FEM FOR COUPLED TRANSPORT EQUATIONS

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Abstract. This is the explanation of the discretization of two transport equations coupled at the boundary (using the finite element method) and of a 2×2 hyperbolic system. The corresponding code is in the files `transport_diag_wf1.m`, as well as `transport_nodiag_wf1.m` and `transport_nodiag_wf2.m`. Our choice of discretization is done in view of extending this method to a more complicated model: the Intrinsic Geometrically Exact Beam model, which is also a first-order system of partial differential equations (treated in the repositories https://github.com/chrdz/GEB-diversSimu and https://github.com/chrdz/GEB-Feedback).

Contents

Notation. Let $m, n \in \{1, 2, \dots\}$. The inner product in $\mathbb{R}^n$ is denoted $\langle \cdot\,, \cdot \rangle$. Here, the identity and null matrices are denoted by $\mathbf{I}_n \in \mathbb{R}^{n \times n}$ and $\mathbf{0}_{n,m} \in \mathbb{R}^{n \times m}$, and we use the abbreviation $\mathbf{0}_n = \mathbf{0}_{n,n}$. If there is no confusion, we omit the subscript and write $\mathbf{I}$ and $\mathbf{0}$ instead.

1. Spatial discretization

We consider a one-dimensional spatial domain $(0, \ell)$ with $\ell > 0$. We place $\mathtt{N_x}$ points $\{x_k\}_{k=1}^{\mathtt{N_x}}$ on the interval $[0, \ell]$, such that $x_1 = 0$ and $x_{\mathtt{N_x}} = \ell$. The number $\mathtt{N_x}$ is even, and each interval $[x_{2e-1}, x_{2e+1}]$ for $e \in \{1, 2, \dots, \mathtt{N_e}\}$ constitute an element, which contains

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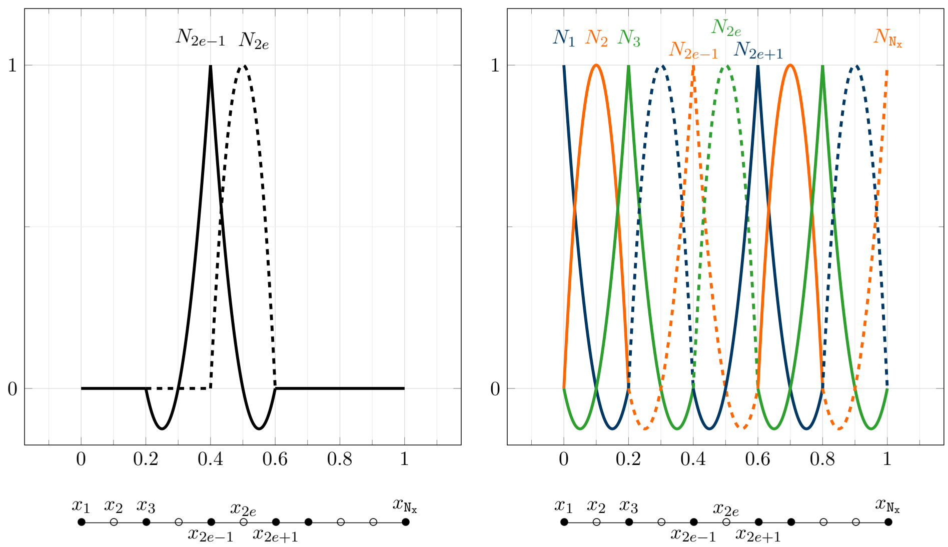

FIGURE 1. Left: two kinds of shape functions. Right: shape functions over the whole interval.

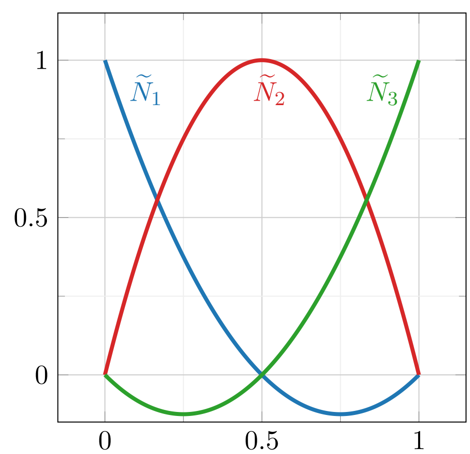

FIGURE 2. Reference element.

the points $x_{2e-1}, x_{2e}, x_{2e+1}$. Thus, the length of the element e is $\mathtt{h_e} = x_{2e+1} - x_{2e-1}$, and $\mathtt{N_x} = 2\mathtt{N_e} + 1$. Let

$$n(i,k) := 2(k-1) + i, \quad \mathtt{N_{tot}} := 2\mathtt{N_x}.$$

Let $\mathbf{N}\colon [0,\ell] \to \mathbb{R}^{2\times \mathtt{N_{tot}}}$ be defined by

$$\begin{bmatrix} \mathbf{N}_1 \\ \mathbf{N}_2 \end{bmatrix} = \mathbf{N} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & \dots & 0 & N_{\mathtt{N_x}} & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & \dots & N_{\mathtt{N_x}-1} & 0 & N_{\mathtt{N_x}} \end{bmatrix}.$$

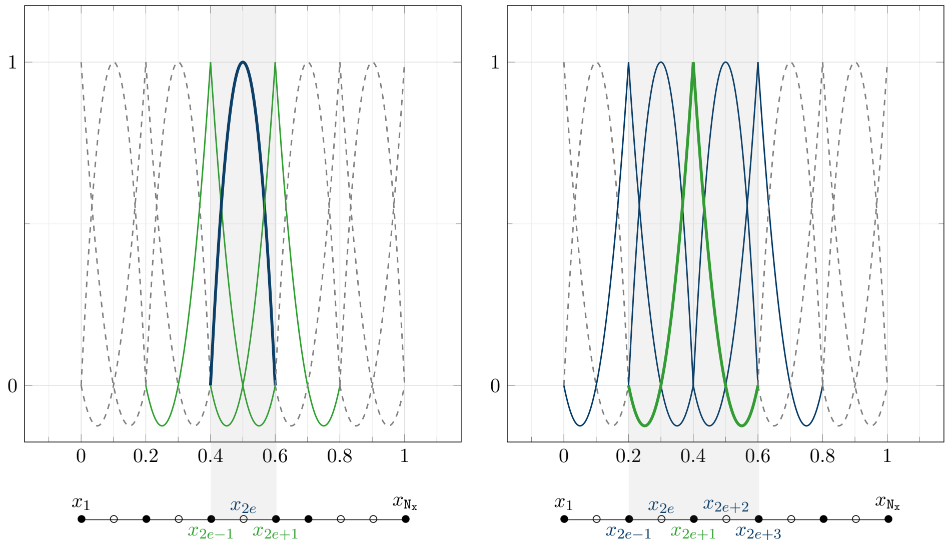

FIGURE 3. Left: shape function whose support intersect that of N_{2e}. Right: shape functions whose support intersect that of N_{2e+1}.

At the moment, we do not take into account the homogeneous Dirichlet boundary condition. We use the approximation

$$\begin{aligned} y(x,t) &\approx \sum_{i=1}^{2} \sum_{k=1}^{\mathtt{N_x}} N_k(x)\bar{\mathbf{e}}_i \mathbf{y}_{n(i,k)}(t) \\ &= \mathbf{N}(x)\mathbf{y}(t), \end{aligned} \tag{1}$$

and similarly $\psi(x) \approx \mathbf{N}(x)\boldsymbol{\psi}$. For any element $\omega^e := [x_{2e-1}, x_{2e+1}]$ for $e \in \{1, \dots, \mathtt{N_e}\}$, only three shape functions are nonzero and are given by

$$[N_{2e-1}, N_{2e}, N_{2e+1}] = \widetilde{\mathbf{N}} \left(\frac{x - x_{2e-1}}{x_{2e+1} - x_{2e-1}} \right),$$

where $\widetilde{\mathbf{N}}$ is defined by

$$\left[\widetilde{N}_1(\xi), \widetilde{N}_2(\xi), \widetilde{N}_3(\xi)\right] = \widetilde{\mathbf{N}}(\xi) = [(1-\xi)(1-2\xi), 4\xi(1-\xi), \xi(2\xi-1)] \,.$$

2. Two transport equations coupled at the boundary

2.1. The model. For $\lambda_1, \lambda_2 > 0$ and $k \in \mathbb{R}$, consider the system

$$\begin{cases} \partial_t y^1 + \lambda_1 \partial_x y^1 = 0 & \text{in } (0,\ell)\times(0,T) \\ \partial_t y^2 - \lambda_2 \partial_x y^2 = 0 & \text{in } (0,\ell)\times(0,T) \\ y^1(0,t) = 0 & t \in (0,T) \\ y^2(\ell,t) = k y^1(\ell,t) & t \in (0,T) \\ (y^1, y^2)(x,0) = (y^{10}, y^{20})(x) & x \in (0,\ell). \end{cases}$$

2.2. The weak formulation. Left-multiply by x-dependent function $\psi^\intercal = (\psi^1, \psi^2)^\intercal$ and integrate over $(0,\ell)$, to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^\ell \left\langle \psi\,, \begin{bmatrix} y^1(t) \\ y^2(t) \end{bmatrix} \right\rangle dx\right) + \int_0^\ell \left\langle \begin{bmatrix} \psi^1 \\ \psi^2 \end{bmatrix}, \begin{bmatrix} \lambda_1 & 0 \\ 0 & -\lambda_2 \end{bmatrix} \partial_x \begin{bmatrix} y^1(t) \\ y^2(t) \end{bmatrix} \right\rangle dx = 0.$$

We use integration by parts on the latter term, to obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^\ell \left\langle \psi\,, \begin{bmatrix} y^1(t) \\ y^2(t) \end{bmatrix} \right\rangle dx\right) + \left\langle \begin{bmatrix} \psi^1(\ell) \\ \psi^2(\ell) \end{bmatrix}, \begin{bmatrix} \lambda_1 & 0 \\ 0 & -\lambda_2 \end{bmatrix} \begin{bmatrix} y^1(\ell,t) \\ y^2(\ell,t) \end{bmatrix} \right\rangle \\ &- \left\langle \begin{bmatrix} \psi^1(0) \\ \psi^2(0) \end{bmatrix}, \begin{bmatrix} \lambda_1 & 0 \\ 0 & -\lambda_2 \end{bmatrix} \begin{bmatrix} y^1(0,t) \\ y^2(0,t) \end{bmatrix} \right\rangle - \int_0^\ell \left\langle \partial_x \begin{bmatrix} \psi^1 \\ \psi^2 \end{bmatrix}, \begin{bmatrix} \lambda_1 & 0 \\ 0 & -\lambda_2 \end{bmatrix} \begin{bmatrix} y^1(t) \\ y^2(t) \end{bmatrix} \right\rangle dx = 0. \end{aligned}$$

Using the boundary conditions satisfied by y, we obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^\ell \left\langle \psi\,, \begin{bmatrix} y^1(t) \\ y^2(t) \end{bmatrix} \right\rangle dx\right) + \left\langle \begin{bmatrix} \psi^1(\ell) \\ \psi^2(\ell) \end{bmatrix}, \begin{bmatrix} \lambda_1 & 0 \\ 0 & -k\lambda_2 \end{bmatrix} \begin{bmatrix} y^1(\ell,t) \\ y^1(\ell,t) \end{bmatrix} \right\rangle \\ &- \left\langle \psi^2(0)\,, -\lambda_2 y^2(0,t) \right\rangle - \int_0^\ell \left\langle \partial_x \begin{bmatrix} \psi^2 \\ \psi^2 \end{bmatrix}, \begin{bmatrix} \lambda_1 & 0 \\ 0 & -\lambda_2 \end{bmatrix} \begin{bmatrix} y^1(t) \\ y^2(t) \end{bmatrix} \right\rangle dx = 0. \end{aligned}$$

which also writes as

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^\ell \left\langle \psi\,, \begin{bmatrix} y^1(t) \\ y^2(t) \end{bmatrix} \right\rangle dx\right) + \left\langle \psi^1(\ell)\lambda_1 - \psi^2(\ell)k\lambda_2, y^1(\ell,t) \right\rangle \\ &- \left\langle \psi^2(0)\,, -\lambda_2 y^2(0,t) \right\rangle - \int_0^\ell \left\langle \partial_x \begin{bmatrix} \psi^1 \\ \psi^2 \end{bmatrix}, \begin{bmatrix} \lambda_1 & 0 \\ 0 & -\lambda_2 \end{bmatrix} \begin{bmatrix} y^1(t) \\ y^2(t) \end{bmatrix} \right\rangle dx = 0. \end{aligned}$$

Consider the functional space $V = \big\{\psi \in H^1(0,\ell;\mathbb{R}^2)\colon \psi^1(0) = 0\big\}$ and let

$$\bar{\mathbf{e}}_1 := \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \bar{\mathbf{e}}_2 := \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

We choose the following weak formulation

$$
\begin{aligned}
&\text{Find } y = (y^1, y^2)^\intercal \in C^0(0,T;V) \text{ such that: for all } \psi \in V,\\
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^\ell \langle \psi\,, y(t)\rangle\, dx\right) + \lambda_1 \langle \bar{\mathbf{e}}_1^\intercal \psi(\ell), \bar{\mathbf{e}}_1^\intercal y(\ell,t)\rangle - k\lambda_2 \langle \bar{\mathbf{e}}_2^\intercal \psi(\ell), \bar{\mathbf{e}}_1^\intercal y(\ell,t)\rangle \\
&+ \lambda_2 \langle \bar{\mathbf{e}}_2^\intercal \psi(0), \bar{\mathbf{e}}_2^\intercal y(0,t)\rangle - \int_0^\ell \left\langle \frac{\mathrm{d}}{\mathrm{d}x}\psi\,, \begin{bmatrix}\lambda_1 & 0\\ 0 & -\lambda_2\end{bmatrix} y(t)\right\rangle dx = 0.
\end{aligned} \tag{2}
$$

2.3. **The semi-discretization.** We semi-discretize in space by using the approximation

$$\mathbf{V} = \left\{\psi \in C^0([0,\ell];\mathbb{R}^2)\colon \psi\big|_{[x_\alpha, x_{\alpha+1}]} \in (\mathbb{P}_2)^2 \text{ for all } \alpha \in \{1,\dots,\mathtt{N_e}-1\},\ \psi^1(0) = 0\right\}.$$

We inject the approximation (1) into the weak form. This yields

$$
\begin{aligned}
&\boldsymbol{\psi}^\intercal \left(\int_0^\ell \mathbf{N}^\intercal\mathbf{N} dx\right) \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{y}(t) + \lambda_1 \boldsymbol{\psi}^\intercal \mathbf{N}(\ell)^\intercal \bar{\mathbf{e}}_1 \bar{\mathbf{e}}_1^\intercal \mathbf{N}(\ell)\mathbf{y}(t) - k\lambda_2 \boldsymbol{\psi}^\intercal \mathbf{N}(\ell)^\intercal \bar{\mathbf{e}}_2 \bar{\mathbf{e}}_1^\intercal \mathbf{N}(\ell)\mathbf{y}(t)\\
&+ \lambda_2 \boldsymbol{\psi}^\intercal \mathbf{N}(0)^\intercal \bar{\mathbf{e}}_2 \bar{\mathbf{e}}_2^\intercal \mathbf{N}(0)\mathbf{y}(t) - \boldsymbol{\psi}^\intercal \left(\int_0^\ell \frac{\mathrm{d}\mathbf{N}}{\mathrm{d}x}^\intercal \begin{bmatrix}\lambda_1 & 0\\ 0 & -\lambda_2\end{bmatrix} \mathbf{N} dx\right)\mathbf{y}(t) = 0,
\end{aligned}
$$

which also writes as

$$\mathcal{M}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{y}(t) + \mathcal{K}\mathbf{y}(t) = 0 \tag{3}$$

for $\mathcal{M}, \mathcal{K} \in \mathbb{R}^{\mathtt{N_{tot}}\times\mathtt{N_{tot}}}$ defined by

$$\mathcal{M} = \int_0^\ell \mathbf{N}^\intercal \mathbf{N} dx \tag{4}$$

and

$$
\begin{aligned}
\mathcal{K} = &\int_0^\ell \frac{\mathrm{d}\mathbf{N}}{\mathrm{d}x}^\intercal \begin{bmatrix}-\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix} \mathbf{N} dx + \lambda_1 \mathbf{N}(\ell)^\intercal \bar{\mathbf{e}}_1 \bar{\mathbf{e}}_1^\intercal \mathbf{N}(\ell)\\
&- k\lambda_2 \mathbf{N}(\ell)^\intercal \bar{\mathbf{e}}_2 \bar{\mathbf{e}}_1^\intercal \mathbf{N}(\ell) + \lambda_2 \mathbf{N}(0)^\intercal \bar{\mathbf{e}}_2 \bar{\mathbf{e}}_2^\intercal \mathbf{N}(0).
\end{aligned}
$$

Note that

$$\mathbf{N}^\intercal \mathbf{N} = \sum_{i=1}^2 \mathbf{N}_i^\intercal \mathbf{N}_i, \qquad \frac{\mathrm{d}\mathbf{N}}{\mathrm{d}x}^\intercal \begin{bmatrix}\lambda_1 & 0\\ 0 & -\lambda_2\end{bmatrix} \mathbf{N} = \sum_{i=1}^2 (-1)^i \lambda_i \frac{\mathrm{d}\mathbf{N}_i}{\mathrm{d}x}^\intercal \mathbf{N}_i,$$

and

$$
\begin{aligned}
\mathbf{N}(x_k)^\intercal \bar{\mathbf{e}}_i \bar{\mathbf{e}}_j^\intercal \mathbf{N}(x_k) &= \mathbf{N}_i(x_k)^\intercal \mathbf{N}_j(x_k)\\
&= \mathbf{e}_{n(i,k)}^\intercal \mathbf{e}_{n(j,k)}.
\end{aligned}
$$

We compute that

$$\mathcal{M} = \sum_{i=1}^2 \sum_{e=1}^{\mathtt{N_e}} \int_{\omega^e} \mathbf{N}_i^\intercal \mathbf{N}_i dx$$

which also writes as

$$
\begin{aligned}
\mathcal{M} = \sum_{i=1}^{2}\sum_{e=1}^{\mathtt{N_e}} & \left(\int_{\omega^e} (N_{2e-1})^2 dx\right) \mathbf{e}_{n(i,2e-1)}\mathbf{e}_{n(i,2e-1)}^{\mathsf{T}} + \left(\int_{\omega^e} (N_{2e})^2 dx\right) \mathbf{e}_{n(i,2e)}\mathbf{e}_{n(i,2e)}^{\mathsf{T}} \\
& + \left(\int_{\omega^e} (N_{2e+1})^2 dx\right) \mathbf{e}_{n(i,2e+1)}\mathbf{e}_{n(i,2e+1)}^{\mathsf{T}} \\
& + \left(\int_{\omega^e} N_{2e-1}N_{2e} dx\right) (\mathbf{e}_{n(i,2e-1)}\mathbf{e}_{n(i,2e)}^{\mathsf{T}} + \mathbf{e}_{n(i,2e)}\mathbf{e}_{n(i,2e-1)}^{\mathsf{T}}) \\
& + \left(\int_{\omega^e} N_{2e-1}N_{2e+1} dx\right) (\mathbf{e}_{n(i,2e-1)}\mathbf{e}_{n(i,2e+1)}^{\mathsf{T}} + \mathbf{e}_{n(i,2e+1)}\mathbf{e}_{n(i,2e-1)}^{\mathsf{T}}) \\
& + \left(\int_{\omega^e} N_{2e}N_{2e+1} dx\right) (\mathbf{e}_{n(i,2e)}\mathbf{e}_{n(i,2e+1)}^{\mathsf{T}} + \mathbf{e}_{n(i,2e+1)}\mathbf{e}_{n(i,2e)}^{\mathsf{T}}).
\end{aligned}
$$

and a change of variable yields

$$
\begin{aligned}
\mathcal{M} = \sum_{i=1}^{2}\sum_{e=1}^{\mathtt{N_e}} \mathtt{h_e} \Bigg[& \left(\int_0^1 (\widetilde{N}_1)^2 dx\right) \mathbf{e}_{n(i,2e-1)}\mathbf{e}_{n(i,2e-1)}^{\mathsf{T}} + \left(\int_0^1 (\widetilde{N}_2)^2 d\xi\right) \mathbf{e}_{n(i,2e)}\mathbf{e}_{n(i,2e)}^{\mathsf{T}} \\
& + \left(\int_0^1 (\widetilde{N}_3)^2 d\xi\right) \mathbf{e}_{n(i,2e+1)}\mathbf{e}_{n(i,2e+1)}^{\mathsf{T}} \\
& + \left(\int_0^1 \widetilde{N}_1\widetilde{N}_2 d\xi\right) (\mathbf{e}_{n(i,2e-1)}\mathbf{e}_{n(i,2e)}^{\mathsf{T}} + \mathbf{e}_{n(i,2e)}\mathbf{e}_{n(i,2e-1)}^{\mathsf{T}}) \\
& + \left(\int_0^1 \widetilde{N}_1\widetilde{N}_3 d\xi\right) (\mathbf{e}_{n(i,2e-1)}\mathbf{e}_{n(i,2e+1)}^{\mathsf{T}} + \mathbf{e}_{n(i,2e+1)}\mathbf{e}_{n(i,2e-1)}^{\mathsf{T}}) \\
& + \left(\int_0^1 \widetilde{N}_2\widetilde{N}_3 d\xi\right) (\mathbf{e}_{n(i,2e)}\mathbf{e}_{n(i,2e+1)}^{\mathsf{T}} + \mathbf{e}_{n(i,2e+1)}\mathbf{e}_{n(i,2e)}^{\mathsf{T}}) \Bigg].
\end{aligned}
$$

Hence, we will construct the mass matrix the following way.

Algorithm 1: Building the matrix $\mathcal{M}$.

Initialize $\mathcal{M}$ as a zero matrix
for $i = 1, 2$ **do**
    for $e = 1, \dots, \mathtt{N_e}$ **do**
        $\mathtt{idx} = [n(i, 2e-1), n(i, 2e), n(i, 2e+1)]$
        $\mathcal{M}(\mathtt{idx}, \mathtt{idx}) = \mathcal{M}(\mathtt{idx}, \mathtt{idx}) + \mathtt{h_e}\mathcal{M}^e$

Here, the element mass matrix $\mathcal{M}^e \in \mathbb{R}^{3\times 3}$ is given by

$$
\mathcal{M}^e = \int_0^1 (\widetilde{\mathbf{N}})^{\mathsf{T}}\widetilde{\mathbf{N}} d\xi = \frac{1}{30}\begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix}
$$

Similarly, the stiffness matrix takes the form

$$\mathcal{K} = \sum_{i=1}^{2}(-1)^i\lambda_i \sum_{e=1}^{\mathtt{N_e}} \int_{\omega^e} \frac{\mathrm{d}\mathbf{N}_i}{\mathrm{d}x} dx + \lambda_1 \mathbf{e}_{n(1,\mathtt{N_x})}\mathbf{e}^{\intercal}_{n(1,\mathtt{N_x})} - k\lambda_2 \mathbf{e}_{n(2,\mathtt{N_x})}\mathbf{e}^{\intercal}_{n(1,\mathtt{N_x})} + \lambda_2 \mathbf{e}_{n(2,1)}\mathbf{e}^{\intercal}_{n(2,1)}$$

which also writes as

$$\begin{aligned}
\mathcal{K} = \sum_{i=1}^{2}(-1)^i\lambda_i \sum_{e=1}^{\mathtt{N_e}} \Bigg[&\left(\int_{\omega^e} \frac{\mathrm{d}N_{2e-1}}{\mathrm{d}x} N_{2e-1} dx\right) \mathbf{e}_{n(i,2e-1)}\mathbf{e}^{\intercal}_{n(i,2e-1)} \\
&+ \left(\int_{\omega^e} \frac{\mathrm{d}N_{2e}}{\mathrm{d}x} N_{2e} dx\right) \mathbf{e}_{n(i,2e)}\mathbf{e}^{\intercal}_{n(i,2e)} + \left(\int_{\omega^e} \frac{\mathrm{d}N_{2e+1}}{\mathrm{d}x} N_{2e+1} dx\right) \mathbf{e}_{n(i,2e+1)}\mathbf{e}^{\intercal}_{n(i,2e+1)} \\
&+ \left(\int_{\omega^e} \frac{\mathrm{d}N_{2e-1}}{\mathrm{d}x} N_{2e} dx\right) \mathbf{e}_{n(i,2e-1)}\mathbf{e}^{\intercal}_{n(i,2e)} + \left(\int_{\omega^e} \frac{\mathrm{d}N_{2e}}{\mathrm{d}x} N_{2e-1} dx\right) \mathbf{e}_{n(i,2e)}\mathbf{e}^{\intercal}_{n(i,2e-1)} \\
&+ \left(\int_{\omega^e} \frac{\mathrm{d}N_{2e-1}}{\mathrm{d}x} N_{2e+1} dx\right) \mathbf{e}_{n(i,2e-1)}\mathbf{e}^{\intercal}_{n(i,2e+1)} + \left(\int_{\omega^e} \frac{\mathrm{d}N_{2e+1}}{\mathrm{d}x} N_{2e-1} dx\right) \mathbf{e}_{n(i,2e+1)}\mathbf{e}^{\intercal}_{n(i,2e-1)} \\
&+ \left(\int_{\omega^e} \frac{\mathrm{d}N_{2e}}{\mathrm{d}x} N_{2e+1} dx\right) \mathbf{e}_{n(i,2e)}\mathbf{e}^{\intercal}_{n(i,2e+1)} + \left(\int_{\omega^e} \frac{\mathrm{d}N_{2e+1}}{\mathrm{d}x} N_{2e} dx\right) \mathbf{e}_{n(i,2e+1)}\mathbf{e}^{\intercal}_{n(i,2e)} \Bigg] \\
&+ \lambda_1 \mathbf{e}_{n(1,\mathtt{N_x})}\mathbf{e}^{\intercal}_{n(1,\mathtt{N_x})} - k\lambda_2 \mathbf{e}_{n(2,\mathtt{N_x})}\mathbf{e}^{\intercal}_{n(1,\mathtt{N_x})} + \lambda_2 \mathbf{e}_{n(2,1)}\mathbf{e}^{\intercal}_{n(2,1)}.
\end{aligned}$$

The definition of the shape functions together with a change of variables yields that

$$\begin{aligned}
\mathcal{K} = \sum_{i=1}^{2}(-1)^i\lambda_i \sum_{e=1}^{\mathtt{N_e}} \Bigg[&\left(\int_0^1 \frac{\mathrm{d}\widetilde{N}_1}{\mathrm{d}\xi} \widetilde{N}_1 d\xi\right) \mathbf{e}_{n(i,2e-1)}\mathbf{e}^{\intercal}_{n(i,2e-1)} \\
&+ \left(\int_0^1 \frac{\mathrm{d}\widetilde{N}_2}{\mathrm{d}\xi} \widetilde{N}_2 d\xi\right) \mathbf{e}_{n(i,2e)}\mathbf{e}^{\intercal}_{n(i,2e)} + \left(\int_0^1 \frac{\mathrm{d}\widetilde{N}_3}{\mathrm{d}\xi} \widetilde{N}_3 d\xi\right) \mathbf{e}_{n(i,2e+1)}\mathbf{e}^{\intercal}_{n(i,2e+1)} \\
&+ \left(\int_0^1 \frac{\mathrm{d}\widetilde{N}_1}{\mathrm{d}\xi} \widetilde{N}_2 d\xi\right) \mathbf{e}_{n(i,2e-1)}\mathbf{e}^{\intercal}_{n(i,2e)} + \left(\int_0^1 \frac{\mathrm{d}\widetilde{N}_2}{\mathrm{d}\xi} \widetilde{N}_1 d\xi\right) \mathbf{e}_{n(i,2e)}\mathbf{e}^{\intercal}_{n(i,2e-1)} \\
&+ \left(\int_0^1 \frac{\mathrm{d}\widetilde{N}_1}{\mathrm{d}\xi} \widetilde{N}_3 d\xi\right) \mathbf{e}_{n(i,2e-1)}\mathbf{e}^{\intercal}_{n(i,2e+1)} + \left(\int_0^1 \frac{\mathrm{d}\widetilde{N}_3}{\mathrm{d}\xi} \widetilde{N}_1 d\xi\right) \mathbf{e}_{n(i,2e+1)}\mathbf{e}^{\intercal}_{n(i,2e-1)} \\
&+ \left(\int_0^1 \frac{\mathrm{d}\widetilde{N}_2}{\mathrm{d}\xi} \widetilde{N}_3 d\xi\right) \mathbf{e}_{n(i,2e)}\mathbf{e}^{\intercal}_{n(i,2e+1)} + \left(\int_0^1 \frac{\mathrm{d}\widetilde{N}_3}{\mathrm{d}\xi} \widetilde{N}_2 d\xi\right) \mathbf{e}_{n(i,2e+1)}\mathbf{e}^{\intercal}_{n(i,2e)} \Bigg] \\
&+ \lambda_1 \mathbf{e}_{n(1,\mathtt{N_x})}\mathbf{e}^{\intercal}_{n(1,\mathtt{N_x})} - k\lambda_2 \mathbf{e}_{n(2,\mathtt{N_x})}\mathbf{e}^{\intercal}_{n(1,\mathtt{N_x})} + \lambda_2 \mathbf{e}_{n(2,1)}\mathbf{e}^{\intercal}_{n(2,1)}.
\end{aligned}$$

So we will construct the stiffness matrix the following way.

Algorithm 2: Building the matrix $\mathcal{K}$.

```
Initialize K as a zero matrix
for i = 1,2 do
    for e = 1,...,N_e do
        idx = [n(i,2e-1), n(i,2e), n(i,2e+1)]
        K(idx,idx) = K(idx,idx) + (-1)^i λ_i K^e
K(n(1,N_x), n(1,N_x)) = K(n(1,N_x), n(1,N_x)) + λ_1
K(n(2,N_x), n(1,N_x)) = K(n(2,N_x), n(1,N_x)) - kλ_2
K(n(2,1), n(2,1)) = K(n(2,1), n(2,1)) + λ_2
```

Here, the element stiffness matrix $\mathcal{K}^e \in \mathbb{R}^{3\times 3}$ is given by

$$\mathcal{K}^e = \int_0^1 \left(\frac{\mathrm{d}\widetilde{\mathbf{N}}}{\mathrm{d}\xi}\right)^\intercal \widetilde{\mathbf{N}} d\xi = \frac{1}{6}\begin{bmatrix} -3 & -4 & 1 \\ 4 & 0 & -4 \\ -1 & 4 & 3 \end{bmatrix}.$$

Up to now, the boundary condition $y^1(0,\cdot) \equiv 0$ has not been taken into account. One will then have to remove the first row and column of the obtained matrices (i.e. extract sub-matrices from $\mathcal{M}$ and $\mathcal{K}$), which amounts to removing the first equation of the obtained ODE. The actual number of unknowns will be $\mathtt{N_f} := 2\mathtt{N_x} - 1$ and the $y(x_k, t)$ will correspond to the $f(i,k)$-th component of the new (smaller) unknown state $\mathbf{y}$, where $f(i,k) := 2(k-1) + i - 1$. More precisely, we proceed as follows.

Algorithm 3: Applying the Dirichlet boundary conditions.

```
dof = 2 : N_tot
M = M(dof, dof)
K = K(dof, dof)
```

3. 2×2 LINEAR HYPERBOLIC SYSTEM

3.1. **The model.** Consider the system

$$\begin{cases} \partial_t y + A\partial_x y = 0 & \text{in } (0,\ell)\times(0,T) \\ y^1(0,t) = 0 & t \in (0,T) \\ y^2(\ell,t) = f(t) & t\in(0,T) \\ (y^1,y^2)(x,0) = (y^{10},y^{20})(x) & x \in (0,\ell), \end{cases}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

3.2. **Working with the physical variables.** In Section 3.2 we work with the unknown state y (rather than diagonalizing the system as in Section 3.3).

3.2.1. *Weak formulation.* The code corresponding to this subsection is given in the file `transport_nodiag_wf1.m`.

We define the matrices Π^1 and Π^2, and space V by

$$\begin{aligned} \Pi^1 &= \begin{bmatrix} 1 & 0 \end{bmatrix}, \Pi^2 = \begin{bmatrix} 0 & 1 \end{bmatrix}, \\ V &= \{\psi \in H^1(0,\ell;\mathbb{R}^2) \colon \Pi^1\psi(0) = 0\}; \end{aligned}$$

Integrating by parts, one has for any $\psi \in V$

$$\int_0^\ell \langle \psi \,, A\partial_x y\rangle dx = -\int_0^\ell \left\langle \frac{\mathrm{d}\psi}{\mathrm{d}x} \,, Ay \right\rangle dx + [\langle \psi \,, Ay\rangle]_0^\ell \,,$$

and one may compute that

$$\begin{aligned} [\langle \psi \,, Ay\rangle]_0^\ell &= \left[\langle \Pi^1\psi \,, \Pi^2 y\rangle + \langle \Pi^2\psi \,, \Pi^1 y\rangle\right]_0^\ell \\ &= \langle \Pi^1\psi(\ell \,, f(t)\rangle + \langle \Pi^2\psi(\ell) \,, \Pi^1 y(\ell)\rangle. \end{aligned}$$

We choose the following weak form: find $y \in C^0(0,T;V)$ such that for all $\psi \in V$ one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^\ell \langle \psi \,, y(t)\rangle \, dx\right) - \int_0^\ell \left\langle \frac{\mathrm{d}\psi}{\mathrm{d}x} \,, Ay \right\rangle dx + \langle \Pi^1\psi(\ell) \,, f(t)\rangle + \langle \Pi^2\psi(\ell) \,, \Pi^1 y(\ell)\rangle = 0.$$

3.2.2. *The semi-discretization.* We inject the approximation (1) into the weak form. This yields the ODE

$$\mathcal{M}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{y}(t) + \mathcal{K}\mathbf{y}(t) + \mathcal{W}f(t) = 0 \tag{5}$$

where $\mathcal{M} \in \mathbb{R}^{\mathtt{N_{tot}}\times\mathtt{N_{tot}}}$ is defined by (4), $\mathcal{K} \in \mathbb{R}^{\mathtt{N_{tot}}\times\mathtt{N_{tot}}}$ is defined by

$$\mathcal{K} = -\int_0^\ell \frac{\mathrm{d}\mathbf{N}^\intercal}{\mathrm{d}x} A\mathbf{N} dx + \langle \Pi^2\mathbf{N}(\ell) \,, \Pi^1\mathbf{N}(\ell)\rangle,$$

and $\mathcal{W} \in \mathbb{R}^{\mathtt{N_{tot}}}$ is defined by

$$\mathcal{W} = (\Pi^1\mathbf{N}(\ell))^\intercal.$$

The matrix $\mathcal{M}$ may be built as before. On the other hand one may compute that $\langle \Pi^2\mathbf{N}(\ell) \,, \Pi^1\mathbf{N}(\ell)\rangle = e_{n(2,Nx)}e_{n(1,Nx)}^\intercal$ and $(\Pi^1\mathbf{N}(\ell))^\intercal = e_{n(1,\mathtt{N_x})}$. Hence, $\mathcal{K}$ and $\mathcal{W}$ can be built as follows.

Algorithm 4: Building the matrices $\mathcal{K}$ and $\mathcal{W}$.

Initialize $\mathcal{K}, \mathcal{W}$ as zero matrices
for $i = 1, 2$ **do**
    for $j = 1, 2$ **do**
        for $e = 1, \dots, \mathtt{N_e}$ **do**
            $\mathtt{idxRow} = [n(i, 2e-1), n(i, 2e), n(i, 2e+1)]$
            $\mathtt{idxCol} = [n(j, 2e-1), n(j, 2e), n(j, 2e+1)]$
            $\mathcal{K}(\mathtt{idxRow}, \mathtt{idxCol}) = \mathcal{K}(\mathtt{idxRow}, \mathtt{idxCol}) - A_{i,j}\mathcal{K}^e$
$\mathcal{K}(n(2, \mathtt{N_x}), n(1, \mathtt{N_x})) = \mathcal{K}(n(2, \mathtt{N_x}), n(1, \mathtt{N_x})) + 1$
$\mathcal{W}(n(1, \mathtt{N_x}), 1) = 1$

Finally, we remove the first equation of the obtained ODE since the value of the state is known to be equal to zero at $x = 0$.

Algorithm 5: Applying the Dirichlet boundary conditions.

$\mathtt{dof} = 2 : \mathtt{N_{tot}}$
$\mathcal{M} = \mathcal{M}(\mathtt{dof}, \mathtt{dof})$
$\mathcal{K} = \mathcal{K}(\mathtt{dof}, \mathtt{dof})$
$\mathcal{W} = \mathcal{W}(\mathtt{dof}, 1)$

3.3. **Working with the diagonal variables.** Let us consider the change of variable $r = Ly$ for $L \in \mathbb{R}^{2\times 2}$ defined by

$$L = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Note that the inverse L^{-1} of this matrix is such that $L^{-1} = L = L^\intercal$. We have chosen this matrix L in such a way that $LAL^\intercal = \mathbf{D}$ for $\mathbf{D} \in \mathbb{R}^{2\times 2}$ defined by

$$\mathbf{D} = \mathrm{diag}(1, -1).$$

Then, the new state $r = (r^-, r^+)^\intercal$ is solution to the equivalent system

$$\begin{cases} \partial_t r + \mathbf{D}\partial_x r = 0 & \text{in } (0,\ell)\times(0,T) \\ r^+(0,t) = r^-(0,t) & t \in (0,T) \\ r^-(\ell,t) = -r^+(\ell,t) + \sqrt{2}f(t) & t \in (0,T) \\ r(x,0) = Ly^0(x) & x \in (0,\ell). \end{cases}$$

3.3.1. *Weak formulation.* To complete.

This time we choose the following functional space V

$$V = H^1(0,\ell;\mathbb{R}^2)$$

where no boundary condition is included in the definition of V.

Using integration by parts, one has for any $\varphi = (\varphi^+, \varphi^-)^\intercal \in V$

$$\int_0^\ell \langle \varphi \,, \mathbf{D}\partial_x r\rangle dx = -\int_0^\ell \left\langle \frac{\mathrm{d}\varphi}{\mathrm{d}x} \,, \mathbf{D}r \right\rangle dx + [\langle \varphi \,, \mathbf{D}r\rangle]_0^\ell \,,$$

Using the boundary condition satisfied by r, we obtain

$$[\langle \varphi \,, \mathbf{D}r\rangle]_0^\ell = ...$$

3.3.2. *The semi-discretization.* To complete.

4. Time discretization

We have obtained the linear ODE

$$\mathcal{M}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{y}(t) + \mathcal{K}\mathbf{y}(t) = 0,$$

and now want to discretize it in time. For the time integration, we use an implicit midpoint rule. Let the time interval be divided into $\mathtt{N_t}$ points $\{t_k\}_{k=1}^{\mathtt{N_t}}$ with $t_1 = 0$ and $t_{\mathtt{N_t}} = T$, and let $\mathtt{h_t} = \frac{T}{\mathtt{N_t}-1}$ be the time step. For an ODE of the form,

$$\frac{\mathrm{d}\mathbf{y}(t)}{\mathrm{d}t} = f(\mathbf{y}(t)),$$

the implicit midpoint rule yields the following time discretization

$$\mathbf{y}^{k+1} = \mathbf{y}^k + \mathtt{h_t} f\left(\frac{\mathbf{y}^k + \mathbf{y}^{k+1}}{2}\right)$$

for $k \in \{1, \ldots, \mathtt{N_t}\}$. Here, $f(\zeta) = -\mathcal{M}^{-1}\mathcal{K}\zeta$. Consequently, the scheme reads

$$\left(\mathcal{M} + \frac{\mathtt{h_t}}{2}\mathcal{K}\right)\mathbf{y}^{k+1} = \left(\mathcal{M} - \frac{\mathtt{h_t}}{2}\mathcal{K}\right)\mathbf{y}^k.$$