FEM FOR COUPLED TRANSPORT EQUATIONS

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Abstract. This is the explanation of the discretization of two transport equations coupled at the boundary (the corresponding code is in `transport_diag_wf1.m`) and of a 2×2 hyperbolic system (the corresponding code is in `transport_nodiag_wf1.m` and `transport_nodiag_wf2.m`). We use the finite element method (quadratic elements).

Our choice of discretization is done in view of extending this method to a more complicated model: the Intrinsic Geometrically Exact Beam model, which is also a first-order system of partial differential equations (treated in the repositories `GEB-diversSimu` and `GEB-Feedback`).

Contents

Notation. Let $m, n \in \{1, 2, \dots\}$. The inner product in $\mathbb{R}^n$ is denoted $\langle \cdot\, , \cdot \rangle$. Here, the identity and null matrices are denoted by $\mathbf{I}_n \in \mathbb{R}^{n \times n}$ and $\mathbf{0}_{n,m} \in \mathbb{R}^{n \times m}$, and we use the abbreviation $\mathbf{0}_n = \mathbf{0}_{n,n}$. If there is no confusion, we omit the subscript and write $\mathbf{I}$ and $\mathbf{0}$ instead.

1. Spatial discretization

We consider a one-dimensional spatial domain $(0, \ell)$ with $\ell > 0$. We place $\mathtt{N_x}$ points $\{x_k\}_{k=1}^{\mathtt{N_x}}$ on the interval $[0, \ell]$, such that $x_1 = 0$ and $x_{\mathtt{N_x}} = \ell$. The number $\mathtt{N_x}$ is even, and each interval $[x_{2e-1}, x_{2e+1}]$ for $e \in \{1, 2, \dots, \mathtt{N_e}\}$ constitute an element, which contains

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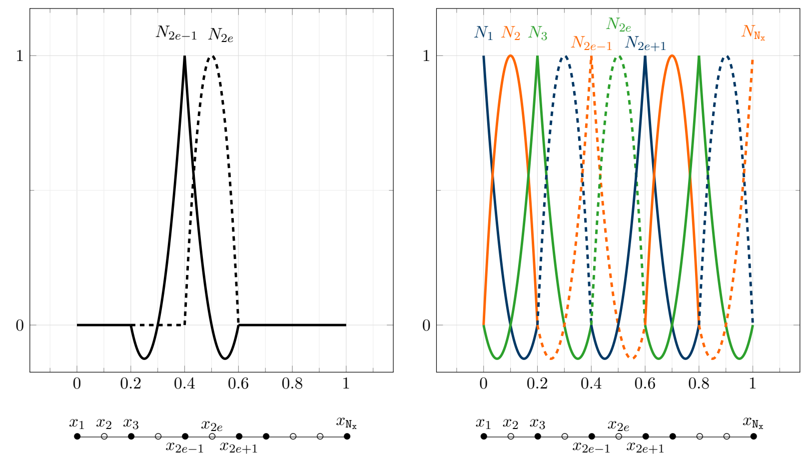

FIGURE 1. Left: two kinds of shape functions. Right: shape functions over the whole interval.

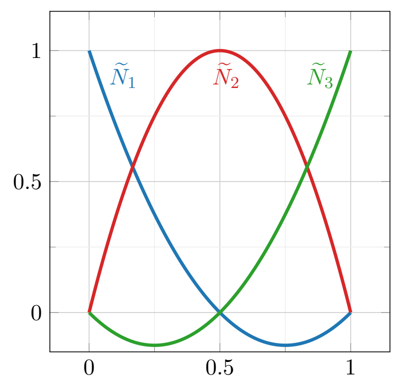

FIGURE 2. Reference element.

the points $x_{2e-1}, x_{2e}, x_{2e+1}$. Thus, the length of the element e is $\mathtt{h_e} = x_{2e+1} - x_{2e-1}$, and $\mathtt{N_x} = 2\mathtt{N_e} + 1$. Let

$$n(i,k) := 2(k-1) + i, \quad \mathtt{N_{tot}} := 2\mathtt{N_x}.$$

Let $\mathbf{N}\colon [0,\ell] \to \mathbb{R}^{2\times \mathtt{N_{tot}}}$ be defined by

$$\begin{bmatrix}\mathbf{N}_1\\ \mathbf{N}_2\end{bmatrix} = \mathbf{N} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & \dots & 0 & N_{\mathtt{N_x}} & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & \dots & N_{\mathtt{N_x}-1} & 0 & N_{\mathtt{N_x}} \end{bmatrix}.$$

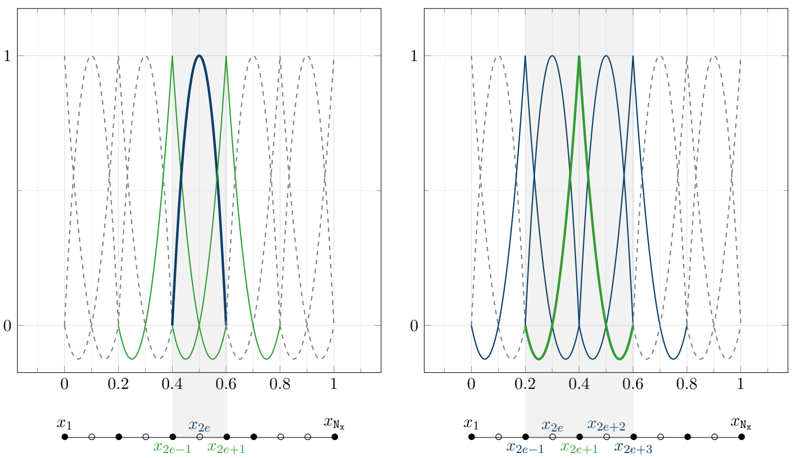

FIGURE 3. Left: shape function whose support intersect that of N_{2e}. Right: shape functions whose support intersect that of N_{2e+1}.

At the moment, we do not take into account the homogeneous Dirichlet boundary condition. We use the approximation

$$\begin{aligned} y(x,t) &\approx \sum_{i=1}^{2} \sum_{k=1}^{\mathtt{N_x}} N_k(x) \bar{\mathbf{e}}_i \mathbf{y}_{n(i,k)}(t) \\ &= \mathbf{N}(x)\mathbf{y}(t), \end{aligned} \tag{1}$$

and similarly $\psi(x) \approx \mathbf{N}(x)\boldsymbol{\psi}$. For any element $\omega^e := [x_{2e-1}, x_{2e+1}]$ for $e \in \{1, \dots, \mathtt{N_e}\}$, only three shape functions are nonzero and are given by

$$[N_{2e-1}, N_{2e}, N_{2e+1}] = \widetilde{\mathbf{N}} \left(\frac{x - x_{2e-1}}{x_{2e+1} - x_{2e-1}} \right),$$

where $\widetilde{\mathbf{N}}$ is defined by

$$\left[\widetilde{N}_1(\xi), \widetilde{N}_2(\xi), \widetilde{N}_3(\xi)\right] = \widetilde{\mathbf{N}}(\xi) = \left[(1-\xi)(1-2\xi), 4\xi(1-\xi), \xi(2\xi-1)\right].$$

2. Two transport equations coupled at the boundary

2.1. The model. For $\lambda_1, \lambda_2 > 0$ and $k \in \mathbb{R}$, consider the system

$$\begin{cases} \partial_t y^1 + \lambda_1 \partial_x y^1 = 0 & \text{in } (0,\ell) \times (0,T) \\ \partial_t y^2 - \lambda_2 \partial_x y^2 = 0 & \text{in } (0,\ell) \times (0,T) \\ y^1(0,t) = 0 & t \in (0,T) \\ y^2(\ell,t) = k y^1(\ell,t) & t \in (0,T) \\ (y^1, y^2)(x,0) = (y^{10}, y^{20})(x) & x \in (0,\ell). \end{cases}$$

2.2. **The weak formulation.** Left-multiply by x-dependent function $\psi^\intercal = (\psi^1, \psi^2)^\intercal$ and integrate over $(0, \ell)$, to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^\ell \left\langle \psi \,, \begin{bmatrix} y^1(t) \\ y^2(t) \end{bmatrix} \right\rangle dx\right) + \int_0^\ell \left\langle \begin{bmatrix} \psi^1 \\ \psi^2 \end{bmatrix} , \begin{bmatrix} \lambda_1 & 0 \\ 0 & -\lambda_2 \end{bmatrix} \partial_x \begin{bmatrix} y^1(t) \\ y^2(t) \end{bmatrix} \right\rangle dx = 0.$$

We use integration by parts on the latter term, to obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^\ell \left\langle \psi \,, \begin{bmatrix} y^1(t) \\ y^2(t) \end{bmatrix} \right\rangle dx\right) + \left\langle \begin{bmatrix} \psi^1(\ell) \\ \psi^2(\ell) \end{bmatrix} , \begin{bmatrix} \lambda_1 & 0 \\ 0 & -\lambda_2 \end{bmatrix} \begin{bmatrix} y^1(\ell,t) \\ y^2(\ell,t) \end{bmatrix} \right\rangle \\ &- \left\langle \begin{bmatrix} \psi^1(0) \\ \psi^2(0) \end{bmatrix} , \begin{bmatrix} \lambda_1 & 0 \\ 0 & -\lambda_2 \end{bmatrix} \begin{bmatrix} y^1(0,t) \\ y^2(0,t) \end{bmatrix} \right\rangle - \int_0^\ell \left\langle \partial_x \begin{bmatrix} \psi^1 \\ \psi^2 \end{bmatrix} , \begin{bmatrix} \lambda_1 & 0 \\ 0 & -\lambda_2 \end{bmatrix} \begin{bmatrix} y^1(t) \\ y^2(t) \end{bmatrix} \right\rangle dx = 0. \end{aligned}$$

Using the boundary conditions satisfied by y, we obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^\ell \left\langle \psi \,, \begin{bmatrix} y^1(t) \\ y^2(t) \end{bmatrix} \right\rangle dx\right) + \left\langle \begin{bmatrix} \psi^1(\ell) \\ \psi^2(\ell) \end{bmatrix} , \begin{bmatrix} \lambda_1 & 0 \\ 0 & -k\lambda_2 \end{bmatrix} \begin{bmatrix} y^1(\ell,t) \\ y^1(\ell,t) \end{bmatrix} \right\rangle \\ &- \left\langle \psi^2(0) \,, -\lambda_2 y^2(0,t) \right\rangle - \int_0^\ell \left\langle \partial_x \begin{bmatrix} \psi^2 \\ \psi^2 \end{bmatrix} , \begin{bmatrix} \lambda_1 & 0 \\ 0 & -\lambda_2 \end{bmatrix} \begin{bmatrix} y^1(t) \\ y^2(t) \end{bmatrix} \right\rangle dx = 0. \end{aligned}$$

which also writes as

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^\ell \left\langle \psi \,, \begin{bmatrix} y^1(t) \\ y^2(t) \end{bmatrix} \right\rangle dx\right) + \left\langle \psi^1(\ell)\lambda_1 - \psi^2(\ell)k\lambda_2, y^1(\ell,t) \right\rangle \\ &- \left\langle \psi^2(0) \,, -\lambda_2 y^2(0,t) \right\rangle - \int_0^\ell \left\langle \partial_x \begin{bmatrix} \psi^1 \\ \psi^2 \end{bmatrix} , \begin{bmatrix} \lambda_1 & 0 \\ 0 & -\lambda_2 \end{bmatrix} \begin{bmatrix} y^1(t) \\ y^2(t) \end{bmatrix} \right\rangle dx = 0. \end{aligned}$$

Consider the functional space $V = \left\{ \psi \in H^1(0,\ell;\mathbb{R}^2) \colon \psi^1(0) = 0 \right\}$ and let the matrices Π^1 and Π^2 be defined by

$$\Pi^1 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \Pi^2 = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

We choose the following weak formulation

$$\begin{aligned} &\text{Find } y = (y^1, y^2)^\intercal \in C^0(0,T;V) \text{ such that: for all } \psi \in V, \\ &\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^\ell \langle \psi \,, y(t) \rangle \, dx\right) + \lambda_1 \left\langle \Pi^1 \psi(\ell), \Pi^1 y(\ell,t) \right\rangle - k\lambda_2 \left\langle \Pi^2 \psi(\ell), \Pi^1 y(\ell,t) \right\rangle \\ &+ \lambda_2 \left\langle \Pi^2 \psi(0), \Pi^2 y(0,t) \right\rangle - \int_0^\ell \left\langle \frac{\mathrm{d}}{\mathrm{d}x}\psi \,, \begin{bmatrix} \lambda_1 & 0 \\ 0 & -\lambda_2 \end{bmatrix} y(t) \right\rangle dx = 0. \end{aligned} \tag{2}$$

2.3. **The semi-discretization.** We discretize with respect to the spatial variable by using the approximation

$$\mathbf{V} = \left\{ \psi \in C^0([0,\ell];\mathbb{R}^2) \colon \psi\big|_{[x_\alpha, x_{\alpha+1}]} \in (\mathbb{P}_2)^2 \text{ for all } \alpha \in \{1, \ldots, \mathtt{N_e} - 1\}, \ \psi^1(0) = 0 \right\} \tag{3}$$

for the space V. We inject the approximation (1) into the weak form. This yields

$$\boldsymbol{\psi}^\intercal \left(\int_0^\ell \mathbf{N}^\intercal \mathbf{N} dx\right) \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{y}(t) + \lambda_1 \boldsymbol{\psi}^\intercal \mathbf{N}(\ell)^\intercal (\Pi^1)^\intercal \Pi^1 \mathbf{N}(\ell)\mathbf{y}(t) - k\lambda_2 \boldsymbol{\psi}^\intercal \mathbf{N}(\ell)^\intercal (\Pi^2)^\intercal \Pi^1 \mathbf{N}(\ell)\mathbf{y}(t)$$
$$+ \lambda_2 \boldsymbol{\psi}^\intercal \mathbf{N}(0)^\intercal (\Pi^2)^\intercal \Pi^2 \mathbf{N}(0)\mathbf{y}(t) - \boldsymbol{\psi}^\intercal \left(\int_0^\ell \frac{\mathrm{d}\mathbf{N}^\intercal}{\mathrm{d}x} \begin{bmatrix} \lambda_1 & 0 \\ 0 & -\lambda_2 \end{bmatrix} \mathbf{N} dx\right) \mathbf{y}(t) = 0,$$

which also writes as

$$\mathcal{M}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{y}(t) + \mathcal{K}\mathbf{y}(t) = 0 \tag{4}$$

for $\mathcal{M}, \mathcal{K} \in \mathbb{R}^{\mathtt{N_{tot}} \times \mathtt{N_{tot}}}$ defined by

$$\mathcal{M} = \int_0^\ell \mathbf{N}^\intercal \mathbf{N} dx \tag{5}$$

and

$$\begin{aligned} \mathcal{K} = \int_0^\ell \frac{\mathrm{d}\mathbf{N}^\intercal}{\mathrm{d}x} \begin{bmatrix} -\lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \mathbf{N} dx + \lambda_1 \mathbf{N}(\ell)^\intercal (\Pi^1)^\intercal \Pi^1 \mathbf{N}(\ell) \\ - k\lambda_2 \mathbf{N}(\ell)^\intercal (\Pi^2)^\intercal \Pi^1 \mathbf{N}(\ell) + \lambda_2 \mathbf{N}(0)^\intercal (\Pi^2)^\intercal \Pi^2 \mathbf{N}(0). \end{aligned}$$

Note that

$$\mathbf{N}^\intercal \mathbf{N} = \sum_{i=1}^{2} \mathbf{N}_i^\intercal \mathbf{N}_i, \qquad \frac{\mathrm{d}\mathbf{N}^\intercal}{\mathrm{d}x} \begin{bmatrix} \lambda_1 & 0 \\ 0 & -\lambda_2 \end{bmatrix} \mathbf{N} = \sum_{i=1}^{2} (-1)^i \lambda_i \frac{\mathrm{d}\mathbf{N}_i}{\mathrm{d}x}^\intercal \mathbf{N}_i,$$

and

$$\begin{aligned} \mathbf{N}(x_k)^\intercal \bar{\mathbf{e}}_i \bar{\mathbf{e}}_j^\intercal \mathbf{N}(x_k) &= \mathbf{N}_i(x_k)^\intercal \mathbf{N}_j(x_k) \\ &= \mathbf{e}_{n(i,k)}^\intercal \mathbf{e}_{n(j,k)}. \end{aligned}$$

We compute that

$$\mathcal{M} = \sum_{i=1}^{2} \sum_{e=1}^{\mathtt{N_e}} \int_{\omega^e} \mathbf{N}_i^\intercal \mathbf{N}_i dx$$

which also writes as

$$\mathcal{M} = \sum_{i=1}^{2}\sum_{e=1}^{\mathtt{N_e}} \left(\int_{\omega^e} (N_{2e-1})^2 dx\right) \mathbf{e}_{n(i,2e-1)}\mathbf{e}_{n(i,2e-1)}^\intercal + \left(\int_{\omega^e} (N_{2e})^2 dx\right) \mathbf{e}_{n(i,2e)}\mathbf{e}_{n(i,2e)}^\intercal$$
$$+ \left(\int_{\omega^e} (N_{2e+1})^2 dx\right) \mathbf{e}_{n(i,2e+1)}\mathbf{e}_{n(i,2e+1)}^\intercal$$
$$+ \left(\int_{\omega^e} N_{2e-1}N_{2e} dx\right) (\mathbf{e}_{n(i,2e-1)}\mathbf{e}_{n(i,2e)}^\intercal + \mathbf{e}_{n(i,2e)}\mathbf{e}_{n(i,2e-1)}^\intercal)$$
$$+ \left(\int_{\omega^e} N_{2e-1}N_{2e+1} dx\right) (\mathbf{e}_{n(i,2e-1)}\mathbf{e}_{n(i,2e+1)}^\intercal + \mathbf{e}_{n(i,2e+1)}\mathbf{e}_{n(i,2e-1)}^\intercal)$$
$$+ \left(\int_{\omega^e} N_{2e}N_{2e+1} dx\right) (\mathbf{e}_{n(i,2e)}\mathbf{e}_{n(i,2e+1)}^\intercal + \mathbf{e}_{n(i,2e+1)}\mathbf{e}_{n(i,2e)}^\intercal).$$

and a change of variable yields

$$\mathcal{M} = \sum_{i=1}^{2}\sum_{e=1}^{\mathtt{N_e}} \mathtt{h_e}\Bigg[\left(\int_0^1 (\widetilde{N}_1)^2 dx\right) \mathbf{e}_{n(i,2e-1)}\mathbf{e}_{n(i,2e-1)}^\intercal + \left(\int_0^1 (\widetilde{N}_2)^2 d\xi\right) \mathbf{e}_{n(i,2e)}\mathbf{e}_{n(i,2e)}^\intercal$$
$$+ \left(\int_0^1 (\widetilde{N}_3)^2 d\xi\right) \mathbf{e}_{n(i,2e+1)}\mathbf{e}_{n(i,2e+1)}^\intercal$$
$$+ \left(\int_0^1 \widetilde{N}_1\widetilde{N}_2 d\xi\right) (\mathbf{e}_{n(i,2e-1)}\mathbf{e}_{n(i,2e)}^\intercal + \mathbf{e}_{n(i,2e)}\mathbf{e}_{n(i,2e-1)}^\intercal)$$
$$+ \left(\int_0^1 \widetilde{N}_1\widetilde{N}_3 d\xi\right) (\mathbf{e}_{n(i,2e-1)}\mathbf{e}_{n(i,2e+1)}^\intercal + \mathbf{e}_{n(i,2e+1)}\mathbf{e}_{n(i,2e-1)}^\intercal)$$
$$+ \left(\int_0^1 \widetilde{N}_2\widetilde{N}_3 d\xi\right) (\mathbf{e}_{n(i,2e)}\mathbf{e}_{n(i,2e+1)}^\intercal + \mathbf{e}_{n(i,2e+1)}\mathbf{e}_{n(i,2e)}^\intercal)\Bigg].$$

Hence, we will construct the mass matrix the following way.

Algorithm 1: Building the matrix $\mathcal{M}$.

1 Initialize $\mathcal{M}$ as a zero matrix
2 **for** $i = 1, 2$ **do**
3   **for** $e = 1, \ldots, \mathtt{N_e}$ **do**
4         $\mathtt{idx} = [n(i, 2e-1), n(i, 2e), n(i, 2e+1)]$
5         $\mathcal{M}(\mathtt{idx}, \mathtt{idx}) = \mathcal{M}(\mathtt{idx}, \mathtt{idx}) + \mathtt{h_e}\mathcal{M}^e$

Here, the element mass matrix $\mathcal{M}^e \in \mathbb{R}^{3\times3}$ is given by

$$\mathcal{M}^e = \int_0^1 (\widetilde{\mathbf{N}})^\intercal \widetilde{\mathbf{N}} d\xi = \frac{1}{30}\begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix}$$

Similarly, the stiffness matrix takes the form

$$\mathcal{K} = \sum_{i=1}^{2}(-1)^i\lambda_i\sum_{e=1}^{\mathtt{N_e}}\int_{\omega^e}\frac{\mathrm{d}\mathbf{N}_i}{\mathrm{d}x}dx + \lambda_1\mathbf{e}_{n(1,\mathtt{N_x})}\mathbf{e}^{\mathsf{T}}_{n(1,\mathtt{N_x})} - k\lambda_2\mathbf{e}_{n(2,\mathtt{N_x})}\mathbf{e}^{\mathsf{T}}_{n(1,\mathtt{N_x})} + \lambda_2\mathbf{e}_{n(2,1)}\mathbf{e}^{\mathsf{T}}_{n(2,1)}$$

which also writes as

$$\begin{aligned}
\mathcal{K} = \sum_{i=1}^{2}(-1)^i\lambda_i\sum_{e=1}^{\mathtt{N_e}}\Bigg[&\left(\int_{\omega^e}\frac{\mathrm{d}N_{2e-1}}{\mathrm{d}x}N_{2e-1}dx\right)\mathbf{e}_{n(i,2e-1)}\mathbf{e}^{\mathsf{T}}_{n(i,2e-1)}\\
&+\left(\int_{\omega^e}\frac{\mathrm{d}N_{2e}}{\mathrm{d}x}N_{2e}dx\right)\mathbf{e}_{n(i,2e)}\mathbf{e}^{\mathsf{T}}_{n(i,2e)} + \left(\int_{\omega^e}\frac{\mathrm{d}N_{2e+1}}{\mathrm{d}x}N_{2e+1}dx\right)\mathbf{e}_{n(i,2e+1)}\mathbf{e}^{\mathsf{T}}_{n(i,2e+1)}\\
&+\left(\int_{\omega^e}\frac{\mathrm{d}N_{2e-1}}{\mathrm{d}x}N_{2e}dx\right)\mathbf{e}_{n(i,2e-1)}\mathbf{e}^{\mathsf{T}}_{n(i,2e)} + \left(\int_{\omega^e}\frac{\mathrm{d}N_{2e}}{\mathrm{d}x}N_{2e-1}dx\right)\mathbf{e}_{n(i,2e)}\mathbf{e}^{\mathsf{T}}_{n(i,2e-1)}\\
&+\left(\int_{\omega^e}\frac{\mathrm{d}N_{2e-1}}{\mathrm{d}x}N_{2e+1}dx\right)\mathbf{e}_{n(i,2e-1)}\mathbf{e}^{\mathsf{T}}_{n(i,2e+1)} + \left(\int_{\omega^e}\frac{\mathrm{d}N_{2e+1}}{\mathrm{d}x}N_{2e-1}dx\right)\mathbf{e}_{n(i,2e+1)}\mathbf{e}^{\mathsf{T}}_{n(i,2e-1)}\\
&+\left(\int_{\omega^e}\frac{\mathrm{d}N_{2e}}{\mathrm{d}x}N_{2e+1}dx\right)\mathbf{e}_{n(i,2e)}\mathbf{e}^{\mathsf{T}}_{n(i,2e+1)} + \left(\int_{\omega^e}\frac{\mathrm{d}N_{2e+1}}{\mathrm{d}x}N_{2e}dx\right)\mathbf{e}_{n(i,2e+1)}\mathbf{e}^{\mathsf{T}}_{n(i,2e)}\Bigg]\\
&+\lambda_1\mathbf{e}_{n(1,\mathtt{N_x})}\mathbf{e}^{\mathsf{T}}_{n(1,\mathtt{N_x})} - k\lambda_2\mathbf{e}_{n(2,\mathtt{N_x})}\mathbf{e}^{\mathsf{T}}_{n(1,\mathtt{N_x})} + \lambda_2\mathbf{e}_{n(2,1)}\mathbf{e}^{\mathsf{T}}_{n(2,1)}.
\end{aligned}$$

The definition of the shape functions together with a change of variables yields that

$$\begin{aligned}
\mathcal{K} = \sum_{i=1}^{2}(-1)^i\lambda_i\sum_{e=1}^{\mathtt{N_e}}\Bigg[&\left(\int_0^1\frac{\mathrm{d}\widetilde{N}_1}{\mathrm{d}\xi}\widetilde{N}_1d\xi\right)\mathbf{e}_{n(i,2e-1)}\mathbf{e}^{\mathsf{T}}_{n(i,2e-1)}\\
&+\left(\int_0^1\frac{\mathrm{d}\widetilde{N}_2}{\mathrm{d}\xi}\widetilde{N}_2d\xi\right)\mathbf{e}_{n(i,2e)}\mathbf{e}^{\mathsf{T}}_{n(i,2e)} + \left(\int_0^1\frac{\mathrm{d}\widetilde{N}_3}{\mathrm{d}\xi}\widetilde{N}_3d\xi\right)\mathbf{e}_{n(i,2e+1)}\mathbf{e}^{\mathsf{T}}_{n(i,2e+1)}\\
&+\left(\int_0^1\frac{\mathrm{d}\widetilde{N}_1}{\mathrm{d}\xi}\widetilde{N}_2d\xi\right)\mathbf{e}_{n(i,2e-1)}\mathbf{e}^{\mathsf{T}}_{n(i,2e)} + \left(\int_0^1\frac{\mathrm{d}\widetilde{N}_2}{\mathrm{d}\xi}\widetilde{N}_1d\xi\right)\mathbf{e}_{n(i,2e)}\mathbf{e}^{\mathsf{T}}_{n(i,2e-1)}\\
&+\left(\int_0^1\frac{\mathrm{d}\widetilde{N}_1}{\mathrm{d}\xi}\widetilde{N}_3d\xi\right)\mathbf{e}_{n(i,2e-1)}\mathbf{e}^{\mathsf{T}}_{n(i,2e+1)} + \left(\int_0^1\frac{\mathrm{d}\widetilde{N}_3}{\mathrm{d}\xi}\widetilde{N}_1d\xi\right)\mathbf{e}_{n(i,2e+1)}\mathbf{e}^{\mathsf{T}}_{n(i,2e-1)}\\
&+\left(\int_0^1\frac{\mathrm{d}\widetilde{N}_2}{\mathrm{d}\xi}\widetilde{N}_3d\xi\right)\mathbf{e}_{n(i,2e)}\mathbf{e}^{\mathsf{T}}_{n(i,2e+1)} + \left(\int_0^1\frac{\mathrm{d}\widetilde{N}_3}{\mathrm{d}\xi}\widetilde{N}_2d\xi\right)\mathbf{e}_{n(i,2e+1)}\mathbf{e}^{\mathsf{T}}_{n(i,2e)}\Bigg]\\
&+\lambda_1\mathbf{e}_{n(1,\mathtt{N_x})}\mathbf{e}^{\mathsf{T}}_{n(1,\mathtt{N_x})} - k\lambda_2\mathbf{e}_{n(2,\mathtt{N_x})}\mathbf{e}^{\mathsf{T}}_{n(1,\mathtt{N_x})} + \lambda_2\mathbf{e}_{n(2,1)}\mathbf{e}^{\mathsf{T}}_{n(2,1)}.
\end{aligned}$$

So we will construct the stiffness matrix the following way.

Algorithm 2: Building the matrix $\mathcal{K}$.

Initialize $\mathcal{K}$ as a zero matrix
for $i = 1, 2$ **do**
  for $e = 1, \dots, \mathtt{N_e}$ **do**
    $\mathtt{idx} = [n(i, 2e-1), n(i, 2e), n(i, 2e+1)]$
    $\mathcal{K}(\mathtt{idx}, \mathtt{idx}) = \mathcal{K}(\mathtt{idx}, \mathtt{idx}) + (-1)^i \lambda_i \mathcal{K}^e$
$\mathcal{K}(n(1, \mathtt{N_x}), n(1, \mathtt{N_x})) = \mathcal{K}(n(1, \mathtt{N_x}), n(1, \mathtt{N_x})) + \lambda_1$
$\mathcal{K}(n(2, \mathtt{N_x}), n(1, \mathtt{N_x})) = \mathcal{K}(n(2, \mathtt{N_x}), n(1, \mathtt{N_x})) - k\lambda_2$
$\mathcal{K}(n(2, 1), n(2, 1)) = \mathcal{K}(n(2, 1), n(2, 1)) + \lambda_2$

Here, the element stiffness matrix $\mathcal{K}^e \in \mathbb{R}^{3\times 3}$ is given by

$$\mathcal{K}^e = \int_0^1 \left(\frac{\mathrm{d}\widetilde{\mathbf{N}}}{\mathrm{d}\xi}\right)^{\intercal} \widetilde{\mathbf{N}} d\xi = \frac{1}{6}\begin{bmatrix} -3 & -4 & 1 \\ 4 & 0 & -4 \\ -1 & 4 & 3 \end{bmatrix}.$$

Up to now, the boundary condition $y^1(0, \cdot) \equiv 0$ has not been taken into account. One will then have to remove the first row and column of the obtained matrices (i.e. extract sub-matrices from $\mathcal{M}$ and $\mathcal{K}$), which amounts to removing the first equation of the obtained ODE. The actual number of unknowns will be $\mathtt{N_f} := 2\mathtt{N_x} - 1$ and the $y(x_k, t)$ will correspond to the $f(i,k)$-th component of the new (smaller) unknown state $\mathbf{y}$, where $f(i, k) := 2(k-1) + i - 1$. More precisely, we proceed as follows.

Algorithm 3: Applying the Dirichlet boundary conditions.

$\mathtt{dof} = 2 : \mathtt{N_{tot}}$
$\mathcal{M} = \mathcal{M}(\mathtt{dof}, \mathtt{dof})$
$\mathcal{K} = \mathcal{K}(\mathtt{dof}, \mathtt{dof})$

3. 2 × 2 linear hyperbolic system

3.1. **The model.** Consider the system

$$\begin{cases} \partial_t y + A\partial_x y = 0 & \text{in } (0, \ell) \times (0, T) \\ y^1(0, t) = 0 & t \in (0, T) \\ y^2(\ell, t) = f(t) & t \in (0, T) \\ (y^1, y^2)(x, 0) = (y^{10}, y^{20})(x) & x \in (0, \ell), \end{cases}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

3.2. **Working with the physical variables.** In Section 3.2 we work with the unknown state y (rather than diagonalizing the system as in Section 3.3).

3.2.1. *Weak formulation.* The code corresponding to this subsection is given in the file `transport_nodiag_wf1.m`. We define the space V by

$$V = \{\psi \in H^1(0,\ell;\mathbb{R}^2) \colon \Pi^1\psi(0) = 0\}.$$

Integrating by parts, one has for any $\psi \in V$

$$\int_0^\ell \langle \psi \,, A\partial_x y(t)\rangle dx = -\int_0^\ell \left\langle \frac{\mathrm{d}\psi}{\mathrm{d}x} \,, Ay(t) \right\rangle dx + [\langle \psi \,, Ay(t)\rangle]_0^\ell \,,$$

where we wrote $y(t) := y(\cdot, t)$ in order to lighten the notation. One may compute that

$$\begin{aligned}[\langle \psi \,, Ay(t)\rangle]_0^\ell &= \left[\langle \Pi^1\psi \,, \Pi^2 y(t)\rangle + \langle \Pi^2\psi \,, \Pi^1 y(t)\rangle\right]_0^\ell \\ &= \langle \Pi^1\psi(\ell \,, f(t)\rangle + \langle \Pi^2\psi(\ell) \,, \Pi^1 y(\ell, t)\rangle.\end{aligned}$$

We choose the following weak form: find $y \in C^0(0,T;V)$ such that for all $\psi \in V$ one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^\ell \langle \psi \,, y(t)\rangle \, dx\right) - \int_0^\ell \left\langle \frac{\mathrm{d}\psi}{\mathrm{d}x} \,, Ay(t) \right\rangle dx + \langle \Pi^1\psi(\ell) \,, f(t)\rangle + \langle \Pi^2\psi(\ell) \,, \Pi^1 y(\ell, t)\rangle = 0.$$

3.2.2. *The semi-discretization.* We inject the approximation (1) into the weak form, using again the approximation (3) for the space V. This yields the ODE

$$\mathcal{M}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{y}(t) + \mathcal{K}\mathbf{y}(t) + \mathcal{W}f(t) = 0 \tag{6}$$

where $\mathcal{M} \in \mathbb{R}^{\mathtt{N_{tot}} \times \mathtt{N_{tot}}}$ is defined by (5), $\mathcal{K} \in \mathbb{R}^{\mathtt{N_{tot}} \times \mathtt{N_{tot}}}$ is defined by

$$\mathcal{K} = -\int_0^\ell \frac{\mathrm{d}\mathbf{N}^\intercal}{\mathrm{d}x} A\mathbf{N} dx + \langle \Pi^2\mathbf{N}(\ell) \,, \Pi^1\mathbf{N}(\ell)\rangle,$$

and $\mathcal{W} \in \mathbb{R}^{\mathtt{N_{tot}}}$ is defined by

$$\mathcal{W} = (\Pi^1\mathbf{N}(\ell))^\intercal.$$

The matrix $\mathcal{M}$ may be built as before. On the other hand one may compute that $\langle \Pi^2\mathbf{N}(\ell) \,, \Pi^1\mathbf{N}(\ell)\rangle = e_{n(2,Nx)}e_{n(1,Nx)}^\intercal$ and $(\Pi^1\mathbf{N}(\ell))^\intercal = e_{n(1,\mathtt{N_x})}$. Hence, $\mathcal{K}$ and $\mathcal{W}$ can be built as follows.

Algorithm 4: Building the matrices $\mathcal{K}$ and $\mathcal{W}$.

Initialize $\mathcal{K}, \mathcal{W}$ as zero matrices
for $i = 1, 2$ **do**
  for $j = 1, 2$ **do**
    for $e = 1, \dots, \mathtt{N_e}$ **do**
      $\mathtt{idxRow} = [n(i, 2e-1), n(i, 2e), n(i, 2e+1)]$
      $\mathtt{idxCol} = [n(j, 2e-1), n(j, 2e), n(j, 2e+1)]$
      $\mathcal{K}(\mathtt{idxRow}, \mathtt{idxCol}) = \mathcal{K}(\mathtt{idxRow}, \mathtt{idxCol}) - A_{i,j}\mathcal{K}^e$
$\mathcal{K}(n(2, \mathtt{N_x}), n(1, \mathtt{N_x})) = \mathcal{K}(n(2, \mathtt{N_x}), n(1, \mathtt{N_x})) + 1$
$\mathcal{W}(n(1, \mathtt{N_x}), 1) = 1$

Finally, we remove the first equation of the obtained ODE since the value of the state is known to be equal to zero at $x = 0$.

Algorithm 5: Applying the Dirichlet boundary conditions.

```
1 dof = 2 : N_tot
2 M = M(dof, dof)
3 K = K(dof, dof)
4 W = W(dof, 1)
```

3.3. **Working with the diagonal variables.** Let us consider the change of variable $r = Ly$ for $L \in \mathbb{R}^{2\times 2}$ defined by

$$L = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Note that the inverse L^{-1} of this matrix is such that $L^{-1} = L = L^\intercal$. We have chosen this matrix L in such a way that $LAL^\intercal = \mathbf{D}$ for $\mathbf{D} \in \mathbb{R}^{2\times 2}$ defined by

$$\mathbf{D} = \mathrm{diag}(1, -1).$$

Then, the new state $r = (r^-, r^+)^\intercal$ is solution to the equivalent system

$$\begin{cases} \partial_t r + \mathbf{D}\partial_x r = 0 & \text{in } (0,\ell)\times(0,T) \\ r^+(0,t) = r^-(0,t) & t \in (0,T) \\ r^-(\ell,t) = -r^+(\ell,t) + \sqrt{2} f(t) & t \in (0,T) \\ r(x,0) = Ly^0(x) & x \in (0,\ell). \end{cases}$$

3.3.1. *Weak formulation.* This time we choose the following functional space V

$$V = H^1(0,\ell;\mathbb{R}^2)$$

where no boundary condition is included in the definition of V.

Using integration by parts, one has for any $\varphi = (\varphi^-, \varphi^+)^\intercal \in V$

$$\int_0^\ell \langle \varphi\,, \mathbf{D}\partial_x r\rangle dx = -\int_0^\ell \left\langle \frac{\mathrm{d}\varphi}{\mathrm{d}x}\,, \mathbf{D}r \right\rangle dx + [\langle \varphi\,, \mathbf{D}r\rangle]_0^\ell\,,$$

Using the boundary condition satisfied by r, we obtain

$$\begin{aligned} [\langle \varphi\,, \mathbf{D}r\rangle]_0^\ell &= \left\langle \varphi(\ell)\,, \mathbf{D}\begin{bmatrix} -r^+(\ell,t) \\ r^+(\ell,t) \end{bmatrix} \right\rangle - \left\langle \varphi(0)\,, \mathbf{D}\begin{bmatrix} r^-(0,t) \\ r^-(0,t) \end{bmatrix} \right\rangle \\ &\quad - \sqrt{2}\varphi^-(\ell) f(t) \\ &= \left\langle \varphi(\ell)\,, \mathbf{D}\begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} r(\ell,t) \right\rangle - \left\langle \varphi(0)\,, \mathbf{D}\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} r(0,t) \right\rangle \\ &\quad - \sqrt{2}\,\Pi^1 \varphi(\ell) f(t). \end{aligned}$$

We choose the following weak form: find $r \in C^0(0,T;V)$ such that for all $\varphi \in V$ one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^\ell \langle \varphi\,, r(t)\rangle\, dx\right) - \int_0^\ell \left\langle \frac{\mathrm{d}\varphi}{\mathrm{d}x}\,, \mathbf{D}r(t)\right\rangle dx$$
$$+ \langle \varphi(\ell)\,, K_1 r(\ell,t)\rangle + \langle \varphi(0)\,, K_2 r(0,t)\rangle - \sqrt{2}\,\Pi^1\varphi(\ell)f(t) = 0,$$

for $K_1, K_2 \in \mathbb{R}^{2\times 2}$ defined by

$$K_1 = \mathbf{D}\begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}, \quad K_2 = -\mathbf{D}\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}.$$

3.3.2. *The semi-discretization.* This time the approximation of the space V is

$$\mathbf{V} = \left\{\psi \in C^0([0,\ell];\mathbb{R}^2) \colon \psi\big|_{[x_\alpha, x_{\alpha+1}]} \in (\mathbb{P}_2)^2 \text{ for all } \alpha \in \{1,\ldots,\mathtt{N_e}-1\}\right\}.$$

We inject the approximation (1) into the weak form. This yields the ODE (6), where where $\mathcal{M} \in \mathbb{R}^{\mathtt{N_{tot}}\times\mathtt{N_{tot}}}$ is defined by (5), $\mathcal{K} \in \mathbb{R}^{\mathtt{N_{tot}}\times\mathtt{N_{tot}}}$ is defined by

$$\mathcal{K} = -\int_0^\ell \frac{\mathrm{d}\mathbf{N}^\intercal}{\mathrm{d}x}\mathbf{D}\mathbf{N} dx + \langle \mathbf{N}(\ell)\,, K_1\mathbf{N}(\ell)\rangle + \langle \mathbf{N}(0)\,, K_2\mathbf{N}(0)\rangle\,,$$

and $\mathcal{W} \in \mathbb{R}^{\mathtt{N_{tot}}}$ are defined by

$$\mathcal{W} = -\sqrt{2}(\Pi^1\mathbf{N}(\ell))^\intercal.$$

Hence, $\mathcal{K}$ and $\mathcal{W}$ may be built as follows.

Algorithm 6: Building the matrices $\mathcal{K}$ and $\mathcal{W}$.

Initialize $\mathcal{K}, \mathcal{W}$ as zero matrices
for $i = 1,2$ **do**
  for $j = 1,2$ **do**
    for $e = 1,\ldots,\mathtt{N_e}$ **do**
      $\mathtt{idxRow} = [n(i,2e-1), n(i,2e), n(i,2e+1)]$
      $\mathtt{idxCol} = [n(j,2e-1), n(j,2e), n(j,2e+1)]$
      $\mathcal{K}(\mathtt{idxRow},\mathtt{idxCol}) = \mathcal{K}(\mathtt{idxRow},\mathtt{idxCol}) - \mathbf{D}_{i,j}\mathcal{K}^e$
    $\mathcal{K}(n(i,\mathtt{N_x}), n(j,\mathtt{N_x})) = \mathcal{K}(n(i,\mathtt{N_x}), n(j,\mathtt{N_x})) + (K_1)_{i,j}$
    $\mathcal{K}(n(i,1), n(j,1)) = \mathcal{K}(n(i,1), n(j,1)) + (K_2)_{i,j}$
$\mathcal{W}(n(1,\mathtt{N_x}),1) = -\sqrt{2}$

Contrary to the previous case, there is no need to remove any equation from the ODE system since no boundary condition has been included in the definition of V.

4. Time discretization

We have obtained the linear ODE

$$\mathcal{M}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{y}(t) + \mathcal{K}\mathbf{y}(t) = 0,$$

and now want to discretize it in time. For the time integration, we use an implicit midpoint rule. Let the time interval be divided into $\mathtt{N_t}$ points $\{t_k\}_{k=1}^{\mathtt{N_t}}$ with $t_1 = 0$ and $t_{\mathtt{N_t}} = T$, and let $\mathtt{h_t} = \frac{T}{\mathtt{N_t}-1}$ be the time step. For an ODE of the form,

$$\frac{\mathrm{d}\mathbf{y}(t)}{\mathrm{d}t} = f(\mathbf{y}(t)),$$

the implicit midpoint rule yields the following time discretization

$$\mathbf{y}^{k+1} = \mathbf{y}^k + \mathtt{h_t} f\left(\frac{\mathbf{y}^k + \mathbf{y}^{k+1}}{2}\right)$$

for $k \in \{1, \dots, \mathtt{N_t}\}$. Here, $f(\zeta) = -\mathcal{M}^{-1}\mathcal{K}\zeta$. Consequently, the scheme reads

$$\left(\mathcal{M} + \frac{\mathtt{h_t}}{2}\mathcal{K}\right)\mathbf{y}^{k+1} = \left(\mathcal{M} - \frac{\mathtt{h_t}}{2}\mathcal{K}\right)\mathbf{y}^k.$$