FEM FOR COUPLED TRANSPORT EQUATIONS

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Abstract. This is the explanation of the discretization of two transport equations coupled at the boundary (using the finite element method). Our choice of discretization is done in view of extending this method to a more complicated model: the Intrinsic Geometrically Exact Beam model, which is also a first-order system of partial differential equations.

Contents

Notation. Let $m, n \in \{1, 2, \dots\}$. The inner product in $\mathbb{R}^n$ is denoted $\langle \cdot\,, \cdot \rangle$. Here, the identity and null matrices are denoted by $\mathbf{I}_n \in \mathbb{R}^{n\times n}$ and $\mathbf{0}_{n,m} \in \mathbb{R}^{n\times m}$, and we use the abbreviation $\mathbf{0}_n = \mathbf{0}_{n,n}$. If there is no confusion, we omit the subscript and write $\mathbf{I}$ and $\mathbf{0}$ instead.

1. The model

For $\lambda_1, \lambda_2 > 0$ and $k \in \mathbb{R}$, consider the system

$$\begin{cases} \partial_t y^1 + \lambda_1 \partial_x y^1 = 0 & \text{in } (0,\ell)\times(0,T) \\ \partial_t y^2 - \lambda_2 \partial_x y^2 = 0 & \text{in } (0,\ell)\times(0,T) \\ y^1(0,t) = 0 & t \in (0,T) \\ y^2(\ell,t) = k y^1(\ell,t) & t \in (0,T) \\ (y^1, y^2)(s,0) = (y^{10}, y^{20})(x) & x \in (0,\ell). \end{cases}$$

2. Spatial discretization

We place $\mathtt{N_x}$ points $\{x_k\}_{k=1}^{\mathtt{N_x}}$ on the interval $[0,\ell]$, such that $x_1 = 0$ and $x_{\mathtt{N_x}} = \ell$. The number $\mathtt{N_x}$ is even, and each interval $[x_{2e-1}, x_{2e+1}]$ for $e \in \{1, 2, \dots, \mathtt{N_e}\}$ constitute an

Date: January 19, 2022.

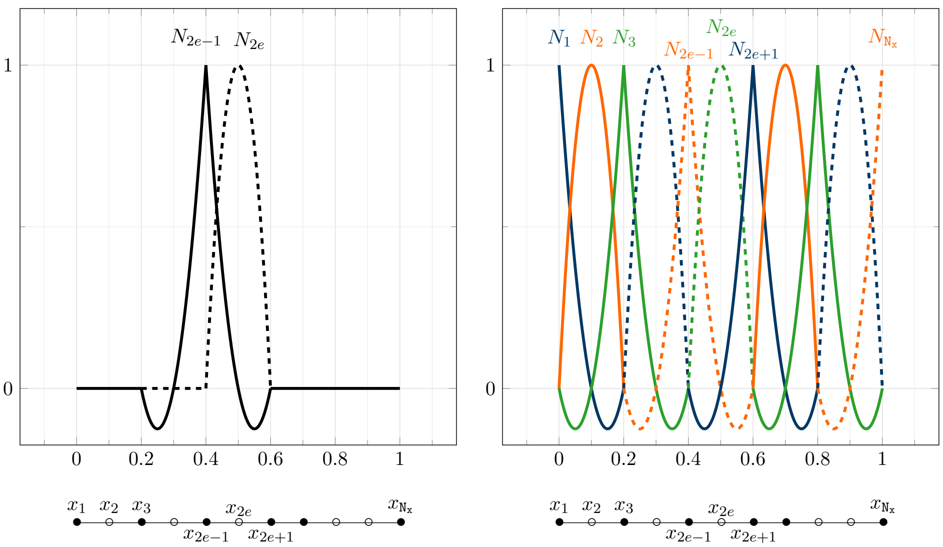

FIGURE 1. Left: two kinds of shape functions. Right: shape functions over the whole interval.

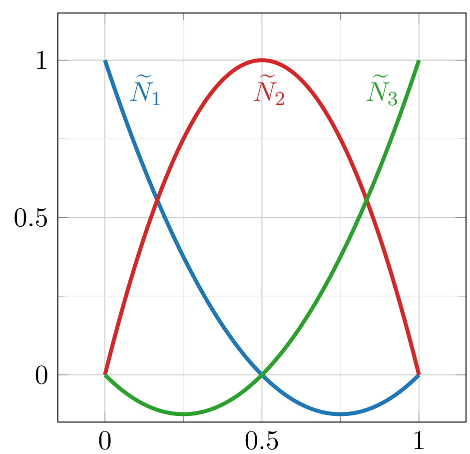

FIGURE 2. Reference element.

element, which contains the points $x_{2e-1}, x_{2e}, x_{2e+1}$. Thus, the length of the element e is $\mathtt{h_e} = x_{2e+1} - x_{2e-1}$, and $\mathtt{N_x} = 2\mathtt{N_e} + 1$. Let

$$n(i,k) := 2(k-1) + i, \quad \mathtt{N_{tot}} := 2\mathtt{N_x}.$$

Let $\mathbf{N}\colon [0,\ell] \to \mathbb{R}^{2\times \mathtt{N_{tot}}}$ be defined by

$$\begin{bmatrix} \mathbf{N}_1 \\ \mathbf{N}_2 \end{bmatrix} = \mathbf{N} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & \dots & 0 & N_{\mathtt{N_x}} & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & \dots & N_{\mathtt{N_x}-1} & 0 & N_{\mathtt{N_x}} \end{bmatrix}.$$

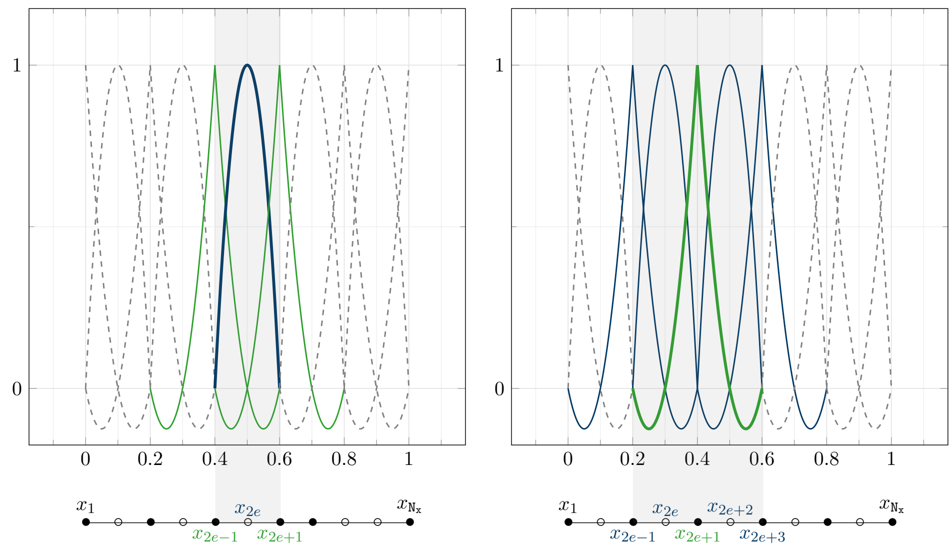

FIGURE 3. Left: shape function whose support intersect that of N_{2e}. Right: shape functions whose support intersect that of N_{2e+1}.

At the moment, we do not take into account the homogeneous Dirichlet boundary condition. We use the approximation

$$\begin{aligned} y(x,t) &\approx \sum_{i=1}^{2} \sum_{k=1}^{\mathtt{N_x}} N_k(x) \bar{\mathbf{e}}_i \mathbf{y}_{n(i,k)}(t) \\ &= \mathbf{N}(x)\mathbf{y}(t), \end{aligned} \tag{1}$$

and similarly $\psi(x) \approx \mathbf{N}(x)\boldsymbol{\psi}$. For any element $\omega^e := [x_{2e-1}, x_{2e+1}]$ for $e \in \{1, \dots, \mathtt{N_e}\}$, only three shape functions are nonzero and are given by

$$[N_{2e-1}, N_{2e}, N_{2e+1}] = \widetilde{\mathbf{N}} \left(\frac{x - x_{2e-1}}{x_{2e+1} - x_{2e-1}} \right),$$

where $\widetilde{\mathbf{N}}$ is defined by

$$\left[\widetilde{N}_1(\xi), \widetilde{N}_2(\xi), \widetilde{N}_3(\xi) \right] = \widetilde{\mathbf{N}}(\xi) = \left[(1-\xi)(1-2\xi), 4\xi(1-\xi), \xi(2\xi-1) \right].$$

3. FIRST WEAK FORM: WITH INTEGRATION BY PARTS

Left-multiply by x-dependent function $\psi^{\intercal} = (\psi^1, \psi^2)^{\intercal}$ and integrate over $(0, \ell)$, to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\int_0^{\ell} \left\langle \psi \, , \begin{bmatrix} y^1(t) \\ y^2(t) \end{bmatrix} \right\rangle dx \right) + \int_0^{\ell} \left\langle \begin{bmatrix} \psi^1 \\ \psi^2 \end{bmatrix} , \begin{bmatrix} \lambda_1 & 0 \\ 0 & -\lambda_2 \end{bmatrix} \partial_x \begin{bmatrix} y^1(t) \\ y^2(t) \end{bmatrix} \right\rangle dx = 0.$$

We use integration by parts on the latter term, to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^\ell \left\langle \psi \,, \begin{bmatrix} y^1(t) \\ y^2(t) \end{bmatrix} \right\rangle dx\right) + \left\langle \begin{bmatrix} \psi^1(\ell) \\ \psi^2(\ell) \end{bmatrix} , \begin{bmatrix} \lambda_1 & 0 \\ 0 & -\lambda_2 \end{bmatrix} \begin{bmatrix} y^1(\ell,t) \\ y^2(\ell,t) \end{bmatrix} \right\rangle$$
$$- \left\langle \begin{bmatrix} \psi^1(0) \\ \psi^2(0) \end{bmatrix} , \begin{bmatrix} \lambda_1 & 0 \\ 0 & -\lambda_2 \end{bmatrix} \begin{bmatrix} y^1(0,t) \\ y^2(0,t) \end{bmatrix} \right\rangle - \int_0^\ell \left\langle \partial_x \begin{bmatrix} \psi^1 \\ \psi^2 \end{bmatrix} , \begin{bmatrix} \lambda_1 & 0 \\ 0 & -\lambda_2 \end{bmatrix} \begin{bmatrix} y^1(t) \\ y^2(t) \end{bmatrix} \right\rangle dx = 0.$$

Using the boundary conditions satisfied by y, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^\ell \left\langle \psi \,, \begin{bmatrix} y^1(t) \\ y^2(t) \end{bmatrix} \right\rangle dx\right) + \left\langle \begin{bmatrix} \psi^1(\ell) \\ \psi^2(\ell) \end{bmatrix} , \begin{bmatrix} \lambda_1 & 0 \\ 0 & -k\lambda_2 \end{bmatrix} \begin{bmatrix} y^1(\ell,t) \\ y^1(\ell,t) \end{bmatrix} \right\rangle$$
$$- \left\langle \psi^2(0) \,, -\lambda_2 y^2(0,t) \right\rangle - \int_0^\ell \left\langle \partial_x \begin{bmatrix} \psi^2 \\ \psi^2 \end{bmatrix} , \begin{bmatrix} \lambda_1 & 0 \\ 0 & -\lambda_2 \end{bmatrix} \begin{bmatrix} y^1(t) \\ y^2(t) \end{bmatrix} \right\rangle dx = 0.$$

which also writes as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^\ell \left\langle \psi \,, \begin{bmatrix} y^1(t) \\ y^2(t) \end{bmatrix} \right\rangle dx\right) + \left\langle \psi^1(\ell)\lambda_1 - \psi^2(\ell)k\lambda_2, y^1(\ell,t) \right\rangle$$
$$- \left\langle \psi^2(0) \,, -\lambda_2 y^2(0,t) \right\rangle - \int_0^\ell \left\langle \partial_x \begin{bmatrix} \psi^1 \\ \psi^2 \end{bmatrix} , \begin{bmatrix} \lambda_1 & 0 \\ 0 & -\lambda_2 \end{bmatrix} \begin{bmatrix} y^1(t) \\ y^2(t) \end{bmatrix} \right\rangle dx = 0.$$

Consider the functional space $V = \{\psi \in H^1(0,\ell;\mathbb{R}^2) \colon \psi^1(0) = 0\}$ and let

$$\bar{\mathbf{e}}_1 := \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \bar{\mathbf{e}}_2 := \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

We choose the following weak formulation

$$\begin{aligned} &\text{Find } y = (y^1, y^2)^\intercal \in C^0(0,T;V) \text{ such that: for all } \psi \in V, \\ &\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^\ell \langle \psi \,, y(t) \rangle \, dx\right) + \lambda_1 \langle \bar{\mathbf{e}}_1^\intercal \psi(\ell), \bar{\mathbf{e}}_1^\intercal y(\ell,t) \rangle - k\lambda_2 \langle \bar{\mathbf{e}}_2^\intercal \psi(\ell), \bar{\mathbf{e}}_1^\intercal y(\ell,t) \rangle \\ &+ \lambda_2 \langle \bar{\mathbf{e}}_2^\intercal \psi(0), \bar{\mathbf{e}}_2^\intercal y(0,t) \rangle - \int_0^\ell \left\langle \frac{\mathrm{d}}{\mathrm{d}x}\psi \,, \begin{bmatrix} \lambda_1 & 0 \\ 0 & -\lambda_2 \end{bmatrix} y(t) \right\rangle dx = 0. \end{aligned} \tag{2}$$

We semi-discretize in space by using the approximation

$$\mathbf{V} = \{\psi \in C^0([0,\ell];\mathbb{R}^2) \colon \psi\big|_{[x_\alpha, x_{\alpha+1}]} \in (\mathbb{P}_2)^2 \text{ for all } i \in \{0,\dots,n\},\ \psi^1(0) = 0\}.$$

We inject the approximation (1) into the weak form. This yields

$$\boldsymbol{\psi}^\intercal \left(\int_0^\ell \mathbf{N}^\intercal \mathbf{N} dx\right) \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{y}(t) + \lambda_1 \boldsymbol{\psi}^\intercal \mathbf{N}(\ell)^\intercal \bar{\mathbf{e}}_1 \bar{\mathbf{e}}_1^\intercal \mathbf{N}(\ell)\mathbf{y}(t) - k\lambda_2 \boldsymbol{\psi}^\intercal \mathbf{N}(\ell)^\intercal \bar{\mathbf{e}}_2 \bar{\mathbf{e}}_1^\intercal \mathbf{N}(\ell)\mathbf{y}(t)$$
$$+ \lambda_2 \boldsymbol{\psi}^\intercal \mathbf{N}(0)^\intercal \bar{\mathbf{e}}_2 \bar{\mathbf{e}}_2^\intercal \mathbf{N}(0)\mathbf{y}(t) - \boldsymbol{\psi}^\intercal \left(\int_0^\ell \frac{\mathrm{d}\mathbf{N}}{\mathrm{d}x}^\intercal \begin{bmatrix} \lambda_1 & 0 \\ 0 & -\lambda_2 \end{bmatrix} \mathbf{N} dx\right) \mathbf{y}(t) = 0,$$

which also writes as

$$\mathcal{M}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{y}(t) + \mathcal{K}\mathbf{y}(t) = 0$$

for $\mathcal{M}, \mathcal{K} \in \mathbb{R}^{\mathtt{N_{tot}} \times \mathtt{N_{tot}}}$ defined by

$$\mathcal{M} = \int_0^\ell \mathbf{N}^\intercal \mathbf{N} dx$$

and

$$\begin{aligned}\mathcal{K} = &\int_0^\ell \frac{\mathrm{d}\mathbf{N}}{\mathrm{d}x}^\intercal \begin{bmatrix} -\lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \mathbf{N} dx + \lambda_1 \mathbf{N}(\ell)^\intercal \bar{\mathbf{e}}_1 \bar{\mathbf{e}}_1^\intercal \mathbf{N}(\ell) \\ &- k\lambda_2 \mathbf{N}(\ell)^\intercal \bar{\mathbf{e}}_2 \bar{\mathbf{e}}_1^\intercal \mathbf{N}(\ell) + \lambda_2 \mathbf{N}(0)^\intercal \bar{\mathbf{e}}_2 \bar{\mathbf{e}}_2^\intercal \mathbf{N}(0).\end{aligned}$$

Note that

$$\mathbf{N}^\intercal \mathbf{N} = \sum_{i=1}^{2} \mathbf{N}_i^\intercal \mathbf{N}_i, \qquad \frac{\mathrm{d}\mathbf{N}}{\mathrm{d}x}^\intercal \begin{bmatrix} \lambda_1 & 0 \\ 0 & -\lambda_2 \end{bmatrix} \mathbf{N} = \sum_{i=1}^{2} (-1)^i \lambda_i \frac{\mathrm{d}\mathbf{N}_i}{\mathrm{d}x}^\intercal \mathbf{N}_i,$$

and

$$\begin{aligned}\mathbf{N}(x_k)^\intercal \bar{\mathbf{e}}_i \bar{\mathbf{e}}_j^\intercal \mathbf{N}(x_k) &= \mathbf{N}_i(x_k)^\intercal \mathbf{N}_j(x_k) \\ &= \mathbf{e}_{n(i,k)}^\intercal \mathbf{e}_{n(j,k)}.\end{aligned}$$

We compute that

$$\mathcal{M} = \sum_{i=1}^{2} \sum_{e=1}^{\mathtt{N_e}} \int_{\omega^e} \mathbf{N}_i^\intercal \mathbf{N}_i dx$$

which also writes as

$$\begin{aligned}\mathcal{M} = \sum_{i=1}^{2} \sum_{e=1}^{\mathtt{N_e}} &\left(\int_{\omega^e} (N_{2e-1})^2 dx \right) \mathbf{e}_{n(i,2e-1)} \mathbf{e}_{n(i,2e-1)}^\intercal + \left(\int_{\omega^e} (N_{2e})^2 dx \right) \mathbf{e}_{n(i,2e)} \mathbf{e}_{n(i,2e)}^\intercal \\ &+ \left(\int_{\omega^e} (N_{2e+1})^2 dx \right) \mathbf{e}_{n(i,2e+1)} \mathbf{e}_{n(i,2e+1)}^\intercal \\ &+ \left(\int_{\omega^e} N_{2e-1} N_{2e} dx \right) (\mathbf{e}_{n(i,2e-1)} \mathbf{e}_{n(i,2e)}^\intercal + \mathbf{e}_{n(i,2e)} \mathbf{e}_{n(i,2e-1)}^\intercal) \\ &+ \left(\int_{\omega^e} N_{2e-1} N_{2e+1} dx \right) (\mathbf{e}_{n(i,2e-1)} \mathbf{e}_{n(i,2e+1)}^\intercal + \mathbf{e}_{n(i,2e+1)} \mathbf{e}_{n(i,2e-1)}^\intercal) \\ &+ \left(\int_{\omega^e} N_{2e} N_{2e+1} dx \right) (\mathbf{e}_{n(i,2e)} \mathbf{e}_{n(i,2e+1)}^\intercal + \mathbf{e}_{n(i,2e+1)} \mathbf{e}_{n(i,2e)}^\intercal).\end{aligned}$$

and a change of variable yields

$$
\begin{aligned}
\mathcal{M} = \sum_{i=1}^{2} \sum_{e=1}^{\mathtt{N_e}} \mathtt{h_e} \Bigg[& \left(\int_0^1 (\widetilde{N}_1)^2 dx \right) \mathbf{e}_{n(i,2e-1)} \mathbf{e}_{n(i,2e-1)}^{\intercal} + \left(\int_0^1 (\widetilde{N}_2)^2 d\xi \right) \mathbf{e}_{n(i,2e)} \mathbf{e}_{n(i,2e)}^{\intercal} \\
& + \left(\int_0^1 (\widetilde{N}_3)^2 d\xi \right) \mathbf{e}_{n(i,2e+1)} \mathbf{e}_{n(i,2e+1)}^{\intercal} \\
& + \left(\int_0^1 \widetilde{N}_1 \widetilde{N}_2 d\xi \right) (\mathbf{e}_{n(i,2e-1)} \mathbf{e}_{n(i,2e)}^{\intercal} + \mathbf{e}_{n(i,2e)} \mathbf{e}_{n(i,2e-1)}^{\intercal}) \\
& + \left(\int_0^1 \widetilde{N}_1 \widetilde{N}_3 d\xi \right) (\mathbf{e}_{n(i,2e-1)} \mathbf{e}_{n(i,2e+1)}^{\intercal} + \mathbf{e}_{n(i,2e+1)} \mathbf{e}_{n(i,2e-1)}^{\intercal}) \\
& + \left(\int_0^1 \widetilde{N}_2 \widetilde{N}_3 d\xi \right) (\mathbf{e}_{n(i,2e)} \mathbf{e}_{n(i,2e+1)}^{\intercal} + \mathbf{e}_{n(i,2e+1)} \mathbf{e}_{n(i,2e)}^{\intercal}) \Bigg].
\end{aligned}
$$

Hence, we will construct the mass matrix the following way.

Algorithm 1: Building the matrices $\mathcal{M}$.

```
for i = 1, 2 do
    for e = 1, ..., N_e do
        idx = [n(i, 2e − 1), n(i, 2e), n(i, 2e + 1)]
        M(idx, idx) = M(idx, idx) + h_e M^e
```

Here, the element mass matrix $\mathcal{M}^e \in \mathbb{R}^{3\times 3}$ is given by

$$
\mathcal{M}^e = \int_0^1 (\widetilde{\mathbf{N}})^{\intercal} \widetilde{\mathbf{N}} d\xi = \frac{1}{30} \begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix}
$$

Similarly, the stiffness matrix takes the form

$$
\mathcal{K} = \sum_{i=1}^{2} (-1)^i \lambda_i \sum_{e=1}^{\mathtt{N_e}} \int_{\omega^e} \frac{\mathrm{d}\mathbf{N}_i}{\mathrm{d}x} dx + \lambda_1 \mathbf{e}_{n(1,\mathtt{N_x})} \mathbf{e}_{n(1,\mathtt{N_x})}^{\intercal} - k\lambda_2 \mathbf{e}_{n(2,\mathtt{N_x})} \mathbf{e}_{n(1,\mathtt{N_x})}^{\intercal} + \lambda_2 \mathbf{e}_{n(2,1)} \mathbf{e}_{n(2,1)}^{\intercal}
$$

which also writes as

$$\begin{aligned}
\mathcal{K} = \sum_{i=1}^{2}(-1)^i\lambda_i \sum_{e=1}^{\mathtt{N_e}} \Bigg[&\left(\int_{\omega^e} \frac{\mathrm{d}N_{2e-1}}{\mathrm{d}x} N_{2e-1} dx\right) \mathbf{e}_{n(i,2e-1)}\mathbf{e}^{\mathsf{T}}_{n(i,2e-1)} \\
&+ \left(\int_{\omega^e} \frac{\mathrm{d}N_{2e}}{\mathrm{d}x} N_{2e} dx\right) \mathbf{e}_{n(i,2e)}\mathbf{e}^{\mathsf{T}}_{n(i,2e)} + \left(\int_{\omega^e} \frac{\mathrm{d}N_{2e+1}}{\mathrm{d}x} N_{2e+1} dx\right) \mathbf{e}_{n(i,2e+1)}\mathbf{e}^{\mathsf{T}}_{n(i,2e+1)} \\
&+ \left(\int_{\omega^e} \frac{\mathrm{d}N_{2e-1}}{\mathrm{d}x} N_{2e} dx\right) \mathbf{e}_{n(i,2e-1)}\mathbf{e}^{\mathsf{T}}_{n(i,2e)} + \left(\int_{\omega^e} \frac{\mathrm{d}N_{2e}}{\mathrm{d}x} N_{2e-1} dx\right) \mathbf{e}_{n(i,2e)}\mathbf{e}^{\mathsf{T}}_{n(i,2e-1)} \\
&+ \left(\int_{\omega^e} \frac{\mathrm{d}N_{2e-1}}{\mathrm{d}x} N_{2e+1} dx\right) \mathbf{e}_{n(i,2e-1)}\mathbf{e}^{\mathsf{T}}_{n(i,2e+1)} + \left(\int_{\omega^e} \frac{\mathrm{d}N_{2e+1}}{\mathrm{d}x} N_{2e-1} dx\right) \mathbf{e}_{n(i,2e+1)}\mathbf{e}^{\mathsf{T}}_{n(i,2e-1)} \\
&+ \left(\int_{\omega^e} \frac{\mathrm{d}N_{2e}}{\mathrm{d}x} N_{2e+1} dx\right) \mathbf{e}_{n(i,2e)}\mathbf{e}^{\mathsf{T}}_{n(i,2e+1)} + \left(\int_{\omega^e} \frac{\mathrm{d}N_{2e+1}}{\mathrm{d}x} N_{2e} dx\right) \mathbf{e}_{n(i,2e+1)}\mathbf{e}^{\mathsf{T}}_{n(i,2e)} \Bigg] \\
&+ \lambda_1 \mathbf{e}_{n(1,\mathtt{N_x})}\mathbf{e}^{\mathsf{T}}_{n(1,\mathtt{N_x})} - k\lambda_2 \mathbf{e}_{n(2,\mathtt{N_x})}\mathbf{e}^{\mathsf{T}}_{n(1,\mathtt{N_x})} + \lambda_2 \mathbf{e}_{n(2,1)}\mathbf{e}^{\mathsf{T}}_{n(2,1)}.
\end{aligned}$$

The definition of the shape functions together with a change of variables yields that

$$\begin{aligned}
\mathcal{K} = \sum_{i=1}^{2}(-1)^i\lambda_i \sum_{e=1}^{\mathtt{N_e}} \Bigg[&\left(\int_0^1 \frac{\mathrm{d}\widetilde{N}_1}{\mathrm{d}\xi} \widetilde{N}_1 d\xi\right) \mathbf{e}_{n(i,2e-1)}\mathbf{e}^{\mathsf{T}}_{n(i,2e-1)} \\
&+ \left(\int_0^1 \frac{\mathrm{d}\widetilde{N}_2}{\mathrm{d}\xi} \widetilde{N}_2 d\xi\right) \mathbf{e}_{n(i,2e)}\mathbf{e}^{\mathsf{T}}_{n(i,2e)} + \left(\int_0^1 \frac{\mathrm{d}\widetilde{N}_3}{\mathrm{d}\xi} \widetilde{N}_3 d\xi\right) \mathbf{e}_{n(i,2e+1)}\mathbf{e}^{\mathsf{T}}_{n(i,2e+1)} \\
&+ \left(\int_0^1 \frac{\mathrm{d}\widetilde{N}_1}{\mathrm{d}\xi} \widetilde{N}_2 d\xi\right) \mathbf{e}_{n(i,2e-1)}\mathbf{e}^{\mathsf{T}}_{n(i,2e)} + \left(\int_0^1 \frac{\mathrm{d}\widetilde{N}_2}{\mathrm{d}\xi} \widetilde{N}_1 d\xi\right) \mathbf{e}_{n(i,2e)}\mathbf{e}^{\mathsf{T}}_{n(i,2e-1)} \\
&+ \left(\int_0^1 \frac{\mathrm{d}\widetilde{N}_1}{\mathrm{d}\xi} \widetilde{N}_3 d\xi\right) \mathbf{e}_{n(i,2e-1)}\mathbf{e}^{\mathsf{T}}_{n(i,2e+1)} + \left(\int_0^1 \frac{\mathrm{d}\widetilde{N}_3}{\mathrm{d}\xi} \widetilde{N}_1 d\xi\right) \mathbf{e}_{n(i,2e+1)}\mathbf{e}^{\mathsf{T}}_{n(i,2e-1)} \\
&+ \left(\int_0^1 \frac{\mathrm{d}\widetilde{N}_2}{\mathrm{d}\xi} \widetilde{N}_3 d\xi\right) \mathbf{e}_{n(i,2e)}\mathbf{e}^{\mathsf{T}}_{n(i,2e+1)} + \left(\int_0^1 \frac{\mathrm{d}\widetilde{N}_3}{\mathrm{d}\xi} \widetilde{N}_2 d\xi\right) \mathbf{e}_{n(i,2e+1)}\mathbf{e}^{\mathsf{T}}_{n(i,2e)} \Bigg] \\
&+ \lambda_1 \mathbf{e}_{n(1,\mathtt{N_x})}\mathbf{e}^{\mathsf{T}}_{n(1,\mathtt{N_x})} - k\lambda_2 \mathbf{e}_{n(2,\mathtt{N_x})}\mathbf{e}^{\mathsf{T}}_{n(1,\mathtt{N_x})} + \lambda_2 \mathbf{e}_{n(2,1)}\mathbf{e}^{\mathsf{T}}_{n(2,1)}.
\end{aligned}$$

So we will construct the stiffness matrix the following way.

Algorithm 2: Building the matrices $\mathcal{K}$.

for $i = 1, 2$ **do**
- **for** $e = 1, \dots, \mathtt{N_e}$ **do**
  - $\mathtt{idx} = [n(i, 2e-1), n(i, 2e), n(i, 2e+1)]$
  - $\mathcal{K}(\mathtt{idx}, \mathtt{idx}) = \mathcal{K}(\mathtt{idx}, \mathtt{idx}) + (-1)^i\lambda_i\mathcal{K}^e$

$\mathcal{K}(n(1, \mathtt{N_x}), n(1, \mathtt{N_x})) = \mathcal{K}(n(1, \mathtt{N_x}), n(1, \mathtt{N_x})) + \lambda_1$

$\mathcal{K}(n(2, \mathtt{N_x}), n(1, \mathtt{N_x})) = \mathcal{K}(n(2, \mathtt{N_x}), n(1, \mathtt{N_x})) - k\lambda_2$

$\mathcal{K}(n(2, 1), n(2, 1)) = \mathcal{K}(n(2, 1), n(2, 1)) + \lambda_2$

Here, the element stiffness matrix $\mathcal{K}^e \in \mathbb{R}^{3\times 3}$ is given by

$$\mathcal{K}^e = \int_0^1 (\frac{\mathrm{d}\widetilde{\mathbf{N}}}{\mathrm{d}\xi})^\intercal \widetilde{\mathbf{N}} d\xi = \frac{1}{6}\begin{bmatrix} -3 & -4 & 1 \\ 4 & 0 & -4 \\ -1 & 4 & 3 \end{bmatrix}.$$

Up to now, the boundary condition $y^1(0,\cdot) \equiv 0$ has not been taken into account. One will then have to remove the first row and column of the obtained ODE (i.e. extract sub-matrices from $\mathcal{M}$ and $\mathcal{K}$. The actual number of unknowns will be $\mathtt{N_f} := 2\mathtt{N_x} - 1$ and the $y(x_k,t)$ will corresponf to the $f(i,k)$-th component of the new (smaller) unknown state $\mathbf{y}$, where $f(i,k) := 2(k-1) + i - 1$.

4. Time discretization

We have obtained the linear ODE

$$\mathcal{M}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{y}(t) + \mathcal{K}\mathbf{y}(t) = 0,$$

and now want to discretize it in time. For the time integration, we use an implicit midpoint rule. Let the time interval be divided into $\mathtt{N_t}$ points $\{t_k\}_{k=1}^{\mathtt{N_t}}$ with $t_1 = 0$ and $t_{\mathtt{N_t}} = T$, and let $\mathtt{h_t} = \frac{T}{\mathtt{N_t}-1}$ be the time step. For an ODE of the form,

$$\frac{\mathrm{d}\mathbf{y}(t)}{\mathrm{d}t} = f(\mathbf{y}(t)),$$

the implicit midpoint rule yields the following time discretization

$$\mathbf{y}^{k+1} = \mathbf{y}^k + \mathtt{h_t} f\left(\frac{\mathbf{y}^k + \mathbf{y}^{k+1}}{2}\right)$$

for $k \in \{1, \dots, \mathtt{N_t}\}$. Here, $f(\zeta) = -\mathcal{M}^{-1}\mathcal{K}\zeta$. Consequently, the scheme reads

$$\left(\mathcal{M} + \frac{\mathtt{h_t}}{2}\mathcal{K}\right)\mathbf{y}^{k+1} = \left(\mathcal{M} - \frac{\mathtt{h_t}}{2}\mathcal{K}\right)\mathbf{y}^k.$$