Quantifier Translation Advice
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I offer below various bits of advice about advanced quantifier translation in Predicate Logic. They are meant to complement (and indeed cover some points also expressed in) *forall* x.

1. Make sure you completely internalize the difference between these four simple sentences:

   $\forall x \neg P(x)$ = Everything is not P; nothing is P.

   $\neg \forall x P(x)$ = Not everything is P.

   $\exists x \neg P(x)$ = Something is not P.

   $\neg \exists x P(x)$ = Nothing is P.

   The first and fourth of these are logically equivalent. The second and third are also logically equivalent. Think about why. I think of this as "If you move the negation in or out, you switch the quantifier."

2. Faced with a compound sentence, first think: Is this an (unquantified) truth-functional compound of simpler sentences? If it's a conjunction, disjunction, conditional, or biconditional, you can divide your task into more-manageable portions.

3. Most quantified compound sentences are either universally-quantified conditionals or existentially quantified conjunctions. (Within these sentences, the antecedent, consequent, or conjuncts may themselves involve quantifiers, but don't let that distract you from identifying the overall shape of the sentence.) Which one?

   (a) Does the sentence express a relationship between sets? Does it say "Things like *this* are like *that*" or "Things in this set are in that set" or "Things with these properties also have these further properties"? If so, its overall structure is probably $\forall x(P(x) \rightarrow Q(x))$

   (b) Does the sentence express that at least one thing, or some things have a property? Does it say that something exists and it's like *this*? If so, its overall structure is probably $\exists x(P(x) \wedge Q(x))$

4. The overall structure of *many* quantified English sentences is one of these four types. They are so common that Aristotle gave them names (A, E, I, and O sentences). To me these names are just one more thing to remember, for no reason. But you *should* recognize their structures. Here they are with sample translations:

   $\forall x(T(x) \rightarrow F(x))$ = All tigers are fierce.

   $\forall x(T(x) \rightarrow \neg F(x))$ = No tigers are fierce.

   $\exists x(T(x) \wedge F(x))$ = Some tigers are fierce.

   $\exists x(T(x) \wedge \neg F(x))$ = Some tigers are not fierce.

5. An alarm should go off in your head if you're ever tempted to write a sentence whose structure is existentially-quantified conditional or universally-quantified conjunction. In the first case, you're saying something strange, like $\exists x(F(x) \rightarrow T(x))$: "Something exists such that if it's fierce, it's a tiger." This sentence is *true* if there are non-fierce chipmunks! That's probably not what you mean. Similarly, $\forall x(T(x) \wedge F(x))$ says that everything is a fierce tiger. It does not say that all tigers are fierce. There are contexts (say, in philosophical arguments) where you might want to say that everything in the domain is a member of two sets, like "Everything is physical and extended." But the alarm should provoke you to make sure that's really what you want to say. Existentially-quantified conditionals should just be avoided. Remember: alarm!

6. Let's put together the ideas in #1 and #4. Using them together, we can see why two common sentence-types are equivalent.

   $\neg\forall x(T(x) \to F(x))$ is logically equivalent to $\exists x(T(x) \wedge \neg F(x))$

   Why? Up in #1 I suggested you think about why "not all" and "something's not" are equivalent. So, $\neg\forall x(T(x) \to F(x))$ is equivalent to $\exists x\neg(T(x) \to F(x))$. Now think back about what a negated conditional is equivalent to: $\neg(X \to Y) \leftrightarrow (X \wedge \neg Y)$. (Remember that the one case in which a conditional is false is where its antecedent is true and its consequent is false.) Substitute the logically-equivalent conjunction and you've got the second sentence.

7. Students sometimes express confusion about when to use "every" and "any." Unfortunately, there is not an exact correlation between these terms in English and FOL quantifiers. You need to think about how many things a claim is about. Here are some examples, using people for a domain:

   Anyone taller than Joe is taller than Mick. $\forall x(T(x,j) \to T(x,m))$
   Everyone taller than Joe is taller than Mick. $\forall x(T(x,j) \to T(x,m))$
   If anyone is taller than Joe, he or she is taller than Mick. $\forall x(T(x,j) \to T(x,m))$
   same FOL sentences

   If anyone's taller than Joe, Paul is. $\exists xT(x,j) \to T(p,j)$
   If everyone's taller than Joe, Paul is. $\forall xT(x,j) \to T(p,j)$
   not the same FOL sentences

   Joe isn't taller than everyone. $\neg\forall xT(j,x)$
   Joe isn't taller than anyone. $\neg\exists xT(j,x)$
   not the same FOL sentences

8. Similarly, *most* occurrences of "some" should be translated with an existential quantifier, like "Some tigers are fierce" and "Someone's taller than Joe." However here is an example of one that's universal:

   If someone's taller than Joe, he or she is taller than Mick. $\forall x(T(x,j) \to T(x,m))$

   Why? Despite the appearance of "someone," notice that this sentence means the same thing as several sentences above about "anyone." How? Notice the reference back to that "someone" in the consequent. This sentence is about the relationship between sets (the set 'taller than Joe' and the set 'taller than Mick'). Consider how different the following sentence is:

   If someone's taller than Joe, someone is taller than Mick. $\exists xT(x,j) \to \exists xT(x,m)$

9. Relatedly, for any sentence P that does not include the variable x (for example, the SL sentence P, or $T(m,j)$, or $\forall yM(y)$), this equivalence holds:

   $\exists xF(x) \to P$ is logically equivalent to $\forall x(F(x) \to P)$

   If that's confusing, read the sentences out loud in logic-ese. They both say "If anything's F, P," as in "If anything's fierce, I'm getting out of here pronto."

10. Another set of sentences whose meanings you should internalize are the four basic sentences with two overlapping quantifiers:

    $\forall x\forall yL(x,y) =$ Everyone loves everyone.

    $\forall x\exists yL(x,y) =$ Everyone loves someone.

    $\exists x\forall yL(x,y) =$ Someone loves everyone.

    $\exists x\exists yL(x,y) =$ Someone loves someone.

11. Then, consider these variants in which the order of variables is reversed:

    $\forall x\exists yL(y,x) =$ Everyone is loved by someone.

    $\exists x\forall yL(y,x) =$ There's someone everyone loves.

    "Everyone is loved by someone" does *not* mean the same thing as "Someone loves everyone." Similarly, "There's someone everyone loves" does *not* mean the same thing as "Everyone loves someone."

12. Sometimes you need to add a quantifier within a sentence. Here is one very typical example of a compound sentence which it would be profitable for you to think about:

    Some non-fierce tigers are hungrier than every fierce tiger.

    Clearly this sentence is an existential conjunction. It says "There are some things like this which are also like that." So it starts out $\exists x(\neg F(x) \wedge T(x))$ (There are non-fierce tigers). It's going to end up saying that those non-fierce tigers (now x) are hungrier than something else y: $H(x, y)$, where $H(x, y)$ means "x is hungrier than y." That will be the last term. But in the middle we have to say something about who it is that they're hungrier than: every fierce tiger. So we want to define y as "all fierce tigers." For that we can write "everything that's fierce and a tiger" and we should start by thinking about $\forall y(F(y) \wedge T(y))$. Wait, you might think: doesn't that say that everything is a fierce tiger? Yes, that sentence does. But we're going to use this conjunction of properties as the antecedent of a conditional, to write that if y is both F and T then it's something the non-fierce tigers are hungrier than. So we're going to extend the scope of the universal quantifier to cover the consequent. Since the universal quantifier will now cover a conditional, there's no more problem here than there is in saying $\forall z(T(z) \rightarrow F(z))$, "All tigers are fierce." We're not saying there that everything is a tiger, just that the set of fierce tigers stands in a certain relation (the predicate H) to the other set. The result is:

    $$\exists x\{(\neg F(x) \wedge T(x)) \wedge \forall y[(F(y) \wedge T(y)) \rightarrow H(x, y)]\}$$

    Or, in logic-ese: there's something that's not fierce and a tiger, and if anything is fierce and a tiger, then the first thing is hungrier than the second thing. That's equivalent to the more natural sentence above.

13. Students sometimes aren't sure whether to put quantifiers at the beginning of a compound sentence or inside it. Here's a recommendation: Give quantifiers the scopes they need to have, and nothing more. Apply them to the shortest sentences you can. For example, if a variable appears only in the consequent of a conditional, quantify it by making its scope just that consequent. Three reasons: (a) This minimizes the chance of saying something you don't mean; (b) This makes sentences much easier to read; (c) When we get to FOL natural deduction (PD), you will be able to apply the rule &E (conjunction elimination) to $\exists x T(x) \wedge \exists y T(y)$, because it's a conjunction, but not to $\exists x \exists y(T(x) \wedge T(y))$. In the latter, you have to deal with the quantifiers. It's a double-quantified sentence, not a conjunction.