

where I is the identity matrix.

4.2. **Implementation.** The program is divided in two parts. The function `power_iteration` takes a matrix as an input and returns the principal eigenvalue of the inverse of this matrix.

```
function [lambda_k, b_k] = power_iteration(B)
  %% return largest (in modulus) eigenvalue of inverse of B
  %% B should be equal to (-diff*Laplacian + c - \mu(x))

   b_k = rand(size(B,1),1); % random vector

   b_k1= zeros(size(B,1),1);  err=12; lambda_k1=0; % initialize
        variables

   [L,U,P,Q]=lu(B); % LU decomposition of A

   while (err>10^(-8))
     lambda_k=lambda_k1; % save previous value

     b_k1 = Q*(U\(L\( P*b_k ))); % i.e. b_k1 = A\b_k;
     b_k = b_k1 / norm(b_k1);
```

```
      y = Q*(U\(L\( P*b_k ))); % i.e. y=A\b_k;
      lambda_k1 = (y')*(b_k);

      err=abs(lambda_k-lambda_k1);
    end
end
```

Then, there is the main part of the program, where we set the variables (matrix, functions and parameters), call the function `power_iteration` and write the fixed-point method. The choice of the variable `mu` allows to select one of the functions μ.

```
clear;

%================== Principal eigenvalue ===================%

% Parameters :
mu=1; % choice of the function \mu

T=10^1; % for sinus function

plus=1; % choose higher value if \mu is very negetive
%(--> convergence of fixed point)

diff=0.1; % diffusion coefficient
% Be carefull not to choose the diffusion coefficient too small,
% if so, the methods solves "(\mu(x)-u)u=0", and lambda_1 seems
% to be "-max(\mu)"

% Mesh :
N=10^3; % number of interior points --> N+2 points
L_1=pi/T*4; % length of interval
h=1/(N+1); % space step

X=linspace(0,L_1,N+2); % space mesh

% Various functions \mu(x) (the derivative of f(x,.)) :

% piecewise constant functions :
p=10;
% "trapz" is an Octave built-in function
% the various disp are meant to test if the functions \mu
% have same integral

% constant :
r1=p*ones(N+2,1);
```

```
disp(['1 : ',num2str(trapz(X,r1')),' vs ',num2str(p*L_1), ' p is ',num2str(p)])

% 2 hills :
x_0=L_1/8; x_1=L_1/8*3; x_2=L_1/8*5; x_3=L_1/8*7;
q=p*L_1/(x_1-x_0+x_3-x_2);
r2=q*(X>=x_0)-q*(X>=x_1)+q*(X>=x_2)-q*(X>=x_3);
disp(['2 : ',num2str(trapz(X,r2)),' vs ',num2str(q*4*L_1/8), ' p is ',num2str(p)])

% 5 hills :
x_0=L_1/10; x_1=L_1/10*2; x_2=L_1/10*3; x_3=L_1/10*4;
x_4=L_1/10*5; x_5=L_1/10*6; x_6=L_1/10*7; x_7=L_1/10*8;
x_8=L_1/10*9; x_9=L_1/10*10;
q=L_1*p/(5*(x_1-x_0));
r3= q*(X>=x_0)-q*(X>=x_1)+q*(X>=x_2)-q*(X>=x_3)+q*(X>=x_4)-q*(X>=x_5)+q*(X>=x_6)-q*(X>=x_7)+q*(X>=x_8)-q*(X>=x_9);
disp(['3 : ',num2str(trapz(X,r3)),' vs ',num2str(q*5*L_1/10), ' p is ',num2str(p)])

% 2 hills + negative parts :
j=5;
x_0=L_1/8; x_1=L_1/8*3; x_2=L_1/8*5; x_3=L_1/8*7;
q=2*p+j;
r4=-j*ones(1,N+2);
r4(X>=x_0 & X<=x_1)=q;
r4(X>=x_2 & X<=x_3)=q;
%r4=q*(X>=x_0)-q*(X>=x_1)+q*(X>=x_2)-q*(X>=x_3);
disp(['4 : ',num2str(trapz(X,r4)),' vs ',num2str(q*4*L_1/8-j*4*L_1/8), ' p is ',num2str(p)])

% 1 hill :
x_0=L_1/8*3; x_1=L_1/8*5; q=L_1*p/(x_1-x_0);
r5=q*(X>=x_0)-q*(X>=x_1);
disp(['5 : ',num2str(trapz(X,r5)),' vs ',num2str(q*2*L_1/8), ' p is ',num2str(p)])

% unfavourable zone in the middle:
y=20;k=L_1/y;
x_0=0*k; x_1=k*9; x_2=k*11; x_3=k*20;
q=p*L_1/(x_1-x_0+x_3-x_2);
r8=q*(X>=x_0)-q*(X>=x_1)+q*(X>=x_2)-q*(X>=x_3);
disp(['8 : ',num2str(trapz(X,r8)),' vs ',num2str(q*8*L_1/10), ' p is ',num2str(p)])
```

```

% Compare sinus and constant function :
r6=p*sin(T*X-L_1)+p/2; % 2 hills
fun1 = @(x) p*sin(T*x-L_1)+p/2;
int_r6 = quad(fun1,0,L_1); % computes integral
% "quad" is an Octave built-in function
p=int_r6/L_1; r7 = p*ones(N+2,1); % constant function
% tests if constant function has same integral as sine :
fun2= @(x) p; quad(fun2,0,L_1);
disp(['6 : (sine) ',num2str(int_r6),' vs ',num2str(quad(fun2,0,L_1)), ' p is ',num2str(p)])
disp(['7 : (trapz - sine) ',num2str(trapz(X,r6)),' vs ',num2str(trapz(X,r7')), ' p is ',num2str(p)])

% Choice of the function \mu(x) :
r=eval(['r',num2str(mu)]);
c=max(r)+2;

% Discrete Laplacian :
A=sparse(1:N+2,1:N+2,2)+sparse(2:N+2,1:N+1,-1,N+2,N+2)+sparse(1:N+1,2:N+2,-1,N+2,N+2);
A(1,N+2)=-1; A(N+2,1)=-1;

% Matrix diff*(1/h^2)*A + c*I - r*I :
B=diff*1/h^2*A + sparse(1:N+2,1:N+2,c) - sparse(1:N+2,1:N+2,r);

% Computation of principal eigenvalue of the inverse of B :
lambda_1=0;
mu_1=power_iteration(B);
lambda_1=1/mu_1 - c

%================== Stationary solution ======================%

u_k=100*rand(N+2,1); u_k1=u_k; err=4; % initialization

% I + diff*(1/h^2)*A :
C=plus*sparse(1:N+2,1:N+2,1) + diff*(1/h^2)*A;

% fixed point method:
while (err>10^(-9))
    u_k=u_k1; % save old value

    D=C+sparse(1:N+2,1:N+2,u_k); % matrix we have to invert

```

```
    % solves D*u_k1 = I*r*u_k + u_k :
    u_k1=D\(sparse(1:N+2,1:N+2,r)*u_k+plus*u_k);

    err=norm(u_k-u_k1); % computes error
end

%==================== ==== Display ==============================%

% Plot of \mu :
scrsz = get(0,'ScreenSize');
figure('Position',[1 1 (2*scrsz(3)/2)/1.2 (2*scrsz(4)/2)/2]) ;

subplot(1,2,1); plot(X, r, 'LineWidth',2);
set(gca,'FontSize',17);
xlim([0 X(N+2)]); ylim([-c c]);
title(['\lambda_1 = ',num2str(lambda_1)]);
xlabel('x'); ylabel('\mu(x)');

% Plot of u :
subplot(1,2,2); plot(X, u_k1,'r', 'LineWidth',2);
set(gca,'FontSize',17);
xlim([0 X(N+2)]); xlabel('x'); ylabel('u(x)');

set(gcf, 'PaperPositionMode', 'auto');
saveas(gcf,[num2str(mu),'.png']); % saves the current figure
```

References

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