Analysis of an elliptic equation from population dynamics

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M2 Analysis, PDE, Probability

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The elliptic equation

$$-\Delta u(x) = f(x, u(x)) \qquad (x \in \mathbb{R}^d). \tag{1}$$

- f is periodic with respect to x, the periodicity cell is

$$\mathcal{O} = (0, L_1) \times (0, L_2) \times \ldots \times (0, L_d)$$

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$$\mathcal{O} = (0, L_1) \times (0, L_2) \times \ldots \times (0, L_d)$$

- $f_u(x,0) \stackrel{\text{def}}{=} \lim_{s\to 0^+} \frac{f(x,s)}{s}$.

Theorem

There is a unique real number λ_1 such that there exists a function $\phi > 0$ periodic with $\|\phi\|_\infty = 1$, which satisfies

$$-\Delta\phi(x) - f_u(x,0)\phi(x) = \lambda_1\phi(x) \qquad (x \in \mathbb{R}^d). \tag{2}$$

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Definition

λ_1 is called the *principal eigenvalue* of the operator $\mathscr{L}_0 \colon \phi \longmapsto -\Delta\phi - f_u(x,0)\phi$.

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Elements of proof: (apply the Krein-Rutman theorem)
$Y \stackrel{\text{def}}{=} \{\varphi \in C^0(\mathcal{O}) : \varphi \text{ periodic}, \varphi \geq 0\}$, $c > 0$

$$T : \begin{cases} Y & \longrightarrow Y \\ \varphi & \longmapsto \big(-\Delta + c - f_u(x,0)\big)^{-1}\varphi \end{cases}$$

is compact and strongly positive.

Theorem

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Elements of proof:
There exists a unique $\lambda > 0$ and a unique $\phi \in Y$ (with $\|\phi\|_\infty = 1$) such that

$$T\phi = \lambda\phi$$

$$\implies \quad -\Delta\phi - f_u(x,0)\phi = \left(\frac{1}{\lambda} - c\right)\phi.$$

Theorem (existence 1)

If there exists $M \geq 0$ such that

$$f(x, s) \leq 0, \ \forall s \geq M, \ \forall x \in \mathbb{R}^d,$$

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Elements of proof: ϕ solution of

$$\begin{cases} -\Delta\phi - f_u(x,0)\phi = \lambda_1\phi \\ \phi \text{ periodic}, \ \phi > 0, \ \|\phi\|_\infty = 1. \end{cases}$$

$\Longrightarrow$ $-\Delta K\phi \leq f(x,K\phi), \quad f(x,M) \leq -\Delta M \quad \text{and} \quad K\phi \leq M$
$\Longrightarrow$ periodic positive solution $K\phi \leq p \leq M$

Theorem (existence 2)

If f satisfies

$$\forall x \in \mathbb{R}^d, \quad s \longmapsto \frac{f(x,s)}{s} \quad \text{is decreasing in } s,$$

and $\lambda_1 \geq 0$, then there is no positive bounded solution of

$$-\Delta u = f(x,u)$$

Which shape of $f_u(x,0)$ maximizes $|\lambda_1|$?

- in 1D:

$$-\varepsilon\partial^2_{x^2}u = u(\mu - u), \qquad \text{in } (0, L_1)$$

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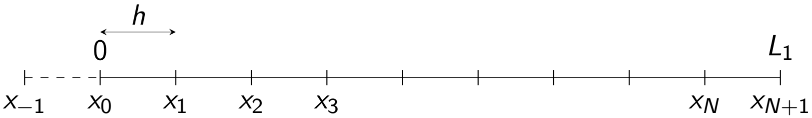
- approximation of λ_1 : *Power iteration method*

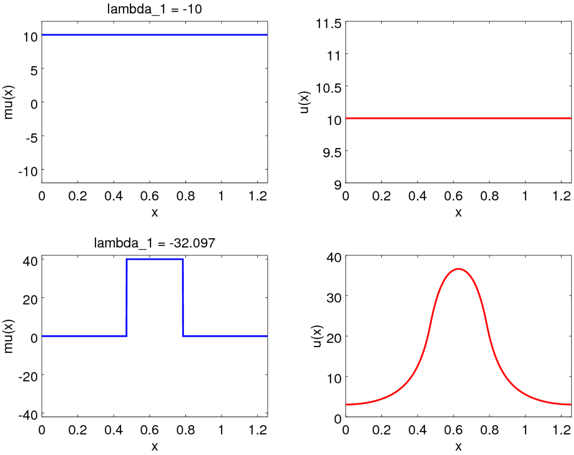
$$T \colon \varphi \longmapsto \big(- \varepsilon\partial^2_{x^2} + c - \mu(x)\big)^{-1}\varphi$$

$$\xrightarrow{\text{power iteration}} \lambda \implies \lambda_1 = \frac{1}{\lambda} - c.$$

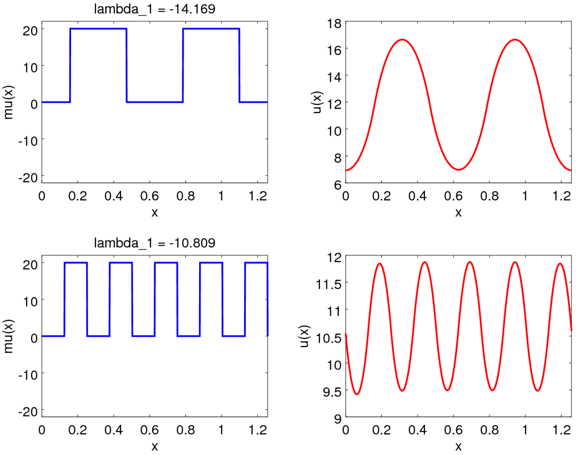


Matrix form:

$$\frac{\varepsilon}{h^2}\begin{bmatrix} 2 & -1 & 0 & \dots & -1 \\ -1 & 2 & -1 & & 0 \\ 0 & -1 & 2 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & -1 \\ -1 & 0 & \dots & -1 & 2 \end{bmatrix} + \begin{bmatrix} c-\mu_0 & 0 & \dots & 0 & 0 \\ 0 & c-\mu_1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & c-\mu_N \end{bmatrix}$$



Function μ and associated solution u of the elliptic problem.



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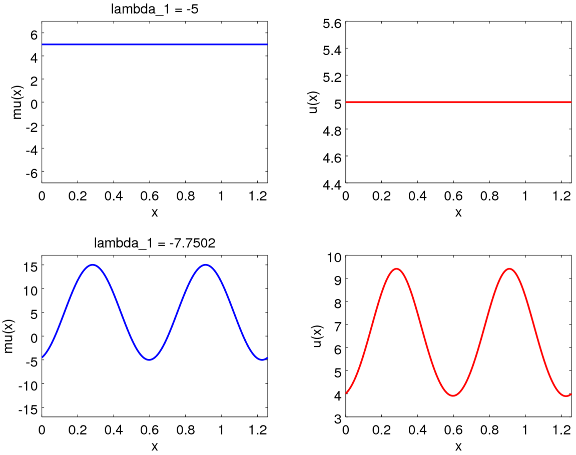

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Thank you for your attention.