CHAPTER 15 Differential Equations

15.1 Linear Homogeneous Equations
15.2 Nonhomogeneous Equations
15.3 Applications of Second-Order Equations

15.1 Linear Homogeneous Equations

We call an equation involving one or more derivatives of an unknown function a **differential equation.** In particular, an equation of the form

$$F\left(x, y, y^{(1)}, y^{(2)}, \ldots, y^{(n)}\right) = 0$$

in which $y^{(k)}$ denotes the kth derivative of y with respect to x, is called an **ordinary differential equation of order *n*.** Examples of differential equations of orders 1, 2, and 3 are

$$y' + 2 \sin x = 0$$

$$\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} - 2y = 0$$

$$\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^2 - e^x = 0$$

If, when $f(x)$ is substituted for y in the differential equation, the resulting equation is an identity for all x in some interval, then $f(x)$ is called a **solution** of the differential equation. Thus, $f(x) = 2 \cos x + 10$ is a solution to $y' + 2 \sin x = 0$ since

$$f'(x) + 2 \sin x = -2 \sin x + 2 \sin x = 0$$

for all x. We call $2 \cos x + C$ the **general solution** of the given equation, since it can be shown that every solution can be written in this form. In contrast, $2 \cos x + 10$ is called a **particular solution** of the equation.

Differential equations appeared earlier in this book, principally in three sections. In Section 3.9, we introduced the technique called *separation of variables* and used it to solve a wide variety of first-order equations. In Section 6.5, we solved the differential equation $y' = ky$ of exponential growth and decay, and in Section 6.6 we studied first-order linear differential equations and some applications.

In this chapter, we consider only ***n*th-order linear** differential equations, that is, equations of the form

$$y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = k(x)$$

where $n \geq 2$. (Note that y and all its derivatives occur to the first power.) This is called a linear equation because, if it is written in operator notation,

$$\left[D_x^n + a_1(x)D_x^{n-1} + \cdots + a_{n-1}(x)D_x + a_n(x)\right]y = k(x)$$

then the operator in brackets is a *linear* operator. Thus, if L denotes this operator and if f and g are functions and c is constant, then

$$L(f + g) = L(f) + L(g)$$
$$L(cf) = cL(f)$$

That L has these properties follows readily from the corresponding properties for the derivative operators $D, D^2, \ldots, D^n$.

Of course, not all differential equations are linear. Many important differential equations, such as

$$\frac{dy}{dx} + y^2 = 0$$

are **nonlinear.** The presence of the exponent 2 on y is enough to spoil the linearity, as you may check. The theory of nonlinear differential equations is both complicated and fascinating, but best left for more advanced courses.

Second-Order Linear Equations A second-order linear differential equation has the form

$$y'' + a_1(x)y' + a_2(x)y = k(x)$$

In this section, we make two simplifying assumptions: (1) $a_1(x)$ and $a_2(x)$ are constants, and (2) $k(x)$ is identically zero. Thus, our initial task is to solve

$$y'' + a_1 y' + a_2 y = 0$$

A differential equation for which $k(x) = 0$ is said to be **homogeneous.**

To solve a first-order equation required one integration and led to a general solution with one arbitrary constant. By analogy, we might expect that solving a second-order equation to involve two integrations and thus that the general solution would have two arbitrary constants. Our expectations are correct. In fact, a second-order homogeneous linear differential equation always has two fundamental solutions $u_1(x)$ and $u_2(x)$, which are **independent** of each other (i.e., neither function is a constant multiple of the other). By the linearity of the operator $D^2 + a_1 D + a_2$,

$$C_1u_1(x) + C_2u_2(x)$$

is also a solution. Moreover, it can be shown that every solution has this form.

The Auxiliary Equation Because $D_x(e^{rx}) = re^{rx}$, it seems likely that e^{rx} will be a solution to our differential equation for an appropriate choice of r. To test this possibility, we first write the equation in the operator form

$$(D^2 + a_1D + a_2)y = 0 \tag{1}$$

Now

$$\begin{aligned}(D^2 + a_1D + a_2)e^{rx} &= D^2(e^{rx}) + a_1D(e^{rx}) + a_2 e^{rx}\\ &= r^2e^{rx} + a_1re^{rx} + a_2e^{rx}\\ &= e^{rx}(r^2 + a_1r + a_2)\end{aligned}$$

The latter expression is zero, provided

$$r^2 + a_1r + a_2 = 0 \tag{2}$$

Equation (2) is called the **auxiliary equation** for (1) (note the similarity in form). It is an ordinary quadratic equation and can be solved by factoring or, if necessary, by the Quadratic Formula. There are three cases to consider, corresponding to whether the auxiliary equation has two distinct real roots, a single repeated root, or two complex conjugate roots.

Theorem A Distinct Real Roots

If r_1 and r_2 are distinct real roots of the auxiliary equation, then the general solution of $y'' + a_1y' + a_2y = 0$ is

$$y = C_1e^{r_1x} + C_2e^{r_2x}$$

EXAMPLE 1 Find the general solution to $y'' + 7y' + 12y = 0$.

SOLUTION The auxiliary equation

$$r^2 + 7r + 12 = (r + 3)(r + 4) = 0$$

has the two roots -3 and -4. Since e^{-3x} and e^{-4x} are independent solutions, the general solution to the differential equation is

$$y = C_1e^{-3x} + C_2e^{-4x}$$ ■

EXAMPLE 2 Find the solution of $y'' - 2y' - y = 0$ that satisfies $y(0) = 0$ and $y'(0) = \sqrt{2}$.

SOLUTION The auxiliary equation $r^2 - 2r - 1 = 0$ is best solved by the Quadratic Formula.

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2 \pm \sqrt{4 + 4}}{2} = 1 \pm \sqrt{2}$$

The general solution to the differential equation is, therefore,

$$y = C_1 e^{(1+\sqrt{2})x} + C_2 e^{(1-\sqrt{2})x}$$

The condition $y(0) = 0$ implies that $C_2 = -C_1$. Then

$$y' = C_1(1 + \sqrt{2})e^{(1+\sqrt{2})x} - C_1(1 - \sqrt{2})e^{(1-\sqrt{2})x}$$

and so

$$\sqrt{2} = y'(0) = C_1(1 + \sqrt{2}) - C_1(1 - \sqrt{2}) = 2C_1\sqrt{2}$$

We conclude that $C_1 = \frac{1}{2}$ and

$$y = \tfrac{1}{2} e^{(1+\sqrt{2})x} - \tfrac{1}{2} e^{(1-\sqrt{2})x}$$ ■

This is all fine if the auxiliary equation has distinct real roots. But what if it has the form

$$r^2 - 2r_1 r + r_1^2 = (r - r_1)^2 = 0$$

Then our method produces the single fundamental solution $e^{r_1 x}$ and we must find another solution independent of this one. Such a solution is $xe^{r_1 x}$, as we now demonstrate.

$$\begin{aligned}(D^2 - 2r_1 D + r_1^2)xe^{r_1 x} &= D^2(xe^{r_1 x}) - 2r_1 D(xe^{r_1 x}) + r_1^2 xe^{r_1 x} \\ &= (xr_1^2 e^{r_1 x} + 2r_1 e^{r_1 x}) - 2r_1(xr_1 e^{r_1 x} + e^{r_1 x}) + r_1^2 xe^{r_1 x} \\ &= 0\end{aligned}$$

Theorem B **A Single Repeated Root**

If the auxiliary equation has the single repeated root r_1, then the general solution of $y'' + a_1 y' + a_2 y = 0$ is

$$y = C_1 e^{r_1 x} + C_2 x e^{r_1 x}$$

EXAMPLE 3 Solve $y'' - 6y' + 9y = 0$.

SOLUTION The auxiliary equation has 3 as a repeated root. Thus,

$$y = C_1 e^{3x} + C_2 x e^{3x}$$ ■

Finally, we consider the case where the auxiliary equation has complex conjugate roots. The simple equation

$$(D^2 + \beta^2)y = 0$$

with auxiliary equation $r^2 + \beta^2 = 0$ and roots $\pm \beta i$ offers a hint. Its fundamental solutions are easily seen to be $\sin \beta x$ and $\cos \beta x$. You can check by direct differentiation that the general situation is as follows.

Theorem C **Complex Conjugate Roots**

If the auxiliary equation has complex conjugate roots $\alpha \pm \beta i$, then the general solution of $y'' + a_1 y' + a_2 y = 0$ is

$$y = C_1 e^{\alpha x} \cos \beta x + C_2 e^{\alpha x} \sin \beta x$$

Summary

Consider the second-order differential equation

$$y'' + a_1 y' + a_2 = 0$$

with auxiliary equation

$$r^2 + a_1 r + a_2 = 0$$

The latter equation may have two real roots r_1 and r_2, one real root r_1, or two complex roots $\alpha \pm \beta i$.

Roots	Solution to Differential Equation
$r_1 \neq r_2$	$y = C_1 e^{r_1 x} + C_2 e^{r_2 x}$
$r_1 = r_2$	$y = C_1 e^{r_1 x} + C_2 xe^{r_1 x}$
$\alpha \pm \beta i$	$y = C_1 e^{\alpha x} \cos \beta x$ $+ C_2 e^{\alpha x} \sin \beta x$

EXAMPLE 4 Solve $y'' - 4y' + 13y = 0$.

SOLUTION The roots of the auxiliary equation $r^2 - 4r + 13 = 0$ are $2 \pm 3i$. Hence, the general solution is

$$y = C_1 e^{2x} \cos 3x + C_2 e^{2x} \sin 3x$$ ■

Higher-Order Equations All of what we have done extends to higher-order linear homogeneous equations with constant coefficients. To solve

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0$$

find the roots of the auxiliary equation

$$r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n = 0$$

and make the obvious generalizations of the second-order case. For example, if the auxiliary equation is

$$(r - r_1)(r - r_2)^3 [r - (\alpha + \beta i)][r - (\alpha - \beta i)] = 0$$

then the general solution to the differential equation is

$$y = C_1 e^{r_1 x} + (C_2 + C_3 x + C_4 x^2)e^{r_2 x} + e^{\alpha x}[C_5 \cos \beta x + C_6 \sin \beta x]$$

EXAMPLE 5 Solve $\dfrac{d^4y}{dx^4} - \dfrac{d^3y}{dx^3} - 20\dfrac{d^2y}{dx^2} = 0$.

SOLUTION The auxiliary equation is

$$r^4 - r^3 - 20r^2 = r^2(r - 5)(r + 4) = 0$$

with roots -4, 5, and a double root of 0. Hence, the general solution is

$$y = C_1 + C_2 x + C_3 e^{5x} + C_4 e^{-4x}$$ ■

Concepts Review

1. The auxiliary equation corresponding to the differential equation $(D^2 + a_1 D + a_2)\, y = 0$ is ________ . This equation may have two real roots, a single repeated root, or ________ .

2. The general solution to $(D^2 - 1)y = 0$ is $y =$ ________ .

3. The general solution to $(D^2 - 2D + 1)\, y = 0$ is $y =$ ________ .

4. The general solution to $(D^2 + 1)\, y = 0$ is $y =$ ________ .

Problem Set 15.1

In Problems 1–16, solve each differential equation.

1. $y'' - 5y' + 6y = 0$

2. $y'' + 5y' - 6y = 0$

3. $y'' + 6y' - 7y = 0$; $y = 0$, $y' = 4$ at $x = 0$

4. $y'' - 3y' - 10y = 0$; $y = 1$, $y' = 10$ at $x = 0$

5. $y'' - 4y' + 4y = 0$

6. $y'' + 10y' + 25y = 0$

7. $y'' - 4y' + y = 0$

8. $y'' + 6y' - 2y = 0$

9. $y'' + 4y = 0$; $y = 2$ at $x = 0$, $y = 3$ at $x = \pi/4$

10. $y'' + 9y = 0$; $y = 3$, $y' = 3$ at $x = \pi/3$

11. $y'' + 2y' + 2y = 0$

12. $y'' + y' + y = 0$

13. $y^{(4)} + 3y''' - 4y'' = 0$

14. $y^{(4)} - y = 0$

15. $(D^4 + 3D^2 - 4)y = 0$

16. $[(D^2 + 1)(D^2 - D - 6)]\, y = 0$

17. Solve $y'' - 4y = 0$ and express your answer in terms of the hyperbolic functions cosh and sinh.

18. Show that the solution of

$$\frac{d^2y}{dx^2} - 2b\frac{dy}{dx} - c^2 y = 0$$

can be written as

$$y = e^{bx}\left(D_1 \cosh \sqrt{b^2 + c^2}\, x + D_2 \sinh \sqrt{b^2 + c^2}\, x\right)$$

19. Solve $y^{(4)} + 2y^{(3)} + 3y'' + 2y' + y = 0$. *Hint:* First show that the auxiliary equation is $(r^2 + r + 1)^2 = 0$.

20. Solve $y'' - 2y' + 2y = 0$ and express your answer in the form $ce^{\alpha x} \sin(\beta x + \gamma)$. *Hint:* Let $\sin \gamma = C_1/c$ and $\cos \gamma = C_2/c$, where $c = \sqrt{C_1^2 + C_2^2}$.

21. Solve $x^2 y'' + 5xy' + 4y = 0$ by first making the substitution $x = e^z$.

22. Show that the substitution $x = e^z$ transforms the Euler equation $ax^2 y'' + bxy' + cy = 0$ to a homogeneous linear equation with constant coefficients.

23. Show that if r_1 and r_2 are distinct real roots of the auxiliary equation, then $y = C_1 e^{r_1 x} + C_2 e^{r_2 x}$ is a solution of $y'' + a_1 y' + a_2 y = 0$.

24. Show that if $\alpha \pm \beta i$ are complex conjugate roots of the auxiliary equation, then $y = C_1 e^{\alpha x} \cos \beta x + C_2 e^{\alpha x} \sin \beta x$ is a solution of $y'' + a_1 y' + a_2 y = 0$.

25. Recall that complex numbers have the form $a + bi$, where a and b are real. These numbers behave much like the real numbers, with the proviso that $i^2 = -1$. Show each of the following:

(a) $e^{bi} = \cos b + i \sin b$ *Hint:* Use the Maclaurin series for e^u, $\cos u$, and $\sin u$.

(b) $e^{a+bi} = e^a(\cos b + i \sin b)$

(c) $D_x e^{(\alpha+\beta i)x} = (\alpha + \beta i)e^{(\alpha+\beta i)x}$

26. Let the roots of the auxiliary equation $r^2 + a_1 r + a_2 = 0$ be $\alpha \pm \beta i$. From Problem 25c, it follows, just as in the real case, that $y = c_1 e^{(\alpha\pm\beta i)x} + c_2 e^{(\alpha-\beta i)x}$ satisfies $(D^2 + a_1 D + a_2)y = 0$. Show that this solution can be rewritten in the form

$$y = C_1 e^{\alpha x} \cos \beta x + C_2 e^{\alpha x} \sin \beta x$$

giving another approach to Theorem C.

CAS *Use a CAS to solve each of the following equations:*

27. $y'' - 4y' - 6y = 0;\ y(0) = 1,\ y'(0) = 2$

28. $y'' + 5y' + 6.25y = 0;\ y(0) = 2,\ y'(0) = -1.5$

29. $2y'' + y' + 2y = 0;\ y(0) = 0,\ y'(0) = 1.25$

30. $3y'' - 2y' + y = 0;\ y(0) = 2.5,\ y'(0) = -1.5$

Answers to Concepts Review: **1.** $r^2 + a_1 r + a_2 = 0$; complex conjugate roots **2.** $C_1 e^{-x} + C_2 e^x$ **3.** $(C_1 + C_2 x)e^x$ **4.** $C_1 \cos x + C_2 \sin x$

15.2 Nonhomogeneous Equations

Consider the general nonhomogeneous linear equation with constant coefficients

$$(1) \qquad y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = k(x)$$

Solving this equation can be reduced to three steps:

1. Find the general solution

   $$y_h = C_1 u_1(x) + C_2 u_2(x) + \cdots + C_n u_n(x)$$

   to the corresponding homogeneous equation (i.e., equation (1) with $k(x)$ being identically zero), as described in Section 15.1.
2. Find a particular solution y_p to the nonhomogeneous equation.
3. Add the solutions from Steps 1 and 2.

We state the result as a formal theorem.

Theorem A

If y_p is any particular solution to the nonhomogeneous equation

$$(2) \qquad L(y) = (D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n)\, y = k(x)$$

and if y_h is the general solution to the corresponding homogeneous equation, then

$$y = y_p + y_h$$

is the general solution of (2).

Proof The linearity of the operator L is the key element in the proof. Let y_p and y_h be as described. Then

$$L(y_p + y_h) = L(y_p) + L(y_h) = k(x) + 0$$

and so $y = y_p + y_h$ is a solution to (2).

Conversely, let y be any solution to (2). Then

$$L(y - y_p) = L(y) - L(y_p) = k(x) - k(x) = 0$$

and so $y - y_p$ is a solution to the homogeneous equation. Consequently, $y = y_p + (y - y_p)$ can be written as y_p plus a solution to the homogeneous equation, as we wished to show. ■

Now we apply this result to second-order equations.

The Method of Undetermined Coefficients The results of the previous section show us how to get the general solution to a homogeneous equation. The work lies in finding a particular solution to the nonhomogeneous equation. One such method of finding such a solution, the method of undetermined coefficients, involves making a conjecture, or educated guess, about the form of y_p, given the form of $k(x)$.

It turns out that the functions $k(x)$ most apt to occur in applications are polynomials, exponentials, sines, and cosines. For these functions, we offer a procedure for finding y_p based on trial solutions.

If $k(x) =$	Try $y_p =$
$b_m x^m + \cdots + b_1 x + b_0$	$B_m x^m + \cdots + B_1 x + B_0$
$be^{\alpha x}$	$Be^{\alpha x}$
$b \cos \beta x + c \sin \beta x$	$B \cos \beta x + C \sin \beta x$

Modification. If a term of $k(x)$ is a solution to the homogeneous equation, multiply the trial solution by x (or perhaps by a higher power of x).

To illustrate the table, we suggest the appropriate trial solution y_p in six cases. The first three are straightforward; the last three are modified because a term on the right side of the differential equation is present in the solution to the homogeneous equation.

1. $y'' - 3y' - 4y = 3x^2 + 2$ $\quad y_p = B_2 x^2 + B_1 x + B_0$
2. $y'' - 3y' - 4y = e^{2x}$ $\quad y_p = Be^{2x}$
3. $y'' + 4y = 2 \sin x$ $\quad y_p = B \cos x + C \sin x$
4. $y'' + 2y' = 3x^2 + 2$ $\quad y_p = B_2 x^3 + B_1 x^2 + B_0 x$
   (2 is a solution to the homogeneous equation)
5. $y'' - 3y' - 4y = e^{4x}$ $\quad y_p = Bxe^{4x}$
   (e^{4x} is a solution to the homogeneous equation)
6. $y'' + 4y = \sin 2x$ $\quad y_p = Bx \cos 2x + Cx \sin 2x$
   ($\sin 2x$ is a solution to the homogeneous equation)

Next we carry out the details in four specific examples.

EXAMPLE 1 Solve $y'' + y' - 2y = 2x^2 - 10x + 3$.

SOLUTION The auxiliary equation $r^2 + r - 2 = 0$ has roots -2 and 1, and so

$$y_h = C_1 e^{-2x} + C_2 e^x$$

To find a particular solution to the nonhomogeneous equation, we try

$$y_p = Ax^2 + Bx + C$$

Substitution of this expression in the differential equation gives

$$2A + (2Ax + B) - 2(Ax^2 + Bx + C) = 2x^2 - 10x + 3$$

Equating coefficients of x^2, x, and 1, we find that

$$-2A = 2, \qquad 2A - 2B = -10, \qquad 2A + B - 2C = 3$$

or $A = -1$, $B = 4$, and $C = -\frac{1}{2}$. Hence,

$$y_p = -x^2 + 4x - \frac{1}{2}$$

and

$$y = -x^2 + 4x - \frac{1}{2} + C_1e^{-2x} + C_2e^x$$

■

EXAMPLE 2 Solve $y'' - 2y' - 3y = 8e^{3x}$.

SOLUTION Since the auxiliary equation $r^2 - 2r - 3 = 0$ has roots -1 and 3, we have

$$y_h = C_1e^{-x} + C_2e^{3x}$$

Note that $k(x) = 8e^{3x}$ is a solution to the homogeneous equation. Thus, we use the *modified* trial solution

$$y_p = Bxe^{3x}$$

Substituting y_p in the differential equation gives

$$(Bxe^{3x})'' - 2(Bxe^{3x})' - 3Bxe^{3x} = 8e^{3x}$$

or

$$(9xBe^{3x} + 6Be^{3x}) - 2(3Bxe^{3x} + Be^{3x}) - 3Bxe^{3x} = 8e^{3x}$$

or, finally,

$$4Be^{3x} = 8e^{3x}$$

We conclude that $B = 2$ and

$$y = 2xe^{3x} + C_1e^{-x} + C_2e^{3x}$$

■

If 3 had been a double root of the auxiliary equation in Example 2, we would have used Bx^2e^{3x} as our trial solution.

EXAMPLE 3 Solve $y'' - 2y' - 3y = \cos 2x$.

SOLUTION The homogeneous equation agrees with that of Example 2, and so

$$y_h = C_1e^{-x} + C_2e^{3x}$$

For the trial solution y_p, we use

$$y_p = B\cos 2x + C\sin 2x$$

Now

$$Dy_p = -2B\sin 2x + 2C\cos 2x$$
$$D^2y_p = -4B\cos 2x - 4C\sin 2x$$

Hence, substitution of y_p in the differential equation gives (after collecting terms)

$$(-7B - 4C)\cos 2x + (4B - 7C)\sin 2x = \cos 2x$$

Thus, $-7B - 4C = 1$ and $4B - 7C = 0$, which imply that $B = -\frac{7}{65}$ and $C = -\frac{4}{65}$. We conclude that

$$y = -\frac{7}{65}\cos 2x - \frac{4}{65}\sin 2x + C_1e^{-x} + C_2e^{3x}$$

■

EXAMPLE 4 Solve $y'' - 2y' - 3y = 8e^{3x} + \cos 2x$.

SOLUTION Combining the results of Examples 2 and 3 and using the linearity of the operator $D^2 - 2D - 3$, we obtain

$$y = 2xe^{3x} - \frac{7}{65}\cos 2x - \frac{4}{65}\sin 2x + C_1e^{-x} + C_2e^{3x}$$ ■

The Method of Variation of Parameters A more general method than that of undetermined coefficients is the method of variation of parameters. If $u_1(x)$ and $u_2(x)$ are independent solutions to the homogeneous equation, then it can be shown (see Problem 23) that there is a particular solution to the nonhomogeneous equation of the form

$$y_p = v_1(x)u_1(x) + v_2(x)u_2(x)$$

where

$$v_1'u_1 + v_2'u_2 = 0$$
$$v_1'u_1' + v_2'u_2' = k(x)$$

We show how this method works in an example.

EXAMPLE 5 Find the general solution of $y'' + y = \sec x$.

SOLUTION The general solution to the homogeneous equation is

$$y_h = C_1 \cos x + C_2 \sin x$$

To find a particular solution to the nonhomogeneous equation, we set

$$y_p = v_1(x)\cos x + v_2(x)\sin x$$

and impose the conditions

$$v_1'\cos x + v_2'\sin x = 0$$
$$-v_1'\sin x + v_2'\cos x = \sec x$$

When we solve this system of equations for v_1' and v_2', we obtain $v_1' = -\tan x$ and $v_2' = 1$. Thus,

$$v_1(x) = \int(-\tan x)\,dx = \ln|\cos x|$$
$$v_2(x) = \int dx = x$$

(We can omit the arbitrary constants in the above integrations, since any solutions v_1 and v_2 will do.) A particular solution is therefore

$$y_p = (\ln|\cos x|)\cos x + x\sin x$$

a result that is easy to check by direct substitution in the original differential equation. We conclude that

$$y = (\ln|\cos x|)\cos x + x\sin x + C_1\cos x + C_2\sin x$$ ■

Concepts Review

1. The general solution to a nonhomogeneous equation has the form $y = y_p + y_h$, where y_p is a ________ and y_h is the general solution to the ________.

2. Thus, after noting that $y'' - y' - 6y = 6$ has the particular solution $y = -1$, we conclude that the general solution is $y =$ ________.

3. The method of undetermined coefficients suggests trying a particular solution of the form $y =$ ________________ for $y'' - y' - 6y = x^2$.

4. The method of undetermined coefficients suggests trying a particular solution of the form $y =$ ________________ for $y'' - y' - 6y = e^{3x}$.

Problem Set 15.2

In Problems 1–16, use the method of undetermined coefficients to solve each differential equation.

1. $y'' - 9y = x$

2. $y'' + y' - 6y = 2x^2$

3. $y'' - 2y' + y = x^2 + x$

4. $y'' + y' = 4x$

5. $y'' - 5y' + 6y = e^x$

6. $y'' + 6y' + 9y = 2e^{-x}$

7. $y'' + 4y' + 3y = e^{-3x}$

8. $y'' + 2y' + 2y = 3e^{-2x}$

9. $y'' - y' - 2y = 2\sin x$

10. $y'' + 4y' = \cos x$

11. $y'' + 4y = 2\cos 2x$

12. $y'' + 9y = \sin 3x$

13. $y'' + 9y = \sin x + e^{2x}$

14. $y'' + y' = e^x + 3x$

15. $y'' - 5y' + 6y = 2e^x$; $y = 1$, $y' = 0$ when $x = 0$

16. $y'' - 4y = 4\sin x$; $y = 4$, $y' = 0$ when $x = 0$

In Problems 17–22, solve each differential equation by variation of parameters.

17. $y'' - 3y' + 2y = 5x + 2$

18. $y'' - 4y = e^{2x}$

19. $y'' + y = \csc x \cot x$

20. $y'' + y = \cot x$

21. $y'' - 3y' + 2y = \dfrac{e^x}{e^x + 1}$

22. $y'' - 5y' + 6y = 2e^x$

23. Let $L(y) = y'' + by' + cy = 0$ have solutions u_1 and u_2, and let $y_p = v_1 u_1 + v_2 u_2$. Show that

$$L(y_p) = v_1(u_1'' + bu_1' + cu_1) + v_2(u_2'' + bu_2' + cu_2) + b(v_1'u_1 + v_2'u_2) + (v_1'u_1 + v_2'u_2)' + (v_1'u_1' + v_2'u_2')$$

Thus, if the conditions of the method of variation of parameters hold,

$$L(y_p) = (v_1)(0) + (v_2)(0) + (b)(0) + 0 + k(x) = k(x)$$

24. Solve $y'' + 4y = \sin^3 x$

Answers to Concepts Review: **1.** particular solution to the nonhomogeneous equation; homogeneous equation **2.** $-1 + C_1e^{-2x} + C_2e^{3x}$ **3.** $y = Ax^2 + Bx + C$ **4.** $y = Bxe^{3x}$

15.3 Applications of Second-Order Equations

Many problems in physics lead to second-order linear differential equations. We first consider the problem of a vibrating spring under various assumptions. Then we return to and generalize an earlier application to electric circuits.

A Vibrating Spring (Simple Harmonic Motion) Consider a coiled spring weighted by an object A and hanging vertically from a support, as in Figure 1a. We wish to consider the motion of the point P if the spring is pulled y_0 units below its equilibrium position (Figure 1b) and given an initial velocity of v_0. We assume friction to be negligible.

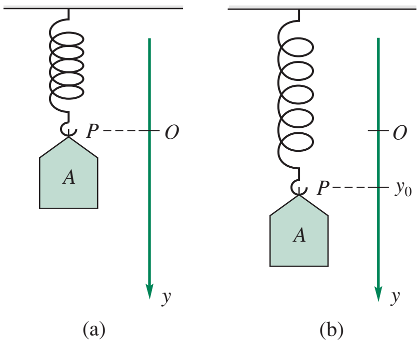

Figure 1

According to Hooke's Law, the force F tending to restore P to its equilibrium position at $y = 0$ satisfies $F = -ky$, where k is a constant depending on the characteristics of the spring and y is the y-coordinate of P. But by Newton's Second Law, $F = ma = (w/g)a$, where w is the weight of the object A, a is the acceleration of P, and g is the constant acceleration due to gravity ($g = 32$ feet per second per second). Thus,

$$\frac{w}{g}\frac{d^2y}{dt^2} = -ky. \qquad k > 0$$

is the differential equation of the motion. The solution y must satisfy the initial conditions $y(0) = y_0$ and $y'(0) = v_0$, where y_0 and v_0 are the initial position and initial velocity, respectively.

If we let $B^2 = kg/w = k/m$, then this equation takes the form

$$\frac{d^2y}{dt^2} + B^2y = 0$$

and has the general solution

$$y = C_1 \cos Bt + C_2 \sin Bt$$

The conditions $y = y_0$ and $y' = v_0$ at $t = 0$ determine the constants C_1 and C_2. If the object is released with an initial velocity of 0, then $C_1 = y_0$ and $C_2 = 0$. Thus,

$$y = y_0 \cos Bt$$

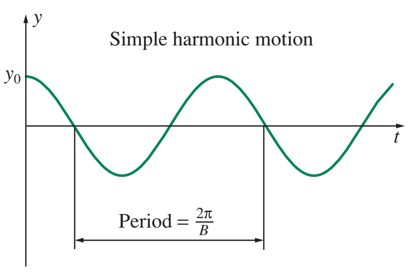

Figure 2

We say that the spring is executing **simple harmonic motion** with amplitude y_0 and period $2\pi/B$ (Figure 2).

EXAMPLE 1 When an object weighing 5 pounds is attached to the lowest point P of a spring that hangs vertically, the spring is extended 6 inches. The 5-pound weight is replaced by a 20-pound weight, and the system is allowed to come to equilibrium. If the 20-pound weight is now pulled downward another 2 feet and then released, describe the motion of the lowest point P of the spring.

SOLUTION The first sentence of the example allows us to determine the spring constant. By Hooke's Law, $|F| = ks$, where s is the amount in feet that the spring is stretched, and so $5 = k(\frac{1}{2})$, or $k = 10$. Now put the origin at the equilibrium point after the 20-pound weight has been attached. From the derivation just before the example, we know that $y = y_0 \cos Bt$. In the present case, $y_0 = 2$ and $B^2 = kg/w = (10)(32)/20 = 16$. We conclude that

$$y = 2 \cos 4t$$

The motion of P is simple harmonic motion, with period $\frac{1}{2}\pi$ and amplitude 2 feet. That is, P oscillates up and down from 2 feet below 0 to 2 feet above 0 and then back to 2 feet below 0 every $\frac{1}{2}\pi \approx 1.57$ seconds. ■

Damped Vibrations So far we have assumed a simplified situation, in which there is no friction either within the spring or resulting from the resistance of the air. We can take friction into account by assuming a retarding force that is proportional to the velocity dy/dt. The differential equation describing the motion then takes the form

$$\frac{w}{g}\frac{d^2y}{dt^2} = -ky - q\frac{dy}{dt}, \qquad k > 0, q > 0$$

By letting $E = qg/w$ and $B^2 = kg/w$, this equation can be written as

$$\frac{d^2y}{dt^2} + E\frac{dy}{dt} + B^2y = 0$$

an equation for which the methods of Section 15.1 apply. The auxiliary equation for this second-order linear differential equation is $r^2 + Er + B^2 = 0$, so the roots are

$$\frac{-E \pm \sqrt{E^2 - 4B^2}}{2}$$

We must consider the cases where $E^2 - 4B^2$ is negative, zero, and positive.

Case 1: $E^2 - 4B^2 < 0$ In this case, the roots are complex:

$$r = -\frac{E}{2} \pm \frac{i}{2}\sqrt{4B^2 - E^2} = -\alpha \pm \beta i$$

Notice that α and β will both be positive. The general solution of the differential equation is thus

$$y = e^{-\alpha t}(C_1 \cos \beta t + C_2 \sin \beta t)$$

which can be written in the form (see Problem 15)

$$y = Ae^{-\alpha t} \sin(\beta t + \gamma)$$

The factor $e^{-\alpha t}$, called the **damping factor,** causes the amplitude of the motion to approach zero as $t \to \infty$ (Figure 3a). ■

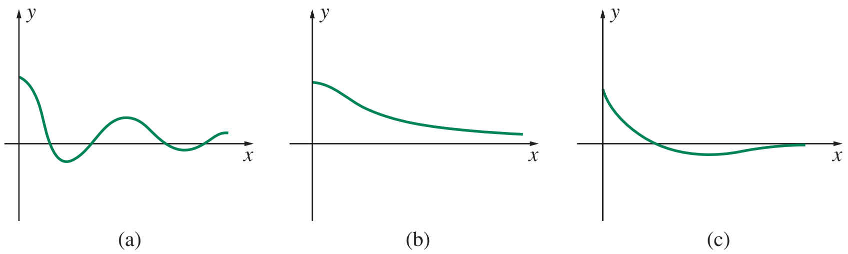

Figure 3

Case 2: $E^2 - 4B^2 = 0$ In this case, the auxiliary equation has the double root $-\alpha$ where $\alpha = E/2$ and the general solution of the differential equation is

$$y = C_1 e^{-\alpha t} + C_2 t e^{-\alpha t}$$

The motion described by this equation is said to be **critically damped.** ■

Case 3: $E^2 - 4B^2 > 0$ The auxiliary equation has roots $-\alpha_1$ and $-\alpha_2$, and the general solution of the differential equation is

$$y = C_1 e^{-\alpha_1 t} + C_2 e^{-\alpha_2 t}$$

It describes a motion that is said to be **overdamped.**

The graphs in the critically damped and overdamped cases cross the t-axis at most once and may look something like Figure 3b or 3c. ■

EXAMPLE 2 If a damping force with $q = 0.2$ is imposed on the system of Example 1, find the equation of motion.

SOLUTION $E = qg/w = (0.2)(32)/20 = 0.32$ and $B^2 = (10)(32)/20 = 16$, so we must solve

$$\frac{d^2y}{dt^2} + 0.32\frac{dy}{dt} + 16y = 0$$

The auxiliary equation $r^2 + 0.32r + 16 = 0$ has roots $r = -0.16 \pm \sqrt{15.9744}i \approx -0.16 \pm 4i$, and thus

$$y = e^{-0.16t}(C_1 \cos 4t + C_2 \sin 4t)$$

When we impose the conditions $y = 2$ and $y' = 0$ at $t = 0$, we find that $C_1 = 2$ and $C_2 = 0.08$. Consequently,

$$y = e^{-0.16t}(2 \cos 4t + 0.08 \sin 4t)$$ ■

Electric Circuits

Consider a circuit (Figure 4) with a resistor (R ohms), an inductor (L henrys), and a capacitor (C farads) in series with a source of electromotive force supplying $E(t)$ volts. The new feature in comparison to the circuits of Section 6.6 is the presence of a capacitor. Kirchhoff's Law in this situation says that the charge Q on the capacitor, measured in coulombs, satisfies

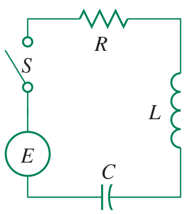

Figure 4

(1) $$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E(t)$$

The current $I = dQ/dt$, measured in amperes, satisfies the equation obtained by differentiating equation (1) with respect to t; that is,

(2) $$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = E'(t)$$

These equations can be solved by the methods of Sections 15.1 and 15.2 for many functions $E(t)$.

EXAMPLE 3 Find the charge Q and the current I as functions of time t in an RCL circuit (Figure 4) if $R = 16$, $L = 0.02$, $C = 2 \times 10^{-4}$, and $E = 12$. Assume that $Q = 0$ and $I = 0$ at $t = 0$ (when the switch is closed).

SOLUTION By Kirchhoff's Law as expressed in equation (1),

$$\frac{d^2Q}{dt^2} + 800\frac{dQ}{dt} + 250{,}000Q = 600$$

The auxiliary equation has roots

$$\frac{-800 \pm \sqrt{640{,}000 - 1{,}000{,}000}}{2} = -400 \pm 300i$$

so

$$Q_h = e^{-400t}(C_1 \cos 300t + C_2 \sin 300t)$$

By inspection, a particular solution is $Q_p = 2.4 \times 10^{-3}$. Therefore, the general solution is

$$Q = 2.4 \times 10^{-3} + e^{-400t}(C_1 \cos 300t + C_2 \sin 300t)$$

When we impose the given initial conditions, we find that $C_1 = -2.4 \times 10^{-3}$ and $C_2 = -3.2 \times 10^{-3}$. We conclude that

$$Q = 10^{-3}\left[2.4 - e^{-400t}(2.4 \cos 300t + 3.2 \sin 300t)\right]$$

and, by differentiation, that

$$I = \frac{dQ}{dt} = 2e^{-400t} \sin 300t$$

Concepts Review

1. A spring that vibrates without friction might obey a law of motion such as $y = 3 \cos 2t$. We say it is executing simple harmonic motion with amplitude ________ and period ________.

2. A spring vibrating in the presence of friction might obey a law of motion such as $y = 3e^{-0.1t} \cos 2t$, called damped harmonic motion. The "period" is still ________ , but now the amplitude ________ as time increases.

3. If the friction is very great, the law of motion might take the form $y = 3e^{-0.1t} + te^{-0.1t}$, the critically damped case, in which y slowly fades to ________ as time increases.

4. Kirchhoff's Law says that a(n) ________ satisfies a second-order linear differential equation.

Problem Set 15.3

1. A spring with a spring constant k of 250 newtons per meter is loaded with a 10-kilogram mass and allowed to reach equilibrium. It is then raised 0.1 meter and released. Find the equation of motion and the period. Neglect friction.

2. A spring with a spring constant k of 100 pounds per foot is loaded with a 1-pound weight and brought to equilibrium. It is then stretched an additional 1 inch and released. Find the equation of motion, the amplitude, and the period. Neglect friction.

3. In Problem 1, what is the absolute value of the velocity of the moving weight as it passes through the equilibrium position?

4. A 10-pound weight stretches a spring 4 inches. This weight is removed and replaced with a 20-pound weight, which is then allowed to reach equilibrium. The weight is next raised 1 foot and released with an initial velocity of 2 feet per second downward. What is the equation of motion? Neglect friction.

5. A spring with a spring constant k of 20 pounds per foot is loaded with a 10-pound weight and allowed to reach equilibrium. It is then displaced 1 foot downward and released. If the weight experiences a retarding force in pounds equal to one-tenth the velocity, find the equation of motion.

6. Determine the motion in Problem 5 if the retarding force equals four times the velocity at every point.

7. In Problem 5, how long will it take the oscillations to diminish to one-tenth their original amplitude?

8. In Problem 5, what will be the equation of motion if the weight is given an upward velocity of 1 foot per second at the moment of release?

9. Using Figure 5, find the charge Q on the capacitor as a function of time if S is closed at $t = 0$. Assume that the capacitor is initially uncharged.

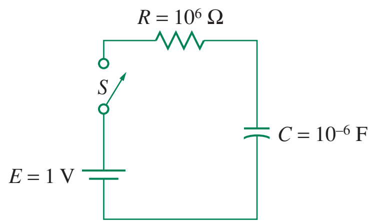

Figure 5

10. Find the current I as a function of time in Problem 9 if the capacitor has an initial charge of 4 coulombs.

11. Use Figure 6.

(a) Find Q as a function of time. Assume that the capacitor is initially uncharged.

(b) Find I as a function of time.

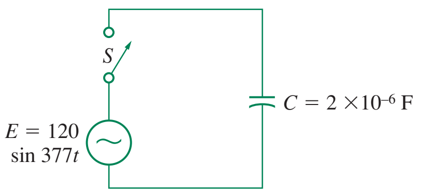

Figure 6

12. Using Figure 7, find the current as a function of time if the capacitor is initially uncharged and S is closed at $t = 0$. *Hint:* The current at $t = 0$ will equal 0, since the current through an inductance cannot change instantaneously.

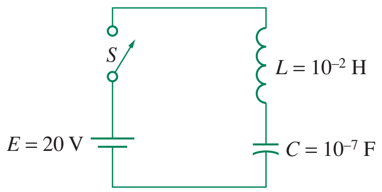

Figure 7

13. Using Figure 8, find the steady-state current as a function of time; that is, find a formula for I that is valid when t is very large ($t \to \infty$).

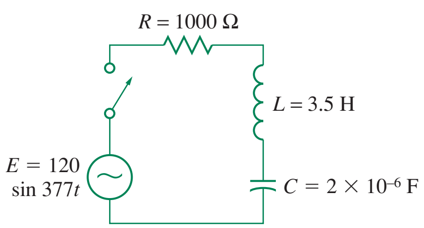

Figure 8

14. Suppose that an undamped spring is subjected to an external periodic force so that its differential equation has the form

$$\frac{d^2 y}{dt^2} + B^2 y = c \sin At, \qquad c > 0$$

(a) Show that the equation of motion for $A \neq B$ is

$$y = C_1 \cos Bt + C_2 \sin Bt + \frac{c}{B^2 - A^2} \sin At$$

(b) Solve the differential equation when $A = B$ (the resonance case).

(c) What happens to the amplitude of the motion in part (b) when $t \to \infty$?

15. Show that $C_1 \cos \beta t + C_2 \sin \beta t$ can be written in the form $A \sin(\beta t + \gamma)$. *Hint:* Let $A = \sqrt{C_1^2 + C_2^2}$, $\sin \gamma = C_1/A$, and $\cos \gamma = C_2/A$.

16. Show that the motion of part (a) of Problem 14 is periodic if B/A is rational.

17. Refer to Figure 9, which shows a pendulum bob of mass m supported by a weightless wire of length L. Derive the equation of motion; that is, derive the differential equation satisfied by θ. *Suggestion:* Use the fact from Section 11.7 that the scalar tangential component of the acceleration is d^2s/dt^2, where s measures arc length in the counterclockwise direction.

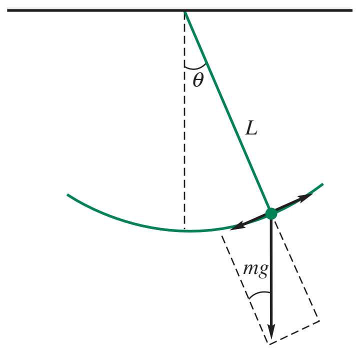

Figure 9

18. The equation derived in Problem 17 is nonlinear, but for small θ it is customary to approximate it by the equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0$$

Here $g = GM/R^2$, where G is a universal constant, M is the mass of the earth, and R is the distance from the pendulum to the center of the earth. Two clocks, with pendulums of length L_1 and L_2 and located at distances R_1 and R_2 from the center of the earth, have periods p_1 and p_2, respectively.

(a) Show that $\dfrac{p_1}{p_2} = \dfrac{R_1\sqrt{L_1}}{R_2\sqrt{L_2}}$.

(b) Find the height of a mountain if a clock that kept perfect time at sea level ($R = 3960$ miles) with $L = 81$ inches had to have its pendulum shortened to $L = 80.85$ inches to keep perfect time at the top of the mountain.

Answers to Concepts Review: **1.** 3; π **2.** π; decreases **3.** 0 **4.** electric circuit

15.4 Chapter Review

Concepts Test

Respond with true or false to each of the following assertions. Be prepared to justify your answer.

1. $y'' + y^2 = 0$ is a linear differential equation.

2. $y'' + x^2 y = 0$ is a linear differential equation.

3. $y = \tan x + \sec x$ is a solution of $2y' - y^2 = 1$.

4. The general solution to $\left[D^2 + aD + b\right]^3 y = 0$ should involve eight arbitrary constants.

5. D^2 is a linear operator.

6. If $u_1(x)$ and $u_2(x)$ are two solutions to $y'' + a_1y' + a_2y = f(x)$, then $C_1u_1(x) + C_2u_2(x)$ is also a solution.

7. The general solution to $y''' + 3y'' + 3y' + y = 0$ is $y = C_1e^{-x} + C_2xe^{-x} + C_3x^2e^{-x}$.

8. If $u_1(x)$ and $u_2(x)$ are solutions to the linear differential equation $L(y) = f(x)$, then $u_1(x) - u_2(x)$ is a solution to $L(y) = 0$.

9. The equation $y'' + 9y = 2 \sin 3x$ has a particular solution of the form $y_p = B \sin 3x + C \cos 3x$.

10. An expression of the form $C_1 \cos \beta t + C_2 \sin \beta t$ can always be written in the form $A \sin(\beta t + \gamma)$.

Sample Test Problems

In Problems 1–13, solve each differential equation.

1. $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} = e^x$ *Suggestion:* Let $u = dy/dx$.

2. $y'' - y = 0$

3. $y'' - 3y' + 2y = 0,\ y = 0,\ y' = 3$ when $x = 0$

4. $4y'' + 12y' + 9y = 0$

5. $y'' - y = 1$

6. $y'' + 4y' + 4y = 3e^x$

7. $y'' + 4y' + 4y = e^{-2x}$

8. $y'' + 4y = 0;\ y = 0,\ y' = 2$ when $x = 0$

9. $y'' + 6y' + 25y = 0$

10. $y'' + y = \sec x \tan x$

11. $y''' + 2y'' - 8y' = 0$

12. $y^{(4)} - 3y'' - 10y = 0$

13. $y^{(4)} - 4y'' + 4y = 0$

14. Suppose that glucose is infused into the bloodstream of a patient at the rate of 3 grams per minute, but that the patient's body converts and removes glucose from its blood at a rate proportional to the amount present (with constant of proportionality 0.02). If $Q(t)$ is the amount present at time t and $Q(0) = 120$,

(a) write the differential equation for Q;

(b) solve this differential equation;

(c) determine what happens to Q in the long run.

15. A spring with a spring constant k of 5 pounds per foot is loaded with a 10-pound weight and allowed to reach equilibrium. It is then raised 1 foot and released. What are the equation of motion, the amplitude, and the period? Neglect friction.

16. In Problem 15, what is the absolute value of the velocity of the moving weight as it passes through the equilibrium position?

17. Suppose the switch of the circuit in Figure 1 is closed at $t = 0$. Find I as a function of time if C is initially uncharged. (The current at $t = 0$ will equal zero, since current through an inductance cannot change instantaneously.)

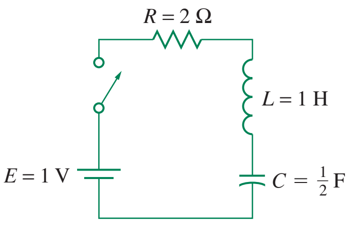

Figure 1